Yuri Manin

Introduction into theory of schemes

Translated from the Russian and edited
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Summary. This book provides one with a concise but extremely lucid exposition of basics of algebraic geometry seasoned by illuminating examples.

The preprints of this book proved useful to students majoring in mathematics and modern mathematical physics, as well as professionals in these fields.

Contents

Editor's preface

It is for more than 40 years now that I wanted to make these lectures — my first love — available to the reader: The preprints of these lectures [Ma1, Ma2] of circulation of mere 500+200 copies became bibliographic rarities almost immediately. Meanwhile the elements of algebraic geometry became everyday language of working theoretical physicists and the need in a concise manual only increased. Various (nice) text-books are usually too thick for anybody who does not want to become a professional algebraic geometer, which makes Manin's lectures even more appealing.

The methods described in these lectures became working tools of theoretical physicists whose subject ranges from high energy physics to solid body physics (see [Del] and [Ef], respectively) — in all questions where supersymmetry naturally arises. In mathematics, supersymmetry in in-build in everything related to homotopy and exterior products of differential forms, hence to (co)homology. In certain places, supersymmetric point of view is *inevitable*, as in the study of integrability of certain equations of mathematical physics ([MaG]). Therefore, preparing proceedings of my *Seminar on Supermanifolds* (see [SoS]) I urgently needed a concise and clear introduction into basics of algebraic geometry.

In 1986 Manin wrote me a letter allowing me to include a draft of this translation as a Chapter in [SoS], and it was preprinted in Reports of Department of Mathematics of Stockholm University. My 1972 definition of superschemes and supermanifolds [L0] was based on these lectures; they are the briefest and clearest source of the background needed for studying those aspects of supersymmetries that can not be reduced to linear algebra.

Written at the same time as Macdonald's lectures [M] and Mumford's lectures [M1, M2, M3] Manin's lectures are more lucid and easier to come to grasps. Later on there appeared several books illustrating the topic from different positions, e.g., [AM], [Sh0], [E1, E2, E3], [H, Kz] and I particularly recommend [Reid] for the first reading complementary to this book.

For preliminaries on algebra, see [vdW, Lang]; on sheaves, see [God, KaS]; on topology, see [K, FFG, RF, Bb3]; on number theory, see [Sh1, Sh2].

In this book, $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$; $\mathbb{F}_m := \mathbb{Z}/m$; $A^\times$ is the group of invertible elements of the ring A. The term "identity" is only applied to relations and maps, the element 1 (sometimes denoted by e) such that $1a = a = a1$ is called unit (sometimes unity). Other notation is standard.

I advise the reader to digress from the main text and skim the section on categories as soon as the word "functor" or "category" appears for the first time.

The responsibility for footnotes is mine.

D. Leites, April 2, 2009.

Author's preface

In 1966–68, at the Department of Mechanics and Mathematics of Moscow State University, I read a two-year-long course in algebraic geometry. The transcripts of the lectures of the first year were preprinted [Ma1, Ma2] (they constitute this book), that of the second year was published in the Russian Mathematical Surveys [Ma3]. These publications bear the remnants of the lecture style with its pros and cons.

Our goal is to teach the reader to practice the geometric language of commutative algebra. The necessity to separately present algebraic material and later "apply" it to algebraic geometry constantly discouraged geometers, see a moving account by O. Zariski and P. Samuel in their preface to [ZS].

The appearance of Grothendieck's scheme theory opened a lucky possibility not to draw any line whatever between "geometry" and "algebra": They appear now as complimentary aspects of a whole, like varieties and the spaces of functions on these spaces in other geometric theories. From this point of view,

> the commutative algebra coincides with (more precisely, is functorially dual to) the theory of local geometric objects — affine schemes.

This book is devoted to deciphering the above claim. I tried to consecutively explain what type of geometric images should be related with, say, primary decomposition, modules, and nilpotents. In A. Weil's words, the spacial intuition is "invaluable if its restrictedness is taken into account". I strived to take into account both terms of this neat formulation.

Certainly, geometric accent considerably influenced the choice of material. In particular, this chapter should prepare the ground for introducing global objects. Therefore, the section on vector bundles gives, on "naive" level, constructions belonging, essentially, to the sheaf theory.

Finally, I wanted to introduce the categoric notions as soon as possible; they are not so important in local questions but play ever increasing role in what follows. I advise the reader to skim through the section "Language of categories" and return to it as needed. [1)]

I am absolutely incapable to edit my old texts; if I start doing it, an irresistible desire to throw everything away and rewrite completely grips me. But to do something new is more interesting. Therefore I wish to heartily thank D. Leites who saved my time.

The following list of sources is by no means exhaustive. It may help the reader to come to grasps with the working aspects of the theory: [Sh0], [Bb1] (general courses); [M1], [M2], [Ma3], [MaG], [S1], [S2] (more special questions).

The approach of this book can be extended, to an extent, to non-commutative geometry [Kas]. My approach was gradually developing in the direction along which the same guiding principle — construct a matrix with commuting elements satisfying only the absolutely necessary commutation relations — turned out applicable in ever wider context of non-commutative geometries, see [MaT], see also [GK], [BM] (extension to operads and further).

Yu. Manin, March 22, 2009.

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[1] See also [McL, GM].

Chapter 1

Affine schemes

1.1. Equations and rings

Study of algebraic equations is an ancient mathematical science. In novel times, vogue and convenience dictate us to turn to rings.

1.1.1. Systems of equations. Let I, J be some sets of indices, let $T = \{T_j\}_{j \in J}$ be indeterminates and $F = \{F_i \in K[T]\}_{i \in I}$ a set of polynomials.

A *system X of equations for unknowns T* is the triple (the ring K, the unknowns T, the functions F), more exactly and conventionally expressed as

$$F_i(T) = 0, \quad \text{where} \quad i \in I. \tag{1.1}$$

Why the *ground ring* or the *ring of constants* K enters the definition is clear: The coefficients of the F_i belong to a fixed ring K. We say that the *system X is defined over K*.

What should we take for a *solution* of (1.1)?

To say, "a solution of (1.1) is a set $t = (t_j)_{j \in J}$ of elements of K such that $F_i(t) = 0$ for all i" is too restrictive: We wish to consider, say, complex roots of equations with real coefficients. The radical resolution of this predicament is to consider solutions in **any** ring, and, "for simplicity", in all of the rings simultaneously.

To consider solutions of X belonging to a ring L, we should be able to substitute the elements of L into F_i, the polynomials with coefficients from K, i.e., we should be able to multiply the elements from L by the elements from K and add the results, that is why L must be a *K-algebra*.

Recall that the set L is said to be a *K-algebra* if L is endowed with the structures of a ring and a K-module, interrelated by the following properties:

1. The multiplication $K \times L \longrightarrow L$ is right and left distributive with respect to addition;
2. $k(l_1 l_2) = (kl_1)l_2$ for any $k \in K$ and $l_1, l_2 \in L$.

A map $f : L_1 \longrightarrow L_2$ is a a *K-algebra homomorphism* if f is simultaneously a map of rings and K-modules.

Let 1_L denote the unity of L. To define a K-algebra structure on L, we need a ring homomorphism (embedding of K) $i\colon K \longrightarrow L$ which sends $k \cdot 1_K \in K$ into $i(k)1_L$.

1.1.1a. Examples. 1) Every ring L is a $\mathbb{Z}$-algebra with respect to the homomorphism $n \mapsto n \cdot 1_L$ for any $n \in \mathbb{Z}$.

2) If $K = \mathbb{F}_p$ or $\mathbb{Q}$ and $L = \mathbb{Z}$ or $\mathbb{F}_{p^2}$, then there are no ring homomorphisms $K \longrightarrow L$.

1.1.1b. Exercise. Prove 2).

1.1.2. Solutions of systems of equations. *A solution of a system* (1.1) *with values in a K-algebra L* is a set $t = (t_j)_{j \in J}$ of elements of L such that $F_i(t) = 0$ for all $i \in I$. The set of all such solutions is denoted by $X(L)$ and each solution is also called an *L-point of the system of equations X*.

As is clear from Example 1.1.1a 1), for any system of equations with *integer* coefficients, we may consider its solution in *any* commutative ring.

Two systems X and Y for the same unknowns given over a ring K are said to be *equivalent*, more precisely *equivalent over* K (and we write $X \sim Y$) if $X(L) = Y(L)$ for any K-algebra L.

Among the system of equations equivalent to a given one, X, there exists the "biggest" one: For the left hand sides of this "biggest" system take the left hand sides of (1.1) that generate the ideal (F) in $K[T]_{j \in J}$.

The *coordinate ring* of the variety $X = V(F)$ is

$$K(X) = K[T]/(F). \tag{1.2}$$

1.1.2a. Proposition. 1) *The system whose left hand sides are elements of the ideal (F) is the largest system equivalent to (F).*

2) *There is a one-to-one correspondence $X(L) \longleftrightarrow \mathrm{Hom}_K(K(X), L)$, where Hom_K denotes the set of K-algebra homomorphisms.*

Proof. 1) Let P be the ideal in $K[T]$, where $T = (T)_{j \in J}$, generated by the left sides of the system of equations (1.1). It is easy to see that the system obtained by equating all elements of P to zero is equivalent to our system, call it (X). At the same time, the larger system is the maximal one since if we add to it any equation $F = 0$ not contained in it, we get a system not equivalent to (X). Indeed, take $L = K[T]/P$. In L, the element t, where $t_j \cong T_j' \mod P$, is a solution of the initial system (X) whereas $F(t) \neq 0$ since $F \notin P$.

2) Let $t = (t_j)_{j \in J} \in X(L)$. There exists a K-algebra homomorphism $K[T] \longrightarrow L$ which coincides on K with the structure homomorphism $K \longrightarrow L$ and sends T_j to t_j. By definition of $X(L)$, the ideal P lies in the kernel of this homomorphism, so we can consider a through homomorphism $A = K[T]/P \longrightarrow L$.

Conversely, any K-algebra homomorphism $A \longrightarrow L$ uniquely determines a through homomorphism $K[T] \longrightarrow A \longrightarrow L$. Let t_j be the image of T_j under this through homomorphism; then $t = (t_j)_{j\in J} \in X(L)$ since all elements of P vanish under this homomorphism.

It is easy to check that the constructed maps $X(L) \longleftrightarrow \mathrm{Hom}_K(K(X), L)$ are inverse to each other. $\square$

For a non-zero K-algebra L, a system X over a ring K is said to be *consistent over* L if $X(L) \neq \emptyset$ and *inconsistent* otherwise. Proposition 1.1.2a shows that X is inconsistent only if its coordinate ring, or rather, algebra, $K(X)$, is zero, in other words, if $1 \in (X)$.[1)]

1.1.3. Examples from arithmetics.

1.1.3a. The language of congruences. Let n be an integer of the form $4m + 3$. The classical proof of the fact that n is not representable as a sum of two perfect squares is as follows: If it were, the congruence

$$T_1^2 + T_2^2 \equiv 3 \pmod 4 \tag{1.3}$$

would have been solvable, whereas a simple case-by-case checking (set $T_1 = 4a + r_1$, $T_2 = 4b + r_2$ and consider the eight distinct values of (r_1, r_2)) establishes that this is not the case.

From our point of view this argument reads as follows: Consider the system

$$T_1^2 + T_2^2 = n, \text{ where } K = \mathbb{Z}. \tag{1.4}$$

The reduction modulo 4, i.e., the map $\mathbb{Z} \longrightarrow \mathbb{Z}/4$, determines a map $X(\mathbb{Z}) \longrightarrow X(\mathbb{Z}/4)$ and, if $X(\mathbb{Z}) \neq \emptyset$, then $X(\mathbb{Z}/4) \neq \emptyset$, which is not the case.

More generally, in order to study $X(\mathbb{Z})$ for any system X with integer coefficients, we can consider sets $X(\mathbb{Z}/m)$ for any m and try to deduce from this consideration some information on $X(\mathbb{Z})$.

Usually,

$$\text{if } X(L) = \emptyset \text{ for a nontrivial (i.e., } 1 \neq 0) \text{ field } L, \text{ then } X(\mathbb{Z}) = \emptyset. \tag{1.5}$$

In practice one usually tests $L = \mathbb{R}$ and the fields $L = \mathbb{F}_m (:= \mathbb{Z}/m)$ for all prime m's.

A number of deepest results of the theory of Diophantine equations are related with the problem: When is the converse statement true? A prototype of these results is

Legendre's theorem ([BSh]). *Let*

$$a_1T_1^2 + a_2T_2^2 + a_3T_3^2 = 0, \quad K = \mathbb{Z}. \tag{1.6}$$

If $X(\mathbb{Z}) = \{(0,0,0)\}$, *then* $X(L) = \{(0,0,0)\}$ *for at least one of the rings* $L = \mathbb{R}$ *or* $L = \mathbb{Z}/m\mathbb{Z}$, *where* $m > 1$.

[1] In school, and even in university, one often omits "over L" in the definition of consistency thus declaring systems inconsistent only partly (over some classes of rings) as totally inconsistent.

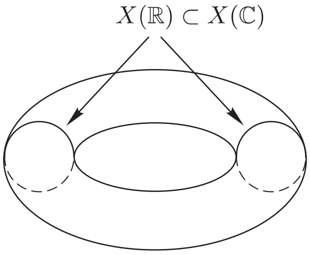
1.1.3b. Equations in prime characteristic. Consider the equation

$$0 \cdot T + 2 = 0, \quad K = \mathbb{Z}. \tag{1.7}$$

Clearly, $X(L) = \emptyset$ if $2 \cdot 1_L \neq 0$ and $X(L) = L$ if $2 \cdot 1_L = 0$.

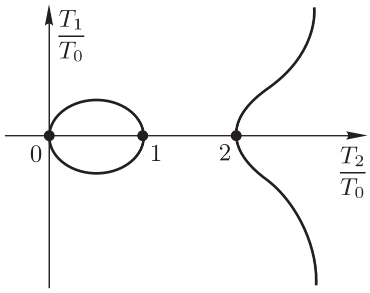
This example seems manifestly artificial; still, we often encounter the like of it in "arithmetical geometry".

1.1.3c. On usefulness of complexification. When studying $X(\mathbb{R}) \subset \mathbb{R}^n$ (for the case where $K = \mathbb{R}$, and the number of unknowns is equal to n), it is expedient to pass to the complexification of $X(\mathbb{R})$, i.e., to $X(\mathbb{C})$. Since $\mathbb{C}$ is algebraically closed, it is usually easier to study $X(\mathbb{C})$ than $X(\mathbb{R})$; this often constitutes the first stage of the investigation even if we are primarily interested in purely real problems. The following example is illuminating.



a torus $X(\mathbb{C})$

Fig. 1



$F = T_0T_1^2 - T_2(T_2 - T_0)(T_2 - 2T_0)$
$X(\mathbb{R})$ on the projective plane

Fig. 2

Harnak's theorem. *Let* $F(T_0, T_1, T_2)$ *be a form* (i.e., a polynomial homogeneous with respect to the degree) *of degree* d *with real coefficients. Let* $X(\mathbb{R})$ *be the curve in* $\mathbb{R}P^2$*, the real projective plane, singled out by the equation* $F = 0$*. Then the number of connected components of* $X(\mathbb{R})$ *does not exceed* $\frac{1}{2}(d-1)(d-2) + 1$.

The method of the proof uses the embedding $X(\mathbb{R}) \longrightarrow X(\mathbb{C})$. For simplicity, let us confine ourselves to the case of a nonsingular $X(\mathbb{C})$, i.e., to the case where $X(\mathbb{C})$ is a compact orientable 2-dimensional manifold. Its genus — the number of handles — is, then, equal to $\frac{1}{2}(d-1)(d-2)$. On Fig. 1, $d = 3$ and $X(\mathbb{C})$ is a torus. The proof is based on the two statements:

First, the complex conjugation can be extended to a continuous action on $X(\mathbb{C})$ so that $X(\mathbb{R})$ is precisely the set of its fixed points.

Second, $X(\mathbb{C})$ being cut along $X(\mathbb{R})$ splits into precisely 2 pieces, cf. Fig. 2 (as, for $d = 1$, the Riemannian sphere — the compactified real plane — does being cut along the real axis). Now, routine topological considerations give Harnak's estimate, see, e.g., [Ch].

1.1.3d. The algebra of mathematical logic in geometric terms. [2] A *Boolean ring* is any ring R (with 1) such that $P^2 = P$ for any $P \in R$. Clearly,

$$P + Q = (P + Q)^2 = P^2 + PQ + QP + Q^2 = P + PQ + QP + Q \tag{1.8}$$

implies $PQ + QP = 0$. Since R is commutative by definition, $2PQ = 0$; moreover,

$$2P = P + P = P^2 + P^2 = 0 \tag{1.9}$$

implies $P = -P$. Therefore every Boolean ring is a commutative ring over $\mathbb{F}_2$.

It is not difficult to show that every prime ideal of a Boolean algebra R is a maximal one, and therefore every element $P \in R$ can be viewed as an $\mathbb{F}_2$-valued function on $\operatorname{Spec} R$.

Given two statements, P and Q, each either true of false, define their *sum* and *product* by setting

$$P + Q = (P \vee Q) \wedge (\bar{P} \vee \bar{Q}), \quad PQ = P \wedge Q, \tag{1.10}$$

where the bar stands for the negation, $\wedge$ for the conjunction and $\vee$ for the disjunction. With respect to the above operations the empty statement $\emptyset$ is the zero, and $\bar{\emptyset}$ is the unit. Clearly, $P^2 = P$ and $2P = P + P = 0$ for any P.

1.1.4. Summary. We have established the *equivalence of the two languages*: That of the systems of equations (which is used in the concrete calculations) and that of the rings and their morphisms. More exactly, we have established the following equivalences:

$$\left\{\begin{array}{c}\text{A system of equations } X\\ \text{over a ring } K\\ \text{for unknowns } \{T_j \mid j \in J\}.\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c}\text{A } K\text{-algebra } K(X) \text{ with a}\\ \text{system of generators}\\ \{t_j \mid j \in J\}.\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{A solution of the system}\\ \text{of equations } X\\ \text{in a } K\text{-algebra } L.\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c}\text{A } K\text{-algebra homomorphism}\\ K(X) \longrightarrow L.\end{array}\right\}$$

Finally, notice that using the language of rings we have no need to consider a fixed system of generators $t = (t_j)_{j \in J}$ of $K(X) = K[T]/(X)$; we should rather identify the systems of equations obtained from each other by any invertible change of unknowns. Every generator of $K(X)$ plays the role of one of "unknowns", and the value this unknown takes at a given solution of the system coincides with its image in L under the corresponding homomorphism.

1.1.5. Exercises. 1) The equation $2T - 4 = 0$ is equivalent to the equation $T - 2 = 0$, if and only if 2 is invertible in K.

[2] For mathematical logic from an algebraist's point of view, see [Ma4].

2) The equation $(T-1)^2=0$ is not equivalent to the equation $T-1=0$.

3) Let the system of equations $\{F_i(X)=0 \mid i\in I\}$, where $X=(X_j)_{j\in J}$, be incompatible. Then it has a finite incompatible subsystem.

4) Let $T_1,\dots,T_n$ be indeterminates; $s_i(T)$ be the i-th elementary symmetric polynomial in them. Determine the rings of constants over which the following systems of equations are equivalent:

$$\begin{aligned}&X_1\colon s_i(T)=0, \qquad i=1,\dots,k\leq n,\\ &X_2\colon \sum_{1\leq j\leq n} T_j^i=0,\ i=1,\dots,k\leq n.\end{aligned}$$

Hint. Use Newton's formulas.

5) Any system of equations over a ring K in a finite number of unknowns is equivalent to a finite system of equations if and only if the ring is *Noetherian.*[3)]

6) Let X be a system of equations over K and A the ring corresponding to X. The maps $L\mapsto X(L)$ and $L\mapsto \operatorname{Hom}_K(A,L)$ determine covariant functors on the category Algs_K of K-algebras with values in Sets. Verify that Proposition 1.1.2a determines an isomorphism of these functors.

1.2. Geometric language: Points

Let, as earlier, K be the main ring, X a system of equations over K for unknowns $T_1,\dots,T_n$.

For any K-algebra L, we realize the set $X(L)$ as a "graph" in L^n, the coordinate space over L. The points of this graph are *solutions of the system* (1.1). Taking into account the results of the preceding section let us introduce the following definition.

1.2.1. The points of a K-algebra A. The *points* of a K-algebra A with values in a K-algebra B (or just B*-points* of A) are the K-homomorphisms $A\longrightarrow B$. Any B-point of A is called *geometric* if B is a field.

Example. (This example shows where the idea to apply the term "point" to a homomorphism comes from.) Let K be a field, V an n-dimensional vector space over K. Let us show that the set of point of V is in on-to-one correspondence with the set of maximal ideals of the ring $K[x]$, where $x=(x_1,\dots,x_n)$, of polynomial functions on V.

Every point (vector) of V is a linear functional on V^*, the space of linear functionals on V with values in K and, as is easy to show ([Lang]), this linear functional can be extended to an algebra homomorphism:

[3] A ring is said to be *Noetherian* if it satisfies any of the following equivalent conditions ([Lang]):

1. Any set of its generators contains a finite subset.
2. Any ascending chain of its (left for non-commutative rings) ideals stabilizes.

$$S^{\bullet}(V^*) = K[x_1, \dots, x_n] \longrightarrow K, \tag{1.11}$$

where $V^* = \mathrm{Span}(x_1, \dots, x_n)$. This homomorphism can be canonically identified with a point (vector) of V.

The next step is to separate the properties of a fixed algebra A from whims of the variable algebra B, namely, instead of homomorphisms $h\colon A \longrightarrow B$ we consider their kernels, the ideals in A.

1.2.2. The spectrum. The kernel of a homomorphism $A \longrightarrow B$ corresponding to a geometric point is, clearly, a prime ideal (to be defined shortly). There are many reasons why one should confine oneself to prime ideals instead of seemingly more natural maximal ones and in the following sections we will give these reasons.

An ideal p of a commutative ring A is said to be *prime* if A/p is an integral domain, i.e., has no zerodivisors (and if we do not forbid $1 = 0$, then the zero ring may not be an integral domain). Equivalently, p is *prime* if $p \neq A$ and

$$a \in A, \quad b \in A, \quad a\, b \in p \Longrightarrow \text{ either } a \in p \text{ or } b \in p. \tag{1.12}$$

The set of all the prime ideals of A is said to be the (prime) *spectrum of* A and is denoted by $\mathrm{Spec}\, A$. The elements of $\mathrm{Spec}\, A$ are called its *points*.

In what follows, we enrich the set $\mathrm{Spec}\, A$ with additional structures making it into a topological space rigged with a sheaf of rings: This will lead to the definition of an affine scheme. Schemes, i.e., topological spaces with sheaves, locally isomorphic to affine schemes, are the main characters of algebraic geometry.

Starting the study of spectra we have to verify, first of all, that there exists indeed what to study.

1.2.3. Theorem. *If* $A \neq 0$*, then* $\mathrm{Spec}\, A \neq \emptyset$*.*

In the proof of this theorem we need the following:

1.2.4. Lemma (Zorn's lemma)**.** *In a partially ordered set M, let every linearly ordered subset $N \subset M$ contain a maximal element. Then M contains a maximal element.*

For proofs of Zorn's lemma, see, e.g., [K] or [Hs], where it is proved together with its equivalence to the choice axiom, the complete order principle, and several other statements. For an interesting new additions to the list of equivalent statements, see, e.g., [Bla]. □

An ordered set satisfying the condition of Zorn's lemma is called an *inductive* set.

Proof of Theorem 1.2.3. Denote by M the set of all the ideals of A different from A. Since M contains (0), it follows that $M \neq \emptyset$. The set M is partially ordered with respect to inclusion. In M, take an arbitrary linearly ordered set $\{P_\alpha\}_{\alpha \in \varLambda}$.

Then $\bigcup\limits_\alpha P_\alpha$ is also an ideal of A (mind the linear order) and this ideal is different from A (since the unit element does not belong to $\bigcup\limits_\alpha P_\alpha$). Therefore M is an inductive set. Denote by p its maximal element; it is a maximal ideal, and therefore a prime one: In A/p, every non-zero ideal, in particular, all the principal ones, coincide with A/p. Therefore every non-zero element of A/p is invertible and A/p is a field. □

Corollary. *Every prime ideal is contained in a maximal ideal.*

This theorem implies, in particular, that the spectrum of every non-zero ring A possesses a geometric point (e.g., any homomorphism $A \longrightarrow A/p$, where $p \subset A$ is a maximal ideal, is one of them).

1.2.5. The center of a geometric point. The *center* of a geometric point $A \longrightarrow L$ is its kernel considered as an element of $\operatorname{Spec} A$. Let $k(x)$ be the *field of quotients* [4] of the ring A/p_x, where $p_x \subset A$ is the ideal corresponding to $x \in \operatorname{Spec} A$.

Proposition. *The geometric L-points of a K-algebra A with center in $x \in \operatorname{Spec} A$ are in one-to-one correspondence with the K-homomorphism $k(x) \longrightarrow L$.*

Proof. Indeed, since L is a field, any homomorphism $A \longrightarrow L$ factorizes as

$$A \longrightarrow A/p_x \longrightarrow k(x) \longrightarrow L. \tag{1.13}$$

The first two arrows of this sequence are rigidly fixed. □

1.2.6. Examples. a) Let K be a perfect [5] field, L an algebraically closed field containing K, let A be a K-algebra, $x \in \operatorname{Spec} A$, and p_x the corresponding ideal.

If $\deg x := [k(x) : K] < \infty$, then, by Galois theory, there are exactly $\deg x$ geometric L-points with center in x.

If the field $k(x)$ is not algebraic over K, and L contains sufficiently many transcendences over K, then there may be infinitely many geometric L-points with center in x.

Here is a very particular case.

b) The set of geometric $\mathbb{C}$-points of $\mathbb{R}[T]$ is the complex line $\mathbb{C}$. The set $\operatorname{Spec} \mathbb{R}[T]$ is the union of the zero ideal (0) and the set of all monic polynomials irreducible over $\mathbb{R}$.

Every such polynomial of degree 2 has two complex conjugate roots corresponding to two distinct geometric points, see Fig. 3.

In general, for any perfect field K, the geometric points of the K-algebra $K[T]$ with values in the algebraic closure $\overline{K}$ are just the elements of $\overline{K}$, and their centers are irreducible polynomials over K, i.e., the sets consisting of all the elements of $\overline{K}$ conjugate over K to one of them. □

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[4] The *field of fractions* or *field of quotients* of a ring is the smallest field in which the ring can be embedded. It will be explicitly constructed in what follows.

[5] Recall that a field K of characteristic p is said to be *perfect* if $K^p := \{x^p \mid x \in K\}$ coincides with K. (The symbol K^p is also used to designate $\underbrace{K \times \cdots \times K}_{p \text{ times}}$.)

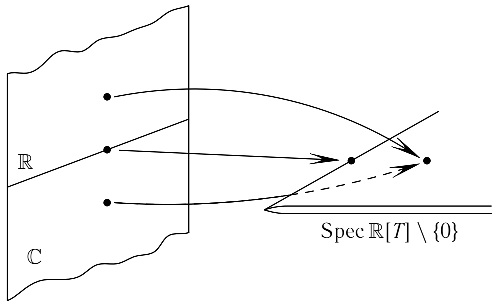

Fig. 3

1.2.7. A duality: Unknowns $\longleftrightarrow$ Coefficients. By considering Spec A we may forget, if needed, that A is a K-algebra; every ideal of A is stable under multiplication by the elements of K. When we are interested in geometric points (or, more generally, in arbitrary L-points) the reference to K is essential, since we have to consider K-homomorphisms $A \longrightarrow L$. Arbitrary ring homomorphisms are, clearly, $\mathbb{Z}$-homomorphisms; therefore this "absolute" case may be considered as a specialization of a "relative" one, the one over K.

For the systems of equations, the passage to the absolute case means that we forget about the difference between "unknowns" and "coefficients" and may vary the values of both. More precisely, consider a system of equations in which the jth equation is

$$\sum_k a_k^{(j)} x^k = 0, \quad \text{where } x^k \text{ runs over monomials in our indeterminates.}$$

Generally, $a_k^{(j)}$ are fixed elements of a ring of constants K whereas "passage to the absolute case" means that we now *write a generating system of all relations between the $a_k^{(j)}$ OVER $\mathbb{Z}$, and add it to our initial system of equations, binding the x_i and the $a_k^{(j)}$ together. And after that we may specialize coefficients as well, retaining only the relations between them.*

1.2.8. Exercise. A weak form of Hilbert's Nullstellensatz (theorem on zeros). Consider a system of equations $\{F_i(T) = 0\}$, where $T = (T_j)_{j\in J}$, over the ring K. Then either this system has a solution with values in a field, or there exist polynomials $G_i \in K[T]$ (finitely many of them are $\neq 0$) such that

$$\sum_i G_i F_i = 1.$$

Hint. Apply Theorem 1.2.3 to the ring corresponding to the system.

1.3. Geometric language, cont.: Functions on spectra

1.3.1. Functions on spectra. Let X be a system of equations over K in unknowns $T = (T_j)_{j\in J}$. Every solution of X in a K-algebra L, i.e., an element of $X(L)$, evaluates the T_j; let $\{t_j \in L\}_{j\in J}$ be the values. Therefore it is natural to consider T_j as a function on $X(L)$ with values in L. Clearly, this function only depends on the class of T_i modulo the ideal generated by the left-hand sides of X. This class is an element of the K-algebra $K(X) := K[T]/(X)$ related with X, and generally all the elements of $K(X)$ are functions on $X(L) = \mathrm{Hom}_K(K(X), L)$: Indeed, for every $\varphi\colon K(X) \longrightarrow L$ and $f \in K(X)$, "the value of f at φ" is by definition $\varphi(f)$.

The classical notation of functions is not well adjusted to reflect the fundamental duality that manifests itself in modern mathematics more and more:

$$\text{“A space} \longleftrightarrow \text{the ring of functions on the space”} \tag{1.14}$$

or, symmetrically, with a different emphasis:

$$\text{“A ring} \longleftrightarrow \text{the spectrum of its ideals of certain type”.} \tag{1.15}$$

When applied to $\operatorname{Spec} A$, this duality leads to consideration of any element $f \in A$ as a function on $\operatorname{Spec} A$. Let $x \in \operatorname{Spec} A$ and p_x the corresponding ideal. Then, by definition, $f(x) = f \pmod{p_x}$ and we assume that $f(x)$ belongs to the field of quotients $k(x)$ of the ring A/p_x.

1.3.1a. Convention. In what follows speaking about *functions* on $\operatorname{Spec} A$ we will always mean the elements of A.

Thus, to every point $x \in \operatorname{Spec} A$, **its own** field $k(x)$ is assigned and the values of the functions on $\operatorname{Spec} A$ belong to these fields.

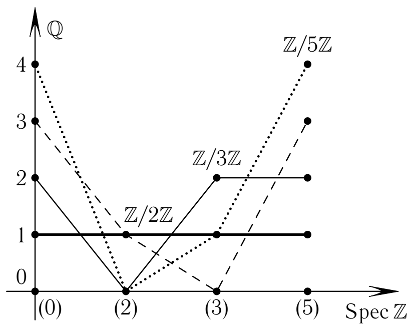

Fig. 4

I tried to plot the first five integers (0, 1, 2, 3, 4) considered as functions on $\operatorname{Spec} \mathbb{Z}$. The picture is not too convincing; besides, for various reasons that lie beyond the scope of these lectures, the "line" over the field $\mathbb{Z}/p$ — the "vertical axis" over the point (p) — should be drawn "coiled into a ring", i.e., the points of this "line" should form vertices of a regular p-gon. This does not simplify the task of an artist.

To distinct elements of A the same functions on the spectrum might correspond; their difference represents the zero function, i.e., belongs to $\bigcap\limits_{x\in\operatorname{Spec} A} p_x$. Clearly, all the nilpotents are contained in this intersection. Let us prove the opposite inclusion. For this, we need a new notion.

The set N of all nilpotent elements of a ring R is an ideal (as is easy to see), it is called the *nilradical* of R.

1.3.2. Theorem. *A function that vanishes at all the points of* $\operatorname{Spec} A$ *is represented by a nilpotent element of A. In other words, the nilradical is the intersection*

$$\bigcap_{p\in\operatorname{Spec} A} p. \tag{1.16}$$

Proof. It suffices to establish that, for every non-nilpotent element, there exists a prime ideal which does not contain it.

Let $h \in A$ and $h^n \neq 0$ for any positive integer n. Let M be the set of all the ideals of A that do not contain h^m for any $m \in M$; then $M \neq \emptyset$ since M contains (0). The inductive property of M can be proved as in Theorem 1.2.3. Let p be a maximal element of M. Let us prove that p is prime.

Let $f, g \in A$ and $f, g \notin p$. Let us prove that $fg \notin p$. Indeed,[6] $p + (f) \supset p$ and $p + (g) \supset p$ (strict inclusions). Since p is maximal in M, we see that $h^n \in p + (f)$ and $h^m \in p + (g)$ for some $m, n \in \mathbb{N}$. Hence, $h^{n+m} \in p + (fg)$; but $h^{n+m} \notin p$, and hence $fg \notin p$. Thus, p is prime. □

This result might give an impression that there is no room for nilpotents in a geometric picture. This would be a very false impression:[7] On the contrary, nilpotents provide with an adequate language for description of differential-geometric notions like "tangency", "the multiplicity of an intersection", "infinitesimal deformation", "the fiber of a map at the points where regularity is violated".

1.3.3. Examples. 1) *A multiple intersection point*. In the affine plane over $\mathbb{R}$, consider the parabola $T_1 - T_2^2 = 0$ and the straight line $T_1 - t = 0$, where $t \in \mathbb{R}$ is a parameter.

Their intersection is given by the system of equations

---

[6] Recall that (f) denotes the ideal generated by an element f. The notation $(f_1, \dots, f_n)$ means — in this context — the finitely generated ideal, not the vector $(f_1, ..., f_n) \in A^n$.

[7] The instances of this sort will be illustrated by the following examples 3.2–3.5 and in the next section to say nothing of the main body of the book, where odd indeterminates, especially odd parameters of representations of Lie supergroups and Lie superalgebras, are the heart (and the *soul*, as B. de Witt might had put it) of the matter.

$$\begin{cases} T_1 - T_2^2 &= 0, \\ T_1 - t &= 0. \end{cases} \tag{1.17}$$

to which the ring $A_t = \mathbb{R}[T_1, T_2]/(T_1 - T_2^2,\ T_1 - t)$ corresponds. An easy calculation shows that

$$A_t \cong \begin{cases} \mathbb{R} \times \mathbb{R} & \text{if } t > 0, \\ \mathbb{R}[T]/(T^2) & \text{if } t = 0, \\ \mathbb{C} & \text{if } t < 0. \end{cases} \tag{1.18}$$

The *geometric* $\mathbb{R}$*-points* of A_t are the following ones: There are two of them for $t > 0$, there is one of them for $t = 0$, there are none for $t < 0$.

The *geometric* $\mathbb{C}$*-points*: There are always two of them except for $t = 0$ (the case of tangency).

In order to be able to state that over $\mathbb{C}$ there are always two intersection points if proper *multiplicities* are ascribed to them, we have to assume that at $t = 0$ the multiplicity of the intersection point is equal to 2.

Observe that $\dim_{\mathbb{R}} A_t = 2$ regardless of the value of t. The equality:

$$\dim_{\mathbb{R}} A_t = \text{the number of intersection points (multiplicity counted)} \tag{1.19}$$

is not accidental, and we will prove the corresponding theorem when we introduce the projective space that will enable us to take into account the points that escaped to infinity.

A singularity like the coincidence of the intersection points corresponding to tangency creates nilpotents in A_0.

2) *One-point spectra.* Let $\operatorname{Spec} A$ consist of one point corresponding to an ideal $p \subset A$. Then A/p is a field and p consists of nilpotents. The ring A is Artinian [8)], hence Noetherian, and the standard arguments show that p is a nilpotent ideal.

Indeed, let $f_1, \dots, f_n$ be its generators and $f_i^m = 0$ for $1 \leq i \leq n$. Then

$$\prod_{j=1}^{mn} \left(\sum_{i=1}^{n} a_{ij} f_i \right) = 0 \tag{1.20}$$

for any $a_{ij} \in A$, where $1 \leq i \leq n$ and $1 \leq j \leq mn$, since, in every monomial of the product, there enters at least one of the f_i raised to the power $\geq m$.

Recall that the *length of a module* M is the length r of the filtration

$$M = M_1 \supset M_2 \supset \cdots \supset M_r = 0. \tag{1.21}$$

The filtration (1.21) of a module M is said to be *simple* if each module M_i/M_{i+1} is simple, whereas as module is said to be *simple* if it contains no proper submodules (different from $\{0\}$ and itself). A module is said to be of *finite length* if either it is $\{0\}$ or admits a simple finite filtration.

[8] Recall that the ring is *Artinian* if *the descending chain condition (DCC) on ideals holds.*

Therefore $p^{mn} = 0$. In the series $A \supset p \supset p^2 \supset \ldots \supset p^{mn} = (0)$, the quotients p^i/p^{i+1} are finite dimensional vector spaces over the field A/p. Therefore, as a module over itself, A is of finite length. In the intersection theory, the length of a local ring A plays the role of the multiplicity of the only point of Spec A, as we have just seen. The multiplicity of the only point of Spec A is equal to 1 if and only if A is a field.

3) *Differential neighborhoods. Jets.* Let $x \in \operatorname{Spec} A$ be a point, p_x the corresponding ideal. In differential geometry, and even in freshmen's calculus courses, we often have to consider the m-th differential neighborhood of x, i.e., take into account not only the values of functions but also the values of its derivatives to the m-th inclusive, in other words, consider the m-jet of the function at x. This is equivalent to considering the function's Taylor series expansion in which the infinitesimals of order greater than m are neglected, i.e., forcefully equated to 0.

Algebraically, this means that we consider the class $f \pmod{p_x^{m+1}}$. The elements from p_x are infinitesimals of order ≥ 1, and, in the ring A/p_x^{m+1}, they turn into nilpotents.

In what follows we will see that the interpretation of Spec A/p_x^{m+1} as the differential neighborhood of x is only natural when p_x is maximal. In the general case of a prime but not maximal p_x, this intuitive interpretation can not *guide* us but still is useful.

4) *Reduction modulo p^N.* Considering Diophantine equations or, equivalently, the quotients of the ring $\mathbb{Z}[T_1, \ldots, T_n]$, one often makes use of the reduction modulo powers of a prime, see sec. 1.1.3. This immediately leads to nilpotents and we see that from the algebraic point of view this process does not differ from the consideration of differential neighborhoods in the above example.

(The congruence $3^5 \cong 7 \mod 5^3$ means that "at point (5) the functions 3^5 and 7 coincide up to the second derivative inclusive". This language does not look too extravagant in the number theory after Hensel had introduced p-adic numbers.)

1.3.4. Exercises. 1) Let $a_1, \ldots, a_n \subset A$ be ideals. Prove that

$$V(a_1 \ldots a_n) = V(a_1 \cap \cdots \cap a_n).$$

2) Let $f_1, \ldots, f_n \in A$, and $m_1, \ldots, m_n$ positive integers. If $(f_1, \ldots, f_n) = A$, then $(f_1^{m_1}, \ldots, f_n^{m_n}) = A$.

3) The elements $f \in A$ not vanishing in any point of Spec A are invertible.

1.4. The Zariski topology on Spec A

The minimal natural condition for compatibility of the topology with "functions" is that the set of zeroes of any function should be closed. The topology on Spec A that satisfies this criterion is called *Zariski topology.* To describe it, for any subset $E \subset A$, denote the variety singled out by E to be

$$V(E) = \{x \in \operatorname{Spec} A \mid f(x) = 0 \text{ for any } f \in E\}. \tag{1.22}$$

1.4.1. Lemma. *The sets $V(E)$, where $E \subset A$, constitute the system of closed sets in a topology of* Spec A.

The topology defined in Lemma 1.4.1 is called the *Zariski topology*.

Proof. Since $\emptyset = V(1)$ and $\operatorname{Spec} A = V(\emptyset)$, it suffices to verify that $V(E)$ is closed with respect to finite unions and arbitrary intersections. This follows from the next statement.

1.4.1a. Exercise. Set $E_1E_2 = \{fg \mid f \in E_1, g \in E_2\}$. Then $V(E_1) \cup V(E_2) = V(E_1E_2)$ and $\bigcap\limits_{i\in I} V(E_i) = V\Big(\bigcap\limits_{i\in I} E_i\Big)$ for any I. □

Using Theorem 1.3.1, we can describe the set of functions that vanish on $V(E)$. Obviously, it contains all the elements of the ideal (E) generated by E and all the elements $f \in A$ such that $f^n \in (E)$ for some n. It turns out that this is all.

The *radical* $\mathfrak{r}(I)$ *of the ideal* $I \subset A$ is the set (actually, an ideal in the ring A) defined to be

$$\mathfrak{r}((E)) = \{f \in A \mid \text{ there exists } n \in \mathbb{N} \text{ such that } f^n \in I\}. \tag{1.23}$$

An ideal coinciding with its own radical is said to be a *radical ideal*.

1.4.2. Theorem. *If $f(x) = 0$ for any $x \in V(E)$, then $f \in \mathfrak{r}((E))$.*

Proof. The condition

$$f(x) = 0 \text{ for all } x \in V(E)$$

means that $f \in \bigcap\limits_{p_x \subset \operatorname{Spec} A/(E)} p_x$, i.e., every element $f \pmod{(E)}$ from $A/(E)$ belongs to the intersection of all prime ideals of $A/(E)$. Therefore $f^n \pmod{(E)} = 0$ for some n, thanks to Theorem 1.3.1. This proves the statement desired. □

1.4.2a. Corollary. *The map $I \longmapsto V(I)$ establishes a one-to-one correspondence between the radical ideals of A and the closed subsets of its spectrum.*

Proof immediately follows from Theorem 1.3.1. □

The topology of the spaces Spec A is very non-classical, in the sense that is very non-Hausdorff (non-separable). We consider typical phenomena specific for algebraic geometry.

1.4.3. Non-closed points. Let us find the closure of a given point $x \in \operatorname{Spec} A$. We have

$$\overline{\{x\}} = \bigcap_{E \subset p_x} V(E) = V\Big(\bigcup_{E \subset p_x} E\Big) = V(p_x) = \{y \in \operatorname{Spec} A \mid p_y \supset p_x\}.$$

In other words,

$\overline{\{x\}} \cong \operatorname{Spec}(A/p_x)$, and

only the points corresponding to the maximal ideals are closed.

A specific relation among points, $y \in \overline{\{x\}}$, is sometimes expressed by saying that y is a *specialization* of x; this is equivalent to the inclusion $p_x \subset p_y$. If A has no zero divisors, then $\{0\} \in \operatorname{Spec} A$ is the point *whose closure coincides with the whole spectrum.*

Therefore Spec A is stratified: The closed points are on the highest level, the preceding level is occupied by the points whose specializations are closed, and so on, the i-th level (from above) is manned by the points whose specializations belong to the level $< i$. The apex of this inverted pyramid is either the *generic point*, (0), if A has no zero divisors, or a finite number of points if A is a Noetherian ring (for proof, see subsec. 1.4.7d).

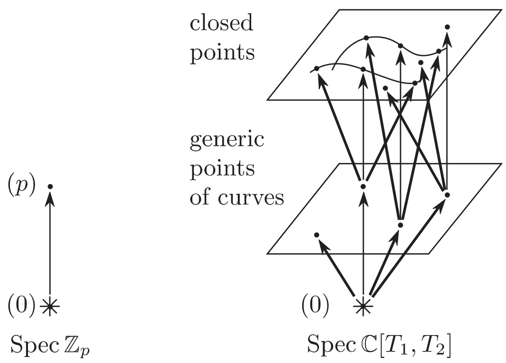

Fig. 5

On Fig. 5 there are plotted two spectra: The spectrum of the ring of integer p-adic numbers, $\mathbb{Z}_p$, and $\operatorname{Spec} \mathbb{C}[T_1, T_2]$. The arrows indicate the specialization relation. The picture of $\operatorname{Spec} \mathbb{Z}_p$ does not require comments; note only that Spec A may be a finite but not discrete topological space. The other picture is justified by the following statement.

Proposition. *Let K be algebraically closed. The following list exhausts the prime ideals of the ring $K[T_1, T_2]$:*

a) *the maximal ideals $(T_1 - t_1, T_2 - t_2)$, where $t_1, t_2 \in K$ are arbitrary;*

b) *the principal ideals* $(F(T_1, T_2))$*, where* F *runs over all irreducible polynomials;*

c) (0).

For proof, see [Reid].

The images influenced by this picture can serve as a base for a working dimension theory of algebraic geometry. We will show this later; at the moment we will confine ourselves to preliminary definitions and two simple examples.

We say that a sequence of points $x_0, x_1, \dots, x_n$ of a topological space X is a *chain of length* n *with the beginning at* x_0 *and the end at* x_n if $x_i \neq x_{i+1}$ and x_{i+1} is a specialization of x_i for all $0 \leq i \leq n-1$.

The *height* $\operatorname{ht} x$ of $x \in X$ is the upper bound of the lengths of the chains with the beginning at x. The *dimension* $\dim X$ of X is the upper bound of the heights of its points.

Example. In the space $X = \operatorname{Spec} K[T_1, \dots, T_n]$, where K is a field, there is a chain of length n corresponding to the chain of prime ideals

$$(0) \subset (T_1) \subset \dots \subset (T_1, \dots, T_n), \tag{1.24}$$

and therefore $\dim X \geq n$.

Similarly, since there is a chain

$$(p) \subset (p, T_1) \subset \dots \subset (p, T_1, \dots, T_n), \tag{1.25}$$

it follows that $\dim \operatorname{Spec} \mathbb{Z}[T_1, \dots, T_n] \geq n + 1$.

As we will see later, in both cases the inequalities are, actually, equalities.

This definition of dimension can be traced back to Euclid: (Closed) points "border" lines that "border" surfaces, and so on. [9)]

1.4.4. Big open sets. For any $f \in A$, set

$$D(f) = \operatorname{Spec} A \setminus V(f) = \{x \mid f(x) \neq 0\}. \tag{1.26}$$

The sets $D(f)$ are called *big open sets*; they constitute a basis of the Zariski topology of $\operatorname{Spec} A$, since

$$\operatorname{Spec} A \setminus V(E) = \bigcap_{f \in E} D(f) \quad \text{for any } E \subset A. \tag{1.27}$$

Consider, for example, $\operatorname{Spec} \mathbb{C}[T]$. Its closed points correspond to the ideals $(T - t)$, where $t \in \mathbb{C}$, and constitute, therefore, the "complex line"; the non-empty open sets are $\{0\}$ and the sets of all the points of the complex line

[9] For discussion of non-integer values of dimension, with examples, see [Mdim]. As Shander and Palamodov showed, in the super setting, the relation of "border"ing is partly reversed: The *odd* codimension of the boundary of a supermanifold might be *negative*, see [SoS].

except a finite number of them. *The closure of any open set coincides with the whole space!*

More generally, if A has no zero divisors and $f \neq 0$, then $D(f)$ is dense in Spec A. Indeed, $D(f) \subset (0)$; hence, $\overline{D(f)} = \overline{(0)} = \operatorname{Spec} A$. Therefore *any non-empty open set of the spectrum of any ring without zero divisors is dense.*

In the analysis of the type of (non-)separability of topological spaces, an important class was distinguished:

1.4.5. Lemma. *The following conditions are equivalent:*

a) *Any non-empty open subset of X is dense.*

b) *Any two non-empty open subsets of X have a non-empty intersection.*

c) *If $X = X_1 \cup X_2$, where X_1, X_2 are closed, then either $X_1 = X$ or $X_2 = X$.*

Proof. a) $\Longleftrightarrow$ b) is obvious. If c) is false, then there exists a representation $X = X_1 \cup X_2$, where X_1, X_2 are proper closed subsets of X; then $X \setminus X_2 = X_1 \setminus (X_1 \cap X_2)$ is non-dense open set; therefore a) fails. This is a contradiction.

Conversely, if a) fails and $U \subset X$ is a non-dense open set, then

$$X = \overline{U} \cup (X \setminus U). \quad \square$$

A topological space X satisfying any of the above conditions is called *irreducible*. Notice that no Hausdorff space with more than one point can be irreducible.

1.4.6. Theorem. *Let A be a ring, N its nilradical. The space* Spec A *is irreducible if and only if N is prime.*

Proof. Let N be prime. Since N is contained in any prime ideal, then Spec A is homeomorphic to Spec A/N, and A/N has no zero divisors.

Conversely, let N be non-prime. It suffices to verify that Spec A/N is reducible, i.e., confine ourselves to the case where A has no nilpotents but contains zero divisors.

Let $f, g \in A$ be such that $g \neq 0$, $f \neq 0$ but $fg = 0$. Obviously,

$$\operatorname{Spec} A = V(f) \cup V(g) = V(fg). \tag{1.28}$$

Therefore f and g vanish on the closed subsets of the whole spectrum that together cover the whole space (this is a natural way for zero divisors to appear in the rings of functions).

We only have to verify that $V(f), V(g) \neq \operatorname{Spec} A$. But this is obvious since f and g are not nilpotents. $\square$

Corollary. *Let $I \subset A$ be an ideal. The closed set $V(I)$ is irreducible if and only if the radical $\mathfrak{r}(I)$ is prime.*

Thus, we obtain a one-to-one correspondence (cf. Corollary 1.4.2a):

$$\boxed{\text{The points of } \operatorname{Spec} A \longleftrightarrow \text{the irreducible closed subsets of } \operatorname{Spec} A.} \quad (1.29)$$

To every point $x \in \operatorname{Spec} A$ the closed set $\overline{\{x\}}$ corresponds; x is called a *generic point* of this closed set. Every irreducible closed subset has only one generic point, obviously.

1.4.7. Decomposition into irreducible components.

1.4.7a. Theorem. *Let A be a Noetherian ring. Then* $\operatorname{Spec} A$ *can be uniquely represented in the form of a finite union $\bigcup\limits_i X_i$, where the X_i are maximal closed irreducible subsets.*

Proof. The theorem is a geometric reformulation of the ascending chain condition on the ideals of A: Every descending chain of closed subsets of $\operatorname{Spec} A$ stabilizes. $\square$

1.4.7b. Noetherian topological space. Since we will encounter spaces with the property described in Theorem 1.4.7 not homeomorphic to the spectrum of any ring, let us introduce a special definition: A given topological space X will be called *Noetherian* if any descending chain of its closed subsets stabilizes (we say: *DCC holds for* X).

Theorem. *Let X be a Noetherian topological space. Then X is finite union of its maximal closed irreducible subsets.*

The maximal closed irreducible subsets of a Noetherian topological space X are called *irreducible components* of X.

Proof. Consider the set of irreducible closed subsets of X ordered with respect to inclusion. Let us prove that the set is an *inductive* one, i.e., if $\{X_\alpha \mid \alpha \in J\}$ is a linearly ordered family of irreducible closed subsets of X, then for its maximal element we may take $\overline{\left(\bigcap\limits_\alpha X_\alpha\right)}$. The irreducibility of the set $\overline{\left(\bigcap\limits_\alpha X_\alpha\right)}$ follows, for example, from the fact that if U_1, U_2 are non-empty open subsets, then $U_1 \cap X_\alpha$ and $U_2 \cap X_\alpha$ are non-empty for some α, and therefore $U_1 \cap U_2$ is non-empty since the X_α are irreducible.

It follows that X is the union of its irreducible components: $X = \bigcup\limits_{i \in I} X_i$.

So far, we have not used the Noetherian property.

Now, let X be Noetherian and $X = X_1 \cup X_2$, where X_1, X_2 are closed. If one of the X_i is reducible, we can represent it in the form of the union of two closed sets, and so on. This process terminates, otherwise we would have got an infinite descending chain of closed sets (the "Noetherian induction"). In the obtained finite union, let us leave only the maximal elements: $X = \bigcup\limits_{1 \leq i \leq n} X_i$. This decomposition coincides with the above one: If Y is an (absolutely) maximal closed subset of X, then $Y \subset \bigcup\limits_{1 \leq i \leq n} X_i$ implies $Y = \bigcup\limits_{1 \leq i \leq n} (X_i \cap Y)$, and therefore $X_i \cap Y = Y$ for some i; hence $Y = X_i$.

If I' is a proper subset of I, then $\bigcup\limits_{i \in I'} X_i$ does not coincide with X: Let X_j be a discarded component, i.e., $j \notin I'$. If $X_j \subset \bigcup\limits_{i \in I'} X_i$ then $X_j = \bigcup\limits_{i \in I'} (X_i \cap X_j)$ and, due to the irreducibility of X_j, we would have $X_i \cap X_j = X_j$ for some $i \in I'$. This is a contradiction. □

1.4.7c. Corollary. *For any Noetherian ring A, the number of its minimal prime ideals is finite.*

Proof. Indeed, the minimal prime ideals of Spec A are the generic points of maximal closed subsets, i.e., the irreducible components of Spec A. □

1.4.7d. Corollary. *Let A be a Noetherian ring. If all the points of* Spec A *are closed, then* Spec A *is a finite and discrete space.*

The rings A with this property (all the points of Spec A are closed) are called *Artinian* ones. We recall that a common definition of an *Artinian ring* is as a one *for which DCC on ideals holds*. By a theorem ([ZS]) *DCC on ideals implies that every prime ideal is a maximal one* we arrive at another formulation:

> *An Artinian ring is a Noetherian ring*
> *all prime ideals of which are maximal.*

The spectra of Artinian rings resemble very much finite sets in the usual topology. As noted in sec. 1.3.3 (and in what follows), every point of the spectrum of an Artinian ring is additionally endowed with a multiplicity.

1.4.8. An interpretation of zero divisors. The following theorem will be refined later.

Theorem. 1) *Any element $f \in A$ that vanishes* (being considered as a function) *on one of irreducible components of* Spec A *is a zero divisor of A.*

2) *Conversely, if f* (mod N) *is a zero divisor in A/N, where N is the nilradical of A, then f vanishes on one of irreducible components of* Spec A.

Remark. The nilpotents cannot be excluded from the second heading of the theorem: If f is a zero divisor only in A, not in A/N, then it is possible for f not to vanish on an irreducible component. Here is an example: Let the group structures of the rings A and $B \oplus I$, where B is the subring of A without zero divisors, be isomorphic, $I \subset A$ an ideal with zero multiplication. Let $I \cong B/p$ as B-modules, where $p \subset B$ is a non-zero prime ideal. Then the elements from p are zero divisors in A, since they are annihilated under multiplication by the elements from I.

On the other hand, clearly, Spec $A \cong$ Spec B, the spectra are irreducible, and the nonzero elements from p cannot vanish on the whole Spec A.

Proof of Theorem. Let Spec $A = X \cup Y$, where X is an irreducible component on which $f \in A$ vanishes, Y the union of the other irreducible components. Since Y is closed and $X \not\subset Y$, there exists $g \in A$ that vanishes on Y but is not identically zero when restricted to X. Then fg vanishes at all the points of Spec A; hence, $(fg)^n = 0$ for some n. It follows, $f(f^{n-1}g^n) = 0$. This does not prove yet that f is a zero divisor since it might happen that $f^{n-1}g^n = 0$; but then we may again separate f and continue in this way until we obtain $f^m g^n = 0$ and $f^{m-1}g^n \neq 0$. This will always be the case eventually because $g^n \neq 0$; otherwise, g would have also vanished on X.

Now, let $\overline{f} = f \pmod{N}$ be a zero divisor in N/M, i.e., $\overline{f}\overline{g} = 0$ for some $\overline{g} = g \pmod{N}$. Then $\operatorname{Spec} A = \operatorname{Spec} A/N = V(f) \cup V(g)$. Splitting $V(f)$ and $V(g)$ into irreducible components we see that at least one of the irreducible components of $V(f)$ is also irreducible in $\operatorname{Spec} A$. Otherwise, all the irreducible components of $\operatorname{Spec} A$ would have been contained in $V(g)$ contradicting the fact that $\overline{g} \neq 0$, i.e., $g \notin N$. Therefore f vanishes on one of the irreducible components of $\operatorname{Spec} A$, as required. □

Examples. 1) Let A be a unique factorization ring, $f \in A$. The space $\operatorname{Spec} A/(f) \cong V(f)$ is irreducible if and only if $f = ep^n$, where p is an indecomposable element and e is invertible. This follows directly from Theorem 1.4.6. In particular, let $A = K[T_1, \dots, T_n]$, where K is a field. Then $V(f)$ corresponds to the hypersurface (in the affine space) singled out by one equation: $f = 0$. We have just obtained a natural criterion for irreducibility of such a hypersurface.

2) Let K be a field, $\operatorname{Char} K \neq 2$, and $f \in K[T_1, \dots, T_n]$ a quadratic form. The equation $f = 0$ determines a reducible set if and only if $\operatorname{rk} f = 2$. Indeed, the reducibility is equivalent to the fact that $f = l_1 \cdot l_2$, where l_1 and l_2 are non-proportional linear forms.

1.4.9. Connected spaces. The definition of connectedness from general topology is quite suitable for us: A space X is said to be *connected* if it can *not* be represented as the union of two non-intersecting non-empty closed subsets. Clearly, any irreducible space is connected.

Any space X can be uniquely decomposed into the union of its maximal connected subspaces which do not intersect ([K]) and are called the *connected components* of X. Every irreducible component of X belongs entirely to one of its connected components. Theorem 1.4.7b implies, in particular, that the number of connected components of a Noetherian space is finite.

The space $\operatorname{Spec} A$ might be disconnected. For the spectral, as well as for the usual (Hausdorff), topology, we, clearly, have:

The ring of continuous function on the disjoint union $X_1 \coprod X_2$ naturally factorizes into the direct product of the rings of functions on X_1 and X_2. (1.30)

Same happens with the spectra.

1.4.10. The decomposition of Spec A corresponding to a factorization of A. Let $A_1, \dots, A_n$ be some rings, and let the product $\prod\limits_{1\leq i\leq n} A_i = A$ be endowed with the structure of a ring with coordinate-wise addition and multiplication. The set

$$a_i = \{x \in A \mid \text{ all coordinates of } x \text{ except the } i\text{-th one are } 0\} \tag{1.31}$$

is an ideal of A, and $a_i a_j = 0$ for $i \neq j$. Set

$$b_i = \sum_{r\neq i} a_r \quad \text{and} \quad X_i = V(b_i) \subset \operatorname{Spec} A. \tag{1.32}$$

Then

$$\bigcup_{1\leq i\leq n} X_i = V(b_1 \dots b_n) = V(0) = \operatorname{Spec} A,$$
$$X_i \cap X_j = V(b_i \cup b_j) = V(A) = \emptyset \ \text{ if } i \neq j.$$

Therefore

$$\begin{gathered}\operatorname{Spec} \prod_{1\leq i\leq n} A_i \text{ splits into the disjoint union of its closed subsets} \\ V(b_i) \cong \operatorname{Spec} A/b_i = \operatorname{Spec} A_i.\end{gathered} \tag{1.33}$$

Note that, for infinite products, the statement (1.33) is false, see Exercise 1.4.14 7e).

1.4.11. The statement converse to (1.30).

Proposition. *Let* $X = \operatorname{Spec} A = \coprod_{1\leq i\leq n} X_i$, *where the* X_i *are closed non-intersecting subsets. Then there exists an isomorphism* $A \cong \prod_{1\leq i\leq n} A_i$ *such that in the notation of the above subsection* $X_i = V(b_i)$.

Proof. We will consider in detail the case $n = 2$. Let $X_i = V(b_i)$. By Corollary 2.4.12 we have

$$X_1 \cup X_2 = X \Longleftrightarrow V(b_1 b_2) = X \Longleftrightarrow b_1 b_2 \subset N,$$
$$X_1 \cap X_2 = \emptyset \Longleftrightarrow V(b_1 + b_2) = \emptyset \Longleftrightarrow b_1 + b_2 = A.$$

Therefore there exist elements $f_i \in b_i$ and an integer $r > 0$ such that $f_1 + f_2 = 1$ and $(f_1 f_2)^r = 0$.

Lemma. *Let* $f_1, \dots, f_n \in A$. *If* $(f_1, \dots, f_n) = A$, *then* $(f_1^{m_1}, \dots, f_n^{m_n}) = A$ *for any positive integers* $m_1, \dots, m_n$.

Proof. By sec. 1.4 we have $(g_1, \dots, g_n) = A$ if and only if $\bigcap_i V(g_i) = \emptyset$. Since $V(g^m) = V(g)$ for $m > 0$ we obtain the statement desired. □

Thanks to Lemma we have $g_1 f_1^r + g_2 f_2^r = 1$ for some $g_i \in A$. Set $e_i = g_i f_i^r$ for $i = 1, 2$. Then $e_1 + e_2 = 1, e_1 e_2 = 0$, and therefore the $e_i \in b_i$ are orthogonal idempotents which determine a factorization of A:

$$A \xrightarrow{\sim} A_1 \times A_2, \quad g \mapsto (ge_1, ge_2). \tag{1.34}$$

It only remains to show that $V(Ae_i) = X_i$. But, clearly, the $V(Ae_i)$ do not intersect, their union is X, and $Ae_i \subset b_i$; hence $X_i \subset V(Ae_i)$ yielding the statement desired.

1.4.11a. Exercise. Complete the proof by induction on n.

□

1.4.12. Example. Let A be an Artinian ring. Since $\operatorname{Spec} A$ is the union of a finite number of closed points, A is isomorphic to the product of the finite number of local Artinian rings. In particular, any Artinian ring is of finite length (cf. Example 1.3.3).

