Representation Theory

v. 2 Nonholonomic distributions in Representation Theory: Quest for simple modular Lie algebras and Lie superalgebras

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Summary. This volume contains

(1) introduction into representation theory of simple Lie superalgebras (by J. Bernstein, P. Grozman, D. Leites, and I. Shchepochkina);

(2) introduction into representation theory of simple modular Lie superalgebras (by S. Bouarroudj, B. Clarke, P. Grozman, A. Lebedev, and D. Leites);

(3) examples of calculations feasible to be performed without computer (E. Poletaeva);

(4) formulation of open problems at course work, M.S. and Ph.D. theses level. Most of the open problems are supposed to be solved with the help of a computer-aided study.

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Editor's preface

In this Volume 2, a sequel to the standard text-book material on the representation theory succinctly given in the first volume, I've collected the information needed to begin his or her own research.

Mathematics is a language of sciences, and any language requires a dictionary. This volume is an analog of an annotated dictionary: It contains few theorems but lots of notions. One should not try to learn all the examples of (simple) Lie superalgebras; for the first reading it is better to concentrate on the simplest examples and most profound notions ($\mathfrak{gl}$ or $\mathfrak{sl}$, $\mathfrak{vect}$, $\mathfrak{q}$) and try to apply the rare theorems to these particular examples.

One of the goals of this volume is to give information needed to discover new simple modular Lie superalgebras and begin studying their representations. In doing so the non-integrable distributions (a.k.a. nonholonomic structures) and deformations are very important. To emphasize this, I used the not very well known term "nonholonomic" (whose meaning is explained in due course) in the title. To see how reasonable is the choice of the material having in mind the task of selecting a reasonably interesting topic for a Ph.D. thesis with severe restriction on time allocated, the reader should have in mind that the chapters written by A. Lebedev and E. Poletaeva contain main parts of their respective Ph.D. theses written within 1.5–2 years. B. Clark's contribution was written within a month. (Poletaeva's thesis took longer but she was working as a TA.)

Computer-aided scientific research. Most of the open problems offered as possible topics for Ph.D. theses in this Volume 2 are easier to solve with the help of the `Mathematica`-based package `SuperLie` for scientific research designed by Pavel Grozman. This feature of these problems encourages to master certain basic skills useful in the modern society in general and for a university professor and researcher in particular.

The references currently IN PREPARATION are to be found in `arXiv`.

I encourage the reader to contact me (mleites@math.su.se) to avoid nuisance of queueing selecting problems and to inform if something is solved. I will be thankful for all remarks that will help to improve the text for the second printing.

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Notation often used

In what follows, the ground field is denoted by $\mathbb{K}$, and its characteristic by p. The parity function on the superspaces is denoted by P or (seldom) by Π but the usual notation of the parity function is p.

The elements of $\mathbb{Z}/n$ are denoted by $\overline{0}$, $\overline{1}$, ..., $\overline{n-1}$ to distinguish them from elements of $\mathbb{Z}$.

The *superdimension* of a given superspace V is sdim $V = \dim V_{\bar 0} + \varepsilon \dim V_{\bar 1}$, where ε is an indeterminate such that $\varepsilon^2 = 1$;

the *supercharacter* of a $\mathbb{Z}$-graded superspace $V = \underset{i\in\mathbb{Z}}{\oplus} V_i$ is

$$\text{sch } V = \sum_{i\in\mathbb{Z}} (\text{sdim } V_i)t^i.$$

The *character* of a $\mathbb{Z}$-graded space (in particular, of a superspace with superstructure forgotten) $V = \underset{i\in\mathbb{Z}}{\oplus} V_i$ is

$$\text{ch } V = \sum_{i\in\mathbb{Z}} (\dim V_i)t^i.$$

For a set $x = (x_1, \dots, x_n)$ of homogeneous with respect to parity indeterminates that span a superspace V, we write

$T[x]$ or $T(V)$ for the tensor algebra of the superspace V,

$S[x]$ or $S(V)$ for the (super)symmetric algebra of the superspace V,

$\Lambda[x]$, or $\Lambda(V)$ or $E(V)$ for the exterior or anti-(super)symmetric algebra of the space V.

The *exterior powers* and *symmetric powers* of the vector space V are defined as quotients of its tensor power

$$\begin{array}{l}
T^0(V) := \wedge^0(V) := S^0(V) := \mathbb{K};\\
T^1(V) := \wedge^1(V) := S^1(V) := V;\\
T^i(V) := \underbrace{V\otimes\dots\otimes V}_{i \text{ factors}} \quad \text{for } i>0;\\
\wedge^\bullet(V) := T^\bullet(V)/(x\otimes x \mid x\in V);\\
S^\bullet(V) := T^\bullet(V)/(x\otimes y + y\otimes x \mid x, y\in V),
\end{array}$$

where $T^\bullet(V) := \oplus T^i(V)$; let $\wedge^i(V)$ and $S^i(V)$ be homogeneous components of degree i. Some authors instead of $\wedge^i(V)$ or $\wedge^\bullet(V)$ write $E^i(V)$ or $E^\bullet(V)$.

Supersymmetrization is performed by means of the Sign Rule (1.1.2).

For any simple Lie algebra $\mathfrak{g}$, we denote the $\mathfrak{g}$-module with the ith fundamental weight π_i by $R(\pi_i)$ (as in [OV, Bou]; these modules are denoted by Γ_i in [FH]).

For $\mathfrak{o}(2k+1)$, the *spinor representation* spin_{2k+1} is *defined* to be the kth fundamental representation, whereas for $\mathfrak{o}_\Pi(2k)$, the *spinor representations* are the kth and the $(k-1)$st fundamental representations. The *realizations* of

these representations and the corresponding modules by means of quantization (as in [LSh3]) can be defined even in the modular cases, since quantization is well-defined for the restricted version of the Poisson algebra. To describe all deformations of the modular Poisson algebras and its super versions (even restricted) is an *Open Problem.*

Let ad denote both the *adjoint* representation and the module in which it acts, let id denote both the *identity* (a.k.a. *standard*) representation of the linear Lie (super)algebra $\mathfrak{g}\subset\mathfrak{gl}(V)$ in the (super)space V and V itself. In particular, having fixed a basis in the n-dimensional space and having realized $\mathfrak{gl}(V)$ as $\mathfrak{gl}(n)$, we write id instead of V, so V does not explicitly appear.

$\mathbb{Z}_+$ is the set of nonnegative integers,

$\mathbb{N}$ is the set of positive integers.

The ground field $\mathbb{K}$ is assumed to be algebraically closed unless specified (certain results are true for perfect but not algebraically closed fields); its characteristic is denoted by p; we assume that $p=2$ unless specified; vector spaces V are finite dimensional unless specified.

We often use the following matrices

$$\begin{array}{l}
J_{2n} = \begin{pmatrix} 0 & 1_n\\ -1_n & 0\end{pmatrix},\\
\Pi_n = \begin{cases} \Pi_{2k} := \text{antidiag}_2(1_k, 1_k) = \begin{pmatrix} 0 & 1_k\\ 1_k & 0\end{pmatrix} & \text{if } n = 2k,\\ \Pi_{2k+1} := \text{antidiag}_3(1_k, 1, 1_k) = \begin{pmatrix} 0 & 0 & 1_k\\ 0 & 1 & 0\\ 1_k & 0 & 0\end{pmatrix} & \text{if } n = 2k+1,\end{cases}\\
S_n = \text{antidiag}_n(1, \dots, 1),\\
Z_{2k} = \text{diag}_k(\Pi_2, \dots, \Pi_2);
\end{array}\tag{0.1}$$

$$1^{(m,n,p)} := \begin{array}{c c} & \begin{array}{cc} p & m-p \end{array}\\ \begin{array}{c} p\\ n-p\end{array} & \begin{pmatrix} 1_p & 0\\ 0 & 0\end{pmatrix}\end{array},\quad 1_{(m,n,q)} = \begin{array}{c c} & \begin{array}{cc} n-q & q \end{array}\\ \begin{array}{c} m-q\\ q\end{array} & \begin{pmatrix} 0 & 0\\ 0 & 1_q\end{pmatrix}\end{array},$$

$$J_{(n,2q)} = \begin{array}{c c} & \begin{array}{cc} n-2q & 2q \end{array}\\ \begin{array}{c} m-2q\\ q\end{array} & \begin{pmatrix} 0 & 0\\ 0 & J_{2q}\end{pmatrix}\end{array};$$

$$\text{Add}(A, \dots, Z) := \text{Ad}_{\text{diag}(A,\dots,Z)}.$$

Let $\mathfrak{o}_I(n)$, $\mathfrak{o}_\Pi(n)$ and $\mathfrak{o}_S(n)$ be Lie algebras that preserve bilinear forms 1_n, Π_n and S_n, respectively.

We identify a given bilinear form with its Gram matrix. Different normal forms of symmetric bilinear forms are used: In some problems, the form 1_n is used; in other problems (usually, mathematical ones) the forms Π_n and S_n

are more preferable (so that the corresponding orthogonal Lie algebra has a Cartan subalgebra consisting of diagonal matrices).

Let $\Pi_{k|k} := \Pi_{2k}$ and $J_{k|k} := J_{2k}$, but considered as supermatrices in the standard format $k|k$.

Let $E^{i,j}$ be the (i,j)th (super)matrix unit.

Any square matrix is said to be *zero-diagonal* if it has only zeros on the main diagonal;

$ZD(n)$ is the space (Lie algebra if $p=2$) of symmetric zero-diagonal $n\times n$-matrices.

For any Lie algebra $\mathfrak{g}$ (or Lie superalgebra and $p\neq 2$), its *derived algebras* are defined to be

$$\mathfrak{g}^{(0)} := \mathfrak{g}, \quad \mathfrak{g}^{(1)} := [\mathfrak{g},\mathfrak{g}], \quad \mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)},\mathfrak{g}^{(i)}].$$

Describing the $\mathfrak{g}_{\bar 0}$-module structure of $\mathfrak{g}_{\bar 1}$ for a Lie superalgebra $\mathfrak{g}$, we write $\mathfrak{g}_{\bar 1}\simeq R(\cdot)$, though it is, actually, $\Pi(R(\cdot))$.

The symbols $A\ltimes B$ and $B\rtimes A$ denote a *semi-direct sum* of modules of which A is a submodule; when dealing with algebras, A is an ideal.

$A^\times$ is the set of invertible elements of the algebra A.

Deform is the result of a deformation, and prolong the result of a prolongation (same as the conventional transform is the result of a transformation).

$\mathfrak{cg}$ or $\mathfrak{c}(\mathfrak{g})$ is the trivial central extension of the Lie superalgebra $\mathfrak{g}$ with the 1-dimensional even center generated by z.

$\langle 1\rangle$ or $\mathbb{1}$ denotes a 1-dimensional trivial module over a Lie (super)algebra considered.

$\overline{1,n}$ denotes the set of integers $\{1,\dots,n\}$.

Part I

Representations of Lie superalgebras over $\mathbb{C}$

Chapter 1

Background over $\mathbb{C}$ (D. Leites)

For further reading, see [D],[LSoS]. The rudiments of algebraic geometry are based on [MaAG].

1.1. Linear algebra in superspaces

1.1.1. Superspaces. A *superspace* is a $\mathbb{Z}/2$-graded space; for any superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$, we denote by $\Pi(V)$ another copy of the same superspace: with the shifted parity, i.e., $(\Pi(V))_{\bar{i}} = V_{\bar{i}+\bar{1}}$. The *superdimension* of V is $\operatorname{sdim} V = p + q\varepsilon$, where $\varepsilon^2 = 1$ and $p = \dim V_{\bar{0}}$, $q = \dim V_{\bar{1}}$. (Usually, $\operatorname{sdim} V$ is expressed as a pair (p, q) or $p|q$; this obscures the fact that $\operatorname{sdim} V \otimes W = \operatorname{sdim} V \cdot \operatorname{sdim} W$.)

A superspace structure in V induces natural superspace structures in the dual space V^* and in the tensor products of superspaces. A *basis of a superspace* is always a basis consisting of *homogeneous* vectors; let $\operatorname{Par} = (p_1, \dots, p_{\dim V})$ be an ordered collection of their parities. We call Par the *format* of (the basis of) V. A square *supermatrix* of format (size) Par is a $\operatorname{sdim} V \times \operatorname{sdim} V$ matrix whose ith row and ith column are of the same parity $p_i \in \operatorname{Par}$. We set $|\operatorname{Par}| = \dim V$ and $s|\operatorname{Par}| = \operatorname{sdim} V$.

Whenever possible, we usually consider one of the simplest formats Par, e.g., the format Par_{st} of the form $(\bar{0}, \dots, \bar{0}; \bar{1}, \dots, \bar{1})$ is called *standard*; that of the form $\operatorname{Par}_{alt} := (\bar{0}, \bar{1}, \bar{0}, \bar{1}, \dots)$ is called *alternating*. Systems of simple roots of Lie superalgebras corresponding to distinct nonstandard formats of supermatrix realizations of these superalgebras are related by *odd reflections*.

The matrix unit E_{ij} is supposed to be of parity $p_i + p_j$.

A *superalgebra* is any superspace A whose multiplication $m : A \otimes A \longrightarrow A$ must be an *even* map. A *superalgebra morphism* is any *parity-preserving* algebra homomorphism.

1.1.2. The Sign Rule. The superbracket of supermatrices (of the same format) is defined by means of the **Sign Rule**:

if something of parity p moves past something of parity q the sign $(-1)^{pq}$ accrues; the expressions defined on homogeneous elements are extended to arbitrary ones via linearity.

Examples of application of the Sign Rule: By setting

$$[X, Y] = XY - (-1)^{p(X)p(Y)}YX$$

we get the notion of the supercommutator and the ensuing notions of *supercommutative* and *superanti-commutative* superalgebras; *Lie superalgebra* is the one which, in addition to superanti-commutativity, satisfies the super Jacobi identity, i.e., the Jacobi identity amended with the Sign Rule; the *superderivation of a given superalgebra* A is a linear map $D\colon A \longrightarrow A$ that satisfies the *super Leibniz rule*

$$D(ab) = D(a)b + (-1)^{p(D)p(a)}aD(b).$$

In particular, let $A = \mathbb{C}[x]$ be the free supercommutative polynomial superalgebra in $x = (x_1, \dots, x_n)$, where the superstructure is determined by the parities of the indeterminates: $p(x_i) = p_i$. Partial derivatives are defined (with the help of super Leibniz Rule) by the formulas

$$\frac{\partial x_i}{\partial x_j} = \delta_{i,j}.$$

Clearly, the collection $\mathfrak{der}\, \mathbb{C}[x]$ of all superderivations of A is a Lie superalgebra whose elements are of the form $\sum f_i(x)\dfrac{\partial}{\partial x_i}$ with $f_i(x) \in \mathbb{C}[x]$ for all i.

Observe that sometimes the Sign Rule requires some dexterity in application. For example, we have to distinguish between superskew- and superanti- although both versions coincide in the non-super case:

$$\begin{array}{ll}
ba = (-1)^{p(b)p(a)}ab & \text{(supercommutativity)}\\
ba = -(-1)^{p(b)p(a)}ab & \text{(anti-supercommutativity)}\\
ba = (-1)^{(p(b)+1)(p(a)+1)}ab & \text{(skew-supercommutativity)}\\
ba = -(-1)^{(p(b)+1)(p(a)+1)}ab & \text{(antiskew-supercommutativity)}
\end{array}$$

In other words, "anti" means the change of the sign, whereas any "skew" notions can be straightened by the change of parity. In what follows, the superanti-symmetric bilinear forms and superanti-commutative superalgebras are named according to the above definitions.

Given the supercommutative superalgebra $\mathcal{F}$ of "functions" in the indeterminates $x = (x_1, \dots, x_{n+m})$ of which n are even and m are odd, define the supercommutative superalgebra Ω of differential forms as polynomial algebra over $\mathcal{F}$ in the dx_i, where $p(d) = \bar{1}$. Since dx_i is even for x_i odd, we can consider not only polynomials in dx_i.

Smooth or analytic functions in the dx_i are called *pseudodifferential forms* on the supermanifold with coordinates x_i, see [BL]. We will need them to

interpret $\mathfrak{b}_\lambda(n)$. The exterior differential is defined on the space of (pseudo) differential forms by the formulas (mind the Super and Leibniz rules):

$$d(x_i) = dx_i \text{ and } d(dx_i) = 0.$$

The *Lie derivative* is defined (minding same Rules) by the formula

$$L_D(df) = (-1)^{p(D)}d(D(f)).$$

In particular,

$$L_D\left((df)^\lambda\right) = \lambda(-1)^{p(D)}d(D(f))(df)^{\lambda-1} \text{ for any } \lambda \in \mathbb{C} \text{ and } f \text{ odd}.$$

Modules over (anti-)supercommutative superalgebras can naturally be endowed with a two-sided module structures. There are, however, some new features as compared with modules over fields, see [LSoS] and these subtleties are vital in computations of (co)homology (e.g., deformations and relations of Lie superalgebras), see [Gr].

1.1.3. Simple, almost simple and semi-simple Lie superalgebras. Recall that the Lie superalgebra $\mathfrak{g}$ without proper ideals and of dimension > 1 is said to be *simple*. **Examples**: $\mathfrak{sl}(m|n)$ for $m > n \geq 1$.

We say that $\mathfrak{h}$ is *almost simple* if it can be sandwiched (non-strictly) between a simple Lie superalgebra $\mathfrak{s}$ and the Lie superalgebra $\mathfrak{der}\, \mathfrak{s}$ of derivations of $\mathfrak{s}$, i.e., $\mathfrak{s} \subset \mathfrak{h} \subset \mathfrak{der}\, \mathfrak{s}$.

By definition, a given Lie superalgebra $\mathfrak{g}$ is said to be *semi-simple* if its radical is zero.

Block described semi-simple Lie algebras over the fields of prime characteristic. Literally following Block's description we describe semi-simple Lie superalgebras as follows:

Let $\mathfrak{s}_1, \dots, \mathfrak{s}_k$ be simple Lie superalgebras, let $n_1, \dots, n_k$ be *pairs* of non-negative integers $n_j = (n_j^{\bar{0}}, n_j^{\bar{1}})$, let $\mathcal{F}(n_j)$ be the supercommutative superalgebra of polynomials in $n_j^{\bar{0}}$ even and $n_j^{\bar{1}}$ odd indeterminates, and $\mathfrak{s} = \bigoplus\limits_j (\mathfrak{s}_j \otimes \mathcal{F}(n_j))$. Then

$$\mathfrak{der}\, \mathfrak{s} = \bigoplus_j \left((\mathfrak{der}\, \mathfrak{s}_j) \otimes \mathcal{F}(n_j) \Subset \mathrm{id}_{\mathfrak{s}_j} \otimes \mathfrak{vect}(n_j)\right). \tag{1.1}$$

Let $\mathfrak{g}$ *be a subalgebra of* $\mathfrak{der}\mathfrak{s}$ *containing* $\mathfrak{s}$. *If*

$$\text{the projection of } \mathfrak{g} \text{ to } \mathrm{id}_{\mathfrak{s}_j} \otimes \mathfrak{vect}(n_j)_{-1} \text{ is } \textbf{onto} \text{ for each } j, \tag{1.2}$$

then $\mathfrak{g}$ *is semi-simple and all semi-simple Lie superalgebras arise in the manner indicated*, i.e., as sums of subalgebras of the summands of (1.1) satisfying (1.2).

1.2. Basics on superschemes (from [LSoS])

1.2.1. The spectrum. An ideal p of a supercommutative superalgebra A is said to be *prime* if A/p is an integral domain (i.e., has no zerodivisors and if we do not forbid $1 = 0$, then the zero ring may not be an integral domain). Equivalently, p is *prime* if $p \neq A$ and

$$a \in A, \quad b \in A, \quad a\, b \in p \Longrightarrow \text{ either } a \in p \text{ or } b \in p. \tag{1.3}$$

The set of all the prime ideals of A is called the (prime) *spectrum of* A and is denoted by $\operatorname{Spec} A$. The elements of $\operatorname{Spec} A$ are called its *points*.

Following Grothendieck, Leites enriched the set $\operatorname{Spec} A$ with additional structure making it into a topological space rigged with a sheaf of superrings [Le0].

1.2.2. Representable functors. For a given *category* C we denote by $\operatorname{Ob}\mathsf{C}$ the set of *objects of* C. (Instead of $X \in \operatorname{Ob}\mathsf{C}$ we often write briefly $X \in \mathsf{C}$.) Given a category C, define its *dual* C° by letting $\operatorname{Ob}\mathsf{C}^\circ$ be a copy of $\operatorname{Ob}\mathsf{C}$ and $\operatorname{Hom}_{C^\circ}(X^\circ, Y^\circ)$ to be in one-to-one correspondence with $\operatorname{Hom}_{\mathsf{C}}(Y, X)$, where $X^\circ \in \operatorname{Ob}\mathsf{C}^\circ$ denotes the object corresponding to $X \in \operatorname{Ob}\mathsf{C}$, so that if a morphism $\varphi^\circ\colon X^\circ \to Y^\circ$ corresponds to a morphism $\varphi\colon Y \to X$, then $\psi^\circ\varphi^\circ = (\varphi\psi)^\circ$ and $\mathrm{id}_{X^\circ} = (\mathrm{id}_X)^\circ$.

Speaking informally, C° is obtained from C by taking the same objects but inverting the arrows.

If our universum is not too large, there exists a category whose objects are categories and morphisms are functors between them. The main example: the category $\mathsf{C}^* = \mathsf{Funct}(\mathsf{C}^\circ, \mathsf{Sets})$ of functors from C° into Sets, where Sets is the category of sets and their maps as morphisms.

1.2.3. Representable functors. Fix any $X \in \mathsf{C}$.

1) Denote by $P_X \in \mathsf{C}^*$ (here: P is for point; usually this functor is denoted by h_X, where h is for homomorphisms) the functor given by

$$P_X(Y^\circ) = \operatorname{Hom}_{\mathsf{C}}(Y, X) \quad \text{for any } Y^\circ \in \mathsf{C}^\circ; \tag{1.4}$$

to any morphism $\varphi^\circ\colon Y_2^\circ \longrightarrow Y_1^\circ$ the functor P_X assigns the map of sets $P_X(Y_2^\circ) \to P_X(Y_1^\circ)$ which sends $\psi\colon Y_2 \longrightarrow X$ into the composition $\varphi\psi\colon Y_1 \longrightarrow Y_2 \to X$.

To any $\varphi \in \operatorname{Hom}_{\mathsf{C}}(X_1, X_2)$, there corresponds a functor morphism $P_\varphi\colon P_{X_1} \longrightarrow P_{X_2}$ which to any $Y \in C$ assigns

$$P_\varphi(Y^\circ)\colon P_{X_1}(Y^\circ) \longrightarrow P_{X_2}(Y^\circ) \tag{1.5}$$

and sends a morphism $\psi \in \operatorname{Hom}_{\mathsf{C}}(Y^\circ, X_1)$ into the composition

$$\varphi\psi\colon Y^\circ \longrightarrow X_1 \longrightarrow X_2. \tag{1.6}$$

Clearly, $P_{\varphi\psi} = P_\varphi P_\psi$.

2) Similarly, define $P^X \in \mathsf{C}^*$ by setting

$$P^X(Y) = \operatorname{Hom}_{\mathsf{C}}(X, Y) \quad \text{for any } Y \in \mathsf{C}; \tag{1.7}$$

to any morphism $\varphi\colon Y_1 \to Y_2$, we assign the map of sets $P^X(Y_1) \longrightarrow P^X(Y_2)$ which sends $\psi\colon X \longrightarrow Y1$ into the composition $\psi\varphi\colon X \longrightarrow Y_1 \longrightarrow Y_2$.

To any $\varphi \in \operatorname{Hom}_{\mathsf{C}}(X_1, X_2)$, there corresponds a functor morphism $P^\varphi\colon P^{X_2} \longrightarrow P^{X_1}$ which to any $Y \in C$ assigns

$$P^\varphi(Y)\colon P^{X_2}(Y) \longrightarrow P^{X_1}(Y) \tag{1.8}$$

and sends a morphism $\psi \in \operatorname{Hom}_{\mathsf{C}}(X_2, Y)$ into the composition

$$\psi\varphi\colon X_1 \longrightarrow X_2 \longrightarrow Y. \tag{1.9}$$

Clearly, $P^{\varphi\psi} = P^\psi P^\varphi$.

A functor $F\colon \mathsf{C}^\circ \longrightarrow \mathsf{Sets}$ (or a functor $F\colon \mathsf{C} \longrightarrow \mathsf{Sets}$) is said to be *(co)representable* if it is isomorphic to a functor of the form P_X (resp. P^X) for some $X \in \mathsf{C}$; then X is called an *object that represents* F.

Theorem. *The map $\varphi \mapsto P_\varphi$ defines an isomorphism of sets*

$$\operatorname{Hom}_{\mathsf{C}}(X, Y) \cong \operatorname{Hom}_{\mathsf{C}^*}(P_X, P_Y). \tag{1.10}$$

This isomorphism is functorial in both X and Y. Therefore, the functor $P\colon \mathsf{C} \to \mathsf{C}^*$ determines an equivalence of C with the full subcategory of C^* consisting of representable functors.

Corollary. *If a functor from C^* is representable, the object that represents it is determined uniquely up to an isomorphism.*

The above Theorem is the source of several important ideas.

1.2.3.1. Passage from the categorical point of view to the structural one. It is convenient to think of P^X as of "the sets of points of an object $X \in \mathsf{C}$ *with values in various objects* $Y \in \mathsf{C}$ or Y*-points*"; notation: $P^X(Y)$ or sometimes $\underline{X}(Y)$. (The sets $P_X(Y)$ are also sometimes denoted by $\underline{X}(Y)$.)

In other words, $P_X = \coprod\limits_{Y \in \mathsf{C}} P_X(Y)$ with an additional structure: the sets of maps $P_X(Y_1) \longrightarrow P_X(Y_2)$ induced by morphisms $Y_2^\circ \longrightarrow Y_1^\circ$ for any $Y_1, Y_2 \in \mathsf{C}$ and compatible in the natural sense (the composition goes into the composition, and so on). The situation with the P^X is similar.

Therefore, in principle, it is always possible to pass from the categorical point of view to the structural one, since all the categorical properties of X are mirrored precisely by the functorial properties of the structure of P_X.

Motivation. Let $*$ be a one-point set. For categories with sufficiently "simple" structure of their objects, such as the category of finite sets or even category of smooth finite dimensional manifolds, $X = P_X(*)$ for every object X, i.e., X is completely determined by its $*$-points or just points.

For varieties (or for supermanifolds), when the object may have either "sharp corners" or "inner degrees of freedom", the structure sheaf may contain nilpotents or zero divisors, and in order to keep this information and be able to completely describe X we need *various types of points*, in particular, Y-points for some more complicated Y's.

1.2.3.2. Replacing X by P_X (resp. by P^X) we may transport conventional set-theoretical constructions to any category: an object $X \in \mathsf{C}$ is a *group, ring*, and so on *in the category* C, if the corresponding structure is given on every set $P_X(Y)$ of its Y-points and is compatible with the maps induced by the morphisms $Y_2^\circ \longrightarrow Y_1^\circ$ (resp. $Y_1 \to Y_2$).

This is exactly the way supergroups are defined and superalgebras *should be* defined. However, it is possible to define superalgebras using just one set-theoretical model and sometimes we have to pay for this deceiving simplicity.

1.2.4. Presheaves. Fix a topological space X. Let $\mathcal{P}$ be a law that to every open set $U \subset X$ assigns a set $\mathcal{P}(U)$ and, for any pair of open subsets $U \subset V$, there is given a *restriction map* $r_U^V \colon \mathcal{P}(V) \longrightarrow \mathcal{P}(U)$ such that

1) $\mathcal{P}(\emptyset)$ consists of one element,
2) $r_U^W = r_U^V \circ r_V^W$ for any open subsets (briefly: *opens*) $U \subset V \subset W$.

Then the system $\{\mathcal{P}(U), r_U^V \mid U, V \text{are opens}\}$ is called a *presheaf* (of sets) on X.

The elements of $\mathcal{P}(U)$, also often denoted by $\Gamma(U, \mathcal{P})$, are called the *sections of* the presheaf $\mathcal{P}$ over U; a section may be considered as a "function" defined over U.

Remark. Axiom 1) is convenient in some highbrow considerations of category theory. Axiom 2) expresses the natural transitivity of restriction.

