NEW $N=6$ INFINITE-DIMENSIONAL SUPERALGEBRA WITH CENTRAL EXTENSION

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Received 27 April 1989

We consider doubly infinite superalgebras describing the $D=2$ field theories with locally gauged super Virasoro algebras Our superalgebras belong to the class of generalized loop algebras $S\to g^{1|N}$, where $g^{1|N}$ is the infinite-dimensional algebra of superdiffeomorphisms of the supercircle $S^{1|N}$ It appears that among such doubly infinite superalgebras only the one with $g^{1|N}=K^{NS}(1|N)$, where $N=6$ ($N=6$ Neveu-Schwarz conformal superalgebra), admits a central extension due to the existence of a Killing form Possible applications are outlined

1. For the two-dimensional world, in particular string theories, the N-extended superconformal algebra (for all N) is infinite-dimensional Usually one constructs the locally superconformal two-dimensional theories by gauging only their finite $OSp(2|N)\oplus OSp(2|N)$ superalgebra [1] Such a result can be understood as imposing the constraints, which trivialize the local invariance with respect to the remaining infinite number of degrees of freedom [2] In this letter we shall consider generalized Kac–Moody algebras with full N-extended Virasoro superalgebras as describing the local gauge degrees of freedom It is known (see e g ref [3]) that the central extension of Kac–Moody algebras is due to the existence of the nondegenerate even symmetric invariant bilinear form, which is the Killing form Interestingly enough, among simple Lie superalgebras of vector-fields on a supercircle, only one, i e the $N=6$ Neveu–Schwarz conformal superalgebra, has nonvanishing Killing form We shall show below, after some mathematical preliminaries, that the "quantum" version, with nonvanishing central extension obtained through the Killing form, exists only for $N=6$ super Virasoro gauge algebra (we consider here only the Neveu–Schwarz boundary conditions)

Let us recall briefly the infinite-dimensional superalgebras of vector fields on the superline $R^{1|N}$ and the supercircle $S^{1|N}$ [4–7]

(i) *Superline* $R^{1|N}$ We denote the polynomial algebra by $R(N)=R[x,\theta_1, \ldots ,\theta_N]$ We define

$$W_N=\mathrm{vect}(1|N) =\left\{\mathrm{der}\, R(N)\quad \mathscr{D}=F\frac{\partial}{\partial x}+F_i\frac{\partial}{\partial\theta_i}\right\} \quad (1a)$$

(general superalgebra of vector fields), with $F, F_1, \ldots, F_N\in R(N)$, $i=1, \ldots, N$ and summation over the repeated indices is understood,

$$S_N=\mathrm{svect}(1|N)=\{\mathscr{D}\in \mathrm{vect}(1|N),\ \mathrm{div}\,\mathscr{D}=0\} \quad (1b)$$

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(North-Holland Physics Publishing Division)

(special or supervolume-preserving superalgebra), where $\mathrm{div}(F\partial/\partial x + F_i\,\partial/\partial\theta_i) = \partial F/\partial x + (-1)^{P(F_i)}\partial F_i/\partial\theta_i$ and $P(F_i)$ is the parity of the superfunction,

$$\mathrm{K}_N = \mathrm{K}(1|N)$$
$$= \left\{ \mathrm{D}_F = \Delta(F)\frac{\partial}{\partial x} + \frac{\partial F}{\partial x}\theta_i\frac{\partial}{\partial\theta_i} + (-1)^{P(F)}\frac{\partial F}{\partial\theta_i}\frac{\partial}{\partial\theta_i} \right\} \quad (1c)$$

(contact superalgebra ≡ O(N)-conformal superalgebra), where $\Delta(F) = 2F - \theta_i\,\partial F/\partial\theta_i$

The superalgebras (1a)–(1c) and their formal or smooth versions [i e R(N) replaced by formal power series or smooth functions] are the Lie superalgebras of vector fields on the superline

(ii) *Supercircles* $S^{1|N}$ *and its Mobius double covering* $S^{1|N-1\,M}$ We describe the superspace of functions on $S^{1|N}$ by Laurent power series $R^{\varrho}(N) = C[x^{-1}, x, \theta_1, \ldots, \theta_N]$, i e by Fourier series ($x = e^{i\varphi}$, where φ is the angle parameter)

If we consider $S^{1|N}$ as a trivial bundle of rank N over S^1, and replace R(N) by $R^{\varrho}(N)$ in the above definitions, we get the superalgebras vect$^{\varrho}(1|N)$, svect$^{\varrho}(1|N)$ and $K^{\varrho}(1|N)$ Unlike svect$(1|N)$, the superalgebras svect$^{\varrho}(1|N)$ admit a deformation, $\mathrm{svect}^{\varrho}_{\lambda}(1|N) = \{\mathscr{D} \in \mathrm{vect}^{\varrho}(1|N), \mathrm{div}\, x^{\lambda}\mathscr{D} = 0\}$, found recently by Schwimmer and Seiberg [8]