1.4.13. Quasi-compactness. The usual term "compact" is prefixed with a *quasi-* to indicate that we are speaking about non-Hausdorff spaces. A topological space X is said to be *quasi-compact* if every of its open coverings contains a finite subcovering.

The following simple result is somewhat unexpected, since it does not impose any finiteness conditions onto A:

Proposition. $\operatorname{Spec} A$ *is quasi-compact for any* A.

Proof. Any cover of $\operatorname{Spec} A$ can be refined to a cover with big open sets: $\operatorname{Spec} A = \bigcup\limits_{i\in I} D(f_i)$. Then $\bigcap\limits_{i} V(f_i) = \emptyset$, so that $(f_i)_{i\in I} = A$. Therefore there exists a partition of unity $1 = \sum\limits_{i\in I} g_i f_i$, where $g_i \neq 0$ for a finite number of indices $i \in J \subset I$. Therefore $\operatorname{Spec} A = \bigcup\limits_{i\in J} D(f_i)$, as desired. □

1.4.14. Exercises. 1) Let $S \subset A$ be a *multiplicative* system[10)]. A given multiplicative system S is said to be *complete*, if $fg \in S \Longrightarrow f \in S$ and $g \in S$. Every multiplicative system S has a uniquely determined completion $\widetilde{S}$: It is the minimal complete multiplicative system containing S.

Show that $D(f) = D(g) \Longleftrightarrow \widetilde{(f^n)}_{n\geq 0} = \widetilde{(g^n)}_{n\geq 0}$.

2) Show that the spaces $D(f)$ are quasi-compact.

3) Are the following spaces connected:

a) $\operatorname{Spec} K[T]/(T^2 - 1)$, where K is a field;

b) $\operatorname{Spec} \mathbb{Z}[T]/(T^2 - l)$?

4) The irreducible components of each of the curves

$$T_1(T_1 - T_2^2) = 0,$$
$$T_2(T_1 - T_2^2) = 0,$$

in $\operatorname{Spec} \mathbb{C}[T_1, T_2]$ consists of the line and a parabola, and hence are pairwise isomorphic. The intersection point in each case is the "apex" of the parabola. Prove that, nevertheless, the rings of functions of these curves are non-isomorphic.

5) Let A be a Noetherian ring. Construct a graph whose vertices are in one-to-one correspondence with the irreducible components of the space $\operatorname{Spec} A$, and any two vertices are connected by an edge if and only if the corresponding components have a non-empty intersection. Prove that the connected components of $\operatorname{Spec} A$ are in one-to-one correspondence with the linearly connected components of the graph.

[10] I.e., $1 \in S$ and $f, g \in S \Longrightarrow fg \in S$.

6) Finish the proof of Proposition 1.4.11. Is the decomposition $A = \prod_{i=1}^{n} A_i$, whose existence is claimed, uniquely defined?

7) Let $(K_i)_{i\in I}$ be a family of fields. Set $A = \prod_{i\in I} K_i$ and let $\pi_i \colon A \longrightarrow K_i$ be the projection homomorphisms.

a) Let $a \subset A$ be a proper ideal. Determine from it a family Φ_a of subsets of I by setting:

$$K \in \Phi_a \iff \begin{array}{l}\text{there exists } f \in a \text{ such that } \pi_i(f) = 0,\\ \text{if and only if } i \in K.\end{array}$$

Show that the subsets K are non-empty and the family Φ_a possesses the following two properties:

$$\begin{aligned}&\alpha)\ K_1 \in \Phi_a \ \&\ K_2 \in \Phi_a \implies K_1 \cap K_2 \in \Phi_a,\\ &\beta)\ K_1 \in \Phi_a \ \&\ K_2 \supset K_1 \implies K_2 \in \Phi_a.\end{aligned}$$

b) The family Φ of non-empty subsets of a set I with properties α) and β) is called a *filter* on the set I.

Let Φ be a filter on I; assign to it the set $a_\Phi \subset A$ by setting:

$$f \in a_\Phi \iff \{i \mid \pi_i(f) = 0\} \in \Phi.$$

Show that a_Φ is an ideal in A.

c) Show that the maps $a \mapsto \Phi_a$ and $\Phi \mapsto a_\Phi$ determine a one-to-one correspondence between the ideals of A and filters on I. Further, $a_1 \subset a_2 \iff \Phi_{a_1} \subset \Phi_{a_2}$. In particular, to the maximal ideals maximal filters correspond; they are called *ultrafilters*.

d) Let $i \in I$, and $\Phi^{(i)} = \{K \subset I \mid i \in K\}$. Show that $\Phi^{(i)}$ is an ultrafilter. Show that if I is a finite set, then any ultrafilter is of the form $\Phi^{(i)}$ for some i. Which ideals in A correspond to filters $\Phi^{(i)}$? What are the quotients of A modulo these ideals?

e) Show that if the set I is infinite, then there exists an ultrafilter on I distinct from $\Phi^{(i)}$. (

Hint. Let $\Phi = \{K \subset I \mid I \setminus K\}$ be finite; let $\overline{\Phi}$ be a maximal filter containing Φ. Verify that $\overline{\Phi} \neq \Phi^{(i)}$ for all $i \in I$.)

f) Let $A = \prod_{q\in I} \mathbb{Z}/q\mathbb{Z}$, where I is the set of all primes. Let $p \subset A$ be the prime ideal corresponding to an ultrafilter distinct from all the $\Phi^{(q)}$. Show that A/p is a field of characteristic 0.

1.5. The affine schemes (a preliminary definition)

To any map of sets $f\colon X \longrightarrow Y$, there corresponds a homomorphism of the rings of functions on these sets $f^*\colon F(Y) \longrightarrow F(X)$ given by the formula

$$f^*(\varphi)(x) = \varphi(f(x)). \tag{1.35}$$

If X, Y are *topological spaces* and $F(X)$, $F(Y)$ are rings not of arbitrary, but of *continuous* functions, then the homomorphism f^* is uniquely recovered from a *continuous* f. Without certain conditions this correspondence does not have to take place, e.g., the homomorphism f^* of rings of, say, continuous functions is **not** uniquely recovered from an arbitrary f.

The prime objects of **our** study are the rings (of “functions”); therefore important maps of the spaces are — for us — only those obtained from ring homomorphisms.

Let $\varphi\colon A \longrightarrow B$ be a ring homomorphism. To every prime ideal $p \subset B$, we assign its pre-image $\varphi^{-1}(p)$. The ideal $\varphi^{-1}(p)$ is prime because φ induces the embedding $A/\varphi^{-1}(p) \hookrightarrow B/p$ and, since B/p has no zero divisors, neither has $A/\varphi^{-1}(p)$. We have determined a map ${}^a\varphi\colon \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$, where the superscript a is for “affine”.

1.5.1. Theorem. 1) *${}^a\varphi$ is continuous as a map of topological spaces (with respect to the Zariski topologies of these spaces).*

2) ${}^a(\varphi\psi) = {}^a\psi \circ {}^a\varphi$.

Proof. 2) is obvious. To prove 1), it suffices to verify that the pre-image of a closed set is closed. Indeed,

$$y \in V(\varphi(E)) \iff \varphi(E) \subset p_y \iff E \subset \varphi^{-1}(p_y) = p_{\,{}^a\varphi(y)}$$

$$\iff {}^a\varphi(y) \in V(E) \iff y \in ({}^a\varphi)^{-1}(V(E)).$$

We have actually proved that $({}^a\varphi)^{-1}(V(E)) = V(\varphi(E))$.

□

Therefore

$$\operatorname{Spec} : \mathsf{Rings}^\circ \longrightarrow \mathsf{Top} \text{ is a functor.}$$

1.5.2. Affine schemes as spaces. An *affine scheme* is a triple (X, α, A), where X is a topological space, A a ring and $\alpha\colon X \xrightarrow{\sim} \operatorname{Spec} A$ an isomorphism of spaces.

A *morphism of affine schemes* $(Y, \beta, B) \longrightarrow (X, \alpha, A)$ is a pair (f, φ), where $f\colon Y \longrightarrow X$ is a continuous map and $\varphi\colon A \longrightarrow B$ is a ring homomorphism, such that the diagram

$$\begin{tikzcd} Y \arrow[r, "\beta"] \arrow[d, "f"'] & \operatorname{Spec} B \arrow[d, "{}^a\varphi"] \\ X \arrow[r, "\alpha"] & \operatorname{Spec} A \end{tikzcd} \tag{1.36}$$

commutes. The composition of morphisms is obviously defined.

To every ring A, there corresponds the affine scheme $(\operatorname{Spec} A, \mathrm{id}, A)$, where id is the identity map; for brevity, we often shorten its name to $\operatorname{Spec} A$. Clearly, any affine scheme is isomorphic to such a scheme; affine schemes constitute a category; the dual category is equivalent to the category of rings.

Our definition is not final since it is ill-adjusted to globalizations, the glueings of general schemes from affine ones. In what follows we will modify it: An additional element of the structure making the *space* $\operatorname{Spec} A$ into the *scheme* $\operatorname{Spec} A$ is a sheaf associated with the ring A. But the above definition will do for a while since

$$\text{\textit{the ring } } A \text{\textit{ and the corresponding sheaf of functions on} } \operatorname{Spec} A \text{\textit{ can be uniquely recovered from each other.}} \tag{1.37}$$

1.5.3. Examples. To appreciate the difference between the set $\operatorname{Hom}(A, B)$, which only is of importance for us, and the set of all the continuous maps $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ consider several simple examples.

1.5.3a. $A = B = \mathbb{Z}$. As we have already observed, $\operatorname{Spec} \mathbb{Z}$ consists of the closed points (p), where p runs over the primes, and (0). The closure of (0) is the whole space; the remaining closed sets consist of a finite number of closed points. There are lots of automorphisms of the space $\operatorname{Spec} \mathbb{Z}$: We may permute the closed points at random; contrariwise $\operatorname{Hom}(\mathbb{Z}, \mathbb{Z})$ contains only the identity map.

1.5.3b. $A = K[T]$, where K is a finite field, $B = \mathbb{Z}$. Obviously, $\operatorname{Spec} A$ and $\operatorname{Spec} B$ are isomorphic as topological spaces, whereas $\operatorname{Hom}(A, B) = \emptyset$.

Examples 1.5.3a and 1.5.3b might make one think that there are much less homomorphisms of rings than there are continuous maps of their spectra. The opposite effect is, however, also possible.

1.5.3c. Let K be a field. Then $\operatorname{Spec} K$ consists of one point, and therefore the set of its automorphism consists of one point, whereas the group of automorpisms of K may be even infinite (a Lie group). Therefore

$$\textbf{one-point spectra may have ``inner degrees of freedom'', like elementary particles.} \tag{1.38}$$

1.5.3d. The spectrum that personifies an "idea" of a vector. ("Combing nilpotents"). Let A be a ring, $B = A[T]/(T^2)$. The natural homomorphism

$$\varepsilon \colon B \longrightarrow A, \quad a + bt \mapsto a, \text{ where } t = T \pmod{(T^2)},$$

induces an isomorphism of topological spaces ${}^a\varepsilon\colon \operatorname{Spec} A \longrightarrow \operatorname{Spec} B$ but by no means that of the schemes.

The scheme $(\operatorname{Spec} B, \mathrm{id}, B)$ is "richer" than $(\operatorname{Spec} A, \mathrm{id}, A)$ by the nilpotents tA. To see how this richness manifests itself, consider arbitrary "projections" ${}^a\pi\colon \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$, i.e., scheme morphisms corresponding to the ring homomorphisms $\pi\colon A \longrightarrow B$ such that $\varepsilon\pi = \mathrm{id}$. Then $\pi(f) - f \in At$. For any such π, define the map $\partial_\pi \colon A \longrightarrow A$ by setting

$$\pi(f) - f = \partial_\pi(f)t. \tag{1.39}$$

Since π is a ring homomorphism, we see that $\partial_\pi(f)$ is a derivation of A, i.e., is linear and satisfies the Leibniz rule:

$$\begin{aligned} \partial_\pi(f+g) &= \partial_\pi(f) + \partial_\pi(g), \\ \partial_\pi(fg) &= \partial_\pi(f)\cdot g + f\cdot\partial_\pi(g). \end{aligned} \tag{1.40}$$

Indeed, the linearity is obvious while the Leibniz rule follows from the identity

$$\pi(fg) = fg + \partial_\pi(fg)t = \pi(f)\pi(g) = (f + \partial_\pi(f)t)(g + \partial_\pi(g)t) \tag{1.41}$$

true thank to the property $t^2 = 0$.

It is easy to see that, the other way round, for any derivation $\partial\colon A \longrightarrow A$ the map $\pi\colon A \longrightarrow B$ determined by the formula $\pi(f) = f + (\partial f)t$ is a ring homomorphism and determines a projection ${}^a\pi$.

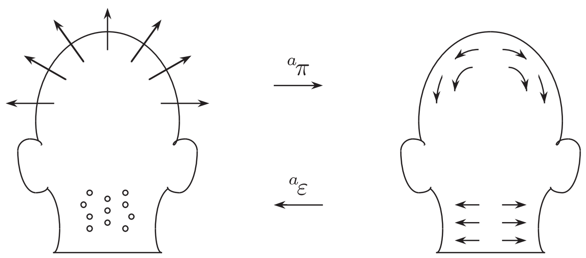

Fig. 6

In Differential Geometry, every derivation of the ring of functions on a manifold is interpreted as a “vector field” on the manifold. One can imagine the scheme $(\operatorname{Spec} B, \mathrm{id})$ endowed with a field of vectors “looking outwards”, as on a hedgehog, as compared with the scheme $(\operatorname{Spec} A, \mathrm{id})$. The morphism ${}^a\pi$ “combs” these vectors turning them into vector fields on $\operatorname{Spec} A$.

In particular, if K is a field, then $\operatorname{Spec} K$ is a point and

$$\text{the scheme } (\operatorname{Spec} K[T]/(T^2), \mathrm{id}) \text{ is an “idea of the vector”} \tag{1.42}$$

with the point $\operatorname{Spec} K$ as the source of the vector.

In what follows we will sometimes plot nilpotents in the form of arrows though it is obvious that even for the simplest scheme like

$$\begin{aligned} &(\operatorname{Spec} K[T_1, T_2]/(T_1^2, T_1T_2, T_2^2), \mathrm{id}); \\ &(\operatorname{Spec} K[T]/(T^n), \mathrm{id}) \quad \text{for } n \geq 3, \\ &\operatorname{Spec}(\mathbb{Z}/(p^2), \mathrm{id}) \quad \text{for } p \text{ prime} \end{aligned} \tag{1.43}$$

such pictures are of limited informative value.

1.5.3e. "Looseness" of affine spaces. Let K be a field (for simplicity's sake), V a linear space over K and $A = S^{\bullet}_K(V)$. Consider the group G of automorphisms of the K-scheme $(\operatorname{Spec} A, \mathrm{id})$. This group consists of transformations induced by those that constitute the group of K-automorphisms of the ring $K[T_1, \dots, T_n]$, where $n = \dim V$. The group G contains as a subgroup the usual Lie group G_0 of invertible affine linear transformations

$$T_i \mapsto \sum_j c_i^j T_j + d_i, \quad \text{where } c_i^j, d_i \in K. \tag{1.44}$$

For $n = 1$, it is easy to see that $G_0 = G$. This is far from so for $n \geq 2$. Indeed, in this case any "triangular" substitution of the form

$$\begin{aligned} &T_1 \mapsto T_1 + F_1, \\ &T_2 \mapsto T_2 + F_2(T_1), \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &T_i \mapsto T_i + F_i(T_1, \dots, T_{i-1}), \end{aligned} \tag{1.45}$$

where $F_i \in K[T_1, \dots, T_{i-1}] \subset K[T_1, \dots, T_n]$, clearly, belongs to G. Therefore the group of automorphisms of the scheme corresponding to the affine space of dimension ≥ 2 contains non-linear substitutions of however high degree. Their existence is used in the proof of *Noether's normalization Theorem.* [11)]

Remark. For $n = 2$, the group G is generated by linear and triangular substitutions (Engel, Shafarevich); for $n \geq 3$, this is not so (*Nagata's conjecture on automorphisms* proved by I. Shestakov and U. Umirbaev [12)]). The explicit description of G is an open (and very tough) problem [13)].

1.5.3f. Linear projections. Let $V_1 \subset V_2$ be two linear spaces over a field K, and let $X_i = \operatorname{Spec} S^{\bullet}_K(V_i)$. The morphism $X_2 \longrightarrow X_1$, induced by the embedding $S^{\bullet}_K(V_1) \hookrightarrow S^{\bullet}_K(V_2)$, is called the *projection of the scheme X_2 onto X_1*. On the sets of K-points it induces the natural map $X_2(K) = V_2^* \longrightarrow V_1^* = X_1(K)$ which restricts every linear functional on V_2 to V_1.

1.6. Topological properties of certain morphisms and the maximal spectrum

In this section we study the most elementary properties of the morphisms

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[11] **Noether's normalization Theorem**: *Let $k[x]$, where $x = (x_1, \dots, x_n)$, be a finitely generated entire ring over a filed k, and assume that $k(x)$ has transcendence degree r. Then there exist elements $y_1, \dots, y_r$ in $k[x]$ such that $k[x]$ is integral over $k[y]$.* ([Lang])

[12] For a generalization of the result of Shestakov and Umirbaev, see [Ku].

[13] This problem is close to a famous and still open *Jacobian problem*, see [EM].

$$ {}^a\varphi\colon \operatorname{Spec} B \longrightarrow \operatorname{Spec} A, \qquad p_x \mapsto \varphi^{-1}(p_x). \tag{1.46}$$

This study gives a partial answer to the question: *What is the structure of the topological space* ${}^a\varphi(\operatorname{Spec} B)$?

As is known (see, e.g., [Lang]), any homomorphism $\varphi\colon A \longrightarrow B$ factorizes into the product of the surjective ring homomorphism $A \longrightarrow A/\operatorname{Ker}\varphi$ and an embedding $A/\operatorname{Ker}\varphi \longrightarrow B$. Let us find out the properties of ${}^a\varphi$ in these two cases.

1.6.1. The properties of ${}^a\varphi$. The first of these cases is very simple.

Proposition. *Let $\varphi\colon A \longrightarrow B$ be a ring epimorphism. Then ${}^a\varphi$ is a homeomorphism of* $\operatorname{Spec} B$ *onto the closed subset $V(\operatorname{Ker}\varphi) \subset \operatorname{Spec} A$.*

1.6.1a. Exercise. Verify that this is a direct consequence of the definitions.

Hint. Prove the continuity of the inverse map

$$({}^a\varphi)^{-1}\colon V(\operatorname{Ker}\varphi) \longrightarrow \operatorname{Spec} B. \quad \square$$

In particular, let A be a *ring of finite type* over a field $\mathbb{K}$ or over $\mathbb{Z}$, i.e., let A be a quotient of either $\mathbb{K}[T_1, \dots, T_n]$ or $\mathbb{Z}[T_1, \dots, T_n]$ for $n < \infty$.

The spectrum of the polynomial ring plays the role of an affine space (over $\mathbb{K}$ or $\mathbb{Z}$, respectively, cf. Example 1.2.1). Therefore spectra of the rings of finite type correspond to affine varieties ("arithmetic affine varieties" if considered over $\mathbb{Z}$): They are embedded into finite dimensional affine spaces.

Thus, ring epimorphisms correspond to embeddings of spaces. Ring monomorphisms do not necessarily induce surjective maps of spectra: Only the closure of ${}^a\varphi(\operatorname{Spec} B)$ coincides with $\operatorname{Spec} A$. This follows from a trifle more general result.

1.6.2. Proposition. *For any ring homomorphism $\varphi\colon A \longrightarrow B$ and an ideal $b \subset B$, we have*

$$\overline{{}^a\varphi(V(b))} = V(\varphi^{-1}(b)).$$

In particular, if $\operatorname{Ker}\varphi = 0$*, then* $\overline{{}^a\varphi(V(0))} = V(0)$*, i.e., the image of* $\operatorname{Spec} B$ *is dense in* $\operatorname{Spec} A$.

Proof. We may assume that b is a radical ideal, since $V(\mathfrak{r}(b)) = V(b)$ and $\varphi^{-1}(\mathfrak{r}(b)) = \mathfrak{r}(\varphi^{-1}(b))$. The set $\overline{{}^a\varphi(V(b))}$ is the intersection of all closed subsets containing ${}^a\varphi(V(b))$, i.e., the set of common zeroes of all the functions $f \in A$ which vanish on ${}^a\varphi(V(b))$. But f vanishes on ${}^a\varphi(V(b))$ if and only if $\varphi(f)$ vanishes on $V(b)$, i.e., if and only if $\varphi(f) \in b$ (since b is a radical ideal) or, finally, if and only if $f \in \varphi^{-1}(b)$. Therefore the closure we are interested in is equal to $V(\varphi^{-1}(b))$. $\square$

Now, let us give examples of ring monomorphisms for which ${}^a\varphi(\operatorname{Spec} B)$ does not actually coincide with $\operatorname{Spec} A$.

1.6.2a. Examples. 1) *The projection of a hyperbola onto a coordinate axis.* For a field K, consider an embedding

$$\varphi\colon A = K[T_1] \hookrightarrow K[T_1, T_2]/(T_1T_2 - 1) = B. \tag{1.47}$$

Then ${}^a\varphi(\operatorname{Spec} B) = D(T_1)$, in accordance with the picture, see Fig. 7.

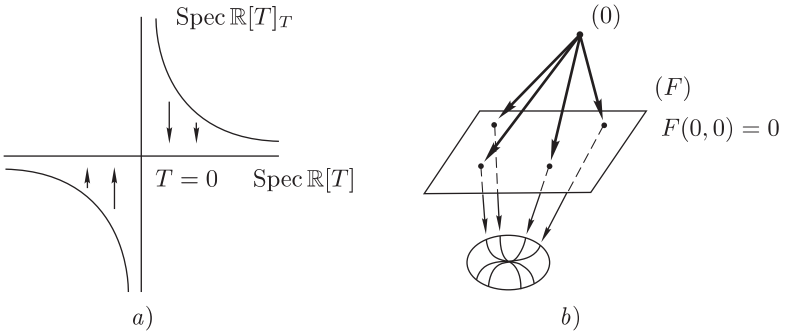


Fig. 7

Indeed, ${}^a\varphi$ maps a generic point into a generic one. The prime ideal $(f(T_1)) \subset A$, where $f \neq cT_1$ is an irreducible polynomial, is the pre-image of the prime ideal $(f(T_1)) \pmod{T_1T_2 - 1} \subset B$. Finally, T_1 and $T_1T_2 - 1$ generate the ideal $(1) = K[T_1, T_2]$, and therefore $(T_1) \notin {}^a\varphi(\operatorname{Spec} B)$. □

In this example ${}^a\varphi(\operatorname{Spec} B)$ is open; but it may be neither open nor closed:

2) *The projection of a hyperbolic paraboloid onto the plane.* Consider the homomorphism

$$\varphi\colon A = K[M, N] \hookrightarrow B = K[M, N, T]/(MT - N).$$

1.6.2b. Exercise. Verify that ${}^a\varphi(\operatorname{Spec} B) = D(M) \cup V(M, N)$, and that this set is actually not open (it is obvious that it is not closed).

□

This example, see also Fig. 8, illustrates the phenomenon noted long ago in the study of equations:

The set ${}^a\varphi(\operatorname{Spec} B)$ is the set of values of the coefficients M, N in a K-algebra for which the equation $MT - N = 0$ is solvable for T. (1.48)

In general, the condition for solvability is the inequality $M \neq 0$, but even for $M = 0$ the solvability is guaranteed if, in addition, $N = 0$.

We may prove that if A is Noetherian and an A-algebra B has a finite number of generators, then ${}^a\varphi(\operatorname{Spec} B)$ is the union of a finite number of *locally*

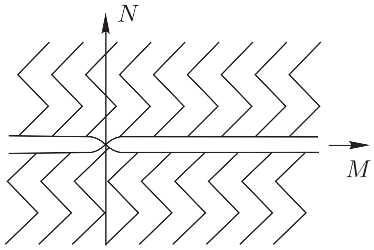


Fig. 8

closed sets [14]. Such unions are called *constructible* sets; the image of a constructible set under ${}^a\varphi$ is always constructible under the described conditions (*Chevalley's theorem* [AM]).

In terms of undetermined coefficients of a (finite) system of equations this means that the condition for the system's compatibility is the following:

$$\text{\textit{The coefficients should satisfy one of a finite number of statements, each statement is a collection of a finite number of polynomial equalities and inequalities} (with the zero right-hand sides).} \tag{1.49}$$

For example, for the equation $MT - N = 0$, there are the two statements:

1. $M \neq 0$;
2. $M = 0$, $N = 0$.

1.6.3. Analogs of "finite-sheeted coverings" of Riemannian surfaces. In the above examples something "escaped to infinity". Let us describe an important class of morphisms ${}^a\varphi$ for which this does not happen.

Let B be an A-algebra. An element $x \in B$ is called *integer over* A if it satisfies an equation of the form

$$x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0, \quad a_0, \ldots, a_{n-1} \in A \tag{1.50}$$

and B is called *integer over* A if all its element are integer over A.

There are two important cases when it is easy to establish whether B is integer over A.

Case 1. *If B has a finite number of generators as an A-module, then B is integer over A.*

Indeed, if A is Noetherian ring, then, for any $g \in B$, the ascending sequence of A-modules

$$B_k = \sum_{0 \leq i \leq k} Ag^i \subset B \tag{1.51}$$

stabilizes. Therefore, for some k, we have $g^k \in \sum\limits_{0 \leq i \leq k-1} Ag^i$, which provides us with an equation of integer dependence.

[14] A *locally closed* set is the intersections of a closed and an open set.

The general case reduces to the above one with the help of the following trick. Let $B = \sum\limits_{1\leq i\leq n} Af_i$. Let

$$f_i f_j = \sum_{1\leq k\leq n} a_{ij}^k f_k, \text{ where } a_{ij}^k \in A, \text{ and } g = \sum_{1\leq i\leq n} g_i f_i, \text{ where } g_i \in A; \tag{1.52}$$

denote by $A_0 \subset A$ the minimal subring containing all the a_{ij}^k and g_i and set $B_0 = \sum\limits_{1\leq i\leq n} A_0 f_i$. Obviously, A_0 is a Noetherian ring, B is an A_0-algebra and $g \in B_0$. Therefore g satisfies the equation of integer dependence with coefficients from A.

Case 2. *Let G be a finite subgroup of the group of automorphisms of B and $A = B^G$ the subring of G-invariant elements. Then B is integer over A.*

Indeed, all the elementary symmetric polynomials in $s(g)$, where $s \in G$, belong to A for any $g \in B$, and g satisfies $\prod\limits_{s\in G}(g - s(g)) = 0$. □

1.6.4. Localization with respect to a multiplicative system. We would like to define the ring of fractions f/g, where g runs over a set S of elements of an arbitrary ring A. When we add or multiply fractions their denominators are multiplied; hence S should be closed with respect to multiplication. Additionally we require S to have the unit element, i.e., to be what is called a *multiplicative system*.

Examples of multiplicative systems S:

(1) $S_f = \{f^n \mid n \in \mathbb{Z}_+\}$ for any $f \in A$,
(2) $S_p := A \setminus p$ for any prime ideal p.

The set S_p is indeed a multiplicative system because S_p consists of "functions" $f \in A$ that do not vanish at $\{p\} \in \operatorname{Spec} A$; now recall the definition of the prime ideal.

For any multiplicative system S of a ring A, we define the *ring of fractions* A_S (also denoted $S^{-1}A$ or $A[S^{-1}]$), or the *localization with respect to* S, as follows. As a set, A_S is the quotient of $A \times S$ modulo the following equivalence relation:

$$\begin{aligned} &(f_1, s_1) \sim (f_2, s_2) \iff \text{there exists } t \in S \text{ such that}\\ &t(f_1 s_2 - f_2 s_1) = 0. \end{aligned} \tag{1.53}$$

Denote by f/s the class of an element (f, s) and define the composition law in A_S by the usual formulas:

$$f/s + g/t = (ft + gs)/st; \quad (f/s)\cdot(g/t) = fg/st. \tag{1.54}$$

Exercise. Verify that the above equivalence relation is well-defined. The unit of A_S is the element $1/1$, and the zero is $0/1$.

Quite often the localizations with respect to S_f and S_p are shortly (and somewhat self-contradictory) denoted by A_f and A_p, respectively.

For the rings without zero divisors, the map $a \mapsto a/1$ embeds A into A_S. In general, however, a nontrivial kernel may appear as is described in the following obvious Lemma.

1.6.4a. Lemma. *Let $j\colon A \longrightarrow A_S$ be the map $a \mapsto a/1$. Then*

1) *j is a ring homomorphism and*

$$\mathrm{Ker}\ \ j = \{f \in A \mid \ \text{there exists } s \in S \text{ such that } sf = 0\};$$

2) *if $0 \notin S$, then all the elements from $j(S)$ are invertible in A_S, otherwise $A_S = 0$;*

3) *every element from A_S can be represented in the form $j(f)/j(s)$ with $s \in S$, $f \in A$.*

Notice how drastically A_S shrinks when we introduce zero divisors in S.

The following theorem is a main fact on rings of fractions, it describes a universal character of localization.

1.6.4b. Theorem. *Let S be a multiplicative system in a ring A and $j\colon a \mapsto a/1$ the canonical homomorphism $A \longrightarrow A_S$. For any ring homomorphism $f\colon A \longrightarrow B$ such that every element from $f(S)$ is invertible, there exists a unique homomorphism $f'\colon A_S \longrightarrow B$ such that $f = f' \circ j$.*

1.6.4c. Exercise. Prove the theorem.

Corollary. *Let A be a ring, T and S its multiplicative systems such that $T \supset S$. Then*

1) *the following diagram commutes:*

$$\begin{tikzcd}
A \arrow[rr, "j_S"] \arrow[dr, "j_T"'] & & A_S \arrow[dl, "j_T(S^{-1})"] & \ni j_S(a)/j_S(t) \arrow[d] \\
& A_T & & \ni j_T(a)/j_T(t)
\end{tikzcd} \tag{1.55}$$

2) $A[T^{-1}] \cong A[S^{-1}][j_S(T)^{-1}]$.

1.6.4d. Theorem. *Let $j\colon A \longrightarrow A_S$ be the canonical homomorphism. Then the induced map ${}^a j\colon \mathrm{Spec}\, A_S \longrightarrow \mathrm{Spec}\, A$ homeomorphically maps $\mathrm{Spec}\, A_S$ onto the subset $\{x \in \mathrm{Spec}\, A \mid p_x \cap S = \emptyset\}$.*

Proof. We may confine ourselves to the case $0 \notin S$. First, let us prove that there is a one-to-one correspondence

$$\mathrm{Spec}\, A_S \longleftrightarrow \{x \in \mathrm{Spec}\, A \mid p_x \cap S = \emptyset\}. \tag{1.56}$$

Let $y \in \mathrm{Spec}\, A_S$ and $x = {}^a j(y)$. Then $p_x \cap S = \emptyset$, since otherwise p_y would have contained the image under j of an element from S, which is invertible, so p_y would have contained the unit.

Second, let $x \in \mathrm{Spec}\, A$ and $p_x \cap S = \emptyset$. Set $p_y = p_x[S^{-1}]$. The ideal p_y is prime. Indeed, let $(fs^{-1})(gt^{-1}) \in p_y$. We have $fg \in p_x$ and, since p_x is prime, then either $f \in p_x$ or $g \in p_x$, and therefore either $fs^{-1} \in p_y$ or $gt^{-1} \in p_y$.

The fact that the maps $p_y \mapsto j^{-1}(p_y)$ and $p_x \mapsto p_x[S^{-1}]$ are mutually inverse may be established as follows.

Consider the set $\{f \in A \mid f/1 \in p_x[S^{-1}]\}$. Let $f/1 = f'/s$, where $f' \in p_x$. Multiplying by some $t \in S$ we obtain $tf \in p_x$ implying $f \in p_x$.

Now, it remains to show that this one-to-one correspondence is a homeomorphism. As had been proved earlier, the map ${}^a j$ is continuous. Therefore it suffices to prove that under this map the image of every closed set is closed, i.e., $j(V(E)) = V(E')$ for some E'. Indeed, take

$$E' = \{\text{denominators of the elements from } E\}. \quad \square$$

1.6.4e. Example. The set $D(f)$ is homeomorphic to $\operatorname{Spec} A_f$. Indeed,

$$p_x \cap \{f^n \mid n \in \mathbb{Z}\} = \emptyset \Longleftrightarrow f \notin p_x. \tag{1.57}$$

Therefore $\operatorname{Spec} A$ splits into the union of an open and a closed set as follows:

$$\operatorname{Spec} A = \operatorname{Spec} A_f \cup \operatorname{Spec} A/(f). \tag{1.58}$$

Here a certain "duality" between the localization and the passage to the quotient ring is reflected. Considering $D(f)$ as $\operatorname{Spec} A_f$ we "send $V(f)$ to infinity".

If S is generated by a finite number of elements $f_1, \dots, f_n$, then

$$p_x \cap S = \emptyset \Longleftrightarrow x \in \bigcap_{1 \le i \le n} D(f_i) \tag{1.59}$$

and in this case the image of $\operatorname{Spec} A_S$ is open in $\operatorname{Spec} A$. However, this is not always the case; cf. Examples 1.6.2.

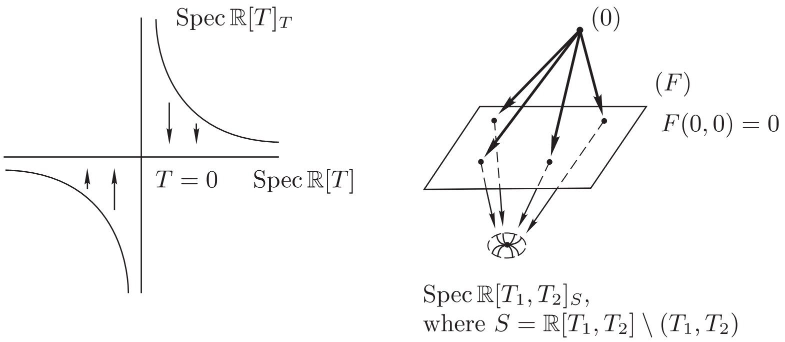
If $S = A \backslash p_x$, then the image of $\operatorname{Spec} A_S$ in $\operatorname{Spec} A$ consists of all the points $y \in \operatorname{Spec} A$ whose specialization is x. It is not difficult to see that in general this set is neither open nor closed in $\operatorname{Spec} A$: *It is the intersection of all the open sets containing x.*

The ring $A_{p_x} = \mathcal{O}_x$ contains the unique maximal ideal p_x whose spectrum describes geometrically the "neighborhood of x" in the following sense: We may trace the behavior of all the irreducible subsets of $\operatorname{Spec} A$ through x in a "vicinity" of x. This is the algebraic version of the *germ of neighborhoods* of x in the following sense: We can trace the behavior of the irreducible subsets of $\operatorname{Spec} A$ passing through x in a "vicinity" of x. $\square$

1.6.5. Theorem. *Let $\varphi\colon A \hookrightarrow B$ be a monomorphism and let B be integer over A. Then ${}^a\varphi(\operatorname{Spec} B) = \operatorname{Spec} A$.*

Proof. We first prove two particular cases; next we reduce the general statement to these cases.

Case 1. *B is a field.* Then ${}^a\varphi(\operatorname{Spec} B) = \{(0)\} \subset \operatorname{Spec} A$ and ${}^a\varphi$ is an epimorphism if and only if A has no other prime ideals, i.e., A is a field. Let us verify this.



a) *b*)

Fig. 9

Let $f\in A$, $f\neq 0$. Let us show that if f is invertible in B, then so it is in A. Since f^{-1} is integer over A, i.e., satisfies an equation

$$f^{-n}+\sum_{1\leq i\leq n-1}a_i f^{-i}=0,\quad a_i\in A. \tag{1.60}$$

Then multiplying by f^{n-1} we obtain

$$f^{-1}=-\sum_{0\leq i\leq n-1}a_i f^{n-i-1}\in A, \tag{1.61}$$

as desired.

Case 2. *A is a local ring.* Then, under the conditions of the theorem, the unique closed point of $\operatorname{Spec}A$ belongs to ${}^a\varphi(\operatorname{Spec}B)$ and, moreover, it is the image under ${}^a\varphi$ of any other closed point of $\operatorname{Spec}B$.

Indeed, let p be a maximal ideal of A, q any maximal ideal of B. Then B/q is a field integer over the subring $A/(A\cap q)$ which by the proved above case 1, should also be a field. This means that $A\cap q$ is a maximal ideal in A, and therefore $A\cap q=\varphi^{-1}(q)=p$.

The general case. Let $p\subset A$ be a prime ideal; we wish to show that there exists an ideal $q\subset B$ such that $q\cap A=p$. Set $S=A\setminus p$.

Considering S as a subset of both A and B, we may construct the rings of quotients $A_S\subset B_S$. Set $p_S=\{f/s\mid f\in p, s\in S\}$. It is easy to see that $p_S\subset A_S$ is a prime ideal. It is maximal since $A_S\setminus p_S$ consists of invertible elements $s/1$.

The ring B_S is integer over A_S since if $f\in B$ satisfies $f^n+\sum\limits_{0\leq i\leq n-1}a_i f^i=0$, then $f/s\in B_S$ satisfies

$$(f/s)^n+\sum_{0\leq i\leq n-1}a_i/s^{n-i}\cdot(f/s)^i=0. \tag{1.62}$$

Therefore, by case 2, there exists a prime ideal $q_S \subset B_S$ such that $A_S \cap q_S = p_S$.

The pre-image of q_S in B with respect to the natural homomorphism $B \longrightarrow B_S$ is a prime ideal. It remains to verify that $A \cap q = p$. The inclusion $p \subset A \cap q$ is obvious.

Let $f \in A \cap q$. There exist $n \in \mathbb{N}$ and $s \in S$ such that $f/s^n \in q_S$. Therefore $f/s^n \in A_S \cap q_S = p_S$, so that $s^m f \in p$ for some $m \geq 0$; hence $f \in p$. □

In this proof the quotient ring A_S appeared as a technical trick which enabled us to "isolate" — localize — the prime ideal $p \subset A$ making from it a maximal ideal p_S in A_S. The term "localization" is applied to the construction of quotient rings with precisely this geometric meaning.

1.6.6. Addendum to Theorem 1.6.4a. Denote by $\operatorname{Spm} A$ the set of *maximal* ideals of A (the "maximal spectrum").

Proposition. *Under conditions of Theorem 1.6.4a we have*

1) ${}^a\varphi(\operatorname{Spm} B) = \operatorname{Spm} A$*;*
2) $({}^a\varphi)^{-1}(\operatorname{Spm} A) = \operatorname{Spm} B$*.*

Proof. Let $p \in \operatorname{Spm} B$; then B/p is a field integer over $A/(A \cap p) = A/\varphi^{-1}(p)$. Thanks to Theorem 1.6.4a (case 1) $A/\varphi^{-1}(p)$ is also a field; therefore $\varphi^{-1}(p)$ is maximal in A. This demonstrates that ${}^a\varphi(\operatorname{Spm} B) \subset \operatorname{Spm} A$.

To prove heading 2), consider a prime ideal $q \subset B$ such that $p = A \cap q \subset A$ is maximal. The ring without zero divisors B/q is integer over the field A/p and we ought to verify that it is also a field. Indeed, any $f \in B/q$, being integer over A/p, belongs to a finite dimensional A/p-algebra generated by the powers of f. Multiplication by f in this algebra is a linear map without kernel, and therefore an epimorphism. In particular, the equation $fu = 1$ is solvable, proving the statement desired. □

1.6.6a. Warning. Let $\varphi\colon A \longrightarrow B$ be a ring homomorphism, and $x \in \operatorname{Spm} B$, $y \in \operatorname{Spm} A$. In general, the point ${}^a\varphi(x)$ is nonclosed, and $({}^a\varphi)^{-1}(y)$ also contains nonclosed points; so Proposition 2.7.6 describes a rather particular situation.

Example. Let $\varphi\colon \mathbb{Z}_p \hookrightarrow \mathbb{Z}_p[T]$ be the natural embedding. Then $p_x = (1 - pT)$ is, clearly, a maximal ideal in $\mathbb{Z}_p[T]$ (the quotient modulo it is isomorphic to $\mathbb{Q}_p$). Obviously,

$$\varphi^{-1}(p_x) = \mathbb{Z}_p \cap (1 - pT) = (0), \tag{1.63}$$

and therefore ${}^a\varphi(x) \notin \operatorname{Spm} \mathbb{Z}_p$, where $x \in \operatorname{Spec} \mathbb{Z}_p[T]$ is the point corresponding to p_x. Moreover, the image of the closed point x is a generic point of $\operatorname{Spec} \mathbb{Z}_p$; this generic point is an open set being the complement to (p).

Now, let $p_y = (p) \subset \mathbb{Z}_p[T]$ and $p_x = (p) \subset \mathbb{Z}_p$. Then $y \in ({}^a\varphi)^{-1}(x)$, and x is closed whereas y is not.

This, however, is not unexpected. Even more transparent is the example of the projection of the plane onto the straight line given by the following embedding of the corresponding rings

$$K[T_1] \hookrightarrow K[T_1, T_2], \quad T_1 \mapsto T_1. \tag{1.64}$$

The pre-image of the point $T_1 = 0$ of the line contains, clearly, the generic point of the T_2-axis which is not closed in the plane.

In particular, $\operatorname{Spm} A$ *is not a functor* in A, unlike $\operatorname{Spec} A$.

1.6.7. Exercises. 1) Let B be an A-algebra. Prove that the elements of B, integer over A, constitute an A-subalgebra of B.

2) Let $A \subset B \subset C$ be rings; let B be integer over A, and C integer over B. Prove that C is integer over A.

3) Let A be a unique factorization ring. Then A is an integrally closed ring in its ring of quotients, i.e., any f/g integer over A belongs to A.

1.7. The closed subschemes and the primary decomposition

In this section we will often denote the subscheme $(V(a), \alpha, S)$ by $\operatorname{Spec} A/a$ and omit the word "closed" because no other subschemes will be considered here.

1.7.1. Reduced schemes. Closed embeddings. Let $X = \operatorname{Spec} A$ be an affine scheme, $a \subset A$ an ideal. The scheme $(V(a), \alpha, A/a)$, where $\alpha\colon V(a) \xrightarrow{\sim} \operatorname{Spec} A/a$ is the canonical isomorphism (see sec. 1.6.1) is said to be a *closed subscheme* of X corresponding to a. Therefore

the closed **subschemes** *of the* **scheme** $X = \operatorname{Spec} A$ *are in 1–1 correspondence with all the ideals of* A, (1.65)

unlike the closed **subsets** of the **space** $\operatorname{Spec} A$, which only correspond to the *radical* ideals.

The *support* of $Y = \operatorname{Spec} A/a \subset X$ is the space $V(a)$; it is denoted by $\operatorname{supp} Y$.

To the projection $A \longrightarrow A/a$ a scheme embedding $Y \longrightarrow X$ corresponds; it is called a *closed embedding* of a subscheme.

For any ring L, we had set

$$X(L) := \operatorname{Hom}(\operatorname{Spec} L, X) := \operatorname{Hom}(A, L), \tag{1.66}$$

and called it the *set of L-points* of X (cf. definition 1.2.1). Clearly, the L-points of a subscheme Y constitute a subset $Y(L) \subset X(L)$ and the functor $L \longrightarrow Y(L)$ is a subfunctor of the functor $L \longrightarrow X(L)$.

There is a natural *order* on the set of closed subschemes of a scheme X: We say that $Y_1 \subset Y_2$ if $a_1 \supset a_2$, where a_i is the ideal that determines Y_i. The use of the inclusion sign is justified by the fact that

$$Y_1 \subset Y_2 \Longleftrightarrow Y_1(L) \subset Y_2(L) \text{ for every ring } L. \tag{1.67}$$

The relation "Y is a closed subscheme of X" is transitive in the obvious sense.

For any closed subset $V(E) \subset X$, there exists a unique minimal closed subscheme with $V(E)$ as the support: It is determined by the ideal $\mathfrak{r}((E))$ and its ring has no nilpotents. Such schemes are called *reduced* ones.

In particular, the subscheme $\operatorname{Spec} A/N$ is the minimal closed subscheme whose support is the whole space $\operatorname{Spec} A$. If $X = \operatorname{Spec} A$, then its reduction $\operatorname{Spec} A/N$ is denoted by X_{red}. Thus, a scheme X is reduced if $X = X_{red}$.

1.7.2. Intersections. The *intersection* $\bigcap\limits_i Y_i$ of a family of subschemes Y_i, where $Y_i = \operatorname{Spec} A/a_i$ is the subscheme determined by the ideal $\sum\limits_i a_i$.

The notation is justified by the fact that, for any ring L, the set of L-points $\big(\bigcap\limits_i Y_i\big)(L)$ of the intersection is naturally identified with $\bigcap\limits_i Y_i(L)$. Indeed, an L-point $\varphi\colon A \longrightarrow L$ belongs to $\bigcap\limits_i Y_i(L)$ if and only if $a_i \subset \operatorname{Ker}\varphi$ for all i, which is equivalent to the inclusion $\sum\limits_i a_i \subset \operatorname{Ker}\varphi$. This argument shows that

$$\operatorname{supp}\Big(\bigcap_i Y_i\Big) = \bigcap_i \operatorname{supp} Y_i. \tag{1.68}$$

1.7.3. Quasiunions. The notion of the *union of a family of subschemes* is *not* defined in a similar way. In general, for given Y_i's, there is no closed subscheme Y such that $Y(L) = \bigcup\limits_i Y_i(L)$ for all L. However, there exists a minimal subscheme Y such that

$$Y(L) \supset \bigcup_i Y_i(L) \text{ for all } L. \tag{1.69}$$

This Y is determined by the ideal $\bigcap\limits_i a_i$. Indeed, if $Y(L) \supset \bigcup\limits_i Y_i(L)$, then the ideal a that determines Y satisfies the condition

$$\text{"any ideal containing one of the } a_i \text{ contains } a\text{."} \tag{1.70}$$

The sum a of all such ideals also satisfies (1.70) and is the unique maximal element of the set of ideal satisfying (1.70); on the other hand, all the elements of this set are contained in a_i, and therefore in $\bigcap\limits_i a_i$.

Having failed to define the union of subschemes, let us introduce a notion corresponding to $\cap a_i$: The *quasiunion* $\bigvee\limits_i Y_i$ of a family of closed subschemes Y_i of a scheme X is the subscheme corresponding to the intersection of all the ideals defining the subschemes Y_i.

It is important to notice that the quasiunion of the subschemes Y_i does not depend on the choice of the closed scheme containing all the Y_i, inside of which we construct this quasiunion.

The main aim of this section is to construct for the Noetherian affine schemes the decomposition theory into "irreducible" in some sense components similar to the one constructed above for Noetherian topological spaces. To this end, we will use the quasi-union; on supports it coincides with the union (for finite families of subschemes).

1.7.3a. Lemma. $\operatorname{supp}\Big(\bigvee\limits_{1\le i\le n} Y_i\Big) = \bigcup\limits_{1\le i\le n} \operatorname{supp} Y_i$.

Proof. The inclusion $\supset$ is already proved. Conversely, if $x \notin \bigcup\limits_{1\le i\le n} \operatorname{supp} Y_i$, then, for every i, there exists an element $f_i \in \mathfrak{a}_i$ such that $f_i(x) \neq 0$, therefore $\Big(\prod\limits_{1\le i\le n} f_i\Big)(x) \neq 0$. Hence, x does not belong to the set of zeroes of all the functions from $\bigcap\limits_{1\le i\le n} \mathfrak{a}_i$, which is $\operatorname{supp}\Big(\bigvee\limits_{1\le i\le n} Y_i\Big)$. Now, apply Exercise 1.3.4 1). □

1.7.4. Irreducible schemes. Now, we have to transport to the subschemes the notion of irreducibility. The first that comes to one's mind, is to try to imitate the definition of irreducibility for spaces.

An affine scheme X is said to be *reducible* if there exists a representation of the form $X = X_1 \vee X_2$, where X_1, X_2 are proper closed subschemes of X; it is said to be *irreducible* otherwise.

An affine scheme X is said to be a *Noetherian* one if its ring of global functions is Noetherian or, equivalently, if X satisfies the descending chain condition on closed subschemes.

1.7.4a. Theorem. *Any Noetherian affine scheme X decomposes into the quasiunion of a finite number of closed irreducible subschemes.*

Proof. The same arguments as at the end of sec. 1.4.7b lead to the result desired. If X is reducible, we write $X = X_1 \vee X_2$ and then decompose, if necessary, X_1 and X_2, and so on. Thanks to the Noetherian property, the process terminates. □

1.7.5. Primary schemes and primary ideals. The above notion of irreducibility turns out to be too subtle. The following notion of primary affine schemes is more useful: An ideal $\mathfrak{q} \subset A$ is called *primary* if any zero divisor in $A/\mathfrak{q}$ is nilpotent. A *closed subscheme* is called *primary* if it is determined by a primary ideal.

Proposition. *Any irreducible Noetherian scheme is primary.*

Remark. The converse statement is false. Indeed, let K be an infinite field. Consider the ring $A = K \times V$, where V is an ideal with the zero product. The ideal (0) is primary and, for any subspace $V' \subset V$, the ideal $(0, V')$ is primary. At the same time, if $\dim_K V > 1$, there exists infinitely many representations of the form $(0) = V_1 \cap V_2$, where $V_i \subset V$ are proper subspaces, i.e., representations of the form $X = Y_1 \vee Y_2$, where $X = \operatorname{Spec} A$. This deprives us of any hope for the uniqueness of the decomposition into quasiunion of irreducible subschemes. Primary subschemes behave a sight more nicely than irreducible subschemes as we will see shortly.

Proof of Proposition. Let us show that a non-primary Noetherian scheme X is reducible. Indeed, in the ring of its global functions A there are two elements f, g such that $fg = 0, g \neq 0$ and f is not nilpotent.

Set $a_k = \operatorname{Ann} f^k := \{h \in A \mid hf^k = 0\}$. The set of ideals a_k ascends, and therefore stabilizes. Let $a_n = a_{n+1}$. Then $(0) = (f^n) \cap (g)$.

Indeed, $h \in (f^n) \cap (g)$; hence, $h = h_1 f^n = h_2 g$; but $h_1 f^{n+1} = h_2 g\, f = 0$, implying $h_1 f^n = 0$, since $a_{n+1} = a_n$. Therefore $Y = Y_1 \vee Y_2$, where Y_1 is determined by (f^n) and Y_2 by (g). □

1.7.6. Remarks. a) The support of a primary Noetherian scheme is irreducible. Indeed: The radical of a primary ideal is a prime ideal.

b) The results of sec. 1.7.4 and 1.7.5 show that an affine Noetherian scheme is a quasi-union of its primary subschemes $X = \bigvee\limits_{1\leq i\leq n} X_i$. We could have left in this decomposition only maximal elements and then try to prove its uniqueness in the same way as we have done for the spaces at the end of sec. 1.4.7b. But these arguments fail twice: First, the formula

$$X \cap \Big(\bigvee_{1\leq i\leq n} Y_i\Big) = \bigvee_{1\leq i\leq n} (X \cap Y_i) \tag{1.71}$$

is generally false, and, second, as we have established, our primary subschemes X_i can be reducible themselves.

Therefore, instead of striking out non-maximal components from $\bigvee\limits_{1\leq i\leq n} Y_i$, we should apply less trivial process, and even after that the uniqueness theorem will be harder to formulate and prove.

1.7.7. Incompressible primary decompositions. A primary decomposition $X = \bigvee\limits_{1\leq i\leq n} X_i$ is called *incompressible* if the following two conditions are satisfied:

(1) $\operatorname{supp} X_i \neq \operatorname{supp} X_j$ if $i \neq j$,
(2) $X_k \not\subset \bigvee\limits_{i\neq k} X_i$ for every $k, 1 \leq k \leq n$.

Theorem. *Every Noetherian affine scheme X decomposes into an incompressible quasiunion of a finite number of its primary closed subschemes.*

Proof. Let us start with a primary decomposition $X = \bigvee\limits_{1\leq i\leq n} X_i$, see Remark 1.7.6 b. Let Y_j, where $j = 1, \dots, m$, be the quasiunions of the subschemes X_i with common support. Then $X = \bigvee\limits_{1\leq i\leq n} Y_j$. If $Y_1 \subset \bigvee\limits_{2\leq j\leq n} Y_j$, let us delete Y_1 from the representation of X. Repeating the process we obtain after a finite number of steps a quasiunion $X = \vee Y_j$ which satisfies the second condition in the definition of incompressibility. It only remains to verify that the subschemes Y_j are primary.

Lemma. *The quasiunion of a finite number of primary subschemes with common support is primary and has the same support.*

Proof. Let $Y = \bigvee_{1\le i\le n} Y_i$, $\operatorname{supp} Y_i = \operatorname{supp} Y_j$ for all i, j. Let Y_i correspond to an ideal a_i in the ring of global functions A on Y. Then $\bigcap_{1\le i\le n} a_i = (0)$. Consider a zero divisor $f \in A$. Let $fg = 0$, where $g \neq 0$ and $g \in a_i$ for some i. Since a_i is primary, $f^n \in a_i$ for some n. But since $V(a_i) = V(0)$, it follows that a_i consists of nilpotents and f is nilpotent; moreover, $\operatorname{supp} Y_i = \operatorname{supp} Y$. $\square$

1.7.8. Theorem (A uniqueness theorem). *Let $X = \bigvee_{1\le i\le n} X_i$ be an incompressible primary decomposition of a Noetherian affine scheme X. The system of generic points of irreducible closed sets* $\operatorname{supp} X_i$ *does not depend on such a decomposition.*

This system of generic points is denoted by $\operatorname{Prime} X$ (or $\operatorname{Prime} A$ if $X = \operatorname{Spec} A$) and is called the *set of prime ideals associated with* X (or A).

We will establish a more precise result giving an invariant characterization of $\operatorname{Prime} X$. Let $X = \operatorname{Spec} A$ and $X_i = \operatorname{Spec}(A/a_i)$ for an ideal $a_i \subset A$.

Proposition. *The following two statements are equivalent for any reduced scheme X* (for the general scheme, the statement is only true when $\operatorname{Ann} f$ in b) *is replaced by $\{g \in A \mid gf \in \bigcap a_i\}$:*

a) *Any prime ideal $p \subset A$ corresponds to a generic point of one of the sets* $\operatorname{supp} X_i$.

b) *There exists an element $f \in A$ such that* $\operatorname{Ann} f := \{g \in A \mid fg = 0\}$ *is primary and p is its radical.*

Proof. a) $\Longrightarrow$ b). Let p_j be the ideal of a generic point of $\operatorname{supp} X_j$. Clearly, $p_j = \mathfrak{r}(a_j)$. Since the representation $X = \bigvee_{1\le j\le n} X_j$ is incompressible, $a_i \not\supset \bigcap_{j\neq i} a_j$, where $1 \le i \le n$. Let us select an element $f \in \bigcap_{j\neq i} a_j \setminus a_i$ and show that $\operatorname{Ann} f$ is primary and p_i its radical.

First of all, $\operatorname{Ann}(f \bmod a_i)$ consists only of nilpotents in A/a_i; therefore $\operatorname{Ann} f \subset p_i$ (since p_i is the pre-image of the nilradical of A/a_j with respect to the natural homomorphism $A \longrightarrow A/a_i$). Besides, $a_i \subset \operatorname{Ann} f$ because, by construction, $fa_i \subset \bigcap_j a_j = (0)$; therefore $\mathfrak{r}(\operatorname{Ann} f) = p_i$.

Now let us verify that all the zero divisors in $A/\operatorname{Ann} f$ are nilpotents. Assume the contrary; then there exist elements $g, h \in A$ such that $gh \in \operatorname{Ann} f$, $h \notin \operatorname{Ann} f$ and g is not nilpotent modulo $\operatorname{Ann} f$; therefore g is not nilpotent modulo a_i as well.

On the other hand, $fgh = 0$; and, since $g \bmod a_i$ is not nilpotent, $fh \bmod a_i = 0$ because a_i is primary, i.e., $fh \in \left(\bigcap_{j\neq i} a_j\right) \cap a_i = 0$, implying $h \in \operatorname{Ann} f$ contrary to the choice of h. This contradiction shows that $\operatorname{Ann} f$ is primary.

b) $\Longrightarrow$ a). Let $f \in A$ be an element such that $\operatorname{Ann} f$ is primary, p its radical. Set $s_i = (a_i : f) = \{g \in A \mid gf \in a_i\}$. Since $\bigcap_i a_i = (0)$, it is easy to see that $\operatorname{Ann} f = \bigcap_i s_i$ and $p = \mathfrak{r}(\operatorname{Ann} f) = \bigcap_i \mathfrak{r}(s_i)$. If $f \in a_i$, then $s_i = \mathfrak{r}(s_i) = A$. If, on the contrary, $f \notin a_i$, then $\mathrm{Ann}(f \bmod a_i)$ consists of nilpotents in A/a_i, so that $\mathfrak{r}(s_i) = p_i$.

Therefore, $p = \bigcap\limits_{i \in I} p_i$, where $I = \{i \mid f \notin a_i\}$, implying that p coincides with one of the p_i. Indeed, $V(p) = \bigcup\limits_{i \in I} V(p_i)$ and $V(p)$ is irreducible. Proposition is proved together with the uniqueness theorem 1.7.8. □

1.7.9. Incompressible primary decompositions, cont. An incompressible primary decomposition $X = \bigvee\limits_i X_i$ drastically differs from a decomposition of $\operatorname{supp} X$ into the union of maximal irreducible components: Though the supports of the subschemes X_i contain all the irreducible components of $\operatorname{supp} X$ only once, the supports may also have another property: $\operatorname{supp} X_i \subset \operatorname{supp} X_j$ for some i, j.

A simple example is given by the ring described in Remark 1.4.8. In notation of Remark 1.4.8, we have $(0) = (0, I) \cap (p, 0)$ in A; so that $X = X_1 \vee X_2$, where $\operatorname{supp} X_i = \operatorname{supp} X$ and $\operatorname{supp} X_2 = V((p, 0))$. The **space** $\operatorname{supp} X_2$ is entirely contained in the space $\operatorname{supp} X_1$, whereas the **subscheme** X_2 is distinguished from the "background" by its nilpotents (see Fig. 10), where $X_1 = \operatorname{Spec} A$ and $X_2 = \operatorname{Spec} B[T]/(T^2)$, where $B = A/p$, and $X = \operatorname{Spec} \mathbb{R}[T_1, T_2] \oplus \mathfrak{a}$, where $\mathfrak{a} = \mathbb{R}[T_1, T_2]/(T_1)$. Clearly, $\mathfrak{a}^2 = (0)$.

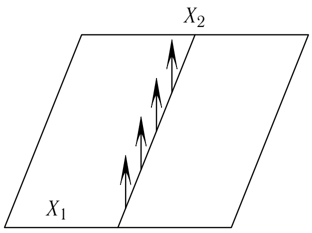


Fig. 10

This remark on nilpotents is of general character. Indeed, let in the incompressible decomposition, $\operatorname{supp} X_i \subset \operatorname{supp} X_j$ and $X_i \not\subset X_j$. Then

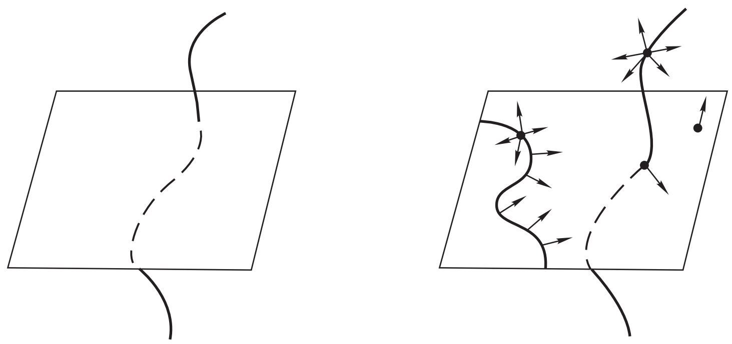
$$\operatorname{supp}(X_i \cap X_j) = \operatorname{supp} X_i, \quad \text{but } X_i \cap X_j \text{ is a proper subscheme in } X_i.$$

This may only happen when there are more nilpotents in the ring of X_i then in the ring of $X_i \cap X_j$ where they are induced by nilpotents from the greater space X_j.

Among the components X_i of the incompressible primary decomposition, those for which $\operatorname{supp} X_i$ is maximal are called *isolated*,, the other ones are called *embedded*. The same terminology is applied to the sets $\operatorname{supp} X_i$ themselves and their generic points, which constitute the set $\operatorname{Prime} X$.

The space of an embedded component may belong simultaneously to several (isolated or embedded) components. Besides, the chain of components subsequently embedded into each other may be however long.

Therefore, innocent at the first glance, the space of an affine scheme may hide in its depth a complicated structure of embedded primary subschemes like the one illustrated on Fig. 11. The reader should become used to the geometric reality of such a structure.



Space Scheme

Fig. 11

Clearly, depicting nilpotents by arrows does not make possible to reflect the details however precisely. It is only obvious that on the embedded components the nilpotents grow thicker, thus giving away their presence.

1.7.10. Incompressible primary decompositions, cont. 2. The finite set of prime ideals Spec A which is invariantly connected with every Noetherian ring A has a number of important properties. In particular, it enables us to refine Theorem 1.4.8

Theorem. *An element $f \in A$ is a zero divisor if and only if it vanishes (as a function) on one of the components of the incompressible primary decomposition of* Spec A.

In other words, the set of zero divisors of A coincides with $\bigcap\limits_{p\in \operatorname{Prime} A} p$.