1.2.5. The category Top_X. The objects of Top_X are open subsets of X and morphisms are inclusions. A *presheaf* of sets on X is a functor $\mathcal{P}\colon \mathsf{Top}_X^\circ \longrightarrow \mathsf{Sets}$.

From genuine functions we can construct their products, sums, and multiply them by scalars; similarly, we may consider presheaves of groups, rings, and so on. A formal definition is as follows:

Let $\mathcal{P}$ be a presheaf of sets on X; if, on every set $P(U)$, there is given an algebraic structure (of a group, ring, A-algebra, and so on) and the restriction maps r_U^V are homomorphisms of this structure, i.e., P is a functor $\mathsf{Top}_X^\circ \longrightarrow \mathsf{Gr}$ (the category of groups), Rings (that of rings, or superrings), A-Algs (that of rings, or A-(super)algebras), and so on, then P is called the *presheaf of groups, rings, A-algebras,* and so on, respectively.

Finally, we may consider exterior composition laws, e.g., a presheaf of modules over a presheaf of rings (given on the same topological space). We leave the task to give a formal definition as an exercise to the reader.

1.2.6. Sheaves. The presheaves of continuous (infinitely differentiable, analytic, and so on) functions on a space X possess additional properties (of "analytic continuation" type) which are axiomized in the following definition.

A presheaf $\mathcal{P}$ on a topological space X is called a *sheaf* if it satisfies the following condition: for any open subset $U \subset X$, its open covering $U = \bigcup\limits_{i\in I} U_i$, and a system of sections $s_i \in \mathcal{P}(U_i)$, where $i \in I$, such that

$$r_{U_i\cap U_j}^{U_i}(s_i) = r_{U_i\cap U_j}^{Uj}(s_j) \text{ for any } i, j \in I, \tag{1.11}$$

there exists a section $s \in \mathcal{P}(U)$ such that $s_i = r_{U_i}^U(s)$ for any $i \in I$, and such a section is unique.

In other words, from a set of compatible sections over the U_i a section over U may be glued and any section over U is uniquely determined by the set of its restrictions onto the U_i.

Remark. If $\mathcal{P}$ is a presheaf of abelian groups, the following reformulation of the above condition is useful:

A presheaf $\mathcal{P}$ is a sheaf if, for any $U = \bigcup\limits_{i\in I} U_i$, the following sequence of abelian groups is exact

$$0 \longrightarrow \mathcal{P}(U) \xrightarrow{\varphi} \prod_{i\in I} \mathcal{P}(U_i) \xrightarrow{\psi} \prod_{i,j\in I} \mathcal{P}(U_i \cap U_j), \tag{1.12}$$

where φ and ψ are determined by the formulas

$$\begin{aligned} \varphi(s) &= (\dots, r_{U_i}^U, (s), \dots) \\ \psi(\dots, s_i, \dots, s_j, \dots) &= (\dots, r_{U_i\cap U_j}^{Ui}(s_i) - r_{U_i\cap U_j}^{U_j}(s_j), \dots). \end{aligned} \tag{1.13}$$

For a generic presheaf of abelian groups, this sequence is only a complex. (Its natural extension determines a Čech cochain complex that will be defined in what follows.)

1.2.7. The structure sheaf $\mathcal{O}_X$ over $X = \operatorname{Spec} A$. We consider the elements from A as functions on $\operatorname{Spec} A$. For every $x \in X$, set $\mathcal{O}_x := A_{A\setminus p_x} = A_{p_x}$ (localization of A with respect to the multiplicative system $A \setminus p_x$). For any open subset $U \subset X$, define the ring of sections of the presheaf $\mathcal{O}_X$ over U to be the subring

$$\mathcal{O}_X(U) \subset \prod_{x\in U} \mathcal{O}_x \tag{1.14}$$

consisting of the elements $(\dots, s_x, \dots)$, where $s_x \in \mathcal{O}_x$, such that for every point $x \in U$ there exists an open neighborhood $D(f_x) \ni x$ (here f_x is a function determined by x) and an element $g \in A_{f_x}$ such that s_y is the image of g under the natural homomorphism $A_{f_x} \longrightarrow \mathcal{O}_y$ for all $y \in U$.

Define the restriction morphisms r_U^V as the homomorphisms induced by the projection $\prod\limits_{x\in V} \mathcal{O}_x \longrightarrow \prod\limits_{x\in U} \mathcal{O}_x$. It is easy to see that $\mathcal{O}_X$ is well-defined and the natural homomorphism $A_{f_x} \longrightarrow \mathcal{O}_y$ is induced by the embedding of multiplicative sets

$$\{f_x^n \mid n \in \mathbb{N}\} \subset A \setminus p_y. \tag{1.15}$$

Theorem. *The presheaf $\mathcal{O}_X$ is a sheaf whose stalk over $x \in X$ is isomorphic to $\mathcal{O}_x$ and r_x^U is the composition*

$$\mathcal{O}_X(U) \longrightarrow \prod_{x' \in U} \mathcal{O}_{x'} \xrightarrow{pr} \mathcal{O}_x \tag{1.16}$$

Furthermore, the ring homomorphism

$$j\colon A_f \longrightarrow \mathcal{O}_X(D(f)),\ j(g/f) = (\dots, j_x(g/f), \dots)_{x \in U}, \tag{1.17}$$

where $j_x\colon A_f \longrightarrow \mathcal{O}_x$ is a natural homomorphism of quotient rings, is an isomorphism.

The sheaf $\mathcal{O}_X$ over the scheme $X = \operatorname{Spec} A$ is called the *structure sheaf* of X.

The above-described sheaf over $X = \operatorname{Spec} A$ will be sometimes denoted by $\widetilde{A}$. The pair $(\operatorname{Spec} A, \widetilde{A})$ consisting of a topological space and a sheaf over it determines the ring A thanks to Theorem 1.2.7: namely, $A = \Gamma(\operatorname{Spec} A, \widetilde{A})$. This pair is the main local object of the algebraic geometry.

1.2.8. Ringed spaces. A *ringed topological space* is a pair $(X, \mathcal{O}_X)$ consisting of a space X and a sheaf of (commutative[1]) rings $\mathcal{O}_X$ over it called the *structure sheaf.*

A *morphism of ringed spaces* $F\colon (X_1, \mathcal{O}_X) \longrightarrow (Y_1, \mathcal{O}_Y)$ is a pair consisting of a morphism $f\colon X \longrightarrow Y$ of topological spaces and the collection of ring homomorphisms

$$\{f_U^*\colon \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U)) \text{ for every open } U \subset Y\} \tag{1.18}$$

that are compatible with restriction maps, i.e., such that

(a) the diagrams

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \xrightarrow{f_U^*} & \mathcal{O}_X(f^{-1}(U)) \\ {\scriptstyle r_V^U}\downarrow & & \downarrow{\scriptstyle r_{f^{-1}(V)}^{f^{-1}(U)}} \\ \mathcal{O}_Y(V) & \xrightarrow{f_V^*} & \mathcal{O}_X(f^{-1}(V)) \end{array} \tag{1.19}$$

[1] Or supercommutative superrings, although nowhere, except in [Le0] that contains only a definition, did anybody consider spaces ringed — not superringed! — by supercommutative superrings. The notion of a **superringed space** introduced below is something more restricted: only *parity preserving* homomorphisms of the rings of sections are considered as morphisms. For more detail, see the Chapter on Volichenko algebras.

commute for every pair of open sets $V \subset U \subset Y$;

(b) for any open $U \subset Y$, and a pair $u \in U$ and $g \in \mathcal{O}_Y(U)$ such that $g(y) = 0$, we have

$$f_U^*(g)(x) = 0 \ \text{ for any } x \text{ such that } f(x) = y. \tag{1.20}$$

Elucidation. If X and Y are Hausdorff spaces, $\mathcal{O}_X$, $\mathcal{O}_Y$ the sheaves of continuous (smooth, analytic, and so on) functions on them, respectively, then to every morphism $f\colon X \longrightarrow Y$ the ring homomorphism $f_U^*\colon \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U))$ corresponds: f_U^* assigns to any function $g \in \mathcal{O}_Y(U)$ the function

$$f_U^*(g)(x) = g(f(x)) \text{ for any } x \in f^{-1}(U); \tag{1.21}$$

i.e., the domain of $f_U^*(g)$ is $f^{-1}(U)$, and $f_U^*(g)$ is constant on the pre-image of every $y \in U$.

In algebraic geometry, the spaces are not Hausdorff ones and their structure sheaves are not readily recognized as sheaves of functions. Therefore

1) the collection of ring homomorphisms $\{f_U^* \mid U \text{ is an open set}\}$ is **not** recovered from f and *must be given separately*;

2) the condition

$$f_U^*(g)(x) = g(f(x)) \text{ for any } x \in f^{-1}(U) \tag{1.22}$$

is replaced by a weaker condition (b1.20).

These two distinctions from the usual functions are caused by the fact that the domains of our "make believe" functions have variable ranges and different sections of the structure sheaf may represent the same function.

A ringed space isomorphic to one of the form $(\operatorname{Spec} A, \widetilde{A})$ is called an *affine scheme.*

1.2.9. Schemes. A ringed topological space $(X, \mathcal{O}_X)$ is called a *scheme* if its every point x has an open neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

One of the methods for explicit description of a global object is just to define the local objects from which it is glued and the method of gluing. Here is the formal procedure.

Proposition. *Let $(X_i, \mathcal{O}_{X_i})_{i \in I}$ be a family of schemes and let in every X_i open subsets U_{ij}, where $i, j \in I$, be given. Let there be given a system of isomorphisms $\theta_{ij}\colon (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \longrightarrow (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ satisfying the cocycle condition*

$$\theta_{ii} = \mathrm{id},\ \theta_{ij} \circ \theta_{ji} = \mathrm{id}, \theta_{ij} \circ \theta_{jk} \circ \theta_{ki} = \mathrm{id}\,. \tag{1.23}$$

Then there exists a scheme $(X, \mathcal{O}_X)$, an open covering $X = \bigcup\limits_{i \in I} X_i'$ and a family of isomorphisms $\varphi_i\colon (X_i', \mathcal{O}_X|_{X_i'}) \longrightarrow (X_i, \mathcal{O}_{X_i})$ such that

$$(\varphi_j|_{X_i \cap X_j})^{-1} \circ \theta_{ij} \circ \varphi_i|_{X_i \cap X_j} = \mathrm{id} \ \textit{for all } i, j. \tag{1.24}$$

1.2.9.1. Superringed spaces. Superschemes. In analogy with the definition of a ringed space, we define a *superringed space* as a pair $(X, \mathcal{F})$, where X is a topological space and $\mathcal{F}$ is a sheaf of supercommutative superrings over X. A *superringed space morphism* $\varphi\colon (X, \mathcal{F}) \to (Y, \mathcal{G})$ is a pair $(\tilde{\varphi}, \{\varphi_U^* \mid U \subset Y\})$ consisting of

a continuous map $\tilde{\varphi}\colon X \to Y$ and

a collection of superring morphisms $\varphi_U^*\colon \mathcal{G}(U) \to \mathcal{F}(\tilde{\varphi}^{-1}(U))$ defined for every open subset $U \subset Y$ and consistent with the restrictions maps, i.e.,

$$r^{\tilde{\varphi}^{-1}(V)}_{\tilde{\varphi}^{-1}(U)}\varphi_V^* = \varphi_U^* r_U^V \quad \text{for any opens } U \subset V \subset X. \tag{1.25}$$

What is the difference between a ringed space and a superringed space? Answer: for the ringed spaces, morphisms φ_U^* are **arbitrary** algebra homomorphisms whereas the morphisms of the same objects considered as superringed spaces must preserve parity.

A superringed space isomorphic to one of the form $(\operatorname{Spec} A, \widetilde{A})$, where A is a supercommutative ring, is said to be an *affine superscheme*. A superringed topological space $(X, \mathcal{O}_X)$ is a *superscheme* if its every point x has an open neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine superscheme.

1.2.10. A group structure on an object of a category. In this subsection, we give definitions and several most important examples of affine group schemes. This notion is not only important by itself, it also lucidly shows the role and possibilities of the "categorical" and "structural" approaches.

We will give consecutively two definitions of a group structure on an object of a category and compare them for the category of schemes.

Let C be a category, $X \in \operatorname{Ob}\mathsf{C}$. A *group structure on* X is said to be given if there are given (set theoretical) group structures on all the sets $P_X(Y) = \operatorname{Hom}_{\mathsf{C}}(Y, X)$, and, for any morphism $Y_1 \longrightarrow Y_2$, the corresponding map of sets $P_X(Y_2) \longrightarrow P_X(Y_1)$ is a group homomorphism.

An object X together with a group structure on it is said to be a *group in the category* C. Let X_1, X_2 be groups in C; a morphism $X_1 \longrightarrow X_2$ in C is said to be a *group morphism in* C if the maps $P_{X_1}(Y) \longrightarrow P_{X_2}(Y)$ are group homomorphisms for any Y.

A group in the category of affine schemes will be called an *affine group scheme* (never an *affine group*: this is a fixed term for a different notion).

Here is the list of the most important examples with their standard notations and names.

Helpful remark. Since $\mathsf{Aff\ Sch}^\circ = \mathsf{Rings}$, instead of studying contravariant functors on $\mathsf{Aff\ Sch}$ represented by an affine group scheme we may discuss the covariant functors on Rings which are simpler to handle.

1.2.11. Examples.

1.2.11.1. The additive group $\mathbb{G}_a = \operatorname{Spec}\mathbb{Z}[T]$. As above, any morphism $\operatorname{Spec} A \longrightarrow \mathbb{G}_a$ is uniquely determined by an element $t \in A_{\bar 0}$, the image of T, which may be chosen at random. The collection of groups with respect to addition $A_{\bar 0} = \mathbb{G}_a(A)$ for the rings $A \in \mathsf{Rings}$ determines the group structure on $\mathbb{G}_a$.

In other words, $\mathbb{G}_a$ represents the functor $\mathsf{Aff\ Sch}^\circ \longrightarrow \mathsf{Gr}$, $\operatorname{Spec} A \mapsto A_{\bar 0}$ or, equivalently, the functor $\mathsf{Rings} \longrightarrow \mathsf{Gr}$, $A \mapsto A^+_{\bar 0}$.

1.2.11.1a. The odd additive supergroup $\mathbb{G}_a^- = \operatorname{Spec}\mathbb{Z}[\theta]$. As above, any morphism $\operatorname{Spec} A \longrightarrow \mathbb{G}_a^-$ is uniquely determined by an element $t \in A_{\bar 1}$, the image of θ, which may be chosen at random. The collection of groups with respect to addition $A_{\bar 1} = \mathbb{G}_a^-(A)$ for the rings $A \in \mathsf{Rings}$ determines the group structure on $\mathbb{G}_a$.

In other words, $\mathbb{G}_a^-$ represents the functor $\mathsf{Aff\ Sch}^\circ \longrightarrow \mathsf{Gr}$, $\operatorname{Spec} A \mapsto A_{\bar 1}$ or, equivalently, the functor $\mathsf{Rings} \longrightarrow \mathsf{Gr}$, $A \mapsto A^+_{\bar 1}$.

1.2.11.2. The multiplicative group $\mathbb{G}_m = \operatorname{Spec}\mathbb{Z}[T, T^{-1}]$. For any superscheme $X = \operatorname{Spec} A$, any morphism $X \longrightarrow \mathbb{G}_m$ is uniquely determined by an element $t \in A_{\bar 0}^\times$, the image of T under the homomorphism $\mathbb{Z}[T, T^{-1}] \longrightarrow A$, where $A_{\bar 0}^\times$ is the group (with respect to multiplication) of invertible elements of $A_{\bar 0}$. Conversely, t corresponds to such a morphism if and only if $t \in A_{\bar 0}^\times$. Therefore

$$P_{\mathbb{G}_m}(\operatorname{Spec} A) = \mathbb{G}_m(A) = A_{\bar 0}^\times, \tag{1.26}$$

and, on the set of A-points, a natural group structure (multiplication) is defined. Furthermore, any ring homomorphism $A \longrightarrow B$ induces, clearly, a group homomorphism $A_{\bar 0}^\times \longrightarrow B_{\bar 0}^\times$ which determines the group structure on $\mathbb{G}_m$.

In other words, $\mathbb{G}_m$ represents the functor $\mathsf{Aff\ Sch}^\circ \longrightarrow \mathsf{Gr}$, $\operatorname{Spec} A \mapsto A_{\bar 0}^\times$ or, equivalently, the functor $\mathsf{Rings} \longrightarrow \mathsf{Gr}$, $A \mapsto A_{\bar 0}^\times$.

1.2.11.2a. The multiplicative group $\mathbb{G}Q_m = \operatorname{Spec}\mathbb{Z}[T, T^{-1}, \theta]$. For any superscheme $X = \operatorname{Spec} A$, any morphism $\varphi\colon X \longrightarrow \mathbb{G}_m$ is uniquely determined by a pair

$$t = \varphi^*(T) \in A_{\bar 0}^\times, \quad \xi = \varphi^*(\theta) \in A_{\bar 1}. \tag{1.27}$$

Therefore

$$P_{\mathbb{G}Q_m}(\operatorname{Spec} A) = \mathbb{G}Q_m(A) = A^\times, \tag{1.28}$$

and, on the set of A-points, a natural group structure (multiplication) is defined. Furthermore, any superring homomorphism $A \longrightarrow B$ induces, clearly, a group homomorphism $A^\times \longrightarrow B^\times$ which determines the group structure on $\mathbb{G}Q_m$.

In other words, $\mathbb{G}Q_m$ represents the functor $\mathsf{Aff\ Sch}^\circ \longrightarrow \mathsf{Gr}$, $\operatorname{Spec} A \mapsto A^\times$ or, equivalently, the functor $\mathsf{Rings} \longrightarrow \mathsf{Gr}$, $A \mapsto A^\times$.

1.2.11.3. The general linear group.

$$\mathrm{GL}(n) = \mathrm{Spec}\,\mathbb{Z}[T_{ij}, T]_{i,j=1}^{n}/(T\det((T_{ij})) - 1). \tag{1.29}$$

It represents the functor $\mathrm{Spec}\,A \mapsto \mathrm{GL}(n; A)$. Obviously, $\mathrm{GL}(1) \simeq \mathbb{G}_m$.

To define *the general linear supergroup* we have to figure out how to express the berezinian by means of polynomials: we do not have any other functions but polynomials in out household. Fortunately, we know that (let us first consider the standard formats)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is invertible} \Longleftrightarrow a \text{ and } d \text{ are invertible} \Longleftrightarrow \mathrm{Ber}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \text{ is invertible.} \tag{1.30}$$

Accordingly, set

$$\begin{aligned}&\mathcal{GL}(n|m) = \mathrm{Spec}\,\mathbb{Z}[T_{ij}, T, U]_{i,j=1}^{n}/(T\mathrm{Ber}(T_{ij}) - 1),\\ &\text{where } T = (T_{ij}) := \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \text{ is the supermatrix of indeterminates}\\ &\text{in the standard format.}\end{aligned} \tag{1.31}$$

This group superscheme represents the functors

$$\mathrm{Spec}\,A \mapsto \mathrm{GL}(n|m; A) \;\text{ and }\; A \mapsto \mathrm{GL}(n|m; A). \tag{1.32}$$

1.2.11.4. The Galois group $\mathrm{Aut}(K'/K)$. Fix a K-algebra K' and let K' be a free K-module of finite rank. The group $\mathrm{Aut}(K'/K)$ of automorphisms of the algebra K' over K is the main object of the study, e.g., in the Galois theory (where the case of fields K, K' is only considered). This group may turn to be trivial if the extension is non-normal or non-separable, and so on.

The functorial point of view suggests to consider all the possible *changes of base* K, i.e., for a variable K-algebra B, consider the group of automorphisms

$$\mathrm{Aut}(B'/B) := \mathrm{Aut}_B(B'), \text{ where } B' = B \bigotimes_K K'. \tag{1.33}$$

We will prove simultaneously that (1) the map $B \mapsto \mathrm{Aut}(B'/B)$ is a functor and (2) this functor is representable.

Select a free basis $e_1, \dots, e_n$ of K' over K. In this basis the multiplication law in K' is given by the formula

$$e_i e_j = \sum_{1\leq k\leq n} c_{ij}^k e_k. \tag{1.34}$$

Denote $e'_i := 1 \bigotimes\limits_K e_i$; then $B' = \bigoplus\limits_{1\leq i\leq n} B\, e'_i$, and any endomorphism t of the B-module B' is given by a matrix (t_{ij}), where $t_{ij} \in B$ and $1 \leq i, j \leq n$. The condition that this matrix determines an endomorphism of an algebra can be expressed as the relations

$$t(e'_i)t(e'_j) = \sum_{1\leq k\leq n} c_{ij}^k t(e'_k). \tag{1.35}$$

Equating the coefficients of e'_k in (1.35) in terms of indeterminates T_{ij} we obtain a system of algebraic relations for T_{ij} with coefficients from K, both necessary and sufficient for (t_{ij}) to define an endomorphism of B'/B.

To obtain *automorphisms*, let us introduce an additional variable t and the additional relation (cf. Example 3) which ensures that $\det(t_{ij})$ does not vanish:

$$t\det(t_{ij}) - 1 = 0. \tag{1.36}$$

The quotient of $K[T, T_{ij}]_{i,j=1}^{n}/(T\det((T_{ij})) - 1)$ is a K-algebra representing the functor

$$B \mapsto \mathrm{Aut}(B'/B). \tag{1.37}$$

This K-algebra replaces the notion of the Galois group of the extension K'/K; it generalizes the notion of the group ring of the Galois group.

1.2.11.4a. Consider the simplest particular case:

$$K' = K(\sqrt{a}), \text{ where } a \in K^\times \setminus (K^\times)^2. \tag{1.38}$$

We may set $e_1 = 1$, $e_2 = \sqrt{a}$; the multiplication table reduces to $e_2^2 = a$.

Let $t(\sqrt{a}) = T_1 + T_2\sqrt{a}$ (obviously, $t(1) = 1$). Since $t(\sqrt{a})^2 = a$, we obtain the equations relating T_1, T_2 and the additional variable T:

$$\begin{cases} T_1^2 + a\,T_2^2 &= a\\ 2T_1\,T_2 &= 0\\ TT_2 - 1 &= 0. \end{cases} \tag{1.39}$$

Now, let us consider separately two cases.

Case 1: $\mathrm{Char}\,K \neq 2$. Hence, 2 is invertible in any K-algebra. The functor of automorphisms is represented by the K-algebra

$$K[T, T_1, T_2]/(T_1^2 + a\,T_2^2 - a,\; T_1T_2,\; TT_2 - 1). \tag{1.40}$$

If B has no zero divisors, then the B-points of this K-algebra have a simple structure: since T_2 must not vanish, T_1 becomes 0 implying that the possible values of T_2 in the quotient ring are ± 1. As the conventional Galois group this group is isomorphic to $\mathbb{Z}/2$; the automorphisms simply change the sign of $\sqrt{a}$.

The following case illustrates that when B does have zero divisors the group of B-points of $\mathrm{Aut}\,K^\times/K$ can be much larger.

Case 2: $\mathrm{Char}\,K = 2$. The functor of automorphisms is represented by the K-algebra

$$K[T, T_1, T_2]/(T_1^2 + aT_2^2 - a, TT_2 - 1). \tag{1.41}$$

In other words, the B-points of the automorphism group are all the B-points of the circle $T_1^2 + aT_2^2 - a = 0$ at which T_2 is invertible!

Let us investigate this in detail. Let B be a field and let (t_1, t_2) be a B-point of the circle at which T_2 is invertible. Then either $t_2 = 1, t_1 = 0$, and we obtain the identity automorphism, or $a = \left(\frac{t_1}{t_2+1}\right)^2$. Therefore there are nontrivial B-points only if $\sqrt{a} \in B$, in which case the equation of the circle turns into the square of a linear one $(T_1 + \sqrt{a}\ \ T_2 + \sqrt{a})^2 = 0$. We have the punctured line (the line without point $T_2 = 0$) of automorphisms!

Obviously, $\mathrm{Aut}(B'/B)$ is isomorphic in this case to $B^\times$ — the multiplicative group of B (under the composition of automorphisms the coefficients of $\sqrt{a}$ are multiplied). So, the non-separable extensions have even more, in a certain sense, automorphisms than separable ones.

The reason why this phenomenon takes place is presence of nilpotents in the algebra $B \bigotimes\limits_K K'$ if $\sqrt{a} \in L$. Indeed, $K(\sqrt{a}) \subset L$, so $K(\sqrt{a}) \otimes_K K(\sqrt{a}) \subset L'$; on the other hand, this product is isomorphic to

$$K(\sqrt{a})[x]/(x^2 - a) \simeq K(\sqrt{a})[y]/(y^2):$$

the automorphisms just multiply y by invertible elements.

One can similarly investigate arbitrary inseparable extensions and construct for them a Galois theory, a generalization of the Jacobson theory.

1.2.11.5. The group μ_n of nth roots of unity. Set

$$\mu_n = \mathrm{Spec}\,\mathbb{Z}[T]/(T^n - 1) = \mathrm{Spec}\,\mathbb{Z}[T, T^{-1}]/(T^n - 1). \tag{1.42}$$

This group represents the functor $\mathrm{Spec}\,A \mapsto \{t \in A_{\bar 0}^\times \mid t^n = 1\}$.

Let X be a closed affine group scheme and Y its closed subscheme Y such that $P_Y(Z) \subset P_X(Z)$ is a subgroup for any Z. We call Y with the induced group structure a *closed subgroup of* X.

Therefore μ_n is a closed subgroup of $\mathbb{G}_m$. Explicitly, the homomorphism $T \mapsto T^n$ determines a group scheme homomorphism $\mathbb{G}_m \longrightarrow \mathbb{G}_m$ of "raising to the power n" and μ_n represents the kernel of this homomorphism.

1.2.11.5a. The supergroup $\mu_{n,m}$ of nth roots of unity. Set

$$\mu_{n,m} = \mathrm{Spec}\,\mathbb{Z}[T, \theta]/(T^n - 1) = \mathrm{Spec}\,\mathbb{Z}[T, T^{-1}, \theta]/(T^n - 1), \tag{1.43}$$

where $\theta = (\theta_1, \dots, \theta_m)$. This supergroup represents the functor

$$\mathrm{Spec}\,A \mapsto \{(t \in A_{\bar 0}^\times, \xi \in A_{\bar 1}^m) \mid t^n = 1\}. \tag{1.44}$$

1.2.11.6. The scheme of a finite group G. Let G be a conventional (set-theoretical) finite group. Set $A = \mathbb{Z}^{(G)} := \prod\limits_{g \in G} \mathbb{Z}$. In other words, A is a free module $\bigoplus\limits_{g \in G} \mathbb{Z}^{(g)}$ ($|G|$ copies of $\mathbb{Z}$) with the multiplication table

$$e_g e_h = \begin{cases} 0 = (0, \dots, 0) & \text{if } h \neq g, \\ e_g & \text{if } h = g. \end{cases} \tag{1.45}$$

The space $X = \mathrm{Spec}\,A$ is disjoint; each of its components is isomorphic to $\mathrm{Spec}\,\mathbb{Z}$ and these components are indexed by the elements of G. For any ring B, whose spectrum is connected, the set of morphisms $\mathrm{Spec}\,B \longrightarrow \mathrm{Spec}\,A$ is, therefore, in the natural one-to-one correspondence with the elements of G.

If $\mathrm{Spec}\,B$ is disjoint, then any morphism $\mathrm{Spec}\,B \longrightarrow \mathrm{Spec}\,A$ is determined by the set of its restrictions onto the connected components of $\mathrm{Spec}\,B$. Let $\mathrm{Conn}\,B$ be the set of these components; then, clearly, the point functor is given by

$$P_X(\mathrm{Spec}\,B) \xrightarrow{\sim} (G)^{\mathrm{Conn}\,B} := \mathrm{Hom}(G, \mathrm{Conn}\,B). \tag{1.46}$$

and therefore X is endowed with a natural group structure called the *scheme of the group* G.

1.2.11.7. The relative case. Let $S = \mathrm{Spec}\,K$. A group object in the category $\mathsf{Aff\ Sch}_S$ of affine schemes over S is said to be an *affine S-group* (or an *affine K-group*). Setting $\mathbb{G}_{m\,/K} = \mathbb{G}_m \times S$ and $\mu_{n\,/K} = \mu_n \times S$, and so on, we obtain a series of groups over an arbitrary scheme S (or a ring K). Each of them represents "the same" functor as the corresponding absolute group, but restricted onto the category of K-algebras.