The contact superalgebras $K^{\varrho}(1|N)$ are known in the physics literature as Neveu–Schwarz (NS) [9] conformal superalgebras ($K^{\varrho}(1|N) \equiv K^{NS}(1|N)$)

One can also consider $S^{1|N-1\,M}$ with the Grassmann sector describing a nontrivial bundle over S^1 It appears that by change of variables, such nontrivial bundles can be described by the Whitney sum of the trivial bundle of rank $N-1$ with the Mobius bundle over S^1 with the coordinate, say, θ_N in the fibre The vector fields (1c) in such a case are modified as follows

$$\mathrm{K}^{\mathrm{R}}(1|N)$$
$$= \left\{ \mathrm{D}^{\mathrm{R}}_F = \Delta(F)\mathscr{D}_x + \mathscr{D}_x F + (-1)^{P(F)}\frac{\partial F}{\partial\theta_i}\frac{\partial}{\partial\theta_i} \right\}, \quad (1d)$$

where $\mathscr{D}_x = \partial/\partial x - (\theta_i/2x)\,\partial/\partial\theta_i$

The superalgebra of vector fields (1d) can be called Ramond [10] conformal superalgebra $K^{R}(1|N)$

Further in this paper we shall discuss only trivial bundles over S^1 (the NS sector of conformal superalgebras)

Recently it has been shown [6,7,11] that the following infinite-dimensional superalgebras admit central extension

$\mathrm{vect}^{\varrho}(1|N)$ $N = 1, 2$,

$\mathrm{svect}^{\varrho}_{\lambda}(1|N)$ $N = 1, 2$,

$K^{NS}(1|N)$ $N = 1, 2, 3, 4$

Our aim here is to propose a new "doubly infinite" superalgebra which admits a central extension due to the existence of a Killing form for the infinite-dimensional target superalgebra

2. In order to construct our example, let us consider the superextensions of loop algebras, describing the smooth mappings $S^1 \to \tilde{G}$, where $\tilde{G}$ is a supergroup and $\tilde{g}$ is the corresponding superalgebra If we introduce the corresponding $D=2$ current algebra with the superalgebra $\tilde{g}$-valued currents $J(\tau,\sigma) \equiv J^a(\tau,\sigma)T_a$, where $[T_a, T_b] = f_{ab}{}^c T_c$ describes the adjoint matrix representation of $\tilde{g}$, the "quantum" extension of the superloop algebra, which is a supersymmetric extension of the Kac–Moody algebra [#1], looks as follows

$$[J_a(\sigma), J_b(\sigma')] = i\delta(\sigma-\sigma')f_{ab}{}^c J_c(\sigma')$$
$$+ \frac{1}{2\pi}\kappa\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma')g_{ab}, \quad (2)$$

where $J_a(\sigma) \equiv J_a(0,\sigma)$ and $g_{ab} = \mathrm{str}(T_a T_b)$ describes the Killing metric on the superalgebra $\tilde{g}$

Below we shall consider the generalized superloop algebras with $\tilde{g}$ describing the *infinite-dimensional* superalgebras of vector fields, discussed in (i) Our aim here is to consider the central extensions for such

[#1] It should be mentioned that the Kac–Moody algebra, describing the smooth mapping $S^1 \to G$, where G is a compact Lie group, can be extended in two ways

(i) by replacing the circle S^1 by a supercircle $S^{1|N}$, parametrized by S^1 and N real Grassmann coordinates It can be argued [12] that such supersymmetric extension of Kac–Moody algebras *with central extension* exists *only* for $N = 1$,

(ii) by replacing the group G by a supergroup $\tilde{G}$, i e by introducing in the loop algebra the structure constants of the Lie superalgebra $\tilde{g}$ (see e g refs [13,14]) The supersymmetric extension described by the formula (2) is of this type generalized superloop algebras, in particular, for the choice $\tilde{g}=K^{NS}(1|N)$ We observe that the vector fields (1c) can be written in the form

$$D_F=2F(X)\frac{\partial}{\partial x}+(-1)^{P(F)}D_iF\,D_i\,, \tag{3}$$

where $X=(x,\theta_1,\theta_2,\ \ ,\theta_n)$ and $D_i=\partial/\partial\theta_i+\theta_i\partial/\partial x$ It is easy to calculate that the superalgebra $K^{NS}(1|N)$ takes the form [6]

$$[D_F, D_G]=D_{\{F\,G\}}\,, \tag{4a}$$

where

$$\{F,G\}=2\left(F\frac{\partial G}{\partial x}-G\frac{\partial F}{\partial x}\right)+(-1)^{P(F)}D_iF\,D_iG \tag{4b}$$