Proof. First, let us show that if $f \notin \bigcap\limits_{p\in \operatorname{Prime} A} p$, then $\operatorname{Ann} f = (0)$.

Let $(0) = \bigcup\limits_i a_i$ be the incompressible primary decomposition, where $p_i = \mathfrak{r}(a_i)$. Let $fg = 0$. Since $f \notin p_i$, it follows that a_i being primary implies that $g \in a_i$. This is true for all i; therefore $g = 0$.

Conversely, let $\operatorname{Ann}(f) = (0)$. Suppose that $f \in p_i$; since A is Noetherian, it follows that $f^k \in p_i^k \subset a_i$ for some $k \geq 1$; i.e., $(a_i : (f^k)) = A$.[15] On the other hand, $\operatorname{Ann}(f^k) = (0)$ implying

$$(0) = \operatorname{Ann}(f^k) = \bigcap_j (a_j : (f^k)) = \bigcap_{j\neq i} (a_j : (f^k)) \supset \bigcap_{j\neq i} a_j. \tag{1.72}$$

[15] The *quotient* $(\mathfrak{a} : \mathfrak{b})$ *of two ideals* in A is defined to be the set $\{x \in A \mid x\mathfrak{b} \subset \mathfrak{a}\}$. It is easy to see that $(\mathfrak{a} : \mathfrak{b})$ is an ideal in A. In particular, the ideal $(0 : \mathfrak{b})$ is called the *annihilator* of $\mathfrak{b}$ and is denoted $\operatorname{Ann}(\mathfrak{b})$.

This contradicts to the fact of incompressibility of the decomposition and completes the proof. □

1.7.11. Incompressible primary decompositions for Noetherian schemes. Finally, notice that the uniqueness theorem 1.7.8 only concerns the supports of primary components of incompressible decompositions, not the components themselves. About them one may only claim the following.

Theorem. *The set of isolated components of an incompressible primary decomposition of a Noetherian scheme* $\operatorname{Spec} A$ *does not depend on the choice of the decomposition.*

For embedded components, this statement is false.

We skip the proof; the reader may find it in [ZS] (v. 1, Theorem 4.5.8) or [Lang].

1.7.12. Exercises. 1) Let K be an algebraically closed field.

a) Describe all primary closed subschemes of the line $\operatorname{Spec} K[T]$.

b) Same for non-closed fields.

c) Same for $\operatorname{Spec} \mathbb{Z}$.

2) Describe up to an isomorphism primary closed subschemes of the plane $\operatorname{Spec} K[T_1, T_2]$ supported at $V(T_1, T_2)$ and whose local rings are of length ≤ 3.

1.8. Hilbert's Nullstellensatz (Theorem on zeroes)

In this section we will establish that the closed subschemes of finite dimensional affine spaces over a field or a ring $\mathbb{Z}$ have many closed points.

1.8.1. Theorem. *Let A be a ring of finite type. Then* $\operatorname{Spm} A$ *is dense in* $\operatorname{Spec} A$.

Corollary. *The intersection* $X \cap \operatorname{Spm} A$ *is dense in X for any open or closed subset* $X \subset \operatorname{Spec} A$.

Indeed, if $X = V(E)$ and we identify X with $\operatorname{Spec} A/(E)$, then $\operatorname{Spm} A \cap V(E)$ coincides with $\operatorname{Spm} A/V(E)$ and $A/(E)$ is a ring of finite type together with A. This easily implies the statement for open sets, too. □

The space $\operatorname{Spm} A$ is easier to visualize since it has no nonclosed points (the "big open sets" still remain nevertheless). On the other hand, Corollary 1.8.1 implies that for rings of finite type the space of $\operatorname{Spec} A$ is uniquely recovered from $\operatorname{Spm} A$ (assuming that the induced topology in $\operatorname{Spm} A$ is given).

The recipe is to use the following statements:

1) The points of $\operatorname{Spec} A$ are in a one-to-one correspondence with the irreducible closed subsets of $\operatorname{Spm} A$. (Therefore to every irreducible closed subset of $\operatorname{Spm} A$, its "generic point" in $\operatorname{Spec} A$ corresponds.)

2) Every closed subset of $\operatorname{Spec} A$ consists of generic points of all the irreducible closed subsets of a closed subset of $\operatorname{Spm} A$.

We advise the reader to prove these statements in order to understand them :-)

Proof of Theorem. We will successively widen the class of rings for which this theorem is true.

a) *Let K be algebraically closed. Then the set of closed points of* $\operatorname{Spec} K[T_1, \dots, T_n]$ *is dense.*

The closure of the set of closed points coincides with the space of zeroes of all the functions which vanish at all the closed points, and therefore it suffices to prove that a polynomial F, which belongs to all the maximal ideals of $K[T_1, \dots, T_n]$, is identically zero. But such a polynomial satisfies $F(t_1, \dots, t_n) = 0$ for all $t_1, \dots, t_n \in K$, and an easy induction on n shows that $F = 0$ (here we actually use not closedness even, but only the fact that K is infinite).

b) *The same as in a) but K is not supposed to be algebraically closed.*

Denote by $\overline{K}$ the algebraic closure of K. We have a natural morphism

$$i\colon A = K[T_1, \dots, T_n] \hookrightarrow \overline{K}[T_1, \dots, T_n] = B. \tag{1.73}$$

The ring B is integer over K, and therefore thanks to the results of 7.4–7.5, we have

$$\overline{\operatorname{Spm} A} = {}^a i(\overline{\operatorname{Spm} B}) = {}^a i(\operatorname{Spec} B) = \operatorname{Spec} A. \tag{1.74}$$

c) *Theorem 1.8.1 holds for the rings A of finite type without zero divisors over K.*

Indeed, by Noether's normalization theorem there exists a polynomial subalgebra $B \subset A$ such that A is integer over B. By the already proved $\overline{\operatorname{Spm} B} = \operatorname{Spec} B$, and the literally same argument as in b) shows that $\overline{\operatorname{Spm} A} = \operatorname{Spec} A$.

d) *Theorem 1.8.1 holds for any rings A of finite type over a field.*

Indeed, any irreducible component of $\operatorname{Spec} A$ is homomorphic to $\operatorname{Spec} A/p$, where p is a prime ideal. The ring A/p satisfies the conditions of c), hence, the closed points are dense in all the irreducible components of $\operatorname{Spec} A$, and therefore in the whole space.

e) *The same as in d) for the rings A of finite type over $\mathbb{Z}$.*

1.8.2. Lemma. *No field of characteristic* 0 *can be a finite type algebra over $\mathbb{Z}$.*

Proof. If a field of characteristic 0 is a finite type algebra over $\mathbb{Z}$, then thanks to Noether's normalization theorem there exist algebraically independent over $\mathbb{Q}$ element $t_1, \dots, t_r \in K$ such that K is integral over $R = \mathbb{Q}[t_1, \dots, t_r]$. By Proposition 1.6.6 the natural map $\operatorname{Spm} K \longrightarrow \operatorname{Spm} \mathbb{Q}[t_1, \dots, t_r]$ is surjective; since $\operatorname{Spec} K$ contains only one closed point, so does $\operatorname{Spec} R$, which is only possible if $R = \mathbb{Q}$. Therefore, K is integral over $\mathbb{Q}$, and hence is a finite extension of the field $\mathbb{Q}$.

Let $x_1, \dots, x_n$ generate K as a $\mathbb{Z}$-algebra. Each of the x_j is a root of a polynomial with rational coefficients. If N is the LCM of the denominators of these coefficients, then, as is easy to see, all the Nx_j are integral over $\mathbb{Z}$; if y is the product of m of the generators $x_1, \dots, x_n$ (perhaps, with multiplicities), then $N^m y$ is integral over $\mathbb{Z}$. Since all elements of K are linear combinations of such products with integer coefficients, it follows that for any $y \in K$, there exists a natural m such that $N^m y$ is integral over $\mathbb{Z}$.

Now, let p be a prime not divisor of N. Since $1/p \in \mathbb{Q} \subset K$, we see that a non-integer rational number N^m/p is integral over $\mathbb{Z}$; Exercise 1.6.7 shows that this is impossible. □

To complete the proof in case e), denote by $\varphi\colon \mathbb{Z} \to A$ a natural homomorphism and show that ${}^a\varphi(\operatorname{Spm} A) \subset \operatorname{Spm} \mathbb{Z}$. Indeed, otherwise there exists a maximal ideal$\mathfrak{p} \subset A$ such that $\varphi^{-1}(\mathfrak{p}) = (0)$; so $\mathbb{Z}$ can be embedded into the field $A/\mathfrak{p}$ (and hence this field is of characteristic 0) which leads to a contradiction with Lemma 1.8.2.

Thus, $\operatorname{Spm} A = \bigcup_p V(p)$, where p runs over primes. Since $V(p)$ is homeomorphic to $\operatorname{Spec} \mathbb{Z}/p$ which is an algebra of finite type over A/pA, the set of closed points of $V(p)$ is dense in $\operatorname{Spec} A/pA$; hence, so is it in $\operatorname{Spec} A$. □

1.8.3. Proposition. *Let $p \subset A$ be a maximal ideal of a ring of finite type over $\mathbb{Z}$ (resp. a field K). Then A/p is a finite field (resp. a finite algebraic extension of K).*

Proof. It follows from step e) of the proof of Theorem 1.8.1 that it suffices to confine ourselves to the case of a ring A over a field K. The quotient ring modulo the maximal ideal A/p, being a field, contains the unique maximal ideal. On the other hand, by Noether's normalization theorem A is an integer extension of the polynomial ring B in n indeterminates over K. The case $n \geq 1$ is impossible, since then B, and therefore A, would have had infinitely many maximal ideals. Therefore $n = 0$, and A is an integer extension of finite type of K. This proves the statement desired. □

1.8.4. Hilbert's Nullstellensatz. Now consider the case of K algebraically closed. By Proposition 1.8.3 in this case the closed points of $\operatorname{Spec} A$ are in one-to-one correspondence with the K-points of the scheme $\operatorname{Spec} A$; the space of the latter ones is called an *affine algebraic variety* over K in the classical sense of the word. The discussion of sec. 1.8.1 shows that in this case $\operatorname{Spec} A$ with the spectral topology and the space of geometric K-points of the scheme $\operatorname{Spec} A$ with Zariski topology are essentially equivalent notions: The passage from one to another does not require any additional data.

Finally, let us give the classical formulation of *Hilbert's Nullstellensatz* in the language of systems of equations.

Theorem (Hilbert's Nullstellensatz). *Let K be algebraically closed and $F = (F_i)_{i\in I}$, where $F_i \in K[T]$ for $T = (T_1, \dots, T_n]$ is a family of polynomials.*

a) *The system $F_i = 0$ for $i \in I$, has a solution in K if and only if the equation $1 = \sum_i F_i X_i$ has no solutions in $K[T_1, \dots, T_n]$, i.e., the ideal (F) does not coincide with the whole ring.*

b) *If a polynomial $G \in K[T_1, \dots, T_n]$ vanishes on all the solutions of the system $F_i = 0$ for $i \in I$, then $G^n = \sum_i F_i G_i$, where $G_i \in K[T_1, \dots, T_n]$, for some positive integer n.*

Proof. a) If (F) does not coincide with the whole ring, then by Theorem 1.2.3 we have $\operatorname{Spec} K[T_1, \dots, T_n]/(F) \neq \emptyset$. Therefore the spectrum contains a maximal ideal the residue field modulo which thanks to Proposition 1.8.3 coincides with K, and the images of T_i in this field give a solution of the system $F_i = 0$ for $i \in I$.

b) If G vanishes at all the solutions of the system, then the image of G in $K[T_1, \dots, T_n]/(F)$ belongs to the intersection of all the maximal ideals of this ring, hence, by Theorem 1.7.10, to the intersection of all the prime ideals. Therefore G is nilpotent thanks to Theorem 1.4.7. □

1.9. The fiber products

This section does not contain serious theorems. We only give a construction of the fiber product of affine schemes. This notion, though a simple one, is one of the most fundamental and explains the popularity of the tensor product in modern commutative algebra. Our main aim is to connect with the fiber products geometrically intuitive images.

We advise the reader to refresh the knowledge of categories (see the last section of this Chapter) before reading this section.

1.9.1. Fiber product. Let C be a category, $S \in \operatorname{Ob}\mathsf{C}$, and $\mathsf{C_S}$ the category of "objects over S". The *fiber product* of two objects $\psi\colon Y \longrightarrow S$ and $\varphi\colon X \longrightarrow S$ over S is their product in $\mathsf{C_S}$.

In other words, the *fiber product* is a triple (Z, π_1, π_2), where $Z \in \operatorname{Ob}\ \mathsf{C}$, $\pi_1\colon Z \longrightarrow X$, $\pi_2\colon Z \longrightarrow Y$, such that

a) the diagram

$$\begin{tikzcd} Z \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X \arrow[d, "\varphi"] \\ Y \arrow[r, "\psi"] & S \end{tikzcd} \tag{1.75}$$

where $(Z, \varphi\,\pi_1) = (Z, \psi\pi_2) \in \operatorname{Ob}\mathsf{C_S}$ and $\pi_1, \pi_2 \in \operatorname{Mor}\mathsf{C_S}$), commutes;

b) for any object $X\colon Z' \longrightarrow S$ (in what follows it will be denoted just by Z') of $\mathsf{C_S}$, the maps induced by the morphisms π_1, π_2 identify the set $\operatorname{Hom}_{\mathsf{C_S}}(Z', Z)$with $\operatorname{Hom}_{\mathsf{C_S}}(Z', X) \times \operatorname{Hom}_{\mathsf{C_S}}(Z', Y)$.

In still other words, (Z, π_1, π_2) is a universal object in the class of all the triples that make the above diagram commuting.

The diagram (1.75) with the properties a) and b) is sometimes called a *Cartesian square*. An object Z in it is usually denoted by $X \underset{S}{\times} Y$ and this object is said to be the *fiber product* of X and Y over S. Using this short notation one should not forget that in it the indication to the four morphisms $X \longrightarrow S$, $Y \longrightarrow S$, $X \underset{S}{\times} Y \longrightarrow X$, and $X \underset{S}{\times} Y \longrightarrow Y$, of which the first two are vital, is omitted.

Speaking formally, the usual — set-theoretical — direct product is not a particular case of a fiber one; but this is so if the category C has a final object F. Then $X \underset{F}{\times} Y$ is actually the same as $X \times Y$.

The fiber product exists in Sets; we will elucidate its meaning with several examples.

Lemma. *Let $\varphi : X \longrightarrow S$ and $\psi\colon Y \longrightarrow S$ be some maps of sets; put*

$$Z = \{(x, y) \in X \times Y \mid \varphi(x) = \psi(y)\} \subset X \times Y \tag{1.76}$$

and define $\pi_1\colon Z \longrightarrow X$ and $\pi_2\colon Z \longrightarrow Y$ as maps induced by the projections $X \times Y \longrightarrow X$, $X \times Y \longrightarrow Y$. Then the triple (Z, π_1, π_2) forms a fiber product of X and Y over S.

Proof is absolutely trivial. $\square$

This construction explains wherefrom stems the name of our operation: Over every point of S the fiber of the map $Z \longrightarrow S$ is the direct product of the fibers X and Y.

1.9.2. Examples. The following notions are encountered everywhere.

Product. In Sets, the one-point set $*$ is a final object, and so

$$X \underset{*}{\times} Y = X \times Y \quad \text{for any } X, Y \in \mathsf{Sets}.$$

Intersection. Let φ, ψ be embeddings of X and Y, as subsets, into S. Then identifying $Z = X \underset{S}{\times} Y$ with a subset of S we see that $Z = X \cap Y$.

The fiber of a map. Let $Y = F$ be a final object, $\psi(F) = s \in S$. Then $Z = \varphi^{-1}(s)$. More generally, if ψ is an embedding, then having identified Y with $\psi(Y) \subset S$ we obtain $Z = \varphi^{-1}(Y)$.

The change of base. This terminology is induced by topology: If $\varphi\colon X \longrightarrow S$ is a bundle (in any sense) and $\psi\colon S' \longrightarrow S$ a morphism of topological spaces, then the bundle $X' = X \underset{S}{\times} S'$ is said to be obtained from $X \longrightarrow S$ by the *change of base* S to S'. The other name for it is the *induced bundle*. $\square$

1.9.3. Fiber products exist. The above examples will serve us as a model for the corresponding notions in the category of schemes (affine ones, for a time being). First of all, let us establish their existence.

Theorem. *Let* $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $S = \operatorname{Spec} C$, *where* A *and* B *are* C*-algebras. The fiber product of* X *and* Y *over* S *exists and is represented by the triple* $(\operatorname{Spec} A \bigotimes_C B, \pi_1, \pi_2)$*, where* π_1 *(resp.* π_2*) is the map induced by the* C*-algebra homomorphism* $A \longrightarrow B \bigotimes_C B$*,* $f \mapsto f \otimes 1$ *(resp.* $B \longrightarrow A \bigotimes_C B$*,* $g \mapsto 1 \otimes g$*).*

For proof, see [Lang], where the fact that the fiber coproducts in the category Rings exist, and are described just as stated, is established. The inversion of arrows gives the statement desired. $\square$

Observe that the category of affine schemes has a final object, $\operatorname{Spec} \mathbb{Z}$. So we may speak about the *absolute product* $X \times Y = \operatorname{Spec} A \bigotimes_{\mathbb{Z}} B$.

1.9.4. Warning. The statement "*the set of points* $|X \underset{S}{\times} Y|$ *of the scheme* $X \underset{S}{\times} Y$ *is the fiber product* $|X| \underset{S}{\times} |Y|$ *of the sets of points* $|X|$ *and* $|Y|$ *over* $|S|$" (where $|X|$ is just the set of points of X, not the cardinality of this set) is only true for the points with values in C-algebras; i.e., when $S = \operatorname{Spec} C$ or, which is the same, for the sets of morphisms over S. Here are typical examples showing what might happen otherwise.

Examples. 1) Let K be a field, $S = \operatorname{Spec} K$, let $X = \operatorname{Spec} K[T_1]$, and $Y = \operatorname{Spec} K[T_2]$ with obvious morphisms between these spectra. Then $X \underset{S}{\times} Y$ is the plane over K; it has a lot of non-closed points: Generic points of irreducible curves non-parallel to the axes, which are not representable by the pairs (x, y), where $x \in X$, $y \in Y$.

2) Let $L \supset K$ be fields.

2a) Let $L \rhd K$ be a finite Galois field extension. Let $X = \operatorname{Spec} L$, $S = \operatorname{Spec} K$. Let us describe $X \underset{S}{\times} X$ or, dually, $L \bigotimes_K L$.

Let us represent the second factor L in the form $L = K[T]/(F(T))$, where $F(T)$ is an irreducible polynomial. In other words, in L, take a primitive element $t = T \bmod (F)$ over K.

It follows from the definition of tensor product that in this case, as L-algebras,

$$L \bigotimes_K L \simeq L[T]/(F(T)) \tag{1.77}$$

if we assume that the L-algebra structure on $L \bigotimes_K L$ is determined by the map $l \mapsto l \otimes 1$. But, by assumption, $F(T)$ factorizes in $L[T]$ into linear factors $F(T) = \prod\limits_{1 \leq i \leq n} (T - t_i)$, where t_i are all the elements conjugate to t over K and $n = [L : K]$.

By the general theorem on the structure of modules over principal ideal rings [Lang], we obtain

$$L \bigotimes_K L \simeq L[T]/(\prod (T - t_i)) \simeq \prod L[T]/(T - t_i) \simeq L^n. \tag{1.78}$$

In particular, $\operatorname{Spec} L \bigotimes_K L \simeq \coprod\limits_{1 \leq i \leq n} \operatorname{Spec} L$: Though $\operatorname{Spec} L$ consists of only one point, there are, miraculously, n of them in $\operatorname{Spec} L \bigotimes_K L$.

2b) The trouble of another nature may happen if for L we take a purely inseparable extension of K. Let, e.g., $F(T) = T^p - g$, where $g \in K \setminus K^p$ and $p = \operatorname{Char} K$ (recall that here: $K^p := \{x^p \mid x \in K\}$). Then in $L[T]$ we have $T^p - g = (T - t)^p$, where $t = g^{1/p}$, so that

$$L \bigotimes_K L \simeq L[T]/((T - t)^p) \simeq L[T]/(T^p); \tag{1.79}$$

therefore we have acquired nilpotents which were previously lacking. The space of $\operatorname{Spec} L \bigotimes_K L$ consists, however, of one point. $\square$

1.9.5. Examples. Let us give examples which are absolutely parallel to the set-theoretical construction.

1) Let $X = \operatorname{Spec} A$, let $Y_1 \xrightarrow{i_1} X$ and $Y_2 \xrightarrow{i_2} X$ be two *closed* subschemes of X determined by ideals $a_1, a_2 \subset A$. Thanks to results of sec. 1.7.2 their

intersection $Y_1 \cap Y_2$ represents the functor $Y_1(Z) \cap Y_2(Z)$, i.e., should coincide with their fiber product over X. This is indeed the case: The corresponding statement on rings is

$$A/(a_1 + a_2) \simeq A/a_1 \bigotimes_A A/a_2 \tag{1.80}$$

and is easy to verify directly.

2) Let $\operatorname{Spec} B = Y \longrightarrow X = \operatorname{Spec} A$ be an affine scheme morphism, $\overline{k(x)}$ the algebraic closure of the field of fractions corresponding to a given point $x \in X$. The natural homomorphism $A \longrightarrow \overline{k(x)}$ represents a geometric point with center at x. The fiber product $Y_x = Y \underset{X}{\times} \operatorname{Spec} \overline{k(x)}$ is said to be the *geometric fiber* of Y over x, and $Y \underset{X}{\times} \operatorname{Spec} k(x)$ the *usual fiber*.

2a) A particular case: $\operatorname{Spec} A/pA$ is the fiber of $\operatorname{Spec} A$ over $(p) \in \operatorname{Spec} \mathbb{Z}$ for any prime ideal p in any ring A.

1.9.6. The diagonal. Let $\operatorname{Spec} B = X \longrightarrow S = \operatorname{Spec} A$ be an affine scheme morphism. The commuting diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & X \arrow[d] \\ X \arrow[r] & S \end{tikzcd} \tag{1.81}$$

defines a morphism $\delta\colon X \longrightarrow X \underset{S}{\times} X$ (see sec. 1.9.1b) which is said to be the *diagonal* one.

Proposition. *δ identifies X with the closed subscheme Δ_X of $X \underset{S}{\times} X$ singled out by the ideal*

$$I_{\Delta_X} = \operatorname{Ker}(\mu\colon B \bigotimes_A B \longrightarrow B), \text{ where } \mu(b_1 \otimes b_2) = b_1 b_2. \tag{1.82}$$

Proof. Writing down all the necessary diagrams we see that $\delta = {}^a\mu$. Since μ is surjective, its kernel determines a closed subscheme isomorphic to the image of δ. □

The scheme Δ_X is said to be the (*relative* over S) *diagonal*.

1.10. The vector bundles and projective modules

1.10.1. Families of vector spaces. Let $\psi\colon Y \longrightarrow X$ be an affine scheme morphism, $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$; let $\varphi\colon A \longrightarrow B$ be the corresponding ring homomorphism. We would like to single out a class of morphisms which is similar to locally trivial vector bundles in topology.

It is convenient to start from a wider notion of "families of vector spaces". Example 2.1 shows that an analogue of a vector space V over a field K is given by the scheme $\operatorname{Spec} S^{\bullet}_K(V^*)$, where $V^* = \operatorname{Hom}_K(V, K)$ and $S^{\bullet}_K(V^*)$ is the symmetric algebra of V^*. Replacing here K by an arbitrary ring A and V by an A-module M we get the following definition.

Under the above notation, let $\chi\colon M \longrightarrow B$ an A-module morphism. Suppose that χ induces an A-algebra isomorphism $S^{\bullet}_A(M) \xrightarrow{\sim} B$. Then the pair (χ, ψ) is said to be a *family of vector spaces over* $X = \operatorname{Spec} A$ and M is said to be the *module that defines the family.*

In other words, an explicit structure of the symmetric algebra over A on B determines a fiber-wise linearization of the morphism χ.

The *morphisms of families of vector spaces* over a fixed base are obviously defined. The category of such families is dual to the category of A-modules; so, in particular, every family of vector spaces is determined by its module M up to an isomorphism.

The following notion is more important for us for the time being.

1.10.2. Families of vector spaces exist. Under the above notation, let there be given a ring homomorphism $A \longrightarrow A'$ which defines a scheme morphism

$$X' = \operatorname{Spec} A' \longrightarrow X = \operatorname{Spec} A. \tag{1.83}$$

Consider a family of vector spaces (χ', ψ'), where

$$\chi' = \operatorname{id} \otimes \chi\colon M' = A' \otimes_A M \longrightarrow A' \otimes_A B \tag{1.84}$$

and ψ' is a morphism $Y' = \operatorname{Spec} A' \otimes_A B \longrightarrow X'$. This family is said to be *induced by the change of base* X' by X.

We see that X' is indeed a family of vector spaces, since there is a canonical isomorphism

$$S^{\bullet}_{A'}(A' \otimes_A M) \xrightarrow{\sim} A' \otimes_A S^{\bullet}_A(M). \tag{1.85}$$

In particular, if A' is a field, Y' is the scheme of the vector space $(A' \otimes_A M)^*$ over A'. This means that all the fibers of the family $\psi\colon Y \longrightarrow X$ over geometric points are vector spaces, which justifies the name "the scheme of the vector space". The dimensions of fibers can, clearly, jump.

Note also that thanks to the isomorphism (1.85) the scheme Y' is identified with the fiber product $X' \underset{X}{\times} Y$, so that our operation of the change of base is an exact analogue of the topological one.

1.10.3. Vector bundles. A family of vector spaces $\chi\colon Y \longrightarrow X$ is said to be *trivial* if its defining A-module is free.

The families of vector spaces trivial in a neighborhood of any point are said to be *vector bundles.* It is not quite clear though how to define the property of local triviality: The neighborhoods of points in $\operatorname{Spec} A$ are just topological spaces, not schemes. Here we first encounter the problem that will be systematically investigated in the next Chapter. For a time being, it is natural to adopt the following preliminary definition.

A family of vector spaces $\chi\colon Y \longrightarrow X$ is said to be *trivial at* $x \in X$ if there exists an open neighborhood $U \ni x$ such that, for any morphism $\psi\colon X' \longrightarrow X$ with $\psi(X') \subset U$, the induced family $Y' \longrightarrow X'$ is trivial.

We will now replace this condition of triviality by another one which is easier to verify. First of all, since the big open sets $D(f)$ constitute a basis of the topology of $\operatorname{Spec} A$, it suffices to consider neighborhoods of the form $D(f)$. They possess the following remarkable property.

Proposition. *Let A be a ring, $f \in A$ not a nilpotent. Set $X_f := \operatorname{Spec} A_f$ and denote by $i^*\colon X_f \longrightarrow X$ the morphism induced by the homomorphism $i\colon A \longrightarrow A_f$, $g \mapsto g/1$. Then*

a) *i^* determines a homeomorphism $X_f \simeq D(f)$*

b) *For any morphism $\psi\colon X' \longrightarrow X$ such that $\psi(X') \subset D(f)$, there exists a unique morphism $\chi\colon X' \longrightarrow X_f$ such that the diagram*

$$\begin{tikzcd} X' \arrow[rr, "\psi"] \arrow[dr, "\chi"'] & & X \\ & X_f \arrow[ur, "i^*"'] & \end{tikzcd} \tag{1.86}$$

commutes.

This implies that the family of vector spaces $Y \longrightarrow X$ is trivial at $x \in \operatorname{Spec} A = X$ if and only if, for any element $f \in A$ such that $f(x) \neq 0$, the family induced over X_f is trivial. Translating this into the language of modules we find a simple condition which will be used in what follows.

Corollary. *An A-module M determines a family of vector spaces trivial at $x \in \operatorname{Spec} A$ if and only if there exists $f \in A$ such that $f(x) \neq 0$ and such that the A_f-module $M_f := A_f \otimes_A M$ is free.*

A module M satisfying conditions of Corollary for all the points $x \in \operatorname{Spec} A$ is said to be *locally free*.

Proof of Proposition. a) This is a particular case of Theorem 1.6.4d

b) This statement expresses the known universal property of the rings of quotients. Indeed, let $\psi\colon A \longrightarrow A'$ be a ring homomorphism such that ${}^a\psi(\operatorname{Spec} A') \subset D(f)$. This means that f does not belong to any of the ideals $\psi^{-1}(p)$, where $p \in \operatorname{Spec} A'$, i.e., $\psi(f)$ does not vanish on $\operatorname{Spec} A'$. Therefore $\psi(f)$ is invertible in A'.

In the category of such A-algebras, the morphism $A \longrightarrow A_f$ is universal object (see 1.6.4b) which proves the statement desired.

In particular, if $D(f) = D(g)$, then the ring A_f is canonically isomorphic to A_g. $\square$

1.10.4. Main definition and main result of this section. A *vector bundle over a scheme* $X = \operatorname{Spec} A$ is a family of vector spaces locally trivial at each point of $\operatorname{Spec} A$.

Unless otherwise stated, until the end of this section we will only consider Noetherian rings and modules.

Recall that a module M is said to be *projective* if it is isomorphic to a direct summand of a free module.

Theorem. *An A-module M determines a vector bundle over* $\operatorname{Spec} A$ *if and only if it is projective.*

The theorem claims that the class of locally free modules coincides with the class of projective modules. This is just the statement we will prove; first the inclusion in one direction and then into the opposite one. We will have to perform rather hard job and we will use the opportunity and establish meanwhile more auxiliary results than is actually needed: They will serve us later.

1.10.5. Localizations of modules. Let $S \subset A$ be a multiplicative system not containing 0, and M an A-module. Set $M_S = A_S \otimes_A M$. Though we only need here the information on $A_f := A_S$ for $S = \{f^n \mid n \in \mathbb{Z}_+\}$ and $M_f := M_S$ for the same S, it is not a problem to extend the result to general multiplicative system S.

1.10.5a. Lemma. *The equality $m/s = 0$ holds if and only if there exists $t \in S$ such that $tm = 0$. In particular, the kernel of the natural homomorphism*

$$M \longrightarrow M_S, \qquad m \mapsto m/1, \tag{1.87}$$

consists of the elements m such that $(\operatorname{Ann} M) \cap S \neq \emptyset$.

Proof. Clearly, $tm = 0 \Longrightarrow tm/ts = 0 = m/s$. To prove the converse implication, consider first a particular case.

a) *M is free.* Let $\{m_i\}_{i\in I}$ be a free A-basis of M. Then $\{m_i = m_i/1\}_{i\in I}$ is a free A_S-basis of M_S. Let $m = \sum f_i m_i$, where $f_i \in A$, be an element of M. If $m/s = 0$, then $f_i/s = 0$ for all i, and hence there exist $t_i \in S$ such that $t_i f_i = 0$. Set $t = \prod t_i$, where i runs over a finite set of indices $I_{0\subset I}$ for which $f_i \neq 0$. Clearly, $tm = 0$ since $tf_i = 0$ for all $i \in I_0$.

b) *The general case.* There exists an exact sequence

$$F_1 \xrightarrow{\varphi} F_0 \xrightarrow{\psi} M \longrightarrow 0, \tag{1.88}$$

where F_0, F_1 are free modules. Tensoring the sequence by A_S we get the exact sequence

$$(F_1)_S \xrightarrow{\varphi_S} (F_0)_S \xrightarrow{\psi_S} M_S \longrightarrow 0 \tag{1.89}$$

(see [Lang]). Here we set $\varphi_S = \mathrm{id}_{A_S} \otimes_A \varphi$, and so on.

Let $m/s = 0$ for $m = \psi(n)$, where $n \in F_0$. Then $\psi_S(n/s) = 0$; this implies

$$n/s = \varphi_S(l/t) = \varphi(l)/t, \text{ where } l \in F_1 \text{ and } t \in S. \tag{1.90}$$

In other words, $(tn - s\varphi(l))/st = 0$ in $(F_0)_S$. Since F_0 is free, there exists $r \in S$ such that $rtn = rs\varphi(l)$ in F_0. Applying ψ to this relation we get

$$rtm = \psi(rtn) = rs\psi \circ \varphi(l) = 0, \tag{1.91}$$

as desired. □

Observe that we never used the Noetherian property.

1.10.5b. Corollary. *Let M be a Noetherian A-module, $f \in A$. There exists an integer $q > 0$ such that $f^q m = 0$ for all $m \in \operatorname{Ker}(M \longrightarrow M_f)$.*

Proof. Select the needed value q_i for every of a finite number of generators of the kernel, and set $q = \max\limits_i q_i$. □

1.10.6. Tensoring exact sequences. In the proof of Lemma 1.10.5 we have used the following general property of the tensor product:

Tensoring sends short exact sequences into the sequences exact everywhere except the leftmost term. (1.92)

Tensoring by A_S, however, possesses a stronger property: It completely preserves exactness; this means A_S is what is called a *flat* A-algebra.

1.10.6a. Proposition. *The sequence $M_S \xrightarrow{\varphi_S} N_S \xrightarrow{\psi_S} P_S$ of A_S-modules is exact for any exact sequence $M \xrightarrow{\varphi} N \xrightarrow{\psi} P$ of A-modules.*

Proof. $\psi \circ \varphi = 0 \Longrightarrow \psi_S \circ \varphi_S = 0 \Longrightarrow \operatorname{Ker} \varphi_S \supset \operatorname{Im} \varphi_S$.

Conversely, let $n/s \in \operatorname{Ker} \psi_S$; then $\psi(n/s) = 0$ implying, thanks to the above, $t\psi(n) = 0$ for some $t \in S$. Therefore $tn = \varphi(m)$ implying

$$n/s = tn/ts = \varphi(m)/ts = \varphi_S(m/ts), \tag{1.93}$$

as desired. □

1.10.7. Lifts of A_f-module homomorphisms. Let $\varphi\colon M \longrightarrow N$ be an A-module homomorphism. For any $f \in A$, we have an induced A_f-module homomorphism $\varphi_f : M_f \longrightarrow N_f$. We will say that $\psi\colon M_f \longrightarrow N_f$ *can be lifted to* $\varphi : M \longrightarrow N$ if $\varphi_f = \psi$.

Lemma. *Let F be a free Noetherian A-module, M a Noetherian A-module, $f \in A$, M_f a free A-module. Then, for any homomorphism $\varphi\colon M_f \longrightarrow F_f$, there exists an integer q such that the homomorphism $f^q\varphi\colon M_f \longrightarrow F_f$ can be lifted to a homomorphism $M \longrightarrow F$.*

Proof. First of all, F_f is free and has a finite number of generators, so that φ is given by a finite number of coordinate A_f-morphisms $M_f \longrightarrow A_f$. If being multiplied by an appropriate power of f each of them can be lifted to a morphism $M \longrightarrow A$, then so does φ. Therefore we can and will assume that $F = A$.

Let m_i, where $i = 1, \dots, n$, be a system of generators of M. Multiplying φ by an appropriate power of f, we may assume that $\varphi(m_i) = g_i/1$, where $g_i \in A$ for all i.

It is tempting to lift $\varphi\colon M_f \longrightarrow A_f$ to a homomorphism $\psi\colon M \longrightarrow A$ by setting $\psi(m_i) = g_i$. This, however, might prove to be impossible since there are relations $\sum f_i m_i = 0$ such that $\sum f_i g_i \neq 0$. But we have the equality $\sum f_i(g_i/1) = 0$, and therefore the set

$$\left\{\sum f_i g_i \mid \sum f_i m_i = 0\right\} \tag{1.94}$$

constitutes, thanks to Lemma 1.10.5, a Noetherian A-submodule, which belongs to $\operatorname{Ker}(A \longrightarrow A_f)$. By Corollary 1.10.5 this submodule is annihilated by f^q for some q. This implies that there exists a homomorphism $f^q\psi\colon M \longrightarrow A$ such that $f^q\psi(m_i) = f^q g_i$ since $(f^q\psi)_f = f^q\varphi$. $\square$

1.10.8. Locally free modules are projective. Now, we can establish the "if" part of Theorem 1.10.4.

Proposition. *Locally free modules are projective.*

Proof. Let M be a Noetherian locally free A-module, $\psi\colon F \longrightarrow M$ an epimorphism, where F is a Noetherian free module. In order to prove that M is a direct summand of F, we have to find a homomorphism $\varphi\colon M \longrightarrow F$ such that $\psi \circ \varphi = \mathrm{id}_M$. More generally, let

$$P = \{\chi \in \operatorname{Hom}_A(M, M) \mid \chi = \psi \circ \varphi;\ \text{for some}\ \ \varphi \in \operatorname{Hom}_A(M, F)\}. \tag{1.95}$$

First, let us show that, for every point $x \in \operatorname{Spec} A$, there exists $f \in A$ such that $f(x) \neq 0$ and $f^q \,\mathrm{id}_M \in P$ for some $q \geq 0$.

Select f so that M_f is A_f-free. Then the epimorphism $\psi_f\colon F_f \longrightarrow M_f$ has a section $\varphi\colon M_f \longrightarrow F_f$. By Lemma 1.10.7 we can lift $f^r\varphi$ to a homomorphism $\chi\colon M \longrightarrow F$ for some $r \geq 0$. Since $\psi_f \circ \varphi = \mathrm{id}_{M_f}$, this implies $(\psi \circ \chi)_f = f^r \,\mathrm{id}_{M_f}$; in particular,

$$(\psi \circ \chi - f^r \,\mathrm{id}_M)_f(m_i) = 0 \tag{1.96}$$

for a finite number of generators m_i of M. Therefore $f^t(\psi \circ \chi - f^r \,\mathrm{id}_M) = 0$ for some $t \geq 0$; hence $f^{r+t} \,\mathrm{id}_M = \psi \circ f^t\chi \in P$.

Now, from a cover of $\operatorname{Spec} A$ with big open sets, select a finite subcovering $\bigcup\limits_{1 \leq i \leq k} D(f_i)$ which is possible because $\operatorname{Spec} A$ is quasi-compact. Find q for which $f_i^q \,\mathrm{id}_M \in P$ for all i. Since $D(f_i^q) = D(f_i)$, it follows that $\{f_i^q\}_{1 \leq i \leq k}$ generates (1.75). The partition of unity $\sum\limits_{1 \leq i \leq k} g_i f_i^q = 1$ implies that

$$\mathrm{id}_M = \left(\sum_{1 \leq i \leq k} g_i f_i^q\right) \mathrm{id}_M \in P. \quad \square \tag{1.97}$$

1.10.9. Nakayama's lemma. Now, we would like to establish that projective modules are locally free. First, we verify this for a stronger localization procedure.

The following simple but fundamental result is called *Nakayama's lemma.*

Lemma (Nakayama's lemma)**.** *Let A be a local ring, $\mathfrak{a} \subset A$ an ideal not equal to A, and M an A-module of finite type. If $M = \mathfrak{a}M$, then $M = \{0\}$.*

Examples illustrating the necessity of finiteness condition.

a) Let A be a ring without zero divisors, M the field of quotients. Obviously, if $\mathfrak{a} \neq \{0\}$, then $\mathfrak{a}M = M$, but $M \neq \{0\}$.

b) Let A be the ring of germs of C^∞-functions in a vicinity of the origin of $\mathbb{R}$ and $\mathfrak{a}$ the ideal of functions that vanish at the origin. Let $M = \bigcap\limits_{n\in\mathbb{Z}_+} \mathfrak{a}^n$ be the ideal of *flat* functions, i.e., the functions that vanish at the origin together with all the derivatives. It is not difficult to establish that $\mathfrak{a}M = M$: This follows from the fact that, for any flat function f and the coordinate function x, the quotient f/x whose value at the origin is defined as zero is a flat function.

Proof of Nakayama's lemma. Let $M \neq \{0\}$. Select a minimal finite system of generators $m_1, \dots, m_2$ of M. Since $M = \mathfrak{a}M$, it follows that $m_1 = \sum\limits_{1\leq i\leq r} f_i m_i$ for $f_i \in \mathfrak{a}$, i.e., $(1 - f_1)m_1 = \sum\limits_{i\geq 2} f_i m_i$. Since f_1 lies in a maximal ideal of A, it follows that $1 - f_1$ is invertible; therefore m_1 can be linearly expressed in terms of $m_2, \dots, m_r$. This contradicts to minimality of the system of generators. □

1.10.9a. Corollary. *Let M be the module of finite type over a local ring A with a maximal ideal p. Let the elements $\overline{m}_i = m_i \pmod{pM}$, where $i = 1, \dots, r$, generate M/pM as a linear space over the field A/p, then the m_i generate the A-module M. In particular, if A is Noetherian, the generators of the A/p-space p/p^2 generate the ideal p.*

Proof. Let $M' = M/(Am_1 + \ldots + Am_r)$. Since $M = pM + Am_1 + \ldots + Am_r$, we see that $M' = pM'$, implying $M' = 0$. □

1.10.10. Proposition. *A projective module M of finite type over a local ring A is free.*

Proof. Let p be a maximal ideal in A. Then M/pM is a finite-dimensional space over A/p; let $\overline{m}_i = m_i \pmod{pM}$, where $i = 1, \dots, r$, be its basis. By the above, the m_i constitute a system of generators of M. Let us show that M is free. Consider an epimorphism $F \longrightarrow M \longrightarrow 0$, where $F = A^r$ is a free module of rank r whose free generators are mapped into $\{m_i\}_{i=1}^r$. Since M is projective, there exists a section $\varphi\colon M \longrightarrow F$ which induces an isomorphism $\overline{\varphi}\colon M/pM \longrightarrow F/pF$ because the dimensions of both the spaces are equal to r. Therefore either $F = \varphi(M) + pF$ or $F/\varphi(M) = p(F/\varphi(M))$. By Nakayama's lemma $F = \varphi(M)$; hence φ is an isomorphism. □

1.10.11. Proof of Theorem 1.10.4, completion.

Proposition. *Any Noetherian projective module M over any Noetherian ring A is locally free.*

Proof. Let $x \in \operatorname{Spec} A$ and $p \subset A$ the corresponding prime ideal, the module $M_p = A_p \otimes M$ is projective, and therefore free thanks to Proposition 1.10.10. Take its A_p-basis. Reducing the elements of the basis to the common denominator we may assume that they are of the form m_i/g, where $m_i \in M$ for $i = 1, \dots, n$ and $g \in A$. Consider a homomorphism $\varphi\colon A_g^n \longrightarrow M_g$ sending the elements of a free basis of $A_{(g)}^n$ into m_i/g; set $K = \operatorname{Ker} \varphi$ and $C = \operatorname{Coker} \varphi$.

Tensoring the exact sequence of $A_{(g)}$-modules

$$0 \longrightarrow K \longrightarrow A_g^n \longrightarrow M_g \longrightarrow C \longrightarrow 0 \tag{1.98}$$

by A_p, which is also the localization of A_g modulo $A_g \setminus p_g$, we get thanks to Theorem 1.10.6 an exact sequence of A_p-modules. Its middle arrow is an isomorphism, and therefore $A_p \otimes_{A_g} K = 0$ and $A_p \otimes_{A_g} C = 0$.

Let $k_2, \dots, k_s$ and $c_1, \dots, c_r$ be bases of the A_g-modules K and C, respectively. By Lemma 1.10.9 there exist $h_i, h'_j \in A_g \setminus p_g$ such that $h_i k_i = 0$ and $h'_j c_j = 0$ for all i, j. In particular,

$$h = \prod_{1 \leq I \leq r} h_i \prod_{1 \leq j \leq s} h'_j \in A_g \setminus p_g \tag{1.99}$$

and h annihilates K and C. Let $h = f/g^k$, where $f \in A \setminus p$. Then $f/1$ annihilates both K and C. Tensoring (1.98) by $(A_g)_{f/1}$ over A_g, and with the A_{fg}-module isomorphism $(M_g)_{f/1} \simeq M_{fg}$ taken into account, we see that there exists an isomorphism $A_{fg}^n \simeq M_{fg}$ because $K_{f/1} = \{0\}$ and $C_{f/1} = \{0\}$. Since $fg(x) \neq 0$, we see that M is locally free at x. □

1.10.12. An example of a non-free projective module. Let A be a ring of real-valued continuous functions on $[0,1]$ such that $f(0) = f(1)$, i.e., A may be viewed as a ring of functions on the circle S^1. The module of sections of the Möbius bundle over S^1 may be described as the A-module M of functions on $[0,1]$ such that $f(0) = -f(1)$.

Theorem. *M is not free, but $M \oplus M \cong A \oplus A$.*

Proof. 1) For any $f_1, f_2 \in M$, we have $f_1 f_2 (f_2) - f_2^2 (f_1) = 0$ and $f_1 f_2, f_2^2 \in A$; hence, any two elements from M are linearly dependent over A. This means that if M is free, $M \cong A$.

But $M \not\cong Am$ for any $m \in M$ since m vanishes somewhere on $[0,1]$ thanks to continuity and since M possesses elements that do not vanish at any prescribed point.

2) The elements $f = (\sin \pi t, \cos \pi t)$ and $g = (-\cos \pi t, \sin d\pi t)$ constitute a free basis of $M \oplus M$ since, for any $(m_1, m_2) \in M \oplus M$, the system

$$\begin{cases} x \sin \pi t - y \cos \pi t &= m_1, \\ x \cos \pi t + y \sin \pi t &= m_2 \end{cases} \tag{1.100}$$

is uniquely solvable in A. □

1.11. The normal bundle and regular embeddings

1.11.1. Conormal module. Let Y be a closed subscheme of an affine scheme $X = \operatorname{Spec} A$ determined by an ideal a. Then the A/a-module a/a^2 is said to be the *conormal module to Y with respect to the embedding* $Y \hookrightarrow X$ or just the *conormal module* to Y and the family of vector spaces $N = \operatorname{Spec} S_{A/a}(a/a^2)$ is said to be the *normal family*.

The following geometric picture illustrates this definition: a is the ideal of functions on X that vanish on Y, a^2 the ideal of functions whose zeroes on Y are of order ≥ 2, and a/a^2 is the module of linear parts of these functions in a neighborhood of Y. A *tangent vector to X at a point $y \in Y$* determines a linear function on such linear parts. A *normal vector to Y* at y (in the absence of a natural metric) is a class of tangent vectors to X at $y \in Y$ modulo those that are tangent to Y, i.e., the ones which vanish on the linear parts of the functions from a. Therefore, in "sufficiently regular" cases, a/a^2 is (locally) the space dual to the space of vectors normal to Y. This explains the term.

1.11.2. Regular embeddings. The conormal module is, in general, neither free nor projective, but it is both free and projective for an important class of subschemes.

A sequence of elements $(f_1, \dots, f_n)$ of a ring A is said to be *regular* (of length n) if, for all $i \leq n$, the element $f_i \mod (f_1, \dots, f_{i-1})$ is not zero divisor in $A/(f_1, \dots, f_{i-1})$; it is convenient to assume that the empty sequence is regular of length 0 and generates the zero ideal.

A closed subscheme $Y \subset X = \operatorname{Spec} A$ is said to be *regularly embedded* or, more often, a *complete intersection* (of co-dimension n), if A contains a regular sequence of length n generating the ideal which singles out Y.

The geometric meaning of complete intersection becomes totally transparent when we recall that we define Y adding one of the equations $f_i = 0$ at a time. Thus, we get a decreasing sequence of subschemes $X \supset Y_1 \supset Y_2 \supset \ldots \supset Y_n = Y$. The complete intersection condition means that Y_i does not contain the whole support of any of the components of the incompressible primary decomposition of Y_{i-1}. In other words, each equation $f_i = 0$ should be "transversal" (in a very weak sense) to all these supports.

Proposition. *Let $Y \hookrightarrow X$ be a complete intersection. Then its conormal module is free.*

In particular, the rank n of the conormal module does not depend on the choice of a regular system of generators of the ideal.

The rank n of the conormal module is said to be the *codimension* of Y in X.

Proof. Let $a=(f_1,\dots,f_n)\subset A$, where $f_1,\dots,f_n$ is a regular sequence. Obviously, the elements $\overline{f}_i=f_i \pmod a$ generate the A/a-module a/a^2. Therefore it suffices to verify that they are linearly independent. This is done by induction on n:

First, let $n=1$. Then $f_1=f$, $\overline{g}=g \pmod{A_f}$. If $\overline{g}\overline{f}=0$, then $gf=hf^2$ for some $h\in A$, and therefore $f(g-hf)=0$; hence $g=hf$, since f is not a zero divisor in A. Therefore $\overline{g}=0$.

Let the result be already proved for a regular sequence $\{f_1,\dots,f_{n-1}\}$. Assume that $\sum\limits_{1\leq i\leq n}\overline{g}_i\overline{f}_i=0$ in a/a^2, where $\overline{g}_i=g_i \pmod a$. We may assume that $\sum\limits_{1\leq i\leq n} g_if_i=0$ in A: Otherwise $\sum\limits_{1\leq i\leq n} g_if_i=\sum\limits_{1\leq i\leq n} u_if_i$, where $u_i\in a$, and we may replace g_i by g_i-u_i without affecting $\overline{g}_i$.

Since the class $\overline{f}_n$ is not a zero divisor in $A/(f_1,\dots,f_{n-1})$, it follows that

$$g_nf_n+\sum_{1\leq i\leq n-1} g_if_i=0 \implies g_n\in(f_1,\dots,f_{n-1}) \tag{1.101}$$

i.e., $g_n=\sum\limits_{1\leq i\leq n-1} h_if_i$, implying $\sum\limits_{1\leq i\leq n-1}(g_i+h_if_n)f_i=0$. By the induction hypothesis this means that $g_i+h_if_n\in(f_1,\dots,f_{n-1})$ for $i=1,\dots,n-1$; hence $g_i\in a$ for all i, i.e., $\overline{g}_i=0$. □

1.11.3. Locally regularly embedded subscheme. A subscheme $Y\hookrightarrow X$ is said to be *locally regularly embedded at* $y\in Y$ if there exists a neighborhood $D(f)\ni y$ such that $Y\cap D(f)$ is regularly embedded into $D(f)$. Obviously, $Y\cap D(f)$ is determined by an ideal $a_f\subset A_f$ and coincides with the fiber product $Y\underset{X}{\times}X_f$.

Statement. *The normal family to a locally regularly embedded subscheme is a vector bundle.*

Indeed, $(a/a^2)_f=a_f/a_f^2$ so that A/a-module a/a^2 is locally free for such a subscheme. □

Remark. It well may happen that a subscheme is regularly embedded locally but not globally. The first example of such a happening was encountered in number theory.

Let $A\supset\mathbb{Z}$ be a ring of integer algebraic numbers of a field K. If the number of classes [16] of K is greater than 1, then A possesses non-principal ideals $a\subset A$ (which are even prime). However, any such ideal, as is known, is "locally" principal. Therefore a determines a locally regularly embedded subscheme of co-dimension 1.

[16] For definition, see [BSh].

1.11.4. The tangent and cotangent spaces. Let $x \in X$ be a closed point: For brevity, we will also denote by x the unique reduced subscheme with support at this point. Let m_x be the maximal ideal corresponding to x. The discussion from sec. 1.11.1 shows that m_x/m_x^2, the *co-normal module* to x, is an analogue of the cotangent space to X at x. This is the *Zariski cotangent space*. Its dual, $(m_x/m_x^2)^*$, is called the *tangent space* at x.

The closed points may be and may not be locally regularly embedded.

For instance, all the closed points of $\mathbb{A}^n = \operatorname{Spec} K[T_1, \ldots, T_n]$, where, for simplicity, K is algebraically closed, correspond to the ideals $(T-t_1, \ldots, T-t_n)$, where $t_i \in K$. The written system of generators of such an ideal (i.e., $T - t_1, \ldots, T - t_n$) is, obviously, a regular sequence.

To get examples of non-locally regularly embedded points, it suffices to consider the spectrum of a local Artinian ring which is not a field: All the elements of its maximal ideal are nilpotents, and therefore there is no system of generators whose first element is not a zero divisor. More meaningful examples are provided by *hypersurfaces*, i.e., subschemes of the affine space $\mathbb{A}^n$ given by one equation.

1.11.5. Example. Let $X \subset \mathbb{A}^n$ be a closed subscheme of an affine space over the field K through the origin 0; let the equation of X be $F = 0$, where $F = F_1 + F_2 + \ldots$ and F_i is a form of degree i in $T_1, \ldots, T_n$.

1.11.5a. Statement. *The point x is locally regularly embedded into X if and only if $F_1(x) = 0$.*

1.11.5b. Corollary. *Let $F(t_1, \ldots, t_n) = 0$, where $t_i \in K$. The point x singled out by the ideal $(\ldots, T - t_i, \ldots)$ is locally regularly embedded into X if and only if there exists an i such that*

$$\frac{\partial F}{\partial T_i}(t_i, \ldots, t_n) \neq 0. \tag{1.102}$$

Proof. Indeed, translate the origin to $(t_1, \ldots, t_n)$; then the linear part of F in a vicinity of the new origin is equal to

$$\sum \frac{\partial F}{\partial T_i}(t_1, \ldots, t_n)(T_i - t_i) \tag{1.103}$$

and it suffices to apply the above statement. □

This differential criterion shows that locally regularly embedded points are exactly the ones which in the classical algebraic geometry are called *nonsingular* ones.

Leaving a systematic theory of such points for future, we confine ourselves here with the general facts needed for studying Example 1.11.5.

Note, first of all, that it suffices to consider the localized rings. More precisely, let $\mathfrak{p} = (T_1, \ldots, T_n) \subset A$, and $\overline{\mathfrak{p}} = \mathfrak{p} \bmod (F) \subset B$. Proof of Proposition 1.11.2 shows that the point x is locally regularly embedded into X if and only if the maximal ideal of the local ring $B_{\overline{\mathfrak{p}}}$ is generated by a regular sequence. Observe that $B_{\overline{\mathfrak{p}}} = A_{\mathfrak{p}}/(F/1)$.

1.11.5c. Lemma. *Under conditions of Example 1.11.5, if $F_1 \neq 0$, then the maximal ideal in $B_{\overline{\mathfrak{p}}}$ is generated by a regular sequence.*

Proof. Indeed, making a non-degenerate linear change of indeterminates we may assume that $F_1 = T_1$. Now, for any $G \in K[T_1, \dots, T_n]$ let $\overline{G}$ denote the class $G/1 \bmod (F/1)$ in the ring $B_{\overline{\mathfrak{p}}}$.

The elements $T_2/1, \dots, T_n/1$ and $F/1$ form a regular system in the ring $A_{\mathfrak{p}}$ since $F \equiv T_1 + a_2T_1^2 + \dots \bmod (T_2, \dots, T_n)$, where $a_i \in K$. Corollary 2.9.8 (to be proved in Chapter 2) implies that $F/1, T_2/1, \dots, T_n/1$ is also a regular sequence. Therefore, $\overline{T}_2/1, \dots, \overline{T}_n/1$ is a regular sequence in $B_{\mathfrak{p}} = A_{\mathfrak{p}}/(F/1)$; clearly this regular sequence generates a maximal ideal in $B_{\mathfrak{p}}$. $\square$

1.11.6. The statement converse to that of Lemma 1.11.5c. To prove this statement, observe that if the origin is locally regularly embedded in X, then the maximal ideal of the local ring $B_{\overline{p}}$ should be generated by a regular sequence. The condition $F_1 = 0$ means that $F/1$ belongs to the square of the maximal ideal in A_p. Therefore it suffices to establish the following Lemma.

1.11.6a. Lemma. *Let A be a local Noetherian ring, $p \subset A$ its maximal ideal generated by a regular sequence of length n. If $f \in p^2$ and f is regular, then the maximal ideal in the local ring $A/(f)$ can not be generated by a regular sequence.*

Proof. Let the maximal ideal in $A/(f)$ be generated by a maximal sequence $\overline{g}_1, \dots, \overline{g}_k$, where $\overline{g}_i = g_i \bmod (f)$, and $g_i \in A$. Then $(f, g_1, \dots, g_k)$ is a regular sequence in A that generates p. Since the length of any such sequence is equal to n (Prop. 1.11.2), we should have $k = n - 1$. But the elements $(f, g_1, \dots, g_k)$ generate in the n-dimensional A/p-space p/p^2 a subspace of dimension $\leq k = n - 1$, since $f \in p^2$. The contradiction obtained proves the lemma and completes the study of the example 1.11.5c. $\square$

1.12. The differentials

1.12.1. The module of universal differentials. Let A and B be commutative rings and B an A-algebra. Set

$$\begin{aligned} I = I_{B/A} = \operatorname{Ker}\mu, \quad &\text{where } \mu\colon B \otimes_A B \longrightarrow B, \\ &\mu(b_1 \otimes b_2) = b_1 b_2 \ \text{ is the multiplication.} \end{aligned} \tag{1.104}$$

Clearly, I is an ideal in $B \otimes_A B$ and $(B \otimes_A B)/I \cong B$.

Consider the B-module:

$$\mathrm{Covect}_{B/A} = I/I^2 \tag{1.105}$$

called the *modules of (relative) differentials* of the A-algebra B.

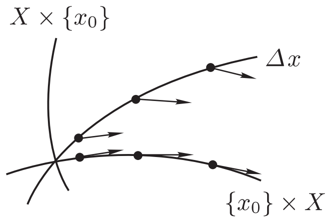


Fig. 12

The B-module $\mathrm{Covect}_{B/A} = I_{B/A}/I^2_{B/A}$ is called the *module of (relative) differentials* of the A-algebra B.[17)]

By Proposition 1.9.6 the ideal I determines the diagonal subscheme $\Delta_X \subset X \underset{S}{\times} X$, where $X = \operatorname{Spec} B$, $S = \operatorname{Spec} A$. Due to the interpretation from sec. 1.11.1 the module $\mathrm{Covect}_{B/A}$ represents the *co-normal to the diagonal.*

In Differential Geometry, the normal bundle to the diagonal Δ is isomorphic to the tangent bundle to the manifold X itself. Indeed, transport a vector field along one of the fibers of the product $X \times X$ "parallel" to the diagonal; we get a vector field everywhere transversal to the diagonal, see Fig. 12. Therefore $\mathrm{Covect}_{B/A}$ is a candidate for the role of the module "cotangent " to X along the fibers of the morphisms $X \longrightarrow S$.

On the other hand, in the interpretation of nilpotents given in §§ 1.5 as an analog of the elements of the "tangent module" to X (over S), we have already considered the B-module $D_{B/A}$ of *derivations* of the A-algebra B (a *vector field on* X, i.e., a section of the tangent bundle can be naturally interpreted as a derivation of the ring of functions on X).

Remark. The higher order "differential neighborhoods of the diagonal" are represented by the schemes $\operatorname{Spec}(B \otimes_A B)/I^n_{B/A}$. They replace the spaces of *jets* of the germs of the diagonal considered in differential geometry.

Define a map $d = d_{B/A}\colon B \longrightarrow \mathrm{Covect}_{B/A}$ by setting

$$d(b) = (b \otimes 1 - 1 \otimes b) \pmod{I^2_{B/A}}. \tag{1.106}$$

In the differential geometry, the tangent and the co-tangent bundles are dual to each other. In the algebraic setting (over finite fields), this is not the case, generally: Only "a half" of the duality is preserved:

$$D_{B/A} = \operatorname{Hom}_B(\mathrm{Covect}_{B/A}, B). \tag{1.107}$$

[17] In the differential and algebraic geometries we also need the **exterior powers** of differential forms, and in **this** context the module of differentials is denoted by $\Omega^1_{B/A}$. For details, see [MaG, SoS].

Therefore, $D_{B/A}$ is recovered from $\mathrm{Covect}_{B/A}$ but not the other way round. This explains the advantage of differentials as compared with derivatives. [18)]

Lemma. 1) *d is an* **even** *A-derivation, and $d(\varphi(a)) = 0$ for any $a \in A$, where $\varphi\colon A \longrightarrow B$ is the morphism that defines the A-algebra structure.*

2) *Let $\{b_i \mid i \in I\}$ be a system of generators of the A-algebra B. Then $\{db_i \mid i \in I\}$ is a system of generators of the B-module* $\mathrm{Covect}_{B/A}$.

Proof. 1) is subject to a straightforward verification.

2) Notice that

$$\begin{gathered}\sum_i b_i \otimes b'_i \in I_{B/A} \iff \sum_i b_i b'_i = 0 \iff \\ \sum_i b_i \otimes b'_i = \sum_i b_i \otimes b'_i - \sum_i b_i b'_i \otimes 1 = \sum_i b_i \otimes 1(1 \otimes b'_i - b'_i \otimes 1).\end{gathered} \tag{1.112}$$

This implies that, as a B-module, $\mathrm{Covect}_{B/A}$ is generated by the elements dB for all $b \in B$. Since d is a derivation that vanishes on the image of A, this easily implies the desired. □

Example. Let $B = A[T_1, \dots, T_n]$. Then $\mathrm{Covect}_{B/A}$ is the free B-module freely generated by dT_i.

1.12.2. Proposition. *For any derivative $d'\colon B \to M$ of B into a B-module M that vanishes on the image of A, there exists a unique B-module homomorphism $\psi\colon$ $\mathrm{Covect}_{B/A} \longrightarrow M$ such that $d' = \psi \circ d_{B/A}$.*

---

[18] Grothendieck showed that one can, however, define *differential operators of order $\leq k$* for any ring R over K or $\mathbb{Z}$ as the K- or $\mathbb{Z}$-linear maps of R-modules $D : M \longrightarrow N$ such that

$$[l_{r_0}, [l_{r_1}, \dots [l_{r_k}, D] \dots]] = 0 \quad \text{for any } \ r_0, r_1, \dots r_k \in R \tag{1.108}$$

where l_r denotes the operator of left multiplication by r in M and in N.