1.2.11.8. Linear algebraic groups. Let K be a field. Any closed subgroup of $\mathrm{GL}_n(A)_{/K}$ is said to be a *linear algebraic group over* K.

In other words, a linear algebraic group is determined by a system of equations

$$F_k(T_{ij}) = 0, \quad \text{for } i, j = 1, \dots, n \text{ and } k \in I \tag{1.47}$$

such that if (t'_{ij}) and (t''_{ij}) are two solutions of the system (1.47) in a K-algebra A such that the corresponding matrices are invertible, then the matrix $(t'_{ij})(t''_{ij})^{-1}$ is also a solution of (1.47)

1.2.12. Linear algebraic groups. The place of linear algebraic groups in the general theory is elucidated by the following fundamental theorem (cf. [OV]).

Theorem. *Let X be an affine group scheme of finite type over K. Then X is isomorphic to a linear algebraic group.*

1.2.13. Statement (Cartier)**.** *Let X be the scheme of a linear algebraic group over a field of characteristic zero. Then X is reduced, i.e., $X = X_{red}$, its ring has no nilpotents.*

If $\mathrm{Char}\,K = p$, then the statement of the theorem is false (cf. also with the group superscheme case) as demonstrated by the following

Example. Set

$$\mu_{p\,/K} = \mathrm{Spec}\,K[T]/(T^{p-1}) = \mathrm{Spec}\,K[T]/((T-1)^p). \tag{1.48}$$

Obviously, $K[T]/((T-1)^p)$ is a local artinian algebra of length p, and its spectrum should be considered as a "point of multiplicity p". This is a nice agreement with our intuition: all the roots of unity of degree p are glued together and turn into one root of multiplicity p.

More generally, set

$$\mu_{p^n\ /K} = \operatorname{Spec} K[T]/((T-1)^{p^n}). \tag{1.49}$$

We see that the length of the nilradical may be however great.

1.2.14. The set-theoretical definition of the group structure. Let a category C contain a final[2)] object E and products. Let X be a group with a unit (identity) 1; let $x, y, z \in X$; then, in the standard notations[3)], we obtain

$$m(x,y) = xy,\ \ i(x) = x^{-1},\ \ u(E) = 1, \tag{1.50}$$

and the conventional axioms of the associativity, the left inverse and the left unit have, respectively, the form

$$(xy)z = x(yz),\ x^{-1}x = 1,\ 1x = x \tag{1.51}$$

The usual set-theoretical definition of the group structure on a set X given above is, clearly, equivalent to the existence of three morphisms

$$\begin{array}{llll} m: & X\times X & \longrightarrow X & (\text{multiplication, } x, y \mapsto xy) \\ i: & X & \longrightarrow X & (\text{inversion, } x \mapsto x^{-1}\ ) \\ u: & E & \longrightarrow X & (\text{unit, the embedding of } E) \end{array} \tag{1.52}$$

that satisfy the axioms of associativity, left inversion and left unit, respectively, expressed as commutativity of the following diagrams:

$$\begin{tikzcd} X\times X\times X \arrow[r, "{(m,\, \mathrm{id}_X)}"] \arrow[d, "{(\mathrm{id}_X, m)}"'] & X\times X \arrow[d, "m"] \\ X\times X \arrow[r, "m"] & X \end{tikzcd} \tag{1.53}$$

$$\begin{tikzcd} X\times X \arrow[rr, "{(i,\, \mathrm{id}_X)}"] & & X\times X \arrow[d, "m"] \\ X \arrow[u, "\delta"] \arrow[r, "\bullet"] & E \arrow[r, "u"] & X \end{tikzcd} \tag{1.54}$$

$$\begin{tikzcd} X\times X \arrow[r, "{(\bullet,\, \mathrm{id}_X)}"] & E\times X \arrow[r, "{(u,\, \mathrm{id}_X)}"] & X\times X \arrow[d, "m"] \\ X \arrow[u, "\delta"] \arrow[r, "\mathrm{id}_X"] & X \arrow[r, "\mathrm{id}_X"] & X \end{tikzcd} \tag{1.55}$$

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[2] An object E is said to be *final* if $\operatorname{card}(\operatorname{Hom}(X, E)) = 1$ for any $X \in \operatorname{Ob} C$.

[3] Here we denote the group unit by 1; it is the image of E.

(In diagrams (1.54), (1.55) the morphism of contraction to a point (E) is denoted by "$\bullet$".)

In the category Sets the axioms (1.53)–(1.55) turn into the usual definition of a group though in an somewhat non-conventional form.

1.2.15. Equivalence of the two definitions of the group structure. Let a group structure in the set-theoretical sense be given on $X \in \operatorname{Ob}\mathsf{C}$. Then, for every $Y \in \operatorname{Ob}\mathsf{C}$, the morphisms m, i, u induce the group structure on the set of Y-points thanks to the above subsection. The verification of the compatibility of these structures with the maps $P_X(Y_1) \longrightarrow P_X(Y_2)$ is left to the reader.

Conversely, let a group structure in the sense of the first definition be given on $X \in \operatorname{Ob}\mathsf{C}$. How to recover the morphisms m, i, u? We do it in three steps:

a) The group $P_X(X\times X)$ contains projections $\pi_1, \pi_2 \colon X\times X \longrightarrow X$. Set $m = \pi_1 \circ \pi_2$ (the product $\circ$ in the sense of the group law).

b) The group $P_X(X)$ contains the element id_X. Denote its inverse (in the sense of the group law) by i.

c) The group $P_X(E)$ has the unit element. Denote it by $u\colon E \longrightarrow X$.

1.2.16. How to describe the group structure on an affine group (super)scheme $X = \operatorname{Spec} A$ in terms of A. We will consider the general, i.e., relative, case, i.e., assume A to be a K-algebra.

The notion of a group G is usually formulated in terms of the states, i.e., points of G. In several questions, however, for example, to quantize it, we need a reformulation in terms of observables, i.e., the functions on G. Since any map of sets $\varphi\colon X \longrightarrow Y$ induces the homomorphism of the algebras of functions $\varphi^*\colon F(Y) \longrightarrow F(X)$, we dualize the axioms of sec. 1.2.14 and obtain the following definition.

A *bialgebra* structure on a K-algebra A is given by three K-algebra homomorphisms:

$$\begin{array}{lll} m^*: & A \longrightarrow A \bigotimes\limits_K A & \text{co-multiplication} \\ i^*: & A \longrightarrow A & \text{co-inversion} \\ u^*: & A \longrightarrow K & \text{co-unit} \end{array} \tag{1.56}$$

which satisfy the axioms of co-associativity, left co-inversion and left co-unit, respectively, expressed in commutativity of the following diagrams:

$$\begin{tikzcd}
A\otimes A\otimes A & A\otimes A \arrow[l, "m^*\otimes \mathrm{id}_A"'] \\
A\otimes A \arrow[u, "\mathrm{id}_A\otimes m^*"] & A \arrow[l, "m^*"] \arrow[u, "m^*"']
\end{tikzcd} \tag{1.57}$$

$$\begin{tikzcd}
A\otimes A \arrow[d, "\mu"'] & & A\otimes A \arrow[ll, "i^*\otimes \mathrm{id}_A"'] \\
A & K \arrow[l] & A \arrow[l, "u^*"'] \arrow[u, "m^*"']
\end{tikzcd} \tag{1.58}$$

(the left vertical arrow is the multiplication $\mu : a\otimes b\mapsto a\ b$ in A, the left horizontal arrow is given by $1\mapsto 1$).

$$\begin{tikzcd}
A\otimes A \arrow[d, "\mu"'] & A \arrow[l] & A\otimes A \arrow[l, "u^*\otimes \mathrm{id}_A"'] \\
A & & A \arrow[ll, "\mathrm{id}_A"'] \arrow[u, "m^*"']
\end{tikzcd} \tag{1.59}$$

(the left arrow in the top line is $a\mapsto 1\otimes a$).

It goes without saying that this definition is dual to that from sec. 1.2.14, and therefore the group structures on the K-scheme $\operatorname{Spec} A$ are in one-to-one correspondence with the co-algebra structures on the K-algebra A.

Example. The homomorphisms m^*, i^*, u^* for the additive group scheme $\mathbb{G}_a = \operatorname{Spec}\mathbb{Z}[T]$ are:

$$m^*(T) = T\otimes 1 + 1\otimes T, \quad i^*(T) = -T, \quad u^*(T) = 0. \tag{1.60}$$

1.2.17. What a Lie superalgebra is. Dealing with superalgebras it sometimes becomes useful to know their definition. Lie superalgebras were distinguished in topology in 1930's, the Grassmann superalgebras half a century earlier. So when somebody offers a "better than usual" definition of a notion which seemed to have been established about 70 year ago this might look strange, to say the least. Nevertheless, the answer to the question "what is a (Lie) superalgebra?" is still not a common knowledge.

So far we defined Lie superalgebras naively: via the Sign Rule (sect. 1.1.2). However, the naive definition suggested above ("apply the Sign Rule to the definition of the Lie algebra") is manifestly inadequate for considering the supervarieties of deformations and for applications of representation theory to mathematical physics, for example, in the study of the coadjoint representation of the Lie supergroup which can act on a supermanifold but never on a superspace — an object from another category. We were just lucky in the case of finite dimensional Lie algebras over $\mathbb{C}$ that the vector spaces can be viewed as manifolds. In the case of spaces over $\mathbb{K}$ and in the super setting, to be able to deform Lie (super)algebras or to apply group-theoretical methods,

we must be able to recover a supermanifold or supervariety from a superspace, and vice versa.

A proper definition of Lie superalgebras is as follows. The *Lie superalgebra* in the category of supervarieties[4)] corresponding to the "naive" Lie superalgebra $L = L_{\bar 0}\oplus L_{\bar 1}$ is a linear supermanifold $\mathcal{L} = (L_{\bar 0}, \mathcal{O})$, where the sheaf of functions $\mathcal{O}$ consists of functions on $L_{\bar 0}$ with values in the Grassmann superalgebra on $L_{\bar 1}^*$; this supermanifold should be such that for "any" (say, finitely generated, or from some other appropriate category) supercommutative superalgebra C, the space $\mathcal{L}(C) = \operatorname{Hom}(\operatorname{Spec} C, \mathcal{L})$, called *the space of C-points of $\mathcal{L}$*, is a Lie algebra and the correspondence $C\longrightarrow \mathcal{L}(C)$ is a functor in C. (A. Weil introduced this approach in algebraic geometry in 1953; in super setting it is called *the language of points* or *families*.) This definition might look terribly complicated, but fortunately one can show that the correspondence $\mathcal{L}\longleftrightarrow L$ is one-to-one and the Lie algebra $\mathcal{L}(C)$, also denoted $L(C)$, admits a very simple description: $L(C) = (L\otimes C)_{\bar 0}$.

A *Lie superalgebra homomorphism* $\rho\colon L_1 \longrightarrow L_2$ in these terms is a functor morphism, i.e., a collection of Lie algebra homomorphisms $\rho_C\colon L_1(C) \longrightarrow L_2(C)$ such that any homomorphism of supercommutative superalgebras $\varphi\colon C \longrightarrow C_1$ induces a Lie algebra homomorphism $\varphi\colon L(C)\longrightarrow L(C_1)$ and products of such homomorphisms are naturally compatible. In particular, a *representation* of a Lie superalgebra L in a superspace V is a homomorphism $\rho\colon L\longrightarrow \mathfrak{gl}(V)$, i.e., a collection of Lie algebra homomorphisms $\rho_C\colon L(C)\longrightarrow (\mathfrak{gl}(V)\otimes C)_{\bar 0}$.

1.2.17.1. Example. Consider a representation $\rho\colon \mathfrak{g}\longrightarrow\mathfrak{gl}(V)$. The space of infinitesimal deformations of ρ is isomorphic to $H^1(\mathfrak{g}; V\otimes V^*)$. For example, if $\mathfrak{g}$ is the $0|n$-dimensional (i.e., purely odd) Lie superalgebra (with the only bracket possible: identically equal to zero), its only irreducible representations are the trivial one, $\mathbb{1}$, and $\Pi(\mathbb{1})$. Clearly, $\mathbb{1}\otimes\mathbb{1}^*\simeq \Pi(\mathbb{1})\otimes\Pi(\mathbb{1})^*\simeq\mathbb{1}$, and, because the Lie superalgebra $\mathfrak{g}$ is commutative, the differential in the cochain complex is zero. Therefore $H^1(\mathfrak{g};\mathbb{1}) = E^1(\mathfrak{g}^*)\simeq\mathfrak{g}^*$, so there are $\dim\mathfrak{g}$ odd parameters of deformations of the trivial representation. If we consider $\mathfrak{g}$ "naively", all of these odd parameters will be lost.

Which of these infinitesimal deformations can be extended to a global one is a separate much tougher question, usually solved *ad hoc*.

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4 That is ringed spaces such that the sections of their sheaves of "functions" form supercommutative superrings and morphisms of supervarieties are only those ring space morphisms that preserve parity of the superrings of sections of the structure sheaves.

Chapter 2

Examples of simple Lie superalgebras and their relatives over $\mathbb{C}$ (D. Leites, I. Shchepochkina)

For details of the proofs, see [LSh].

2.1. On setting of the problem

Selection of Lie algebras with reasonably nice properties is a matter of taste and is influenced by the underlying "meta"-problem. One of the usual choices is the class of *simple* algebras: they have a richer structure, and therefore are easier to study than other types of Lie algebras; they also illuminate important symmetries. Even representatives of the "complementary" type — solvable algebras (and their particular case, nilpotent algebras) — are often of real interest only when subalgebras of simple algebras.

Of simple Lie algebras, the finite dimensional ones are naturally the first to study. Over $\mathbb{C}$, they constitute three "classical" series ($\mathfrak{sl}$, $\mathfrak{o}$ and $\mathfrak{sp}$) and five exceptional algebras; all are neatly encoded by Cartan matrices or, in an even more graphic way, by very simple graphs — Dynkin diagrams.

Next on the agenda are $\mathbb{Z}$-*graded Lie algebras of polynomial growth* (let us call them ZGLAPGs for short). Some of these algebras have Cartan matrix; they resemble finite dimensional simple Lie algebras more than others, and have proved very useful in various branches of mathematics and theoretical physics, cf. [K3].

Let us recall the meaning of the terms entering ZGLAPG. A $\mathbb{Z}$-graded algebra $A = \oplus A_i$ is said to be *simple graded* if it has no *homogeneous* or better say *graded* ideals (i.e., no ideals $I = \oplus I_i$ such that $I_i = I \cap A_i$; a simple graded algebra can be non-simple as an abstract algebra: e.g., loop algebras with values in a simple Lie algebra are simple graded but not simple). If $\dim A_i < \infty$ for all i, one can define the *growth* of A to be

$$\mathrm{gth}(A) = \overline{\lim_{n\longrightarrow\infty}} \frac{\ln \dim \underset{|i|\leq n}{\oplus} A_i}{\ln n}.$$

We say that A is of *polynomial growth* if $\operatorname{gth}(A) < \infty$, i.e., $\dim \bigoplus\limits_{|i|\leq n} A_i \sim n^r$ as $n \longrightarrow \infty$.

Around 1966, V. Kac and B. Weisfeiler started to study simple *filtered* Lie algebras of polynomial growth, i.e., Lie algebras such that the associated graded algebra is of polynomial growth. The first step in the study was, of course, the *simple graded* algebras. Kac classified simple ZGLAPGs **provided they are generated by elements of degree** -1 **and** 1.

Among simple ZGLAPGs, only $\mathfrak{vect}(1) = \mathfrak{der}\ \mathbb{C}[x]$ and

$$\mathfrak{vect}^L(1) := \mathfrak{der}\ \mathbb{C}[x^{-1}, x]$$

are not generated by elements of degree ± 1 with respect to any grading. Kac conjectured that these two examples, plus his list, exhaust all the simple ZGLAPGs. Twenty years after, O. Mathieu proved this conjecture in a string of rather complicated papers culminating in [M].

It goes without saying that we look at the *simple* Lie algebras as a preliminary (though vital) material. The central extensions, deformations and algebras of outer derivations of simple Lie superalgebras are often more interesting in applications (e.g., affine Kac–Moody algebras are "more useful" than loop algebras; likewise, Poisson algebras and their deformations are often no less appealing to the heart of the physicist than even the Lie algebras of Hamiltonian vector fields, vital in the classical mechanics). So the real object of interest is the answer to the following **Main Problem**:

2.1.1. Problem. Starting with a simple Lie algebra of class ZGLAPG, list the results of iterated combinations of the following operations:

a) $\mathfrak{g} \longrightarrow \mathfrak{e}(\mathfrak{g})$, the nontrivial central extension of $\mathfrak{g}$ (example: central extensions of loop algebras; the Poisson algebra);

b) $\mathfrak{g} \longrightarrow \mathfrak{der}(\mathfrak{g})$, the whole derivation algebra of $\mathfrak{g}$ or a subalgebra of $\mathfrak{der}(\mathfrak{g})$ (example: affine Kac–Moody algebras);

c) deformations of $\mathfrak{g}$ (although they sometimes lead out of the class ZGLAPG, such deformed Lie algebras are often no less important in applications than the original algebras; examples: the result of the *quantization* of either the Poisson algebra or of the algebra of Hamiltonian vector fields; the Krichever-Novikov algebras);

d) all filtered completions of $\mathfrak{g}$;

e) forms of these algebras over non-closed fields.

We will loosely refer to the results of the iterated procedures described in Main Problem as *classical* Lie algebras and call the algebras obtained from a simple one, $\mathfrak{g}$, by the iterations of the above procedures a) and b) (and sometimes even by all the procedures a) –d)) the *relatives* of $\mathfrak{g}$ and each other (from the next of kin to distant ones). To list them, their counterparts in the super setting and over fields of prime characteristic, and ALL their various gradings, is the main strategic goal of Leites's *Seminar on Supersymmetries* ([LSoS]).

2.1.1.1. Remark. The reader should not think, as was customary 30 years ago, that algebras "bigger" than the $\mathbb{Z}$-graded of polynomial growth type are useless or too difficult to study. For example, various versions of $\mathfrak{gl}(\infty)$ and algebras studied by Borcherds or Gritsenko and Nikulin are, though huge, of huge interest and (some of) their representations are describable.

2.1.2. Types of the classical Lie algebras. Let us qualitatively describe the simple Lie algebras of polynomial growth to better visualize them. They break into the disjoint union of the following types:

1) *Finite dimensional* algebras (growth 0).

2) *Loop algebras*, perhaps twisted (of growth 1); more important in applications are their non-simple "relatives" called *affine Kac–Moody* algebras, cf. [K3].

3) *Vectorial algebras*, i.e., Lie algebras of vector fields [1)] with polynomial coefficients (growth is equal to the number of indeterminates) or their completions with formal power series as coefficients.

4) *Stringy algebra* [2)] $\mathfrak{vect}^L(1) = \mathfrak{der}\ \mathbb{C}[t^{-1}, t]$, where the superscript stands for Laurent.

This class consists of one algebra. It is often called *Witt algebra* and denoted $\mathfrak{witt}$ in honor of Witt who considered its analog over fields of prime characteristic, and by physicists *the centerless Virasoro algebra* because its nontrivial central extension (discovered by Fuchs and Gelfand), $\mathfrak{vir}$, is called the *Virasoro algebra* in honor of Virasoro who rediscovered the corresponding central extension and indicated its importance in physical models.

Strictly speaking, stringy algebras are also vectorial, but we retain the generic term *vectorial* for algebras with *polynomial* or *formal* coefficients.

The above are graded algebras. Filtered algebras are not classified. Among the known examples of filtered algebras, we distinguish various deformations of the above-listed graded algebras, several more examples listed in the next item, and random examples to Main Problem given above.

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[1] These algebras are sometimes known under a clumsy name "algebras of Cartan type" introduced in the 1960s: just imagine "the Cartan subalgebra in the Lie superalgebra of Cartan type with Cartan matrix" ($\mathfrak{svect}^L_\alpha(1|2)$ is an example of a Lie superalgebra of Cartan type with Cartan matrix).

[2] The term "stringy algebra" is induced by the lingo of imaginative physicists who now play with the idea that an elementary particle is not a point but rather a slinky springy string. (We share with these physicists the sensation of beauty of stringy superalgebras [GSW] and urge to investigate them, even if beware of the social dangers in overvaluing their importance [WL].) The term "stringy algebra" means "pertaining to string theory" but also mirrors their structure as a collection of several strings — the $\mathfrak{witt}$-modules. In our works "stringy" means either "simple vectorial superalgebra on a supermanifold whose underlying manifold is a circle **or a relative** (in the sense of Main Problem) of such algebra". Other appellations, not quite equivalent, are "vertex superalgebras" and somewhat self-contradictory "superconformal algebras".

5) *Lie algebra of matrices of complex size and its generalizations*, cf. [GL2, LSe, DGS]. These algebras are simple filtered Lie algebras of polynomial growth whose associated graded algebras are not simple. Up to some twists, the examples known to us are multiparameter deformations of the Poisson algebras of functions on the orbits in the coadjoint representation of simple finite dimensional or Lie algebras of matrices over the rings of differential operators. Super version of this class is even wider due to S. Montgomery's and Konstein's constructions ([GL2, KV]). Though they are much more numerous than simple *graded* ones, we still hope that it is possible to distill a tame subproblem. Still more examples are given by the current algebras and vectorial algebras on varieties M, cf. [Le4]; such examples are lately known for compact Riemann surfaces M as Krichever–Novikov algebras.

The algebras of the above classes 1) – 5) can be characterized by other properties that are sometimes taken for their definition, cf. [FSS], making the nomenclature rather complicated. We think that the situation is, actually, rather simple: There are only two major types of Lie superalgebras ("symmetric" and "skew"):

(SY) For symmetric algebras, related with a Cartan subalgebra (or a maximal toral subalgebra which might be smaller than Cartan subalgebra) is a root decomposition such that

$$\operatorname{sdim}\mathfrak{g}_\alpha = \operatorname{sdim}\mathfrak{g}_{-\alpha} \quad \text{for any root} \quad \alpha; \tag{2.1}$$

(SK) For skew algebras, related with a Cartan subalgebra (or a maximal torus) is a root decomposition such that (2.1) fails. (Usually, skew algebras can be realized as **vectorial** Lie superalgebras — subalgebras of the Lie superalgebra of vector fields $\mathfrak{vect}(n|m) = \mathfrak{der}\mathbb{K}[x,\theta]$, where $x = (x_1, \dots, x_n)$ are even indeterminates and $\theta = (\theta_1, \dots, \theta_m)$ are odd indeterminates.

(Of course, symmetric algebras can sometimes be realized as subalgebras of $\mathfrak{vect}(n|m)$, but this is beside the point.)

Algebras of classes 1) and 2) have an additional property: they have a Cartan matrix (the algebras of class 5) also have a Cartan matrix, albeit in a very generalized sense, see [SV]). For Lie algebras of classes 1) and 2), this Cartan matrix is always symmetrizable and, moreover, not only with integer entries but of so simple form that it can be encoded by a simple graph (Dynkin graph). Generally, Lie algebras with a symmetrizable Cartan matrix and "most" of the algebras of class 5) possess an invariant symmetric non-degenerate bilinear form — a powerful tool for solving numerous problems, see *Dynkin index* in [CES].

2.1.3. Superization. Even before Wess and Zumino made importance of supersymmetries manifest in physics, cf. [D, GSW, WZ], the definition of what was later called superschemes (1972) made it manifest that most of notions of differential and algebraic geometry have a super counterpart. In 1972, after Leites gave a talk on supermanifolds and supergroups at the seminar of

Vinberg and Onishchik, the leaders of the seminar addressed Kac and Leites with an assignment: Classify simple (finite dimensional) Lie superalgebras. It immediately turned out that the above types of simple Lie algebras and their properties become intermixed under superization.

• *Finite dimensional superalgebras*. They split into the following two types: <u>"symmetric"</u> ones, subdivided into
SYCM, the class of algebras **with Cartan matrix**;
SYWCM, the class of algebras **without Cartan matrix** in the conventional sense (only, perhaps, in the sense of Saveliev and Vershik, [SV]) but with a *non-degenerate supersymmetric bilinear form* — an analog of the Killing form — that can be either even or odd;
<u>"skew" or vectorial</u> types: with more positive roots than it has negative ones.

2.1.3.1. Remark. *A posteriori* we see that the matrix is symmetrizable; since the Cartan matrix is symmetrizable, these algebras possess an *invariant non-degenerate even supersymmetric bilinear form* (not necessarily related with any representation.

• *(Twisted) loop superalgebras*. It is obvious that loop superalgebras with values in simple Lie superalgebras are also simple (as graded algebras). Unlike Lie algebras, some of the loop superalgebras have no Cartan matrix, some have a non-symmetrizable Cartan matrix, and so on, so classification of (twisted) loop superalgebras splits into several very different cases.

The very first examples showed that even if a finite dimensional Lie superalgebra $\mathfrak{g}$ possesses a Cartan matrix, the twisted loop superalgebra $\mathfrak{g}_\varphi^{(k)}$ (for notations and theory in non-super case, see [K3]) may possess no Cartan matrix and *vice versa*.

An intrinsic characterization of loop superalgebras without appeal to Cartan matrix (deduced from [M]) is given in [GLS1] together with an intrinsic characterization of stringy superalgebras: both types of algebras are $\mathbb{Z}$-graded $\mathfrak{g} = \mathop{\oplus}\limits_{i=-d}^{\infty} \mathfrak{g}_i$ of infinite depth $d = \infty$ but in the adjoint representation they act differently:

for the loop-type algebras, every root vector corresponding to any real root acts locally nilpotently in the adjoint representation; for the stringy algebras, this is not so.

Recall that a root is said to be *real*, if only finitely many of its multiples are roots (otherwise the root is said to be *imaginary*. Recall also that the action of an operator X on a space V is said to be *nilpotent* if $X^N(v) = 0$ for any $v \in V$ and some N; the action is said to be *locally nilpotent* N depends on v.

Before we came to the above intrinsic definition, Leites conjectured that, as for Lie algebras, simple twisted loop superalgebras correspond to outer automorphisms of $\varphi \in \operatorname{Aut}\mathfrak{g}/\operatorname{Int}\mathfrak{g}$, where $\operatorname{Aut}\mathfrak{g}$ is the group of all automorphisms $\mathfrak{g}$

and $\mathfrak{Int}\,\mathfrak{g}$ is the subgroup of the inner automorphisms. Serganova listed these automorphisms and amended the conjecture: to get distinct algebras, one should factorize Aut $\mathfrak{g}$ modulo a group somewhat larger than $\mathfrak{Int}\,\mathfrak{g}$, namely, the connected component of the unit, see [Se], [FLS]. An *a priori* classification of Lie superalgebras of polynomial growth with symmetrizable Cartan matrix is due to J. van de Leur [vdL]; his classification and Serganova's recently published classification of Lie superalgebras of polynomial growth with *non-symmetrizable* Cartan matrix [HS] cited in [GLS1] support the Leites-Serganova conjecture on the completeness of the list of simple loop-type superalgebras with Cartan matrix given in [FLS].

• *Stringy Lie superalgebras*. For the conjectural list of simple ones, see [GLS1]. Partly (with several extra conditions) this conjecture is proved in [KvdL], [K4]. Observe that some of the stringy superalgebras possess Cartan matrix, though non-symmetrizable ones [GLS1], and some of them have deformations whereas their vectorial namesakes are rigid.

We suggested the term *stringy* for the general class of algebras inside of which some are conformal, some are simple, and so on.

Some of the simple stringy superalgebras ([GLS1]) are *distinguished*: they admit a nontrivial central extensions. In [GLS1], there are also indicated exceptional stringy superalgebras and occasional isomorphisms unnoticed in previous papers and often ignored in later ones.

In this paper we consider the remaining type of simple $\mathbb{Z}$-graded Lie superalgebras of polynomial growth:

• *Vectorial Lie superalgebras*. The main examples to look at with the mind's eye are the Lie algebra $L = \mathfrak{der}\ \mathbb{C}[x]$ of polynomial vector fields with grading and filtration given by setting $\deg x_i = 1$ for all i, and its (x)-adic completion, the filtered Lie algebra $\mathcal{L} = \mathfrak{der}\ \mathbb{C}[[x]]$ of formal vector fields.