3. There are two ways of writing down our superextension

(i) To consider the supergroup indices a, b and c in (2) as describing the infinite discrete basis of $\mathrm{vect}^{\varrho}(1|N)$, $\mathrm{svect}^{\varrho}_{\lambda}(1|N)$, or $K^{NS}(1|N)$, obtained e g by taking the Laurent expansions on the supercircle $S^{1|N}$ For example, for $K^{NS}(1|N)$ the supergroup indices are described by 2^N-triples of integer numbers $(n_1,\ \ ,n_r)$ with $r=2^N$

(ii) One can equivalently replace the discrete indices (a,b,c) in eq (2) by a continuous index $X=(x,\theta_1,\theta_2,\ \ ,\theta_N)$ describing the parametrization of the supercircle $S^{1|N}$, e g in the case of $\tilde{g}=K^{NS}(1|N)$ by a substitution

$$J_a(\sigma)\to J(\sigma,X)\equiv J(\sigma,x,\theta_1,\ \ ,\theta_N) \tag{5}$$

in eq (2), with the structure constants $f_{ab}{}^c$ replaced by the structure distributions of the $K^{NS}(1|N)$ superalgebra written in a local supercurrent form (see e g refs [7,12]) We propose the following superalgebra for the supercurrents $J(\sigma, X)$

$$\begin{aligned}
&[J(\sigma,X),J(\sigma',X')]\\
&\quad=i\delta(\sigma-\sigma')\,[D_i\delta(X-X')D_iJ(\sigma',X')\\
&\quad-2\delta(X-X')\;J'(\sigma',x')\\
&\quad-(4-N)\delta'(X-X')J(\sigma',X')]\\
&\quad+\frac{1}{2\pi}\kappa\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma')g(X,X')\,,
\end{aligned} \tag{6}$$

where $J'=(\partial/\partial x)J$, $\delta'=(\partial/\partial x)\delta$

It is easy to see that the supercurrents $J(\sigma; X)$ in eq (6) have the conformal dimension [#2] $d=3-N/2$, while the dimension of the symmetric metric $g(x,X')$ is equal to $4-N$ There are the following two local choices of $g(X,X')$ consistent with the choice of dimensionless parameter κ

$$N=2\quad g(X,X')=\delta''(X-X')\,, \tag{7a}$$

$$N=6\quad g(X,X')=\delta(X-X') \tag{7b}$$

One can show, however, that only the choice (7b) is consistent, i e defines an invariant bilinear symmetric nondegenerate (Killing) form on $K^{NS}(1|N)$ Indeed, if we introduce for $N=6$ the scalar product

$$\langle F,G\rangle=\int \mathrm{d}v^N_{x\,\theta}\,F(X)G(X)\,, \tag{8}$$

where $\mathrm{d}v^N_{x\,\theta}$ is the volume element in the coordinates $x,\theta_1,\ \ ,\theta_N$ [15], one can show that

$$\langle\delta F,G\rangle+\langle F,\delta G\rangle=0\,, \tag{9a}$$

or equivalently,

$$\langle\{F,G\},H\rangle=\langle F,\{G,H\}\rangle\,, \tag{9b}$$

where { , } is defined by (4b)

It follows from our construction that for $N=6$ the superalgebra (6) with $g(X,X')$ given by (7b) satisfies the Jacobi identity, or equivalently, the central extension term satisfies the 2-cocycle integrability condition

The superalgebra (6) for $N=6$ can be described by an infinite sequence of $N=6$ superconformal currents $J_n(X)$ $[n=0,\pm1,\pm2,\ \ ,X=(x,\theta_1,\ \ ,\theta_6)]$, where

$$J(\sigma,X)=\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\mathrm{e}^{-in\sigma}J_n(X) \tag{10}$$

and

$$\begin{aligned}
&[J_n(X),J_m(X')]\\
&\quad=i\,[D_i\delta(X-X')D_iJ_{n+m}(X')\\
&\quad-2\delta(X-X')J'_{n+m}(X')+2\delta'(X-X')J_{n+m}(X')]\\
&\quad+\kappa n\delta_{n+m\,0}\delta(X-X')\,,
\end{aligned} \tag{11}$$

[#2] We recall that in the mass dimensions, $\dim\delta(\sigma)$, $\delta(x)$, $\partial/\partial\sigma$, $\partial/\partial x=1$, $\dim\theta=-\frac{1}{2}$, $\dim \mathrm{d}\theta=\frac{1}{2}$ and $\dim D_i=\frac{1}{2}$

[with $J' \equiv (\partial/\partial x)J$ and $\delta' \equiv (\partial/\partial x)\delta$]