Denote by $\mathrm{Diff}_k(M,N)$ the R-module of differential operators of degree $\leq k$; set

$$\mathrm{Diff}(M,N) = \varinjlim \mathrm{Diff}_k(M,N). \tag{1.109}$$

Define the R-module of *symbols of differential operators* to be the graded space associated with $\mathrm{Diff}(M,N)$

$$\mathrm{Smbl}(M,N) = \oplus\, \mathrm{Smbl}_k(M,N), \tag{1.110}$$

where

$$\mathrm{Smbl}_k(M,N) = \mathrm{Diff}_k(M,N)/\,\mathrm{Diff}_{k-1}(M,N). \tag{1.111}$$

If $M = N$, then, clearly, $\mathrm{Diff}\ (M) = \mathrm{Diff}\ (M,M)$ is an associative algebra with respect to the product and a Lie algebra — denoted by $\mathfrak{diff}(M)$ — with respect to the bracket of operators. $\mathrm{Smbl}(M)$ is a commutative R-algebra with respect to the product. The bracket in $\mathfrak{diff}(M)$ induces a Lie algebra structure in $\mathrm{Smbl}(M)$ (the Poisson bracket), as is not difficult to see, this Lie algebra is isomorphic to the Poisson Lie algebra in $\dim M$ variables over R.

(Applying this result to $M = B$ we get (1.107)).

Proof. The *uniqueness* of ψ follows immediately from the fact that $d'b = \psi(db)$ for all $b \in B$, so that ψ is uniquely determined on the system of generators of $\mathrm{Covect}_{B/A}$.

To prove the *existence*, let us first define a group homomorphism

$$\chi\colon B \otimes_A B \longrightarrow M \quad \chi(b \otimes b') = bd'b'. \tag{1.113}$$

This homomorphism vanishes on $I^2_{B/A}$. Indeed, first notice that χ is a B-module homomorphism with respect to the B-action on $B \otimes_A B$ given by $b \mapsto b \otimes 1$. Furthermore, as shown above, the elements $b \otimes 1 - 1 \otimes b$ generate the B-module $I_{B/A}$, and therefore the products

$$(b_1 \otimes 1 - 1 \otimes b_1)(b_2 \otimes 1 - 1 \otimes b_2) \tag{1.114}$$

generate the B-module $I^2_{B/A}$. Therefore, it suffices to verify that χ vanishes on such products. This is straightforward; hence, we see that χ induces a map $\varphi\colon I/I^2 \longrightarrow M$. We get

$$\varphi(d\,b) = \varphi(b \otimes 1 - 1 \otimes b) = d'b, \tag{1.115}$$

completing the proof. $\square$

1.12.3. Conormal bundles. Now, let $i\colon Y \hookrightarrow X$ be a closed embedding of schemes. In models of differential geometry the restriction of the tangent bundle to X onto Y contains, under certain regularity conditions, the tangent bundle to Y, and the quotient bundle is the *normal bundle* to Y. We would like to find out to what extent similar statement is true for the schemes.

Let us translate the problem into the algebraic language.

Let B be an A-algebra, $b \subset B$ an ideal. Then $\overline{B} = B/b$ is also an A-algebra, and the relative (over $\operatorname{Spec} A$) cotangent sheaves on *SpecB* and $\operatorname{Spec} \overline{B}$ are represented by $\mathrm{Covect}_{B/A}$ and $\mathrm{Covect}_{\overline{B}/A}$, respectively. On the other hand, the *conormal bundle* to the embedding $\operatorname{Spec} \overline{B} \longrightarrow \operatorname{Spec} B$ is represented by the B/b-module b/b^2.

An analogue of the classical situation is given by the following Proposition. Let $\delta\overline{e} = 1 \otimes_B d_{B/A}(e)$ for $\overline{e} \in b/b^2$ be represented by an element $e \in b$. Since a map $d'\colon B \longrightarrow \mathrm{Covect}_{\overline{B}/A}$ such that $d'f = d_{\overline{B}/A}(f \bmod b)$ is an A-derivation, it factorizes through a uniquely determined B-module homomorphism $\mathrm{Covect}_{B/A} \longrightarrow \mathrm{Covect}_{\overline{B}/A}$; since the target is annihilated by multiplication by b, this homomorphism determines a B/b-module morphism $u\colon B/b \otimes_B \mathrm{Covect}_{B/A} \longrightarrow \mathrm{Covect}_{\overline{B}/A}$.

Proposition. *There exists an exact sequence of B/b-modules:*

$$b/b^2 \xrightarrow{\delta} B/b \otimes_B \mathrm{Covect}_{B/A} \xrightarrow{u} \mathrm{Covect}_{\overline{B}/A} \longrightarrow 0, \tag{1.116}$$

where $\delta\overline{e} = 1 \otimes_B d_{B/A}(e)$ for $\overline{e} \in b/b^2$ is represented by an element $e \in b$.

Proof. If $\overline{e} = 0$, i.e., $e \in b^2$, then $de \in bd_{B/A}b$, so that $1_{\overline{B}} \otimes d_{B/A}(e) = 0$. Hence, $\delta(\overline{e})$ does not depend on the choice of e. It is obvious that δ is a group homomorphism, and the compatibility with the B/b-action follows from the fact that

$$\begin{aligned}&\delta(\overline{f}\overline{e}) = 1 \otimes_B d(fe) = 1 \otimes_B (edf + fde) = \overline{f} \otimes_B de = \overline{f}\delta(\overline{e})\\ &\text{for any } \overline{f} = f \bmod b.\end{aligned} \tag{1.117}$$

It is easy to see that $u(1 \otimes_B d_{B/A} f) = d_{\overline{B}/A}(f \bmod b)$ and, therefore, u is an epimorphism.

It is easy to see that $u \circ \delta = 0$:

$$u \circ \delta(\overline{e}) = u(1 \otimes de) = d(e \bmod b) = 0. \tag{1.118}$$

Let us verify the exactness in the middle term. Construct a homomorphism

$$v\colon \operatorname{Covect}_{B/A} \longrightarrow \overline{B} \otimes_B \operatorname{Covect}_{B/A} / \operatorname{Im} \delta \tag{1.119}$$

such that u and v are mutually inverse. For this, first define a derivation

$$d' : (\overline{B} \to \overline{B} \otimes_B \operatorname{Covect}_{B/A}) / \operatorname{Im} \delta \tag{1.120}$$

by setting

$$d'(\overline{f}) = 1 \otimes_B d_{B/A}(f) \bmod \operatorname{Im} \delta, \quad \text{ for any } \overline{f} = f \bmod b. \tag{1.121}$$

The result does not depend on the choice of a representative of $\overline{f}$ since

$$1 \otimes d_{B/A}(e) \in \operatorname{Im} \delta \quad \text{ for any } e \in b. \tag{1.122}$$

This derivation determines the homomorphism v. Since

$$u \circ v(df) = df \;\text{ and }\; (v \circ u)(1 \otimes df \bmod \operatorname{Im} \delta) = 1 \otimes df \bmod \operatorname{Im} \delta, \tag{1.123}$$

it follows that v and u are mutually inverse in some cases of our modules, proving the desired. □

1.12.3a. Remark. The difference of the above constructions from similar ones in the differential geometry is crucial: It well may happen that $\operatorname{Ker} \delta \neq 0$ even if the subscheme $Y \hookrightarrow X$ is regularly embedded. For example, let $X = \operatorname{Spec} \mathbb{Z}$ and $Y = \operatorname{Spec} \mathbb{Z}/p$, where p is a prime, $S = \operatorname{Spec} \mathbb{Z}$. Then $\operatorname{Covect}_{X/S} = 0$ and $\operatorname{Covect}_{Y/S} = 0$; whereas $(p)/(p)^2$ is a one-dimensional linear space over $\mathbb{Z}/p$.

Informally speaking, [19]

it is impossible to differentiate in the "arithmetic direction".

[19] See also the previous footnote.

1.13. Digression: Serre's problem and Seshadri's theorem

Serre posed the following problem: *Over n-dimensional affine space, are there non-trivial vector bundles?*

In other words, is the following statement true?

Any projective Noetherian module over $K[T_1, \dots, T_n]$, where K is a field, is free.

For $n = 1$, the ring $K[T]$ is an integral principal ideal ring. Therefore, any Noetherian torsion-free $K[T]$-module (in particular, any projective Noetherian module) is free ([Lang]).

For $n = 2$, there are no non-trivial bundles, either. This theorem is due to Seshadri; this section is devoted to its proof.

For $n \geq 3$, the answer to Serre's question remained unknown for some time [20]. The problem is vary attractive and has all features of a classical one: it is natural, pertains to the fundamental objects and is difficult (at least, for ten years since it was posed there appeared no essential results on modules over polynomial rings apart from Seshadri's theorem and the following fact due to Serre himself.

1.13.1. Theorem. *Let P be a projective Noetherian module over $K[T]$, where $T = (T_1, \dots, T_n)$. Then there exists a free Noetherian module F such that $P \oplus F$ is free.*

In terms of topologists, vector bundles over affine spaces are *stably free.*

Proof easily follows from Hilbert's syzygies theorem which will be given in an appropriate place.

Therefore we confine ourselves to Seshadri's theorem. It involves a class of rings containing, in addition to $K[T_1, T_2]$, e.g., $\mathbb{Z}[T]$.

1.13.2. Theorem. *Let A be an integral principal ideal ring. Then any projective Noetherian $A[T]$-module P is free.*

Proof will be split into a series of lemmas. Its driving force is a simple remark that if A is a field, the statement is true. One can cook a field of A in two ways: pass from A to its field of quotients K or to the quotient field $k = A/(p)$, where p is any prime element. Accordingly, the modules $K[T] \otimes A[T]P$ and $k[T] \otimes A[T]P$ are free. Let us use these circumstances in turn.

1.13.3. Lemma. *There exists an exact sequence of $A[T]$-modules*

$$0 \to F \to P \to P/F \to 0 \tag{1.124}$$

with the following properties:

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[20] The affirmative answer (Any projective Noetherian $K[T]$-module is free) is due to Suslin and D. Quillen[VSu, Su]. L. Vassershtein later gave a simpler and much shorter proof of the theorem which can be found in Lang's book [Lang].

a) *F is a maximal $A[T]$-free submodule of P;*
b) $\operatorname{Ann} P/F \cap A \neq \{0\}$.

Proof. Let $m'_1, \dots, m'_r$ be a free $K[T]$-basis of the module $K[T] \otimes P$. There exists an element $0 \neq f \in A$ such that $m_i = fm' \in P_i \hookrightarrow K[T] \otimes P$. The submodule $F' \subset P$ generated by the elements m_i, where $i = 1, \dots, r$, is free. On the other hand, any element of a finite fixed system of generators of the module P is represented in $K[T] \otimes P$ as a linear combination $\sum_i F_{ij}(t) m_i$, where $F_{ij}(T) \in K[T]$. The common denominator of all coefficients of all polynomials $F_{ij}(T)$ A annihilates P/F'. Now, for the role of F we may take a maximal free submodule in P containing F': it exists thanks to Noetherian property. Clearly, $\operatorname{Ann}(P/F) \supset \operatorname{Ann}(P/F')$, so $\operatorname{Ann}(P/F) \cap A \neq \{0\}$. □

We retain notation of Lemma 1.13.3 and *intend to deduce a contradiction from the assumption that* $F \neq P$. In this case $\operatorname{Ann}(P/F) \cap A = (f) \subset A$, where f is non-invertible (since A is a principal ideal ring). Let p be a prime element of A dividing f. Set $k = A/(p)$ and tensor the exact sequence (1.124) by $k[T]$ over $A[T]$, having set $\overline{F} = F/pF = k[T] \otimes A[T]F$ and so on:

$$\overline{F} \xrightarrow{i} \overline{P} \longrightarrow P/F \longrightarrow 0.$$

Let $\overline{F}_1 = \operatorname{Ker} i$, $\overline{F}_2 = \operatorname{Im} i$. Since $\overline{P}$ is projective over $k[T]$, it follows that $\overline{F}_2$ is torsion-free, and hence is free. Therefore $\overline{F}_1$ is also free and is singled out in $\overline{F}$ as a direct summand, so there is defined a split sequence of free $k[T]$-modules:

$$0 \longrightarrow \overline{F}_1 \xrightarrow{j} \overline{F} \xrightarrow{i} \overline{F}_2 \longrightarrow 0. \tag{1.125}$$

1.13.4. Lemma. $\overline{F}_1 \neq 0$.

Proof. Indeed, $j(\overline{F}_1) = pP \cap F/pF$. Let $f = pg$. Since $g \notin \operatorname{Ann} P/F \cap A$, we have $gP \not\subset F \implies pgP \not\subset pF$ (because p is torsion-free). But $pgP = fP \subset pP \cap F$, so, moreover, $pP \cap F \not\subset pF$. □

The last step requires some additional arguments.

1.13.5. Lemma. *There exists a free $A[T]$-submodule $F_1 \subset F$ with a free direct complement and such that $k[T] \otimes F_1 = j(\overline{F}_1)$.*

Speaking informally the sequence (1.125) can be lifted to a split exact sequence of free $A[T]$-modules.

1.13.6. Deduction of Theorem 1.13.2. Let $F_1 \subset F$ be a submodule whose existence is claimed in Lemma 1.13.5, $F_2 \subset F$ its free direct complement. Since $F_1/pF_1 = \operatorname{Ker} i$, all elements $F_1 \subset F \subset P$ are divisible by p inside P. Set

$$F'_1 = \{m \in P \mid pm \in F_1\}.$$

Clearly, F'_1 is free (the multiplication by p determines an isomorphism $F'_1 \simeq F_1$) and is strictly larger than F_1 (by Lemma 1.13.4). Therefore the module $F' = F_1' \oplus F_2 \subset P$ is free and contains F as a proper submodule contradicting maximality of F and completing the proof of Seshadri's theorem. $\square$

1.13.7. Proof of Lemma 1.13.5. Any automorphism φ of the module F induces an automorphisms φ of the module $\overline{F}$. We need the following auxiliary statement:

1.13.7a. Lemma. *The map* $\varphi \mapsto \overline{\varphi}$: $\mathrm{SL}(n, A[T]) \longrightarrow \mathrm{SL}(n, k[T])$ *is surjective.*

Proof. We use a classical result on reduction of the matrix with elements over the Euclidean ring $k[T]$ to the diagonal form by "admissible transformations". For this result, see the book [vdW], § 85, where it is given in terms of bases. To formulate it, denote by I_n the unit $(n \times n)$-matrix over $k[T]$, let $I_{(ij)}$ (resp. $I^{(ij)}$) be the matrix obtained from I by transposition of the i-th and j-th rows (resp. columns), let E_{ij} be the matrix with a 1 on the (ij)-the slot and 0 elsewhere.

Proof of the *Theorem on elementary divisors* in the book [vdW] shows, in particular, that in a fixed basis of F any automorphism with determinant 1 can be represented as the product of matrices of the following types:

a) $I + fE_{ij}$, $f \in k[T]$;
b) $I_{(ij)}$;
c) $I^{(ij)}$;
d) diagonal matrices with elements of k and with determinant 1.

The matrices of the first three types can obviously be lifted to elements of $\mathrm{SL}(n, A[T])$. The matrices of the fourth type can be factorized in the product of diagonal matrices with determinant 1 and such that only two of their diagonal elements are $\neq 1$. Therefore we have reduced the problem to lifting matrices of the form $\begin{pmatrix} \overline{f} & 0 \\ 0 & \overline{f}^{-1} \end{pmatrix} \in \mathrm{SL}(2, k)$ to matrices of $\mathrm{SL}(2, A)$.

This can be done in a completely elementary way. First, select an element $f \in A$ such that $\overline{f} \equiv f \mod (p)$, next an element $g \in A$ such that $\overline{f}^{-1} \equiv g \mod (p)$ and $(g, f) = 1$ (this is possible thanks to the Chinese remainder theorem). Now we have $fg = 1 + ph$. Solve in A the equation $fx + gy = h$; then

$$(f - py)(g - px) \equiv 1 + p^2xy,$$

so the matrix $\begin{pmatrix} f - py & px \\ py & g - px \end{pmatrix}$ is a solution desired. $\square$

Now, return to the proof of Lemma 1.13.5. Select a free $A[T]$-basis $(m_i)_{i \in I}$ of the module F; its reduction modulo p is a free $k[T]$-basis $(\overline{m}_i)_{i \in I}$ of the module F. Further, select a free $k[T]$-basis $(\overline{n}_i)_{i \in I}$ of the module $\overline{F}$ compatible with the split sequence 1.13.2 (in the sense that the first $\mathrm{rk}\, \overline{F}_1$ of its elements constitute a basis of $\overline{F}_1$). We may assume that the matrix $\overline{M} \in \mathrm{GL}(n, k[T])$ sending the set $(\overline{m}_i)_{i \in I}$ into the set $(\overline{n}_i)_{i \in I}$ belongs to $\mathrm{SL}(n, k[T])$: If not, it suffices to replace $\overline{n}_1$ by $(\det \overline{M})^{-1}\overline{n}_1$. Let us now lift $\overline{M}$ to $M \in \mathrm{SL}(n, A[T])$ and let $(n_i)_{i\in I}$ be an $A[T]$-basis of the M-module F. Further, let F_1 be a submodule of F generated by the first $\operatorname{rk} \overline{F}_1$ elements of the basis $(n_i)_{i\in I}$, and F_2 the submodule generated by the remaining elements. The construction of these submodules shows that they satisfy Lemma 1.13.5 completing the proof. □

1.14. Digression: ζ-function of a ring

1.14.1. An overview. The rings of finite type over a field are called *geometric rings*, those over $\mathbb{Z}$ *arithmetic ones.* These two types of rings have a nonzero intersection: The rings of finite type over finite fields. Such rings (and their spectra) enjoy a blend of arithmetic and geometric properties as demonstrated by A. Weil in his famous conjectures on ζ-functions proved by P. Deligne.

Here we will introduce ζ-functions of arithmetic rings and indicate their simplest properties. A motivation of introducing the ζ-function: The closed points x in the spectrum of an arithmetic ring have a natural "norm" $N(x)$ equal to the number of elements in the finite field $k(x)$, and the number of points of given norm is finite. It is natural to assume that directly counting such points we get an interesting invariant of the ring.

Let A be an arithmetic ring. Let $n(p^a)$ be the number of closed points $x \in \operatorname{Spec} A$ such that $N(x) = p^a$, and $\nu(p^a)$ be the number of geometric $\mathbb{F}_{p^a}$-points of A.

Lemma. *The numbers $\nu(p^a)$ and $n(p^a)$ are finite and related with each other as follows:*

$$\nu(p^a) = \sum_{b|a} bn(p^b). \tag{1.126}$$

Proof. Every geometric $\mathbb{F}_{p^a}$-point of A is by definition a homomorphism $A \longrightarrow \mathbb{F}_{p^a}$. Consider all the geometric points with the same center x; then $\mathfrak{p}_x \subset A$ is the kernel of the corresponding homomorphism and its image coincides with the unique subfield $\mathbb{F}_{p^b} \hookrightarrow \mathbb{F}_{p^a}$, where $p^b = N(x)$. There are exactly b homomorphisms with fixed kernel and image, since $\mathbb{F}_{p^b}/\mathbb{F}_{p^a}$ is a Galois extension of degree b. Therefore

$$\nu(p^a) = \sum_{b|a} b\Bigg(\sum_{\{x|N(x)=p^b\}} 1\Bigg) = \sum_{b|a} bn(p^b). \tag{1.127}$$

(This equality is obviously well-defined even if it is not known that $\nu(p^a)$ and $n(p^b)$ are finite).

In particular, $n(p^a) \leq \nu(p^a)$ and it suffices to prove that $\nu(p^a)$ is finite. We identify $\operatorname{Spec} A$ with a closed subset in $\operatorname{Spec} \mathbb{Z}[T_1, \dots, T_n]$; then $N(x)$ does not depend on whether we consider x as belonging to $\operatorname{Spec} A$ or to $\operatorname{Spec} \mathbb{Z}[T_1, \dots, T_n]$. Therefore, in obvious notation,

$$\nu(p^a) \leq \nu_{\mathbb{Z}[T_1,\dots,T_n]}(p^a) = p^{na} \tag{1.128}$$

where, clearly, p^{na} is just the number of geometric points of the n-dimensional affine space over a field of p^a elements. □

1.14.2. ζ-functions of arithmetic rings. Define the ζ-function of any arithmetic ring A first formally, by setting

$$\zeta_A(s) = \prod_{x \in \operatorname{Spm} A} \frac{1}{1 - N(x)^{-s}}. \tag{1.129}$$

Clearly, for $A = \mathbb{Z}$, we get the usual Euler function

$$\zeta_{\mathbb{Z}}(s) = \prod_p \frac{1}{1 - p^{-s}} = \zeta(s). \tag{1.130}$$

The relation of the ζ-function with $n(p^a)$ and $\nu(p^a)$ is given by the following obvious identity

$$\zeta_A(s) = \prod_p \prod_{1 \leq a < \infty} \frac{1}{(1 - p^{-as})^{n(p^a)}} = \prod_p \zeta_{A/pA}(s) \tag{1.131}$$

and another, a trifle less obvious one,

$$\ln \zeta_A(s) = \sum_p \sum_a \nu(p^a) \frac{1}{ap^{as}} \ . \tag{1.132}$$

Proof of eq.(1.132) (use the above lemma):

$$\begin{aligned} \ln \zeta_A(s) &= -\sum_p \sum_a \ln(1 - p^{-bs}) n(p^b) = -\sum_p \sum_{b \in \mathbb{N}} \sum_{k \in \mathbb{N}} n(p^b) \frac{1}{kp^{bks}} \\ &= \sum_p \sum_{a \in \mathbb{N}} \sum_{b|a} \frac{b}{ap^{as}} n(p^b) = \sum_p \sum_a \nu(p^a) \frac{1}{a^s p^{as}}. \quad \square \end{aligned} \tag{1.133}$$

Therefore the calculation of the ζ-function is equivalent to that of $n(p^a)$ or $\nu(p^a)$ for all p, a. □

Eq. (1.131) shows that $\zeta_A(s)$ factorizes into the product of ζ-functions of the rings of finite type over finite fields. This, however, does not mean in the least that the study of ζ-functions reduces to the cases of such rings and the example of the Riemann ζ-function shows how nontrivial the behavior of the global ζ-function can be even for simplest local factors.

Even to separate p-factors can be a sufficiently complicated task if A is nontrivial.

A part of Weil's conjectures proved by Dwork [21)] shows, however, that $\zeta(s)$ is a rational function in p^{-s} for any ring A of finite type over a field

[21] See [Kz]. By early 1970s Weil's conjectures (and even more difficult statements) were proved by Grothendieck and Deligne, see [SGA4, SGA4.5, Dan].

of characteristic p. For such rings, it is convenient to change the variable by setting $p^{-s} = t$ and set $\zeta_A(s) = Z_A(t)$. Then eq.(1.132) shows that

$$\ln Z_A(t) = \sum_a \frac{\nu(p^a)t^a}{a} \tag{1.134}$$

or

$$\frac{Z'_A(t)}{Z_A(t)} = \sum_a \nu(p^a)t^{a-1} \tag{1.135}$$

In particular, the rationally of $Z_A(t)$ establishes that the sequence $\nu_a = \nu(p^a)$ should satisfy a recurrent relation of type

$$\nu_{a+n} = \sum_{0\leq i\leq n-1} r_i \nu_{a+i} \tag{1.136}$$

for sufficiently large a with some fixed constants n, r_i. Since the ν_a are the numbers of solutions of a system of equations with values in finite fields of growing degree, the statement on rationality bears a direct arithmetic meaning.

1.14.3. Frobenius morphism. In any study of the ζ-function of a ring A over a field k of characteristic p the following circumstance is of fundamental importance: $\nu(p^k)$ can be viewed as the number of fixed points of a power of a certain map F acting on the set of geometric points of A.

A *Frobenius morphism* $F : A \longrightarrow A$ is the map $g \mapsto g^p$ for any $g \in A$, where $p = \operatorname{Char} k$.

The same term — *Frobenius morphism* — is applied to the corresponding morphism of spectra, ${}^aF$, to its powers, F^n, to $({}^aF)^n$, and to the maps of some other objects induced by these maps. In particular, let $\bar{\mathbb{F}}_p$ be the algebraic closure of the Galois field of characteristic p. Then F induces a map ${}^aF : A(\bar{\mathbb{F}}_p) \longrightarrow A(\bar{\mathbb{F}}_p)$ of $\bar{\mathbb{F}}_p$-points of A into itself.

Proposition. *$A(\mathbb{F}_{p^b})$ coincides with the set of fixed points of F^b.*

Proof. Let $\varphi \in A(\bar{\mathbb{F}}_p)$ and let φ be represented by $\varphi\colon A \longrightarrow \bar{\mathbb{F}}_p$; let $F^b(\varphi)$ be represented by $f \mapsto \varphi(f)^{p^b}$ for any $f \in A$.

The condition $\varphi \in A(\mathbb{F}_{p^b})$ means that $\operatorname{Im}\varphi \subset \mathbb{F}_{p^b} \subset \bar{\mathbb{F}}_{p^b}$, i.e., $\varphi(f)^{p^b} = \varphi(f)$ for all f. Therefore ${}^aF\varphi = \varphi$.

The converse statement follows from the Galois theory: $\mathbb{F}_{p^b}$ is the field of invariants for F^b. □

1.14.4. Lefschetz formula. If an endomorphism F acts on a compact topological space V, then the number $\nu(F)$ of its fixed points (appropriately defined) satisfies the following famous *Lefschetz formula*:

$$\nu(F) = \sum_{0\leq i\leq \dim V} (-1)^i \operatorname{tr} F|_{H^i(V)}, \tag{1.137}$$

where the summands are the traces of linear operators induced by F on the spaces of cohomology of V with complex coefficients.

The role of compact topological spaces V is played in our setting by smooth projective schemes. For them, the essential part of Weil's conjectures states that the numbers $\nu(p^a)$ are always expressed by Lefschetz type formulas. So far, we have dealt with Euler's products and Dirichlet series purely formally. Now, let us study a little their convergence.

Let A be an arithmetic ring, $\{x_i\}_{i\in I}$ the set of generic points of its irreducible components. Define the *dimension* of A setting

$$\dim A = \begin{cases} \max_i(\text{tr. deg. } k(x_i)) + 1 & \text{if } \mathbb{Z} \subset A \\ \max_i(\text{tr. deg. } k(x_i)) & \text{if } \operatorname{Char} A > 0 \end{cases} \tag{1.138}$$

(The transcendence degree is calculated over a prime subring of $k(x_i)$. The dimension thus introduced was considered already by Kronecker.)

Theorem. *The Euler product* $\prod\limits_{x\in \operatorname{Spm}\ A} \dfrac{1}{1-N(x)^{-s}}$ *converges absolutely for* $\operatorname{Re}\ s > \dim A$.

Proof. We will verify the theorem consecutively extending the class of rings considered. We assume that for the Riemannian ζ-function $\zeta_{\mathbb{Z}}(s)$ the convergence of the product is known.

a) Let $A = \mathbb{F}_p[T_1, \dots, T_n]$. It is subject to a direct verification that eq.(1.132) converges absolutely to $\ln(1-p^{n-s})^{-1}$ for $\operatorname{Re}\ s > n = \dim A$ since $\nu(p^a) = p^{an}$. This implies that under the same conditions the Euler's product for A absolutely converges to $\dfrac{1}{1-p^{n-s}}$.

b) Let A be a ring without zero divisors and of finite type over $\mathbb{F}_p$. Let us apply Noether's normalization theorem and find a polynomial subring $B = \mathbb{F}_p[T_1, \dots, T_n] \subset A$ such that A is a B-module with a finitely many generators. There exists a constant d such that, over every geometric $\bar{\mathbb{F}}_p$-point of B, not more than d geometric points of A are situated.

Indeed, let a homomorphism $A \longrightarrow \bar{\mathbb{F}}_p$ be given on B. To extend it over A, we have to define in $\bar{\mathbb{F}}_p$ the images of a finite number of generators of B over A each of which is a root of an integer equation with coefficients from A. The images of these coefficients are already defined, and therefore the roots of equations are determined in finitely many ways.

This implies that $\nu_A(p^a) \leq \alpha\nu_B(p^a) = \alpha p^{na}$, and therefore $\zeta_A(s)$ converges absolutely for $\operatorname{Re}\ s > n = \dim A$, as above. Moreover, in this domain, we have

$$|\ln \zeta_A(s)| \leq \alpha \ln(1-p^{n-\sigma})^{-1}, \text{ where } \sigma = \operatorname{Re}\ s. \tag{1.139}$$

c) Let A be an arbitrary ring of finite type over $\mathbb{F}_p$. Let the $p_i \subset A$ be all the minimal prime ideals of A, and $A_i = A/p_i$. Every geometric point of $\operatorname{Spec} A$ belongs to an irreducible component, and therefore

$$\nu_A(p^a) \le \sum_i \nu_{A_i}(p^a),$$

so the Euler product for A converges if $\mathrm{Re}\ s > \max_i \dim A_i$ and in this domain satisfies

$$|\ln \zeta_A(s)| \le \sum_i \alpha \ln(1 - p^{n_i - \sigma})^{-1}, \text{ where } n_i = \dim A_i. \tag{1.140}$$

d) $A = \mathbb{Z}[T_1, \dots, T_n]$. It follows from the calculations from a) that in this case

$$\zeta(s) = \prod_p \frac{1}{1 - p^{n-s}} = \zeta(s - n) \tag{1.141}$$

is the usual ζ-function with shifted argument whose Euler product, as is well known, converges absolutely for $\mathrm{Re}\ (s - n) > 1$, i.e., $\mathrm{Re}\ s > n + 1 = \dim A$.

e) Let A be a ring without zero divisors containing $\mathbb{Z}$. If we can find a subring $\mathbb{Z}[T_1, \dots, T_n]$ of A, over which A is integer, the arguments from b) bring about the result. Regrettably, this is not always possible; we can, however, remedy the situation for the price of localization modulo a finite number of primes.

More exactly, let us apply Noether's normalization theorem to $A' = \mathbb{Q} \otimes_{\mathbb{Z}} A$ and find a subring $\mathbb{Q}[T_1, \dots, T_n]$ in A' over which A is integer. Multiplying, if necessary, T_i by integers, we can assume that $T_i \in A$. Any element of A over $\mathbb{Z}[T_1, \dots, T_n]$ satisfies an equation whose highest coefficient is an integer. Consider the set of prime divisors of all such highest coefficients for a finite system of generators of A over $\mathbb{Z}[T_1, \dots, T_n]$ and denote by S the multiplicative system generated by this set. Then A_S is integer over $\mathbb{Z}_S[T_1, \dots, T_n]$, and

$$\zeta_A(s) = \prod_{p \in S} \zeta_{A/(p)}(s) \prod_{p \notin S} \zeta_{A/(p)}(s). \tag{1.142}$$

The set $\{p \mid p \in S\}$ is finite and from the above $\zeta_{A/(p)}(s)$ converges uniformly for $\mathrm{Re}\ s > \dim A/(p) \ge \dim A - 1$.

For $p \notin S$, we have $\zeta_{A/(p)}(s) = \zeta_{A_S/(p)}(s)$, and the constant α for the pair of rings $\mathbb{Z}_S[T_1, \dots, T_n]/(p) \subset A_S/(p)$ determined in b) can be chosen to be independent of p. Indeed, the class modulo p of the fixed system of integer generators A_S over $\mathbb{Z}_S[T_1, \dots, T_n]$ gives a system of generators of $A_S/(p)$ for all $p \notin S$. Therefore the second (infinite) product in (5) for $\sigma = \mathrm{Re}\ s > \dim A/(p)$ is majored by

$$\prod_{p \notin S} (1 - p^{n - \sigma})^{-\alpha},$$

and therefore converges uniformly for $\sigma > n + 1 = \dim A$.

f) Finally, we trivially reduce the general case to the above-considered ones by decomposing $\operatorname{Spec} A$ into irreducible components as in c). $\square$

1.14.5. Exercises. 1) Express $n(p^a)$ in terms of $\nu(p^b)$.

2) Calculate the number of irreducible polynomials in one indeterminate of degree d over the field of q elements.

3) Calculate $\zeta_A(s)$, where $A = \mathbb{Z}[T_1, \dots, T_n]/(F)$, and F is a quadratic field.

4) Let A be a ring of finite type over $\mathbb{Z}$, and P the set of prime numbers; let

$$S = \{p \in P \mid \text{there exists } x \in \operatorname{Spec} A, \text{ such that } \operatorname{Char} k(x) = p\}.$$

Prove that either S or $P \setminus S$ is finite. For the integer domain not of finite type over $\mathbb{Z}$, give an example when both S and $P \setminus S$ are infinite.

1.15. The affine group schemes

In this section we give definitions and several most important examples of affine group schemes. This notion is not only important by itself, it also lucidly shows the role and possibilities of the “categorical” and “structural” approaches.

We will give two definitions of a group structure on an object of a category and compare them for the category of schemes.

1.15.1. A group structure on an object of a category. Let C be a category, $X \in \operatorname{Ob} \mathsf{C}$. A *group structure on* X is said to be given if there are given (set theoretical) group structures on all the sets $P_X(Y) := \operatorname{Hom}_{\mathsf{C}}(Y, X)$, and, for any morphism $Y_1 \longrightarrow Y_2$, the corresponding map of sets $P_X(Y_2) \longrightarrow P_X(Y_1)$ is a group homomorphism.

An object X together with a group structure on it is said to be a *group in the category* C. Let X_1, X_2 be groups in C; a morphism $X_1 \longrightarrow X_2$ in C is a *group morphism in* C if the maps $P_{X_1}(Y) \longrightarrow P_{X_2}(Y)$ are group homomorphisms for any Y.

A group in the category of affine schemes will be called an *affine group scheme*[22)]

Here is the list of the most important examples with their standard notation and names.

Helpful remark. Since $\mathsf{Aff\ Sch}^\circ = \mathsf{Rings}$, instead of studying contravariant functors on $\mathsf{Aff\ Sch}$ represented by an affine group scheme we may discuss the covariant functors on Rings which are simpler to handle.

1.15.2. Examples.

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[22] Never an *affine group*: This is a fixed term for a different notion.

1.15.2a. The additive group $\mathbb{G}_a = \operatorname{Spec}\mathbb{Z}[T]$. As above, a morphism $\operatorname{Spec} A \longrightarrow \mathbb{G}_a$ is uniquely determined by an element $t \in A$, the image of T, which may be chosen at random. The collection of groups with respect to addition $A = \mathbb{G}_a(A)$ for the rings $A \in \mathsf{Rings}$ determines the group structure on $\mathbb{G}_a$.

In other words, $\mathbb{G}_a$ represents the functor $\mathsf{Aff\ Sch}^\circ \longrightarrow \mathsf{Gr}$, $\operatorname{Spec} A \mapsto A$ or, equivalently, the functor $\mathsf{Rings} \longrightarrow \mathsf{Gr}$, $A \mapsto A^+$.

1.15.2b. The multiplicative group $\mathbb{G}_m = \operatorname{Spec}\mathbb{Z}[T, T^{-1}]$. For any superscheme $X = \operatorname{Spec} A$, a morphism $X \longrightarrow \mathbb{G}_m$ is uniquely determined by an element $t \in A^\times$, the image of T under the homomorphism $\mathbb{Z}[T, T^{-1}] \longrightarrow A$. Conversely, t corresponds to such a morphism if and only if $t \in A^\times$, where $A_{\bar{0}}^\times$ is the group (with respect to multiplication) of invertible elements of A. Therefore

$$P_{\mathbb{G}_m}(\operatorname{Spec} A) = \mathbb{G}_m(A) = A^\times, \tag{1.143}$$

and on the set of A-points the natural group structure (multiplication) is defined. Furthermore, any ring homomorphism $A \longrightarrow B$ induces, clearly, a group homomorphism $A^\times \longrightarrow B_{\bar{0}}^\times$ which determines the group structure on $\mathbb{G}_m$.

In other words, $\mathbb{G}_m$ represents the functor $\mathsf{Aff\ Sch}^\circ \longrightarrow \mathsf{Gr}$, $\operatorname{Spec} A \mapsto A^\times$ or, equivalently, the functor $\mathsf{Rings} \longrightarrow \mathsf{Gr}$, $A \mapsto A^\times$.

1.15.2c. The general linear group.

$$\mathrm{GL}(n) = \operatorname{Spec}\mathbb{Z}[T_{ij}, T]_{i,j=1}^n/(T\det(T_{ij}) - 1). \tag{1.144}$$

It represents the functor $\operatorname{Spec} A \mapsto \mathrm{GL}(n; A)$. Obviously, $\mathrm{GL}(1) \simeq \mathbb{G}_m$.

1.15.2d. The Galois group $\mathrm{Aut}(K'/K)$. Fix a K-algebra K' and let K' be a free K-module of finite rank. The group $\mathrm{Aut}(K'/K)$ of automorphisms of the algebra K' over K is the main object of the study, e.g., in the Galois theory (where the case of fields K, K' is only considered). This group may turn to be trivial if the extension is non-normal or non-separable, and so on.

The functorial point of view suggests to consider all the possible *changes of base* K, i.e., for a variable K-algebra B, consider the group of automorphisms

$$\mathrm{Aut}(B'/B) := \mathrm{Aut}_B(B'), \text{ where } B' = B \bigotimes_K K'. \tag{1.145}$$

We will prove simultaneously that (1) the map $B \mapsto \mathrm{Aut}(B'/B)$ is a functor and (2) this functor is representable.

Select a free basis $e_1, \dots, e_n$ of K' over K. In this basis the multiplication law in K' is given by the formula

$$e_i e_j = \sum_{1 \le k \le n} c_{ij}^k e_k. \tag{1.146}$$

Denote $e_i' := 1 \bigotimes_K e_i$; then $B' = \bigoplus_{1 \le i \le n} B\, e_i'$, and any endomorphism t of the B-module B' is given by a matrix (t_{ij}), where $t_{ij} \in B$ and $1 \le i, j \le n$. The condition that this matrix determines an endomorphism of an algebra can be expressed as the relations

$$t(e_i')t(e_j') = \sum_{1\le k\le n} c_{ij}^k t(e_k'). \tag{1.147}$$

Equating the coefficients of e_k' in (1.147) in terms of indeterminates T_{ij} we obtain a system of algebraic relations for T_{ij} with coefficients from K, both necessary and sufficient for (t_{ij}) to define an endomorphism of B'/B.

To obtain *automorphisms*, let us introduce an additional variable t and the additional relation (cf. Example 3) which ensures that $\det(t_{ij})$ does not vanish:

$$t\det(t_{ij}) - 1 = 0. \tag{1.148}$$

The quotient of $K[T, T_{ij}]_{i,j=1}^n/(T\det(T_{ij})_{i,j=1}^n - 1)$ is a K-algebra representing the functor

$$B \mapsto \mathrm{Aut}(B'/B). \tag{1.149}$$

This K-algebra replaces the notion of the *Galois group* of the extension K'/K; it generalizes the notion of the group ring of the Galois group.

1.15.2e. Consider the simplest particular case:

$$K' = K(\sqrt{a}), \text{ where } a \in K^\times \setminus (K^\times)^2. \tag{1.150}$$

We may set $e_1 = 1$, $e_2 = \sqrt{a}$; the multiplication table reduces to $e_2^2 = a$.

Let $t(\sqrt{a}) = T_1 + T_2\sqrt{a}$ (obviously, $t(1) = 1$). Since $t(\sqrt{a})^2 = a$, we obtain the equations relating T_1, T_2 and the additional variable T:

$$\begin{cases} T_1^2 + a\,T_2^2 &= a \\ 2T_1\,T_2 &= 0 \\ TT_2 - 1 &= 0. \end{cases} \tag{1.151}$$

Now, let us consider separately two cases.

Case 1: $\mathrm{Char}\,K \neq 2$. Hence, 2 is invertible in any K-algebra. The functor of automorphisms is represented by the K-algebra

$$K[T, T_1, T_2]/(T_1^2 + a\,T_2^2 - a,\ T_1T_2,\ TT_2 - 1). \tag{1.152}$$

If B has no zero divisors, then the B-points of this K-algebra have a simple structure: Since T_2 must not vanish, T_1 vanishes implying that the possible values of T_2 in the quotient ring are ± 1. As the conventional Galois group this group is isomorphic to $\mathbb{Z}/2$; the automorphisms simply change the sign of $\sqrt{a}$.

The following case illustrates that when B does have zero divisors the group of B-points of $\mathrm{Aut}\,K^\times/K$ can be much larger.

Case 2: $\mathrm{Char}\,K = 2$. The functor of automorphisms is represented by the K-algebra

$$K[T,T_1,T_2]/(T_1^2+aT_2^2-a, TT_2-1). \tag{1.153}$$

In other words, the B-points of the automorphism group are all the B-points of the circle $T_1^2+aT_2^2-a=0$ at which T_2 is invertible!

Let us investigate this in detail. Let B be a field and let (t_1,t_2) be a B-point of the circle at which T_2 is invertible. Then either $t_2=1, t_1=0$, and we obtain the identity automorphism, or $a=\left(\frac{t_1}{t_2+1}\right)^2$. Therefore there are nontrivial B-points only if $\sqrt{a}\ \in B$, in which case the equation of the circle turns into the square of a linear one $(T_1+\sqrt{a}\ \ T_2+\sqrt{a})^2=0$. We have the punctured line (the line without point $T_2=0$) of automorphisms!

Obviously, $\operatorname{Aut}(B'/B)$ is isomorphic in this case to $B^\times$ — the multiplicative group of B (under the composition of automorphisms the coefficients of $\sqrt{a}$ are multiplied). So, the non-separable extensions have even more, in a certain sense, automorphisms than separable ones.

The reason why this phenomenon takes place is presence of nilpotents in the algebra $B\bigotimes\limits_K K'$. Indeed, $K(\sqrt{a})\subset L$, so $K(\sqrt{a})\otimes_K K(\sqrt{a})\subset L'$; on the other hand, this product is isomorphic to

$$K(\sqrt{a})[x]/(x^2-a)\simeq K(\sqrt{a})[y]/(y^2):$$

The automorphisms just multiply y by invertible elements.

One can similarly investigate arbitrary inseparable extensions and construct for them an analog of the Galois theory.

1.15.2f. The group $\boldsymbol{\mu_n}$ of $\boldsymbol{n}$th roots of unity. Set

$$\mu_n=\operatorname{Spec}\mathbb{Z}[T]/(T^n-1)=\operatorname{Spec}\mathbb{Z}[T,T^{-1}]/(T^n-1). \tag{1.154}$$

This group represents the functor $\operatorname{Spec}A\mapsto\{t\in A^\times\mid t^n=1\}$.

Let X be a closed affine group scheme and Y its closed subscheme Y such that $P_Y(Z)\subset P_X(Z)$ is a subgroup for any Z. We call Y with the induced group structure a *closed subgroup of* X.

Therefore μ_n is a closed subgroup of $\mathbb{G}_m$. Explicitly, the homomorphism $T\mapsto T^n$ determines a group scheme homomorphism $\mathbb{G}_m\longrightarrow\mathbb{G}_m$ of "raising to the power n" and μ_n represents the kernel of this homomorphism.

1.15.2g. The scheme of a finite group $\boldsymbol{G}$. Let G be a conventional (set-theoretical) finite group. Set $A=\mathbb{Z}^{(G)}:=\prod\limits_{g\in G}\mathbb{Z}$. In other words, A is a free module $\bigoplus\limits_{g\in G}\mathbb{Z}^{(g)}$ ($|G|$ copies of $\mathbb{Z}$) with the multiplication table

$$e_ge_h=\begin{cases}0=(0,\dots,0) & \text{if } h\neq g,\\ e_g & \text{if } h=g.\end{cases} \tag{1.155}$$

The space $X=\operatorname{Spec}A$ is disjoint; each of its components is isomorphic to $\operatorname{Spec}\mathbb{Z}$ and these components are indexed by the elements of G. For any ring B, whose spectrum is connected, the set of morphisms $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ is, therefore, in the natural one-to-one correspondence with the elements of G.

If $\operatorname{Spec} B$ is disjoint, then a morphism $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ is determined by the set of its restrictions onto the connected components of $\operatorname{Spec} B$. Let $\operatorname{Conn} B$ be the set of these components; then, clearly, the point functor is given by

$$P_X(\operatorname{Spec} B) \xrightarrow{\sim} (G)^{\operatorname{Conn} B} := \operatorname{Hom}(G, \operatorname{Conn} B). \tag{1.156}$$

and therefore X is endowed with a natural group structure called the *scheme of the group* G.

1.15.2h. The relative case. Let $S = \operatorname{Spec} K$. A group object in the category $\mathsf{Aff\ Sch}_S$ of affine schemes over S is said to be an *affine* S*-group* (or an *affine* K*-group*). Setting $\mathbb{G}_{m\ /K} = \mathbb{G}_m \times S$ and $\mu_{n\ /K} = \mu_n \times S$, and so on, we obtain a series of groups over an arbitrary scheme S (or a ring K). Each of them represents "the same" functor as the corresponding absolute group, but restricted onto the category of K-algebras.

1.15.2i. Linear algebraic groups. Let K be a field. Any closed subgroup of $\mathbb{GL}_n(A)_{/K}$ is said to be a *linear algebraic group over* K.

In other words, a linear algebraic group is determined by a system of equations

$$F_k(T_{ij}) = 0, \quad \text{ for } i, j = 1, \dots, n \text{ and } k \in I \tag{1.157}$$

such that if (t'_{ij}) and (t''_{ij}) are two solutions of the system (1.157) in a K-algebra A such that the corresponding matrices are invertible, then the matrix $(t'_{ij})(t''_{ij})^{-1}$ is also a solution of (1.157)

The place of linear algebraic groups in the general theory is elucidated by the following fundamental theorem (cf. [OV]).

Theorem. *Let X be an affine group scheme of finite type over K. Then X is isomorphic to a linear algebraic group.*

1.15.3. Statement (Cartier). *Let X be the scheme of a linear algebraic group over a field of characteristic zero. Then X is reduced, i.e., $X = X_{red}$, its ring has no nilpotents.*

For proof of this theorem, see, e.g., [M1]. □

The statement of the theorem is false if [23] $\operatorname{Char} K = p$ as demonstrated by the following

Example. Set

$$\mu_{p\ /K} = \operatorname{Spec} K[T]/(T^{p-1}) = \operatorname{Spec} K[T]/((T-1)^p). \tag{1.158}$$

Obviously, $K[T]/((T-1)^p)$ is a local Artinian algebra of length p, and its spectrum should be considered as a "point of multiplicity p". This is a nice

[23] And also for the group superschemes over any fields.

agreement with our intuition: All the roots of unity of degree p are glued together and turn into one root of multiplicity p.

More generally, set

$$\mu_{p^n\ /K} = \operatorname{Spec} K[T]/((T-1)^{p^n}). \tag{1.159}$$

We see that the length of the nilradical may be however great.

1.15.4. The set-theoretical definition of the group structure. Let a category C contain a final[24)] object E and products. Let X be a group with a unit 1; let $x, y, z \in X$; then, in the standard notation[25)], we obtain

$$m(x,y) = xy, \quad i(x) = x^{-1}, \quad u(E) = 1, \tag{1.160}$$

and the conventional axioms of the associativity, the left inverse and the left unit have, respectively, the form

$$(xy)z = x(yz), \ x^{-1}x = 1, \ 1x = x \tag{1.161}$$

The usual set-theoretical definition of the group structure on a set X given above is, clearly, equivalent to the existence of three morphisms

$$\begin{array}{rlll} m: & X \times X & \longrightarrow X & (\text{multiplication, } x, y \mapsto xy) \\ i: & X & \longrightarrow X & (\text{inversion, } x \mapsto x^{-1}\ ) \\ u: & E & \longrightarrow X & (\text{unit, the embedding of } E) \end{array} \tag{1.162}$$

that satisfy the axioms of associativity, left inversion and left unit, respectively, expressed as commutativity of the following diagrams:

$$\begin{tikzcd} X \times X \times X \arrow[r, "{(m,\, \mathrm{id}_X)}"] \arrow[d, "{(\mathrm{id}_X, m)}"'] & X \times X \arrow[d, "m"] \\ X \times X \arrow[r, "m"] & X \end{tikzcd} \tag{1.163}$$

$$\begin{tikzcd} X \times X \arrow[r, "{(i,\, \mathrm{id}_X)}"] & X \times X \arrow[d, "m"] \\ X \arrow[u, "\delta"] \arrow[r, "\bullet"] & E \arrow[r, "u"] & X \end{tikzcd} \tag{1.164}$$

$$\begin{tikzcd} X \times X \arrow[r, "{(\bullet,\, \mathrm{id}_X)}"] & E \times X \arrow[r, "{(u,\, \mathrm{id}_X)}"] & X \times X \arrow[d, "m"] \\ X \arrow[u, "\delta"] \arrow[r, "\mathrm{id}_X"] & X \arrow[r, "\mathrm{id}_X"] & X \end{tikzcd} \tag{1.165}$$

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[24] An object E is said to be *final* if $\operatorname{card}(\operatorname{Hom}(X,E)) = 1$ for any $X \in \operatorname{Ob} C$.

[25] Here we denote the group unit by 1; it is the image of E.

(In diagrams (1.164), (1.165) the morphism of contraction to a point (E) is denoted by "$\bullet$".)

In the category Sets the axioms (1.163)–(1.165) turn into the usual definition of a group though in an somewhat non-conventional form.

Exercise. Formulate the commutativity axiom for the group.

1.15.5. Equivalence of the two definitions of the group structure. Let a group structure in the set-theoretical sense be given on $X \in \operatorname{Ob} \mathsf{C}$. Then, for every $Y \in \operatorname{Ob} \mathsf{C}$, the morphisms m, i, u induce the group structure on the set $P_X(Y)$ of Y-points thanks to the above subsection.

1.15.5a. Exercise. Verify compatibility of these structures with the maps $P_X(Y_1) \longrightarrow P_X(Y_2)$ induced by the maps $Y_1 \longrightarrow Y_2$.

Conversely, let a group structure in the sense of the first definition be given on $X \in \operatorname{Ob} \mathsf{C}$. How to recover the morphisms m, i, u? We do it in three steps:

a) The group $P_X(X \times X)$ contains projections $\pi_1, \pi_2 \colon X \times X \longrightarrow X$. Set $m = \pi_1 \circ \pi_2$ (the product $\circ$ in the sense of the group law).

b) The group $P_X(X)$ contains the element id_X. Denote its inverse (in the sense of the group law) by i.

c) The group $P_X(E)$ has the unit element. Denote it by $u \colon E \longrightarrow X$.

Exercise. 1) Prove that m, i, u satisfy the axioms of the second definition.

2) Verify that the constructed maps of sets

$$\left\{\begin{array}{r}\text{group structures on } X\\ \text{with respect to}\\ \text{the first definition}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{group structures on } X\\ \text{with respect to}\\ \text{the second definition}\end{array}\right\} \tag{1.166}$$

are mutually inverse.

1.15.6. How to describe the group structure on an affine group scheme $X = \operatorname{Spec} A$ in terms of A. We will consider the general, i.e., relative, case, i.e., assume A to be a K-algebra.

The notion of a group G is usually formulated in terms of the states, i.e., points of G. In several questions, however, for example, to quantize it, we need a reformulation in terms of observables, i.e., the functions on G. Since any map of sets $\varphi \colon X \longrightarrow Y$ induces the homomorphism of the algebras of functions $\varphi^* \colon F(Y) \longrightarrow F(X)$, we dualize the axioms of sec. 1.15.4 and obtain the following definition.

A *bialgebra* structure on a K-algebra A is given by three K-algebra homomorphisms:

$$\begin{array}{lll} m^* : A \longrightarrow A \bigotimes\limits_K A & \text{co-multiplication} \\ i^* : \ A \longrightarrow A & \text{co-inversion} \\ u^* : \ A \longrightarrow K & \text{co-unit} \end{array} \tag{1.167}$$

which satisfy the axioms of co-associativity, left coinversion and left counit, respectively, expressed in commutativity of the following diagrams:

$$\begin{tikzcd} A\otimes A\otimes A & & A\otimes A \arrow[ll, "m^*\otimes \mathrm{id}_A"'] \\ A\otimes A \arrow[u, "\mathrm{id}_A \otimes m^*"] & & A \arrow[ll, "m^*"] \arrow[u, "m^*"'] \end{tikzcd} \tag{1.168}$$

$$\begin{tikzcd} A\otimes A \arrow[d, "\mu"'] & & A\otimes A \arrow[ll, "i^*\otimes \mathrm{id}_A"'] \\ A & K \arrow[l] & A \arrow[l, "u^*"'] \arrow[u, "m^*"'] \end{tikzcd} \tag{1.169}$$

(the left vertical arrow is the multiplication $\mu : a\otimes b \mapsto a\ b$ in A, the left horizontal arrow is given by $1 \mapsto 1$).

$$\begin{tikzcd} A\otimes A \arrow[d, "\mu"'] & A \arrow[l] & A\otimes A \arrow[l, "u^*\otimes \mathrm{id}_A"'] \\ A & & A \arrow[ll, "\mathrm{id}_A"'] \arrow[u, "m^*"'] \end{tikzcd} \tag{1.170}$$

(the left arrow in the top line is $a \mapsto 1\otimes a$).

It goes without saying that this definition is dual to that from sec. 1.15.4, and therefore the group structures on the K-scheme $\operatorname{Spec} A$ are in one-to-one correspondence with the co-algebra structures on the K-algebra A.

Example. The homomorphisms m^*, i^*, u^* for the additive group scheme $\mathbb{G}_a = \operatorname{Spec} \mathbb{Z}[T]$ are:

$$m^*(T) = T\otimes 1 + 1\otimes T, \quad i^*(T) = -T, \quad u^*(T) = 0. \tag{1.171}$$

Exercise. Write explicitly the homomorphisms m^*, i^*, u^* for the Examples 1.15.2.

1.16. Appendix: The language of categories. Representing functors

1.16.1. General remark. The language of categories is an embodiment of a "sociological" approach to a mathematical object: A group, a manifold or a space are considered not as a set with an intrinsic structure but as a member of a community of similar objects.

The 'structural' and 'categorical' descriptions of an object via the functor it represents are complementary. The role of the second description increases nowadays in various branches of mathematics (especially in algebraic geometry) [26] although its richness of content was first demonstrated in topology, by $K[\Pi, n]$ spaces.

The proposed gist of definitions and examples purports to be an abridged phraseological dictionary of the language of categories (ordered logic-wise, however, not alphabetically). [27]

1.16.2. Definition of categories. A *category* C is a collection of the following data:

a) A set $\operatorname{Ob}\mathsf{C}$ whose elements are called *objects of* C. (Instead of $X \in \operatorname{Ob}\mathsf{C}$ we often write briefly $X \in \mathsf{C}$.)

b) For every ordered pair $X, Y \in \mathsf{C}$, there is given a (perhaps empty) collection (a set or a class) $\operatorname{Hom}_{\mathsf{C}}(X, Y)$ (or shortly $\operatorname{Hom}(X, Y)$) whose elements are called *morphisms* or *arrows* from X into Y. Notation: $\varphi \in \operatorname{Hom}(X, Y)$ is expressed as $\varphi\colon X \mapsto Y$ and $\operatorname{Mor}\mathsf{C} = \coprod\limits_{X,Y\in\mathsf{C}} \operatorname{Hom}(X, Y)$ is a collection of morphisms.

c) For every ordered triple $X, Y, Z \in \mathsf{C}$, there is given a map

$$\operatorname{Hom}(X, Y) \times \operatorname{Hom}(Y, Z) \to \operatorname{Hom}(X, Z) \tag{1.172}$$

assigning to morphisms $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ a morphism $\psi\varphi$ called their *composition*.

The data a)–c) should satisfy the following two axioms:

Associativity: $(\chi\psi)\varphi = \chi(\psi\varphi)$ for any $\varphi\colon X \to Y$, $\psi\colon Y \to Z$, $\chi\colon Z \to V$.

Identity morphisms: for every $X \in \mathsf{C}$, there exists a morphism $\mathrm{id}_X\colon X \to X$ such that $\mathrm{id}_X \circ \varphi = \varphi$, $\psi \circ \mathrm{id}_X = \psi$ whenever the compositions are defined. (Clearly, id_X is uniquely defined.)

A morphism $\varphi\colon X \to Y$ is called an *isomorphism* if there exists a morphism $\psi\colon Y \to X$ such that $\psi\varphi = \mathrm{id}_X$, $\varphi\psi = \mathrm{id}_Y$.

Given two categories C and D such that $\operatorname{Ob}\mathsf{C} \subset \operatorname{Ob}\mathsf{D}$ and

$$\operatorname{Hom}_{\mathsf{C}}(X, Y) = \operatorname{Hom}_{D}(X, Y) \quad \text{for any } X, Y \in \operatorname{Ob}\mathsf{C} \tag{1.173}$$

[26] To say nothing of supersymmetric theories of theoretical physics, where the language of representable functors is a part of the working language (albeit often used sub- or un-consciously).

[27] An excellent textbook is [McL]. New trends are reflected in [GM], see also [Ke]; see also Molotkov's paper [MV] on glutoses — generalized toposes).

and, moreover, such that the compositions of morphisms in C and in D coincide, C is called a *full sub-category* of D.

A category C is said to be *small* if $\operatorname{Ob}\mathsf{C}$ and collections $\operatorname{Hom}_{\mathsf{C}}(X,Y)$ are sets. A category C is said to be *big* if $\operatorname{Mor}\mathsf{C}$ is a proper class [28)] and, for any objects $X, Y \in \mathsf{C}$, the class of morphisms $\operatorname{Hom}_{\mathsf{C}}(X,Y)$ is a set.

Sometimes, e.g., if we wish to consider a category of functors on a category C, such a category is impossible to define if $\operatorname{Ob}\mathsf{C}$ is a class; on the other hand, in the framework of small categories we cannot consider, say, the category of "all" sets, which is highly inconvenient.

P. Gabriel suggested a way out of this predicament: He introduced a *universum*, a large set of sets stable under all the operations needed; hereafter the categories will be only considered belonging to this universum. For a list of axioms a universum should satisfy see, e.g., Gabriel's thesis [Gab]. We will also assume the existence of a universum.

However, at the present state of foundations of mathematics and its consistency the whole problem seems to me an academic one. My position is close to that of an experimental physicists not apt to fetishize nor destroy his gadgets while they bring about the results.

Nicolas Bourbaki's opinion [Bb2] on this occasion reflects once again his Gaullean common sense and tolerance:

"Mathematicians, it seems, agree that there is but a slight resemblance between our 'intuitive' perceptions of sets and numbers and the formalisms purported to describe them. The disputes only reflect the preference of choice."

1.16.3. Examples. (Some of these examples are defined later in this book.) For convenience we have grouped them:

The first group of examples. The objects in this series of examples are sets endowed with a structure, the morphisms are maps of these sets preserving this structure. (A purist is referred for a definition of a *structure* to, e.g., [EM].)

- Sets, or Ens for French-speaking people, the category of sets and their maps;
- Top, the category of topological spaces and their continuous maps;
- Gr, the category of groups and their homomorphisms;

[28] By definition a *class* X is said to be a *set* if there exist Y such that $X \in Y$. Classes appear in BG (Bernais-Gedel (or Morse)) set theory; in the original ZF (Zermelo-Frenkel) set theory only sets exist. But, if one adds to ZF an axiom of existence for any set X of a universum U, containing X (as Grothendieck did in [EGA, Gab]), then BG theory is modeled inside ZF with this additional axiom: The sets of BG are modeled by sets X of ZF+Bourbaki belonging to some universum U big enough to contain the set of natural numbers, whereas classes of BG are subsets of U.

But what is a universum? It is any set closed with respect to ordinary set-theoretic operations: Unions of (a family of) sets, intersections, power set, etc. (for precise definition, see [EGA, Gab]).

• Gr_f, the subcategory of Gr, whose objects are finite (as the subscript indicates) groups;

• Ab, the subcategory of Gr of Abelian groups;

• $\mathsf{Aff\ Sch}$, the category of affine schemes and their morphisms;

• Rings, the category of (commutative) rings (with unit) and their unit preserving homomorphisms;

• A-Algs, the category of algebras over an algebra A;

• A-Mods, the category of (left) A-modules over a given algebra A;

• Man, the category of manifolds and their morphisms.

The second group of examples. The objects in this series of examples are still structured sets but the morphisms are no longer structure-preserving maps of these sets. (No fixed name for some of these categories).

• The main category of homotopic topology: Its object are topological spaces, the morphisms are the homotopy classes of continuous maps (see, e.g., [FFG]).

• *Additive relations* (See, e.g., [GM]): Its objects are Abelian groups. A morphism $f\colon X \to Y$ is any subgroup of $X \times Y$ and the composition of $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ is given by the relation

$$\begin{aligned}\psi\varphi =\{(x,z) \in X \times Z \mid \ &\text{there exists } y \in Y \text{ such that}\\ &(x,y) \in \varphi, (y,z) \in \psi\}.\end{aligned} \tag{1.174}$$

(No fixed name for this category).

The third group of examples. This group of examples is constructed from some classical structures that are *sometimes* convenient to view as categories.