É. Cartan came to the classification problem of simple vectorial algebras from geometrical problems in which "primitive", rather than simple, Lie algebras naturally arise. Infinite dimensional primitive Lie algebras are isomorphic to the algebras of derivations of simple ones, and since $\dim \mathfrak{der}\ \mathfrak{g}/\mathfrak{g} \leq 1$, to classify infinite dimensional *primitive* algebras is practically the same as to classify the simple vectorial ones. (The list of finite dimensional primitive algebras is much longer than that of simple ones but still manageable: [O].) Contrariwise, the classification problem of primitive Lie *super*algebras is wild, as shown in [ALSh].

2.1.4. Classification problems. Consider infinite dimensional complex filtered Lie superalgebras $\mathcal{G}$ with decreasing filtration of the form

$$\mathcal{G} = \mathcal{G}_{-f} \supset \mathcal{G}_{-f+1} \supset \cdots \supset \mathcal{G}_0 \supset \mathcal{G}_1 \supset \dots \tag{2.2}$$

where *depth* f is finite and where

1) $\mathcal{G}_0$ is a maximal subalgebra (usually, of finite codimension);
2) $\mathcal{G}_0$ does not contain ideals of the whole $\mathcal{G}$.

The algebra $\mathcal{G}$ with such a subalgebra $\mathcal{G}_0$ is called a *primitive Lie superalgebra*.

We assume that these Lie superalgebras $\mathcal{G}$ are complete with respect to a natural topology whose basis of neighborhoods of zero is formed by the spaces of finite codimension, e.g., the $\mathcal{G}_i$. (In the absence of odd indeterminates this topology is a most natural one: we consider two vector fields *k-close* if their coefficients coincide up to terms of degree $\leq k$.) This topology is naturally (see. §1) called, "briefly", *projective limit topology* but even more unfortunate term "linearly compact topology" is also used. No other topology will be encountered, so all topological terms (complete, open, closed) refer to this topology.

Observe that the very term "filtered algebra" implies that $[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}$. Set $G_i = \mathcal{G}_i/\mathcal{G}_{i+1}$. Conditions 1) and 2) manifestly imply that $\dim G_i < \infty$ for all i and the $\mathbb{Z}$-graded Lie superalgebra $G = \underset{k\geq -f}{\oplus} G_k$ associated with $\mathcal{G}$ grows *polynomially*, i.e., $\dim \underset{k\leq n}{\oplus} G_k$ grows as a polynomial in n.

Weisfeiler endowed every such filtered Lie algebra $\mathcal{G}$ with another, refined, filtration (here $\mathcal{G} = \mathcal{L}$ as abstract algebras):

$$\mathcal{L} = \mathcal{L}_{-d} \supset \mathcal{L}_{-d+1} \supset \cdots \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \dots \tag{2.3}$$

by setting $\mathcal{L}_0 = \mathcal{G}_0$ and letting $\mathcal{L}_{-1}$ to be a minimal $\mathcal{L}_0$-invariant subspace strictly containing $\mathcal{L}_0$, different from $\mathcal{L}_0$, and $\mathcal{L}_0$-invariant, the other terms being defined by the formula (for $i \geq 1$):

$$\mathcal{L}_{-i-1} = [\mathcal{L}_{-1}, \mathcal{L}_{-i}] + \mathcal{L}_{-i} \quad \text{and} \quad \mathcal{L}_i = \{D \in \mathcal{L}_{i-1} \mid [D, \mathcal{L}_{-1}] \subset \mathcal{L}_{i-1}\}. \tag{2.4}$$

The d in (2.3) is called the *depth* of $\mathcal{L}$ and of the associated graded Lie superalgebra L.

An advantage of the *Weisfeiler filtrations* is that for the corresponding regraded Lie superalgebra $\mathcal{L}$ the L_0-action on L_{-1} is irreducible. These refined filtrations are called, after [W], *Weisfeiler filtrations* and the term is applied even to Lie superalgebras, where Weisfeiler's construction is literally applied; we will shortly write *W-filtrations* and call the gradings associated with W-filtrations *W-gradings*.

Now, observe that condition 2) on filtrations considered is equivalent to the following condition which is sometimes more convenient to use. Set $G_k = \mathcal{G}_k/\mathcal{G}_{k+1}$. We have:

2′) For any non-zero $x \in G_k$, where $k \geq 0$, there exists $y \in G_{-1}$ such that $[x, y] \neq 0$.

When the L_0-module L_{-1} is faithful, as is always the case for the *simple* Lie superalgebras $\mathcal{L}$, such filtered Lie superalgebras $\mathcal{L}$ (and the associated with them graded ones, L) can be realized by *vector fields* on the supermanifold corresponding to the linear superspace $(\mathcal{L}/\mathcal{L}_0)^*$ with formal (resp. polynomial) coefficients. So, being primarily interested in simple Lie superalgebras, we assume that the L_0-module L_{-1} is faithful.

The following problems arise:

A′) Classify simple W-graded vectorial Lie superalgebras as abstract ones, i.e., distinction in grading disregarded.

We immediately see that this problem is very unnatural: quite distinct algebras become equivalent. Indeed, as early as in [ALSh], we observed that $\mathfrak{vect}(1|1)$, the Lie algebra of all vector fields on $(1|1)$-dimensional superspace, is isomorphic as an abstract algebra to $\mathfrak{k}(1|2)$, the Lie algebra of *contact* vector fields on $(1|2)$-dimensional superspace (and also to $\mathfrak{m}(1)$, another, "odd" type of contact Lie superalgebra). A more natural formulation is, therefore, the following one:

A) Classify simple W-graded vectorial Lie superalgebras.

In some applications it suffices to confine ourselves to graded algebras ([Le5]), but in applications (e.g., in the study of representations, *complete* algebras are usually more important and more natural than the associated with them graded ones. So the following problem might look like as a reasonable goal:

B) Classify simple W-filtered complete vectorial Lie superalgebras.

Observe natural ("trivial") examples of complete algebras: the algebras $\hat{L}$ obtained from the list of examples L that answers to Problem A by taking formal series, rather than polynomials, as coefficients. In addition to these examples there might occur nontrivial *filtered deforms* (as Gerstenhaber calls the **result** of a deformation), i.e., complete algebras $\tilde{\mathcal{L}}$ such that the graded algebras associated with $\hat{\mathcal{L}}$ and $\tilde{\mathcal{L}}$ are isomorphic (to L). Examples of filtered deforms will be given later.

On the road to solution of Problem B the following problem *seems* to be a natural step:

B′) Classify simple W-filtered complete vectorial Lie superalgebras as abstract ones, i.e., distinction in filtrations disregarded.

Observe that whereas the difference between Problems A′ and B′ is as large as that between Problems A and B, the difference between Problems B and B′ is negligible. Actually, it is impossible to solve Problem B′ and not solve Problem B. Indeed, various filtered deformations are not *a priori* isomorphic as abstract algebras, so, in order to solve Problem B′, we have to know first all the W-filtrations or the associated W-gradings and then describe filtered deformations for *every* W-grading.

However, even Problem B is not the most natural one: there exist deforms of algebras from class B which do not lie in class B: e.g., the result of the factorization of the quantized Poisson algebra modulo center. So the true problem one should solve is

C) Describe all the deformations (not only filtered ones) of the simple W-graded vectorial Lie superalgebras.

2.2. The linear Lie superalgebras

2.2.1. General linear Lie superalgebras.

2.2.1.1. The straightforward superization. The *general linear* Lie superalgebra of all supermatrices of size Par is denoted by $\mathfrak{gl}(\mathrm{Par})$, where $\mathrm{Par} = (p_1, \dots, p_{|\mathrm{Par}|})$ is the ordered collection of parities of the rows identical to that of the columns; usually, for the *standard* (simplest) format, $\mathfrak{gl}(\bar 0, \dots, \bar 0, \bar 1, \dots, \bar 1)$ is abbreviated to $\mathfrak{gl}(\dim V_{\bar 0}|\dim V_{\bar 1})$. Any nonzero supermatrix from $\mathfrak{gl}(\mathrm{Par})$ can be uniquely expressed as the sum of its even and odd parts; in the standard format this is the following block expression:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \quad p\left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right) = \bar 0,\ p\left(\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right) = \bar 1.$$

The *supertrace* is the map

$$\mathfrak{gl}(\mathrm{Par}) \longrightarrow \mathbb{C}, \quad (A_{ij}) \mapsto \sum (-1)^{p_i} A_{ii}, \quad \text{where } \mathrm{Par} = (p_1, \dots, p_{|\mathrm{Par}|}).$$

Since (this is a characteristic property of the (super)traces — to vanish on the derived algebra)

$$\mathrm{str}[x, y] = 0, \tag{2.5}$$

the subsuperspace of supertraceless matrices constitutes the *special linear* Lie subsuperalgebra $\mathfrak{sl}(\mathrm{Par})$.

2.2.1.2. The queer superization. There are, however, at least two super versions of $\mathfrak{gl}(n)$, not one. The other version — $\mathfrak{q}(n)$ — is called the *queer* Lie superalgebra and is defined as the one that preserves the complex structure given by an *odd* operator J, i.e., $\mathfrak{q}(n)$ is the centralizer $C(J)$ of J:

$$\mathfrak{q}(n) = C(J) = \{X \in \mathfrak{gl}(n|n) \mid [X, J] = 0\}, \text{ where } J^2 = -\mathrm{id}.$$

It is clear that by a change of basis we can reduce J to the form $J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ in the standard format and then $\mathfrak{q}(n)$ takes the form

$$\mathfrak{q}(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \;\middle|\; A, B \in \mathfrak{gl}(n) \right\}. \tag{2.6}$$

The nonstandard formats $\mathfrak{q}(\mathrm{Par})$ of $\mathfrak{q}(n)$ for $|\mathrm{Par}| = n$ are of the form (2.6) with $A, B \in \mathfrak{gl}(\mathrm{Par})$.

On $\mathfrak{q}(n)$, the *queertrace* is defined:

$$\mathrm{qtr}\colon \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mapsto \mathrm{tr}\, B. \tag{2.7}$$

Denote by $\mathfrak{sq}(n)$ the Lie superalgebra of *queertraceless* matrices.

Observe that the standard representations of $\mathfrak{q}(V)$ and $\mathfrak{sq}(V)$ in V, though irreducible in super sense (no invariant subsuperspaces), are not irreducible in the ungraded sense: take homogeneous (with respect to parity) and linearly independent vectors $v_1, \dots, v_n$ from V; then $\mathrm{Span}(v_1+J(v_1), \dots, v_n+J(v_n))$ is an invariant subspace of V which is not a subsuperspace. In particular, the irreducible representation of the least dimension of $\mathfrak{q}(\mathrm{Par})$ is of dimension $|\,\mathrm{Par}\,| = n$.

A representation is said to be *irreducible* of *general type* or just of *G-type* if there is no invariant subspaces, an *irreducible representation of Q-type* has no invariant sub*super*spaces but **has** an invariant subspace.

Both Lie superalgebras $\mathfrak{q}(n)$ and $\mathfrak{sq}(n)$ contain a center consisting of scalar matrices; factorizing them by the center we get algebras $\mathfrak{pq}(n)$ and $\mathfrak{psq}(n)$. The Lie superalgebra $\mathfrak{psq}(n)$ is simple.

2.2.2. Lie superalgebras that preserve bilinear forms: two types. To the linear map $F\colon V \longrightarrow W$ of superspaces there corresponds the dual map $F^*\colon W^* \longrightarrow V^*$ between the dual superspaces. In a basis consisting of the vectors v_i of format Par, the formula

$$F(v_j) = \sum_i v_i A_{ij}$$

assigns to F the supermatrix A. In the dual bases, the *supertransposed* supermatrix A^{st} corresponds to F^*:

$$(A^{st})_{ij} = (-1)^{(p_i+p_j)(p_i+p(A))} A_{ji}. \tag{2.8}$$

The supermatrices $X \in \mathfrak{gl}(\mathrm{Par})$ such that

$$X^{st}B + (-1)^{p(X)p(B)}BX = 0 \quad \text{for a fixed } B \in \mathfrak{gl}(\mathrm{Par}) \tag{2.9}$$

constitute the Lie superalgebra $\mathfrak{aut}(B)$ that preserves the bilinear form B^f on V whose matrix B is given by the formula

$$B_{ij} = (-1)^{p(B^f)p(v_i)} B^f(v_i, v_j)$$

for the basis vectors v_i.

The *supersymmetry* of the homogeneous bilinear form B^f means that its matrix $B = \begin{pmatrix} R & S \\ T & U \end{pmatrix}$ satisfies the condition

$$B^u = B, \quad \text{where } B^u = \begin{pmatrix} R^t & (-1)^{p(B)}T^t \\ (-1)^{p(B)}S^t & -U^t \end{pmatrix}.$$

Similarly, *anti-supersymmetry* of B means that $B^u = -B$. Thus, we see that the *upsetting* of bilinear forms $u\colon \mathrm{Bil}(V,W) \longrightarrow \mathrm{Bil}(W,V)$, which for the *spaces* and the case where $V = W$ is expressed on matrices in terms of the transposition, is a new operation.

Most popular canonical forms of the even non-degenerate supersymmetric form are the ones whose supermatrices in the standard format are the following canonical ones, B_{ev} or B'_{ev}:

$$B'_{ev}(m|2n) = \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad \text{where } J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

or

$$B_{ev}(2k|2n) = \begin{pmatrix} \Pi_{2k} & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad \text{where } \Pi_{2k} = \begin{pmatrix} 0 & 1_k \\ 1_k & 0 \end{pmatrix},$$

$$B_{ev}(2k+1|2n) = \begin{pmatrix} \Pi_{2k+1} & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad \text{where } \Pi_{2k+1} = \begin{pmatrix} 0 & 0 & 1_k \\ 0 & 1 & 0 \\ 1_k & 0 & 0 \end{pmatrix}.$$

The *ortho-symplectic Lie superalgebra* $\mathfrak{aut}(B_{ev}(m|2n))$ is usually denotated $\mathfrak{osp}(m|2n)$; sometimes we will write more precisely, $\mathfrak{osp}^{sy}(m|2n)$. Observe that the passage from V to $\Pi(V)$ sends the supersymmetric forms to superanti-symmetric ones, preserved by the "symplectico-orthogonal" Lie superalgebra, $\mathfrak{spo}(2n|m)$ or, more prudently, $\mathfrak{osp}^a(m|2n)$, which is isomorphic to $\mathfrak{osp}^{sy}(m|2n)$ but has a different matrix realization.

In the standard format the matrix realizations of these algebras are:

$$\mathfrak{osp}(m|2n) = \left\{ \begin{pmatrix} E & Y & X^t \\ X & A & B \\ -Y^t & C & -A^t \end{pmatrix} \right\}; \quad \mathfrak{osp}^a(m|2n) = \left\{ \begin{pmatrix} A & B & X \\ C & -A^t & Y^t \\ Y & -X^t & E \end{pmatrix} \right\},$$
$$\text{where } \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{sp}(2n), \quad E \in \mathfrak{o}(m).$$

A given non-degenerate supersymmetric odd bilinear form $B_{odd}(n|n)$ can be reduced to a canonical form whose matrix in the standard format is J_{2n}. A canonical form of the superanti-symmetric odd non-degenerate form in the standard format is Π_{2n}. The usual notation for $\mathfrak{aut}(B_{odd}(\mathrm{Par}))$ is $\mathfrak{pe}(\mathrm{Par})$.

The passage from V to $\Pi(V)$ gives an isomorphism $\mathfrak{pe}^{sy}(\mathrm{Par}) \cong \mathfrak{pe}^a(\mathrm{Par})$. These isomorphic Lie superalgebras are called, as A. Weil suggested, *periplectic*. The matrix realizations in the standard format of these superalgebras is:

$$\mathfrak{pe}^{sy}\ (n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where } B = -B^t,\ C = C^t \right\};$$
$$\mathfrak{pe}^a(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where } B = B^t,\ C = -C^t \right\}.$$

Observe that, despite the isomorphisms $\mathfrak{osp}^{sy}(m|2n) \simeq \mathfrak{osp}^a(m|2n)$ and $\mathfrak{pe}^{sy}(n) \simeq \mathfrak{pe}^a(n)$, the difference between the different incarnations is sometimes crucial, e.g., their Cartan prolongs are totally different.

The *special periplectic* superalgebra is

$$\mathfrak{spe}(n) = \{X \in \mathfrak{pe}(n) \mid \mathrm{str}\, X = 0\}.$$

Of particular interest to us will be also the Lie superalgebras

$$\mathfrak{spe}(n)_{a,b} = \mathfrak{spe}(n) \Subset \mathbb{C}(az+bd), \text{ where } z = 1_{2n},\ d = \mathrm{diag}(1_n, -1_n) \tag{2.10}$$

and the nontrivial central extension $\mathfrak{as}$ of $\mathfrak{spe}(4)$ that will be described after some preparation.

2.2.3. Projectivization. If $\mathfrak{s}$ is a Lie algebra of scalar matrices, and $\mathfrak{g} \subset \mathfrak{gl}(n|n)$ is a Lie subsuperalgebra containing $\mathfrak{s}$, then the *projective* Lie superalgebra of type $\mathfrak{g}$ is $\mathfrak{pg} = \mathfrak{g}/\mathfrak{s}$.

Projectivization sometimes leads to new Lie superalgebras, for example: $\mathfrak{pq}(n)$, $\mathfrak{psq}(n)$; $\mathfrak{pgl}(n|n)$, $\mathfrak{psl}(n|n)$; whereas $\mathfrak{pgl}(p|q) \cong \mathfrak{sl}(p|q)$ if $p \neq q$.

2.2.4. A. Sergeev's central extension. In 1970's, A. Sergeev proved that there is just one nontrivial central extension of $\mathfrak{spe}(n)$ for $n > 2$. It exists only for $n = 4$ and we denote it by $\mathfrak{as}$. Let us represent an arbitrary element $A \in \mathfrak{as}$ as a pair $A = x + d \cdot z$, where $x \in \mathfrak{spe}(4)$, $d \in \mathbb{C}$ and z is the central element. The bracket in $\mathfrak{as}$ is

$$\left[\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} + d \cdot z, \begin{pmatrix} a' & b' \\ c' & -a'^t \end{pmatrix} + d' \cdot z\right] = \left[\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & -a'^t \end{pmatrix}\right] + \operatorname{tr}\, c\tilde{c}' \cdot z, \tag{2.11}$$

where $\tilde{}$ is extended via linearity from matrices $c_{ij} = E_{ij} - E_{ji}$ on which $\tilde{c}_{ij} = c_{kl}$ for any even permutation $(1234) \mapsto (ijkl)$.

The Lie superalgebra $\mathfrak{as}$ can also be described with the help of the spinor representation. For this, we need several vectorial superalgebras. Consider $\mathfrak{po}(0|6)$, the Lie superalgebra whose superspace is the Grassmann superalgebra $\Lambda(\xi, \eta)$ generated by $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$ and the bracket is the Poisson bracket (2.36).

Recall that $\mathfrak{h}(0|6) = \operatorname{Span}(H_f \mid f \in \Lambda(\xi, \eta))$. Now, observe that $\mathfrak{spe}(4)$ can be embedded into $\mathfrak{h}(0|6)$. Indeed, setting $\deg \xi_i = \deg \eta_i = 1$ for all i we introduce a $\mathbb{Z}$-grading on $\Lambda(\xi, \eta)$ which, in turn, induces a $\mathbb{Z}$-grading on $\mathfrak{h}(0|6)$ of the form $\mathfrak{h}(0|6) = \underset{i \geq -1}{\oplus} \mathfrak{h}(0|6)_i$. Since $\mathfrak{sl}(4) \cong \mathfrak{o}(6)$, we can identify $\mathfrak{spe}(4)_0$ with $\mathfrak{h}(0|6)_0$.

It is not difficult to see that the elements of degree -1 in the standard gradings of $\mathfrak{spe}(4)$ and $\mathfrak{h}(0|6)$ constitute isomorphic $\mathfrak{sl}(4) \cong \mathfrak{o}(6)$-modules. It is subject to a direct verification that it is possible to embed $\mathfrak{spe}(4)_1$ into $\mathfrak{h}(0|6)_1$.

Sergeev's extension $\mathfrak{as}$ is the result of the restriction to $\mathfrak{spe}(4) \subset \mathfrak{h}(0|6)$ of the cocycle that turns $\mathfrak{h}(0|6)$ into $\mathfrak{po}(0|6)$. The quantization deforms $\mathfrak{po}(0|6)$ into $\mathfrak{gl}(\Lambda(\xi))$; the through maps

$$T_\lambda \colon \mathfrak{as} \longrightarrow \mathfrak{po}(0|6) \longrightarrow \mathfrak{gl}(\Lambda(\xi))$$

are representations of $\mathfrak{as}$ in the $4|4$-dimensional modules $\operatorname{spin}_\lambda$ isomorphic to each other for all $\lambda \neq 0$. The explicit form of T_λ is as follows:

$$T_\lambda \colon \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} + d \cdot z \mapsto \begin{pmatrix} a & b - \lambda\tilde{c} \\ c & -a^t \end{pmatrix} + \lambda d \cdot 1_{4|4}, \tag{2.12}$$

where $1_{4|4}$ is the unit matrix and $\tilde{c}$ is defined in eq. (2.11). Clearly, T_λ is an irreducible representation for any λ.

2.3. Vectorial Lie superalgebras

2.3.1. The standard realization. The elements of the Lie algebra $\mathcal{L} = \mathfrak{der}\, \mathbb{C}[[u]]$ are considered as vector fields. The Lie algebra $\mathcal{L}$ has only one maximal subalgebra $\mathcal{L}_0$ of finite codimension (consisting of the fields that vanish at the origin). The subalgebra $\mathcal{L}_0$ determines a filtration of $\mathcal{L}$: set

$$\mathcal{L}_{-1} = \mathcal{L} \text{ and } \mathcal{L}_i = \{D \in \mathcal{L}_{i-1} \mid [D, \mathcal{L}] \subset \mathcal{L}_{i-1}\} \text{ for } i \geq 1. \tag{2.13}$$

The associated graded Lie algebra $L = \underset{i \geq -1}{\oplus} L_i$, where $L_i = \mathcal{L}_i/\mathcal{L}_{i+1}$, consists of the vector fields with *polynomial* coefficients.

Superization. For a simple Lie superalgebra $\mathcal{L}$ (for example, take $\mathcal{L} = \mathfrak{der}\, \mathbb{C}[u, \xi]$), suppose $\mathcal{L}_0 \subset \mathcal{L}$ is a maximal subalgebra of finite codimension. Let $\mathcal{L}_{-1}$ be a minimal subspace of $\mathcal{L}$ containing $\mathcal{L}_0$, different from $\mathcal{L}_0$ and $\mathcal{L}_0$-invariant. A *Weisfeiler filtration* of $\mathcal{L}$ is determined by setting for $i \geq 1$:

$$\mathcal{L}_{-i-1} = [\mathcal{L}_{-1}, \mathcal{L}_{-i}] + \mathcal{L}_{-i} \text{ and } \mathcal{L}_i = \{D \in \mathcal{L}_{i-1} \mid [D, \mathcal{L}_{-1}] \subset \mathcal{L}_{i-1}\}. \tag{2.14}$$

Since the codimension of $\mathcal{L}_0$ is finite, the filtration takes the form

$$\mathcal{L} = \mathcal{L}_{-d} \supset \dots \supset \mathcal{L}_0 \supset \dots \tag{2.15}$$

for some d. This d is called the *depth* of $\mathcal{L}$ and of the associated graded Lie superalgebra L.

Considering the subspaces (2.13) as the basis of a topology, we can complete the graded or filtered Lie superalgebras L or $\mathcal{L}$; the elements of the completion are the vector fields with formal power series as coefficients. Though the structure of the graded algebras is easier to describe, in applications the completed Lie superalgebras are usually needed.

2.3.1.1. Remarks. 1) Not all filtered or graded Lie superalgebras of finite depth are *vectorial*, i.e., realizable with vector fields on a supermanifold of the same dimension as that of $\mathcal{L}/\mathcal{L}_0$; only those with faithful L_0-action on $L_- = \underset{i<0}{\oplus} L_i$ are.

2) Unlike Lie algebras, simple vectorial Lie *super*algebras possess *several* non-isomorphic maximal subalgebras of finite codimension, see sec. 1.3.

1) General algebras. Let $x = (u_1, \dots, u_n, \theta_1, \dots, \theta_m)$, where the u_i are even indeterminates and the θ_j are odd ones. Set $\mathfrak{vect}(n|m) = \mathfrak{der}\, \mathbb{C}[x]$; it is called *the general vectorial Lie superalgebra.*

On vectorial Lie superalgebras, there are **two** *analogs of trace.*

More precisely, there are *traces*, and there are their Cartan prolongations, called *divergencies*. On any Lie (super)algebra $\mathfrak{g}$ over a field $\mathbb{K}$, a *trace* is any map $\operatorname{tr} : \mathfrak{g} \longrightarrow \mathbb{K}$ such that

$$\operatorname{tr}([\mathfrak{g},\mathfrak{g}])=0. \tag{2.16}$$

The straightforward analogs of the trace are, therefore, the linear functionals that vanish on $\mathfrak{g}'$, the first derived of $\mathfrak{g}$; the number of linearly independent traces is equal to codim $\mathfrak{g}'$, these traces (or supertraces if $\mathfrak{g}$ is a Lie superalgebra) can be even or odd. Obviously, each trace is defined up to a non-zero scalar factor selected *ad lib*.

Let now $\mathfrak{g}$ be a $\mathbb{Z}$-graded vectorial Lie superalgebra with $\mathfrak{g}_- := \bigoplus\limits_{i<0}\mathfrak{g}_i$ generated by $\mathfrak{g}_{-1}$, and let tr be a (super)trace on $\mathfrak{g}_0$. The *divergence* $\operatorname{div}:\mathfrak{g}\longrightarrow\mathcal{F}$ is an $\operatorname{ad}_{\mathfrak{g}_{-1}}$-invariant prolongation of the trace satisfying the following conditions:

$$\begin{array}{l}
\operatorname{div}:\mathfrak{g}\longrightarrow\mathcal{F}\quad \text{preserves the degree, i.e., } \deg\operatorname{div}=0;\\
X_i(\operatorname{div}D)=\operatorname{div}[X_i,D]\quad \text{for all elements } X_i \text{ that span } \mathfrak{g}_{-1};\\
\operatorname{div}|_{\mathfrak{g}_0}=\operatorname{tr};\\
\operatorname{div}|_{\mathfrak{g}_-}=0.
\end{array}$$

By construction, the Lie (super)algebra $\mathfrak{sg}:=\operatorname{Ker}\operatorname{div}|_{\mathfrak{g}}$ of divergence-free elements of $\mathfrak{g}$ is the complete prolong of $(\mathfrak{g}_-,\operatorname{Ker}\operatorname{tr}|_{\mathfrak{g}_0})$. This fact explains why we say that div is the prolongation of the trace.

Strictly speaking, divergences are not traces (they do not satisfy (2.16)) but for vectorial Lie (super)algebras they embody the idea of the trace (understood as property (2.16)) better than the traces. We denote the *special* (*divergence free*) subalgebra of a vectorial algebra $\mathfrak{g}$ by $\mathfrak{sg}$, e.g., $\mathfrak{svect}(n|m)$. If there are several traces on $\mathfrak{g}_0$, there are several types of special subalgebras of $\mathfrak{g}$ and we need a different name for each.

2) Special algebras. The *divergences* (depending on a fixed volume element) belong to the other type of traces. Accordingly, the *special* (*divergence free*) subalgebra of a vectorial algebra $\mathfrak{g}$ is denoted by $\mathfrak{sg}$, e.g., $\mathfrak{svect}(n|m)$; the superscript $'$ denotes the derived algebra.

The *divergence* of the field $D=\sum\limits_i f_i\dfrac{\partial}{\partial u_i}+\sum\limits_j g_j\dfrac{\partial}{\partial\theta_j}$ corresponding to the volume element with constant coefficient is the function (in our case: a polynomial, or a series)

$$\operatorname{div}D=\sum_i\frac{\partial f_i}{\partial u_i}+\sum_j(-1)^{p(g_j)}\frac{\partial g_i}{\partial\theta_j}. \tag{2.17}$$

• The Lie superalgebra $\mathfrak{svect}(n|m)=\{D\in\mathfrak{vect}(n|m)\mid \operatorname{div}D=0\}$ is called the *special* (or *divergence-free*) *vectorial superalgebra*.

It is clear that it is also possible to describe $\mathfrak{svect}(n|m)$ as

$$\{D\in\mathfrak{vect}(n|m)\mid L_D\operatorname{vol}_x=0\},$$

where vol_x is the volume form with constant coefficients in coordinates x and L_D the Lie derivative with respect to D.