If we expand the supercurrent $J_n(X)$ in the Laurent modes on $S^{1|N}$, we obtain the description of our superalgebra in a discrete basis [see (i) in this section]

4. It is known that the Poisson–Lie brackets described by the N-extended super Virasoro algebra provides the supersymmetric extensions of the KdV equation [16,12] In order to describe an analogous hamiltonian flow on coadjoint orbits of our new superalgebra (6) for $N=6$, one should classify the hamiltonians with conformal dimension three, corresponding to the bilinear hamiltonian of the second KdV structure As the simplest choice, we obtain

$$H=\int dv^6_{\sigma\, x\, \theta}\, \partial_\sigma J(\sigma, X)\, \partial_\sigma J(\sigma, X) \tag{12}$$

It is interesting to observe that for the $N=6$ superalgebra (6) one can write the physical hamiltonian, of dimension one, generalizing in the simplest way the Sugawara hamiltonian $H=\int d\sigma\, j^a j_a$, as follows

$$H=\frac{\pi}{\kappa}\int dv^6_{\sigma\, x\, \theta}\, J(\sigma, X) J(\sigma, X) \tag{13}$$

The supersymmetric equation of motion, with one-dimensional $N=6$ supersymmetry (in the variable X), describing the time evolution in the space with three bosonic variables (τ, σ, x) and six Grassmann variables, which corresponds to the choice of Poisson–Lie bracket given by (6) (with $N=6$) and the hamiltonian (13), looks surprisingly simple as follows

$$\partial_\tau \mathscr{J}(\tau, \sigma, X) + \partial_\sigma \mathscr{J}(\tau, \sigma, X) = \frac{2\pi}{\kappa}(6-N)\mathscr{J}\partial_\sigma \mathscr{J} = 0\,, \tag{14}$$

where $\mathscr{J}(\tau, \sigma, X)|_{\tau=0} \equiv J(\sigma, X)$ and the relation $(D_i \mathscr{J})^2 = 0$ has been used The general solution of (14) is given by $\mathscr{J}=f(\tau-\sigma, X)$, where f is an arbitrary $D=2$, $N=6$ superfield

5 Our basic superalgebra (6) describes the supersymmetric two-dimensional local gauge theory with infinite-dimensional N-extended superconformal algebra as the graded gauge algebra We see the following several possible applications

(a) Following ref [2], if one restricts the representations of the superalgebra (6) by suitable constraints, one can obtain $D=2$, N-extended superconformal gravity, as well as other locally N-extended superconformal field theories in two dimensions In such an approach, only a finite number of $D=2$ gauge fields are not pure gauge degrees of freedom

(b) One can also consider the models with nontrivial gauging of infinite number of supercurrents $J_n(X)$ [see (10)] In such a case, our basic superalgebra (6) describes "membrane-like" supersymmetry, with asymmetric treatment of two continuous variables one, denoted by x, is supersymmetrized and carries the representations of $K^{NS}(1|N)$ (N-extended superconformal symmetry), while the second one, denoted by σ, is invariant under supersymmetry and describes the local density of the internal symmetry charges We recall that in the light-cone gauge the (super)membrane with spherical topology has been described as a one-dimensional (super) Yang–Mills theory with infinite-dimensional gauge group of symplectic (area-preserving) diffeomorphisms [17,18] Our choice, described by eq (11), corresponds to a different type of supermembrane solutions, describing the $N=6$ supermembrane with the topology $S^1 \times S^{1|6}$ as an infinite family of $N=6$ superstrings

It should also be mentioned that

(c) One can show that for other infinite-dimensional superalgebras of vector fields on $S^{1|N}$, i e for $\mathrm{vect}^{\varrho}(1|N)$ and $\mathrm{svect}^{\varrho}_{\lambda}$ the Killing form does not exist for any N ($N=0, 1, 2,$) We see, therefore, that the generalized superloop algebras with $\tilde{g}$ described by $\mathrm{vect}^{\varrho}(1|N)$ and $\mathrm{svect}^{\varrho}_{\lambda}(1|N)$ do not permit a central extension via the Killing form In that sense the choice of $K^{NS}(1|6)$ corresponding to our algebra (6), is unique

(d) The central extension of generalized superloop algebra can be also obtained if the target algebra has an extension The choice of superalgebras $\tilde{g}$ for which such a central extension exists are listed at the end of section 1

(e) The existence of the nondegenerate symmetric form for our $N=6$ superalgebra permits to consider the solutions of the Yang–Baxter equation [19,20] with values in the $N=6$ super Virasoro algebra (see ref [21])

Two of us (M C and J L) are grateful to Z Hasıewıcz and W Troost for several clarıfyıng dıscussıons

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