• For a (partially) ordered set I, the category $\mathsf{C}(I)$ is given by $\operatorname{Ob}\mathsf{C}(I) = I$, where

$$\operatorname{Hom}_{\mathsf{C}(I)}(X,Y) = \begin{cases}\text{one element if } X \le Y,\\ \emptyset \qquad\qquad\quad \text{otherwise.}\end{cases} \tag{1.175}$$

The main example: I is the set of indices of an *inductive system* of groups. [29)]

[29] Let $(I, \le)$ be a directed poset (not all authors require I to be directed, i.e., to be nonempty together with a reflexive and transitive binary relation $\le$, with the additional property that every pair of elements has an upper bound). Let $(G_i)_{i\in I}$ be a family of groups and

1. we have a family of homomorphisms $f_{ij} : G_i \longrightarrow G_j$ for all $i \le j$ with the following properties:
2. f_{ii} is the unit in G_i,
3. $f_{ik} = f_{ij} \circ f_{jk}$ for all $i \le j \le k$.

Then the set of pairs $(G_i, f_{ij})_{i,j\in I}$ is called an *inductive system* (or *direct system*) of groups and morphisms over I.
If in (1) above we have a family of homomorphisms $f_{ij} : G_j \longrightarrow G_i$ for all $i \le j$ (note the order) with the same properties, then the set of pairs $(G_i, f_{ij})_{i,j\in I}$ is called an *inverse system* (or *projective system*) of groups and morphisms over I.

• Let X be a topological space. Denote by Top_X the category whose objects are the open sets of X and morphisms are their natural embeddings. This trivial reformulation conceives a germ of an astoundingly deep generalization of the notion of a topological space, the *Grothendieck topologies* or *toposes*, see [J].[30)]

• Categories associated with a scheme of a diagram.

A *scheme of a diagram* is (according to Grothendieck) a triple consisting of the two sets V (vertices), A (arrows) and the map $e\colon A \to V \times V$ that to every arrow from A assigns its ends, i.e., an ordered pair of vertices: The source and the target of the arrow.

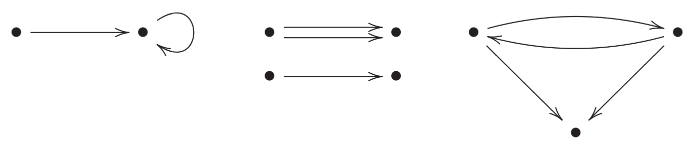

Fig. 13

Let (V, A, e) be a graph. Define the category $\mathsf{D} = \mathsf{D}(V, A, e)$ setting $\operatorname{Ob}\mathsf{D} = V$, $\operatorname{Hom}_{\mathsf{D}}(X, Y) = \{\text{paths from } X \text{ to } Y \text{ along the arrows}\}$.

More precisely, if $X \neq Y$, then any element from $\operatorname{Hom}_{\mathsf{D}}(X, Y)$ is a sequence $f_1, \dots, f_k \in A$ such that the source of f_1 is X, the target of f_i is the source of f_{i+1} and the target of f_k is Y. If $X = Y$, then $\operatorname{Hom}_{\mathsf{D}}(X, X)$ must contain the identity arrow. The compositions of morphisms is defined in an obvious way as the composition of passes.

• Define the category $\mathsf{D}_C = \mathsf{D}_C(V, A, e)$ setting $\operatorname{Ob}\mathsf{D}_C = V$ and

$$\operatorname{Hom}_{\mathsf{D}_C}(X, Y) = \begin{cases} \text{one element if } \operatorname{Hom}_{\mathsf{D}}(X, Y) \neq \emptyset \\ \emptyset \qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{1.176}$$

Intuitively, D_C is the category corresponding to a commutative diagram: All the paths from X to Y define the same morphism.

1.16.4. Examples of constructions of categories. There are several useful formal constructions of new categories. We will describe here only three of them.

1.16.4a. The dual category. Given a category C, define its *dual* C° setting $\operatorname{Ob}\mathsf{C}^\circ$ to be a copy of $\operatorname{Ob}\mathsf{C}$ and $\operatorname{Hom}_{C^\circ}(X^\circ, Y^\circ)$ to be in 1-1 correspondence with $\operatorname{Hom}_{\mathsf{C}}(Y, X)$, where $X^\circ \in \operatorname{Ob}\mathsf{C}^\circ$ denotes the object corresponding to $X \in \operatorname{Ob}\mathsf{C}$, so that if a morphism $\varphi^\circ\colon X^\circ \to Y^\circ$ corresponds to a morphism $\varphi\colon Y \to X$, then $\psi^\circ\varphi^\circ = (\varphi\psi)^\circ$ and $\mathrm{id}_{X^\circ} = (\mathrm{id}_X)^\circ$.

[30] A. Rosenberg used it to construct spectra of noncommutative rings in his noncommutative books [R1], [R2].

Speaking informally, C° is obtained from C by taking the same objects but inverting the arrows.

This construction is interesting in two opposite cases. If C° highly resembles C, e.g., is equivalent (the definition of equivalence of categories will be given a little later) to C (as is the case of, say Ab_f, the category of finite Abelian groups), this situation provides us with a stage where different duality laws perform.

Conversely, if C° is very un-similar to C, then the category C° might have new and nice properties as compared with C; e.g., for $\mathsf{C} = \mathsf{Rings}$, the dual category $\mathsf{C}^\circ = \mathsf{Aff\ Sch}$ has "geometric" properties enabling us to glue global objects from local ones — the operation appallingly unnatural and impossible even to imagine inside Rings.

1.16.4b. The category of objects over a given base. Given a category C and its object S, define the category C_S by setting $\operatorname{Ob}\mathsf{C}_S = \{\varphi \in \operatorname{Hom}_{\mathsf{C}}(X,S)\}$. For any $\varphi\colon X \to S$ and $\psi\colon Y \to S$, define:

$$\operatorname{Hom}_{\mathsf{C}_S}(\varphi,\psi) = \{\chi \in \operatorname{Hom}_{\mathsf{C}}(X,Y) \mid \varphi = \psi\chi\}, \tag{1.177}$$

i.e., $\operatorname{Hom}_{\mathsf{C}_S}(\varphi,\psi)$ is the set of commutative diagrams

$$\begin{tikzcd} X \arrow[rr, "\chi"] \arrow[dr, "\varphi"'] & & Y \arrow[dl, "\mu"] \\ & S & \end{tikzcd} \tag{1.178}$$

The composition of morphisms in C_S is induced by the composition in C.

Examples. 1) $\mathsf{Rings}_R = R\text{-}\mathsf{Algs}$, the category of R-algebras, where R is a fixed ring;

2) Vebun_M, the category of vector bundles over a fixed base M; this is a subcategory of the category Bun_M whose fibers are arbitrary manifolds;

3) the category $(\mathsf{C}_S)^\circ$ that deals with morphisms $S \to X$ for a fixed $S \in \mathsf{C}$.

1.16.4c. The product of categories. Given a family of categories C_i, where $i \in I$, define their product $\prod\limits_{i\in I} \mathsf{C}_i$ by setting

$$\begin{aligned} &\operatorname{Ob}\prod \mathsf{C}_i = \prod \operatorname{Ob} C_i;\\ &\operatorname{Hom}_{\prod \mathsf{C}_i}\left(\prod X_i, \prod Y_i\right) = \prod \operatorname{Hom}_{\mathsf{C}_i}(X_i,Y_i) \end{aligned} \tag{1.179}$$

with coordinate-wise composition.

1.16.5. Functors. A *covariant functor*, or just a *functor*, $F\colon \mathsf{C} \to \mathsf{D}$ from a category C to a category D is a collection of maps $(F, F_{X,Y})$, usually abbreviated to F, where $F\colon\ \operatorname{Ob}\mathsf{C} \to \operatorname{Ob}\mathsf{D}$ and

$$F_{X,Y}\colon\ \operatorname{Hom}_{\mathsf{C}}(X,Y) \to \operatorname{Hom}_{\mathsf{D}}(F(X),F(Y)) \quad \text{for any } X, Y \in \mathsf{C}, \tag{1.180}$$

such that $F_{X,Z}(\varphi\psi) = F_{Y,Z}(\varphi)F_{X,Y}(\psi)$ for any $\varphi, \psi \in \operatorname{Hom} \mathsf{C}$ provided $\varphi\psi$ is defined.

A functor from C° into D is called a *contravariant functor* from C into D. A functor $F\colon \mathsf{C}_1 \times \mathsf{C}_2 \to \mathsf{D}$ is called a *bifunctor*, and so on.

Given categories C, D, E and functors $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{E}$, define $GF\colon \mathsf{C} \to \mathsf{E}$ composing the constituents of F and G in the usual set-theoretical sense.

The most important examples of functors are just "natural constructions": (Co)homology and homotopy are functors $\mathsf{Top} \to \mathsf{Ab}$; characters of finite groups constitute a functor $\mathsf{Gr}_f \to \mathsf{Rings}$. These examples are too meaningful to discuss them in passing.

1.16.6. Examples of presheaves. A *presheaf of sets* (groups, rings, algebras, superalgebras, R-modules, and so on) is a contravariant functor from the category Top_X of open sets of a topological space X with values in Sets (or Gr, Rings, Algs, Salgs, R-Mods, and so on, respectively).

Let (V, A, e) be a scheme of a diagram, D and D_C the associated categories. A functor from D into a category E is called a *diagram of objects from* E (of type (V, A, e)). A functor from D_C into E is a *commutative* diagram of objects.

For an ordered set I considered as a category, a functor from I into a category C is a family of objects from C indexed by the elements from I and connected with morphisms so that these objects constitute either a projective or an inductive system in C.

Given two functors $F, G\colon \mathsf{C} \to \mathsf{D}$, a *functor morphism* $f\colon F \to G$ is a set of morphisms $f(X)\colon F(X) \to G(X)$ (one for each $X \in \mathsf{C}$) such that for any $\varphi \in \operatorname{Hom}_{\mathsf{C}}(X, Y)$ the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "f(X)"] \arrow[d, "F(\varphi)"'] & G(X) \arrow[d, "G(\varphi)"] \\ F(Y) \arrow[r, "f(Y)"] & G(Y). \end{tikzcd} \tag{1.181}$$

The *composition of functor morphisms* is naturally defined.

A functor morphism f is called a *functor isomorphism* if $f(\varphi) \in \operatorname{Mor} \mathsf{D}$ are isomorphisms for all $X \in \mathsf{C}$. The functors from C to D are objects of a category denoted by $\mathsf{Funct}(\mathsf{C}, \mathsf{D})$.

A functor $F\colon \mathsf{C} \to \mathsf{D}$ is called an *equivalence of categories* if there exists a functor $G\colon \mathsf{D} \to \mathsf{C}$ such that $GF \cong \mathrm{id}_{\mathsf{C}}$, $FG \cong \mathrm{id}_{\mathsf{D}}$ and in this case C is said to be *equivalent* to D.

Examples. 1) $(\mathsf{Ab}_f)^\circ \cong \mathsf{Ab}_f$; $G \longleftrightarrow \mathfrak{X}(G)$, the character group of G;

2) Rings° is equivalent to the category $\mathsf{Aff\ Sch}$ of affine schemes.

1.16.7. The category $\mathsf{C}^* = \mathsf{Funct}(\mathsf{C}^\circ, \mathsf{Sets})$. If our universum is not too large, there exists a category whose objects are categories and morphisms are functors between them. The main example: The category $\mathsf{C}^* = \mathsf{Funct}(\mathsf{C}^\circ, \mathsf{Sets})$ of functors from C° into Sets.

1.16.8. Representable functors. Fix any $X \in \mathsf{C}$.

1) Denote by $P_X \in \mathsf{C}^*$ (here: P is for point; usually this functor is denoted by h_X, where h is for homomorphisms) the functor given by

$$P_X(Y^\circ) = \mathrm{Hom}_{\mathsf{C}}(Y, X) \quad \text{for any } Y^\circ \in \mathsf{C}^\circ; \tag{1.182}$$

to any morphism $\varphi^\circ\colon Y_2^\circ \longrightarrow Y_1^\circ$ the functor P_X assigns the map of sets $P_X(Y_2^\circ) \to P_X(Y_1^\circ)$ which sends $\psi\colon Y_2 \longrightarrow X$ into the composition $\varphi\psi\colon Y_1 \longrightarrow Y_2 \to X$.

To any $\varphi \in \mathrm{Hom}_{\mathsf{C}}(X_1, X_2)$, there corresponds a functor morphism $P_\varphi\colon P_{X_1} \longrightarrow P_{X_2}$ which to any $Y \in C$ assigns

$$P_\varphi(Y^\circ)\colon P_{X_1}(Y^\circ) \longrightarrow P_{X_2}(Y^\circ) \tag{1.183}$$

and sends a morphism $\psi \in \mathrm{Hom}_{\mathsf{C}}(Y^\circ, X_1)$ into the composition

$$\varphi\psi\colon Y^\circ \longrightarrow X_1 \longrightarrow X_2. \tag{1.184}$$

Clearly, $P_{\varphi\psi} = P_\varphi P_\psi$.

2) Similarly, define $P^X \in \mathsf{C}^*$ by setting

$$P^X(Y) := \mathrm{Hom}_{\mathsf{C}}(X, Y) \quad \text{for any } Y \in \mathsf{C}; \tag{1.185}$$

to any morphism $\varphi\colon Y_1 \to Y_2$, we assign the map of sets $P^X(Y_1) \longrightarrow P^X(Y_2)$ which sends $\psi\colon X \longrightarrow Y_1$ into the composition $\psi\varphi\colon X \longrightarrow Y_1 \longrightarrow Y_2$.

To any $\varphi \in \mathrm{Hom}_{\mathsf{C}}(X_1, X_2)$, there corresponds a functor morphism $P^\varphi\colon P^{X_2} \longrightarrow P^{X_1}$ which to any $Y \in C$ assigns

$$P^\varphi(Y)\colon P^{X_2}(Y) \longrightarrow P^{X_1}(Y) \tag{1.186}$$

and sends a morphism $\psi \in \mathrm{Hom}_{\mathsf{C}}(X_2, Y)$ into the composition

$$\psi\varphi\colon X_1 \longrightarrow X_2 \longrightarrow Y. \tag{1.187}$$

Clearly, $P^{\varphi\psi} = P^\psi P^\varphi$.

A functor $F\colon \mathsf{C}^\circ \longrightarrow \mathsf{Sets}$ (resp. a functor $F\colon \mathsf{C} \longrightarrow \mathsf{Sets}$) is said to be *representable*(resp. *corepresentable* if it is isomorphic to a functor of the form P_X (resp. P^X) for some $X \in \mathsf{C}$; then X is called an *object that represents* F.

1.16.8a. Theorem. 1) *The map* $\varphi \mapsto P_\varphi$ *defines an isomorphism of sets*

$$\mathrm{Hom}_{\mathsf{C}}(X, Y) \cong \mathrm{Hom}_{\mathsf{C}^*}(P_X, P_Y). \tag{1.188}$$

This isomorphism is functorial in both X *and* Y. *Therefore, the functor* $P\colon \mathsf{C} \to \mathsf{C}^*$ *determines an equivalence of* C *with the full subcategory of* C^* *consisting of representable functors.*

2) *If a functor from* C^* *is representable, the object that represents it is determined uniquely up to an isomorphism.*

The above Theorem is the source of several important ideas.

1.16.8b. The first direction. It is convenient to think of P^X as of "the sets of points of an object $X \in \mathsf{C}$ *with values in various objects* $Y \in \mathsf{C}$ or Y*-points*"; notation: $P^X(Y)$ or sometimes $\underline{X}(Y)$. (The sets $P_X(Y)$ are also sometimes denoted by $\underline{X}(Y)$.)

In other words, $P_X = \coprod\limits_{Y \in \mathsf{C}} P_X(Y)$ with an additional structure: The sets of maps $P_X(Y_1) \longrightarrow P_X(Y_2)$ induced by morphisms $Y_2^\circ \longrightarrow Y_1^\circ$ for any $Y_1, Y_2 \in \mathsf{C}$ and compatible in the natural sense (the composition goes into the composition, and so on). The situation with the P^X is similar.

> **Therefore, in principle, it is always possible to pass from the categorical point of view to the structural one, since all the categorical properties of X are mirrored precisely by the categorical properties of the structure of P_X.**

Motivation. Let $*$ be a one-point set. For categories with sufficiently "simple" structure of their objects, such as the category of finite sets or even category of smooth finite dimensional manifolds, $X = P_X(*)$ for every object X, i.e., X is completely determined by its $*$-points or just points.

For varieties (or for supermanifolds), when the object may have either "sharp corners" or "inner degrees of freedom", the structure sheaf may contain nilpotents or zero divisors, and in order to keep this information and be able to completely describe X we need *various types of points*, in particular, Y-points for some more complicated Y's.

1.16.8c. The second direction. Replacing X by P_X (resp. by P^X) we may transport conventional set-theoretical constructions to any category: An object $X \in \mathsf{C}$ is a *group, ring*, and so on *in the category* C, if the corresponding structure is given on every set $P_X(Y)$ of its Y-points and is compatible with the maps induced by the morphisms $Y_2^\circ \longrightarrow Y_1^\circ$ (resp. $Y_1 \to Y_2$). [31)]

1.16.8d. The third direction. Let C be a category of structures of a given type. Among the functors $\mathsf{C}^\circ \to \mathsf{Sets}$, i.e., among objects from C^*, some natural functors may exist which a priori are constructed not as P_X or P^X but eventually prove to be representable. (*Examples*: The functor $X \mapsto H^*(X; G)$ on the homotopy category; $C \mapsto \mathrm{GL}(m|n; C)$ for any supercommutative superalgebra C.)

In such cases, it often turns out that the properties of the functor representable by such an object are exactly the most important properties of the object itself and its structural description only obscures this fact.

It may well happen that some natural functors $\mathsf{C} \to \mathsf{Sets}$ are not representable, though it is highly desirable that they would have been. The most frequent occurrence of such a situation is when we try to generalize to C some set-theoretical construction, e.g., factorization modulo the group action or

[31] This is exactly the way supergroups are defined and superalgebras *should be* defined. However, it is possible to define superalgebras using just one set-theoretical model and sometimes we have to pay for this deceiving simplicity.

modulo a more general relation. **In such a case it may help to add to C, considered embedded into C^*, the appropriate functors.**

In algebraic geometry of 1970s this way of reasoning lead to monstrous creatures which B. Moisheson called *minischemes*, M. Artin *étale schema*, and A. Grothendieck just *varietées* [32] and lately to *stacks*.

1.16.9. Proof of Theorem 1.16.8. Let us construct a map

$$i\colon \operatorname{Hom}_{\mathsf{C}^*}(P_X, P_Y) \to \operatorname{Hom}_{\mathsf{C}}(X, Y) \tag{1.189}$$

which to every functor morphism $P_X \to P_Y$ assigns the image of $\mathrm{id}_X \in P_X(X)$ in $P_Y(X)$ under the map $P_X(X) \to P_Y(X)$ defined by this functor morphism. Let us verify that i and $\varphi \mapsto P_\varphi$ are mutually inverse.

1) $i(P_\varphi) = P_\varphi(\mathrm{id}_X) = \varphi$ by definition of P_φ.

2) Conversely, given a functor morphism $g\colon P_X \to P_Y$, we have by definition the maps $g(Z)\colon P_X(Z) \to P_Y(Z)$ for all $Z \in \operatorname{Ob}\mathsf{C}$. By definition $i(g) = g(X)(\mathrm{id}_X)$ and we have to verify that $P_{i(g)}(Z) = g(Z)$.

By definition, $P_{i(g)}(Z)$ to every morphism $g\colon Z \to X$ assigns the composition $i(g)\circ\varphi\colon Z \to X \to Y$, and therefore we have to establish that $g(Z)\varphi = i(g)\circ\varphi$.

Now, let us use the commutative diagram

$$\begin{tikzcd} P_X(X) \arrow[r, "g(x)"] \arrow[d, "P_X(\varphi)"'] & P_Y(X) \arrow[d, "P_Y(\varphi)"] \\ P_X(Z) \arrow[r, "g(z)"] & P_Y(Z). \end{tikzcd} \tag{1.190}$$

Thus, we have verified that $\operatorname{Im} P$ is a full subcategory of C^*, and therefore is equivalent to C. The remaining statements are easily established. □

It is worth mentioning that if we add representable functors to the full subcategory of functors P_X from C^*, i.e., if we add functors isomorphic to the ones the category already possesses, we obtain an equivalent category.

1.16.10. The object of inner homomorphisms in a category. All morphisms of a set into another set constitute a set; morphisms of an Abelian group into another Abelian group constitute an Abelian group, there are many more similar examples. A natural way to axiomize the situation when for $X, Y \in \operatorname{Ob}\mathsf{C}$ there is an *object of inner homomorphisms* $\underline{\operatorname{Hom}}(Y, X) \in \operatorname{Ob}\mathsf{C}$, is to define the corresponding representable functor $\underline{\operatorname{Hom}}$. Quite often it is determined by the formula

$$\operatorname{Hom}_{\mathsf{C}}(X, \underline{\operatorname{Hom}}(Y, Z)) = \operatorname{Hom}_{\mathsf{C}}(X * Y, Z) \tag{1.191}$$

for a convenient operation $*$. [33]

[32] In supermanifold theory, this is one of the ways to come to *point-less* (or, perhaps more politely, *point-free*) supermanifolds.

[33] For examples of categories with interesting objects of inner homomorphisms, we can take the categories C of superspaces, superalgebras, supermanifolds and supergroups.

1.17. Solutions of selected problems of Chapter 1

Exercise 1.3.4. 1) Since $a_1 \dots a_n \subset a_1 \cup \dots \cup a_n$, it follows that $V(a_1 \dots a_n) \supset V(a_1 \cup \dots \cup a_n)$. The other way round, let p_x be a prime ideal. Then

$$p_x \in V(a_1 \dots a_n) \Longleftrightarrow x \in \bigcap_i V(a_i) \longrightarrow x \in V(a_i) \text{ for some } i, \tag{1.192}$$

hence, $p_x \supset a_i \supset a_1 \cup \dots \cup a_n$ for this i, i.e., $x \in V(a_1 \cup \dots \cup a_n)$.

2) By Lemma 1.4.1 we have:

$$(g_1, \dots, g_n) = A \Longleftrightarrow \bigcap_{i=1}^{n} V(g_i) = \emptyset.$$

Besides, $V(g^m) = V(g)$ for $m > 0$ implying the desired

Exercise 1.6.7. 3) Let f/g, where $f, g \in A$, be such that

$$(f/g)^n + a_{n-1}(f/g)^{n-1} + \dots + a_0 = 0, \quad \text{where } a_i \in A.$$

Then $f^n + fa_{n-1}f^{n-1} + \dots + g^n a_0 = 0$ implying $g|f^n$ and if $(g, f) = 1$, then g is invertible in A. Therefore, $f/g \in A$.

Chapter 2

Sheaves, schemes, and projective spaces

2.1. Basics on sheaves

The topological space $\operatorname{Spec} A$ is by itself a rather coarse invariant of A, see examples 1.5.3. Therefore it is natural to take for a "right" geometric object corresponding to A, the pair $(\operatorname{Spec} A, \widetilde{A})$ consisting of the space $\operatorname{Spec} A$ and the set of elements of A considered, more or less adequately, as functions on $\operatorname{Spec} A$, and so we did up to now.

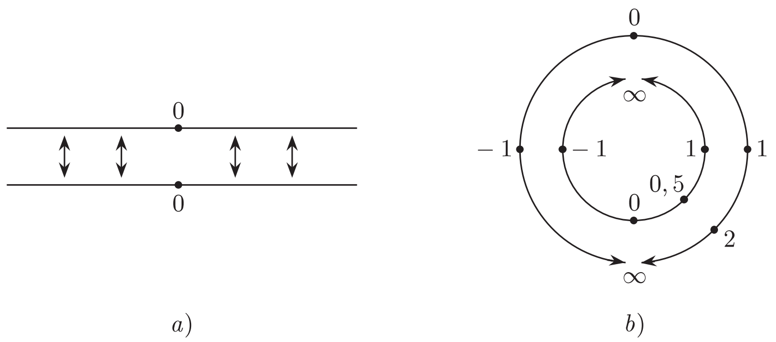

Fig. 1

But, on the other hand, only *local* geometric objects are associated with rings; so in order to construct, say, a projective space we have to glue it from affine ones. Let us learn how to do so.

The gluing procedure we are interested in may be described, topologically, as follows: Let X_1, X_2 be two topological spaces, $U_1 \subset X_1$, $U_2 \subset X_2$ their open subsets and $f\colon U_1 \longrightarrow U_2$ an isomorphism. Then we construct a set

$$X = (X_1 \cup X_2)/R_f,$$

where R_f is the equivalence relation which identifies the points that correspond to each other with respect to f. On X, a natural topology is induced, and we say that X *is the result of gluing* X_1 *with* X_2 *by means of* f.

On Fig. 1, the two ways to glue two affine lines, $X_1 = X_2 = \mathbb{R}$ with $U_1 = U_2 = \mathbb{R} \setminus \{0\}$, are illustrated. The results are:

a) The line with a double point (here $f = \mathrm{id}$);
b) $\mathbb{P}^1$, the projective line (here $f(x) = x^{-1}$). Clearly, $\mathbb{P}^1$ is homeomorphic to the circle S^1.

When we try to apply this construction to the spectra of rings we immediately encounter the above-mentioned circumstance, namely that the topological structure of the open sets reflects but slightly the algebraic data which we would like to preserve and which is carried by the ring A itself. The theories of differentiable manifolds and analytic varieties suggest a solution.

In order to glue a differentiable manifold from two balls U_1 and U_2, we require that the isomorphism $f\colon U_1 \longrightarrow U_2$ which determines the gluing should not be just an isomorphism of topological spaces but should also preserve the differentiable structure. This means that the map f^*, sending continuous functions on U_2 into continuous functions on U_1, must induce an isomorphism of subrings of differentiable functions, otherwise the gluing would not be "smooth".

The analytic case is similar.

Therefore we have to consider the functions of a certain class which are defined on various open subsets of the topological space X.

The relations between the functions on different open sets are axiomized by the following definition.

2.1.1. Presheaves. Fix a topological space X. Let $\mathcal{P}$ be a law that to every open set $U \subset X$ assigns a set $\mathcal{P}(U)$ and, for any pair of open subsets $U \subset V$, there is given a *restriction map* $r_U^V\colon \mathcal{P}(V) \longrightarrow \mathcal{P}(U)$ such that

1) $\mathcal{P}(\emptyset)$ consists of one element,
2) $r_U^W = r_U^V \circ r_V^W$ for any open subsets (briefly: *opens*) $U \subset V \subset W$.

Then the system $\{\mathcal{P}(U), r_U^V \mid U, V \text{ are opens}\}$ is called a *presheaf* (of sets) on X.

The elements of $\mathcal{P}(U)$, also often denoted by $\Gamma(U, \mathcal{P})$, are called the *sections of* the presheaf $\mathcal{P}$ over U; a section may be considered as a "function" defined over U.

Remark. Axiom 1) is convenient in some highbrow considerations of category theory. Axiom 2) expresses the natural transitivity of restriction.

2.1.2. The category Top_X. The objects of Top_X are open subsets of X and morphisms are inclusions. A *presheaf* of sets on X is a functor $\mathcal{P}\colon \mathsf{Top}_X^\circ \longrightarrow \mathsf{Sets}$.

From genuine functions we can construct their products, sums, and multiply them by scalars; similarly, we may consider presheaves of groups, rings, and so on. A formal definition is as follows:

Let $\mathcal{P}$ be a presheaf of sets on X; if, on every set $P(U)$, there is given an algebraic structure (of a group, ring, A-algebra, and so on) and the restriction maps r_U^V are homomorphisms of this structure, i.e., P is a functor $\mathsf{Top}_X^\circ \longrightarrow \mathsf{Gr}$ (Ab, Rings, A-Algs, and so on), then P is called the *presheaf of groups, rings, A-algebras,* and so on, respectively.

Finally, we may consider exterior composition laws, e.g., a presheaf of modules over a presheaf of rings (given on the same topological space).

2.1.2a. Exercise. Give a formal definition of such exterior composition laws.

2.1.3. Sheaves. The presheaves of continuous (infinitely differentiable, analytic, and so on) functions on a space X possess additional properties (of "analytic continuation" type) which are axiomized in the following definition.

A presheaf $\mathcal{P}$ on a topological space X is called a *sheaf* if it satisfies the following condition: For any open subset $U \subset X$, its open cover $U = \bigcup\limits_{i\in I} U_i$, and a system of sections $s_i \in \mathcal{P}(U_i)$, where $i \in I$, such that

$$r^{U_i}_{U_i\cap U_j}(s_i) = r^{Uj}_{U_i\cap U_j}(s_j) \text{ for any } i, j \in I, \tag{2.1}$$

there exists a section $s \in \mathcal{P}(U)$ such that $s_i = r^U_{U_i}(s)$ for any $i \in I$, and such a section is unique.

In other words, from a set of compatible sections over the U_i a section over U may be glued and any section over U is uniquely determined by the set of its restrictions onto the U_i.

Remark. If $\mathcal{P}$ is a presheaf of Abelian groups, the following reformulation of the above condition is useful:

A presheaf $\mathcal{P}$ is a sheaf if, for any $U = \bigcup\limits_{i\in I} U_i$, the following sequence of Abelian groups is exact

$$0 \longrightarrow \mathcal{P}(U) \xrightarrow{\varphi} \prod_{i\in I} \mathcal{P}(U_i) \xrightarrow{\psi} \prod_{i,j\in I} \mathcal{P}(U_i \cap U_j), \tag{2.2}$$

where φ and ψ are determined by the formulas

$$\begin{aligned} \varphi(s) &= (\dots, r^U_{U_i}, (s), \dots) \\ \psi(\dots, s_i, \dots, s_j, \dots) &= (\dots, r^{Ui}_{U_i\cap U_j}(s_i) - r^{U_j}_{U_i\cap U_j}(s_j), \dots). \end{aligned} \tag{2.3}$$

For a generic presheaf of Abelian groups, this sequence is only a complex. (Its natural extension determines a Čech cochain complex that will be defined in what follows.)

2.1.4. The relation between sheaves and presheaves. The natural objects are sheaves but various constructions with them often lead to presheaves which are not sheaves.

Example. (This example is of a fundamental importance for the cohomology theory). Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two sheaves of Abelian groups and $\mathcal{F}_1(U) \subset \mathcal{F}_2(U)$ with compatible restrictions, i.e., $\mathcal{F}_1 \subset \mathcal{F}_2$. As is easy to see, the set of groups $\mathcal{P}(U) = \mathcal{F}_1(U)/\mathcal{F}_2(U)$ and natural restrictions is a presheaf but, in general, not a sheaf. Here is a particular case:

Let X be a circle, $\mathcal{O}$ the sheaf over X for which $\mathcal{O}(U)$ is the group of $\mathbb{R}$-valued continuous functions on U, and $\widetilde{\mathbb{Z}} \subset \mathcal{O}$ the "constant" presheaf for which $\widetilde{\mathbb{Z}}(U) = \mathbb{Z}$ for each non-empty U.

The presheaf for which

$$\mathcal{P}(U) = \mathcal{O}(U)/\widetilde{\mathbb{Z}}(U) \text{ for every open set } U \in X \tag{2.4}$$

is not a sheaf by the following reason.

Consider two connected open sets $U_1, U_2 \subset X$ such that $X = U_1 \cup U_2$ and $U_1 \cap U_2$ is the union of two connected components V_1, V_1 (e.g., U_1, U_2 are slightly enlarged half-circles). Let $f_1 \in \mathcal{O}(U_1), f_2 \in \mathcal{O}(U_2)$ be two continuous functions such that

$$r^{U_1}_{V_1}(f_1) = r^{U_2}_{V_1}(f_2) = 0, \quad r^{U_1}_{V_2}(f_1) - r^{U_2}_{V_2}(f_2) = 1. \tag{2.5}$$

Then the classes $f_1 \bmod \mathbb{Z} \in \mathcal{O}(U_1)/\widetilde{\mathbb{Z}}(U_1)$ and $f_2 \bmod \mathbb{Z} \in \mathcal{O}(U_2)/\widetilde{\mathbb{Z}}(U_2)$ are compatible over $U_1 \cap U_2$.

On the other hand, the sheaf $\mathcal{O}$ over X has no section whose restrictions onto U_1 and U_2 are $f_1 \bmod \mathbb{Z}$ and $f_2 \bmod \mathbb{Z}$, respectively, since it is impossible to remove the incompatibility on V_2. The cause is, clearly, in the non-simple-connectedness of the circle.

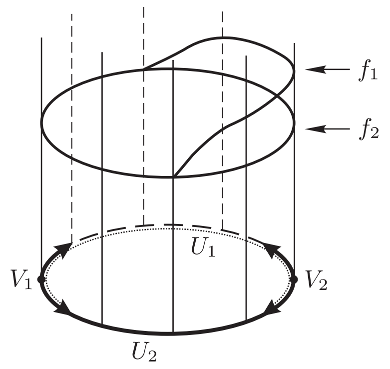

Fig. 2

2.1.5. Gluing sheaves from presheaves. With every presheaf $\mathcal{P}$ there is associated a certain sheaf $\mathcal{P}^+$. We construct the sections of $\mathcal{P}^+$ from the sections of $\mathcal{P}$ with the help of two processes.

The first one *diminishes* the number of sections in $\mathcal{P}(U)$ identifying those which start to coincide being restricted onto a sufficiently fine cover $U = \bigcup\limits_i U_i$.

The second process *increases* the number of sections in $\mathcal{P}(U)$ adding the sections glued from the compatible sets of sections on the coverings of U.

Clearly, a passage to limits is required. A notion, technically convenient here and in other problems, is the following one. Let $\mathcal{P}$ be a presheaf over X. The *stalk* $\mathcal{P}_x$ of $\mathcal{P}$ over a point $x \in X$ is the set $\varinjlim \mathcal{P}(U)$, where the inductive limit [1] is taken over the directed system of open neighborhoods of x.

The elements from $\mathcal{P}_x$ are called the *germs* of sections over x. A *germ* is an equivalence class in the set of sections over different open neighborhoods U containing x modulo the equivalence relation which identifies sections $s_1 \in \mathcal{P}(U_1)$ and $s_2 \in \mathcal{P}(U_2)$ if their restriction onto a common subset $U_3 \subset U_1 \cap U_2$ containing x coincide.

For any point x and an open neighborhood $U \ni x$, a map $r_x^U : \mathcal{P}(U) \longrightarrow \mathcal{P}_x$ is naturally defined.

Clearly, $\mathcal{P}_x$ carries the same structure as the sets $\mathcal{P}(U)$; i.e., is a group, ring, module, and so on, provided $\mathcal{P}$ is a presheaf of groups, rings, modules, and so on.

The idea behind constructing a sheaf $\mathcal{P}^+$ from a presheaf $\mathcal{P}$ is that we define the sections, i.e., elements of $\mathcal{P}^+(U)$, as certain sets of germs of sections, i.e., as elements of $\prod\limits_{x\in U} \mathcal{P}_x$, which are compatible in a natural sense:

Let $\mathcal{P}$ be a presheaf on a topological space X. For every nonempty open subset $U \subset X$, define the subset $\mathcal{P}^+(U) \subset \prod\limits_{x\in U} \mathcal{P}_x$ as follows:

---

[1] Recall the definition (see. [StE]). Let the objects of a category be sets with algebraic structures (such as groups, rings, modules (over a fixed ring), and so on) and morphisms be the morphisms of these structures. Let $(I, \leq)$ be an inductive set. Let $\{A_i \mid i \in I\}$ be a family of objects enumerated by elements of I and for all $i \leq j$, let a family of homomorphisms $f_{ij} \colon A_i \longrightarrow A_j$ be given having the following properties:

a) $f_{ii} = \mathrm{id}|_{\{A_i\}}$,

b) $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$.

Then the pair (A_i, f_{ij}) is said to be *directed (inductive) system over* I. The set of *inductive* (or *direct*) *limit* A of the inductive system (A_i, f_{ij}) is defined as the quotient of the disjoint union of the sets A_i' modulo an equivalence relation $\sim$:

$$\varinjlim A_i = \coprod_i A_i \Big/ \sim,$$

were the equivalence relation $\sim$ is defined as follows: If $x_i \in A_i$ and $x_j \in A_j$, then $x_i \sim x_j$ if, for some $k \in I$, we have $f_{ik}(x_i) = f_{jk}(x_j)$.

$$\mathcal{P}^+(U) = \left\{ \begin{array}{l} (\dots, s_x, \dots) \mid \text{ for any } x \in U, \text{ there exists} \\ \text{its open neighborhood } V \subset U \text{ and a section } s \in \mathcal{P}(V) \\ \text{such that } s_y = r_y^V(s) \text{ for any } y \in V. \end{array} \right\} \quad (2.6)$$

Further, for any pair of open subsets $V \subset U$, define the restriction map $\mathcal{P}^+(U) \longrightarrow \mathcal{P}^+(V)$ as the one induced by the projection $\prod\limits_{x\in U} \mathcal{P}_x \longrightarrow \prod\limits_{x\in V} \mathcal{P}_x$.

2.1.6. Theorem. *The family of sets $\mathcal{P}^+(U)$ together with the described restriction maps is a sheaf on X and $\mathcal{P}^+_x = \mathcal{P}_x$ for any $x \in X$.*

The sheaf $\mathcal{P}^+$ is called the *sheaf associated with* $\mathcal{P}$. Clearly, the algebraic structures are transplanted from $\mathcal{P}$ to $\mathcal{P}^+$.

Proof is a straightforward verification of definitions; so we leave it to the reader.

2.1.7. Another definition of sheaves. Being equivalent to the previous one, given in sec. 2.1.3, it is sometimes more convenient to grasp intuitively.

A *sheaf* $\mathcal{F}$ over a topological space X is a pair $(Y_{\mathcal{F}}, r)$, where $Y_{\mathcal{F}}$ is a topological space and $r\colon Y_{\mathcal{F}} \longrightarrow X$ is an open continuous map onto X such that, for any $y \in Y_{\mathcal{F}}$, there exists its open neighborhood (in $Y_{\mathcal{F}}$) and r is a local homeomorphism in this neighborhood.

The relation between this definition and the previous one is as follows. Giving $r\colon Y_{\mathcal{F}} \longrightarrow X$, we define a sheaf as in sec. 2.1.3: By setting $\mathcal{F}(U)$ to be the set of local sections of $Y_{\mathcal{F}}$ over U, i.e., the maps $s\colon U \longrightarrow Y_{\mathcal{F}}$ such that $r \circ s = \mathrm{id}|_U$.

Conversely, if $\mathcal{F}$ is given by its sections over U for all $U \subset X$, set

$$Y_{\mathcal{F}} = \bigcup_{x\in X} \mathcal{F}_x, \tag{2.7}$$

let r be the map $r\colon \mathcal{F}_x \longrightarrow x$. Define the topology on $Y_{\mathcal{F}}$ considering various sections $s \in \mathcal{F}(U)$ as open sets, and set

$$s = \bigcup_{x\in U} r_x^U(s) \subset \bigcup_{x\in U} \mathcal{F}_x \subset \bigcup_{x\in X} \mathcal{F}_x \tag{2.8}$$

One should bear in mind that, even for Hausdorff spaces X, the spaces $Y_{\mathcal{F}}$ are not Hausdorff, generally.

Fig. 3 depicts a part of the space $Y_{\mathcal{F}}$ corresponding to the sheaf of continuous functions on the segment $[0,1]$. To the graph of every continuous function an open subset of $Y_{\mathcal{F}}$ corresponds and the sections corresponding to the graphs of functions do not intersect in $Y_{\mathcal{F}}$ unless their graphs intersect.

If the functions f_1, f_2 coincide over a (necessarily *closed*!) set Y, then the corresponding sections of $Y_{\mathcal{F}}$ coincide over the *interior* of Y. In the space $Y_{\mathcal{F}}$, the points over a (or b) belonging to sections 3 and 4 are distinct, but any two neighborhoods of these points intersect. □

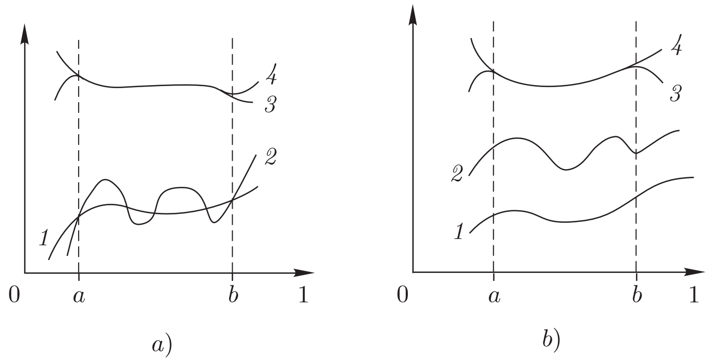

Fig. 3

2.2. The structure sheaf on Spec A

We consider the elements from A as functions on Spec A. Now, if we wish to consider restrictions of these functions, we should construct from them functions with lesser domains. The only such process which does not require limits is to consider quotients f/g since $f(x)/g(x)$ makes sense for all $x \notin V(g)$. This suggests the following definition.

2.2.1. The case of rings without zero divisors. Let A be a ring without zero divisors, $X = \operatorname{Spec} A$, K the field of fractions of A. For any nonempty open subset $U \subset X$, denote:

$$\begin{aligned}&\mathcal{O}_X(U) = \{a \in K \mid a \text{ can be represented in the form } f/g,\\ &\text{where } g(x) \neq 0 \text{ for any } x \in U\}.\end{aligned} \tag{2.9}$$

If $U \subset V$ denote by r_U^V the obvious embedding $\mathcal{O}_X(V) \hookrightarrow \mathcal{O}_X(U)$.

2.2.1a. Theorem. 1) *The above-defined presheaf* $\mathcal{O}_X$ *is a sheaf of rings.*

2) *The stalk of* $\mathcal{O}_X$ *at* x *is*

$$\mathcal{O}_x = \{f/g \mid f, g \in A,\ g \notin \mathfrak{p}_x\} \tag{2.10}$$

and $\Gamma(D(g), \mathcal{O}_X) = \{f/g^n \mid f \in A,\ n \geq 0\}$ *for any nonzero* $g \in A$.

Remarks. Heading 2) is less trivial. Intuitively it reflects two things:

2a) If an "algebraic function" is defined at points where $g \neq 0$, then the worst what may happen with it on $V(g)$ is a pole of finite order: It has no essentially singular points;

2b) over big open sets there is no need to consider "glued" sections: They are given by one equation.

On the other hand, the description of the stalks $\mathcal{O}_x$ enables us to identify Spec A with a set of subrings of the field K. This is the point of view pursued in works by Chevalley and Nagata.

Proof. The fact that $\mathcal{O}_X$ is a sheaf is established by trivial reference to definitions, and $\mathcal{O}_x$ is similarly calculated. The reason for that is that all the $\mathcal{O}_x$ are embedded into K, so that the relations of "extension" and "restriction" are induced by the identity relation in K. (In what follows, we will give a longer definition suitable for any rings, even when K, the localization of A, does not exist.)

Here we will prove heading (2) only for $g = 1$ to illustrate the main idea. The general case is treated in the following subsection.

Therefore we would like to show that if an element of the quotient field K can be represented as a rational fraction so that its denominator does not belong to any prime ideal of A, then this element belongs to A, i.e., there is no denominator at all. This is obvious for a unique factorization ring.

In the general case, the arguments (due to Serre) are suggested by analogy with differentiable manifolds and use the "partition of unity".

Let $f \in \Gamma(X, \mathcal{O}_X) \subset K$. For any point $x \in \operatorname{Spec} A$, set $f = g_x/h_x$, where $g_x, h_x \in A$ and $h_x \notin \mathfrak{p}_x$. Let $U_x = D(h_x)$. Obviously, U_x is an open neighborhood of x.

Now, apply Proposition 1.4.13 on quasi-compactness to the cover $X = \bigcup\limits_x D(h_x)$.

Let $x_i \in X$, and $a_i \in A$ for $i = 1, \ldots, n$. Set $h_i = h_{x_i}$ and $g_i = g_{x_i}$. $1 = \sum\limits_{1 \leq i \leq n} a_i h_i$ in A. Then $X = \bigcap\limits_{1 \leq i \leq n} D(h_i)$ and

$$f = \sum_{1 \leq i \leq n} a_i h_i f = \sum_{1 \leq i \leq n} a_i h_i (g_i h_i) = \sum_{1 \leq i \leq n} a_i g_i \in A. \quad \square \tag{2.11}$$

Let us give an example of calculation of $\mathcal{O}_X(U)$ for open sets other than the big ones.

2.2.2. Example. In $\operatorname{Spec} K[T_1, T_2]$, where K is a field, let $U = D(T_1) \cup D(T_2)$. Therefore U is the complement to the origin. Since $K[T_1, T_2]$ has no zero divisors, we obtain

$$\mathcal{O}_X(U) = K[T_1, T_2, T_1^{-1}] \cap K[T_1, T_2, T_2^{-1}]. \tag{2.12}$$

By the unique factorization property in the polynomial rings we immediately see that $\mathcal{O}_X(U) = K[T_1, T_2]$, and therefore a function on a plane cannot have a singularity supported at one closed point: The function is automatically defined at it.

Similar arguments are applicable to a multidimensional affine space: If $n \geq 2$ and $F_1, \ldots, F_n \in K[T_1, \ldots, T_n]$ are relatively prime, then

$$\mathcal{O}_X\Big(\bigcup_{1 \leq i \leq n} D(F_i)\Big) = K[T_1, \ldots, T_n] : \tag{2.13}$$

The support of the set of singularities of a rational function cannot be given by more than one equation.

2.2.3. The structure sheaf on Spec A: The general case. If A has zero divisors, it has no quotient field. Therefore an algebraic formalism necessary for a correct definition of quotient rings and relations between them becomes more involved. Nevertheless, a sheaf is actually introduced in the same way as for the ring without zero divisors and with the same results.

Constructing a sheaf on Spec A we have to study the dependence of A_S on S, and the ring homomorphisms $A_S \longrightarrow A_{S'}$ for different S and S'. Theorem 1.6.4b stating a universal character of localization, is foundational in what follows.

2.2.4. The structure sheaf $\mathcal{O}_X$ over $X = \operatorname{Spec} A$. For every $x \in X$, set (see sec. 1.6.2)

$$\mathcal{O}_x := A_{A \setminus p_x} = A_{p_x}.$$

For any open subset $U \subset X$, define the ring of sections of the presheaf $\mathcal{O}_X$ over U to be the subring

$$\mathcal{O}_X(U) \subset \prod_{x \in U} \mathcal{O}_x \tag{2.14}$$

consisting of the elements $(\dots, s_x, \dots)$, where $s_x \in \mathcal{O}_x$, such that for every point $x \in U$, there exists an open neighborhood $D(f_x) \ni x$ (here f_x is a function corresponding to x) and an element $g \in A_{f_x}$ such that s_y is the image of g under the natural homomorphism $A_{f_x} \longrightarrow \mathcal{O}_y$ for all $y \in U$.

Define the restriction morphisms r_U^V as the homomorphisms induced by the projection $\prod\limits_{x \in V} \mathcal{O}_x \longrightarrow \prod\limits_{x \in U} \mathcal{O}_x$. It is easy to see that $\mathcal{O}_X$ is well-defined and the natural homomorphism $A_{f_x} \longrightarrow \mathcal{O}_y$ is induced by the embedding of multiplicative sets

$$\{f_x^n \mid n \in \mathbb{N}\} \subset A \setminus p_y. \tag{2.15}$$

Theorem. *The presheaf $\mathcal{O}_X$ is a sheaf whose stalk over $x \in X$ is isomorphic to $\mathcal{O}_x$ and r_x^U is the composition*

$$\mathcal{O}_X(U) \longrightarrow \prod_{x' \in U} \mathcal{O}_{x'} \xrightarrow{pr} \mathcal{O}_x \tag{2.16}$$

Furthermore, the ring homomorphism

$$j \colon A_f \longrightarrow \mathcal{O}_X(D(f)),\ j(g/f) = (\dots, j_x(g/f), \dots)_{x \in U}, \tag{2.17}$$

where $j_x \colon A_f \longrightarrow \mathcal{O}_x$ is a natural homomorphism of quotient rings, is an isomorphism.

The sheaf $\mathcal{O}_X$ over the scheme $X = \operatorname{Spec} A$ is called the *structure sheaf* of X.

Proof. The fact that $\mathcal{O}_X$ is a sheaf follows immediately from the definitions and compatibility of natural homomorphisms of quotient rings. By definition, the stalk $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ over x is equal to

$$\varinjlim \mathcal{O}_X(U) = \varinjlim_{f(x)\neq 0} \mathcal{O}_X(D(f)), \tag{2.18}$$

since the neighborhoods $D(f)$ constitute a basis of neighborhoods of x. The natural homomorphisms $A_f \longrightarrow \mathcal{O}_X(D(f)) \longrightarrow \mathcal{O}_x$ determine the homomorphism $\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_x$. It is an epimorphism, since any element from $\mathcal{O}_x$ is of the form g/f, where $f(x) \neq 0$, and therefore is an image of the corresponding element from $\mathcal{O}_X(D(f))$.

Moreover, the kernel of this homomorphism is trivial: If g/f goes into $0 \in \mathcal{O}_x$, then by Lemma 1.6.4a $f_1 g = 0$ for some f_1 such that $f_1(x) \neq 0$, and therefore the image of g/f in A_{ff_1} and, with even more reason, in the inductive limit $\varinjlim \mathcal{O}_X(D(f))$ is zero.

It remains to demonstrate the "furthermore" part of the theorem; notice that it gives a "finite" description of rather cumbersome and bulky rings $\mathcal{O}_X(U)$ for big open sets U.

First of all, $\operatorname{Ker} j = 0$. Indeed, if $j(g/f) = 0$, then, for every point $x \in D(f)$, there exists an element t_x such that $t_x(x) \neq 0$ and $t_x \in \operatorname{Ann} g = \{a \mid ag = 0\}$. But this means that $\operatorname{Ann} g \not\subset p_x$; i.e., $x \notin V(\operatorname{Ann} g)$ for $x \in D(f)$; and therefore $V(\operatorname{Ann} g) \subset V(f)$; i.e., $f^n \in \operatorname{Ann} g$ for some n. Hence, $g/f = 0$ in A_f.

Now, let us prove that j is an epimorphism. Let $s \in \mathcal{O}_X(D(f))$ be a section. By definition there exists a cover $D(f) = \bigcup\limits_{x\in D(f)} D(h_x)$ such that s is induced over $D(h_x)$ by an element g_x/h_x. As in sec. 2.2.1, construct a partition of unity, or rather not of unity but of an invertible on $D(f)$ function f^n :

$$V(f) = \bigcap_{x\in D(f)} V(h_x) = V\Big(\bigcap_{x\in D(f)} \{h_x\}\Big) \tag{2.19}$$

implying $f^n = \sum\limits_{x\in D(f)} a_x h_x$. Since $a_x = 0$ for almost all x, we may denote the incompressible decomposition by

$$D(f) = \bigcup_{1\leq i\leq r} D(h_i), \text{ where } f^n = \sum_{1\leq i\leq r} a_i h_i, \text{ where } a_i = a_{x_i}, h_i = h_{x_i} \tag{2.20}$$

Now, consider our section s glued together from the g_i/h_i. The fact that of g_i/h_i and g_j/h_j are compatible on $D(h_i) \cap D(h_j) = D(h_i h_j)$ means that the images of g_i/h_i and g_j/h_j coincide in all the rings $\mathcal{O}_x$, where $x \in D(h_i h_j)$.

By the proved above $g_i/h_i - g_j/h_j = 0$ in $A_{h_i h_j}$; i.e., for some m (which may be chosen to be independent of indices since the cover is finite) we have $(h_i h_j)^m (g_i h_j - g_j h_i) = 0$. Replacing h_i with h_i^{m+1} and g_i with $g_i h_i^m$ we may assume that $m = 0$.

Now, the compatibility conditions take the form $g_i h_j = g_j h_i$. Therefore

$$f^n g_j = \sum a_i h_i g_j = \Big(\sum_{1\leq i\leq r} a_i g_i\Big) h_j \tag{2.21}$$

implying that the image of $\sum\limits_{1\leq i\leq r} a_ig_i/f^n$ in A_{h_j} is precisely g_j/h_j.

Therefore the compatible local sections over $D(h_j)$ are the restrictions of one element from A_f, as required. □

The sheaf over $X = \operatorname{Spec} A$ described above will be sometimes denoted by $\widetilde{A}$. The pair $(\operatorname{Spec} A, \widetilde{A})$ consisting of a topological space and a sheaf over it determines the ring A thanks to Theorem 2.2.4: Namely, $A = \Gamma(\operatorname{Spec} A, \widetilde{A})$. This pair is the main local object of the algebraic geometry.

2.3. The ringed spaces. Schemes

2.3.1. Ringed spaces. A *ringed topological space* is a pair $(X, \mathcal{O}_X)$ consisting of a space X and a sheaf of rings $\mathcal{O}_X$ over it called the *structure sheaf.*

A *morphism of ringed spaces* $F\colon (X_1, \mathcal{O}_X) \longrightarrow (Y_1, \mathcal{O}_Y)$ is a pair consisting of a morphism $f\colon X \longrightarrow Y$ of topological spaces and the collection of ring homomorphisms

$$\{f^*_U\colon \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U)) \text{ for every open } U \subset Y\} \tag{2.22}$$

that are compatible with restriction maps, i.e., such that

(a) the diagrams

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \xrightarrow{f^*_U} & \mathcal{O}_X(f^{-1}(U)) \\ {\scriptstyle r^U_V}\downarrow & & \downarrow{\scriptstyle r^{f^{-1}(U)}_{f^{-1}(V)}} \\ \mathcal{O}_Y(V) & \xrightarrow{f^*_V} & \mathcal{O}_X(f^{-1}(V)) \end{array} \tag{2.23}$$

commute for every pair of open sets $V \subset U \subset Y$;

(b) for any open $U \subset Y$, and a pair $u \in U$ and $g \in \mathcal{O}_Y(U)$ such that $g(y) = 0$, we have

$$f^*_U(g)(x) = 0 \text{ for any } x \text{ such that } f(x) = y. \tag{2.24}$$

Elucidation. If X and Y are Hausdorff spaces, $\mathcal{O}_X$, $\mathcal{O}_Y$ the sheaves of continuous (smooth, analytic, and so on) functions on them, respectively, then to every morphism $f\colon X \longrightarrow Y$ the ring homomorphism $f^*_U\colon \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U))$ corresponds: f^*_U assigns to any function $g \in \mathcal{O}_Y(U)$ the function

$$f^*_U(g)(x) = g(f(x)) \text{ for any } x \in f^{-1}(U); \tag{2.25}$$

i.e., the domain of $f^*_U(g)$ is $f^{-1}(U)$, and $f^*_U(g)$ is constant on the pre-image of every $y \in U$.

In algebraic geometry, the spaces are not Hausdorff ones and their structure sheaves are not readily recognized as sheaves of functions. Therefore

1) the collection of ring homomorphisms $\{f_U^* \mid U \text{ is an open set}\}$ is **not** recovered from f and *must be given separately*;
2) the condition

$$f_U^*(g)(x) = g(f(x)) \text{ for any } x \in f^{-1}(U) \tag{2.26}$$

is replaced by a weaker condition (b).

These two distinctions from the usual functions are caused by the fact that the domains of our "make believe" functions have variable ranges and different sections of the structure sheaf may represent the same function.

A ringed space isomorphic to one of the form $(\operatorname{Spec} A, \widetilde{A})$ is called an *affine scheme*. We will prove in next subsections the equivalence of this definition of affine scheme to that given earlier (sec. 1.5.2).

2.3.2. Schemes. A ringed topological space $(X, \mathcal{O}_X)$ is said to be a *scheme* if its every point x has an open neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

One of the methods for explicit description of a global object is just to define the local objects from which it is glued and the method of gluing. Here is the formal procedure.

Proposition. *Let $(X_i, \mathcal{O}_{X_i})_{i\in I}$, be a family of schemes and let in every X_i open subsets U_{ij}, where $i, j \in I$, be given. Let there be given a system of isomorphisms $\theta_{ij}\colon (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \longrightarrow (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ satisfying the cocycle condition*

$$\theta_{ii} = \mathrm{id},\ \theta_{ij} \circ \theta_{ji} = \mathrm{id}, \theta_{ij} \circ \theta_{jk} \circ \theta_{ki} = \mathrm{id}\,. \tag{2.27}$$

Then there exists a scheme $(X, \mathcal{O}_X)$, an open cover $X = \bigcup\limits_{i\in I} X_i'$ and a family of isomorphisms $\varphi_i\colon (X_i', \mathcal{O}_X|_{X_i'}) \longrightarrow (X_i, \mathcal{O}_{X_i})$ such that

$$(\varphi_j|_{X_i\cap X_j})^{-1} \circ \theta_{ij} \circ \varphi_i|_{X_i\cap X_j} = \mathrm{id} \ \textit{for all } i, j. \tag{2.28}$$

2.3.2a. Exercise. Prove this theorem.

Hint. For any open subset U, the space $(U, \mathcal{O}_X|_U)$ is also a scheme. Indeed, let $x \in U$; then x has a neighborhood $U_x \subset X$ such that $(U_x, \mathcal{O}_X|_{U_x})$ is isomorphic to $(\operatorname{Spec} A, \widetilde{A})$. Then, $U \cap U_x$ is a nonempty open subset of $\operatorname{Spec} A$ and, since the big open sets $D(f)$, where $f \in A$, constitute a basis of Zariski topology and $(D(f), \widetilde{A}|_{D(f)}) \cong (\operatorname{Spec} A_f, \widetilde{A}_f)$, we can find an affine neighborhood of x contained inside U.

2.3.3. Examples. In these examples, $(X_i, \mathcal{O}_{X_i})$ are most often affine schemes.

2.3.3a. Projective spaces. Let K be a ring. Define the scheme $\mathbb{P}^n_K$, the n-dimensional projective space over K.

Let $T_0, \dots, T_n$ be independent variables. Set

$$U_i = \operatorname{Spec} K\left[\frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}\right], \quad U_{ij} = \operatorname{Spec} K\left[\frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}\right]_{T_j/T_i} \subset U_i, \tag{2.29}$$

and determine a scheme isomorphism $\theta_{ij} : U_{ij} \longrightarrow U_{ji}$ identifying naturally the quotient ring whose elements are of the form $\dfrac{f(T_0, \dots, T_n)}{T_i^a T_j^b}$, where f is a form (homogeneous polynomial) of degree $a+b$ with coefficients in K.

It is not difficult to verify that all the conditions of Proposition 12.2 are satisfied, and therefore $n+1$ affine spaces U_i may be glued together.

2.3.3b. Monoidal transformation. In notation of sec. 2.3.3a, set

$$\begin{aligned} U_i &= \operatorname{Spec} K\left[T_0, \dots, T_n; \frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}\right], \\ U_{ij} &= \operatorname{Spec} K\left[T_0, \dots, T_n; \frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}\right]_{T_j/T_i}. \end{aligned} \tag{2.30}$$

As above, the rings of functions over U_{ij} and U_{ji} can be identified with the ring whose elements are of the form $f(T_0, \dots, T_n)/T_i^a T_j^b$, where f is now an inhomogeneous polynomial, the power of its lowest term being $a+b$.

Denote by X the scheme obtained after gluing up the U_i and identify the U_i and U_{ij} with the corresponding open sets in X. We have

$$X = \bigcup_{0 \le i \le n} U_i, \quad U_{ij} = U_i \cap U_j. \tag{2.31}$$

Consider the structure of X in detail. The monomorphism

$$K[T_0, \dots, T_n] \longrightarrow K\left[T_0, \dots, T_n, \frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}\right] \tag{2.32}$$

determines the projection of the U_i to $\mathbb{A}^{n+1}_K = \operatorname{Spec} K[T_0, \dots, T_n]$. Obviously, these projections are compatible on U_{ij}. In U_i, single out the open subset $D_i = D(T_i)$. Since

$$K[T_0, \dots, T_n]_{T_i} = K\left[T_0, \dots, T_n; \frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}\right], \tag{2.33}$$

then D_i is isomorphically mapped onto the complement to the "coordinate hyperplane" $V(T_i)$ in $\mathbb{A}^{n+1}_K$. Therefore X has an open subset $\bigcup\limits_{0 \le i \le n} D_i$ isomorphic to $\mathbb{A}^{n+1}_K \setminus V(T_0, \dots, T_n)$, and, if K is a field, this is just the complement to the origin of the $(n+1)$-dimensional affine space $\mathbb{A}^{n+1}_K$.

What is the structure of $X \setminus \bigcup\limits_{0 \le i \le n} D_i$? We have

$$X \setminus \bigcup_{0\le i\le n} D_i = \bigcup_{0\le i\le n} V(T_i), \text{ where } \mid V(T_i) \subset U_i. \tag{2.34}$$

Furthermore,

$$V(T_i) = \operatorname{Spec} K\left[T_0, \dots, T_n, \frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}\right]/(T_i) \tag{2.35}$$

The ring $K\left[T_0, \dots, T_n, \frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}\right]$ consists of the elements of the form

$$\begin{aligned}&\frac{f(T_0, \dots, T_n)}{T_i^a}, \text{ where } f \text{ is a polynomial}\\ &\text{the degree of its lowest terms being } \ge a.\end{aligned} \tag{2.36}$$

Hence the ideal (T_i) of this ring consists of the same elements, but with the degree of the lowest terms of the numerator being $\ge a\ +1$. Therefore it is easy to see that

$$K\left[T_0, \dots, T_n, \frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}\right]/(T_i) \simeq K\left[\frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}\right]$$

and we have:

$$V(T_i) = \operatorname{Spec} K\left[\frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}\right]. \tag{2.37}$$

The affine schemes $V(T_i)$ are glued together as in the preceding example; therefore, from the set-theoretical point of view, $X = \Big(\bigcup\limits_{0\le i\le n} D_i\Big) \cup \mathbb{P}^n_K$.