• The Lie superalgebra $\mathfrak{svect}'(1|m)=[\mathfrak{svect}(1|m),\mathfrak{svect}(1|m)]$ is said to be the *traceless special vectorial superalgebra*.

• The Lie superalgebra

$$\mathfrak{svect}_\lambda(0|m)=\{D\in\mathfrak{vect}(0|m)\mid \operatorname{div}(1+\lambda\theta_1\cdot\dots\cdot\theta_m)D=0\},$$

where $p(\lambda)\equiv m \pmod 2$, — the deform of $\mathfrak{svect}(0|m)$ — is called the *deformed special* (or *divergence-free*) *vectorial superalgebra*. Clearly,

$$\mathfrak{svect}_\lambda(0|m)\cong\mathfrak{svect}_\mu(0|m)\text{ for }\lambda\mu\neq0.$$

So we briefly denote these deforms by $\widetilde{\mathfrak{svect}}(0|m)$.

Observe that for m odd, the parameter of deformation, λ, is odd.

2.3.1.2. Remark. As is customary in differential geometry, we sometimes write $\mathfrak{vect}(x)$ or $\mathfrak{vect}(V)$ if $V=\operatorname{Span}(x)$ and use similar notations for the subalgebras of $\mathfrak{vect}$ introduced below. Some algebraists sometimes abbreviate $\mathfrak{vect}(n)$ and $\mathfrak{svect}(n)$ to $W(n)$ (in honor of Witt) and $S(n)$, respectively.

3) The algebras that preserve Pfaff equations and differential 1- and 2-forms.

• Set $u=(t,p_1,\dots,p_n,q_1,\dots,q_n)$; let

$$\tilde\alpha_1=dt+\sum_{1\le i\le n}(p_idq_i-q_idp_i)+\sum_{1\le j\le m}\theta_jd\theta_j\quad\text{and}\quad\tilde\omega_0=d\tilde\alpha_1\,. \tag{2.18}$$

The form $\tilde\alpha_1$ is called *contact*, the form $\tilde\omega_0$ is called *symplectic*. Sometimes it is more convenient to redenote the θ's and set $\Theta=(\xi,\eta)$, or $\Theta=(\xi,\eta,\theta)$, where

$$\begin{array}{l}
\xi_j=\dfrac{1}{\sqrt2}(\theta_j-i\theta_{k+j});\quad \eta_j=\dfrac{1}{\sqrt2}(\theta_j+i\theta_{k+j})\ \text{ for } j\le k=\left[\dfrac{m}{2}\right];\\
\text{here } i^2=-1,\ \theta=\theta_{2k+1}\quad\text{for } m=2k+1
\end{array} \tag{2.19}$$

and in place of $\tilde\omega_0$ or $\tilde\alpha_1$ take α_1 and $\omega_0=d\alpha_1$, respectively, where

$$\alpha_1=dt+\sum_{1\le i\le n}(p_idq_i-q_idp_i)+\sum_{1\le j\le k}(\xi_jd\eta_j+\eta_jd\xi_j)\begin{cases} & \text{if } m=2k\\ +\theta d\theta & \text{if } m=2k+1.\end{cases} \tag{2.20}$$

The Lie superalgebra that preserves the *Pfaff equation*

$$\alpha_1(X)=0\ \text{ for } X\in\mathfrak{vect}(2n+1|m),$$

i.e., the superalgebra

$$\mathfrak{k}(2n+1|m)=\{D\in\mathfrak{vect}(2n+1|m)\mid L_D\alpha_1=f_D\alpha_1\text{ for some } f_D\in\mathbb{C}[t,p,q,\theta]\}, \tag{2.21}$$

is called the *contact superalgebra*. The Lie superalgebra

$$\mathfrak{po}(2n|m) = \{D \in \mathfrak{k}(2n+1|m) \mid L_D\alpha_1 = 0\} \tag{2.22}$$

is called the *Poisson* superalgebra. (An interpretation of the Poisson superalgebra: it is the Lie superalgebra that preserves the connection with form α in the line bundle over a symplectic supermanifold with the symplectic form $d\alpha$.)

The above "symmetric" expression of α_1 is popular among algebraists; due to its symmetry it is convenient in computations. In mechanics and differential geometry, however, the following expression of the form α_1 (without odd coordinates, of course) is natural (the passage from one form to the other one can be performed by an invertible change of indeterminates):

$$\alpha_{1(2)} = dt - \sum_{1\leq i\leq n} p_i dq_i + \sum_{1\leq j\leq k} \xi_j d\eta_j \begin{cases} & \text{if } m = 2k \\ +\theta d\theta & \text{if } m = 2k+1. \end{cases} \tag{2.23}$$

The form $\alpha_{1(2)}$ is the only reasonable shape of the contact form over the fields of characteristic 2 whereas the symmetric expression of the contact forms are unnatural: $d\alpha_1 = 2\omega = 0$.

• Similarly, set $u = q = (q_1, \dots, q_n)$, let $\theta = (\xi_1, \dots, \xi_n; \tau)$ be odd. Set

$$\alpha_0 = d\tau + \sum_i (\xi_i dq_i + q_i d\xi_i), \qquad \omega_1 = d\alpha_0 \tag{2.24}$$

and call these forms, as A. Weil advised, the *pericontact* and *periplectic*, respectively. The periplectic form is odd.

In characteristic 2, we should take one of the following forms, where $0 \leq r \leq n$ (the passage from one form to the other one can be performed by an invertible change of indeterminates):

$$\alpha_{0(2,r)} = d\tau + \sum_{1\leq i\leq r} \xi_i dq_i + \sum_{r+1\leq i\leq n} q_i d\xi_i. \tag{2.25}$$

The Lie superalgebra that preserves the Pfaff equation

$$\alpha_0(X) = 0 \quad \text{for } X \in \mathfrak{vect}(n|n+1),$$

i.e., the superalgebra

$$\mathfrak{m}(n) = \{D \in \mathfrak{vect}(n|n+1) \mid L_D\alpha_0 = f_D \cdot \alpha_0 \text{ for some } f_D \in \mathbb{C}[q, \xi, \tau]\} \tag{2.26}$$

is called the *pericontact superalgebra.* [3)]

The Lie superalgebra

$$\mathfrak{b}(n) = \{D \in \mathfrak{m}(n) \mid L_D\alpha_0 = 0\} \tag{2.27}$$

[3] The unfortunate term $\mathfrak{m}$ "odd contact superalgebra" (still used sometimes) is misleading since the even part of $\mathfrak{m}$ is nonzero.

is called the *Buttin* superalgebra. (A geometric interpretation of the Buttin superalgebra: it is the Lie superalgebra that preserves the connection with form α_0 in the line bundle of superrank $\varepsilon = (0|1)$ over a *periplectic* supermanifold, i.e., over a supermanifold with the periplectic form $d\alpha_0$.)

The Lie superalgebras

$$\begin{aligned} \mathfrak{sm}(n) &= \{D \in \mathfrak{m}(n) \mid \operatorname{div} D = 0\}, \\ \mathfrak{sb}(n) &= \{D \in \mathfrak{b}(n) \mid \operatorname{div} D = 0\} \end{aligned} \tag{2.28}$$

are called the *divergence-free* (or *special*) *pericontact* and *special Buttin* superalgebras, respectively.

2.3.1.3. Remark. A relation with finite dimensional geometry is as follows. Clearly, $\ker \alpha_1 = \ker \tilde{\alpha}_1$. The restriction of ω_0 to $\ker \alpha_1$ is the ortho-symplectic form $B_{ev}(m|2n)$; the restriction of $\tilde{\omega}_0$ to $\ker \tilde{\alpha}_1$ is $B'_{ev}(m|2n)$. Similarly, the restriction of ω_1 to $\ker \alpha_0$ is $B_{odd}(n|n)$.

2.3.1.4. Generating functions. A laconic way to describe $\mathfrak{k}$, $\mathfrak{m}$ and their subalgebras is via generating functions. There are several standard realizations of the Lie algebra of contact vector fields, all are usually given in an "unnatural" basis of partial derivatives which suffices, however, for calculations. We will also give a representation of the contact fields in "natural" bases.

• Odd form α_1. For any $f \in \mathbb{C}[t, p, q, \theta]$, set :

$$K_f = (2 - E)(f)\frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t}E, \tag{2.29}$$

where $E = \sum\limits_i y_i \dfrac{\partial}{\partial y_i}$ (here the y_i are all the coordinates except t) is the *Euler operator*, and H_f is the hamiltonian vector field with Hamiltonian f that preserves $d\tilde{\alpha}_1$:

$$H_f = \sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right) - (-1)^{p(f)}\left(\sum_{j\leq m}\frac{\partial f}{\partial \theta_j}\frac{\partial}{\partial \theta_j}\right). \tag{2.30}$$

The choice of the form α_1 instead of $\tilde{\alpha}_1$ only affects the shape of H_f that we give for $m = 2k+1$:

$$H_f = \sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right) - (-1)^{p(f)}\sum_{j\leq k}\left(\frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j}\frac{\partial}{\partial \xi_j} + \frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}\right). \tag{2.31}$$

The expression of the contact field corresponding to the form α_1 or $\tilde{\alpha}_1$ is as follows:

$$K_f = (2 - E)(f)\frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t}E, \tag{2.32}$$

where $E = \sum\limits_i p_i \dfrac{\partial}{\partial p_i} + \sum\limits_j \xi_j \dfrac{\partial}{\partial \xi_j}$, and H_f is the hamiltonian vector field with Hamiltonian f that preserves $d\tilde{\alpha}_1$, see (2.30).

• Even form α_0. For any $f \in \mathbb{C}[q, \xi, \tau]$, define the (peri)*contact* vector field to be

$$M_f = (2 - E)(f)\frac{\partial}{\partial\tau} - Le_f - (-1)^{p(f)}\frac{\partial f}{\partial\tau}E, \tag{2.33}$$

where $E = \sum\limits_i y_i \dfrac{\partial}{\partial y_i}$ (here the y_i are all the coordinates except τ), and where the *periplectic* vector field is

$$\mathrm{Le}_f = \sum_{i\leq n}\left(\frac{\partial f}{\partial q_i}\ \frac{\partial}{\partial \xi_i} + (-1)^{p(f)}\frac{\partial f}{\partial \xi_i}\ \frac{\partial}{\partial q_i}\right). \tag{2.34}$$

Since

$$\begin{aligned} L_{K_f}(\alpha_1) &= 2\frac{\partial f}{\partial t}\alpha_1 = K_1(f)\alpha_1,\\ L_{M_f}(\alpha_0) &= -(-1)^{p(f)}2\frac{\partial f}{\partial \tau}\alpha_0 = -(-1)^{p(f)}M_1(f)\alpha_0, \end{aligned} \tag{2.35}$$

it follows that $K_f \in \mathfrak{k}(2n+1|m)$ and $M_f \in \mathfrak{m}(n)$. Observe that

$$p(\mathrm{Le}_f) = p(M_f) = p(f) + \bar{1}.$$

• To the (super)commutators $[K_f, K_g]$ or $[M_f, M_g]$ there correspond *contact brackets* of the generating functions:

$$\begin{aligned} [K_f, K_g] &= K_{\{f,\,g\}_{k.b.}};\\ [M_f, M_g] &= M_{\{f,\,g\}_{m.b.}}. \end{aligned}$$

The explicit expressions for the contact brackets are as follows. Let us first define the brackets on functions that do not depend on t (resp. τ).

The *Poisson bracket* $\{\cdot,\cdot\}_{P.b.}$ (in the realization with the form $\widetilde{\omega}_0$) is given by the equation

$$\{f, g\}_{P.b.} = \sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\ \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right) - (-1)^{p(f)}\sum_{j\leq m}\frac{\partial f}{\partial \theta_j}\ \frac{\partial g}{\partial \theta_j} \quad \text{for any } f, g \in \mathbb{C}[p, q, \theta] \tag{2.36}$$

and in the realization with the form ω_0 for $m = 2k+1$ it is given by the formula

$$\begin{aligned} \{f, g\}_{P.b.} = &\sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\ \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\ \frac{\partial g}{\partial p_i}\right) - \\ &(-1)^{p(f)}\left(\sum_{j\leq m}\left(\frac{\partial f}{\partial \xi_j}\ \frac{\partial g}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j}\ \frac{\partial g}{\partial \xi_j}\right) + \frac{\partial f}{\partial \theta}\ \frac{\partial g}{\partial \theta}\right) \text{ for } f, g \in \mathbb{C}[p, q, \xi, \eta, \theta]. \end{aligned} \tag{2.37}$$

The *Buttin bracket* $\{\cdot,\cdot\}_{B.b.}$ is given by the formula

$$\{f, g\}_{B.b.} = \sum_{i\leq n}\left(\frac{\partial f}{\partial q_i}\ \frac{\partial g}{\partial \xi_i} + (-1)^{p(f)}\frac{\partial f}{\partial \xi_i}\ \frac{\partial g}{\partial q_i}\right) \text{ for any } f, g \in \mathbb{C}[q, \xi]. \tag{2.38}$$

2.3.1.5. Remark. What Leites christened in [Le2] the "Buttin bracket" was discovered in pre-super era by Schouten (in the particular case of the supermanifold ringed by the sheaf of sections of the Grassmann algebra of the tangent bundle and with the Lie superalgebra structure on the same space of sections but with the opposite parity); Buttin was the first to prove that this bracket establishes a Lie superalgebra structure. The interpretations of the Buttin superalgebra similar to that of the Poisson algebra and of the elements of $\mathfrak{le}$ as analogs of Hamiltonian vector fields was given in [Le2]. The Buttin bracket and "odd mechanics" introduced in [Le2] was rediscovered by Batalin with Vilkovisky (and, even earlier, by Zinn-Justin, but his papers went mainly unnoticed); it gained a great deal of currency under the name *antibracket*. The *Schouten bracket* was originally defined on the superspace of polyvector fields on a manifold, i.e., on the superspace of sections of the exterior algebra (over the algebra $\mathcal{F}$ of functions) of the tangent bundle, $\Gamma(\Lambda^{\boldsymbol{\cdot}}(T(M))) \cong \Lambda^{\boldsymbol{\cdot}}_{\mathcal{F}}(Vect(M))$. The explicit expression of the Schouten bracket (in which the hatted slot should be ignored, as usual) is

$$\begin{aligned} &[X_1 \wedge \dots \wedge \dots \wedge X_k, Y_1 \wedge \dots \wedge Y_l] = \\ &\textstyle\sum_{i,j}(-1)^{i+j}[X_i, Y_j] \wedge X_1 \wedge \dots \wedge \hat{X}_i \wedge \dots \wedge X_k \wedge Y_1 \wedge \dots \wedge \hat{Y}_j \wedge \dots \wedge Y_l. \end{aligned} \tag{2.39}$$

With the help of the Sign Rule (sect. 1.1.2) we easily superize eq. (2.39), i.e., replace M by a supermanifold $\mathcal{M}$. Let x and ξ be the even and odd coordinates on $\mathcal{M}$. By setting

$$\theta_i = \Pi\left(\frac{\partial}{\partial x_i}\right) = \check{x}_i, \quad q_j = \Pi\left(\frac{\partial}{\partial \xi_j}\right) = \check{\xi}_j \tag{2.40}$$

we get an identification of the Schouten bracket of polyvector fields on $\mathcal{M}$ with the Buttin bracket of functions on the supermanifold $\check{\mathcal{M}}$ with coordinates x, ξ and $\check{x}$, $\check{\xi}$, and the transformation rule of the checked variables induced by that of unchecked ones via (2.40).

In terms of the Poisson and Buttin brackets, respectively, the contact brackets are

$$\{f, g\}_{k.b.} = (2 - E)(f)\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}(2 - E)(g) - \{f, g\}_{P.b.} \tag{2.41}$$

and

$$\{f, g\}_{m.b.} = (2 - E)(f)\frac{\partial g}{\partial \tau} + (-1)^{p(f)}\frac{\partial f}{\partial \tau}(2 - E)(g) - \{f, g\}_{B.b.}. \tag{2.42}$$

The Lie superalgebras of *Hamiltonian vector fields* (or *Hamiltonian superalgebra*) and its special subalgebra (defined only if $n = 0$) are

$$\begin{aligned} \mathfrak{h}(2n|m) &= \{D \in \mathfrak{vect}(2n|m) \mid L_D\omega_0 = 0\},\\ \mathfrak{h}'(m) &= \{H_f \in \mathfrak{h}(0|m) \mid \textstyle\int f \operatorname{vol}_\theta = 0\}. \end{aligned} \tag{2.43}$$

The "odd" analogues of the Lie superalgebra of Hamiltonian fields are the Lie superalgebra of vector fields Le_f introduced in [Le2] and its special subalgebra:

$$\begin{aligned} \mathfrak{le}(n) &= \{D \in \mathfrak{vect}(n|n) \mid L_D\omega_1 = 0\},\\ \mathfrak{sle}(n) &= \{D \in \mathfrak{le}(n) \mid \operatorname{div} D = 0\}. \end{aligned} \tag{2.44}$$

It is not difficult to prove the following isomorphisms (as superspaces):

$$\begin{aligned} \mathfrak{k}(2n+1|m) &\cong \operatorname{Span}(K_f \mid f \in \mathbb{C}[t,p,q,\xi]);\\ \mathfrak{le}(n) &\cong \operatorname{Span}(Le_f \mid f \in \mathbb{C}[q,\xi]);\\ \mathfrak{m}(n) &\cong \operatorname{Span}(M_f \mid f \in \mathbb{C}[\tau,q,\xi]);\\ \mathfrak{h}(2n|m) &\cong \operatorname{Span}(H_f \mid f \in \mathbb{C}[p,q,\xi]). \end{aligned} \tag{2.45}$$

We have

$$\begin{aligned} \mathfrak{po}'(m) &= \{K_f \in \mathfrak{po}(0|m) \mid \textstyle\int f vol_\xi = 0\};\\ \mathfrak{h}'(m) &= \mathfrak{po}'(m)/\mathbb{C}\cdot K_1. \end{aligned}$$

2.3.1.6. Divergence-free subalgebras. Since, as is easy to calculate,

$$\operatorname{div} K_f = (2n+2-m)K_1(f), \tag{2.46}$$

it follows that the divergence-free subalgebra of the contact Lie superalgebra either coincides with it (for $m = 2n+2$) or is the Poisson superalgebra. For the pericontact series, the situation is more interesting: the divergence free subalgebra is simple and new (as compared with the above list).

Since

$$\operatorname{div} M_f = (-1)^{p(f)}2\left((1-E)\frac{\partial f}{\partial \tau} - \sum_{i\le n}\frac{\partial^2 f}{\partial q_i \partial \xi_i}\right), \tag{2.47}$$

it follows that the divergence-free subalgebra of the pericontact superalgebra is

$$\mathfrak{sm}(n) = \operatorname{Span}\left(M_f \in \mathfrak{m}(n) \mid (1-E)\frac{\partial f}{\partial \tau} = \sum_{i\le n}\frac{\partial^2 f}{\partial q_i \partial \xi_i}\right). \tag{2.48}$$

In particular,

$$\operatorname{div} Le_f = (-1)^{p(f)}2\sum_{i\le n}\frac{\partial^2 f}{\partial q_i \partial \xi_i}. \tag{2.49}$$

The odd analog of the Laplacian, namely, the operator

$$\Delta = \sum_{i\le n}\frac{\partial^2}{\partial q_i \partial \xi_i} \tag{2.50}$$

on a periplectic supermanifold appeared in physics under the name of *BRST operator*, cf. [GPS], or Batalin-Vilkovysky operator. Observe that Δ is just the Fourier transform (with respect to the "ghost indeterminates" $\check{x}$ (the odd ones, if considered on manifolds) of the exterior differential d.

The divergence-free vector fields from $\mathfrak{sle}(n)$ are generated by *harmonic* functions, i.e., such that $\Delta(f) = 0$.

Lie superalgebras $\mathfrak{sle}(n)$, $\mathfrak{sb}(n)$ and $\mathfrak{svect}(1|n)$ have traceless ideals $\mathfrak{sle}'(n)$, $\mathfrak{sb}'(n)$ and $\mathfrak{svect}'(n)$ of codimension 1 defined from the exact sequences

$$\begin{gathered} 0 \longrightarrow \mathfrak{sle}'(n) \longrightarrow \mathfrak{sle}(n) \longrightarrow \mathbb{C}\cdot Le_{\xi_1\dots\xi_n} \longrightarrow 0,\\ 0 \longrightarrow \mathfrak{sb}'(n) \longrightarrow \mathfrak{sb}(n) \longrightarrow \mathbb{C}\cdot M_{\xi_1\dots\xi_n} \longrightarrow 0,\\ 0 \longrightarrow \mathfrak{svect}'(n) \longrightarrow \mathfrak{svect}(1|n) \longrightarrow \mathbb{C}\cdot \xi_1\dots\xi_n\frac{\partial}{\partial t} \longrightarrow 0. \end{gathered} \tag{2.51}$$

2.3.2. The Cartan prolongs. We will repeatedly use the Cartan prolongation. So let us first recall its definition and then that of its generalization.

Let $\mathfrak{g}$ be a Lie algebra, V a $\mathfrak{g}$-module, S^i the operator of the ith symmetric power. Set $\mathfrak{g}_{-1} = V$ and $\mathfrak{g}_0 = \mathfrak{g}$.

Recall that, for any (finite dimensional) vector space V, we have

$$\operatorname{Hom}(V, \operatorname{Hom}(V, \dots, \operatorname{Hom}(V,V)\dots)) \simeq L^i(V,V,\dots,V;V),$$

where L^i is the space of i-linear maps and we have $(i+1)$-many V's on both sides. Now, we recursively define, for any $i > 0$:

$$\begin{aligned} &\mathfrak{g}_i = \{X \in \operatorname{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{i-1}) \mid X(v_1)(v_2,v_3,\dots,v_{i+1}) = X(v_2)(v_1,v_3,\dots,v_{i+1})\\ &\quad \text{where } v_1,\dots,v_{i+1} \in \mathfrak{g}_{-1}\}. \end{aligned} \tag{2.52}$$

The space $\mathfrak{g}_i$ is said to be the ith *Cartan prolong* (the result of the Cartan *prolongation*) of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$.

Equivalently, let

$$i\colon S^{k+1}(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1} \longrightarrow S^k(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1} \tag{2.53}$$

and

$$j\colon S^k(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_0 \longrightarrow S^k(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1} \tag{2.54}$$

be the natural maps. Then $\mathfrak{g}_k = i(S^{k+1}(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1}) \cap j(S^k(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_0)$.

The *Cartan prolong* of the pair $(V, \mathfrak{g})$ is $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathop{\oplus}\limits_{k\ge -1} \mathfrak{g}_k$.

(In what follows $\cdot$ in superscript denotes, as is now customary, the collection of all degrees, while $*$ is reserved for dualization; in the subscripts we retain the old-fashioned $*$ instead of $\cdot$ to avoid too close a contact with the punctuation marks.)

Suppose that the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is *faithful*. Then, clearly,

$$\begin{aligned} &(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \subset \mathfrak{vect}(n) = \mathfrak{der}\, \mathbb{C}[x_1,\dots,x_n], \quad \text{where } n = \dim\, \mathfrak{g}_{-1} \text{ and}\\ &\quad \mathfrak{g}_i = \{D \in \mathfrak{vect}(n) \mid \deg D = i, [D,X] \in \mathfrak{g}_{i-1} \text{ for any } X \in \mathfrak{g}_{-1}\}. \end{aligned} \tag{2.55}$$

It is subject to an easy verification that the Lie algebra structure on $\mathfrak{vect}(n)$ induces a Lie algebra structure on $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$; actually, $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ possesses a Lie algebra structure even if the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is not faithful.

Of the four simple vectorial Lie algebras, three are Cartan prolongs:

$$\begin{aligned} \mathfrak{vect}(n) &= (\mathrm{id}, \mathfrak{gl}(n))_*,\\ \mathfrak{svect}(n) &= (\mathrm{id}, \mathfrak{sl}(n))_*, \\ \mathfrak{h}(2n) &= (\mathrm{id}, \mathfrak{sp}(n))_*. \end{aligned} \tag{2.56}$$

The fourth one — $\mathfrak{k}(2n+1)$ — is the result of a trifle more general construction described as follows.

2.3.2.1. A generalization of the Cartan prolong: The Tanaka–Shchepochkina prolong. Let $\mathfrak{g}_- = \bigoplus\limits_{-d\leq i\leq -1} \mathfrak{g}_i$ be a nilpotent $\mathbb{Z}$-graded Lie algebra and $\mathfrak{g}_0 \subset \mathfrak{der}_0\mathfrak{g}$ a Lie subalgebra of the $\mathbb{Z}$-grading-preserving derivations. Let

$$i\colon S^{k+1}(\mathfrak{g}_-)^*\otimes \mathfrak{g}_- \longrightarrow S^k(\mathfrak{g}_-)^*\otimes \mathfrak{g}_-^*\otimes \mathfrak{g}_- \tag{2.57}$$

and

$$j\colon S^{k}(\mathfrak{g}_-)^*\otimes \mathfrak{g}_0 \longrightarrow S^k(\mathfrak{g}_-)^*\otimes \mathfrak{g}_-^*\otimes \mathfrak{g}_- \tag{2.58}$$

be the natural maps similar to (2.53) and (2.54), respectively. For $k>0$, define the kth prolong of the pair $(\mathfrak{g}_-, \mathfrak{g}_0)$ to be:

$$\mathfrak{g}_k = (j(S^{\bullet}(\mathfrak{g}_-)^*\otimes \mathfrak{g}_0)\cap i(S^{\bullet}(\mathfrak{g}_-)^*\otimes \mathfrak{g}_-))_k\,, \tag{2.59}$$

where the subscript k in the right hand side singles out the component of degree k.

Set $(\mathfrak{g}_-, \mathfrak{g}_0)_* = \bigoplus\limits_{i\geq -d} \mathfrak{g}_i$; then, as is easy to verify, $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ is a Lie algebra.

What is the Lie algebra of contact vector fields in these terms? Denote by $\mathfrak{hei}(2n)$ the Heisenberg Lie algebra: its space is $W\oplus\mathbb{C}\cdot z$, where W is a $2n$-dimensional space endowed with a non-degenerate anti-symmetric bilinear form B and the bracket in $\mathfrak{hei}(2n)$ is given by the following relations:

$$z \text{ is in the center and } [v,w] = B(v,w)\cdot z \text{ for any } v,w\in W. \tag{2.60}$$

Clearly, $\mathfrak{k}(2n+1)\cong(\mathfrak{hei}(2n), \mathfrak{csp}(2n))_*$.

2.3.2.2. Lie superalgebras of vector fields as Cartan prolongs. The superization of the constructions of Cartan prolongations are straightforward: via Sign Rule (1.1.2). We thus get infinite dimensional Lie superalgebras

$$\begin{aligned} \mathfrak{vect}(m|n) &= (\mathrm{id}, \mathfrak{gl}(m|n))_*;\\ \mathfrak{svect}(m|n) &= (\mathrm{id}, \mathfrak{sl}(m|n))_*;\\ \mathfrak{h}(2m|n) &= (\mathrm{id}, \mathfrak{osp}^a(m|2n))_*;\\ \mathfrak{le}(n) &= (\mathrm{id}, \mathfrak{pe}^a(n))_*;\\ \mathfrak{sle}(n) &= (\mathrm{id}, \mathfrak{spe}^a(n))_*. \end{aligned} \tag{2.61}$$

2.3.2.3. Remark. Observe that the Cartan prolongs $(\mathrm{id}, \mathfrak{osp}^{sy}(m|2n))_*$ and $(\mathrm{id}, \mathfrak{pe}^{sy}(n))_*$ are finite dimensional.

Observe that there are **two** superizations of the contact series: $\mathfrak{k}$ and $\mathfrak{m}$.

• Define the Lie superalgebra $\mathfrak{hei}(2n|m)$ on the direct sum of a $(2n,m)$-dimensional superspace W endowed with a non-degenerate anti-symmetric bilinear form B and a $(1,0)$-dimensional space spanned by z by (2.60).

Clearly, we have

$$\mathfrak{k}(2n+1|m) = (\mathfrak{hei}(2n|m), \mathfrak{cosp}^a(m|2n))_* \tag{2.62}$$

and, given $\mathfrak{hei}(2n|m)$ and a subalgebra $\mathfrak{g}$ of $\mathfrak{cosp}^a(m|2n)$, we call $(\mathfrak{hei}(2n|m), \mathfrak{g})_*$ the *k-prolong* of $(W,\mathfrak{g})$, where W is the standard $\mathfrak{osp}^a(m|2n)$-module.

2.3.2.3a. The Fock space. Let $\mathfrak{hei}(2m|2n) = \mathrm{Span}(p,q,\xi,\eta,z)$, with the odd elements $\xi=(\xi_1,\dots,\xi_n)$ and $\eta=(\eta_1,\dots,\eta_n)$, and the even elements $p=(p_1,\dots,p_m)$, $q=(q_1,\dots,q_m)$ and z, and where the brackets are

$$[p_i,q_j]=\delta_{ij}z,\quad [\xi_i,\eta_j]=\delta_{ij}z,\quad [z,\mathfrak{hei}(2m|2n)]=0. \tag{2.63}$$

In what follows we will need the Lie superalgebra $\mathfrak{hei}(2m|2n)$ (for the cases where $mn=0$) and its only (up to the change of parity) non-trivial irreducible representation, called the *Fock space*, which is $\mathbb{K}[q,\xi]$ on which the elements q_i and ξ_j act as operators of left multiplication by q_i and ξ_j, respectively, whereas p_i and η_j act as $h\partial_{q_i}$ and $h\partial_{\xi_j}$, where $h\in\mathbb{K}\setminus\{0\}$ can be fixed to be equal to 1 by a change of the basis.

• The "odd" analog of $\mathfrak{k}$ is associated with the following "odd" analog of $\mathfrak{hei}(2n|m)$. Denote by $\mathfrak{ab}(n)$ the *antibracket* Lie superalgebra: its space is $W\oplus\mathbb{C}\cdot z$, where W is an $n|n$-dimensional superspace endowed with a non-degenerate anti-symmetric odd bilinear form B; the bracket in $\mathfrak{ab}(n)$ is given by the following relations:

$$z \text{ is odd and lies in the center; } [v,w]=B(v,w)\cdot z \text{ for any } v,w\in W. \tag{2.64}$$

Clearly,

$$\mathfrak{m}(n) = (\mathfrak{ab}(n), \mathfrak{cpe}^a(n))_* \tag{2.65}$$

and, given $\mathfrak{ab}(n)$ and a subalgebra $\mathfrak{g}$ of $\mathfrak{cpe}^a(n)$, we call $(\mathfrak{ab}(n),\mathfrak{g})_*$ the *m-prolong* of $(W,\mathfrak{g})$, where W is the standard $\mathfrak{pe}^a(n)$-module.

Generally, given a non-degenerate form B on a superspace W and a Lie subsuperalgebra $\mathfrak{g}\subset\mathfrak{aut}(B)$, we refer to the above generalized prolongations as to *mk-prolongation* of the pair $(W,\mathfrak{g})$.

2.3.2.4. Partial Cartan prolongs: Prolongations of a positive part. Let $\mathfrak{h}_1\in\mathfrak{g}_1$ be a $\mathfrak{g}_0$-submodule such that $[\mathfrak{g}_{-1},\mathfrak{h}_1]=\mathfrak{g}_0$. If such $\mathfrak{h}_1$ exists (usually, $[\mathfrak{g}_{-1},\mathfrak{h}_1]\subset\mathfrak{g}_0$), define the 2nd prolongation of $(\bigoplus\limits_{i\leq 0}\mathfrak{g}_i, \mathfrak{h}_1)$ to be

$$\mathfrak{h}_2 = \{D\in\mathfrak{g}_2 \mid [D,\mathfrak{g}_{-1}]\in\mathfrak{h}_1\}. \tag{2.66}$$

The terms $\mathfrak{h}_i$, where $i>2$, are similarly defined. Set $\mathfrak{h}_i=\mathfrak{g}_i$ for $i<0$ and $\mathfrak{h}_*=\oplus\mathfrak{h}_i$ and call this Lie superalgebra the *partial Cartan prolong*.

2.3.2.5. Examples. The Lie superalgebra $\mathfrak{vect}(1|n;n)$ is a subalgebra of $\mathfrak{k}(1|2n;n)$. The former is obtained as the Cartan prolong of the same non-positive part as $\mathfrak{k}(1|2n;n)$ and a submodule of $\mathfrak{k}(1|2n;n)_1$. The simple exceptional superalgebra $\mathfrak{kas}$ is another example.

2.3.2.6. Remark. In non-super setting, the generalized Cartan prolongation was first introduced by Tanaka [T], see a clarifying review [Y]. Different from ours emphasis of these papers delayed our recognition of similarity of constructions. The partial prolong was first discovered in [ALSh].

2.3.2.7. The exceptional Lie superalgebras as Cartan prolongs. The five families of exceptional Lie superalgebras are given below in their minimal realizations as Cartan prolongs $(\mathfrak{g}_{-1},\mathfrak{g}_0)_*$ or generalized (see sec. 2.5) Cartan prolongs $(\mathfrak{g}_-,\mathfrak{g}_0)_*^{mk}$.

For depth ≤ 2, for $\mathfrak{g}_- = \bigoplus\limits_{-2\leq i\leq -1}\mathfrak{g}_i$, we write $(\mathfrak{g}_{-2},\mathfrak{g}_{-1},\mathfrak{g}_0)_*^{mk}$ instead of $(\mathfrak{g}_-,\mathfrak{g}_0)_*^{mk}$. In the table below (2.67), indicated is also one of the Lie superalgebras from the list of series (2.93) – (2.95) as an ambient which contains the exceptional one as a maximal subalgebra. The W-graded superalgebras of depth 3 appear as regradings of the listed ones at certain values of r; the corresponding terms $\mathfrak{g}_i$ for $i\leq 0$ will be given below.

$\mathfrak{vle}(4\|3;r)=(\Pi(\Lambda(3))/\mathbb{C}\cdot 1,\mathfrak{cvect}(0\|3))_*\subset\mathfrak{vect}(4\|3;R)$	$r=0,1,K$
$\mathfrak{vas}(4\|4)=(\mathrm{spin},\mathfrak{as})_*\subset\mathfrak{vect}(4\|4)$	
$\mathfrak{kas}^\xi(1\|6;r)\subset\mathfrak{k}(1\|6;r)$ $\mathfrak{kas}^\xi(1\|6;3\eta)=(\mathrm{Vol}_0(0\|3),\mathfrak{c}(\mathfrak{vect}(0\|3)))_*\subset\mathfrak{svect}(4\|3)$	$r=0,1\xi,3\xi$
$\mathfrak{mb}(4\|5;r)=(\mathfrak{ab}(4),\mathfrak{cvect}(0\|3))^m_*\subset\mathfrak{m}(4\|5;R)$	$r=0,1,K$
$\mathfrak{ksle}(9\|6;r)=(\mathfrak{hei}(8\|6),\mathfrak{svect}(0\|4)_{3,4})^k_*\subset\mathfrak{k}(9\|6;r)$ $\mathfrak{ksle}(9\|6;K)=(\mathrm{id}_{\mathfrak{sl}(5)},\Lambda^2(\mathrm{id}^*_{\mathfrak{sl}(5)}),\mathfrak{sl}(5))^k_*\subset\mathfrak{svect}(5\|10;R)$	$r=0,2,\mathrm{CK}$

(2.67)

In (2.67), most of the regradings R of the ambients are the same as that of the embedded algebra, i.e., $R(r)=r$. Certain regradings $R(r)$ of the ambients are so highly nonstandard that even the homogeneous fibers are of infinite dimension.

For $\mathfrak{kas}$, the notations $r=0$, 1ξ, 3ξ are clear: none, or one or three of the ξ's have degree 0 (and the corresponding η's acquire degree 2). The following table describes these regradings: (the degrees of the even indeterminates | the degrees of the odd indeterminates; after semicolon stands the degree of t (resp. τ)):

$\mathfrak{vle}(4\|3)$	$R(K)=(2220\|111)$, $R(0)=(1111\|111)$, $R(1)=(2110\|011)$
$\mathfrak{mb}(4\|5)$	$R(K)=(0222\|3111;3)$, $R(0)=(1111\|1111;2)$, $R(1)=(0211\|2011;2)$
$\mathfrak{ksle}(9\|6)$	$R(K)=(22222\|11111111)$

(2.68)

2.3.3. The modules of tensor fields. To advance further, we have to recall the definition of the modules of tensor fields over $\mathfrak{vect}(m|n)$ and its subalgebras, see [BL2], [Le3].

Let $\mathfrak{g}=\mathfrak{vect}(m|n)$ and $\mathfrak{g}_\geq = \bigoplus\limits_{i\geq 0}\mathfrak{g}_i$. For any other $\mathbb{Z}$-graded vectorial Lie superalgebra, the construction is identical.

Clearly, $\mathfrak{vect}_0(m|n)\cong\mathfrak{gl}(m|n)$. Let V be the $\mathfrak{gl}(m|n)$-module with the *lowest* weight $\lambda=\mathrm{lwt}(V)$. Make V into a $\mathfrak{g}_\geq$-module setting $\mathfrak{g}_+\cdot V=0$ for $\mathfrak{g}_+=\bigoplus\limits_{i>0}\mathfrak{g}_i$. Let us realize $\mathfrak{g}$ by vector fields on the $m|n$-dimensional linear supermanifold $\mathcal{C}^{m|n}$ with coordinates $x=(u,\xi)$. The superspace $T(V)=\mathrm{Hom}_{U(\mathfrak{g}_\geq)}(U(\mathfrak{g}),V)$ is isomorphic, due to the Poincaré–Birkhoff–Witt theorem, to $\mathbb{C}[[x]]\otimes V$. Its elements have a natural interpretation as formal *tensor fields of type* V. When $\lambda=(a,\dots,a)$ we will simply write $T(\mathbf{a})$ instead of $T(\lambda)$. We will usually consider $\mathfrak{g}$-modules induced from irreducible $\mathfrak{g}_0$-modules.

2.3.4. Examples. As $\mathfrak{vect}(m|n)$- and $\mathfrak{svect}(m|n)$-module, $\mathfrak{vect}(m|n)$ is $T(\mathrm{id})$. More examples:

$T(\mathbf{0})$ is the superspace of functions;

$\mathrm{Vol}(m|n)=T(1,\dots,1;-1,\dots,-1)$ (the semicolon separates the first m ("even") coordinates of the weight with respect to the matrix units E_{ii} of $\mathfrak{gl}(m|n)$) is the superspace of *densities* or *volume forms*.

We denote the generator of $\mathrm{Vol}(m|n)$ corresponding to the ordered set of coordinates x by $\mathrm{vol}(x)$. The space of λ-densities — called *weighted densities of weight* λ — is denoted by $\mathrm{Vol}^\lambda(m|n)=T(\lambda,\dots,\lambda;-\lambda,\dots,-\lambda)$. In particular, $\mathrm{Vol}^\lambda(m|0)=T(\boldsymbol{\lambda})$ but $\mathrm{Vol}^\lambda(0|n)=T(\overrightarrow{-\lambda})$. We set:

$$\mathrm{Vol}_0(0|m)=\{v\in\mathrm{Vol}\mid\int v=0\}\text{ and }T_0(\mathbf{0})=\Lambda(m)/\mathbb{C}\cdot 1. \tag{2.69}$$

If the generator vol of Vol is fixed, then $\mathrm{Vol}\cong T(\mathbf{0})$, as $\mathfrak{svect}(m|n)$-modules. Denote the $\mathfrak{svect}(0|m)$-module $\mathrm{Vol}_0(0|m)/\mathbb{C}\ \mathrm{vol}(\xi)$ by $T_0^0(\mathbf{0})$.

2.3.4.1. Remark. To view the volume element as "$d^mud^n\xi$" is totally wrong: the superdeterminant can never appear as a factor under the changes of variables. We can try to use the usual notations of differentials provided *all* the differentials anticommute. Then at least the linear transformations that do not intermix the even u's with the odd ξ's multiply the volume element $\mathrm{vol}(x)$, viewed as the fraction $\dfrac{du_1\cdot\ldots\cdot du_m}{d\xi_1\cdot\ldots\cdot d\xi_n}$, by the correct factor, the Berezinian of the transformation. But how to justify this? Let $X=(x,\xi)$. If we consider the usual, exterior, differential forms, then the dX_i's super anti-commute, hence the $d\xi_i$ commute; whereas if we consider the *symmetric* product of the differentials, as in the metrics, then the dX_i's supercommute, hence the dx_i commute. So, neither exterior nor symmetric product is what we need: All factors should anti-commute.

However, from transformations' point of view, $\dfrac{\partial}{\partial\xi_i}=\dfrac{1}{d\xi_i}$, and the $\dfrac{\partial}{\partial\xi_i}$ anticommute. The notation, $du^m\left(\dfrac{\partial}{\partial\xi}\right)^n := du_1\cdot\ldots\cdot du_m\cdot\dfrac{\partial}{\partial\xi_1}\cdot\ldots\cdot\dfrac{\partial}{\partial\xi_n}$, is, nevertheless, still wrong: the generic transformation $A\colon (u,\xi)\mapsto(v,\eta)$ sends $du_1\cdot\ldots\cdot du_m\cdot\frac{\partial}{\partial\xi_1}\cdot\ldots\cdot\frac{\partial}{\partial\xi_n}$ to the correct element, $\operatorname{ber}(A)(du^m\cdot\frac{\partial}{\partial\xi_1}\cdot\ldots\cdot\frac{\partial}{\partial\xi_n})$, **plus extra terms**. Indeed, the fraction $du_1\cdot\ldots\cdot du_m\cdot\frac{\partial}{\partial\xi_1}\cdot\ldots\cdot\frac{\partial}{\partial\xi_n}$ is the highest weight vector of an *indecomposable* $\mathfrak{gl}(m|n)$-module and $\mathrm{vol}(x)$ is the notation of the image of this vector in the 1-dimensional quotient module modulo the invariant submodule that consists precisely of all the extra terms.

2.3.5. Deformations of the Buttin superalgebra. As is clear from the definition of the Buttin bracket, there is a regrading (namely, $\mathfrak{b}(n;n)$ given by $\deg\xi_i=0, \deg q_i=1$ for all i) under which $\mathfrak{b}(n)$, initially of depth 2, takes the form $\mathfrak{g}=\underset{i\geq -1}{\oplus}\mathfrak{g}_i$ with $\mathfrak{g}_0=\mathfrak{vect}(0|n)$ and $\mathfrak{g}_{-1}\cong\Pi(\mathbb{C}[\xi])$. Replace now the $\mathfrak{vect}(0|n)$-module $\mathfrak{g}_{-1}$ of functions (with inverted parity) by the module of λ-densities, i.e., set $\mathfrak{g}_{-1}\cong\Pi(\mathrm{Vol}(0|n)^\lambda)$, where

$$L_D(vol_\xi)^\lambda=\lambda\operatorname{div}D\cdot vol_\xi^\lambda \quad\text{and}\quad p(vol_\xi)^\lambda=\bar 1. \tag{2.70}$$

Define $\mathfrak{b}_\lambda(n;n)$ to be the Cartan prolong

$$(\mathfrak{g}_{-1},\mathfrak{g}_0)_*=(\Pi(\mathrm{Vol}(0|n)^\lambda),\mathfrak{vect}(0|n))_*. \tag{2.71}$$

Clearly, this is a deform of $\mathfrak{b}(n;n)$. The collection of these $\mathfrak{b}_\lambda(n;n)$ for all λ's is called the *main deformation*, the other deformations, defined in what follows, will be called *singular*.

The deform $\mathfrak{b}_\lambda(n)$ of $\mathfrak{b}(n)$ is a regrading of $\mathfrak{b}_\lambda(n;n)$ described as follows. For $\lambda=\frac{2a}{n(a-b)}$, set

$$\mathfrak{b}_{a,b}(n)=\{M_f\in\mathfrak{m}(n)\mid a\ \operatorname{div}M_f=(-1)^{p(f)}2(a-bn)\frac{\partial f}{\partial\tau}\}. \tag{2.72}$$

For future use, we will denote the operator that singles out $\mathfrak{b}_\lambda(n)$ in $\mathfrak{m}(n)$ as follows:

$$\operatorname{div}_\lambda=(bn-aE)\frac{\partial}{\partial\tau}-a\Delta,\quad\text{for }\lambda=\frac{2a}{n(a-b)}\text{ and }\Delta=\sum_{i\leq n}\frac{\partial^2}{\partial q_i\partial\xi_i}. \tag{2.73}$$

Taking into account the explicit form of the divergence of M_f we get

$$\begin{aligned}\mathfrak{b}_{a,b}(n)=\{M_f\in\mathfrak{m}(n)\mid(bn-aE)\frac{\partial f}{\partial\tau}=a\Delta f\}=\\ \{D\in\mathfrak{vect}(n|n+1)\mid L_D(\mathrm{vol}^a_{q,\xi,\tau}\,\alpha_0^{a-bn})=0\}.\end{aligned} \tag{2.74}$$

It is subject to a direct verification that $\mathfrak{b}_{a,b}(n)\simeq\mathfrak{b}_\lambda(n)$ for $\lambda=\frac{2a}{n(a-b)}$. This isomorphism shows that λ actually runs over $\mathbb{C}P^1$, not $\mathbb{C}$.

As follows from the description of $\mathfrak{vect}(m|n)$-modules ([BL2]) and the criteria for simplicity of $\mathbb{Z}$-graded Lie superalgebras ([K2]), the Lie superalgebras

$\mathfrak{b}_\lambda(n)$ are simple for $n>1$ and $\lambda\neq 0, 1, \infty$. It is also clear that the $\mathfrak{b}_\lambda(n)$ are non-isomorphic for distinct λ's, bar occasional isomorphisms (2.97).

The Lie superalgebra $\mathfrak{b}(n)=\mathfrak{b}_0(n)$ is not simple: it has an ε-dimensional, i.e., $(0|1)$-dimensional, center. At $\lambda=1$ and ∞ the Lie superalgebras $\mathfrak{b}_\lambda(n)$ are not simple either: they has an ideal of codimension ε^n and ε^{n+1}, respectively. The corresponding exact sequences are

$$\begin{gathered}0\longrightarrow\mathbb{C}M_1\longrightarrow\mathfrak{b}(n)\longrightarrow\mathfrak{le}(n)\longrightarrow 0,\\ 0\longrightarrow\mathfrak{b}_1'(n)\longrightarrow\mathfrak{b}_1(n)\longrightarrow\mathbb{C}\cdot M_{\xi_1\ldots\xi_n}\longrightarrow 0,\\ 0\longrightarrow\mathfrak{b}_\infty'(n)\longrightarrow\mathfrak{b}_\infty(n)\longrightarrow\mathbb{C}\cdot M_{\tau\xi_1\ldots\xi_n}\longrightarrow 0.\end{gathered} \tag{2.75}$$

Clearly, at the exceptional values of λ, i.e., 0, 1, and ∞, the deformations of $\mathfrak{b}_\lambda(n)$ should be investigated extra carefully. As we will see immediately, it pays: at each of exceptional points we find extra deformations. An exceptional deformation at $\lambda=-1$ remains inexplicable. Other exceptional values ($\lambda=\frac{1}{2}$ and $-\frac{3}{2}$) come from the isomorphisms $\mathfrak{b}_{1/2}(2;2)\cong\mathfrak{h}_{1/2}(2|2)=\mathfrak{h}(2|2)$ and $\mathfrak{h}_\lambda(2|2)\cong\mathfrak{h}_{-1-\lambda}(2|2)$, see (2.97).

For $\mathfrak{g}=\mathfrak{b}_\lambda(n)$, set $H=H^2(\mathfrak{g};\mathfrak{g})$.

2.3.5.1. Theorem. 1) $\operatorname{sdim}\ H=(1|0)$ *for* $\mathfrak{g}=\mathfrak{b}_\lambda(n)$ *unless* $\lambda=0, -1, 1, \infty$ *for* $n>2$. *For* $n=2$, *in addition to the above,* $\operatorname{sdim}\ H\neq(1|0)$ *at* $\lambda=\frac{1}{2}$ *and* $\lambda=-\frac{3}{2}$.

2) *At the exceptional values of* λ *listed in heading* 1) *we have*

$\operatorname{sdim}\ H=(2|0)$ *at* $\lambda=\pm1$ *and* n *odd, or* $\lambda=\infty$ *and* n *even, or* $n=2$ *and* $\lambda=\frac{1}{2}$ *or* $\lambda=-\frac{3}{2}$.

$\operatorname{sdim}\ H=(1|1)$ *at* $\lambda=0$, *or* $\lambda=\infty$ *and* n *odd, or* $\lambda=\pm1$ *and* n *even. The corresponding cocycles* C *are given by the following nonzero values in terms of the generating functions* f *and* g, *where* $d_{\bar 1}(f)$ *is the degree of* f *with respect to odd indeterminates only (here* $k=(k_1,\ldots,k_n)$*; we set* $q^k=q_1^{k_1}\ldots q_n^{k_n}$ *and* $|k|=\sum k_i$*):*

$\mathfrak{b}_\lambda(n)$	$p(C)$	$C(f,g)$
$\mathfrak{b}_0(n)$	*odd*	$(-1)^{p(f)}(d_{\bar 1}(f)-1)(d_{\bar 1}(g)-1)fg$
$\mathfrak{b}_{-1}(n)$	$n+1 \pmod 2$	$f=q^k,\ g=q^l\mapsto(4-\lvert k\rvert-\lvert l\rvert)q^{k+l}\xi_1\ldots\xi_n+\tau\Delta(q^{k+l}\xi_1\ldots\xi_n)$
$\mathfrak{b}_1(n)$	$n+1 \pmod 2$	$f=\xi_1\ldots\xi_n,\ g\mapsto\begin{cases}(d_{\bar 1}(g)-1)g & \text{if } g\neq af,\ a\in\mathbb{C}\\ 2(n-1)f & \text{if } g=f \text{ and } n \text{ is even}\end{cases}$
$\mathfrak{b}_\infty(n)$	$n \pmod 2$	$f=\tau\xi_1\ldots\xi_n,\ g\mapsto\begin{cases}(d_{\bar 1}(g)-1)g & \text{if } g\neq af,\ a\in\mathbb{C}\\ 2f & \text{if } g=f \text{ and } n \text{ is odd}\end{cases}$
$\mathfrak{b}_{\frac{1}{2}}(2)$	*even*	*described below*

On $\mathfrak{b}_{\frac{1}{2}}(2) \simeq \mathfrak{b}_{-\frac{3}{2}}(2) \simeq \mathfrak{h}(2|2;1)$, the cocycle C is the one induced on $\mathfrak{h}(2|2) = \mathfrak{po}(2|2)/\mathbb{C}\, K_1$ by the usual deformation (quantization) of $\mathfrak{po}(2|2)$: we first quantize $\mathfrak{po}(2|2)$ and then take the quotient modulo the center (generated by constants).

3) *The space H is diagonalizable with respect to the Cartan subalgebra of $\mathfrak{der}\ \mathfrak{g}$. Let the cocycle M corresponding to the main deformation be one of the eigenvectors. Let C be another eigenvector in H, it determines a singular deformation from heading 2). The only cocycles $kM + lC$, where $k, l \in \mathbb{C}$, that can be extended to a global deformation are those for $kl = 0$, i.e., either M or C.*

All the singular deformations of the bracket $\{\cdot,\cdot\}_{old}$ in $\mathfrak{b}_\lambda(n)$, except the one for $\lambda = \frac{1}{2}$ (or $\lambda = -\frac{3}{2}$) and $n = 2$, have the following very simple form even for the even $\hbar$:

$$\{f, g\}_\hbar^{sing} = \{f, g\}_{old} + \hbar \cdot C(f, g) \text{ for any } f, g \in \mathfrak{b}_\lambda(n). \tag{2.76}$$

Since the elements of $\mathfrak{b}_\lambda(n)$ are encoded by functions (for us: polynomials) in τ, q and ξ subject to one relation with an odd left hand side in which τ enters, it seems plausible that the bracket in $\mathfrak{b}_\lambda(n)$ can be, at least for generic values of parameter λ, expressed solely in terms of q and ξ. Indeed, here is the explicit expression (in which $\{f, g\}_{B.b.}$ is the usual antibracket and $\Delta = \sum\limits_{i\leq n} \frac{\partial^2}{\partial q_i \partial \xi_i}$):

$$\begin{aligned}\{f_1, f_2\}_\lambda^{main} = \{f_1, f_2\}_{B.b.} + &\\ \lambda(c_\lambda(f_1, f_2) f_1 \Delta f_2 + (-1)^{p(f_1)} c_\lambda(f_2, f_1)(\Delta f_1) f_2),&\end{aligned} \tag{2.77}$$

where

$$c_\lambda(f_1, f_2) = \frac{\deg f_1 - 2}{2 + \lambda(\deg f_2 - n)}$$

and deg is computed with respect to the standard grading $\deg q_i = \deg \xi_i = 1$.

2.3.5.2. Deformations of $\mathfrak{g} = \mathfrak{b}_{1/2}(n; n)$. Clearly, $\mathfrak{g}_{-1}$ is isomorphic to $\Pi(\sqrt{Vol})$. Therefore, there is an embedding

$$\mathfrak{b}_{1/2}(n; n) \subset \begin{cases} \mathfrak{h}(2^{n-1}|2^{n-1}) & \text{for } n \text{ even} \\ \mathfrak{le}(2^{n-1}) & \text{for } n \text{ odd.} \end{cases} \tag{2.78}$$

It is tempting to determine quantizations of $\mathfrak{g}$ in addition to those considered by Kochetkov, as the composition of embedding (2.78) and the subsequent quantization.

For $n = 2$, when (2.78) is not just an embedding but an isomorphism, this idea certainly works and we get the following extra quantization of the antibracket described in Theorem 2.3.5.1: We first deform the antibracket to the point $\lambda = \frac{1}{2}$ (or $\lambda = -\frac{3}{2}$) along the main deformation, and then quantize it as the quotient of the Poisson superalgebra. This scheme fails to give new algebras for $n = 2k > 2$:

2.3.5.3. Theorem. *For $n = 2k > 2$, the image of $\mathfrak{b}_{1/2}(n; n)$ under embedding (2.78) is rigid under the quantization of the ambient.*

2.3.6. Nonstandard realizations. The following nonstandard gradings exhaust all the W-gradings of all the simple vectorial Lie superalgebras. In particular, the gradings in the series $\mathfrak{vect}$ induce the gradings in the series $\mathfrak{svect}$, $\mathfrak{svect}'$ and the exceptional families $\mathfrak{vle}(4|3)$ and $\mathfrak{vas}(4|4)$; the gradings in $\mathfrak{m}$ induce the gradings in $\mathfrak{b}_\lambda$, $\mathfrak{le}$, $\mathfrak{sle}$, $\mathfrak{sle}'$, $\mathfrak{b}$, $\mathfrak{sb}$, $\mathfrak{sb}'$ and the exceptional family $\mathfrak{mb}$; the gradings in $\mathfrak{k}$ induce the gradings in $\mathfrak{po}$, $\mathfrak{h}$, $\mathfrak{h}'$ and the exceptional families $\mathfrak{kas}$ and $\mathfrak{ksle}$.

Lie superalgebra	its $\mathbb{Z}$-grading
$\mathfrak{vect}(n\vert m; r)$,	$\deg u_i = \deg \xi_j = 1$ for any i, j $(*)$
$0 \leq r \leq m$	$\deg \xi_j = 0$ for $1 \leq j \leq r$; $\deg u_i = \deg \xi_{r+s} = 1$ for any i, s
$\mathfrak{m}(n; r)$,	$\deg \tau = 2$, $\deg q_i = \deg \xi_i = 1$ for any i $(*)$
$0 \leq r < n - 1$	$\deg \tau = \deg q_i = 2$, $\deg \xi_i = 0$ for $1 \leq i \leq r < n-1$; $\deg q_{r+j} = \deg \xi_{r+j} = 1$ for any j
$\mathfrak{m}(n; n)$	$\deg \tau = \deg q_i = 1$, $\deg \xi_i = 0$ for $1 \leq i \leq n$
$\mathfrak{k}(2n+1\vert m; r)$,	$\deg t = 2$, whereas, for any i, j, k, $\deg p_i = \deg q_i = \deg \xi_j = \deg \eta_j = \deg \theta_k = 1$ $(*)$
$0 \leq r \leq \left[\frac{m}{2}\right]$ $r \neq k-1$ for $m = 2k$ and $n = 0$	$\deg t = \deg \xi_i = 2$, $\deg \eta_i = 0$ for $1 \leq i \leq r \leq \left[\frac{m}{2}\right]$; $\deg p_i = \deg q_i = \deg \theta_j = 1$ for $j \geq 1$ and all i
$\mathfrak{k}(1\vert 2m; m)$	$\deg t = \deg \xi_i = 1$, $\deg \eta_i = 0$ for $1 \leq i \leq m$

Here we consider $\mathfrak{k}(2n+1|m)$ as preserving the Pfaff equation $\tilde{\alpha}(X) = 0$ for $X \in \mathfrak{vect}(2n+1|m)$, where (see (2.18))

$$\tilde{\alpha} = dt + \sum_{i\leq n}(p_i dq_i - q_i dp_i) + \sum_{j\leq r}(\xi_j d\eta_j + \eta_j d\xi_j) + \sum_{k\geq m-2r} \theta_k d\theta_k.$$

The standard realizations correspond to $r = 0$, they are marked by an $(*)$. Observe that the codimension of $\mathcal{L}_0$ attains its minimum in the standard realization.