Thus, X is obtained from the $(n+1)$-dimensional affine space

$$\mathbb{A}^{n+1}_K = \operatorname{Spec} K[T_0, \dots, T_n]$$

by pasting $V(T_0, \dots, T_n)$ in $\mathbb{P}^n_K$ instead of the “origin”.

Exercise. 1) Prove that $\Gamma(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}) = K$. Calculate $\Gamma(X, \mathcal{O}_X)$ for the scheme X constructed in Example 2.3.3b.

2) Prove that if $\operatorname{Spec} K$ is irreducible, then so is $\mathbb{P}^n_K$.

2.3.4. Condition necessary for gluing spectra. Let us give a simple algebraic condition necessary for the possibility to glue $\operatorname{Spec} A$ and $\operatorname{Spec} B$:

Proposition. *Let A and B be the rings without zero divisors. If there is an open subset U of* $\operatorname{Spec} A$ *such that $(U, \widetilde{A}|U)$ is isomorphic to $(W, \widetilde{B}|_W)$, where W is an open subset of* $\operatorname{Spec} B$*, then the quotient fields of A and B are isomorphic.*

If A and B are rings of finite type over a field or $\mathbb{Z}$, then the converse is also true.

Proof. Consider an isomorphism $(U, \widetilde{A}|_U) \xrightarrow{\sim} (W, \widetilde{B}|_W)$. Generic points of $\operatorname{Spec} A$ and $\operatorname{Spec} B$ are mapped into each other (they are contained in U, and therefore in W). The stalks of the structure sheaves at these points are the quotient fields of A and B respectively.

To prove the converse statement notice first of all that if A has no zero divisors, then $\operatorname{Spec} A$ and $\operatorname{Spec} A\left[\frac{f}{g}\right]$, where $f, g \in A$, have isomorphic big open sets:

$$A\left[\frac{1}{fg}\right] = A\left[\frac{1}{f}, \frac{1}{g}\right] = \left(A\left[\frac{f}{g}\right]\right)_{f/1}. \tag{2.38}$$

Now, if A is generated (over K or $\mathbb{Z}$) by elements $x_1, \dots, x_n$ and B by elements $y_1, \dots, y_n$, and the quotient fields of A and B are isomorphic, then we may pass from A (resp. B) to the ring $A[y_1, \dots, y_n] = B[x_1, \dots, x_n]$ by a finite number of steps adjoining each time one element from the quotient field and at each step the spectra of the considered rings have isomorphic open sets. Taking into account the irreducibility of $\operatorname{Spec} A$ and $\operatorname{Spec} B$ we obtain the statement desired. □

One of the immediate corollaries of this Proposition is a promised equivalence of the category $\mathsf{Aff\ Sch}$ of affine schemes as defined in sec. 1.5.2 with the category of affine schemes as defined in sec.2.3.2 — the full subcategory of the category of schemes Sch.

The schemes X and Y are said to be *birationally equivalent* if there exist everywhere dense open subsets $U \subset X$ and $V \subset Y$ such that $(U, \mathcal{O}_X|_U)$ is isomorphic to $(V, \mathcal{O}_X|_V)$.

The origin of the term "birational equivalence" is as follows. On the spectrum of any ring without zero divisors, the elements of its field of quotients can be considered as "rational functions". An isomorphism of open sets of open subsets of $\operatorname{Spec} A$ and $\operatorname{Spec} B$ was interpreted as "not everywhere" defined map determined by rational functions.

2.3.4a. Example. $\operatorname{Spec} k[T]$ (the line) and $\operatorname{Spec} k[T_1, T_2]/(T_1^2 + T_2^2 - 1)$ (the circle) are birationally equivalent if $\operatorname{Char} k \neq 2$. (How to establish that they are non-isomorphic?)

Indeed, the classical parametrization

$$t_1 = \frac{T^2 - 1}{T^2 + 1}, \quad t_2 = \frac{2T}{T^2 + 1}$$

and its inversion $T = \dfrac{t_2}{1 + t_1}$ establish an isomorphism of rings of quotients

$$k[T]_{T^2+1} = k[T_1, T_2]/(T_1^2 + T_2^2 - 1)_{1-t_1},$$

where $t_i = T_i \pmod{T_1^2 + T_2^2 - 1}$.

2.3.4b. Example. A generalization of the construction of the previous example. Let $f(T_1, \dots, T_n)$ be an indecomposable quadratic polynomial over a field k of characteristic $\neq 2$, and with a zero over k. The spaces $X = \operatorname{Spec} k[T_1, \dots, T_n]/(f)$ and $Y = \operatorname{Spec} k[T'_1, \dots, T'_{n-1}]$ are birationally equivalent. We will describe a parametrization geometrically, leaving its detalization and description of isomorphic open sets as an exercise for the reader.

Consider X as a subspace of the affine space $E = \operatorname{Spec} k[T_1, \dots, T_n]$; let us embed Y into E by means of the ring homomorphism

$$T_i \mapsto T_i' \text{ for } i = 1, \dots, n-1, \text{ and } T_n \mapsto 0.$$

On geometric k-points of the spaces X and Y, the correspondence given by this parametrization is described as follows.

Let us fix a k-point x of the quadric X. We may assume that it does not lie on Y; otherwise we can modify the embedding of Y. Let us draw lines in E through the fixed point $x \in X$ and a variable point $y \in Y$. To each point $y \in Y$ we assign a distinct from x point of intersection z of the line $\overline{xy}$ with Y. The point z exists and is defined uniquely if y is contained in a non-empty open subset of Y.

2.3.4c. Spec $\boldsymbol{k[T]}$ and Spec $\boldsymbol{k[T_1, T_2]/(T_1^3 + T_2^3 - 1)}$ are birationally equivalent. To prove this, it suffices to establish that the equation $X^3 + Y^3 = Z^3$ has no solutions in $k[T]$, except those proportional to "constant" solutions (i.e., with $X, Y, Z \in k$). We may assume that the cubic roots of unity lie in k and apply the classical Fermat's descent (on degree of the polynomial) using the unique factorization property of $k[T]$.

2.4. The projective spectra

In what follows we introduce a very important class of schemes — projective spectra of $\mathbb{Z}$-graded rings. This class contains analogues of classical projective varieties and in particular, projective spaces.

2.4.1. $\mathbb{Z}$-graded rings. First, recall the definition of a $\mathbb{Z}$-graded ring R (commutative and with a 1, as always in these lectures). Let $R = \bigoplus\limits_{i \in \mathbb{Z}} R_i$ be the direct sum of the commutative subgroups with respect to addition and let $R_i R_j \subset R_{i+j}$, in the sense that $r_i r_j \in R_{i+j}$ for any $r_i \in R_i$ and $r_j \in R_j$. The elements $r \in R_i$ are said to be *homogeneous of degree* i. The function $\deg \colon R \longrightarrow \mathbb{Z}$ is defined by the formula

$$\deg r = i \Longleftrightarrow r \in R_i. \tag{2.39}$$

Clearly, any nonzero $r \in R$ can be uniquely represented in the form $r = \Sigma r_i$ with $r_i \in R_i$.

An ideal $I \subset R$ is said to be *homogeneous* if $I = \oplus(I \cap R_i)$. For any homogeneous ideal I, the quotient ring is naturally $\mathbb{Z}$-graded: $R/I = \oplus(R_i/I_i)$. Clearly, R_0 is a subring and if $R_i = 0$ for $i < 0$, then $R_+ = \oplus_{i>0} R_i$ is a homogeneous ideal of R.

Example. The *standard grading* of $R = K[T_0, \dots, T_n]$: Let $\deg f = 0$ for any $f \in K$ and $\deg T_i = 1$ for all i.

2.4.2. Projective spectra. The *projective spectrum* of a $\mathbb{Z}$-graded ring R with $R_i = 0$ for $i < 0$ is the topological space

$$\operatorname{Proj} R = \{\text{homogeneous prime ideal } p \text{ of } R \mid p \not\supset R_+ = \bigoplus_{i>0} R_i\} \tag{2.40}$$

with the topology induced by the Zariski topology of $\operatorname{Spec} R$.

2.4.2a. A geometric interpretation. In the classical projective geometry, the projective variety X over an algebraically closed field K is given by a system of homogeneous equations

$$F_i(T_0, \dots, T_n) = 0, \quad \text{where } i \in I. \tag{2.41}$$

We associate with X the graded ring $R = K[T_0, \dots, T_n]/(F_i)_{i \in I}$. Let us list several schemes that can be constructed from $K[T_0, \dots, T_n]$ are related with the following geometric objects:

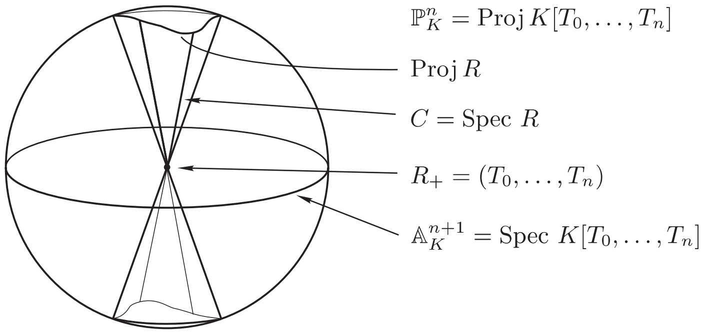

Fig. 4

• $\mathbb{A}^{n+1}_K = \operatorname{Spec} K[T_0, \dots, T_n]$. It is the $(n+1)$-dimensional *affine space* over K with a fixed coordinate system.

• $C = \operatorname{Spec} R$, the subscheme of $\mathbb{A}^{n+1}_K$. It is the cone with the vertex at the origin. Indeed, the characteristic property of the cone is that together with every (geometric) point it contains a *generator* — the straight line through this point — and the vertex of the cone. The straight line connecting point $(t_0, \dots, t_n)$ with the origin consists of the points $(tt_0, \dots, tt_n)$ for any $t \in K$, and all of these points belong to C, since C is given by homogeneous equations. Changing t we move along the generator. Moreover, every *non-zero*value of t determines an automorphism of R which multiplies the homogeneous elements of degree i by t^i. From this it is easy to derive that, conversely, every cone is given by homogeneous equations.

• $\mathbb{P}^n_K = \operatorname{Proj} K[T_0, \dots, T_n]$. By the above to the points of $\mathbb{P}^n_K$ the irreducible cones in $\mathbb{A}^{n+1}_K$ correspond; in particular, to the closed points there correspond straight lines through the origin.

This is the usual definition of the projective space. Though the origin in $\mathbb{A}_K^{n+1}$ is a homogeneous prime ideal, $R_+ = (T_0, \dots, T_n)$, it only contains the vertex of the cone, and therefore is excluded from the definition of points of $\mathbb{P}_K^n$.

It is convenient to assign to every straight line through the origin its infinite point. Then, $\mathbb{P}_K^n$ can be interpreted as a *hyperplane in* $\mathbb{A}_K^{n+1}$ *moved to infinity.*

• $\operatorname{Proj} R$. The above implies that $\operatorname{Proj} R$ corresponds to the base of C, which belongs to the hyperplane through infinity, $\mathbb{P}_K^n$. □

2.4.3. The scheme structure on Proj *R*. Let us define the structure sheaf on $\operatorname{Proj} R$ and show that the obtained ringed space is locally affine.

For every $f \in R$, set $D_+(f) = D(f) \cap \operatorname{Proj} R$. Clearly, if $f = \sum\limits_{i \in \mathbb{Z}_+} f_i$, where f_i are homogeneous elements of degree i, then $D_+(f) = \bigcup\limits_{i \in \mathbb{Z}_+} D_+(f_i)$, and therefore the sets $D_+(f)$, where f runs over homogeneous elements of R, form a basis of a topology of $\operatorname{Proj} R$. For any homogeneous $f \in R$, the localization R_f, is, clearly, graded:

$$\deg(g/f^k) = \deg(g) - k \deg(f). \tag{2.42}$$

Denote by $(R_f)_0$ the component of degree 0. This component is most important in the projective case, since, unlike the affine case, only the elements from $(R_f)_0$ may pretend for the role of "functions" on $D_+(f)$.

Proposition. *Let f, g be homogeneous elements of R. Then*

a) $D_+(f) \cap D_+(g) = D_+(fg)$;

b) *there exists a system of homeomorphisms* $\psi_f : D_+(f) \longrightarrow \operatorname{Spec}(R_f)_0$ *such that all the diagrams*

$$\begin{tikzcd} D_+(f) \arrow[r, "\Psi_f"] & \operatorname{Spec}(R_f)_0 \\ D_+(fg) \arrow[u] \arrow[r, "\Psi_{fg}"] & \operatorname{Spec}(R_{fg})_0 \arrow[u] \end{tikzcd} \tag{2.43}$$

commute.

(Here the left vertical arrow is the natural embedding, and the right one is induced by the natural ring homomorphism $R_f \longrightarrow R_{fg}$.)

Corollary. *The sheaves $\psi_f^*(\widetilde{R_f})_0)$ transported onto $D_+(f)$ via ψ_f are glued together and determine a scheme structure on* $\operatorname{Proj} R$.

Proof of Proposition. Since $D(f) \cap D(g) = D(fg)$, we obtain a).

To prove b), define ψ_f as the through map

$$D_+(f) \longrightarrow D(f) \longrightarrow \operatorname{Spec} R_f \longrightarrow \operatorname{Spec}(R_f)_0, \tag{2.44}$$

where the first arrow is the natural embedding, the second one is the isomorphism and the last one is induced by the ring monomorphism.

Clearly, ψ_f is continuous. Let us show that it is one-to-one: construct the inverse map $\varphi\colon \operatorname{Spec}(R_f)_0 \longrightarrow D_+(f)$. Let $p \in \operatorname{Spec}(R_f)_0$. Set

$$\varphi(p)_n = \{x \in R_n \mid x^d/f^n \in p\}, \text{ where } f \in R_d. \tag{2.45}$$

First, let us verify that $\varphi(p) = \bigoplus\limits_n \varphi(p)_n$ is a homogeneous prime ideal. Let $x, y \in \varphi(p)_n$. Let us establish that $\varphi(p)_n$ is closed with respect to addition, since the other properties of the ideal are even easier to establish.

Clearly, if x^d/f^n and $y^d/f^n \in p$, then $(x+y)^{2d}/f^{2n} \in p$; and therefore $(x+y)^d/f^n \in p$, so $x+y \in \varphi(p)$. We have taken into account that p is a prime ideal in $(R_f)_0$. Furthermore, if $x^d/f^n,\ y^d/f^m \in \varphi(p)_{m+n}$, then $(xy)^d \in p$, implying either $x \in p$ or $y \in p$, since p is prime; therefore either x^d/f^n or y^d/f^m belongs to $\varphi(p)$; hence $\varphi(p)$ is prime.

2.4.3a. Exercise. Verify that φ and ψ_f are mutually inverse.

Now, let us prove that ψ_f is a homeomorphism. It suffices to show that ψ_f is an open map, since its continuity is already established. Let $g \in R_f$. We have to verify that the image of $D_+(f) \cap D_+(g)$ under ψ_f is open:

$$\begin{aligned} D_+(f) \cap D_+(g) = D_+(fg) \longrightarrow \operatorname{Spec}(R_{fg})_0 = \\ \operatorname{Spec}((R_f)_0)_{(g^d/f^e)} \longrightarrow \operatorname{Spec}(R_f)_0 \end{aligned} \tag{2.46}$$

which also shows the possibility of gluing $\operatorname{Spec}(R_f)_0$ and $\operatorname{Spec}(R_g)_0$ along $D(fg)$ since

$$\begin{aligned} \operatorname{Spec}(R_f)_0 \supset D_+(fg) = \operatorname{Spec}(R_{fg})_0 = \operatorname{Spec}((R_f)_0)_{(g^d/f^e)} \\ = \operatorname{Spec}((R_g)_0)_{(f^e/g^d)} = D_+(gf) \subset \operatorname{Spec}(R_g)_0. \quad \square \end{aligned} \tag{2.47}$$

2.4.4. Examples. 1) Proj $K[x_0, \dots, x_n]$ is the projective space $\mathbb{P}^n_K$ constructed in sec. 2.3.3a

2) The scheme X from Example 2.3.3b could have been obtained as follows. In $K[T_0, \dots, T_n]$, consider the ideal $I = (T_0, \dots, T_n)$. Denote by $R_k = I^k T^k$ the set of degree k monomials in T with coefficients from the k-th power of I; set $R = \oplus R_k$. Clearly, $X = \operatorname{Proj} R$.

A generalization of this construction is the following one.

3) **The monoidal transformation with center in an ideal**. Let R be a ring and I its ideal. Construct the graded subring in $R[T]$:

$$R = \bigoplus_{k \geq 0} R_k, \quad \text{where } R_k = I^k T^k, \tag{2.48}$$

i.e., the elements of R are the polynomials $\Sigma a_k T^k$ such that $a_k \in I^k$.

We say that $\operatorname{Proj} R$ is the result of a *monoidal transformation with center in I* applied to $\operatorname{Spec} R$. $\square$

2.4.5. Two essential distinctions between affine and projective spectra. 1) Not every homomorphism of graded rings $f\colon R \longrightarrow R'$, not even homogeneous one, induces a map $\operatorname{Proj} R' \longrightarrow \operatorname{Proj} R$. For example, consider a monomorphism

$$K[T_0, T_1] \longrightarrow K[T_0, T_1, T_2], \qquad T_i \mapsto T_i. \tag{2.49}$$

Then, in $\operatorname{Proj}\ K[T_0, T_1]$, the ideal $(T_0, T_1) \subset K[T_0, T_1, T_2]$ has no pre-image.

The corresponding geometric picture is the following one: The projection of the plane onto the straight line given by the formula $(t_0, t_1, t_2) \mapsto (t_0, t_1)$ is not defined at $(0, 0, 1)$ since point $(0, 0)$ does not exist on the projective line.

2) We have established a one-to-one correspondence between rings and affine schemes: From A we recover the scheme $(\operatorname{Spec} A, \widetilde{A})$ and from an affine scheme $(X, \mathcal{O}_X)$ we recover the ring of global functions: $A = \Gamma(X, \mathcal{O}_X)$. The analogue of this statement fails for projective spectra: It may very well happen that $\operatorname{Proj} R_1 \simeq \operatorname{Proj} R_2$ for quite different rings R_1 and R_2. (Examples of such phenomena are given in the next subsection.)

2.4.6. Properties of R which reflect certain properties of $\operatorname{Proj} R$. A very extravagant from an algebraic point of view relation

$$R_1 \sim R_2 \Longleftrightarrow \operatorname{Proj} R_1 \cong \operatorname{Proj} R_2 \tag{2.50}$$

adds geometric flavor to our algebra.

Here are two ways to vary R while preserving $X = \operatorname{Proj} R$:

1) For any $\mathbb{Z}$-graded ring R, define its *d-th Veronese ring* $R^{(d)}$, where $d \in \mathbb{N}$, by setting $(R^{(d)})_i = R_{di}$.
2) Take $R \subset R'$ such that $R_i = R'_i$ for all $i \geq i_0$.

Lemma. a) $\operatorname{Proj} R' \cong \operatorname{Proj} R$.
b) $\operatorname{Proj} R^{(d)} \cong \operatorname{Proj} R$.

Proof. a) Indeed, $\operatorname{Proj} R = \bigcup\limits_{\deg f \geq i_0} D_+(f) = \bigcup\limits_{\deg f \geq i_0} \operatorname{Spec}(R_f)_0$ and, besides, any element of $(R_f)_0$ can be represented in the form g/f^n with $\deg g \geq i_0$ (multiply both g and f^n by a sufficiently high power of f).

b) Define the homeomorphism $\operatorname{Proj} R \cong \operatorname{Proj} R^{(d)}$ by setting $p \mapsto p \cap R^{(d)}$. For the sheaves, see the argument from the proof of a). □

2.5. Algebraic invariants of graded rings

Unless otherwise stated we will only consider the following simplest case of $\mathbb{Z}$-graded rings R:

1) $R_i = 0$ for $i < 0$ and $R_0 = K$ is a field;
2) R_1 is a finite dimensional space over K;
3) R_1 generates the K-algebra R.

The space $\operatorname{Proj} R$ for such rings R is most close to the classical notion of a projective algebraic variety over K. In particular, if $\dim_K R_1 = r+1$, then the epimorphism of $\mathbb{Z}$-graded rings $K[T_0, \dots, T_r] \longrightarrow R$ sending $T_0, \dots, T_r$ into a K-basis of R_1 determines an embedding

$$\operatorname{Proj} R \hookrightarrow \operatorname{Proj} K[T_0, \dots, T_r] = \mathbb{P}^r_K.$$

We will introduce certain invariants of R following a simple and beautiful idea of Hilbert to study $h_r(n) = \dim_K R_n$ as a function of n.

2.5.1. $\dim_K R_n$ is a polynomial with rational coefficients for some $n \geq n_0 = n_0(R)$. We will prove a more general statement making use of the following notions. An R-module M is said to be a $\mathbb{Z}$-*graded* one if $M = \bigoplus_{i \in \mathbb{Z}} M_i$ and $R_i M_j \subset M_{i+j}$. A homomorphism $f\colon M \longrightarrow N$ of $\mathbb{Z}$-graded modules is called *homogeneous of degree* d if $f(M_i) \subset N_{i+d}$.

2.5.2. Theorem. *Let M be a $\mathbb{Z}$-graded R-module with finitely many generators. Then* $\dim_K M_n = h_M(n)$ *for* $n \geq n_0 = n_0(M)$*, where* $h_M(n)$ *is a polynomial with rational coefficients.*

Proof. Induction on $\dim_K R_1 = r$.

For $r = 0$, we have $R = K$ and M is a usual finite dimensional linear space. Clearly, in this case, for a sufficiently large n (greater than the maximal degree of generators of M), we have $\dim_K M_n = 0$, and the desired polynomial is zero.

The induction step: Let the statement hold for $\dim_K R_1 \leq r - 1$. Let x be a nonzero element from R_1. Then the action $l_x\colon M \longrightarrow M$ of x on M is a homomorphism of degree 1. Consider an exact sequence, where K_n and C_n are the degree n homogeneous components of the kernel and cokernel of x, respectively:

$$0 \longrightarrow K_n \longrightarrow M_n \xrightarrow{x} M_{n+1} \longrightarrow C_{n+1} \longrightarrow 0 \tag{2.51}$$

Clearly, $K = \operatorname{Ker} x = \oplus K_n$ and $C = \operatorname{Coker}\ x = \bigoplus_n C_n$ are $\mathbb{Z}$-graded $R/(x)$-modules. We have

$$\dim(R/(x))_1 = \dim R_1/Kx = \dim R_1 - 1. \tag{2.52}$$

By the induction hypothesis

$$h_M(n+1) - h_M(n) = h_C(n+1) - h_K(n) = h'(n). \tag{2.53}$$

Summing up the identities (2.53) starting from some $n = n_0$ we get the result desired if we take into account the following elementary result: The sum $\sum\limits_{n=n_0}^{N} n^i$, considered as the function of the upper limit of summation, is a polynomial in N (of degree $i+1$). $\square$

The polynomial $h_M(n)$ is called the *Hilbert polynomial* of the R-module M. In particular, for $M = R$, we get the *Hilbert polynomial* of R.

The number $\deg h_R(n)$ is called the *dimension* of $X = \operatorname{Proj} R$. (We do not claim yet that it only depends on X, not on R.)

2.5.3. Lemma. *Let $h(x) \in \mathbb{Q}[x]$ take integer values at integer x. Then*

$$h(x) = \sum_{i\geq 0} a_i \frac{x(x-1)\dots(x-i+1)}{i!}, \quad \textit{where } a_i \in \mathbb{Z}. \tag{2.54}$$

Proof. Every polynomial from $\mathbb{Q}[x]$ can be represented in the form

$$h(x) = \sum_{i\geq 0} a_i \frac{x(x-1)\dots(x-i+1)}{i!}, \tag{2.55}$$

where $a_i \in \mathbb{Q}$ and where $a_0 = h(0) \in \mathbb{Z}$. By induction on i we get

$$h(i) = a_i + \sum_{j\leq i-1} a_j \frac{i\dots(i-j+1)}{i!} \tag{2.56}$$

implying $a_i \in \mathbb{Z}$. □

2.5.4. Hilbert polynomial of the projective space. Let us apply Theorems 2.7.4 and 2.7.5 to calculation of the Hilbert polynomial for the projective space over a field and prove that it does not depend on the representation $\mathbb{P}^r_K = \operatorname{Proj} R$.

Let $\mathbb{P}^r_K = \operatorname{Proj} R$; and let temporarily $\mathcal{O}_r(1)$ denote the invertible sheaf on $\mathbb{P}^r_K$ constructed with the help of $K[T_0, \dots, T_r]$, i.e., from the standard representation $\mathbb{P}^r_K$ as $\operatorname{Proj} K[T_0, \dots, T_r]$. Then by Theorem 2.7.5 we have $\mathcal{O}_r(1) \simeq \mathcal{O}(d)$ for some $d \in \mathbb{Z}$.

For $r \geq 1$, we have $d > 0$, since the rank of the space of sections of $\mathcal{O}_r(n)$ grows as $n \longrightarrow \infty$.

On the other hand, due to a (yet not proved!) part of Theorem 2.7.4 for sufficiently large n the map

$$\alpha_n \colon R_n \longrightarrow \Gamma(\mathbb{P}^r_K, \mathcal{O}_r(n)) = \Gamma(\mathbb{P}^r_K, \mathcal{O}(nd)) \tag{2.57}$$

is an isomorphism, and therefore

$$h_r(n) = \binom{nd+r}{r}. \tag{2.58}$$

In particular, the degree and the constant term of the Hilbert polynomial do not depend on R.

The highest coefficient of the polynomial h_r is called the *degree of the projective spectrum* $\operatorname{Proj} R$ and $\deg h_r$ is called the *dimension* of $X = \operatorname{Proj} R$.

The constant term $h_R(0)$ is called the *characteristic* of $X = \operatorname{Proj} R$ and is denoted by $\chi(X)$.

The *arithmetic genus* of X is $p_a(X) = (-1)^{\dim X}(\chi(X) - 1)$.

Exercise. $\dim \mathbb{P}^r_K = r,\ \deg \mathbb{P}^r_K = 1,\ p_a(\mathbb{P}^r_K) = 0.$

Remark. The degree is the most important projective invariant which together with the dimension participate in the formulation of the Bezout theorem. The geometric meaning of other coefficients is unknown except for the following result.

R. Hartshorn proved a beautiful theorem according to which the Hilbert polynomial is the only "discrete" projective invariant in the following sense:

Two projective schemes X and Y have the same Hilbert polynomials if and only if they can be "algebraically deformed" into each other inside a given projective space.

2.5.5. Properties of the degree. If h_r vanishes identically, set $\deg h_r = -1$.

2.5.5a. Lemma. *The following statements are equivalent:*

1) $\dim X = -1$*;*
2) $X = \emptyset$*;*
3) $R_1 \subset \mathfrak{n}(R)$*, where* $\mathfrak{n}(R)$ *is the nilradical of* R*.*

Proof. 1) $\Longrightarrow$ 3) is obvious since in this case $R_1^n = 0$ for $n \geq n_0$.

3) $\Longrightarrow$ 1) follows from the fact that $\dim_K R_1 < \infty$ and R is generated by R_1.

3) $\Longrightarrow$ 2) follows from the equalities $X = \bigcup\limits_{f\in R_+} D_+(f) = \bigcup\limits_{f\in R_+} \operatorname{Spec}(R_f)_0$ and the fact that $\operatorname{Spec}(R_f)_0 = \emptyset$, since the localization with respect to any multiplicative system containing a nilpotent element is 0.

2) $\Longrightarrow$ 3) is obvious since in this case $\operatorname{Spec}(R_f)_0 = \emptyset$ for any $f \in R_+$ which means exactly that $(R_f)_0 = 0$, i.e., $f^n = 0$ for some n and every f. $\square$

2.5.5b. Lemma. *Let R be an arbitrary $\mathbb{Z}$-graded ring, and $\{f_i\}_{i\in I}$ a collection of its elements. Then the following statements are equivalent:*

1) $X = \bigcup\limits_i D_+(f_i)$*;*
2) $g^n \in \sum\limits_i Rf_i$ *for any* $g \in R_+$ *and some* n*.*

Proof. Let $X = \bigcup\limits_i D_+(f_i)$ and let $g \in R_+$ be a homogeneous element. Then $D_+(g) = \bigcup\limits_i D_+(f_i g)$ by definition of D_+. Furthermore,

$$\operatorname{Spec}(R_g)_0 = \bigcup_i \operatorname{Spec}((R_g)_0) f_i^d/g^l \tag{2.59}$$

implying

$$\begin{aligned} 1/1 &= \textstyle\sum (f_i^d/g^l)a_i, \text{ where } a_i \in (R_g)_0;\\ g^n &= \textstyle\sum f_i^{m_i} b_i, \qquad \text{where } b_i \in R. \end{aligned} \tag{2.60}$$

Since all these arguments are reversible, 1) and 2) are equivalent. $\square$

2.5.5b.i. Corollary. *If R is generated by R_1 as an R_0-algebra, then, for any system of generators $\{f_i\}_{i\in I}$ of the R_0-module R_1, we have* $\operatorname{Proj} R = \bigcup D_+(f_i)$.

2.5.5c. Proposition. *Let R satisfy the condition formulated at the beginning of this section.*

1) *The following conditions are equivalent:*

$$\dim \operatorname{Proj} R = 0 \Longleftrightarrow \operatorname{Proj} R \text{ is finite.} \tag{2.61}$$

2) *If these conditions hold, then* $\operatorname{Proj} R = X$ *is endowed with the discrete topology, and* $\operatorname{Proj} R \cong \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$ *as schemes, where* $\Gamma(X, \mathcal{O}_X)$ *is a K-algebra of finite rank. In this case*

$$\chi(X) := \deg X = \dim_K \Gamma(X, \mathcal{O}_X). \tag{2.62}$$

Proof. First, let us show that if $\dim X = 0$, then X is finite and

$$\dim_K R_n = \dim \Gamma(X, \mathcal{O}_X) \text{ for } n \geq n_0. \tag{2.63}$$

Indeed, if $\dim X = 0$, then $\dim R_n = d \neq 0$ for $n \geq n_0$ implying that $\dim_K R_{(f)} < \infty$ for every $f \in R_1$. Otherwise there would have existed $g \in R_1$ such that g/f, $(g/f)^2$, $\dots$, $(g/f)^n$ were linearly independent over K which is impossible, since then $g^i f^{n-i}$ for $1 \leq i \leq n$ should be linearly independent in R_n for however great n, in particular, for $n > d$.

Since $\dim_K(R_f)_0 < \infty$, we deduce that $\operatorname{Spec}(R_f)_0$ is finite and discrete. Indeed, any prime ideal in $(R_f)_0$ is maximal since the quotient modulo it is a finite dimensional algebra over K without zero divisors, i.e., a field. Therefore any prime ideal of $(R_f)_0$ is minimal and since, clearly, $(R_f)_0$ is Noetherian, there are only finitely many minimal ideals.

The space $\operatorname{Proj} R$ can be covered by finitely many discrete open spaces $\operatorname{Spec}(R_f)_0$, where f runs over a K-basis of R_1, and therefore $\operatorname{Proj} R$ is finite and discrete. It follows that $\Gamma(X, \mathcal{O}_X) = \prod\limits_{x\in X} \mathcal{O}_x$ and $\dim_K \mathcal{O}_x < \infty$. This immediately implies an isomorphism, as ringed spaces:

$$X \simeq \operatorname{Spec} \Gamma(X, \mathcal{O}_X).$$

Let us show that

$$\dim_K \Gamma(X, \mathcal{O}_X) = \deg X = \dim_K R_n \text{ for any } n \geq n_0 \text{ for some } n_0.$$

First of all, there exists a homogeneous element $f \in R_+$ such that $D_+(f) = X$. Indeed, assume the contrary. Then every element from R_+ vanishes at one of the points of X. Select a minimal subset of points $Y \subset X$ such that every element from R_+ vanishes at one of the points of Y. Since the ideals — points of $\operatorname{Proj} R$ — do not contain R_+, it follows that Y contains more than one point. For every $y \in Y$, there exists $a_y \in R_+$ which does not vanish anywhere on Y except at y. Then $b_x = \prod\limits_{y\in Y\setminus\{x\}} a_y$ vanishes everywhere on Y except at x, whereas $\sum\limits_{x\in X} b_x$ does not vanish anywhere on Y. This is a contradiction.

Now, let $D_+(f) = X$; then $\Gamma(X, \mathcal{O}_X) = R_{(f)}$ and it suffices to establish that

$$\dim_K R_{(f)} = \dim_K R_n \text{ for all } n \geq n_0. \tag{2.64}$$

We have $(R_f)_0 = \bigcup R_n/f^n$, and $R_n/f^n \subset R_{n+1}/f^{n+1}$. Since $\dim_K(R_f)_0 < \infty$, we see that $(R_f)_0 = R_n/f^n$ for all $n \geq n_0$, and therefore

$$\dim_K(R_f)_0 = \dim_K R_n. \tag{2.65}$$

To prove this equality, it suffices to verify that $\dim_K \bigcup\limits_m \operatorname{Ann} f^m < \infty$; then, for large n, the map $g \mapsto g/f^n$, where $g \in R_n$, is an isomorphism.

Indeed, since $X = D_+(f)$, then, for any $g \in R_+$, there exists a k such that $g^k \in fR$. Since $\dim_K R < \infty$, we can choose k independent of g, and therefore $R_n = fR_{n-1}$ for a sufficiently large n. Since $\dim_K R_n = \dim_K R_{n-1}$, no power of f is annihilated by elements of sufficiently high degree.

To complete the proof, it remains to verify that X is infinite if $\dim X > 0$. Indeed, if $\dim X > 0$, then $\dim_K R_n \longrightarrow \infty$ as $n \longrightarrow \infty$, implying that $\dim_K(R_f)_0 = \infty$ for some $f \in R_1$. Otherwise the same arguments as above lead to a contradiction.

Therefore it suffices to show that $\operatorname{Spec} A$ is infinite for any K-algebra A with finitely many generators such that $\dim_K A = \infty$. By Noether's normalization theorem A is a finitely generated module over its subalgebra isomorphic to $K[T_1, \ldots, T_d]$; since $\dim_K A = \infty$, we have $d > 0$. Now, Theorem 1.6.5 and the fact that $\operatorname{card}(\operatorname{Spec} K[T_1, \ldots, T_d]) = \infty$ imply that $\operatorname{card}(\operatorname{Spec} A) = \infty$. □

2.5.6. Characteristic functions and Bezout's theorem. Let $h_r(n)$ be the Hilbert polynomial. Consider the generating function

$$F_r(t) = \sum_{n\geq 0} h_r(n)t^n. \tag{2.66}$$

Proposition. *$F_r(t)$ is a rational function in t;*

1) *the degree of the pole of $F_r(t)$ at $t = 1$ is equal to $1 + \dim \operatorname{Proj} R$;*

2) $\chi(\operatorname{Proj} R) = -\operatorname*{Res}\limits_{t=1} \dfrac{F_r(t)}{t} dt$;

3) $\deg \operatorname{Proj} R = \lim\limits_{t\longrightarrow 1} (t-1)^{\dim ProjR+1} F_r(t)$.

Proof. For $k \geq 1$, we have

$$\sum_{n\in\mathbb{Z}_+} n^k t^n = \left(t\frac{d}{dt}\right) \sum_{n\in\mathbb{Z}_+} n^{k-1} t^n = \ldots = \left(t\frac{d}{dt}\right)^k \frac{1}{1-t} \tag{2.67}$$

implying

$$\sum_{n\in\mathbb{Z}_+} h_r(n)t^n = h_r\left(t\frac{d}{dt}\right)\frac{1}{1-t}. \tag{2.68}$$

The induction on k shows that

$$\left(t\frac{d}{dt}\right)^k\left(\frac{1}{1-t}\right) = \frac{k!}{(1-t)^{k+1}} + \dots, \tag{2.69}$$

where dots stand for the terms with the poles of orders $\leq k$ at $t = 1$. This and definitions of deg and χ imply 1) and 3).

To prove 2), observe that

$$\operatorname*{Res}_{t=1}\Bigl(-\frac{h_R(0)}{t(1-t)}\,dt\Bigr) = h_R(0) = \chi(\operatorname{Proj} R),$$

and for $h_R(n) = n^k$, where $k > 0$, we have

$$F_R(t) = \left(t\frac{d}{dt}\right)^k\frac{1}{1-t},$$

implying

$$\operatorname*{Res}_{t=1}\frac{F_R(t)}{t}dt = \operatorname*{Res}_{t=1} d\left(\left(t\frac{d}{dt}\right)^{k-1}\frac{1}{1-t}\right) = 0.$$

□

2.5.7. Example. Let $f \in R_d$ be not a zero divisor. Given $F_r(t)$, it is easy to calculate $F_{R/fR}(t)$:

$$h_{R/fR}(n) = h_R(n) - h_R(n-d);$$

implying

$$F_{R/fR}(t) = \sum_{n=0}^{\infty}(h_R(n) - h_R(n-d))t^n = (1-t^d)F_R(t) + P(t),$$

where $P(t)$ is a polynomial. (Without assumption that f is not a zero divisor we only get a coefficient-wise inequality $F_r/_{fR}(t) \geq (1-t^d)F_r(t) + P(t)$.) In particular,

$$\begin{aligned} \dim\operatorname{Proj} R/fR &= \dim\operatorname{Proj} R - 1,\\ \chi(\operatorname{Proj}(R/fR)) &= \chi(\operatorname{Proj} R) - h_r(-d),\\ \deg\operatorname{Proj}(R/fR) &= d\deg\operatorname{Proj} R.\end{aligned} \tag{2.70}$$

The scheme $Y = \operatorname{Proj} R/fR$ can be naturally embedded into $\operatorname{Proj} R$ (as $V_+(f)$); since it is given by one equation, it is called a *hypersurface* in $\operatorname{Proj} R$.

A particular case: Let $R = K[T_0, \dots, T_r]$, where the $T_i \in R_1$, and let $\mathbb{P}^r = \operatorname{Proj} R$. The induction on r gives

$$F_{\mathbb{P}^r}(t) = \frac{1}{1-t} F_{\mathbb{P}^{r-1}}(t) = \ldots = \frac{1}{(1-t)^{r+1}} \tag{2.71}$$

implying

$$\begin{aligned} h_{\mathbb{P}^r}(n) &= \frac{1}{n!}\frac{d^n}{dt^n}\frac{1}{(1-t)^{r+1}} = \binom{n+r}{r}; \\ \dim \mathbb{P}^r &= r, \\ \deg \mathbb{P}^r &= 1, \\ \chi(\mathbb{P}^r) &= 1. \end{aligned} \tag{2.72}$$

2.5.7a. Theorem (Bezout's theorem). *Let $f_1, \ldots, f_s \in R = K[T]$, where $T = (T_0, \ldots, T_r)$, be homogeneous polynomials of degrees $d_1, \ldots, d_s$, respectively. Let $Y = \operatorname{Proj} R/(f_1, \ldots, f_s)$. Then*

$$\begin{aligned} &\dim Y \geq r - s \quad \textit{and} \deg Y \geq d_1 \ldots d_s, \\ &\textit{where} \deg Y = d_1 \ldots d_s \textit{ for} \dim Y = r - s. \end{aligned} \tag{2.73}$$

If f_{i+1} is not a zero divisor in $R/(f_1, \ldots, f_i)$ for all $i = 1, \ldots s-1$ (recall that such Y is called a *complete intersection*) *the inequalities* (2.73) *turn into equalities.*

In particular, if Y is a zero-dimensional complete intersection, i.e., $r = s$, then $\deg Y = d_1 \ldots d_s$.

Proof. Induction on s. □

2.5.7b. A geometric interpretation of complete intersections. Since, set-theoretically, we have $\operatorname{Proj} R/(f_1, \ldots, f_s) = \bigcap\limits_{1 \leq i \leq s} V_+(f_i)$, one should visualize Y as the intersection of hypersurfaces singled out by equations $f_i = 0$ in $\mathbb{P}^r_K$.

The condition "f_{i+1} is not a zero divisor in $R/(f_1, \ldots, f_i)$" geometrically means that the $(i+1)$-th hypersurface is in "general position" with the intersection of the preceding i hypersurfaces, i.e., it does not entirely contain any of the components of this intersection.

When $\dim Y = 0$, the formula

$$\deg Y = \dim_K \Gamma(Y, \mathcal{O}_Y) = \sum_{y \in Y} \dim_K \mathcal{O}_y \tag{2.74}$$

replaces the notion of "the number of intersection points multiplicities counted".

If K is algebraically closed, the multiplicity of $y \in Y$ is by definition equal to the rank of the local ring.

The term "complete intersection" is connected with the following images. In $\mathbb{P}^3$ (over $\mathbb{R}$ or $\mathbb{C}$), let f_1 determine a non-degenerate quadric, f_2 be its tangent plane at some point x. The intersection of $f_1 = 0$ and $f_2 = 0$ is the set of two straight lines through x (the generators of the hyperboloid). These two straight lines constitute a complete intersection of the quadric and the plane. If we wish to single out one of them, we have to take for f_3, say, the equation of the plane through the line, and it is not difficult to see that f_3 is a zero divisor in $R/(f_1, f_2)$. This straight line is not a complete intersection inside of the quadric because a complete intersection should be of degree ≥ 2 and the degree of the straight line is 1.

2.5.7c. A geometric complete intersection. There is an interesting version of the notion of complete intersection. Let, for definiteness, $R = K[T_0, \dots, T_r]$ and $p \subset R$ a homogeneous prime ideal. The scheme $X = \operatorname{Proj} R/p$ is called a *geometric complete intersection* if there exists an ideal $p' \subset p$ such that $X' = \operatorname{Proj} R/p'$ is a complete intersection and $\mathfrak{r}(p') = p$. The latter condition means that the space of X' is the same as that of X and the only difference is in the presence of nilpotents in the structure sheaf of X'.

Is it true that any scheme of the form $X = \operatorname{Proj} R/p$ is a geometric complete intersection?

2.5.7d. Problem. The answer is unknown even for the curves in the three-dimensional space: Is it possible to define any irreducible curve by two equations? [2)]

If $\dim X = r - 1$, where $X \subset \mathbb{P}^r$, the answer is positive:

Proposition. *If $X \subset \mathbb{P}^r$ and the dimension of every irreducible component of X is $r-1$, then X is a geometric complete intersection, i.e., X is given by one equation.*

Proof. It suffices to consider the case where X is irreducible.

Let $X = \operatorname{Proj} K[T_0, \dots, T_r]/p$, where p is a prime ideal, and $f \in P$ an irreducible homogeneous element (obviously, it always exists). Then

$$R = K[T_0, \dots, T_r]/(f), \quad \text{i.e., } p = fK[T_0, \dots, T_r]\ . \tag{2.75}$$

Indeed, there exists a natural epimorphism

$$K[T_0, \dots, T_r]/(f) \longrightarrow K[T_0, \dots, T_r]/p. \tag{2.76}$$

If it had a nontrivial kernel, then, since f is irreducible, we would have had

$$\dim K[T_0, \dots, T_r]/p < \dim K[T_0, \dots, T_r]/(f) \tag{2.77}$$

in contradiction with $\dim X = r - 1$. □

The above proposition implies, in particular, the following description of points in $\mathbb{P}^r_K$. They are of three types:

a) The generic point of $\mathbb{P}^r_K$ corresponding to the zero ideal of $K[T_0, T_1, T_2]$.

[2] To define the curve by 3 equations is relatively easy: Ex. 9 in §6 of [Sh0], v.1.

b) The generic points of irreducible "curves", one-dimensional irreducible sets. They are in one-to-one correspondence with irreducible forms in three indeterminates (up to a nonzero constant factor).

c) Closed points. As follows from Hilbert's Nullstellensatz, if K is algebraically closed, such points are in one-to-one correspondence with nonzero triples $(t_1 : t_2 : t_3)$ of elements from K determined up to a nonzero constant factor.

Let f_1, f_2 be two forms; then f_2 is not a zero divisor in R/f_1R if and only if f_1 and f_2 are coprime, i.e., if and only if the curves $f_1 = 0$ and $f_2 = 0$ have no common irreducible components.

Comparing the results of sec. 2.5.5c and sec. 2.5.7a we get the classical Bezout's theorem for $\mathbb{P}^2$:

Theorem. *If two curves on $\mathbb{P}^2$ have no common irreducible components, then the number of their intersection points (multiplicities counted) is equal to the product of their degrees.*

2.6. The presheaves and sheaves of modules

The sheaves of modules over schemes arise naturally in algebraic geometry as a generalization of the notion "module over a commutative ring"; a more accurate analysis of this correspondence leads to distinguishing of *quasi-coherent* sheaves of modules. As we will show, over $\operatorname{Spec} A$ the quasi-coherent sheaves are indeed in one-to-one correspondence with A-modules.

From the geometric point of view the sheaves of modules over a ringed space X embody the intuitive notion of a continuous system of linear spaces parameterized by X. If the sheaf is isomorphic to the direct sum of finitely many copies of $\mathcal{O}_X$, then this system is "constant" and the total space of the corresponding bundle is the direct product of X by the fiber. If the sheaf is locally isomorphic to such a direct sum, then it corresponds to a locally trivial vector bundle. In the general case even the dimensions of the fibers may vary.

2.6.1. Presheaves and sheaves of modules over a ringed space $(X, \mathcal{O}_X)$. Here we describe some main notions and results of the sheaf theory which do not depend on the assumption that X is a scheme; owing to their generality they are not deep.

A *presheaf of modules over* $(X, \mathcal{O}_X)$, or a *presheaf of* $\mathcal{O}_X$*-modules*, is a presheaf $\mathcal{P}$ of Abelian groups such that a $\Gamma(U, \mathcal{O}_X)$-module structure is given on each group $\Gamma(U, \mathcal{P})$ and these structures are compatible with restrictions:

$$r^U_V(sp) = r^U_V(s) r^U_V(p) \quad \text{for any } U \supset V,\ p \in \mathcal{P}(V),\ s \in \mathcal{O}_X(V). \tag{2.78}$$

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two presheaves of modules over $(X, \mathcal{O}_X)$. A *presheaf morphism* $f\colon \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ is a set of $\Gamma(U, \mathcal{O}_X)$-module homomorphisms

$f(U)\colon \mathcal{P}_1(U) \longrightarrow \mathcal{P}_2(U)$ commuting with restriction maps given for any open set $U \subset X$.

It is not difficult to verify that *the presheaves of modules over* $(X, \mathcal{O}_X)$ *constitute an Abelian category.* In particular, for any presheaf morphism $f\colon \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ both presheaves $\mathcal{K}er f$ and $\mathcal{C}oker f$ exist and are described by their groups of sections calculated locally:

$$(\mathcal{K}er f)(U) = \mathcal{K}er f(U), \qquad (\mathcal{C}oker f)(U) = \mathcal{C}oker f(U) \tag{2.79}$$

(with obviously determined restriction maps).

Given two presheaves of $\mathcal{O}_X$-modules $\mathcal{P}_1$ and $\mathcal{P}_2$, define their *tensor product* $\mathcal{P}_1 \otimes_{\mathcal{O}_X} \mathcal{P}_2$ setting:

$$(\mathcal{P}_1 \otimes_{\mathcal{O}_X} \mathcal{P}_2)(U) = \mathcal{P}_1(U) \otimes_{\mathcal{O}_X(U)} \mathcal{P}_2(U) \tag{2.80}$$

(with obviously defined restriction maps).

Given a set I, define the *direct sum* (*direct product* if $|I| \geq |\mathbb{Z}|$) of $|I|$ copies of a presheaf $\mathcal{P}$ by setting

$$\mathcal{P}^{(I)}(U) = \prod_{i \in I} \mathcal{P}_i(U), \quad \text{where } \mathcal{P}_i \text{ is the } i\text{th copy of } \mathcal{P}. \tag{2.81}$$

The *direct sum* (product) of distinct presheaves is obviously defined. If $|I| = n \in \mathbb{N}$, we write $\mathcal{O}_x^{(n)}$ instead of $\mathcal{O}_x^{(I)}$.

A presheaf of $\mathcal{O}_X$-modules $\mathcal{P}$ is called a *sheaf of* $\mathcal{O}_X$*-modules* if $\mathcal{P}$ is a sheaf. If $\mathcal{P}$ is a presheaf of $\mathcal{O}_X$-modules and $\mathcal{P}^+$ the associated sheaf, then $\mathcal{P}^+$ also acquires the natural structure of a sheaf of $\mathcal{O}_X$-modules: Originally, $\mathcal{P}^+(U)$ are only determined as Abelian groups; however, the multiplication by the sections of $\mathcal{O}_X$ commutes with both the limits from the definition of the sheaf $\mathcal{P}^+$.

There is determined a canonical morphism of presheaves of modules $\mathcal{P} \longrightarrow \mathcal{P}^+$ because the image of $\mathcal{P}(U)$ under the homomorphism $\mathcal{P}(U) \longrightarrow \prod_{x \in U} \mathcal{P}_x$ belongs to $\mathcal{P}^+(U)$.

Every sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ can be considered as a presheaf; the presheaf obtained in this way from a sheaf $\mathcal{F}$ will be denoted by $i(\mathcal{F})$. Defining a sheaf morphism $\mathcal{F}_1 \longrightarrow \mathcal{F}_2$ as a morphism of corresponding presheaves we can consider i as a functor which embeds the category of sheaves of $\mathcal{O}_X$-modules into the category of presheaves. This "tautological" functor is associated with a far less trivial functor $\mathcal{P} \mapsto \mathcal{P}^+$ acting in the opposite direction as follows: For any presheaf $\mathcal{P}$ and any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, we have a natural isomorphism

$$\operatorname{Hom}(i(\mathcal{F}), \mathcal{P}) \xrightarrow{\sim} \operatorname{Hom}(\mathcal{F}, \mathcal{P}^+) \tag{2.82}$$

which to each element $i(\mathcal{F}) \longrightarrow \mathcal{P}$ from the left-hand side group assigns the through map $i(\mathcal{F}) \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}^+$ from the right-hand side group (the presheaf morphism $\mathcal{P} \longrightarrow \mathcal{P}^+$ is described in sec. 2.1.5).

Therefore i and $+$ are adjoint functors.

By means of i and $+$ we can define the tensor operators over sheaves of modules. The general rule is as follows: Perform the corresponding operation over presheaves and make the result into a sheaf with the help of $+$. In particular, given sheaves of $\mathcal{O}_X$-modules $\mathcal{F}_1$, $\mathcal{F}_2$ we define:

$$\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2 = (i(\mathcal{F}_1) \otimes_{\mathcal{O}_X} i(\mathcal{F}_2))^+, \tag{2.83}$$

and, for any sheaf morphism $f\colon \mathcal{F}_1 \longrightarrow \mathcal{F}_2$, we set

$$\mathcal{K}er f = (\mathcal{K}er i(f))^+ \quad \text{and} \quad \mathcal{C}oker f = (\mathcal{C}oker(f))^+. \tag{2.84}$$

Actually, as is not difficult to show, $\operatorname{Ker} i(f)$ is automatically a sheaf, whereas for $\operatorname{Coker} i(f)$ this is not true, as is shown in Example 2.6.2 below. Somewhat later we will encounter examples showing that $i(\mathcal{F}_1 \otimes_{\mathcal{O}_X} i(\mathcal{F}_2)$ is not a sheaf either; these examples demonstrate the importance of the functor $+$.

Statement. *The category of sheaves of modules is Abelian.*

2.6.2. Quasi-coherent sheaves. The category of all the sheaves of modules is usually too large. In what follows we will use two notions which distinguish a class of important sheaves needed in what follows: Quasi-coherent and coherent sheaves. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is said to be *quasi-coherent* if it is locally isomorphic to the cokernel of a homomorphism of free sheaves.

More precisely, $\mathcal{F}$ is *quasi-coherent* if, for every $x \in X$, there exists a neighborhood $x \in U$, two sets of indices I and J and a homomorphism of sheaves of $\mathcal{O}_X|_U$-modules $f\colon \mathcal{O}_X^{(I)}|_U \longrightarrow \mathcal{O}_X^{(J)}|_U$ such that $\mathcal{F}|_U \simeq \mathcal{C}oker f$.

To elucidate the meaning of the quasi-coherent property, recall that the sheaves $\mathcal{O}_X^{(I)}$ correspond to "trivial bundles". The property to be isomorphic to the cokernel of a morphism of trivial bundles is a continuity-type condition: The jumps of the fibers should not be "too local", they should mirror a global picture over open sets.

2.6.2a. Example. Consider a case where X has the simplest non-trivial structure: $X = \operatorname{Spec} \mathbb{Z}_p$. Then $X = \{(0), (p)\}$, and the open sets are just $X, \{(0)\}$ and $\emptyset$ with the structure sheaf described by the following diagram, where $\mathbb{Z}_p \longrightarrow \mathbb{Q}_p$ is the natural embedding into the field of quotients:

$$\begin{array}{ccc} \mathit{\Gamma}(X, \mathcal{O}_X) & = & \mathbb{Z}_p \\ \big\downarrow & & \big\downarrow \\ \mathit{\Gamma}(\{(0)\}, \mathcal{O}_X) & = & \mathbb{Q}_p \end{array} \tag{2.85}$$

Any presheaf of modules over $(X, \mathcal{O}_X)$ is determined by a $\mathbb{Z}_p$-module F_1, $\mathbb{Q}_p$-module F_2 and a $\mathbb{Z}_p$-homomorphism $F_1 \longrightarrow F_2$; moreover, every presheaf is a sheaf.

Since the only open neighborhood of the point (p) is X, the sheaf $\mathcal{F} = (F_1, F_2)$ is quasi-coherent if and only if there is an exact sequence of the form $\mathcal{O}_X^{(I)} \longrightarrow \mathcal{O}_X^{(J)} \longrightarrow \mathcal{F} \longrightarrow 0$. In terms of (F_1, F_2) this can be expressed as two exact sequences forming a commutative diagram:

$$\begin{tikzcd} \mathbb{Z}_p^{(I)} \arrow[r] \arrow[d] & \mathbb{Z}_p^{(J)} \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & 0 \\ \mathbb{Q}_p^{(I)} \arrow[r] & \mathbb{Q}_p^{(J)} \arrow[r] & F_2 \arrow[r] & 0 \end{tikzcd} \tag{2.86}$$

This immediately implies that $F_2 \simeq F_1 \bigotimes\limits_{\mathbb{Z}_p} \mathbb{Q}_p$. It is not difficult to see that this condition is also sufficient for quasi-coherentness of $\mathcal{F} = (F_1, F_2)$.

Thus, a quasi-coherent sheaf in this case is uniquely determined by the module of global sections $F_1 = \Gamma(X, \mathcal{O}_X)$; while F_2 and the homomorphism $F_1 \longrightarrow F_2$ are recovered from F_1. Without this quasi-coherentness condition we have a greater freedom in defining both F_2 and the homomorphism $F_1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow F_2$: Now the sheaf may suffer a jump at a generic point as compared with the quasi-coherent case.

In sec. 2.6.4 the result of this example will be generalized to general affine schemes.

2.6.3. Coherent sheaves. A restriction of "finite type" distinguishes *coherent* sheaves from quasi-coherent ones.

A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is said to be a *sheaf of finite type* if it is locally isomorphic to the image of $\mathcal{O}_X^{(n)}$ for some n (in other words, for every $x \in X$, there exists an open neighborhood $\mathbb{U} \ni x$ and a sheaf epimorphism $\mathcal{O}_X^{(n)}|_U \longrightarrow \mathcal{F}|_U \longrightarrow 0$). A sheaf of $\mathcal{O}_X$-modules is said to be *coherent* if it is of finite type and, for every open U and every sheaf morphism

$$\varphi\colon \mathcal{O}_X^{(n)}|_U \longrightarrow \mathcal{F}|_U \longrightarrow 0, \tag{2.87}$$

the sheaf $\mathcal{K}er\varphi$ is of finite type.

The general properties of coherent sheaves were first derived in Serre's thesis. We confine ourselves to listing them, cf. [KaS]:

a) *A subsheaf of finite type of a coherent sheaf is coherent.*

b) *If, in an exact sequence of sheaves* $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$, *any two of the three sheaves are coherent, then the third one is also coherent. In particular, the direct sum of coherent sheaves and also the kernel, cokernel and the image of any morphism of coherent sheaves are coherent.*

c) *The tensor product of coherent sheaves is coherent.*

d) *If the structure sheaf* $\mathcal{O}_X$ *is coherent, then a sheaf of* $\mathcal{O}_X$*-modules* $\mathcal{F}$ *is coherent if and only if it is locally isomorphic to the cokernel of a morphism of the form* $\mathcal{O}_X^{(p)} \longrightarrow \mathcal{O}_X^{(q)}$. □

2.6.4. Quasi-coherent sheaves over affine schemes. Since the notion of quasi-coherentness is local, it suffices to describe quasi-coherent sheaves over affine schemes. Let A be a ring and $(X, \mathcal{O}_X) = \operatorname{Spec} A$. Let M be an A-module. Our aim is to construct a sheaf $\widetilde{M}$ for which M is the module of global sections.

Let us proceed as in sec 2.2.3: For any multiplicative system $S \subset A$, define the localization M_S (sometimes denoted by $M[S^{-1}]$) to be the set $\{(m,s) \mid m \in M,\ s \in S\}/R$, where the relation R is given by the formula

$$(m,s) \sim (m',s') \iff t(s'm - sm') = 0 \text{ for some } t \in S. \tag{2.88}$$

The A-module M_S is naturally endowed with an A_S-module structure such that

$$\begin{gathered}\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{s_2 m_1 + s_1 m_2}{s_1 s_2}; \quad \frac{a}{s}\frac{m}{t} = \frac{am}{st}\\ \text{for any } s_i \in S,\ a \in A,\ m_i, m \in M.\end{gathered} \tag{2.89}$$

Lemma (Cf. 1.6.4b)**.** *$M_S \cong A_S \otimes_A M$, in particular, for every multiplicative subsystem $T \subset S$, there is defined a natural homomorphism $M_T \longrightarrow M_S$.*

For the natural homomorphism $f\colon M \longrightarrow M_S$, where $f(m) = m/1$, we have

$$\mathcal{K}er F = \{m \in M \mid sm = 0 \textit{ for some } s \in S\}. \tag{2.90}$$

For any A-module homomorphism $f\colon M \longrightarrow N$, the map $f_S\colon M_S \longrightarrow N_S$ given by the formula $f_S(m/s) = f(m)/s$ is an A_S-module homomorphism.

Proof is easy, we will only give one hint: define the isomorphism

$$\begin{aligned}&i : A_S \otimes_A M \cong M_S\\ &i\left(\frac{a}{s} \otimes m\right) = \frac{am}{s}. \quad \square\end{aligned} \tag{2.91}$$

Set $M_x = M_{A \setminus \mathfrak{p}_x}$ and define

$$m^x = (\dots, m_x, \dots) \in \prod_{x \in U} M_x = \Gamma(U, \widetilde{M}) \tag{2.92}$$

as an element such that, for every $x \in U$, there exists a neighborhood of the form $D(f)$ such that, for every $y \in D(f)$, the y-th component of m is the image of some $y \in M_f$ under the morphism $M_f \longrightarrow M_y$.

For example, $\widetilde{A} \simeq \mathcal{O}_X$.

2.6.5. Theorem. *$\Gamma(D(f), \widetilde{M}) \simeq M_f$, and the stalk of M over a point x is isomorphic to M_x.*

Hint. . Define the isomorphism $\varphi\colon M_f \longrightarrow \Gamma(D(f), \widetilde{M})$ setting

$$\varphi(m/f) = (\dots, m/f, \dots, m/f, \dots) \in \prod_{x \in D(f)} M_x. \quad \square \tag{2.93}$$

The natural map $M \mapsto \widetilde{M}$ from the category of A-modules to the category of sheaves of $\mathcal{O}_X$-modules is, actually, a functor. Indeed, to every morphism $f\colon M \longrightarrow N$ a morphism $\tilde{f}\colon \widetilde{M} \longrightarrow \widetilde{N}$ corresponds which over each $D(f)$ is just the localization. The equality $\widetilde{fg} = \tilde{f}\tilde{g}$ is directly verified. $\square$

Proposition. *For any exact sequence of A-modules $M \xrightarrow{f} N \xrightarrow{g} P$, the sequence of sheaves $\widetilde{M} \xrightarrow{f} \widetilde{N} \xrightarrow{g} \widetilde{P}$ is exact.*

Proof. It suffices to verify the statement "stalk-wise" and apply Lemma 2.6.4. $\square$

2.6.6. Proposition. a) $M \cong \Gamma(\operatorname{Spec} A, \widetilde{M})$. (This means that not only $\widetilde{M}$ is recovered from M, but also M is uniquely recovered from $\widetilde{M}$.)

b) $\operatorname{Hom}_A(M, N) \cong \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$.

Proof. a) is a particular case of b). To prove b), observe that the localization determines a natural map $\operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$.

On the other hand, a morphism $\widetilde{M} \longrightarrow \widetilde{N}$ is a collection of morphisms $\widetilde{M}(U) \longrightarrow \widetilde{N}(U)$ among which there is a morphism

$$M \cong \Gamma(X, \widetilde{M}) \longrightarrow \Gamma(X, \widetilde{N}) \cong N. \tag{2.94}$$

This determines a map $\operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \longrightarrow \operatorname{Hom}_A(M, N)$.