2.3.6.1. The exceptional nonstandard regrading $\mathrm{Reg}_{\mathfrak{b}}$. This is a regrading of $\mathfrak{b}_{a,b}(2)$ given by the formulas:

$$\deg \tau = 0;\ \ \deg \xi_1 = \deg \xi_2 = -1;\ \ \deg q_1 = \deg q_2 = 1. \tag{2.79}$$

We have the following two cases:

1) $\underline{b=0 \text{ or } a=b}$: $\mathfrak{b}_{a,0}(2;\mathrm{Reg}_{\mathfrak{b}}) \cong \mathfrak{le}(2)$ and $\mathfrak{b}_{a,a}(2;\mathrm{Reg}_{\mathfrak{b}}) \cong \mathfrak{b}'_\infty(2)$,
in particular, $\mathfrak{g}_{-2}=0$;
2) a and b generic:
$$\mathfrak{g}_{-2}=\mathrm{Span}\{\mathrm{Le}_{\xi_1\xi_2}\} \text{ and } \mathfrak{g}_{-1}=\mathrm{Span}\{\mathrm{Le}_{\xi_1},\mathrm{Le}_{\xi_2},\mathrm{Le}_{Q_1},\mathrm{Le}_{Q_2}\}, \tag{2.80}$$

where $Q_1=A\xi_1\xi_2q_1+B\tau\xi_2$, $Q_2=A\xi_1\xi_2q_2-B\tau\xi_1$ and where A and B are some coefficients determined by a and b. The bracket on $\mathfrak{g}_{-1}$ is determined by the odd form $\omega=c\sum dQ_id\xi_i$, so $\mathfrak{g}_0$ must be contained in $\mathfrak{m}(2)_0$. Direct calculations show that $\mathrm{sdim}\,\mathfrak{g}_0=4|4$ and

$$\mathfrak{g}_0=\mathfrak{spe}(2)\oplus\mathbb{C}X, \text{ where } X=\mathrm{Le}_{a\tau+b\sum q_i\xi_i}. \tag{2.81}$$

Indeed,

$$\mathfrak{spe}(2)_0\cong\mathfrak{sl}(2)=\mathrm{Span}\{\mathrm{Le}_{q_1\xi_2},\mathrm{Le}_{q_2\xi_1},\mathrm{Le}_{q_1\xi_1-q_2\xi_2}\},\ \mathfrak{spe}(2)_{-1}=\mathbb{C}\cdot\mathrm{Le}_1, \tag{2.82}$$

and

$$\mathfrak{spe}(2)_1=\mathbb{C}\,\mathrm{Le}_{\alpha\xi_1\xi_2P(q)+\beta\tau\Delta(\xi_1\xi_2P(q))}, \tag{2.83}$$

where $P(q)$ is a monomial of degree 2 and α,β are some constants and where $\Delta=\sum\dfrac{\partial^2}{\partial q_i\partial\xi_i}$, see (2.50).

The eigenvalues of X on $\mathfrak{g}_{-1}$ are: $-a+b$ on the even part and $a+b$ an the odd part. So

$$\mathfrak{b}_{a,b}(2;\mathrm{Reg}_{\mathfrak{b}})\cong\mathfrak{b}_{-b,-a}(2)\cong\mathfrak{b}_{b,a}(2). \tag{2.84}$$

For $n>2$, as well as for $\mathfrak{m}(n)$ with $n>1$, similar regradings are not Weisfeiler ones, as is not difficult to see.

The exceptional grading $\mathrm{Reg}_{\mathfrak{b}}$ of $\mathfrak{b}_\lambda(2)$ induces the exceptional grading $\mathrm{Reg}_{\mathfrak{h}}$ of the isomorphic algebra $\mathfrak{h}_\lambda(2|2)$, see (2.97).

Thus, the exceptional regradings $\mathrm{Reg}_{\mathfrak{b}}$ or $\mathrm{Reg}_{\mathfrak{h}}$ do not provide with new W-graded vectorial algebras. Still, they are important for description of automorphisms.

2.3.6.2. The fifteen W-regradings of exceptional algebras.

2.3.6.3. Theorem ([Sh, CK])**.** *The W-regradings of the exceptional simple vectorial Lie superalgebras are given by the following regradings of their "standard" ambients listed in* (2.67)*:*

1) $\mathfrak{vle}(4|3;r)=(\Pi(\Lambda(3)/\mathbb{C}\cdot 1),\mathfrak{cvect}(0|3))_*\subset\mathfrak{vect}(4|3)$ for $r=0,1,K$;
$\underline{r=0}$: $\deg y=\deg u_i=\deg\xi_i=1$;
$\underline{r=1}$: $\deg y=\deg\xi_1=0$, $\deg u_2=\deg u_3=\deg\xi_2=\deg\xi_3=1$, $\deg u_1=2$;
$\underline{r=K}$: $\deg y=0$, $\deg u_i=2$; $\deg\xi_i=1$.

2) $\mathfrak{vas}(4|4)=(\mathrm{spin},\mathfrak{as})_*\subset\mathfrak{vect}(4|4)$;

3) $\mathfrak{kas}\subset\mathfrak{k}(1|6;r)$ for $r=0,1,3\xi$;

$\underline{r=0}$: $\deg t=2$, $\deg\eta_i=1$; $\deg\xi_i=1$;
$\underline{r=1}$: $\deg\xi_1=0$, $\deg\xi_2=\deg\xi_3=\deg\eta_2=\deg\eta_3=1$, $\deg\eta_1=\deg t=2$;
$\underline{r=3\xi}$: $\deg\xi_i=0$, $\deg\eta_i=\deg t=1$;
$\underline{r=3\eta}$: $\deg\eta_i=0$, $\deg\xi_i=\deg t=1$;

warning: $\mathfrak{kas}(1|6;3\eta)\subset\mathfrak{svect}(4|3)$.

4) $\mathfrak{mb}(4|5;r)=(\mathfrak{ab}(4),\mathfrak{cvect}(0|3))^m_*\subset\mathfrak{m}(4)$ for $r=0,1,K$;
$\underline{r=0}$: $\deg\tau=2$, $\deg u_i=\deg\xi_i=1$ for $i=0,1,2,3$;
$\underline{r=1}$: $\deg\tau=\deg\xi_0=\deg u_1=2$, $\deg\xi_1=\deg u_0=0$; $\deg u_2=\deg u_3=\deg\xi_2=\deg\xi_3=1$;
$\underline{r=K}$: $\deg\tau=\deg\xi_0=3$, $\deg u_0=0$, $\deg u_i=2$; $\deg\xi_i=1$ for $i>0$.

5) $\mathfrak{ksle}(9|6;r)=(\mathfrak{hei}(8|6),\mathfrak{svect}_{3,4}(4))^k_*\subset\mathfrak{k}(9|6)$ for $r=0,\ 2,\ \mathrm{CK}$;
$\underline{r=0}$: $\deg t=2$, $\deg p_i=\deg q_i=\deg\xi_i=\deg\eta_i=1$;
$\underline{r=2}$: $\deg t=\deg q_3=\deg q_4=\deg\eta_1=2$,
$\deg q_1=\deg q_2=\deg p_1=\deg p_2=\deg\eta_2=\deg\eta_3=\deg\zeta_2=\deg\zeta_3=1$;
$\deg p_3=\deg p_4=\deg\zeta_1=0$;
$\underline{r=CK}$: $\deg t=\deg q_1=3$, $\deg p_1=0$;
$\deg q_2=\deg q_3=\deg q_4=\deg\zeta_1=\deg\zeta_2=\deg\zeta_3=2$;
$\deg p_2=\deg p_3=\deg p_4=\deg\eta_1=\deg\eta_2=\deg\eta_3=1$;
$\underline{r=K}$: $\deg t=\deg q_i=2$, $\deg p_i=0$; $\deg\zeta_i=\deg\eta_i=1$;

warning: $\mathfrak{ksle}(9|6;K)\subset\mathfrak{svect}(5|10;R)$.

Thus, from the point of view of classification of the W-filtered Lie superalgebras, there are five *families* of exceptional algebras consisting of 15 individual algebras.

2.3.6.4. Several first terms that determine the Cartan and *mk*-prolongations. To facilitate the comparison of various vectorial superalgebras, we offer the following Table. The most interesting phenomena occur for extremal values of parameter r and small values of superdimension $m|n$.

The central element $z\in\mathfrak{g}_0$ is supposed to be chosen so that it acts on $\mathfrak{g}_k$ as $k\cdot\mathrm{id}$.

Let $\Lambda(r)=\mathbb{C}[\xi_1,\dots,\xi_r]$ be the Grassmann superalgebra generated by the ξ_i, each of degree 0. We set (for more elucidations, see sec. 2.6):

$$\begin{aligned}
\mathfrak{vect}(0|m)\text{-modules}:\ & \Lambda(0)=\mathbb{C},\qquad T(\mathbf{0}):=\Lambda(m);\\
& T_0(\mathbf{0}):=\Lambda(m)/\mathbb{C}\cdot 1;\\
& \mathrm{Vol}_0(0|m):=\{v\in\mathrm{Vol}(0|m)\mid \textstyle\int v=0\};\\
\mathfrak{svect}(0|m)\text{-modules}:\ & T^0_0(\mathbf{0}):=\mathrm{Vol}_0(0|m)/\mathbb{C}\cdot 1\ .
\end{aligned}\tag{2.85}$$

Over $\mathfrak{svect}(0|m)$, it is convenient to consider $\mathrm{Vol}_0(0|m)$ as a submodule of $\Lambda(m)$.

Recall that the range of the parameter r (see sec. 1.3 and 2.11) is the set of integers from $[0, m]$, where m is the number of odd indeterminates; for the series $\mathfrak{k}$ and $\mathfrak{h}$ the range is $[0, [\frac{m}{2}]]$; for the series $\mathfrak{m}(n)$, $\mathfrak{b}_\lambda(n)$, and related subalgebras, the range is the set of integers from $[0, n]$ with $r = n-1$ excluded.

Recall that we exclude certain values of r, namely, $r = k-1$ for $\mathfrak{k}(1|2k)$, as well as $r = n-1$ for $\mathfrak{m}(n)$ and its subalgebras, because, for these values of r, the corresponding grading is not a W-grading: the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is reducible.

The $\mathfrak{g}_0$-modules $\mathfrak{g}_{-1}$ for $N = 5', 5''$ and 9 are described as $\mathfrak{vect}(0|2)$-modules.

N	$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$
1	$\mathfrak{vect}(n\vert m; r)$	–	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{gl}(n\vert m-r)\otimes\Lambda(r)\ltimes\mathfrak{vect}(0\vert r)$
2	$\mathfrak{vect}(1\vert m; m)$	–	$\Lambda(m)$	$\Lambda(m)\ltimes\mathfrak{vect}(0\vert m)$
3	$\mathfrak{svect}(n\vert m; r), n\neq 1$	–	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{sl}(n\vert m-r)\otimes\Lambda(r)\ltimes\mathfrak{vect}(0\vert r)$
4	$\mathfrak{svect}'(1\vert m; r), r\neq m$	–	$\mathrm{id}\otimes\mathrm{Vol}_0(0\vert r)$	$\mathfrak{sl}(n\vert m-r)\otimes\Lambda(r)\ltimes\mathfrak{vect}(0\vert r)$
5	$\mathfrak{svect}'(1\vert m; m)$	–	$\mathrm{Vol}_0(0\vert m)$	$\Lambda(m)\ltimes\mathfrak{svect}(0\vert m)$
$5'$	$\mathfrak{svect}'(1\vert 2)$	–	$T_0(\mathbf{0})$	$\mathfrak{sl}(1\vert 2)\cong\mathfrak{vect}(0\vert 2)$
$5''$	$\mathfrak{svect}(2\vert 1)$	–	$\Pi(T_0(\mathbf{0}))$	$\mathfrak{sl}(2\vert 1)\cong\mathfrak{vect}(0\vert 2)$

6	$\mathfrak{h}(2n\vert m)$	–	id	$\mathfrak{osp}(m\vert 2n)$
7	$\mathfrak{h}(2n\vert m; r)$	$T_0(\mathbf{0})$	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{osp}(m-2r\vert 2n)\otimes\Lambda(r)\ltimes\mathfrak{vect}(0\vert r)$
8	$\mathfrak{h}(2n\vert 2r; r)$	–	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{sp}(2n)\otimes\Lambda(r)\ltimes\mathfrak{vect}(0\vert r)$
9	$\mathfrak{h}_\lambda(2\vert 2)$	–	$\Pi(\mathrm{Vol}^\lambda(0\vert 2))$	$\mathfrak{osp}(2\vert 2)\cong\mathfrak{vect}(0\vert 2)$
10	$\mathfrak{h}_\lambda(2\vert 2; 1)$	–	$\mathrm{id}_{\mathfrak{sp}(2)}\otimes\mathrm{Vol}^\lambda(0\vert 1)$	$\mathfrak{sp}(2)\otimes\Lambda(1)\ltimes\mathfrak{vect}(0\vert 1)$
11	$\mathfrak{k}(2n+1\vert m; r)$	$\Lambda(r)$	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{cosp}(m-2r\vert 2n)\otimes\Lambda(r)\ltimes\mathfrak{vect}(0\vert r)$
12	$\mathfrak{k}(1\vert 2m; m)$	–	$\Lambda(m)$	$\Lambda(m)\ltimes\mathfrak{vect}(0\vert m)$
13	$\mathfrak{k}(1\vert 2m+1; m)$	$\Lambda(m)$	$\Pi(\Lambda(m))$	$\Lambda(m)\ltimes\mathfrak{vect}(0\vert m)$

In what follows, for $N = 16$, we set $\widetilde{p(\mu)} \equiv n \pmod 2$, so μ can be odd indeterminate. The Lie superalgebras $\widetilde{\mathfrak{svect}}_\mu(0|n)$ are isomorphic for nonzero μ's; and therefore so are the algebras $\widetilde{\mathfrak{sb}}_\mu(2^{n-1}-1|2^{n-1})$. So, for n even, we can set $\mu = 1$, whereas if μ is odd, we should consider it as an additional indeterminate on which the coefficients depend.

N	$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$
14	$\mathfrak{m}(n; r)$	$\Pi(\Lambda(r))$	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{cpe}(n-r)\otimes\Lambda(r)\ltimes\mathfrak{vect}(0\vert r)$
15	$\mathfrak{m}(n; n)$	–	$\Pi(\Lambda(n))$	$\Lambda(n)\ltimes\mathfrak{vect}(0\vert n)$
16	$\widetilde{\mathfrak{sb}}_\mu(2^{n-1}-1\vert 2^{n-1})$	–	$\dfrac{\Pi(\mathrm{Vol}(0\vert n))}{\mathbb{C}(1+\mu\xi_1\dots\xi_n)\,\mathrm{vol}(\xi)}$	$\widetilde{\mathfrak{svect}}_\mu(0\vert n)$

In what follows $\lambda = \dfrac{2a}{n(a-b)} \neq 0, 1, \infty$; the three exceptional cases (corresponding to the "drop-outs" $\mathfrak{le}(n)$, $\mathfrak{b}'_1(n)$ and $\mathfrak{b}'_\infty(n)$, respectively) are considered separately.

The irreducibility condition of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ for $\mathfrak{g} = \mathfrak{b}'_\infty$ excludes $r = n-1$.

The case $r = n-2$ is extra exceptional, so in the following tables, unless specified, we assume that

$$0 < r < n-2; \text{ additionally } a \neq b \text{ and } (a, b) \neq \alpha(n, n-2) \text{ for any } \alpha\in\mathbb{C}. \tag{2.86}$$

To further clarify the following tables, denote the superspace of the standard $k|k$-dimensional representation of $\mathfrak{spe}(k)$ by V; let

$$d = \mathrm{diag}(1_k, -1_k) \in \mathfrak{pe}(k).$$

Let $W = V\otimes\Lambda(r)$ and $D\in\mathfrak{vect}(0|r)$. Let $\Xi = \xi_1\cdots\xi_n \in \Lambda(\xi_1, \dots, \xi_n)$.

Denote by T^r the representation of $\mathfrak{vect}(0|r)$ in $\mathfrak{spe}(n-r)\otimes\Lambda(r)$ given by the formula

$$T^r(D) = 1\otimes D + d\otimes\frac{1}{n-r}\,\mathrm{div}\, D. \tag{2.87}$$

$\underline{\mathfrak{g} = \mathfrak{sle}'(n; r)_0 \text{ for } r\neq n-2}$. For $\mathfrak{g}_0$, we have:

$$\begin{gathered}\mathfrak{vect}(0|r) \text{ acts on the ideal } \mathfrak{spe}(n-r)\otimes\Lambda(r) \text{ via } T^r;\\ \text{any } X\otimes f\in\mathfrak{spe}(n-r)\otimes\Lambda(r) \text{ acts in } \mathfrak{g}_{-1} \text{ as } \mathrm{id}\otimes f \text{ and in } \mathfrak{g}_{-2} \text{ as } 0;\\ \text{any } D\in\mathfrak{vect}(0|r) \text{ acts in } \mathfrak{g}_{-1} \text{ via } T^r \text{ and in } \mathfrak{g}_{-2} \text{ as } D.\end{gathered} \tag{2.88}$$

$\underline{\mathfrak{g} = \mathfrak{sle}'(n; n-2)}$. For $\mathfrak{g}_0$, we observe:

$$\mathfrak{spe}(2) \cong \mathbb{C}(\mathrm{Le}_{q_1\xi_1 - q_2\xi_2}) \ltimes \mathbb{C}\,\mathrm{Le}_{\xi_1\xi_2},$$

whereas $\mathfrak{g}_{-2}$ and $\mathfrak{g}_{-1}$ are as above, for $r < n-2$. Set $\mathfrak{h} = \mathbb{C}(\mathrm{Le}_{q_1\xi_1 - q_2\xi_2})$. In this case,

$$\mathfrak{g}_0 \cong \underline{(\mathfrak{h}\otimes\Lambda(n-2)\ltimes\mathbb{C}\,\mathrm{Le}_{\xi_1\xi_2}\otimes(\Lambda(n-2)\setminus\mathbb{C}\xi_3\cdots\xi_n))} \ltimes T^1(\mathfrak{vect}(0|n-2)). \tag{2.89}$$

The action of $\mathfrak{vect}(0|n-2)$, the quotient of $\mathfrak{g}_0$ modulo the underlined ideal, is performed via (2.88). In the subspace $\xi_1\xi_2\otimes\Lambda(n-2)\subset\mathfrak{g}_0$ this action is the same as in the space of volume forms. So we can throw away Ξ, or, speaking more correctly, take the irreducible submodule of functions with integral 0.

For $N = 21, 22$, the terms "$\mathfrak{g}_{-i}$" denote the superspace isomorphic to the one in quotation marks but with the action given by formulas (2.88) and (2.89).

N	$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$
17	$\mathfrak{le}(n)$	$-$	id	$\mathfrak{pe}(n)$
18	$\mathfrak{le}(n;r)$	$\Pi(T_0(\mathbf{0}))$	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{pe}(n-r)\otimes\Lambda(r)\subplus\mathfrak{vect}(0\vert r)$
19	$\mathfrak{le}(n;n)$	$-$	$\Pi(T_0(\mathbf{0}))$	$\mathfrak{vect}(0\vert n)$
20	$\mathfrak{sle}'(n)$	$-$	id	$\mathfrak{spe}(n)$
21	$\mathfrak{sle}'(n;r)$	"$\Pi(T_0(\mathbf{0}))$"	"$\mathrm{id}\otimes\Lambda(r)$"	$\mathfrak{spe}(n-r)\otimes$ "$\Lambda(r)$" $\subplus T^1(\mathfrak{vect}(0\vert r))$
22	$\mathfrak{sle}'(n;n-2)$	"$\Pi(T_0(\mathbf{0}))$"	"$\mathrm{id}\otimes\Lambda(r)$"	see (2.89)
23	$\mathfrak{sle}'(n;n)$	$-$	$\Pi(T_0^0(\mathbf{0}))$	$\mathfrak{svect}(0\vert n)$

Next, in the table below, we consider $\mathfrak{b}_{a,b}(n;r)$ for $0<r<n-2$ and $ar-bn\neq 0$; in particular, this excludes $\mathfrak{b}'_\infty(n;n)=\mathfrak{b}'_{a,a}(n;n)$ and $\mathfrak{b}'_1(n;n-2)=\mathfrak{b}'_{n,n-2}(n;n-2)$.

If z is the central element of $\mathfrak{cspe}(n-r)$ that acts on $\mathfrak{g}_{-1}$ as $-\mathrm{id}$, then

$$z\otimes\psi \text{ acts on } \mathfrak{g}_{-1} \text{ as } -\mathrm{id}\otimes\psi, \text{ and on } \mathfrak{g}_{-2} \text{ as } -2\,\mathrm{id}\otimes\psi. \tag{2.90}$$

Set

$$c=\frac{a}{ar-bn}. \tag{2.91}$$

Let $a\cdot\mathrm{str}\otimes\mathrm{id}$ be the representation of $\mathfrak{pe}(n-r)=\mathfrak{spe}(n-r)\subplus\mathbb{C}d$ which is $\mathrm{id}_{\mathfrak{spe}(n-r)}$ on $\mathfrak{spe}(n-r)$ and sends d to $2a\cdot\mathrm{id}$. Recall that $\mathbb{C}[k]$ is the 1-dimensional representation of $\mathfrak{g}_0$ where k is the value of the central element z from $\mathfrak{g}_0$, where z is chosen so that $z|_{\mathfrak{g}_i}=i\cdot\mathrm{id}_{\mathfrak{g}_i}$.

For $N=27$, 32, 33, the terms "$\mathfrak{g}_i$" denote the superspace isomorphic to the one in quotation marks but with the action given by eq. (2.90).

In the exceptional case $ar=bn$, i.e., $\lambda=\dfrac{2}{n-r}$, we see that the $\mathfrak{vect}(0|r)$-action on the ideal $\mathfrak{cspe}(n-r)\otimes\Lambda(r)\subplus\mathfrak{vect}(0|r)$ of $\mathfrak{g}_0$, and on $\mathfrak{g}_-$, is the same as for $\mathfrak{sle}'$, see (2.88).

2.3.6.5. The exceptional Lie subsuperalgebras. In the table below are the terms $\mathfrak{g}_i$ for $-2\leq i\leq 0$ of the 15 exceptional W-graded algebras.

Observe that none of the simple W-graded vectorial Lie superalgebras is of depth >3 and only two algebras are of depth 3: $\mathfrak{mb}(4|5;K)$, for which we have

$$\mathfrak{mb}(4|5;K)_{-3}\cong\Pi(\mathrm{id}_{\mathfrak{sl}(2)}),$$

and another one, $\mathfrak{ksle}(9|6;CK)=\mathfrak{ksle}(9|11)$, for which we have

$$\mathfrak{ksle}(9|11)_{-3}\simeq\Pi(\mathrm{id}_{\mathfrak{sl}(2)}\otimes\mathbb{C}[-3]).$$

2.3.7. The exceptional Lie subsuperalgebra $\mathfrak{kas}$ of $\mathfrak{k}(1|6)$. Like $\mathfrak{vect}(1|m;m)$, the Lie superalgebra $\mathfrak{kas}$ is not determined by its non-positive part and requires a closer study. The Lie superalgebra $\mathfrak{g}=\mathfrak{k}(1|2n)$ is generated by the functions from $\mathbb{C}[t,\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_n]$. The standard $\mathbb{Z}$-grading of $\mathfrak{g}$ is induced by the $\mathbb{Z}$-grading of $\mathbb{C}[t,\xi,\eta]$ given by

$$\deg K_f=\deg f-2, \text{ where } \deg t=2,\quad \deg\xi_i=\deg\eta_i=1.$$

Clearly, in this grading, $\mathfrak{g}$ is of depth 2. Let us consider the functions that generate several first homogeneous components of $\mathfrak{g}=\underset{i\geq -2}{\oplus}\mathfrak{g}_i$:

component	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$	$\mathfrak{g}_1$
its generators	1	$\Lambda^1(\xi,\eta)$	$\Lambda^2(\xi,\eta)\oplus\mathbb{C}\cdot t$	$\Lambda^3(\xi,\eta)\oplus t\Lambda^1(\xi,\eta)$

As one can prove directly, the component $\mathfrak{g}_1$ generates the whole subalgebra $\mathfrak{g}_+=\underset{i>0}{\oplus}\mathfrak{g}_i$. The component $\mathfrak{g}_1$ splits into two $\mathfrak{g}_0$-modules $\mathfrak{g}_{11}=\Lambda^3$ and $\mathfrak{g}_{12}=t\Lambda^1$. It is obvious that $\mathfrak{g}_{12}$ is always irreducible and the component $\mathfrak{g}_{11}$ is trivial for $n=1$.

The partial Cartan prolongs of $\mathfrak{g}_{11}$ and $\mathfrak{g}_{12}$ are well-known:

$$(\mathfrak{g}_-\oplus\mathfrak{g}_0,\mathfrak{g}_{11})_*^{mk}\cong\mathfrak{po}(0|2n)\oplus\mathbb{C}\cdot K_t\cong\mathfrak{d}(\mathfrak{po}(0|2n));$$
$$(\mathfrak{g}_-\oplus\mathfrak{g}_0,\mathfrak{g}_{12})_*^{mk}=\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_{12}\oplus\mathbb{C}\cdot K_{t^2}\cong\mathfrak{osp}(2n|2).$$

Observe a remarkable property of $\mathfrak{k}(1|6)$: only for $n=3$ the component $\mathfrak{g}_{11}$ splits into 2 irreducible modules; we will denote $\mathfrak{g}_{11}^{\xi}$ the one that contains $\xi_1\xi_2\xi_3$, let $\mathfrak{g}_{11}^{\eta}$ be the other one, that contains $\eta_1\eta_2\eta_3$.

Observe further, that $\mathfrak{g}_0=\mathfrak{co}(6)\cong\mathfrak{gl}(4)$. As $\mathfrak{gl}(4)$-modules, $\mathfrak{g}_{11}^{\xi}$ and $\mathfrak{g}_{11}^{\eta}$ are the symmetric squares $S^2(\mathrm{id})$ and $S^2(\mathrm{id}^*)$ of the standard 4-dimensional representation and its dual, respectively.

2.3.8. Theorem ([Sh]). *The partial Cartan prolongs*

$$\mathfrak{kas}^{\xi}=(\mathfrak{g}_-\oplus\mathfrak{g}_0,\mathfrak{g}_{11}^{\xi}\oplus\mathfrak{g}_{12})_*^{mk}\simeq\mathfrak{kas}^{\eta}=(\mathfrak{g}_-\oplus\mathfrak{g}_0,\mathfrak{g}_{11}^{\eta}\oplus\mathfrak{g}_{12})_*^{mk}$$

are infinite dimensional, simple and isomorphic.

When it does not matter which of isomorphic algebras $\mathfrak{kas}^{\xi}\simeq\mathfrak{kas}^{\eta}$ to take, we will simply write $\mathfrak{kas}$.