The verification of the fact that the constructed maps are mutually inverse is trivial. $\square$

Theorem. *A sheaf $\mathcal{F}$ over $X = \operatorname{Spec} A$ is quasi-coherent if and only if $\mathcal{F} = \widetilde{M}$ for some A-module M.*

Proof. a) Let us prove that if $\mathcal{F} = \widetilde{M}$, then $\mathcal{F}$ is quasi-coherent. Let us represent M as a cokernel of a free A-module morphism:

$$A^{(I)} \longrightarrow A^{(J)} \longrightarrow M \longrightarrow 0. \tag{2.95}$$

This gives the exact sequence of sheaves $\widetilde{A}^{(I)} \longrightarrow \widetilde{A}^{(J)} \longrightarrow \widetilde{M} \longrightarrow 0$ which implies the quasi-coherentness of $\widetilde{M}$ since $\widetilde{A} = \mathcal{O}_X$.

b) Let $\mathcal{F}$ be quasi-coherent. Every point has a neighborhood of the form $D(f)$ over which $\mathcal{F}$ is isomorphic to the cokernel of a free sheaf morphism. Let $\{D(f_i) \mid 1 \leq i \leq n\}$ be an open cover of $X = \operatorname{Spec} A$ by such neighborhoods, and let

$$\begin{aligned}\mathcal{F}|_{D(f_i)} &= \operatorname{Coker}\left(\mathcal{O}_X^{(I)}|_{\mathcal{D}(f_i)} \longrightarrow \mathcal{O}_X^{(J)}|_{D(f_i)}\right)\\ &= \operatorname{Coker}\left(\widetilde{A}_{f_i}^{(I)} \longrightarrow \widetilde{A}_{f_j}^{(J)}\right) \simeq \operatorname{Coker}\left(A_{f_i}^{(I)} \longrightarrow A_{f_i}^{(J)}\right) = \widetilde{M}_i,\end{aligned} \tag{2.96}$$

where M_i is an A_{f_i}-module. Notice that M_i can be also considered as an A-module (thanks to the localization homomorphism $A \longrightarrow A_{f_i}$). Further on, let $M_{ij} = (M_i)_{f_j/1} = \Gamma(D(f_i f_j), \mathcal{F})$, where $i, j = 1, \ldots, n$. Now set

$$M = \operatorname{Ker}(\varphi\colon \prod_{1\le i\le n} M_i \longrightarrow \prod_{1\le i,j\le n} M_{ij}), \tag{2.97}$$

where φ is given by the formula

$$\varphi((\dots, m_i, \dots)) = (\dots, m_{ij}, \dots), \quad \text{where } m_{ij} = m_i/1 - m_j/1. \tag{2.98}$$

Let us prove that $\widetilde{M} \simeq \mathcal{F}$. It suffices to verify that

$$\Gamma(D(g), \mathcal{F}) = \Gamma(D(g), \widetilde{M}) \text{ for any } g \in A, \tag{2.99}$$

and this case easily reduces to $g = 1$ by replacing A by A_g and localizing all the modules with respect to $\{g^n \mid n \in \mathbb{Z}_+\}$.

Therefore it suffices to show that $\Gamma(\operatorname{Spec} A, \mathcal{F}) \simeq M$; but by definition of a sheaf

$$\Gamma(\operatorname{Spec} A, \mathcal{F}) = \operatorname{Ker}(\prod_{1\le i\le n} \Gamma(D(f_i), \mathcal{F}) \longrightarrow \prod_{1\le i,j\le n} \Gamma(D(f_i\, f_j), \mathcal{F})) \tag{2.100}$$

and by definition of M we have $\widetilde{M}|_{D(f_i)} = \widetilde{M_i}$ and $\widetilde{M_{ij}}|_{D(f_i\, f_j)} = \widetilde{M_{ij}}$. It only remains to apply Theorem 2.6.5 and the definition of M. □

2.6.7. Example. Let $(X, \mathcal{O}_X)$ be a scheme, $J_X \subset \mathcal{O}_X$ a quasi-coherent sheaf of ideals. The quotient sheaf $\mathcal{O}_X/J_X$ is obviously quasi-coherent. Define the *support of* $\mathcal{O}_X/J_X$ by setting

$$\operatorname{supp} \mathcal{O}_X/J_X := \{x \in X \mid \mathcal{O}_X/J_{X,x} \neq \{0\}\}. \tag{2.101}$$

2.6.8. Lemma. *If* J_X *is quasi-coherent, then* $\operatorname{supp} \mathcal{O}_X/J_X$ *is closed in* X *and the ringed space* $(\operatorname{supp} \mathcal{O}_X/J_X, \mathcal{O}_X/J_X|_{\operatorname{supp} \mathcal{O}_X/J_X})$ *is a scheme.*

Proof. Consider an affine neighborhood $X \supset U \ni x$; i.e., let $U = \operatorname{Spec} A$. Denote $J := \Gamma(U, J_X) \subset A$. Obviously, $\operatorname{supp} \mathcal{O}_X/J_X \cap U = V(F)$. □

2.7. The invertible sheaves and the Picard group

How to characterize in inner terms the projective spectra of $\mathbb{Z}$-graded rings? The question is not very precise, still, at any rate, a step towards its answer will be done in this section. We will show that the existence of a grading in a ring R enables one to define on $\operatorname{Proj} R = X$ a particular quasi-coherent sheaf $\mathcal{O}_X(1)$. We will introduce certain invariants of R and show that they actually characterize the pair $(X, \mathcal{O}_X(1))$.

2.7.1. Invertible sheaves. A sheaf of modules L over a ringed space $(X, \mathcal{O}_X)$ is said to be *invertible* if it is locally isomorphic, as a sheaf of $\mathcal{O}_X$-modules, to $\mathcal{O}_X$. The following statement is immediate:

Statement. *For any invertible sheaf $\mathcal{L}$ over $(X, \mathcal{O}_X)$, set*

$$\mathcal{L}^{-1} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X). \tag{2.102}$$

a) *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are invertible sheaves, then $\mathcal{L}_1 \oplus_{\mathcal{O}_X} \mathcal{L}_2$ is invertible.*
b) $\mathcal{L}^{-1} \oplus_{\mathcal{O}_X} \mathcal{L} \simeq \mathcal{O}_X$.

Corollary. *The isomorphism classes of invertible sheaves over $(X, \mathcal{O}_X)$ constitute a commutative group with respect to tensoring over $\mathcal{O}_X$.*

This group is called the *Picard group* and denoted by $\operatorname{Pic} X$.

2.7.2. A cohomologic description of Pic X. Let $\mathcal{L}$ be an invertible sheaf, $X = \bigcup\limits_{i\in I} U_i$ an open cover of X sufficiently fine to satisfy $\mathcal{L}|_{U_i} \simeq \mathcal{O}_X|_{U_i}$ for all i. Fix an isomorphism $\varphi_i\colon \mathcal{L}_{U_i} \longrightarrow \mathcal{O}_X|_{U_i}$ and consider the restriction maps $r_{ij}\colon \mathcal{L}|_{U_i} \longrightarrow \mathcal{L}|_{U_i\cap U_j}$ and $r_{ji}\colon \mathcal{L}|_{U_j} \longrightarrow \mathcal{L}|_{U_i\cap U_j}$.

The elements $\varphi_j^{-1}(1) = u_j \in \Gamma(U_j, \mathcal{L})$ completely determine the isomorphisms φ_j. Since $r_{ij}(U_i)$ and $r_{ji}(U_j)$ are generators of $\Gamma(U_i \cap U_j, \mathcal{L})$, the elements s_{ij} determined from the equations $r_{ij}(U_i) = s_{ij} r_{ji}(U_j)$ are invertible, i.e.,

$$s_{ij} \in (\Gamma(U_i \cap U_j, \mathcal{O}_X))^\times. \tag{2.103}$$

Let $\Gamma(U, \mathcal{O}_X^\times) = \Gamma(U, \mathcal{O}_X)^\times$. Then, to a cover of X and an invertible sheaf $\mathcal{L}$ trivial on the elements from this cover, we have assigned the set

$$\{s_{ij} \in (\Gamma(U_i \cap U_j, \mathcal{O}_X^\times) \ \text{ for } i, j \in I\}. \tag{2.104}$$

Obviously, the elements s_{ij} satisfy the following conditions:

$$\begin{aligned} s_{ij}s_{ji} &= 1 \quad \text{if } \ i \neq j \\ s_{ij}s_{jk}s_{ki} &= 1 \quad \text{if } \ i \neq j \neq k \neq i. \end{aligned} \tag{2.105}$$

All such sets (2.104) constitute a group with respect to multiplication called the *group of 1-dimensional Čech cocycles of the cover $(U_i)_{i\in I}$ with coefficients in the sheaf $\mathcal{O}_X^\times$* and denoted by $Z^1((U_i)_{i\in I}, \mathcal{O}_X^\times)$.

Two cocycles $(s_{ij}), (s'_{ij}) \in Z^1$ are said to be *equivalent* if there exist $t_i \in \mathcal{O}_X^\times|_{U_i}$ such that $s'_{ij} = t_i s_{ij} t_j^{-1}$. The elements $t_i t_j^{-1}$ obviously constitute a subgroup $B^1((U_i)_{i\in I}, \mathcal{O}_X^\times) \subset Z^1$ of *coboundaries.*

The corresponding quotient group is called the *first Čech cohomology group of X with coefficients in $\mathcal{O}^\times$* and denoted by $H^1((U_i)_{i\in I}, \mathcal{O}_X^\times)$.

The above constructed Čech cocycles for an invertible sheaf is multiplied by a coboundary if we change isomorphisms $\{\varphi_i\}$. Indeed, let $\varphi'_i\colon \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_X|_{U_i}$ be another set of isomorphisms. Since $\varphi_i {\varphi'}_i^{-1} \in \mathrm{Aut}(\mathcal{L}|_{U_i})$, we get $\varphi'_i = t_i\varphi_i$ implying $s'_{ij} = t_i s_{ij} t_j^{-1}$.

Proposition. *The above map from the set of invertible sheaves $\mathcal{L}$ on X trivial on a given cover $(U_i)_{i\in I}$ into the set of first Čech cohomology $H^1((U_i)_{i\in I}, O_X^\times)$ determined from the same cover is one-to-one.*

Exercise. Prove this.

Thus, we have obtained a group monomorphism

$$H^1((U_i)_{i\in I}, \mathcal{O}_X^\times) \longrightarrow \operatorname{Pic} X \tag{2.106}$$

whose image is the set of classes of sheaves trivial over all the U_i.

Let $(U'_j)_{j\in J}$ be a finer cover. Then a natural monomorphism

$$H^1((U_i)_{i\in I}, \mathcal{O}_X^\times) \longrightarrow H^1((U'_j)_{j\in J}, \mathcal{O}_X^\times) \tag{2.107}$$

arises (we leave the task to precisely formulate its definition to the reader). Since every invertible sheaf is trivial on elements of a sufficiently fine cover, we get

$$\operatorname{Pic} X \simeq \varinjlim H^1((U_i)_{i\in I}, \mathcal{O}_X^\times) = H^1(X, \mathcal{O}_X^\times), \tag{2.108}$$

where the inductive limit is taken with respect to an ordered system of coverings.

2.7.3. Example. On $X = \operatorname{Proj} R$, where R is generated by R_1 over R_0, consider a cover $X = \bigcup\limits_{f\in R} U_f$, where $U_f = D_+(f)$, and a cocycle $s_{fg} \in Z^1(U_f, \mathcal{O}_X^\times)$ given by

$$s_{fg} = (f/g)^n \in \Gamma(U_f \cap U_g, \mathcal{O}_X^\times). \tag{2.109}$$

The invertible sheaf determined with the help of this cocycle is denoted by $\mathcal{O}_X(n)$; obviously, we have

$$\mathcal{O}_X(n) \simeq \begin{cases} \mathcal{O}_X(1)^{\otimes n} & \text{if } n \geq 0, \\ \mathcal{O}_X(-1)^{\otimes n} & \text{if } n \leq 0, \end{cases} \tag{2.110}$$

where $\mathcal{O}_X(-1) = \mathcal{O}_X(1)^{-1}$.

These sheaves are constructed from R; the other way round, R can be recovered to an extent from X and $\mathcal{O}_X(1)$. Here we will only prove a part of the result; the second part will be proved with the help of the cohomology technique in what follows.

First, notice that, for every invertible sheaf $\mathcal{L}$ over a ringed space $(X, \mathcal{O}_X)$, there is a natural structure of a $\mathbb{Z}$-graded ring on $\bigoplus\limits_{n\in\mathbb{Z}} \Gamma(X, \mathcal{L}^n)$ with the product of homogeneous elements determined from the map

$$\Gamma(X, \mathcal{L}^n) \times \Gamma(X, \mathcal{L}^m) \longrightarrow \Gamma(X, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{L}^m) \cong \Gamma(X, \mathcal{L}^{n+m}). \tag{2.111}$$

2.7.4. Theorem. *Let $X = \operatorname{Proj} R$, where R_0 is a Noetherian ring and R_1 is a Noetherian R_0-module that generates R, and $\mathcal{L} = \mathcal{O}_X(1)$. Then there exists a homogeneous homomorphism of graded rings $\alpha\colon R \longrightarrow \bigoplus\limits_{n\in\mathbb{Z}} \Gamma(X, \mathcal{L}^n)$ and $n_0 \in \mathbb{Z}$ such that the maps $\alpha_n\colon R_n \longrightarrow \Gamma(X, \mathcal{L}^n)$ are group isomorphisms for $n \geq n_0$.*

Proof. Let us construct α and show that its kernel is only supported in small degrees. The statement on isomorphism will be proved in what follows.

Let $h \in R_n$. For any $f \in R_1$, set

$$\alpha(h)|_{D_+(f)} = \frac{h}{f^n} \in \Gamma(D_+(f), \mathcal{O}_X). \tag{2.112}$$

Obviously, the sections $\dfrac{h}{f^n}$ are glued together with the help of the cocycle $(f/g)^n$ into a section of $\mathcal{L}^n$ over the whole X; denote this section by $\alpha(h)$. Clearly, α is a homomorphism of graded rings.

Let $h \in R_n \cap \operatorname{Ker} \alpha$. This means that $\dfrac{h}{f^n} = 0$ for all $f \in R_1$. Since R_1 is Noetherian, there exists an integer m_0 such that $R_m h = 0$ for $m \geq m_0$. Consider an arbitrary h whose annihilator contains $\bigoplus\limits_{m \geq m_0(h)} R_m$. Obviously, all such elements constitute an ideal $J \subset R$.

There are finitely many generators of J since R is Noetherian, and therefore we can choose one m_0 for all generators h; so that $J_m = 0$ for $m \geq m_0$. □

2.7.5. Picard groups: Examples.

Proposition. *If A is a unique factorization ring, then* $\operatorname{Pic}(\operatorname{Spec} A) = \{0\}$.

Proof. In the system of all open coverings the finite coverings of the form $\bigcup\limits_{i \in J} D(f_i)$ constitute a *cofinal subsystem*[3)], and therefore it suffices to verify that

$$H^1(D(f_i), \mathcal{O}_X^\times) = \{0\}. \tag{2.113}$$

Let $s_{ij} \in Z^1(D(f_i), \mathcal{O}_X^\times)$. Let us represent all s_{ij} for $i \neq j$ in the form t'_i/t'_j, where the t'_i are elements from the quotient field K of A. It is easy to see that this is indeed possible: Since $s_{ij}s_{jk}s_{ki} = 1$ for any k, it follows that $s_{ij} = s_{ik}/s_{jk}$ because $s_{ik}s_{ik} = 1$.

Now let p be a prime element of A and $v_p(a)$ the exponent with which p enters the decomposition of $a \in K$. Up to multiplication by invertible elements, the set P of primes $p \in A$ such that $v_p(t'_i) \neq 0$ for some i is only finite.

Fix $p \in P$, and divide all the f_i into two groups: The one with p dividing f_i for $i \in J_1$ and the other with $f_i \not\vdots\, p$ if $i \in J_2$. Since the f_i are coprime, $J_1 \neq \emptyset$.

Since s_{ij} is invertible an $A_{f_i f_j}$, we see that $v_p(s_{ij}) = 0$ if $f_{ij} \not\vdots\, p$ and $v_p(t'_i)$ takes the same value — a_p — for all $i \in J_1$. Set

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[3] Recall that a subset B of a partially ordered set A is said to be *cofinal* if, for every $a \in A$, there exists $b \in B$ such that $a = b$.

Also, a sequence or net of elements of A is said to be *cofinal* if its image is cofinal in A.)

$$t_i = \left(\prod_{p\in P} p^{-a_p}\right) t_i' \tag{2.114}$$

Obviously, $s_{ij} = t_i/t_j$; on the other hand, $t_i \in \Gamma(D(f_i), \mathcal{O}_X) = A_f^\times$. Indeed, if $f_i \not\vdots\, p$, then $v_p(t_i) = 0$ and t_i only factorizes into the product of the prime divisors of f_i, which implies that t_i is invertible in A_{f_i}. $\square$

Remark. In terms of cohomology with coefficients in sheaves, we can interpret this proof as follows. The exact sequence of sheaves of Abelian groups on X

$$1 \longrightarrow \mathcal{O}_X^\times \longrightarrow \widetilde{K}^\times \xrightarrow{p} \widetilde{K}^\times/\mathcal{O}_X^\times \longrightarrow 1, \tag{2.115}$$

where $\widetilde{K}^\times$ is the constant sheaf (i.e., $\Gamma(U, \widetilde{K}^\times) = K^\times$ for any U), induces the exact sequence of cohomology groups

$$\Gamma(X, \widetilde{K}^\times) \xrightarrow{p_0^*} \Gamma(X, \widetilde{K}^\times/\mathcal{O}_X^\times) \longrightarrow H^1(X, \mathcal{O}_X) \xrightarrow{p_1^*} H^1(X, \widetilde{K}^\times) \tag{2.116}$$

The first step of the above proof establishes that $p_1^* = 0$ (actually, the same argument shows that $H^1(X, \widetilde{K}^\times) = 0$). The second step shows that p_0^* is an epimorphism; and it is only here that we have used the fact that A is a unique factorization ring, which, in particular, implies that $\widetilde{K}^\times/\widetilde{A}^\times \simeq \bigoplus\limits_{p\in\operatorname{Spec} A} \widetilde{\mathbb{Z}}$.

Here is an important application of the above statement:

2.7.5a. Theorem. *Let A be a unique factorization ring. Then* $\operatorname{Pic}\mathbb{P}_A^r$ *for $r \geq 1$ is an infinite cyclic group with the class of $\mathcal{O}(1)$ as its generator.*

Proof. Recall that $\mathbb{P}_A^r = \operatorname{Proj} A[T_0, \dots, T_r]$. By the above proposition, any invertible sheaf $\mathcal{L}$ over $\mathbb{P}_A^r$ is trivial on $D_+(T_i) = \operatorname{Spec} A\left[\frac{T_0}{T_i}, \dots, \frac{T_r}{T_i}\right]$, since by a theorem of Gauss (cf. [Pr]) the polynomial ring over A preserves the unique factorization property of A.

Now let $(s_{ij} \mid 0 \leq i, j \leq r)$ be a cocycle defining $\mathcal{L}$ for the cover $(D_+(T_i) \mid 0 \leq i \leq r)$. Since s_{ij} is homogeneous of degree 0 and only factorizes in the product of the divisors of T_iT_j (use the unique factorization property of $A[T_0, \dots, T_r]$), we have

$$s_{ij} = \varepsilon_{ij}\left(\frac{T_i}{T_j}\right)^{n_{ij}}, \quad \text{where } \varepsilon_{ij} \in A^\times. \tag{2.117}$$

Since $s_{ij}s_{ji} = 1$ and $s_{ij}s_{jk}s_{ki} = 1$, it follows that $n_{ij} = n$ (does not depend on i, j), and therefore ε_{ij} is a cocycle.

In actual fact, ε_{ij} is automatically a coboundary, since $\varepsilon_{ij} = \varepsilon_{ik}/\varepsilon_{jk}$ for any k and $\varepsilon_{ik} \in \Gamma(\mathbb{P}_A^r, \mathcal{O}_X^\times)$. Therefore (s_{ij}) is cohomologic to the cocycle $(T_i/T_j)^n$ defining $\mathcal{O}(n)$.

We will now get theorem's statement if we prove that all the sheaves $\mathcal{O}(n)$ are non-isomorphic. This is true for any A as shown by the following statement:

Lemma.

$$\Gamma(\mathbb{P}^r_{\mathbb{A}}, \mathcal{O}(n)) = \begin{cases} 0 & \text{if } n < 0 \\ \bigoplus\limits_{a_0+\ldots+a_r=n} AT_0^{a_0}\ldots T_r^{a_r} & \text{if } n \geq 0. \end{cases} \tag{2.118}$$

Proof. Let $R = A[T_0, \ldots, T_r]$, $\deg T_i = 1$ for all i. Let us show that the homomorphism $\alpha_n \colon R_n \longrightarrow \Gamma(\mathbb{P}^r_A, \mathcal{O}(n))$ is an isomorphism for $n \geq 0$.

Recall that $\alpha_n(f)|_{D_+(T_i)} = f/T_i^n$ for any $f \in R_n$. Now the fact that the T_i are not zero divisors immediately implies that α_n is a monomorphism.

Let us prove that α_n is an epimorphism. A section of the sheaf $\mathcal{O}^\times$ over $\mathbb{P}^r_A$ is represented by the set

$$\left\{ f_i \in A\left[\frac{T_0}{T_i}, \ldots, \frac{T_r}{T_i}\right] \mid 0 \leq i \leq n, \text{ and } f_i\left(\frac{T_i}{T_j}\right)^n = f_j \right\}. \tag{2.119}$$

Since T_i are not zero divisors, the compatibility conditions imply that $f_iT_i^n$ do not depend on i. Obviously, $f_iT_i^n$ is a polynomial since its denominator can only be a power of T_i.

Now let $n < 0$; then, in the same notation, we get

$$f_i/T_i^{-n} = f_j/T_j^{-n}, \tag{2.120}$$

and similar divisibility considerations show that this is only possible for $f_i = 0$.

□

Theorem is also proved. □

2.7.5b. Corollary. *$\mathcal{O}(n) \not\cong \mathcal{O}(m)$ if $n \neq m$.*

Proof. This is an immediate corollary of the above Lemma since the ranks of the A-modules of sections of a certain power of these sheaves are distinct.

□

2.7.6. Hilbert polynomial of the projective space. As an application of Theorems 2.7.5a and 2.7.4 we can now compute Hilbert polynomials of the projective spaces over a given field and look which of the numerical characteristics of $\mathbb{P}^r_R$ introduced in § 2.5 do not depend on the representation of $\mathbb{P}^r_R$ in the form $\operatorname{Proj} R$.

Indeed, let $\mathbb{P}^r_k = \operatorname{Proj} R$; temporarily, denote by $\mathcal{O}_R(1)$ the invertible sheaf on $\mathbb{P}^r_k$ constructed with the help of R, and let $\mathcal{O}(1)$ be the invertible sheaf constructed by means of the standard representation $\mathbb{P}^r_k = \operatorname{Proj} k[T_0, \ldots, T_r]$.

By Theorem 2.7.5a we have

$$\mathcal{O}_R(1) \simeq \mathcal{O}_R(d)$$

for some $d \in \mathbb{Z}$; for $r \geq 1$, we have $d > 0$ since the rank of the space of sections of the sheaf $\mathcal{O}_R(n)$ grows as $n \longrightarrow \infty$.

On the other hand, by (yet unproven!) part of Theorem **??** for n sufficiently large, the map

$$\alpha_n\colon R_n \longrightarrow \Gamma(\mathbb{P}^r_k, \mathcal{O}_k(n)) = \Gamma(\mathbb{P}^r_k, \mathcal{O}(nd))$$

is an isomorphism. Hence

$$h_R(n) = \binom{nd+r}{r}.$$

In particular, the *degree* and the *constant term* of the Hilbert polynomial do not depend on R, as claimed.

2.7.7. Exercises. 1) Prove that the curve in $\mathbb{P}^r_k$ can be isomorphic to $\mathbb{P}^1_k$ only if its degree is equal to 1 or 2.

Hint. By definition, any curve in $\mathbb{P}^r_k$ is of the form $\operatorname{Proj} k[T_0, T_1, T_2]/(f)$, where f — is a form. Its Hilbert polynomial is computed in § 2.5.

Try to prove that the curve determined by a quadratic form f, is isomorphic to $\mathbb{P}^1_k$ if and only if the following two conditions are fulfilled: 1) rank $f = 3$; 2) The equation $f = 0$ has a non-zero solution in k.

2) Let $r \geq 1$; prove that any automorphism $f\colon \mathbb{P}^r_k \xrightarrow{f} \mathbb{P}^r_k$ over k is linear (what does it mean?).

Hint. Look how f acts on invertible sheaves and on the Picard group.

2.8. The Čech cohomology

2.8.1. The Čech complex. Let X — be a topological space and $\mathcal{F}$ — a sheaf of abelean groups on X. Let$U = \bigcup\limits_{i=1}^{r} U_i$ be a finite open cover of X. In this situation we give the following

A *Čech complex* is a complex whose homogeneous components $C^p(U, \mathcal{F})$ (called *groups of Čech p-cochains*) and the differential are as follows:

Let $[1, r]^{p+1}$ be the $(p+1)$-fold direct product of the set of integers $1, \dots, r$. The elements of $C^p(U, \mathcal{F})$ are the functions

$$s(i_0, \dots, i_p) \in \Gamma(U_{i_0} \dots U_{i_p}, \mathcal{F}), \text{ where } U_{i_0 \dots i_p} := U_{i_0} \cap \dots \cap U_{i_p}$$

"skew-symmetric" in the following sense:

$$s(\sigma(i_0), \dots, \sigma(i_p)) = \operatorname{sgn}\sigma \cdot s(i_0, \dots, i_p) \text{ for any } \sigma \in S_p$$

and

$$s(i_0, \dots, i_p) = 0 \text{ if among the indices } i_0, \dots, i_p \text{ at least two coincide.}$$

In particular, $C^p(U, \mathcal{F}) = 0$ for $p \geq r$; besides, $C^0(U, \mathcal{F}) = \prod\limits_{i=1}^{r} \Gamma(U_i, \mathcal{F})$.

The differential in the Čech complex is as follows (hereafter in similar sums, the hatted argument should be ignored):

$$(ds)(i_0,\ldots,i_{p+1}) = \sum_{k=0}^{p+1}(-1)^k rs(i_0,\ldots,\widehat{i}_k,\ldots,i_{p+1}),$$

where $r\colon \mathit{\Gamma}(U_{i_0\ldots\widehat{i}_k\ldots i_{p+1}},\mathcal{F}) \longrightarrow \mathit{\Gamma}(U_{i_0\ldots i_{p+1}},\mathcal{F})$ — is the restriction homomorphism (which, obviously, depends on $i_0,\ldots,i_{p+1}$ and k).

The cohomology groups of this complex are called *Čech cohomology groups* of the cover U with coefficients in the sheaf $\mathcal{F}$ and are denoted $\check{H}^p(U,\mathcal{F})$. Cohomology with coefficients in the sheaf defined *a la* Čech are convenient for calculations. However, they characterize, in a sense, the cover U of the space X and sections of the sheaf $\mathcal{F}$ over its charts, rather than the space X itself and the sheaf $\mathcal{F}$ itself.

Grothendieck suggested an axiomatic definition of cohomology of the space X with coefficients in a sheaf $\mathcal{F}$. By this definition *the p-th cohomology $H^p(X,\mathcal{F})$ of X with coefficients in a sheaf* $\mathcal{F}$ is the right derived functors of the functor that to every sheaf of Abelian groups $\mathcal{F}$ on X assigns its group of sections $\mathit{\Gamma}(X,\mathcal{F})$.

Let us formulate without proof a theorem offering a sufficient condition for coincidence of Grothendieck's $H^p(X,\mathcal{F})$ with Čech's $\check{H}^p(U,\mathcal{F})$.

2.8.2. Theorem (Cartan). *Let V be a family of quasi-compact open subsets of a topological space X forming a basis of topology of X and such that $H^p(U,\mathcal{F})$ for all $p \geq 1$ and $\bigcup\limits_{i=1}^{r} U_i \in V$ for all finite coverings $U = (U_i)_{1=1}^r$ such that $U_i \in V$. Then the cohomological functor $\mathcal{F} \longrightarrow \check{H}^*(X,\mathcal{F})$ is equivalent to $H^*(X,\mathcal{F})$ for any finite cover U of the space X by elements of V.*

Remark. The theorem is not formulated in full generality, but it suffices for our nearest purposes. For its proof, see Th. 5.9.2 in Ch. 2 of Godeman's book [God].

We will apply this theorem to the schemes X considering as V the families of affine open sets and taking any quasi-coherent sheaf as $\mathcal{F}$. Let us establish that in this situation the conditions of Cartan's theorem are fulfilled. For this, it suffices to prove the following result.

2.8.3. Proposition. *Let $X = \operatorname{Spec} A$, and $\mathcal{F} = \widetilde{M}$, where M is an A-module. Let $U_i = D(f_i)$, where $i = 1,\ldots,r$; let $X = \cup U_i$, and $U = (U_i)_{1=1}^r$. Then*

$$H^p(U,\mathcal{F}) = 0, \text{ for } p \geq 1.$$

2.8.3a. Corollary. *For any affine scheme X and a quasi-coherent sheaf $\mathcal{F}$ on it, we have $H^p(X,\mathcal{F}) = 0$ for any $p \geq 1$.*

Proof. Apply Cartan's theorem to X and the family of big open sets $D(f)$. □

2.8.3b. Corollary. *For any scheme X, its finite cover U by affine schemes, and a quasi-coherent sheaf $\mathcal{F}$ on X, we have*

$$\check{H}^p(U,\mathcal{F}) = H^p(X,\mathcal{F}).$$

Proof. Apply Cartan's theorem to X and the family of affine open subsets of X using Corollary 2.8.3a. □

Observe without proof that Serre proved the inversion of Corollary 2.8.3a: *if for a scheme X and any quasi-coherent sheaf of ideals J on X, we have $H^1(X,J)=0$, then X is an affine scheme.*

2.8.4. Properties of Čech complexes. Retain notation of Proposition 2.8.3. Let $U_{i_0\dots i_p} = U_{i_0}\cap\dots\cap U_{i_p} = D(f_{i_0}\dots f_{i_p})$. We have

$$\Gamma(U_{i_0\dots i_p},\widetilde{M}) \simeq M_{f_{i_0}\dots f_{i_p}}.$$

Each cochain of $C^p(U,\widetilde{M})$ for $p<r$ can be represented by a collection of $\binom{r}{p+1}$ elements of different localizations of the module M:

$$s(i_0,\dots,i_p)\in M_{f_{i_0}\dots f_{i_p}}.$$

We wish to prove that the Čech complex is *acyclic* in dimensions $p\geq 1$, i.e., the cohomology of this complex are 0 in these dimensiona. The standard method of proving acyclicity is to construct a *homotopy operator* or, more precisely a series of *chain homotopy operators* proving an equivalence of the given complex and an acyclic one. [4] For the Čech complex we can not construct such an operator, but we will get round this difficulty as follows. We construct a chain of complexes $(C^p_n(M))$, where n is the number of a complex, and homomorphisms between them so that:

a) *the Čech complex is the inductive limit of the complexes $C^p_n(M)$.*
b) *The complexes $C^p_n(M)$ are acyclic.*

[4] Let (A,d) and (A',d') be chain complexes and $f: A\to A'$, $g: A\to A'$ be chain maps. A *chain homotopy* D between f and g is a sequence of homomorphisms $\{D_n : A_n \to A'_{n+1}\}$ so that $d'_{n+1}\circ D_n + D_{n-1}\circ d_n = f_n - g_n$ for each n. Thus, we have the following diagram:

$$\begin{tikzcd}
A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}-g_{n+1}"'] & A_n \arrow[r, "d_n"] \arrow[dl, "D_n"] \arrow[d] & A_{n-1} \arrow[dl, "D_{n-1}"'] \arrow[d, "f_{n-1}-g_{n-1}"] \\
A'_{n+1} \arrow[r, "d'_{n+1}"'] & A'_n \arrow[r, "d'_n"'] & A'_{n-1}
\end{tikzcd}$$

If there exists a chain homotopy between f and g, then f and g are said to be *chain homotopic*. The complex chain homotopic to the one with zero cohomology is said to be *acyclic*.

Since the passage to the inductive limit commutes with computing cohomology, we see that

$$\check{H}^p(U, \widetilde{M}) = H(C^p(U, \widetilde{M})) = 0.$$

Let us pass to item a) of this program.

First, let us prove that for any ring A, any A-module M, and any $g \in A$, the module M_g can be naturally represented as an inductive limit.

First of all, by setting

$$M_g^{(n)} = \left\{ \frac{m}{g^n} \mid m \in M \right\},$$

we see that $M_g = \bigcup\limits_{n=0}^{\infty} M_g^{(n)}$. Each space $M_g^{(n)}$ is an A-module and we can replace the union by the inductive limit by considering the system

$$\dots \longrightarrow M_g^{(n)} \longrightarrow M_g^{(n+1)} \longrightarrow \dots, \tag{2.121}$$

in which the homomorphisms are described by the formula $\dfrac{m}{g^n} \mapsto \dfrac{mg}{g^{n+1}}$. If g is not a zero divisor in M, then the A-module $M_g^{(n)}$ is isomorphic to M for all n with respect to the map $m \longrightarrow \dfrac{m}{g^n}$. The inductive system (2.121) turns now into

$$\dots \xrightarrow{g} M^{(n)} = M \xrightarrow{g} M^{(n+1)} = M \xrightarrow{g} \dots, \tag{2.122}$$

where each slot is occupied by a copy of the A-module M, and each homomorphism is multiplication by g.

2.8.4a. Lemma. *The inductive limit of the system* (2.122) *is always isomorphic to M_g (even if g is a zero divisor in M).*

Proof. Consider the homomorphisms

$$M^{(n)} \longrightarrow M_g, \quad m \longrightarrow \frac{m}{g^n}.$$

They are compatible with homomorphisms of the system (2.122) and therefore define the homomorphism of its limit:

$$\varinjlim M^{(n)} \longrightarrow M_g.$$

Its cokernel is, clearly, zero. Any element of the kernel is represented by a chain of elements gm_n, g^2m_n, …, where $m_n \in M^{(n)} = M$, such that $m_n/g^n = 0$ in M_g; this means that $g^{n+k}m_n = 0$ for k sufficiently large, and therefore the whole chain represents the zero class. □

2.8.4b. Now, in the whole complex $(C^p(U,\widetilde{M}))$, replace the localizations of M by their "approximations" $M^{(n)}$ and appropriately define the cochain operators.

To make the correct expression of the cochains graphic, we first assume that the elements $f_{i_0},\dots,f_{i_p}$ are not zero divisors in M. Let $C^p_n(M)$ denote the subgroup of $C^p(U,\widetilde{M})$ consisting of cochains such that

$$s(i_0,\dots,i_p)\in M^{(n)}_{f_{i_0}\dots f_{i_p}}=\Big\{\frac{m}{(f_{i_0}\dots f_{i_p})^n}\mid m\in M\Big\}.$$

Let

$$s(i_0,\dots,i_p)=\frac{m_{i_0\dots i_p}}{(f_{i_0}\dots f_{i_p})^n}.$$

Then

$$ds(i_0,\dots,i_{p+1})=\frac{m_{i_0\dots i_{p+1}}}{(f_{i_0}\dots f_{i_{p+1}})^n}=\sum_{k=0}^{p+1}(-1)^k\frac{m_{i_0\dots\widehat{i}_k\dots i_{p+1}}}{(f_{i_0}\dots\widehat{f}_{i_k}\dots f_{i_{p+1}})^n},$$

implying that

$$m_{i_0\dots i_{p+1}}=\sum_{k=0}^{p+1}(-1)^k f^n_{i_k}m_{i_0\dots\widehat{i}_k\dots i_{p+1}}. \tag{2.123}$$

The embedding homomorphism $C^p_n(M)\longrightarrow C^p_{n+1}(M)$ is described by the formula

$$m_{i_0\dots i_p}\longrightarrow f_{i_0}\dots f_{i_p}m_{i_0\dots i_p}. \tag{2.124}$$

We use formulas (2.123) and (2.124) to define both the differential in the complex $C^p_n(M)$ when the condition on zero divisors is not satisfied, and complex homomorphism $C^p_n(M)\longrightarrow C^p_{n+1}(M)$.

In the general case, denote by $C^p_n(M)$ the group of skew-symmetric functions on $[1,r]^{p+1}$ with values in M and define the coboundary operator $C^p_n(M)\longrightarrow C^{p+1}_n(M)$ by setting:

$$(dm)(i_0,\dots,i_{p+1})=\sum_{k=0}^{p+1}(-1)^k f^n_{i_k}m(i_0,\dots,\widehat{i}_k,\dots,i_{p+1}). \tag{2.125}$$

Define the group homomorphism $\varphi_n=C^p_n(M)\longrightarrow C^p_{n+1}(M)$ by the formula:

$$(\varphi_n m)(i_0,\dots,i_p)=f_{i_0}\dots,f_{i_p}m(i_0,\dots,i_p). \tag{2.126}$$

2.8.4c. Lemma. 1) *The collection of sets $(C^p_n(M))$ for n fixed is not a complex for any n.*

2) *The collection of homomorphisms φ_n is a complex homomorphism.*

3) $\varinjlim C^p_n(M)=C^p(U,\widetilde{M})$*; the inductive limit of differentials is the differential in the inductive limit.*

The first two statements are verified by trivial calculations; the third one follows from definitions and Lemma 2.8.4c.

This is the end of stage a) of computing cohomology of the Čech complex, i.e., its approximating by complexes $C^p_n(M)$ which are easier to deal with.

2.8.4d. Stage b) of the program. Let us now pass to the proof of acyclic property of the complex $C_n^p(M)$. In the construction of this complex there are involved the ring A, the A-module M, the elements $f_1^n, \ldots, f_r^n \in A$ that determine the cover $\{D(f_i^n)\}_{i=1}^r = \{D(f_i)\}_{i=1}^r$, and the differential given by formula (2.125).

Since the complex $C_n^p(M)$ is important in various problems of algebraic geometry, we will study it in more detail than is strictly necessary for our purposes.

2.8.5. Koszul complex. Let A be a ring, $f = (f_1, \ldots, f_r)$ a collection of its elements; set $f^n := (f_1^n, \ldots, f_r^n)$.

Consider a free A-module $Ae_1 \oplus \ldots \oplus Ae_r = A^r$ of rank r and its exterior powers $K_p = \Lambda_A^p A^r$; by definition, $K_0 = A$.Clearly, K_p is a free A-module of rank $\binom{r}{p}$; the elements $e_{i_1} \wedge \ldots \wedge e_{i_p}$, where $i_1 < \ldots < i_p$, constitute its basis. Define the differential $d\colon K_{p+1} \longrightarrow K_p$ by setting:

$$d(e_{i_1} \wedge \ldots \wedge e_{i_{p+1}}) = \sum_{k=1}^{p+1} (-1)^{k+1} f_{i_k} e_{i_1} \wedge \ldots \wedge \widehat{e}_{i_k} \wedge \ldots \wedge e_{i_{p+1}}.$$

(**NB**: -1 is raised to power $k+1$ in order for the first term enter with its initial sign.) It is trivial job to verify that $d^2 = 0$; let $K_p(f, M)$ or briefly $K_p(f)$ be the complex obtained. (Observe that it is a *chain* complex, whereas the Čech complex is a *cochain* one.)

The relation between complexes $K_p(f)$ and $C_n^p(M)$ is as follows.

2.8.5a. Lemma. *For $p \geq 0$, we have*

$$C_n^p(M) \simeq K^{p+1}(f^n, M) := \operatorname{Hom}(K_{p+1}(f^n), M);$$

and the isomorphism can be selected to be compatible with the differentials.

Proof. We assign to any cochain $m = (m(i_0, \ldots, i_p)) \in C_n^p(M)$ the homomorphism

$$\begin{aligned} &K_{p+1}(f^n) \longrightarrow M, \\ &g_m \colon e_{i_0} \wedge \ldots \wedge e_{i_p} \longmapsto m(i_0, \ldots, i_p). \end{aligned}$$

The differential of g_m considered as an element of $\operatorname{Hom}(K_{p+2}(f^n), M)$, is given by the formula:

$$\begin{aligned} (dg_m)(e_{i_0} \wedge \ldots \wedge e_{i_{p+1}}) &= g_m(d(e_{i_0} \wedge \ldots \wedge e_{i_{p+1}}) = \\ &= g_m\left(\sum_{k=0}^{p+1} (-1)^k f_{i_k}^n m(i_0, \ldots, \widehat{i}_k, \ldots, i_{p+1}\right) = g_{dm}(e_{i_0} \wedge \ldots \wedge e_{i_{p+1}}), \end{aligned}$$

which proves the desired. $\square$

Therefore, we have extracted the dependence of $C^p_n(M)$ on M. The complex $K_p(f)$ can also be further "dismantled"; this is convenient in the proofs based on induction on r.

2.8.6. Lemma. $K_0(f) \simeq K_0(f_1) \otimes \ldots \otimes K_0(f_r)$.

Proof. First of all, recall that the *tensor product of two chain complexes* K_0 and L, is the complex such that

$$(K \otimes L)_p = \bigoplus_{i+j} K_i \otimes L_j,$$
$$d(k \otimes l) = dk \otimes l + (-1)^r k \otimes dl, \text{ where } k \in K_i;$$

and, generally, for $K^{(1)} \otimes \ldots \otimes K^{(r)}$, we have

$$d(k_1 \otimes \ldots \otimes k_r) = \sum_{j=1}^{r} (-1)^{d_1+\ldots+d_{j-1}} k_1 \otimes \ldots \otimes \widehat{k}_j \otimes \ldots \otimes k_r,$$
$$\text{where } k_j \in K^{(j)}_{d_j} \text{ for } j = 1, \ldots, r.$$

Let now $K^{(i)}_0 = A$, and $K^i_1 = Ae_i$ for all i. Construct a complex by setting

$$0 \longleftarrow A \longleftarrow Ae_i, \quad d(e_i) = f_i.$$

Then $(K^{(1)} \otimes \ldots \otimes K^{(r)})_p$ is a free A-module with a basis $e_{i_1} \otimes \ldots \otimes e_{i_p}$, where $1 \leq i_1 < \ldots < i_r \leq p$, and the differential

$$d(e_{i_1} \otimes \ldots \otimes e_{i_p}) = \sum_{k=1}^{p} (-1)^{k+1} f_{i_k} e_{i_1} \otimes \ldots \otimes \widehat{e}_{i_k} \otimes \ldots \otimes e_{i_p}.$$

This shows that it is isomorphic to the Koszul complex $K_0(f)$. We can now prove that the complex $C^p_n(M)$ is acyclic in dimensions ≥ 1. For this, it suffices to construct a homotopy for $K_p(f)$. $\square$

2.8.7. Proposition. *Let $g_1, \ldots, g_r$ be an arbitrary collection of r elements of A. Let $h\colon K_p(f) \longrightarrow K_{p+1}(f)$ be the exterior multiplication on the left by $\sum\limits_{i=1}^{r} g_i e_i$. Then*

$$hd + dh = \sum_{i=1}^{r} f_i g_i$$

(*i.e., the multiplication by the sum on the right*).

Proof. Fix a set of indices (without repetitions) $i_1, \ldots, i_{p+1} \in [1, r]$ and let $j_1, \ldots, j_{r-p-1}$ be a complementary set of indices. We have:

$$dh(e_{i_1}\wedge\ldots\wedge e_{i_{p+1}}) =$$
$$d\Bigg(\Bigg(\sum_{k=1}^{r} g_k e_k\Bigg)\wedge e_{i_1}\wedge\ldots\wedge e_{i_{p+1}}\Bigg) = d\Bigg(\sum_{k=1}^{r=p-1} g_{j_k} e_{j_k}\wedge e_{i_1}\wedge\ldots\wedge e_{i_{p+1}}\Bigg) =$$
$$\sum_{k=1}^{r=p-1}(g_{j_k} f_{j_k} e_{i_1}\wedge\ldots\wedge e_{i_{p+1}}) + g_{j_k}\sum_{l=1}^{p+1}(-1)^l f_{i_l} e_{j_k}\wedge\ldots\wedge\widehat{e}_{i_l}\wedge\ldots\wedge e_{i_{p+1}}.$$

On the other hand:

$$hd(e_{i_1}\wedge\ldots\wedge e_{i_{p+1}}) = h\Bigg(\sum_{l=1}^{p+1}(-1)^{l+1} f_{i_l} e_{i_1}\wedge\ldots\wedge\widehat{e}_{i_l}\wedge\ldots\wedge e_{i_{p+1}}\Bigg) =$$
$$= \sum_{l=1}^{p+1}(-1)^{l+1} f_{i_l}(g_{i_l} e_{i_l}\wedge e_{i_1}\wedge\ldots\wedge\widehat{e}_{i_l}\wedge\ldots\wedge e_{i_{p+1}} +$$
$$+\sum_{k=1}^{p+1} g_{j_k} e_{j_k}\wedge e_{i_1}\wedge\ldots\wedge\widehat{e}_{i_l}\wedge\ldots\wedge e_{i_{p+1}} =$$
$$= \sum_{l=1}^{p+1}(-1)^{l+1} f_{i_l}(-1)^{l-1} g_{i_l} e_{i_1}\wedge\ldots\wedge e_{i_{p+1}} +$$
$$+\sum_{k=1}^{p+1} g_{j_k} e_{j_k}\wedge e_{i_1}\wedge\ldots\wedge\widehat{e}_{i_l}\wedge\ldots\wedge e_{i_{p+1}}.$$

Adding up these two expressions we get the desired. □

2.8.7a. Corollary. *If the ideal of A generated by the elements f_i, where $i = 1, \ldots, r$, is equal to A, then the complexes $C_n^p(M)$ and $\overline{C(U, \mathcal{F})}$ are acyclic in dimensions ≥ 1.*

(Indeed, then $\sum\limits_{i=1}^{r} f_i g_i = 1$ for certain g_i, and hence h is the homotopy operator.)

This completes the proof of Proposition 2.8.3 and its corollaries. □

2.8.7b. Remark. In the general case: $C_n^{p+1}(M) = K^{p+1}(f^n, M)$ for $p \geq 0$. By definition we have an exact sequence

$$0 \longrightarrow H^0(f^n, M) \longrightarrow K^0(f^n, M) \xrightarrow{d} Z^1(f^n, M) \longrightarrow H^1(f^n, M) \longrightarrow 0.$$

The limit $\varinjlim$ as $n \longrightarrow \infty$ gives:

$$0 \longrightarrow H^0((f), M) \longrightarrow K^0((f), M) \xrightarrow{d} Z^1((f), M) \longrightarrow H^1((f), M) \longrightarrow 0,$$

and

$$H^0(U, \mathcal{F}) = Z^1((f), M),$$
$$K^0((f), M) = K^0((f^n), M) = \mathrm{Hom}(A, M) = M,$$

so we have an exact sequence:

$$0 \longrightarrow H^0((f), M) \longrightarrow M \xrightarrow{d} H^0(U, \mathcal{F}) \longrightarrow H^1((f), M) \longrightarrow 0.$$

2.9. Cohomology of the projective space

2.9.1. Let A be a fixed ring, $\mathbb{P}_A^{r-1} = \operatorname{Proj} R$, where $R = A[T_1, \dots, T_r]$ with the standard grading, $U_i = D_+(T_i)$, $U = (U_i)$.

In this section we compute $H^p(\mathbb{P}_A^{r-1}, \mathcal{O}(n))$ for any p, n and r. This computation, due to Serre, is the base of the proof (in the next section) of main results on cohomology of coherent sheaves on projective schemes.

Thanks to results of § 2.8, we have

$$H^p(\mathbb{P}_A^{r-1}, \mathcal{O}(n)) = \check{H}^p(U, \mathcal{O}(n)).$$

Therefore we can compute the cohomology of the Čech complex of the cover U.

Since $U_{i_0,\dots,i_p} = D_+(T_{i_0} \dots T_{i_p}) = \operatorname{Spec} R_{(T_{i_0} \dots T_{i_p})}$, we have, for the usual description of the sheaf $\mathcal{O}(n)$:

$$\Gamma(U_{i_0 \dots i_p}, \mathcal{O}(n)) = \Big\{ \frac{m(i_0, \dots, i_p)}{(T_{i_0} \dots T_{i_p})^k} \mid k \in \mathbb{Z},\ m \in R_{k(p+1)+n} \Big\}.$$

This implies that

$$\bigoplus_{n \in \mathbb{Z}} \Gamma(U_{i_0 \dots i_p}, \mathcal{O}(n)) = \{ s_{i_0 \dots i_p} \mid s_{i_0 \dots i_p} \in R_{T_{i_0 \dots i_p}} \}.$$

This formula indicates that it is convenient to compute the direct sum of Čech complexes $\bigoplus\limits_{n \in \mathbb{Z}} C^0(U, \mathcal{O}(n))$, and its cohomology, tracing the natural grading, and at the end separate the homogeneous components in the answer.

2.9.2. $C_k^p(U, \mathcal{O}(n))$ Fix $k \in \mathbb{Z}$. Denote by $C_k^p(U, \mathcal{O}(n))$ the subgroup of chains whose components can be represented as $\dfrac{m(i_0, \dots, i_p)}{(T_{i_0} \dots T_{i_p})^k}$, where m is a form. As p varies, these groups form a complex $\bigoplus\limits_{n \in \mathbb{Z}} C_k^p(U, \mathcal{O}(n))$; computing the action of its differential on the numerators of the cochain's component we easily obtain:

Lemma. 1) *The complex* $\bigoplus\limits_{n \in \mathbb{Z}} C_k^p(U, \mathcal{O}(n))$ *with its grading is isomorphic to the Koszul complex* $K^{p+1}(T_1^k, \dots, T_r^k; R)$ *with the grading in which the elements* $g \in \operatorname{Hom}(K_{p+1}(T^k), R)$ *such that*

$$g(e_{i_0} \wedge \dots \wedge e_{i_p}) \in R_{k(p+1)+n}$$

are homogeneous of degree n.

2) *The map*

$$m(i_0,\dots,i_p)\longmapsto T_{i_0}\dots T_{i_p}\cdot m(i_0,\dots,i_p)$$

determines homogeneous homomorphisms of graded complexes

$$\bigoplus_{n\in\mathbb{Z}} C^p_k(U,\mathcal{O}(n))\longrightarrow \bigoplus_{n\in\mathbb{Z}} C^p_{k+1}(U,\mathcal{O}(n))$$

and

$$\oplus C^p(U,\mathcal{O}(n))=\varinjlim_k \oplus C^p_k(U,\mathcal{O}(n))$$

relative this system of homomorphisms.

Observe that condition 1) uniquely determines on K^{p+1} the structure of a graded R-module.

Lemma 2.9.2 illustrates the necessity to study homology of the Koszul complex $K^p(T^k,R)$. The method of chain homotopy operator is inapplicable since the elements $(T_1^k,\dots,T_n^k)$ generate a non-trivial ideal; the Koszul complex is not, actually, acyclic in one dimension. [5] Therefore, another approach is needed here.

We return, temporarily, to notation of § 2.8: Let A be a ring, $f_1,\dots,f_r$ a collection of its elements. First, observe a duality:

2.9.3. Lemma. *Define an A-homomorphism*

$$\varphi\colon K_{r-p}(f,A)\longrightarrow K^p(f,A)$$

by setting:

$$\varphi(e_{i_1}\wedge\dots\wedge e_{i_{r-p}})(e_{j_1}\wedge\dots\wedge e_{j_p})=\\ =\begin{cases}0, & \textit{if } (i)\cap(j)\neq\emptyset,\\ \varepsilon(i_1,\dots,i_{r-p};j_1,\dots,j_p), & \textit{if } (i)\cap(j)=\emptyset.\end{cases}$$

Then φ is a complex isomorphism up to a sign of the differentials.

Proof. It suffices to find how φ commutes with the differentials. We have:

$$\varphi(d(e_{i_1}\wedge\dots\wedge e_{i_{r-p}}))(e_{j_1}\wedge\dots\wedge e_{j_{p+1}})=\\ =\varphi\left(\sum_{k=1}^{p}(-1)^{k+1}f_{i_k}e_{i_1}\wedge\dots\wedge\widehat{e}_{i_k}\wedge\dots\wedge e_{i_{r-p}}\right)(e_{j_1}\wedge\dots\wedge e_{j_{p+1}})=\\ =\begin{cases}0, & \text{if } |(i)\cap(j)|>1,\\ (-1)^{k+1+\sigma}f_{i_k}, & \text{if } i_k=(i)\cap(j),\end{cases}$$

where

$$(-1)^\sigma=\varepsilon(i_1\dots\widehat{i}_k\dots i_{r-p};j_1\dots j_{p+1}).$$

[5] **Exercise**. Determine in which one.

On the other hand:

$$\begin{aligned}&\varphi(e_{i_1}\wedge\ldots\wedge e_{i_{r-p}})(d(e_{j_1}\wedge\ldots\wedge e_{j_{p+1}}))=\\&\quad=\varphi(e_{i_1}\wedge\ldots\wedge e_{i_{r-p}})\left(\sum_{l=1}^{p+1}(-1)^{l+1}e_{j_1}\wedge\ldots\wedge e_{j_l}\wedge\ldots\wedge e_{j_{p+1}}\right)=\\&\qquad\qquad=\begin{cases}0, & \text{if } |(i)\cap(j)|>1,\\ (-1)^{l+1+\tau}f_{j_l}, & \text{if } j_l=(i)\cap(j),\end{cases}\end{aligned}$$

where

$$(-1)^\tau=\varepsilon(i_1,\ldots,i_{r-p};j_1,\ldots,\widehat{j_l},\ldots,j_{p+1}).$$

Comparing these answers we see how φ commutes with the differentials. $\square$

Next, we use Lemma 2.8.5a which implies that

$$K_0(f_1,\ldots,f_r)=K_0(f_1,\ldots,f_{r-1})\otimes K_0(f_r).$$

The following result is a reason for computing homology of K_0 by induction on r.

For any A-module M and element $f\in A$, let ${}_fM$ denote the kernel of the homomorphism $M\xrightarrow{f}M$ of multiplication by f.

2.9.4. Lemma. *Let L be a chain complex of A-modules, $i\geq 1$. Then there is an exact sequence of A-modules*

$$0\longrightarrow H_i(L)/f\cdot H_i(L)\longrightarrow H_i(L\otimes K(f))\longrightarrow {}_fH_{i-1}(L)\longrightarrow 0.$$

Proof. The complex $K(f)$ is of the form

$$0\longrightarrow Ae_1\xrightarrow{d}Ae_0\longrightarrow 0,\quad de_1=fe_0.$$

Therefore for $i\geq 1$, we have

$$(L\otimes K(f))_i=L_ie_0\oplus L_{i-1}e_1$$

and

$$d(l_ie_0+l_{i-1}e_1)=(dl_i+(-1)^{i-1}fl_{i-1})e_0+dl_{i-1}e_1.$$

This implies a commutative diagram with exact rows

$$\begin{tikzcd}0 \arrow[r] & L_i \arrow[r]\arrow[d,"d"] & (L\otimes K(f))_i \arrow[r]\arrow[d,"d"] & L_{i-1} \arrow[r]\arrow[d,"d"] & 0\\ 0 \arrow[r] & L_{i-1} \arrow[r] & (L\otimes K(f))_{i-1} \arrow[r] & L_{i-2} \arrow[r] & 0\end{tikzcd}$$

All these diagrams can be united into a sequence of complexes. This sequence is exact in dimensions ≥ 1:

$$0 \longrightarrow L \longrightarrow (L \otimes K(f)) \longrightarrow L[-1] \longrightarrow 0, \tag{2.127}$$

where $L[-1]_i := L_{i-1}$ (the complex L, shifted by 1 to the right). In its turn, this sequence leads to an exact sequence of homology groups

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H_{i+1}(L[-1]) & \xrightarrow{\delta} & H_i(L) & \longrightarrow & H_i(L \otimes K(f)) & \longrightarrow & H_i(L[-1]) & \xrightarrow{\delta} \cdots \\ & \| & & & & & & \| & \\ & H_i(L) & & & & & & H_{i-1}(L) & \end{array}$$

Let us show that the diagram

$$\begin{tikzcd} H_{i+1}(L[-1]) \arrow[r, "\delta"] \arrow[d, equal] & H_i(L) \\ H_i(L) \arrow[ur, "{[-1]^i f}"'] & \end{tikzcd}$$

is commutative. This follows from calculations: If $z \in Z_{i+1}(L[-1])$, then

$$\begin{aligned} &d(z \otimes e_1) = dz \otimes e_1 + (-1)^i z \otimes f e_0 = (-1)^i z f \otimes e_0, \\ &\delta(\mathrm{class}(z)) = (-1)^i \,\mathrm{class}(fz). \end{aligned}$$

Thus the following sequence is exact:

$$0 \longrightarrow H_i(L)/fH_i(L) \longrightarrow H_i(L \otimes K(f)) \longrightarrow fH_{i-1}(L) \longrightarrow 0.$$

2.9.4a. Corollary. *If $H_i(L) = 0$ for $i \geq 1$, then $H_i(L \otimes K(f)) = 0$ for $i \geq 2$.*

Being applied to the Koszul complex, Lemma 2.9.4 and its Corollary yield the following result:

2.9.5. Proposition. *Let $f_1, \dots, f_r \in A$ be a sequence of elements such that f_i is not a zero divisor in $A/(f_1, \dots, f_{i-1})$ for all $i \geq 1$. Then $H_i(K(f)) = 0$ for $i \geq 1$. Besides, we always have*

$$H_0(K(f_1, \dots, f_r)) = A/(f_1, \dots, f_r)A.$$

Proof immediately follows by induction on r. Indeed, if the statement holds for the sequences of length $i-1$, then Corollary 2.9.4 immediately shows acyclic property in dimensions ≥ 2 for any sequence of length i, whereas the exact sequence (2.127) shows acyclic property in dimension 1 as well. The claim on H_0 is obvious from definition. □

2.9.6. A regular sequence. Any sequence $f_1, \dots, f_r \in A$ satisfying conditions of Proposition 2.9.5 is said to be *regular*.

The result of the above subsection shows that if the elements $f_1, \dots, f_r$ form a regular sequence, then the Koszul complex is a free resolution of the A-module $A/(f_1, \dots, f_r)$.

In several important cases the statement converse to Proposition 2.9.5 holds:

2.9.7. Proposition. *Let A be a local Noetherian ring, $\mathfrak{p}$ its maximal ideal, and $f_1, \dots, f_r \in \mathfrak{p}$. If $H_i(K(f_1, \dots, f_r)) = 0$ for $i \geq 1$ (or even if only $H_1(K(f_1, \dots, f_r)) = 0$), then $f_1, \dots, f_r$ is a regular sequence.*

Proof. For $r = 1$, Proposition obviously holds (for any A and $f_1 \in A$). Let the statement is proved for $r = n - 1$, let $f_1, \dots, f_n \in \mathfrak{p}$, and $H_1(K(f_1, \dots, f_n)) = 0$. By Lemma 2.9.4 for $f = f_n$, and $i = 1$, we get an exact sequence

$$\begin{aligned} 0 \longrightarrow H_1(K(f_1, \dots, f_{n-1}))/fH_1(K(f_1, \dots, f_{n-1})) \longrightarrow \\ \longrightarrow H_1(K(f_1, \dots, f_n)) \longrightarrow {}_fH_1(K(f_1, \dots, f_{n-1})) \longrightarrow 0, \end{aligned}$$

which implies that

$$\begin{aligned} H_1(K(f_1, \dots, f_{n-1}))/fH_1(K(f_1, \dots, f_{n-1})) = 0, \\ {}_fH_0(K(f_1, \dots, f_{n-1})) = 0. \end{aligned}$$

Since $H_1(K(f_1, \dots, f_{n-1}))$ in conditions of Proposition is a Noetherian A-module, the first equality and Nakayama's lemma imply that

$$H_1(K(f_1, \dots, f_{n-1})) = 0,$$

so $f_1, \dots, f_{n-1}$ is a regular sequence by induction hypothesis. Since

$$H_0(K(f_1, \dots, f_{n-1})) = A/(f_1, \dots, f_{n-1}),$$

the second equality implies that f_n is not a zero divisor in $A/(f_1, \dots, f_{n-1})$. Hence $f_1, \dots, f_n$ is a regular sequence. $\square$

2.9.8. Corollary. *If $(f_1, \dots, f_n)$ is a regular sequence of elements of the maximal ideal of a Noetherian local ring, the sequence obtained from this one by any permutation is also regular.*

Proof. It suffices to observe that the Koszul complexes corresponding to the sequences that only differ by a permutation are isomorphic to each other. $\square$

Now we are able to formulate the main result on cohomology of the projective space.

2.9.9. Theorem. a) *$H^p(\mathbb{P}_A^{r-1}, \mathcal{O}(n)) = 0$ for $p \neq 0,\ r - 1$.*

b) *$H^0(\mathbb{P}_A^{r-1}, \mathcal{O}(n)) = 0$ for $n < 0$ and is a free A-module of rank $\binom{n+r-1}{r-1}$ for $n \geq 0$.*

c) *$H^{r-1}(\mathbb{P}_A^{r-1}, \mathcal{O}(n)) = 0$ for $n \geq -r + 1$ and is a free A-module of rank $\binom{-n-1}{r-1}$ for $n \leq -r$.*

On the (p, n)-plane mark points where $H^p(\mathbb{P}_A^{r-1}, \mathcal{O}(n)) \neq 0$. Obviously, the diagram is central symmetric, and this symmetry $(p, n) \longrightarrow (r-1-p, -r-n)$ preserves the rank of A-modules of cohomology. In a deeper theory, this symmetry is explained by the duality theorem for cohomology of coherent sheaves.

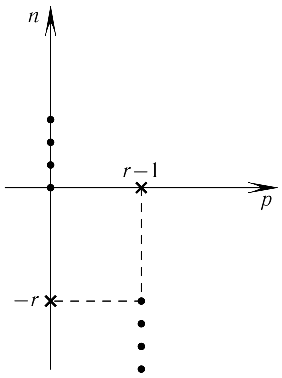

Fig. 5

Proof. We only have to trace homomorphisms connecting $H^p(\mathbb{P}_A^{r-1}, \mathcal{O}(n))$ with Koszul complexes.

The fact that Koszul complex $K_0(T^k, R)$ is acyclic (Proposition 2.9.5) and duality described in Lemma 2.9.3 immediately imply statement) if we realize that the role of ring A is played now by $R = A[T_1, \dots, T_r]$, and the role of the f_i are played by the T_i.

A somewhat more tedious count with grading and explicit form of homomorphisms $C_k \longrightarrow C_{k+1}$ (see Lemma 2.9.2) taken into account allows one to establish statements b) and c). We leave this count to the reader. □

2.10. Serre's theorem

2.10.1. Theorem (Serre). *Let R be a graded ring: $R = \bigoplus\limits_{n=0}^{\infty} R_n$, where $R_0 = A$ is a Noetherian ring and R is generated by a Noetherian A-module R_1. Let $X = \operatorname{Proj} R$ and $\mathcal{F}$ a coherent sheaf on X. Then the following statements hold*:

a) $H^q(X, \mathcal{F}) = 0$ *if $q+1$ is greater than the number of generators of the A-module R_1.*

b) $H^q(X, \mathcal{F})$ *is a Noetherian A-module for any q.*

c) $H^q(X, \mathcal{F}(n)) = 0$ *for $q \geq 1$ and $n \geq n_0(\mathcal{F})$, where $n_0(\mathcal{F})$ is a number depending on $\mathcal{F}$.*

Observe that this theorem allows one to introduce important invariants of the scheme X. For example, if A is a field and $\mathcal{F}$ is the structure sheaf, then the cohomology spaces $H^q(X, \mathcal{O}_X)$ are determined by their dimensions recovered from X.

Proof. First of all reduce Serre's theorem to the case where $X = \mathbb{P}_A^r$, where $r+1$ is the cardinality of a system of generators of the A-module R_1. Consider the ring $A[T_1, \dots, T_{r+1}]$, whose projective spectrum is $\mathbb{P}_A^r$, and construct an homogeneous epimorphism of graded rings $A[T_1, \dots, T_{r+1}] \longrightarrow R$ sending T_i into the i-th generator of R_1. This epimorphism induces a closed embedding of spectra $j\colon \operatorname{Proj} R \longrightarrow \mathbb{P}^r_A$ which, in turn, enables one to continue the sheaf $\mathcal{F}$ on $\operatorname{Proj} R$, given in conditions of the theorem to the sheaf $j_*(\mathcal{F})$ on $\mathbb{P}^r_A$. The sheaf $j_*(\mathcal{F})$ is determined by the following rule

$$\varGamma(U, j_*(\mathcal{F})) = \varGamma(U \cap j(X), \mathcal{F}) \text{ for any open set } U \subset \mathbb{P}^r_A.$$

The following properties of the continuation of the sheaf operation are rather obvious.

First,

$$H^q(X, \mathcal{F}) = H^q(\mathbb{P}^r_A, j_*(\mathcal{F})) \quad \text{for any } q,$$

which is easy to see on the level of Čech complexes: in this case they are just isomorphic modules with differentials. (For opens on $\mathbb{P}^r_A$ we take, as always, the sets $D_+(T_i)$; for opens in $\operatorname{Proj} R$ we take, in order to establish an isomorphism desired of complexes, the sets $D_+(t_i)$, where t_i is the generator of R_1 corresponding to T_i.) We may also use of the fact that $\mathcal{F} \longrightarrow j_*(\mathcal{F})$ is a fully faithful functor.

Second, we have $j_*(\mathcal{F}(n)) = j_*(\mathcal{F})(n)$, where $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}_X(n)$. Now, heading) of Serre's theorem immediately follows from skew-symmetry of Čech cochains and the fact that Čech cohomology for the cover $(D_+(T_i))$ coincide with the usual cohomology.

To prove headings b) and c) of Serre's theorem we use the following technical result.

2.10.1a. Lemma. *For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r_A$, there exists an integer m such that for a natural p we have the following exact sequence*

$$\mathcal{O}^p_{\mathbb{P}^r_A} \longrightarrow \mathcal{F}(m) \longrightarrow 0.$$

From this Lemma the needed facts are established by a simple descending induction on q.

Let us define the coherent sheaf $\mathcal{E}$ from the exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}^p_{\mathbb{P}^r_A} \longrightarrow \mathcal{F}(m) \longrightarrow 0.$$

We tensor it by $\mathcal{O}(n)$, where $n \in \mathbb{Z}$:

$$0 \longrightarrow \mathcal{E}(n) \longrightarrow \mathcal{O}^p_{\mathbb{P}^r_A}(n) \longrightarrow \mathcal{F}(m+n) \longrightarrow 0.$$

This gives rise to the exact cohomology sequence:

$$\dots \longrightarrow H^q(\mathbb{P}^r_A, \mathcal{O}(n)) \longrightarrow H^q(\mathbb{P}^r_A, \mathcal{F}(m+n)) \longrightarrow H^{q+1}(\mathbb{P}^r_A, \mathcal{E}(n)) \longrightarrow \dots \tag{2.128}$$

For $q = r+1$, heading a) shows that properties b) and c) obviously hold. The induction hypothesis: let headings b) and c) of Serre's theorem hold for the cohomology of all coherent sheaves in dimension $q+1$.

In the exact cohomology sequence (2.128), the A-module $H^{q+1}(\mathbb{P}^r_A, \mathcal{E}(n))$ is Noetherian and vanishes for $n \geq n_0$ by induction hypothesis; for $H^q(\mathbb{P}^r_A, \mathcal{O}(n))$ the same is true thanks to Theorem 2.9.9. This implies the desired for $H^q(\mathbb{P}^r_A, \mathcal{F}(m+n))$. $\square$

It remains to prove Lemma 2.10.1a.

Proof of Lemma 2.10.1a. We may assume that $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^r_A$. For the standard cover $U_i := D_+(T_i)$, having identified $\mathcal{F}|_{U_i}$ with $\mathcal{F}(m)|_{U_i}$, we see that $\mathcal{F}(m)$ is glued of the $\mathcal{F}|_{D_+(T_i)}$ by means of the cocycle:

$$\left(\frac{T_i}{T_j}\right)^m : (\mathcal{F}|_{U_i})|_{U_i \cap U_j} \xrightarrow{\sim} (\mathcal{F}|_{U_j})|_{U_j \cap U_i}.$$

Each section $s \in \varGamma(\mathbb{P}^r_A, \mathcal{F}(m))$ is then given by its "components" $s \in \varGamma(U_i, \mathcal{F})$: There is an embedding

$$\begin{aligned} &\varGamma(\mathbb{P}^r_A, \mathcal{F}(m)) \hookrightarrow \prod \varGamma(U_i, \mathcal{F}): \\ &s \mapsto (\dots s_i \dots), \text{ where } s_i \cdot \left(\frac{T_i}{T_j}\right)^m = s_j. \end{aligned}$$

To construct the epimorphism $\mathcal{O}^p \longrightarrow \mathcal{F}(m)$, it suffices to establish that it is possible to construct a finite number of global sections of the sheaf $\mathcal{F}(m)$ whose restrictions onto U_i generate the $\varGamma(U_i, \mathcal{O}_{\mathbb{P}^r_A})$-module $\varGamma(U_i, \mathcal{F})$.

To this end, it suffices to be able to establish, for any section $s \in \varGamma(U_i, \mathcal{F})$ and all m sufficiently large, that s is a component of a sheaf $\mathcal{F}(m)$, i.e., to be able to "extend s". Having extended in this way (finitely many) generators of all modules $\varGamma(U_i, \mathcal{F})$ and having selected a common m sufficiently large, we get the desired.

2.10.2. Extending a section $s_0 \in \varGamma(U_0, \mathcal{F})$. Being multiplied by $\left(\frac{T_0}{T_i}\right)^p$ (for a certain p), the section $s_0|_{U_0 \cap U_i}$ can be extended to U_i:

$$s_0 \left(\frac{T_0}{T_i}\right)^p = s'_i|_{U_0 \cap U_i}, \text{ where } s'_i \in \varGamma(U_i, \mathcal{F}),$$

for a p is common for all U_i, and however large.

On $U_0 \cap U_i \cap U_j$, we have:

$$s'_i \left(\frac{T_i}{T_j}\right)^p \Big|_{U_0 \cap U_i \cap U_j} = s'_j|_{U_0 \cap U_i \cap U_j}.$$

Therefore, for q sufficiently large (and also common for all i), we have

$$\left(s'_i \left(\frac{T_i}{T_j}\right)^p - s'_j\right) \left(\frac{T_0}{T_j}\right)^q = 0. \tag{2.129}$$

Set

$$s''_i = s'_i \left(\frac{T_0}{T_i}\right)^q.$$

Clearly,

$$s_0'' = s_0.$$

Besides, s_0'' enters as a component into the section $(\dots s_i'' \dots)$ of the sheaf $\mathcal{F}(p+q)$. Verification (see (2.129)):

$$s_i''\left(\frac{T_i}{T_j}\right)^{p+q} = s_j'' \Longleftrightarrow s_i'\left(\frac{T_0}{T_i}\right)^q\left(\frac{T_i}{T_j}\right)^{p+q} = s_j'\left(\frac{T_0}{T_i}\right)^q. \qquad \square$$

2.10.3. Comments to Serre's theorem. 1) The role of coherent property of $\mathcal{F}$: otherwise b) fails. Statement b) is non-trivial: The modules C^q in the Čech complex are not of finite type: Noetherean property appears only after passage to cohomology.

2) The nature of the proof: there are "plenty" sheaves $\mathcal{O}_{\mathbb{P}^r}(n)$ with known cohomology; everything can be reduced to them with the help of an exact cohomology sequence. Lemma 2.10.1a explains the meaning of the term "plenty".

Cp. with the following algebraic fact: every module is the image of a free one; here the situation is global, and "twisting" by means of $\mathcal{O}_{\mathbb{P}^r}(1)$ is essential.

3) What is $n_0(\mathcal{F})$, starting with which c) holds? The question is a difficult one; for the sheaves of ideal we have, nevertheless, a "rough" answer: If the Hilbert polynomial h is known, then $n_0(\mathcal{F})$ is a universal polynomial in coefficients of h. Generally, the numbers $n_0(\mathcal{F})$ depend on $\mathcal{F}$; for example, if $\mathcal{F} = \mathcal{O}_{\mathbb{P}^r}(-N)$, then $H^r(\mathbb{P}^r_A, \mathcal{F}(n)) = 0$ only for $n \geq N - r$.

2.11. Sheaves on Proj *R* and graded modules

2.11.1. The main fact of the theory of quasi-coherent sheaves over affine schemes $X = \mathrm{Spec}\, A$ The main fact in question is the statement

The functor $\mathcal{F} \longrightarrow \Gamma(X, \mathcal{F})$ determines the equivalence of the category of sheaves with the category of A-modules. (2.130)

The purpose of this section is to show the existence of an analogous correspondence between the category of quasi-coherent sheaves on Proj R and the category of *graded* R-modules.

This correspondence is not, however, that simple and straightforward as in the affine case; in particular, the graded modules that differ in only finitely many components lead to isomorphic sheaves. For a precise formulation, see sect. 2.11.6.

2.11.2. Let $X = \mathrm{Proj}\, R$, where the ring R is generated by the set R_1 over R_0; let $\mathcal{F}$ be a quasi-coherent sheaf over X. Set

$$\Gamma_*(X, \mathcal{F}) := \bigoplus_{n=0}^{\infty} \Gamma(X, \mathcal{F}(n)).$$

If $\mathcal{F} = \mathcal{O}_X$, then on $\Gamma_*(X, \mathcal{O}_X)$ there exists a natural structure of a graded ring with multiplication induced by the homomorphisms $\mathcal{O}_X(n) \times \mathcal{O}_X(m) \longrightarrow \mathcal{O}_X(m+n)$.

More generally, the multiplications $\mathcal{O}_X(n) \times \mathcal{F}(m) \longrightarrow \mathcal{F}_X(m+n)$ determine on $\Gamma_*(X, \mathcal{F})$ the structure of a graded $\Gamma_*(X, \mathcal{O}_X)$-module.

Now, recall that, for any n, there are defined group homomorphisms

$$\alpha_n \colon R_n \longrightarrow \Gamma(X, \mathcal{O}_X(n)).$$

(see Theorem 2.7.4). The definition makes clear that the homomorphisms are compatible with multiplication and hence provide with a homogeneous homomorphism of graded rings

$$\alpha \colon R \longrightarrow \Gamma_*(X, \mathcal{O}_X).$$

Therefore, *on $\Gamma_*(X, \mathcal{F})$, there is a canonical structure of a graded R-module.*

2.11.3. Conversely, let M be a graded R-module. Let us construct from it a quasi-coherent sheaf $\widetilde{M}$ on X by setting

$$\Gamma(D_+(f), \widetilde{M}) = M_{(f)} \quad \text{for any } f \in R_1$$

and defining the restriction homomorphisms as in § 2.4 for $M = R$.

The construction of the homomorphism $\alpha \colon M \longrightarrow \Gamma_*(X, \widetilde{M})$ is easy to be translated, component-wise, to this case:

$$\alpha_n \colon M_n \longrightarrow \Gamma_*(X, \widetilde{M}(n)), \quad \alpha_n(m)|_{D_+(f)} = m/f^n \in M_{(f)}.$$

2.11.3a. Exercise. Verify compatibilities.

2.11.4. Proposition. *Any quasi-coherent sheaf $\mathcal{F}$ on X is isomorphic to a sheaf of the form $\widetilde{M}$, where $M = \Gamma_*(X, \mathcal{F})$.*

Proof. First of all let us construct an isomorphism

$$\beta \colon \widetilde{M} = \widetilde{\Gamma_*(X, \mathcal{F})} \longrightarrow \mathcal{F}.$$

It suffices to construct this isomorphism for the sections of the corresponding sheaves over opens $D_+(f)$. We have

$$\Gamma(D_+(f), \widetilde{M}) = \widetilde{M}_{(f)} = \left[\bigoplus_{n=0}^{\infty} \Gamma(X, \mathcal{F}(n))\right]_{(f)} = \left\{\frac{m}{f^n} \mid m \in \Gamma(X, \mathcal{F}(n))\right\}.$$

Set

$$\beta\left(\frac{m}{f^n}\right) = \alpha(m)|_{D_+(f)} \cdot \alpha(f)^{-n}|_{D_+(f)} \in \Gamma(D_+(f)).$$

2.11.4a. Exercise. Verify that the notion is well defined.

It only remains to establish that β is an isomorphism. In the same way as for the structure sheaf, one can prove that $\operatorname{Ker} \alpha_n = 0$ for n sufficiently large. This implies that β is a monomorphism; indeed,

$$\alpha(m)\alpha(f)^{-n} = 0 \Longrightarrow \alpha(f^e m) = 0, \quad e \geq e_0 \Longrightarrow m/f^n = 0.$$

Lemma 2.10.1a immediately implies that it is an epimorphism: indeed, to extend the section $s \in \Gamma(D_+(f), \mathcal{F})$ to a section of $\mathcal{F}(n)$ over this set means precisely to represent s as an image of a m/f^n. □

The following result, analogous to a result of sects. 2.11.2–2.11.3 is deeper, but concerns only the homomorphism α.

2.11.5. Theorem. *Let the ring R satisfy the conditions of Serre's theorem 2.10.1, M a graded Noetherian R-module, $\mathcal{F} = \widetilde{M}$. Then the sheaf $\mathcal{F}$ is coherent and the map*

$$\alpha_n \colon M_n \longrightarrow \Gamma(X, \mathcal{F}(n))$$

is an isomorphism for n sufficiently large.

Proof. First of all, as in Serre's theorem, it is easy to see that it suffices to carry out the proof for the case $X = \mathbb{P}^r_A$. Let us have this in mind in what follows.

There exists an exact sequence of the form

$$L \xrightarrow{f} L' \longrightarrow M \longrightarrow 0,$$

where L and L' are free graded R-modules, i.e., direct sums of R-modules $R(n)$.

This gives an exact sequence of sheaves

$$\widetilde{L}(n) \xrightarrow{\tilde{f}(n)} \widetilde{L}'(n) \longrightarrow \widetilde{M}(n) \longrightarrow 0,$$

which immediately implies that the sheaf $\mathcal{F}$ is coherent and with the help of which one constructs the exact cohomology sequence in the bottom line of the following commutative diagram:

$$\begin{tikzcd}
L_n \arrow[r] \arrow[d, "\alpha"'] & L'_n \arrow[r] \arrow[d, "\alpha"'] & M_n \arrow[r] \arrow[d, "\alpha"'] & 0 \arrow[d] \\
\Gamma(X, \widetilde{L}(n)) \arrow[r] & \Gamma(X, \widetilde{L}'(n)) \arrow[r] & \Gamma(X, \mathcal{F}(n)) \arrow[r] & H^1(X, f(\widetilde{L})(n)).
\end{tikzcd}$$

By Serre's theorem $H^1(X, f(\widetilde{L})(n)) = 0$ for $n \geq n_0$, and since $X = \mathbb{P}^r_A$, Lemma 2.7.5a shows that the first two vertical arrows are isomorphisms. Hence, $\alpha \colon M_n \longrightarrow \Gamma(X, \mathcal{F}(n))$ is also an isomorphism. □

2.11.6. Main theorem on correspondence $\mathcal{F} \longrightarrow \Gamma_*(X, \mathcal{F})$. We formulate it without proof: A good deal of it is already verified; to prove the rest of it does not require any new ideas.

Let R be a graded ring satisfying the conditions of Serre's theorem. Let GM_R denote the *category* defined as follows:

the objects of GM_R are graded Noetherian R-modules;

the morphisms in GM_R: for any two R-modules $M = \oplus M_i$ and $N = \oplus N_i$ we set

$$\mathrm{Hom}(M, N) = \varinjlim_{i_0} \mathrm{Hom}_R\Big(\bigoplus_{i \geq i_0} M_i, \bigoplus_{i \geq i_0} N_i\Big),$$

where the sign $\varinjlim$ is applied to the group Hom_R of R-homomorphisms of graded R-modules.

Informally speaking, in the category GM_R the homomorphism $M \longrightarrow N$ defined only in degrees sufficiently high is represented; two homomorphisms coinciding in high degrees are equal.

Modules M and N, isomorphic as objects of GM_R, are said to be *TN-isomorphic.*

Theorem. *The functor $\mathcal{F} \longrightarrow \Gamma_*(X, \mathcal{F})$ determines an equivalence of the category of coherent sheaves on $X = \mathrm{Proj}\, R$ with GM_R; its inverse is the functor*

$$M \longrightarrow \widetilde{M}.$$

2.12. Applications to the theory of Hilbert polynomials

2.12.1. The results of the later sections allow one to give a "geometric" definition of the Hilbert polynomial and prove invariance of a number of numerical characteristics introduced in § 2.5.

In what follows, R is a ring satisfying the conditions of Serre's theorem and such that, moreover, $R_0 = k$ is a *field* (this is needed to count dimensions of homogeneous components; all the results to follow can be easily generalized to the case where R_0 is an Artinian ring by considering the lengths of the modules instead of the dimensions of linear spaces.

2.12.2. Theorem. *Let $X = \mathrm{Proj}\, R$, M a Noetherian R-module, $\mathcal{F} = \widetilde{M}$, $h_M(n)$ the Hilbert polynomial of M. Then for all $n \in \mathbb{Z}$, we have*

$$h_M(n) = \sum_{i=0}^{\infty} (-1)^i \dim H^i(X, \mathcal{F}(n)) \stackrel{def}{=} \chi(\mathcal{F}(n)).$$

The number $\chi(\mathcal{F})$ is called the *Euler characteristic* of the sheaf $\mathcal{F}$.

Proof. Serre's theorem and Theorem 2.11.4 imply that

$$h_M(n) = \dim M_n = \dim H^0(X, \mathcal{F}(n)) = \chi(\mathcal{F}(n)) \text{ for } n \geq n_0.$$

Therefore, the result desired is obtained if we establish that $\chi(\mathcal{F}(n))$ can be represented as a polynomial in n *for all* n. The idea is the same as in the proof of existence of Hilbert polynomial. It is bases on the following lemma.

2.12.2a. Lemma. *Let*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow 0$$

be an exact sequence of coherent sheaves on X*. Then*

$$\chi(\mathcal{F}) = \chi(\mathcal{F}_1) + \chi(\mathcal{F}_2).$$

Proof. Let $r = \dim R_1$, then $H^i(X, \mathcal{F}) = 0$ for $i \geq r$. The exact cohomology sequence

$$0 \longrightarrow H^0(X, \mathcal{F}_1) \longrightarrow H^0(X, \mathcal{F}) \longrightarrow \dots \longrightarrow H^{r-1}(X, \mathcal{F}) \longrightarrow H^{r-1}(X, \mathcal{F}_2) \longrightarrow 0$$

implies that the alternated sum of dimensions of these cohomology spaces vanishes. This proves the desired. □

2.12.2b. Corollary. *For any exact sequence*

$$0 \longrightarrow \mathcal{F}_0 \longrightarrow \dots \longrightarrow \mathcal{F}_k \longrightarrow 0$$

of coherent sheaves on X *we have:*

$$\sum_{i=0}^{k} (-1)^i \chi(\mathcal{F}_i) = 0.$$

Now, by induction on r we prove that $\chi(\mathcal{F}(n))$ is a polynomial in n. A standard reduction makes it possible to assume that $X = \mathbb{P}_k^{r-1} = \operatorname{Proj}\ k[T_1, \dots, T_r]$. Let $M \xrightarrow{T_r} M(1)$ be a homomorphism of multiplication by T_r (it preserves the degree: $M(1)_i = M_{i+1}$). If K is the kernel and C the cokernel of this homomorphism, then the following sequence is exact:

$$0 \longrightarrow K \longrightarrow M \xrightarrow{T_r} M(1) \longrightarrow C \longrightarrow 0.$$

We derive from here an exact sequence of sheaves

$$0 \longrightarrow \widetilde{K} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}(1) \longrightarrow \widetilde{C} \longrightarrow 0$$

and an exact sequence of twisted sheaves

$$0 \longrightarrow \widetilde{K}(n) \longrightarrow \mathcal{F}(n) \longrightarrow \mathcal{F}(n+1) \longrightarrow \widetilde{C}(n) \longrightarrow 0.$$

By Corollary 2.12.2b we have

$$\chi(\mathcal{F}(n)) - \chi(\mathcal{F}(n+1)) = \chi(\widetilde{K}(n)) - \chi(\widetilde{C}(n)).$$

The sheaves $\widetilde{K}$ and $\widetilde{C}$ correspond to $k[T_1, \dots, T_{r-1}]$-modules, i.e., are "concentrated" on $\mathbb{P}_k^{r-1}$. This allows one to make the induction step: For $r = 0$. the statement is trivial. □

2.12.3. Comment. Theorem 2.12.2 shows that the Hilbert polynomial is invariantly determined by the triple of geometric objects $(X, \mathcal{L}, \mathcal{F})$, where $X = \operatorname{Proj} R$, $\mathcal{L} = \mathcal{O}_X(1)$, and $\mathcal{F}$ is a coherent sheaf on X. Let us show that the *degree* of this polynomial and its *constant term* only depend, actually, on $(X, \mathcal{F})$, but not on $\mathcal{L}$.

In particular, the *dimension* $\dim X$ and *characteristic* $\chi(X)$, introduced above, do not depend on representation of X as $\operatorname{Proj} R$.

The claim on constant term is obvious:

$$h_M(0) = \chi(\mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i \dim H^i(X, \mathcal{F}).$$

2.12.4. Theorem. *Set*

$$\dim \mathcal{F} = \deg \chi(\mathcal{F}(n)).$$

Then $\dim \mathcal{F}$ *does not depend on the choice of invertible sheaf* $\mathcal{L}$.

Proof. The formulation of the theorem assumes that $\mathcal{L}$ is *very ample*, i.e., is of the form $\mathcal{O}(1)$ for a representation of X in the form $\operatorname{Proj} R$. We have no means for characterization of such sheaves apart from Serre's theorem; let us use it.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two very ample sheaves on X. For any $n \geq n_0$, we have $h_i(n) = \dim H^0(X, \mathcal{F} \otimes \mathcal{L}_i^n)$, where $i = 1, 2$. The sheaf $\mathcal{M} = \mathcal{L}_1^{-1} \otimes \mathcal{L}_2^N$ for N sufficiently large is generated by its sections, as follows from Lemma 2.10.1a.

Since $(\mathcal{F} \otimes \mathcal{L}_1^n) \otimes \mathcal{M}^n = \mathcal{F} \otimes \mathcal{L}_2^{nN}$, we have an isomorphism of groups of global sections

$$H^0(X, \mathcal{F} \otimes \mathcal{L}_1^n \otimes \mathcal{M}^n) \simeq H^0(X, \mathcal{F} \otimes \mathcal{L}_2^{nN}).$$

The canonical map

$$H^0(X, \mathcal{F} \otimes \mathcal{L}_1^n) \otimes S \longrightarrow H^0(X, \mathcal{F} \otimes \mathcal{L}_1^n \otimes \mathcal{M}^n)$$

is injective for the zero section $S \in H^0(X, \mathcal{M}^n)$ because $\mathcal{M}^n$ is locally free. But $\dim H^0(X, \mathcal{M}) \geq 1$, so $\dim H^0(X, \mathcal{M}^n) \geq 1$. This implies that

$$\dim H^0(X, \mathcal{F} \otimes \mathcal{L}_1^n) \leq \dim H^0(X, \mathcal{F} \otimes \mathcal{L}_2^{nN})$$

or

$$h_1(n) \leq h_2(nN) \quad \text{for any } n \geq n_0.$$

By symmetry, $h_2(n) \leq h_1(nN')$, so $\deg h_1 = \deg h_2$. Theorem is proved. $\square$

Let us give now another description of the dimension, often useful. Recall that $f \in R$ is said to be an *essential zero divisor* in the graded R-module M, if the kernel of multiplication by $f \colon M \xrightarrow{f} M$ has infinitely many non-zero homogeneous components, i.e., is non-trivial in the category GM_R.

2.12.5. An *M*-sequence. Let M be a graded R-module. A finite set of homogeneous and non-invertible elements $\{f_0, \dots, f_d\}$ in R such that f_i is not an essential zero divisor in $M/(f_0, \dots, f_{i-1})$ for all $i \geq 0$ will be called an *M-sequence.*

The number d is the *length* of the M-sequence $\{f_0, \dots, f_d\}$. A given M-sequence is said to be *maximal* if it is impossible to add to it any element of R and get an M-sequence.

The symbol $d(M)$ will denote the *length of the shortest of the maximal M-sequences*.

2.12.6. Theorem. $d(M) = \dim \widetilde{M}$.

We will need the following Lemma.

2.12.7. Lemma. *If* $\dim \widetilde{M} \geq 0$*, then there exists an element* $f \in R$ *such that* $\{f\}$ *is an* M*-sequence.*

We will prove Lemma later, let us now use it to prove Theorem 2.12.6.

2.12.8. Proof of Theorem 2.12.6. We induct on $d(M)$, starting from $d(M) = -1$ in which case there are no M-sequences.

For $d(M) = -1$, Lemma 2.12.7 immediately implies that $\dim \widetilde{M} = -1$.

Let Theorem be proved for all M with $d(M) \leq d-1$ and let $d(M) = d$. Let $\{f_0, \dots, f_d\}$ be the shortest maximal M-sequence. Then $d(M/f_0M) = d - 1$ by induction hypothesis $\dim (\widetilde{M/f_0M}) = d - 1$. Consider an exact sequence

$$0 \longrightarrow N \longrightarrow M \xrightarrow{f_0} M(k) \longrightarrow M/f_0M \longrightarrow 0,$$

where $k = \deg f_0$ and $M \xrightarrow{f_0} M(k)$ is the grading-preserving homomorphism of multiplication by f_0. Since $f_0 \in R$ is an inessential zero divisor, we have $N_n = 0$ for $n \geq n_0$ and $\dim M(k)_n - \dim M_n = \dim(M/f_0M)_n$. Therefore $\dim \widetilde{M} = (d-1) + 1 = d$, proving the desired. □

2.12.9. Proof of Lemma 2.12.7. In R, we have to find an inessential zero divisor in M.

First of all let us show that there exists a sequence of graded modules M_i such that

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M, \quad M_i/M_{i+1} \simeq (R/p_i)_{n_i},$$

where p_i are prime graded ideals.

Since M is a Noetherian module, it suffices to find in it a non-trivial graded submodule $M_1 \subset M$ such that $M_1 \simeq (R/p)_n$, where p is a prime graded ideal.

Let S be the set of graded ideal p_m in R for each of which there exists a homogeneous element $m \in M$ whose annihilator is p_m. Since R is Noetherian, there is a maximal element in S; denote it p. Clearly, $R_m \simeq (R/p)_n$, where $p = \operatorname{Ann} m$, $n = \deg m$. Let us prove that p is prime. Indeed, let $ab \in p$, $a \notin p$.

Then the inclusion $\operatorname{Ann}(bm) \supseteq (a, \operatorname{Ann} m)$ is strict by maximality, so $bm = 0$ and $b \in \operatorname{Ann} m$. So p is prime.

(Observe that the same argument without taking grading into account proves a similar result for non-graded Noetherian modules.)

We have thus constructed a sequence of modules desired. Let us use it to find in R an inessential zero divisor in M. Let q be a maximal graded prime ideal. If $q \not\subset \cup p$, then, for an element of R to be found, we can take any element of q not lying in $\cup p_i$. (Since it lies in q, it is non-invertible.) If, however, $q \subset \cup p_i$ for any maximal ideal q, then $q \subset p_j$ for some j since the ideals p_i are prime, so the ideals p_i exhaust the set of maximal ideals. Since there are finitely many of them, $\dim R = 0$.

In this case $R \approx \Gamma(X, \mathcal{O}_X)[T]$ is the polynomial ring in one indeterminate T ("$\approx$" means *isomorphism up to a finite number of homogeneous components*) because $\dim R = \text{const}$ for $n \geq n_0$. Therefore multiplication by T has no kernel in all sufficiently large dimensions, and T is the inessential zero divisor to be found.

Both Lemma and Theorem are completely proved. □

In the course of the proof we have obtained a number of useful statements; we separate them for convenience of references.

2.12.10. Corollary. 1) $\operatorname{Ann} M = \cup p_i$, *where* $(p_i)_{i \in I}$ *is a finite family of prime ideals in* R.

2) *Let* $\mathcal{F}$ *be a coherent sheaf on* $X = \operatorname{Proj} R$. *Then there exists a sequence of coherent pre-sheaves*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F},$$

such that $\mathcal{F}_{i+1}/\mathcal{F}_i \simeq \mathcal{O}_X/J_i(n_i)$, *where* J_i *are coherent sheaves of prime ideals on* X.

2.12.11. Theorem. $H^q(X, \mathcal{F}) = 0$ *for* $q > \dim \mathcal{F}$.

Proof. Let the theorem be established for some sheaves $\mathcal{F}'$ and $\mathcal{F}''$ entering the exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

Then

$$\chi(\mathcal{F}(n)) = \chi(\mathcal{F}'(n)) + \chi(\mathcal{F}''(n)),$$

and hence $\dim \mathcal{F} = \max(\dim \mathcal{F}', \dim \mathcal{F}'')$. The exact cohomology sequence easily implies the validity of the statement of the theorem for $\mathcal{F}$: It suffices to consider the terms

$$\ldots \longrightarrow H^q(X, \mathcal{F}') \longrightarrow H^q(X, \mathcal{F}) \longrightarrow H^q(X, \mathcal{F}'') \longrightarrow 0,$$

where $q > \dim \mathcal{F}$.

Set $\mathcal{F} = \mathcal{O}_X/\mathcal{J}(n)$, where $\mathcal{J} \subset \mathcal{O}_X$ is a coherent sheaf of ideals. The sheaf $\mathcal{J}$ determines a closed subscheme $Y \subset X$, where $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{J}$, which can be considered as the projective spectrum of the graded ring $\bigoplus_{n=0}^{\infty} \Gamma(X, \mathcal{O}_X/\mathcal{J}(n))$. Therefore, and thanks to heading 2) of Corollary 2.12.10 it suffices to verify the statement for $\mathcal{O}_Y(n) = \mathcal{F}$. Set $J = \operatorname{Im} \Gamma_*(X, \mathcal{J}) \subset R$.

Let $d = \dim Y$. By Theorem 2.12.6 there exists a maximal R/J-sequence $f_0, \dots, f_d$, where $f_i \in R$ such that $\dim_k R/(f_0, \dots, f_d, J) < \infty$. So setting $\overline{f}_i = f_i \pmod J$ we have $(R/J)_n \subset (\overline{f}_0, \dots, \overline{f}_d)$ for $n \geq n_0$ in R/J. Geometrically, this means that $Y = \bigcup_{i=0}^{d} D_+(\overline{f}_i)$. Computing Čech cohomology immediately yields the statement of Theorem. $\square$

2.12.12. Remark. Since $\dim \mathcal{F} \leq \dim X$ for any $\mathcal{F}$ (as follows from the proof of Theorem), then, in particular, $H^q(X, \mathcal{F}) = 0$ for any $q > \dim X$.

2.13. The Grothendieck group: First notions

2.13.1. The classical "Riemann-Roch problem" is to compute $\dim H^0(X, \mathcal{F})$, where X is a projective scheme over a field, and $\mathcal{F}$ is a coherent sheaf on it. The main qualitative information about this function is the statement

$$\dim H^0(X, \mathcal{F}(n)) \textit{ is a polynomial, } \chi, \textit{ for } n \geq n_0(\mathcal{F}). \tag{2.131}$$

In practice, therefore, the Riemann-Roch problem is usually subdivided into two questions solved by distinct approaches:

a) Describe the coefficients of the Hilbert polynomial of $\mathcal{F}$ in "geometric" terms.
b) Find a "good" estimate of the number $n_0(\mathcal{F})$.

For an example of the answer to question a) we offer a derivation of the degree of a given projective scheme $X \subset \mathbb{P}^r$ by means of Bezout's theorem: This degree is equal to the number of intersection points of X with "sufficiently general" linear submanifold of $\mathbb{P}^r_k$ of complementary dimension. The general answer is given by the Riemann-Roch-Hirzebruch-Grothendieck-... theorem.

In these lectures we will not touch question b); the rest of the chapter is devoted to the study of the characteristic $\chi(\mathcal{F})$.

Lemma 2.12.2 describes its main property. Axiomatizing this property we introduce the following definition. Let X be a scheme, Coh_X the category of coherent sheaves on it.

2.13.2. Additive functions on Abelian categories. Let G be an Abelian group; a map $\psi\colon \mathsf{Coh}_X \longrightarrow G$ is said to be an *additive function on* Coh_X (or any other Abelian category) *with values in* G if, for any exact sequence

$$0 \longrightarrow \widetilde{\mathcal{F}}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow 0$$

of sheaves from the category Coh_X, we have

$$\psi(\widetilde{\mathcal{F}}) = \psi(\widetilde{\mathcal{F}}_1) + \psi(\widetilde{\mathcal{F}}_2).$$

Any additive function has the following properties.

2.13.3. Lemma. *Let ψ be an additive function on Coh_X, and $\widetilde{\mathcal{F}}_i$ be objects of Coh_X.*

a) *For any exact sequence*

$$0 \longrightarrow \widetilde{\mathcal{F}}_1 \longrightarrow \dots \longrightarrow \mathcal{F}_K \longrightarrow 0,$$

we have

$$\sum_{i=1}^{K} (-1)^i \psi(\widetilde{\mathcal{F}}_i) = 0.$$

b) *Let $0 = \widetilde{\mathcal{F}}_0 \subset \widetilde{\mathcal{F}}_1 \subset \dots \subset \widetilde{\mathcal{F}}_K = \widetilde{\mathcal{F}}$. Then*

$$\psi(\widetilde{\mathcal{F}}) = \sum_{i \geq 1} \psi(\widetilde{\mathcal{F}}_i / \widetilde{\mathcal{F}}_{i-1}).$$

2.13.3a. Exercise. Prove Lemma.

It is not difficult to see that even for the simplest schemes X (e.g., for $X = \operatorname{Spec} K$) there are plenty of additive functions on Coh_X. Still, the whole totality of them is easy to overview thanks to the existence of a "universal" additive function $k\colon \mathsf{Coh}_X \longrightarrow K(X)$ with values in a universal group $K(X)$.

Let $\mathbb{Z}[\mathsf{Coh}_X]$ be the free Abelian group generated by symbols $[\widetilde{\mathcal{F}}]$ corresponding to classes of isomorphic coherent sheaves on X. Let $J \subset \mathbb{Z}[\mathsf{Coh}_X]$ be the subgroup generated by the elements

$$[\widetilde{\mathcal{F}}] - [\widetilde{\mathcal{F}}_1] - [\widetilde{\mathcal{F}}_2],$$

one for each exact sequence

$$0 \longrightarrow \widetilde{\mathcal{F}}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow 0.$$

2.13.4. $K(X)$, the Grothendieck group. The group $K(X) = \mathbb{Z}[\mathsf{Coh}_X]/J$ is called the *Grothendieck group* (of the category Coh_X or the scheme X).

2.13.4a. Proposition. *The function*

$$k\colon \mathsf{Coh}_X \longrightarrow K(X), \tag{2.132}$$

for which $k(\mathcal{F}) = [\mathcal{F}] \pmod J$ is additive; the image $k(\mathsf{Coh}_X)$ generates the group $K(X)$, and for any additive function $\psi\colon \mathsf{Coh}_X \longrightarrow G$, there exists a uniquely determined homomorphism $\varphi\colon K(X) \longrightarrow G$ such that $\psi = \varphi \circ k$.

Proof is trivial.

From the point of view of the group $K(X)$, the *Riemann-Roch problem* is the problem of describing the function $\chi\colon K(X) \longrightarrow \mathbb{Z}$.

The advantage of such a reformulation is as follows. As we will shortly see, the group $K(X)$ is endowed with a reach collection of additional structures. Sometimes it can be explicitly and completely computed. For example, for $X = \mathbb{P}^r_k$ and then we describe χ using this information. In the general case, there are sufficiently many known data on $K(X)$ to geometrically interpret $\chi(\mathcal{F})$.

Lemma 2.13.3 gives an approach to computing $K(X)$; sometimes it allows one to indicate a smaller system of generators than the set of elements $k(\mathcal{F})$ for *all* coherent sheaves $\mathcal{F}$. Let us illustrate this by examples.

2.13.5. Examples. 1) Let A be any Noetherian ring, $X = \operatorname{Spec} A$. In subsect. 2.12.9 we have established that any Noetherian A-module has a composition series with factors isomorphic to A/p, where $p \subset A$ is a prime ideal. The generators of $K(X)$ are, therefore, the elements $k(A/p)$, where k is the function from eq. (2.132). To describe all relations is a bit more difficult. We confine ourselves to the case where A is an *Artinian* ring.

In this case, the Jordan-Hoelder theorem can be interpreted as computation of $K(X)$: *the map*

$$j\colon K(X) \longrightarrow \mathbb{Z}[X],$$

where $\mathbb{Z}[X]$ *is a free Abelian group generated by the points of the scheme* X*, and*

$$j(k(\mathcal{F})) = \sum_{x\in X} (\operatorname{length}_{\mathcal{O}_x} \mathcal{F}_x)_x$$

is a group isomorphism.

2) Let A be a principal ideal ring, $X = \operatorname{Spec} A$. Then any Noetherian A-module M has a *free projective* resolution of length 1:

$$0 \longrightarrow A^r \longrightarrow A^s \longrightarrow M \longrightarrow 0.$$

This immediately implies that the group $K[X]$ is cyclic and generated by the class of ring A. The order of this class is equal to infinity which is easy to see passing to linear spaces $M \bigotimes\limits_A K$ over the ring of quotients K of A; and hence

$$K(X) \simeq \mathbb{Z}.$$

More generally, *the same is true for any affine scheme* $\operatorname{Spec} A$ *provided any Noetherian A-module has a free resolution of finite length.*

This condition is still too strong to lead to interesting notions; however, even a slight slackening of it defines a very important class of schemes.

2.13.6. Smooth schemes. Let X be a Noetherian scheme, $\mathcal{F}$ a coherent sheaf on it. Let, for any point $x \in X$, there exists an open neighborhood $U \ni x$ such that the sheaf $\mathcal{F}|_U$ has in this neighborhood a free resolution consisting of "free" sheaves $\mathcal{O}^r_X|_U$. Then the scheme X is said to be *smooth.*

In the next section we will prove that the two types of schemes are smooth: the projective spaces over fields and spectra of local rings whose maximal ideals are generated by regular sequences.

Smoothness of projective spaces immediately follows from the following classical *Hilbert's theorem on syzygies.* [6)]

2.13.7. Theorem. *Let $R = K[T_0, \dots, T_r]$. Any graded R-module graded free projective resolution of length $\leq r+1$.*

Proof will be given at the end of the chapter; now we use this theorem to compute $K(\mathbb{P}^r_K)$.

2.13.8. Theorem. *The map $x^i \longmapsto k(\mathcal{O}(i))$, where k is the function from eq.* (2.132)*, determines an isomorphism of Abelian groups*

$$\mathbb{Z}[x]/((x-1)^{r+1}) \longrightarrow K(\mathbb{P}^r_K).$$

In particular, this group is free of rank $r+1$.

Proof. Translating the first statement of Theorem 2.13.7 into the language of sheaves by means of Theorem 2.11.6 we find that any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r_K$ possesses a resolution the terms of which are direct sums of sheaves $\mathcal{O}(n)$, where $n \in \mathbb{Z}$. Lemma 2.13.3 shows that the elements $k(\mathcal{O}(n))$ generate the group $K(\mathbb{P}^r_K)$.

Obviously, these generators are not independent. At least one relation is obtained if, using Proposition 2.9.5, we consider the Koszul complex $K_0(T_0, \dots, T_R; R)$ which is a resolution of the R-module $K = R/(T_0, \dots, T_n)$:

$$\dots \longrightarrow R^{\binom{r+1}{i}} \longrightarrow \dots \longrightarrow R^{\binom{r+1}{2}} \longrightarrow R^{r+1} \longrightarrow R \longrightarrow K \longrightarrow 0,$$
$$\text{where } \; R^{\binom{r+1}{i}} = \wedge^i(Re_0 + \dots + Re_r) = K_i(T, R).$$

We can consider this resolution as an exact sequence of graded modules if we consider the elements $e_{i_1} \wedge \dots \wedge e_{i_k}$ homogeneous of degree k. Applying to this resolution the *sheafification functor* we get the exact sequence

$$0 \longrightarrow \dots \longrightarrow \mathcal{O}(-i)^{\binom{r+1}{i}} \longrightarrow \dots \longrightarrow \mathcal{O}(-2)^{\binom{r+1}{2}} \longrightarrow \mathcal{O}(-1)^{r+1} \longrightarrow \mathcal{O}_{\mathbb{P}^r} \longrightarrow 0$$

(take into account that the R-module K is TN-isomorphic to 0, i.e., $\widetilde{K} = 0$). This exact sequence can be tensored by $\mathcal{O}(n)$ for any $n \in \mathbb{Z}$ without violating the exactness. Therefore, there are the following relations in $K(\mathbb{P}^r_K)$:

---

[6] In broadest terms, *syzygy* is a kind of unity, especially through coordination or alignment, most commonly used in the astronomical and/or astrological sense. In mathematics, a syzygy is a relation between the generators of a module M. The set of all such relations is called the *first syzygy module of M*. A relation between generators of the first syzygy module is called a *second syzygy* of M, and the set of all such relations is called the *second syzygy module of M*, and so on.

$$\sum_{i=0}^{r+1} \binom{r+1}{i} k(\mathcal{O}(n-i)) = 0.$$

This clearly implies that the kernel of the homomorphisms of additive groups

$$\mathbb{Z}[x, x^{-1}] \longrightarrow K(X); \quad x^i \mapsto k(\mathcal{O}(i)) \text{ for } i \in \mathbb{Z}$$

contains an ideal generated by the polynomial $(x^{-1} - 1)^{r+1}$ or, equivalently, by $(x-1)^{r+1}$.

Since we have already established epimorphic property of this homomorphism, to complete the proof of Theorem, it suffices to verify that the elements $k(\mathcal{O})$, $k(\mathcal{O}(1))$, ..., $k(\mathcal{O}(r))$ are linearly independent over $\mathbb{Z}$.

Obviously, the functions $\chi_n(\mathcal{F}) = \chi(\mathcal{F}(n))$ are additive on $S_{\mathbb{P}^r}$ for any $n \in \mathbb{Z}$. Therefore, if there existed a non-trivial linear dependence

$$\sum_{i=0}^{r} a_i k(\mathcal{O}(i)) = 0, \text{ where } a_i \in \mathbb{Z},$$

it would imply that

$$\sum_{i=0}^{r} a_i \chi_n(\mathcal{O}(i)) = \sum_{i=0}^{r} a_i \binom{n+i+r}{r} = 0,$$

which is only possible if $a_i = 0$ for all i since, as an easy verification shows (say, by induction on r), the polynomials $\binom{n+i+r}{r}$ in n are linearly independent for $i = 0, \dots, r$. $\square$

2.13.9. The group $K(\mathbb{P}^r_K)$ in formulation of Theorem 2.13.8 turned out endowed with a ring structure. The multiplication in this ring possesses an invariant meaning: Indeed, as is easy to see

$$k(\mathcal{F})k(\mathcal{O}_{\mathbb{P}^r}(i)) = k(\mathcal{F}(i)) = k(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^r}(i)),$$

so this multiplication corresponds, at least sometimes, to tensor products of sheaves. There are, however, examples for which $k(\mathcal{F})k(\mathcal{G}) \neq k(\mathcal{F} \otimes \mathcal{G})$, So the general description of multiplication can not be that simple. This question is studied in detail in the second part of the course [Ma3].

Meanwhile, using our description of $K(\mathbb{P}^r_K)$, we give a (somewhat naive) form of the Riemann-Roch theorem for the projective space.

The idea is to select some simple additive function on $K(\mathbb{P}^r_K)$, and then "propagate" it using ring multiplication.

Any element of $K(\mathbb{P}^r_K)$ can be uniquely represented, thanks to Theorem 2.13.8, as a polynomial $\sum\limits_{i=0}^{r} a_i (l-1)^i$, where $l = k(\mathcal{O}(1))$. Introduce an additive function $\varkappa_r \colon K(\mathbb{P}^r_K) \longrightarrow \mathbb{Z}$ by setting

$$\varkappa_r\left(\sum_{i=0}^{r} a_i (l-1)^i\right) = a_r.$$

2.13.10. Lemma. *For any additive function*

$$\psi\colon K(\mathbb{P}^r_K) \longrightarrow \mathbb{Z},$$

there exists a unique element $t(\psi) \in K(\mathbb{P}^r_K)$ such that

$$\psi(y) = \varkappa_r(t(\psi)y), \textit{ for any } y \in K(\mathbb{P}^r_K).$$

Proof. The function ψ is a linear combination of coefficients a_i in the representation $\sum\limits_{i=0}^{r} a_i(l-1)^i$, where these coefficients are considered as functions on $K(\mathbb{P}^r_K)$; for these coefficients, we have:

$$a_i(y) = \varkappa_r((l-1)^{r-i}y). \qquad \square$$

2.13.11. Theorem. *Let $\chi\colon K(\mathbb{P}^r_K) \longrightarrow \mathbb{Z}$ be the Euler characteristic. Then*

$$\chi(y) = \varkappa_r(l^2 y), \quad \textit{i.e.}, \ t(\chi) = l^2. \tag{2.133}$$

Proof. It suffices to verify the coincidence of the left side with the right side for the elements l^i, where $i = 0, 1, \dots, r$, constituting a $\mathbb{Z}$-basis of the group $K(\mathbb{P}^r_K)$. We have:

$$\begin{aligned}
\chi(l^i) &= \chi(\mathcal{O}(i)) = \dim H^0(\mathbb{P}^r_K, \mathcal{O}(i)) = \binom{r+i}{i},\\
\varkappa_r(l^{r+i}) &= \varkappa_r((1+(l-1))^{r+i}) = \binom{r+i}{i}.
\end{aligned} \qquad \square$$

2.13.12. Remark. The usage of $\varkappa_r$ as a simplest additive function was not yet motivated. Besides, it is clear that to apply Theorem 2.13.11 is not that easy: To compute $\chi(\mathcal{F})$ with its help, we have to first know what is the class of the sheaf $\mathcal{F}$ in $K(\mathbb{P}^r_K)$. The only means known to us at the moment is to consider the resolution of the sheaf $\mathcal{F}$, but this is not "geometric", besides, this makes our formula useless: If we know the resolution, we can calculate $\chi(\mathcal{F})$ just by additivity.

Nevertheless, formula (2.133) is very neat; I consider this as a serious argument in its favor.

2.14. Resolutions and smoothness

Let us show, first of all, that smoothness of a given scheme X is a property of the totality of its local rings $\mathcal{O}_x$.

2.14.1. Theorem. *A scheme X is smooth if and only if* $\operatorname{Spec} \mathcal{O}_x$ *is a smooth scheme at all points $x \in X$.*

Proof. Let X be smooth. Consider an arbitrary coherent sheaf on $\operatorname{Spec} \mathcal{O}_x$. This sheaf is determined by a $\mathcal{O}_x$-module $\mathcal{F}_x$. Let us show that, in a neighborhood $U \ni x$, there exists a sheaf $\mathcal{F}$ whose fiber at x coincides with $\mathcal{F}_x$. Indeed, consider the exact sequence

$$\mathcal{O}_x^r \xrightarrow{f} \mathcal{O}_x^s \longrightarrow \mathcal{F}_x \longrightarrow 0,$$

where the homomorphism f is determined by a $r \times s$ matrix whose entries are the germs of sections of the structure sheaf of X at point x. There exists an affine neighborhood $\operatorname{Spec} A$ of this point onto which the elements of this matrix can be continued. Let

$$f \colon A^r \longrightarrow A^s$$

be the homomorphism determined by this continuation. Set $M = \operatorname{Coker} f$ and let $\mathcal{F}$ be the sheaf $\widetilde{M}$ considered on $\operatorname{Spec} A$. Its fiber at point x is isomorphic to $\mathcal{F}_x$. Since the scheme X is smooth, we can assume that $U = \operatorname{Spec} A$ is so small that there is a finite resolution in it:

$$0 \longrightarrow \mathcal{O}_x^{r_n}|_U \longrightarrow \ldots \longrightarrow \mathcal{O}_x^{r_0}|_U \longrightarrow \mathcal{F} \longrightarrow 0.$$

Passing to fibers at point x we obtain a finite resolution of the sheaf $\widetilde{\mathcal{F}}_x$ on $\operatorname{Spec} \mathcal{O}_x$; this proves that $\operatorname{Spec} \mathcal{O}_x$ is smooth.

Conversely, let $\operatorname{Spec} \mathcal{O}_x$ be a smooth scheme, $\mathcal{F}$ a coherent sheaf on X. Since the only neighborhood of a closed point x in $\operatorname{Spec} \mathcal{O}_x$ is the whole spectrum, there exists a resolution of the fiber

$$0 \longrightarrow \mathcal{O}_x^{r_n} \longrightarrow \ldots \longrightarrow \mathcal{O}_x^{r_0} \longrightarrow \mathcal{F}_x \longrightarrow 0.$$

The argument analogous to the above one allows us to continue this sequence onto an open neighborhood U of the point x:

$$0 \longrightarrow \mathcal{O}_x^{r_n}|_U \longrightarrow \ldots \longrightarrow \mathcal{O}_x^{r_0}|_U \longrightarrow \mathcal{F}|_U \longrightarrow 0.$$

Since this sequence is exact at the point x, it remains exact in a (perhaps, smaller than U) neighborhood of x, proving the desired. $\square$

This result indicates the importance of study of the smooth spectra of local rings. For this, we will need several elementary results from homological algebra.

The next result allows one to overview the family of resolutions of a given A-module M.

2.14.2. Lemma. *Let A be a ring, M an A-module; P and P' either projective or free A-modules. Let*

$$0 \longrightarrow S \longrightarrow P \xrightarrow{f} M \longrightarrow 0,$$
$$0 \longrightarrow S' \longrightarrow P' \xrightarrow{f} M \longrightarrow 0$$

be two exact sequences. Then $S \oplus P' \simeq S' \oplus P$.

Proof. Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] \arrow[dr, "g"'] & P \oplus P' \arrow[r, "f+f'"] \arrow[d, "pr"] & M \\ & & P' & \end{tikzcd}$$

where $T = \operatorname{Ker}(f + f')$. Clearly, g is an epimorphism; since P' is projective, there exists a "section" $s\colon P' \longrightarrow T$ (i.e., a homomorphism such that $g \circ s = 1_T$) and $T \simeq \operatorname{Ker} g \oplus P'$. But

$$\operatorname{Ker} g = \{(p, 0) \mid f(p) = 0\} \simeq S,$$

so $T \simeq S \oplus P'$. By symmetry, $T \simeq S' \oplus P$, proving Lemma. □

This result is a reason to introduce the following definition.

2.14.3. Modules equivalent projectively or freely. The two A-modules S and S' are said to be *projectively* (respectively *freely*) *equivalent*, if there exist projective (resp., free) A-modules P and P' such that

$$S \oplus P' \simeq S' \oplus P.$$

We can now sharpen Lemma 2.14.2 as follows.

2.14.3a. Lemma. *Let A-modules M and M' be projectively* (*resp., freely*) *equivalent and the following exact sequences*

$$\begin{gathered} 0 \longrightarrow S \longrightarrow P \longrightarrow M \longrightarrow 0, \\ 0 \longrightarrow S' \longrightarrow P' \longrightarrow M' \longrightarrow 0, \end{gathered}$$

be given in which P and P' are projecte (*resp., free*) *modules. Then S and S' are projectively* (*resp., freely*) *equivalent.*

Proof. Let $M \oplus Q \simeq M' \oplus Q'$, where Q and Q' are projective (resp., free). Then the following sequences are exact (with obvious homomorphisms)

$$\begin{gathered} 0 \longrightarrow S \longrightarrow P \oplus Q \longrightarrow M \oplus Q \longrightarrow 0, \\ 0 \longrightarrow S' \longrightarrow P' \oplus Q' \longrightarrow M' \oplus Q' \longrightarrow 0, \end{gathered}$$

implying that $S \oplus P' \oplus Q' \simeq S \oplus P \oplus Q$ by Lemma 2.14.2. □

2.14.3b. Corollary. *Let there is given the beginning part of a projective* (*resp., free*) *resolution of an A-module M*

$$P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow M.$$

Then the class up to projective (*resp., free*) *equivalence of the module* $\operatorname{Ker} d_n$ *is uniquely determined and does not depend on the choice of the resolution. In particular, it there exists a projective resolution of length n of the module M, then* $\operatorname{Ker} d_n$ *is projective.*

For modules over local rings, we construct a special type of resolutions, so-called *minimal resolutions*; Corollary 2.14.3a enables to derive from these minimal resolutions information about arbitrary resolutions.

Let A be a local Noetherian ring, $\mathfrak{m}$ its maximal ideal, M an A-module of finite type, F a free A-module.

An epimorphism $F \to M \longrightarrow 0$ is said to be *minimal*, if it induces an isomorphism $F/\mathfrak{m}F \longrightarrow M/\mathfrak{m}M$.

The minimal epimorphism is determined uniquely in the following sense. If $F' \longrightarrow M \longrightarrow 0$ is another minimal epimorphism, then there exists a commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & M \arrow[r] & 0 \\ F' \arrow[ur] & & \end{tikzcd}$$

in which the vertical arrow is an isomorphism. Indeed $\operatorname{rk} F = \operatorname{rk} F'$ by Corollary 1.10.9a and the pre-images of any basis of $M/\mathfrak{m}M$ in F and F' are bases in F and F', respectively.

2.14.4. Minimal resolutions. Iterating minimal epimorphisms, we arrive at the notion of the *minimal resolution* (here A is supposed to be *Noetherian*)

$$\dots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \dots \longrightarrow F_0 \longrightarrow M \longrightarrow 0;$$

this resolution is minimal if $F_n \xrightarrow{d_n} d_n(F_n)$ *is a minimal epimorphism* for all n.

Claim: *A resolution is minimal if and only if*

$$d_n(F_n) \subset mF_{n-1} \quad \text{for all} \quad n \geq 1.$$

2.14.4a. Exercise. Prove that this claim.

2.14.5. Examples. 1) *The minimal resolution.* Let $f_1, \dots, f_r \subset \mathfrak{m}$ be a regular sequence of elements. Then the Koszul complex $K_0(f, A)$ is a minimal resolution of the A-module $A/(f_1, \dots, f_r)$.

2) *An infinite minimal resolution.* Consider the local ring $A = k + t^2k[[t]]$, and let $M = \mathfrak{m} = t^2k[[t]] \subset A$. Then $M = At^2 + At^3$, and $\mathfrak{m}^2 = \mathfrak{m}M = t^4k[[t]]$, and hence, $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2$.Here is the beginning of the minimal resolution:

$$Af_1 \oplus Af_2 \xrightarrow{d_1} Ae_1 \oplus Ae_2 \xrightarrow{\varepsilon} M \longrightarrow 0;$$

$$\varepsilon(e_i) = t^{i+1};$$

$$d_1(f_1) = t^3e_1 - t^2e_2;$$

$$d_1(f_2) = t^4e_1 - t^3e_2.$$

It is easy to see that $\operatorname{Ker} d_1 \simeq M$; so in the next part of the resolution this segment will be periodically repeated again and again.

Let us prove now the following theorem on smooth local rings.

2.14.6. Theorem. *Let A be a local Noetherian whose maximal ring is generated by a regular sequence $x_1, \dots, x_d$. Then any Noetherian A-module M has a free resolution of length $\leq d$.*

2.14.6a. Lemma. *Under assumptions of Theorem, construct a minimal resolution (F_n, d_n) of the module M and set*

$$S^n_A(M) = \operatorname{Ker} d_n.$$

Then for any $x \in m$ which is not a zero divisor in A or in M, we have:

$$S^n_{A/xA}(M/xM) \simeq S^n(M)/xS^n(M).$$

Proof. We may confine ourselves to the case where $n = 1$. Consider the commutative diagram

$$\begin{tikzcd}
F \arrow[r, "\varphi"] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
F/xF \arrow[r, "\psi"] & M/xM \arrow[r] & 0
\end{tikzcd}$$

where φ is the minimal epimorphism. The definition easily implies that ψ is also minimal (as an epimorphism of A/xA-modules). Therefore $S^1_A(M) = \operatorname{Ker}\varphi$, and $S^1_{A/xA}(M/xM) = \operatorname{Ker}\psi$.

There exists a unique homomorphism θ for which the diagram

$$\begin{tikzcd}
S^1_A(M) \arrow[r] \arrow[d, "\theta"] & F \arrow[r, "\varphi"] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
S^1_{A/xA}(M/xM) \arrow[r] & F/xF \arrow[r, "\psi"] & M/xM \arrow[r] & 0
\end{tikzcd}$$

is commutative.

Let us show that θ is an epimorphism. Let $f + xF \in \operatorname{Ker}\psi$. Then $\varphi(f) \in xM$, and hence

$$\varphi(f - xf_0) = 0 \Longrightarrow f \in xf_0 + S^1(M) \Longrightarrow f + xf \in \theta(S^1(M)).$$

It remains to verify that $\operatorname{Ker}\theta = xS^1(M)$. Indeed, $\operatorname{Ker}\theta = S^1(M) \cap xF$; but if $\varphi(xf) = 0$, then $\varphi(f) = 0$ since x is not zero divisor in M. □

2.14.7. Proof of Theorem 2.14.6. Let $\mathfrak{m} = Ax_1 + \ldots + Ax_d$, where $(x_1, \dots, x_d)$ is a regular sequence. Let us show by induction on d that $S^{d+1}_A(M) = 0$ for all M.

If $d = 0$, then A is a field and all is evident.

Let the statement be true for $d - 1$. We have

$$S_A^{d+1}(M) = S_A^d(M'), \text{ where } M' = S^1(M).$$

Since M' is a submodule of a free module, it follows that x_1 is not zero divisor in M'. By induction hypothesis and thanks to Lemma 2.14.6 we have

$$0 = S^d_{A/x_1A}(M'/x_1M') = S^d_A(M')/x_1S^d_A(M').$$

The Nakayama lemma implies that

$$S^d_A(M') = 0. \quad \square$$

Remark. The number d is an invariant of the ring A: Indeed, it can be defined as the length of the maximal resolution (*Koszul complex*) of the A-module A/m.

Finally, let us sketch a proof of Hilbert's syzygies theorem 2.13.7.

Consider a graded ring R instead of the local ring A, the ideal $R_+ = \bigoplus\limits_{i\geqslant 1} R_i$ instead of the maximal ideal $\mathfrak{m}$, and assume that "module" means "a graded R-module". Then Nakayama's lemma and Corollary 1.10.9a tho notions of a minimal epimorphism and a minimal resolution and Theorem 2.14.6 are applicable to the new situation. All arguments can be repeated literally, except for the formulation and proof of Nakayama's lemma: We have to replace "the ideal distinct from the whole A" by "the ideal containing in R_+" and the argument with inversion of $1 - f_1$ by a remark that multiplying by an element of R_+ we enlarge by 1 the number of the first non-zero component.

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