N	$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$
24	$\mathfrak{b}_\lambda(n)$	$\Pi(\mathbb{C}[-2])$	id	$\mathfrak{spe}(n)\subplus\mathbb{C}(az+bd)$
25	$\mathfrak{b}_\lambda(n;r)$	$\Pi((-c)\,\mathrm{str})\otimes(\mathrm{Vol}(0\vert r))^{2c}$	$\left((-\frac{c}{2})\,\mathrm{str})\otimes\mathrm{id}\right)\otimes(\mathrm{Vol}(0\vert r))^{c}$	$(\mathfrak{pe}(n-r)\otimes\Lambda(r))\subplus\mathfrak{vect}(0\vert r)$
26	$\mathfrak{b}_\lambda(n;n)$	$-$	$\Pi(\mathrm{Vol}^\lambda(0\vert n))$	$\mathfrak{vect}(0\vert n)$
27	$\mathfrak{b}_{2/(n-r)}(n;r)$	$\Pi(\mathbb{C})\otimes\Lambda(r)$	$\mathrm{id}\otimes$“$\Lambda(r)$”	$\mathfrak{cpe}(n-r)\otimes$ “$\Lambda(r)$” $\subplus T^r(\mathfrak{vect}(0\vert r))$
28	$\mathfrak{b}'_\infty(n)$	$\Pi(\mathbb{C})$	id	$\mathfrak{spe}(n)_{a,a}$
29	$\mathfrak{b}'_\infty(n;r)$	$\Pi(\mathbb{C})\otimes\Lambda(r)$	$\mathrm{id}\otimes\Lambda(r)$	$((\mathfrak{spe}(n-r)_{a,a})\otimes\Lambda(r))\subplus\mathfrak{vect}(0\vert r)$
30	$\mathfrak{b}'_\infty(n;n), n>2$	$-$	$\Pi(\Lambda(n))$	$(\Lambda(n)\setminus\mathbb{C}\Xi)\subplus\mathfrak{svect}(0\vert n)$
31	$\mathfrak{b}'_1(n)$	$\Pi(\mathbb{C})$	id	$\mathfrak{spe}(n)_{n,n-2}$
32	$\mathfrak{b}'_1(n;r)$	“$\Pi(\mathrm{Vol}_0(0\vert r))$”	$\mathrm{id}\otimes$“$\Lambda(r)$”	$((\mathfrak{spe}(n-r)_{n,n-2})\otimes$ “$\Lambda(r)$”$)\subplus T^r(\mathfrak{vect}(0\vert r))$
33	$\mathfrak{b}'_1(n;n-2)$	“$\Pi(T_0(\mathbf{0}))$”	$\mathrm{id}\otimes$“$\Lambda(r)$”	(2.89) for the above line with $\mathfrak{cspe}(2)$ instead of $\mathfrak{spe}(2)$
34	$\mathfrak{b}'_1(n;n)$	$-$	$\Pi(\mathrm{Vol}_0(0\vert n))$	$\mathfrak{vect}(0\vert n)$

$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$	sdim $\mathfrak{g}_-$
$\mathfrak{vle}(4\vert 3)$	$-$	$\Pi(\Lambda(3)/\mathbb{C}1)$	$\mathfrak{c}(\mathfrak{vect}(0\vert 3))$	$4\vert 3$
$\mathfrak{vle}(4\vert 3;1)$	$\mathbb{C}[-2]$	$\mathrm{id}\otimes\Lambda(2)$	$\mathfrak{c}(\mathfrak{sl}(2)\otimes\Lambda(2)\subplus T^{1/2}(\mathfrak{vect}(0\vert 2))$	$5\vert 4$
$\mathfrak{vle}(4\vert 3;K)$	$\mathrm{id}_{\mathfrak{sl}(3)}\otimes\mathbb{C}[-2]$	$\mathrm{id}^*_{\mathfrak{sl}(3)}\otimes\mathrm{id}_{\mathfrak{sl}(2)}\otimes\mathbb{C}[-1]$	$\mathfrak{sl}(3)\oplus\mathfrak{sl}(2)\oplus\mathbb{C}z$	$3\vert 6$
$\mathfrak{vas}(4\vert 4)$	$-$	spin	$\mathfrak{as}$	$4\vert 4$
$\mathfrak{kas}$	$\mathbb{C}[-2]$	$\Pi(\mathrm{id})$	$\mathfrak{co}(6)$	$1\vert 6$
$\mathfrak{kas}(;1\xi)$	$\Lambda(1)$	$\mathrm{id}_{\mathfrak{sl}(2)}\otimes\mathrm{id}_{\mathfrak{gl}(2)}\otimes\Lambda(1)$	$(\mathfrak{sl}(2)\oplus\mathfrak{gl}(2)\otimes\Lambda(1))\subplus\mathfrak{vect}(0\vert 1)$	$5\vert 5$
$\mathfrak{kas}(;3\xi)$	$-$	$\Lambda(3)$	$\Lambda(3)\oplus\mathfrak{sl}(1\vert 3)$	$4\vert 4$
$\mathfrak{kas}(;3\eta)$	$-$	$\mathrm{Vol}_0(0\vert 3)$	$\mathfrak{c}(\mathfrak{vect}(0\vert 3))$	$4\vert 3$
$\mathfrak{mb}(4\vert 5)$	$\Pi(\mathbb{C}[-2])$	$\mathrm{Vol}(0\vert 3)$	$\mathfrak{c}(\mathfrak{vect}(0\vert 3))$	$4\vert 5$
$\mathfrak{mb}(4\vert 5;1)$	$\Lambda(2)/\mathbb{C}1$	$\mathrm{id}_{\mathfrak{sl}(2)}\otimes\Lambda(2)$	$\mathfrak{c}(\mathfrak{sl}(2)\otimes\Lambda(2)\subplus T^{1/2}(\mathfrak{vect}(0\vert 2))$	$5\vert 6$
$\mathfrak{mb}(4\vert 5;K)$	$\mathrm{id}_{\mathfrak{sl}(3)}\otimes\mathbb{C}[-2]$	$\Pi(\mathrm{id}^*_{\mathfrak{sl}(3)}\otimes\mathrm{id}_{\mathfrak{sl}(2)}\otimes\mathbb{C}[-1])$	$\mathfrak{sl}(3)\oplus\mathfrak{sl}(2)\oplus\mathbb{C}z$	$3\vert 8$
$\mathfrak{ksle}(9\vert 6)$	$\mathbb{C}[-2]$	$\Pi(T^0_0(\mathbf{0}))$	$\mathfrak{svect}(0\vert 4)_{3,4}$	$9\vert 6$
$\mathfrak{ksle}(9\vert 6;2)$	$\Pi(\mathrm{id}_{\mathfrak{sl}(1\vert 3)})$	$\mathrm{id}_{\mathfrak{sl}(2)}\otimes\Lambda(3)$	$(\mathfrak{sl}(2)\otimes\Lambda(3))\subplus\mathfrak{sl}(1\vert 3)$	$11\vert 9$
$\mathfrak{ksle}(9\vert 6;K)$	id	$\Pi(\Lambda^2(\mathrm{id}^*))$	$\mathfrak{sl}(5)$	$5\vert 10$
$\mathfrak{ksle}(9\vert 6;CK)$	$\mathrm{id}^*_{\mathfrak{sl}(3)}\otimes\Lambda(1)$	$\mathrm{id}_{\mathfrak{sl}(2)}\otimes\left(\mathrm{id}_{\mathfrak{sl}(3)}\otimes\Lambda(1)\right)$	$\mathfrak{sl}(2)\oplus(\mathfrak{sl}(3)\otimes\Lambda(1)\subplus\mathfrak{vect}(0\vert 1))$	$9\vert 11$

2.3.9. Grozman's theorems and a description of $\mathfrak{g}$ as $\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$. It is convenient to describe the Lie superalgebra $\mathfrak{g}$ of twisted polyvector fields as $\mathfrak{vect}$-module. Similarly, in [CK] the exceptional algebras are described as $\mathfrak{g}=\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$. For most of the series, such description is of little value because each homogeneous component $\mathfrak{g}_{\bar 0}$ and $\mathfrak{g}_{\bar 1}$ has a complicated structure. For the exceptions (when $\mathfrak{g}_{\bar 0}$ is simple, or almost), the situation is totally different! Observe that apart from being beautiful, such a description is useful for the construction of simple *Volichenko algebras*, i.e., inhomogeneous with respect to parity subalgebras of simple Lie superalgebras, cf. [LSa2].

Recall in relation to this a theorem of Grozman. He completely described bilinear differential operators acting in the spaces of tensor fields and invariant under all changes of coordinates. Miraculously, almost each of the first order invariant operators determine a Lie superalgebra on its domain. Some of these superalgebras turn out to be very close to simple. In the constructions below we use some of these invariant operators.

Let ρ be an irreducible representation of the group $GL(n)$ in a finite dimensional vector space V and $\lambda=(\lambda_1,\dots,\lambda_n)$ its lowest weight. The *tensor field of type* ρ or V on an n-dimensional connected manifold M is any section t of the locally trivial vector bundle over M with fiber V such that, under the change of coordinates,

$$t(y(x))=\rho\left(\frac{\partial y}{\partial x}\right)t(x).$$

The space of tensor field of type ρ or V will be denoted by $T(\rho)$ or $T(V)$ or even $T(\lambda)$.

Denote by $T(\mu,\dots,\mu)=\mathrm{Vol}^{\mu}$ the space of μ-densities. Other important examples: Ω^r, the space of differential r-forms, is $T(0,\dots,0,1,\dots,1)$ with r-many 1's; in particular, $T(0)=\Omega^0$ is the space of functions; $\mathfrak{vect}(n)=T(-1,0,\dots,0)$; set

$$L^r_{\Omega^0}(\mathfrak{vect}(n))=T(-1,\dots,-1,0,\dots,0)$$

with r-many -1's, this is the space of r-vector fields, i.e., the rth exterior power of $\mathfrak{vect}(n)$; hereafter in this sec., tensor, exterior, and symmetric powers are taken over the algebra of functions.

The spaces of twisted r-forms and twisted r-vector fields with twist μ are defined to be, respectively,

$$\Omega^r_\mu=\Omega^r\otimes_{\Omega^0}\mathrm{Vol}^{\mu}\ \text{ and }\ L^r_\mu=L^r\otimes_{\Omega^0}\mathrm{Vol}^{\mu}.$$

Obviously, $L^r_\mu\simeq\Omega^{n-r}_{\mu-1}$ and $\mathrm{Vol}^1=\Omega^n$.

The following statements are excerpts from Grozman's difficult result [G]. To describe one of the operators, P_4, we need the *Nijenhuis bracket*, originally defined by the formula

$$\begin{aligned}&\omega^k\otimes\xi,\omega^l\otimes\eta\mapsto(\omega^k\wedge\omega^l)\otimes[\xi,\eta]+\\&\left(\omega^k\wedge L_\xi(\omega^l)+(-1)^k d\omega^k\wedge\iota(\xi)(\omega^l)\right)\otimes\eta+\\&\left(-L_\eta(\omega^k)\wedge\omega^l+(-1)^l\iota(\eta)(\omega^k)\wedge d\omega^l\right)\otimes\xi,\end{aligned}$$

where ι is the inner product and L_X is the Lie derivative with respect to the field X. The Nijenhuis bracket has the following interpretation which implies its invariance: the invariant operator $D\colon(\Omega^k\otimes\mathfrak{vect}(M),\Omega^\bullet)\longrightarrow\Omega^\bullet$ given by the formula

$$\begin{aligned}&D(\omega^k\otimes\xi,\omega)=\\&d\left(\omega^k\wedge\iota(\xi)(\omega)+(-1)^k\omega^k\wedge\iota(\xi)(d\omega)\right)=d\omega^k\wedge\iota(\xi)(\omega)+(-1)^k\omega^k\wedge L_\xi(\omega)\end{aligned}$$

is, for a fixed $\omega^k\otimes\xi$, a superderivation of the supercommutative superalgebra $\Omega^\bullet$ and the Nijenhuis bracket is just the supercommutator of these superderivations. So we can identify $\Omega^\bullet\otimes\mathfrak{vect}(M)$ with the Lie subsuperalgebra $C(d)\subset\mathfrak{vect}(\hat M)$, where $\hat M$ is the supermanifold $(M,\Omega^\bullet(M))$, i.e., $C(d)$ is the centralizer of the exterior differential on M:

$$C(d)=\{D\in\mathfrak{vect}(\hat M)\mid[D,d]=0\}.$$

2.3.9.1. Dualizations. To any map $F\colon T(V)\longrightarrow T(W)$, the dual map $F^*\colon(T(W))^*\longrightarrow(T(V))^*$ corresponds. If we consider tensors with compact support, so integration can be performed, we can identify $(T(V))^*$ with $T(V^*)\otimes\mathrm{Vol}\simeq T(V^*\otimes\mathrm{tr})$, where str is the 1-dimensional $\mathfrak{gl}$-module given by the trace or supertrace, if M is a supermanifold. We will **formally define** $(T(V))^*$ to be $T(V^*\otimes\mathrm{str})$.

Given a bilinear map $F\colon T(V_1)\otimes T(V_2)\longrightarrow T(W)$, we can dualize it with respect to each argument:

$$\begin{aligned}F^{*1}&\colon T(W^*\otimes\mathrm{str})\otimes T(V_2)\longrightarrow T(V_1^*\otimes\mathrm{str}),\\F^{*2}&\colon T(V_1)\otimes T(W^*\otimes\mathrm{str})\longrightarrow T(V_2^*\otimes\mathrm{str}).\end{aligned}$$

2.3.9.2. Theorem ([G]). *Irreducible differential bilinear operators*

$$D\colon T(\rho_1)\otimes T(\rho_2)\longrightarrow T(\rho_3)$$

of order 1 *invariant under arbitrary changes of variables are, up to dualizations and permutation of arguments, only the following ones:*

$$\mathbf{P_1}\colon\Omega^r\otimes T(\rho_2)\longrightarrow T(\rho_3),\qquad(w,t)\mapsto Z(dw,t),$$

where Z is the zeroth order operator, the extension of the projection $\rho_1\otimes\rho_2\longrightarrow\rho_3$ onto any of the irreducible components;

$$\mathbf{P_2}\colon Vect\otimes T(\rho)\longrightarrow T(\rho),\qquad\underline{\textit{the Lie derivative}};$$

$$\mathbf{P_3}\colon T(S^p(\mathrm{id}^*))\otimes T(S^q(\mathrm{id}^*))\longrightarrow T(S^{p+q-1}(\mathrm{id}^*)),\qquad\underline{\textit{the Poisson bracket}};$$

$\mathbf{P_4}$: *On manifolds, the bracket in $C(d)$ is called the* <u>*Nijenhuis bracket*</u>*. This bracket is a linear combination of operators P_1, P_1^{*1}, their composition* *with the permutation operator* $T(V) \otimes T(W) \longrightarrow T(W) \otimes T(V)$, *and a new, irreducible, operator which Grozman denoted* P_4;

$\mathbf{P_5}: \Omega^p \otimes \Omega^q \longrightarrow \Omega^{p+q+1}; \quad \omega_1, \omega_2 \mapsto (-1)^{p(\omega_1)} a(d\omega_1\omega_2) + b(\omega_1 d\omega_2)$, *where* $a, b \in \mathbb{C}$;

$\mathbf{P_6}: \Omega^p_\mu \otimes \Omega^q_\nu \longrightarrow \Omega^{p+q+1}_{\mu+\nu}$, *where* $|\mu|^2 + |\nu|^2 \neq 0$ *and* $p+q<n$;
$\omega_1 \operatorname{vol}^\mu, \omega_2 \operatorname{vol}^\nu \mapsto (\nu(-1)^{p(\omega_1)} d\omega_1\omega_2 - \mu\omega_1 d\omega_2) \operatorname{vol}^{\mu+\nu}$;

$\mathbf{P_7}: L^p \otimes L^q \longrightarrow L^{p+q-1}$ *the* <u>*Schouten bracket*</u>;

$\mathbf{P_8}: L^p_\mu \otimes L^q_\nu \longrightarrow L^{p+q-1}_{\mu+\nu}$, *a generalization of the Schouten bracket given by the next formula on manifolds for* $p+q \leq n$ *and on supermanifolds of superdimension* $n|1$ *for* $p, q \in \mathbb{C}$:

$$\begin{aligned} X \operatorname{vol}^\mu, Y \operatorname{vol}^\nu \mapsto &((\nu-1)(\mu+\nu-1) \operatorname{div} X \cdot Y + \\ &(-1)^{p(X)}(\mu-1)(\mu+\nu-1) X \operatorname{div} Y - \\ &(\mu-1)(\nu-1)\operatorname{div}(XY)) \operatorname{vol}^{\mu+\nu}, \end{aligned} \tag{2.92}$$

where the divergence of a polyvector field f *is defined to be*

$$\operatorname{div}(f) = \sum_{i \leq n} \frac{\partial^2 f}{\partial_{x_i} \partial_{\check{x}_i}}$$

in local coordinates $(x, \check{x})$ *on the supermanifold* $\check{M}$ *associated to the sheaf of sections of the exterior algebra of the tangent bundle on any supermanifold* M: *the checked coordinates on* $\check{M}$ *being*

$$\theta_i = \Pi(\frac{\partial}{\partial x_i}) = \check{x}_i.$$

2.3.10. Theorem ([G]). *The following natural invariant operators determine associative or Lie superalgebras on their domains. Some of these Lie superalgebras are close to simple ones:*

P_3, P_8 *and* P_4 *(by definition). The vectorial Lie superalgebra* $C(d)$ *is not, however, transitive.*

P_5: *For* $ab = 0$, *it determines an associative superalgebra structure.*

For $a = b$, *it determines the structure of a nilpotent Lie superalgebra on* $\Pi(\Omega^\bullet)$.

The bracket given by $d\omega_1\omega_2 - \omega_1 d\omega_2$ *determines a Lie algebra structure on the space* $\Omega^\bullet$.

P_6 *multiplied by* $\frac{\mu-\nu}{\mu\nu}$ *determines structures of nilpotent Lie superalgebras on the superspaces*

$$\Pi(\underset{\lambda\in\mathbb{C}}{\oplus} \Omega^\bullet \otimes \operatorname{Vol}^\lambda)/d\Omega^\bullet,$$
$$\Omega^\bullet_+ = \Pi(\underset{\lambda>0; \lambda\in\mathbb{R}}{\oplus} \Omega^\bullet \otimes \operatorname{Vol}^\lambda),$$
$$\Omega^\bullet_- = \Pi(\underset{\lambda<0; \lambda\in\mathbb{R}}{\oplus} \Omega^\bullet \otimes \operatorname{Vol}^\lambda).$$

P_6 *multiplied by* $\frac{\mu-\nu}{\mu+\nu}$ *determines a nilpotent Lie superalgebra structure on*

$$\Pi(d\Omega^\bullet \oplus \underset{\lambda\neq 0; \lambda\in\mathbb{R}}{\oplus} \Omega^\bullet \otimes \operatorname{Vol}^\lambda).$$

P_8 *multiplied by* $\frac{1}{\mu\nu}$ *determines a nilpotent Lie superalgebra structures on* $\Omega^\bullet_+$ *and* $\Omega^\bullet_-$.

2.3.10.1. Remark. The operator P_8 is a deformation of the Schouten bracket considered here as the multiplication in $\mathfrak{b}_\lambda(n)$ and P_6 is its "Fourier transform".

2.3.11. W-like gradings. It is sometimes convenient to consider gradings of our vectorial Lie superalgebras which lead them out of the polynomial growth class but still preserve the finiteness of depth and preserve the two characteristic features of the W-gradings: for these grading $\oplus\mathfrak{g}_i$, the subalgebra $\mathfrak{g}_{\geq 0} := \underset{i\geq 0}{\oplus} \mathfrak{g}_i$ is a maximal subalgebra and the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible. Namely, we set $\deg x_i = 0$ for r even indeterminates x; we denote this grading $\bar{r}$. Then the exceptional Lie superalgebras and $\mathfrak{b}_\lambda(n; \bar{n})$ look as follows:

<u>$\mathfrak{g} = \mathfrak{ksle}(5|10)$</u>: $\mathfrak{g}_{\bar 0} = \mathfrak{svect}(5|0) \simeq d\Omega^3$, $\mathfrak{g}_{\bar 1} = \Pi(d\Omega^1)$ with the natural $\mathfrak{g}_{\bar 0}$-action on $\mathfrak{g}_{\bar 1}$ and the bracket of any two odd elements being their product; we identify

$$dx_i \wedge dx_j \wedge dx_k \wedge dx_l \otimes \operatorname{vol}^{-1} = \operatorname{sign}(ijklm)\frac{\partial}{\partial x_m}$$

for any permutation $(ijklm)$ of (12345).

<u>$\mathfrak{g} = \mathfrak{vas}(4|4)$: $\mathfrak{g}_{\bar 0} = \mathfrak{vect}(4|0)$, and $\mathfrak{g}_{\bar 1} = \Omega^1 \otimes \operatorname{Vol}^{-1/2}$ with the natural $\mathfrak{g}_{\bar 0}$-action on $\mathfrak{g}_{\bar 1}$</u> and the bracket of odd elements being given by

$$\left[\frac{\omega_1}{\sqrt{\operatorname{vol}}}, \frac{\omega_2}{\sqrt{\operatorname{vol}}}\right] = \frac{d\omega_1 \wedge \omega_2 + \omega_1 \wedge d\omega_2}{\operatorname{vol}},$$

where we identify

$$\frac{dx_i \wedge dx_j \wedge dx_k}{\operatorname{vol}} = \operatorname{sign}(ijkl)\frac{\partial}{\partial x_l} \text{ for any permutation } (ijkl) \text{ of } (1234).$$

<u>$\mathfrak{g} = \mathfrak{vle}(3|6)$</u>: $\mathfrak{g}_{\bar 0} = \mathfrak{vect}(3|0) \oplus \mathfrak{sl}(2)^{(1)}_{\geq 0}$, where $\mathfrak{g}^{(1)}_{\geq 0} = \mathfrak{g} \otimes \mathbb{C}[x_1, x_2, x_3]$, and

$$\mathfrak{g}_{\bar 1} = \left(\Omega^1 \otimes \operatorname{Vol}^{-1/2}\right) \otimes \operatorname{id}_{\mathfrak{sl}(2)^{(1)}_{\geq 0}} \text{ with the natural } \mathfrak{g}_{\bar 0}\text{-action on } \mathfrak{g}_{\bar 1}.$$

Recall that $\mathrm{id}_{\mathfrak{sl}(2)}$ is the irreducible $\mathfrak{sl}(2)$-module L^1 with highest weight 1; its tensor square splits into the symmetric square $L^2 \simeq \mathfrak{sl}(2)$ and the anti-symmetric square — the trivial module L^0; accordingly, denote by $v_1 \wedge v_2$ and $v_1 \bullet v_2$ the projections of $v_1 \otimes v_2 \in L^1 \otimes L^1$ onto the anti-symmetric and symmetric components, respectively. For any $f_1, f_2 \in \Omega^0$, $\omega_1, \omega_2 \in \Omega^1$ and $v_1, v_2 \in L^1$, we set

$$\left[\frac{\omega_1 \otimes v_1}{\sqrt{\mathrm{vol}}}, \frac{\omega_2 \otimes v_2}{\sqrt{\mathrm{vol}}}\right] = \frac{(\omega_1 \wedge \omega_2) \otimes (v_1 \wedge v_2) + (d\omega_1 \wedge \omega_2 + \omega_1 \wedge d\omega_2) \otimes (v_1 \bullet v_2)}{\mathrm{vol}},$$

where we naturally identify Ω^0 with $\Omega^3 \otimes_{\Omega^0} \mathrm{Vol}^{-1}$ and $\Omega^2 \otimes_{\Omega^0} \mathrm{Vol}^{-1}$ with $\mathfrak{vect}(3|0)$ by setting

$$\frac{dx_i \wedge dx_j}{\mathrm{vol}} = \mathrm{sign}(ijk)\frac{\partial}{\partial x_k} \text{ for any permutation } (ijk) \text{ of } (123).$$

$\underline{\mathfrak{g} = \mathfrak{mb}(3|8)}$: $\mathfrak{g}_{\bar 0} = \mathfrak{vect}(3|0) \oplus \mathfrak{sl}(2)^{(1)}_{\geq 0}$, as for $\mathfrak{vle}(3|6)$, while $\mathfrak{g}_{\bar 1} = \mathfrak{g}_{11} \oplus \mathfrak{g}_{12}$, where

$$\mathfrak{g}_{11} = \left(\mathrm{Vol}^{-1/2}\right) \otimes \mathrm{id}_{\mathfrak{sl}(2)^{(1)}_{\geq 0}} \text{ and } \mathfrak{g}_{12} = \left(\Omega^2 \otimes \mathrm{Vol}^{-1/2}\right) \otimes \mathrm{id}_{\mathfrak{sl}(2)^{(1)}_{\geq 0}}.$$

Multiplication is similar to that of $\mathfrak{g} = \mathfrak{vle}(3|6)$. For any $f_1, f_2 \in \Omega^0$, $\omega_1, \omega_2 \in \Omega^1$ and $v_1, v_2 \in L^1$, we set

$$\begin{aligned}
&\left[\frac{\omega_1 \otimes v_1}{\sqrt{\mathrm{vol}}}, \frac{\omega_2 \otimes v_2}{\sqrt{\mathrm{vol}}}\right] = 0,\\
&\left[\frac{f_1 \otimes v_1}{\sqrt{\mathrm{vol}}}, \frac{f_2 \otimes v_2}{\sqrt{\mathrm{vol}}}\right] = \frac{(df_1 \wedge df_2) \otimes (v_1 \wedge v_2)}{\mathrm{vol}},\\
&\left[\frac{f_1 \otimes v_1}{\sqrt{\mathrm{vol}}}, \frac{\omega_1 \otimes v_2}{\sqrt{\mathrm{vol}}}\right] = \frac{f_1\omega_1 \otimes (v_1 \wedge v_2) + (df_1\omega_1 + f_1 d\omega_1) \otimes (v_1 \bullet v_2)}{\mathrm{vol}}.
\end{aligned}$$

$\underline{\mathfrak{g} = \mathfrak{kas}}$: $\mathfrak{g}_{\bar 0} = \mathfrak{vect}(1|0) \ni \mathfrak{sl}(4)^{(1)}_{\geq 0}$, where $\mathfrak{sl}(4)^{(1)}_{\geq 0} = \mathfrak{sl}(4) \otimes \mathbb{C}[x]$, and $\mathfrak{g}_{\bar 1} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$, where $\mathfrak{g}_{-1} = \Lambda^2(\mathrm{id}_{\mathfrak{sl}(4)^{(1)}_{\geq 0}})$ and $\mathfrak{g}_1 = S^2(\mathrm{id}^*_{\mathfrak{sl}(4)^{(1)}_{\geq 0}})$; clearly, one can interchange $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$.

The multiplication is natural: the bracket of $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ (or the other way round) is the product of the anti-symmetric matrix by symmetric one (or vice versa); the action of $\mathfrak{sl}(4)^{(1)}_{\geq 0}$ on $\mathfrak{g}_{\pm 1}$ is the natural action in the space of bilinear forms (or its dual), $\mathfrak{vect}(1|0)$ acts on functions-coefficients.

$\underline{\mathfrak{g} = \mathfrak{b}_\lambda(n;\bar n)}$: here $\bar n$ denotes the non-Weisfeiler grading given by the formulas $\underline{\deg q_i = 0,}$ $\deg \xi_i = 1$ for $i = 1, \dots, n$. Then we have

$$\mathfrak{g}_i = \left(\Pi(\Lambda^{i-1}(\mathfrak{vect}(n|0))\right) \otimes \mathrm{Vol}^{-(i-1)\lambda}$$

for $i = -1, 0, \dots, n-1$. The multiplication is given by Grozman's operator P_8, see (2.92).

Consider $n = 2$ more attentively. Clearly, one can interchange $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$; this possibility explains the isomorphism

$$\mathfrak{b}_\lambda(2;\bar 2) \simeq \mathfrak{b}_{1-\lambda}(2;\bar 2),$$

and hence the mysterious isomorphism $\mathfrak{h}_\lambda(2|2) \simeq \mathfrak{h}_{-1-\lambda}(2|2)$ mentioned in (2.97).

In particular, we have additional outer automorphisms of $\mathfrak{b}_{-1/2}(2;\bar 2)$, whereas for $\lambda = \frac{1}{2}$ (and $\lambda = -\frac{3}{2}$) there is a nontrivial central extension missed in [Ko1].

Therefore to the generic exceptional values of $\lambda = 0$, 1 and ∞, we should add, for $n = 2$, also $\lambda = -1$ and -2.

2.3.12. The structures preserved. It is always desirable to find the structure preserved by the Lie superalgebra under the study. To see what do the vectorial superalgebras in nonstandard realizations preserve, we have to say, first of all, what is the structure that $\mathfrak{g}_0 = \mathfrak{vect}(0|n)$ preserves on $\mathfrak{g}_{-1} = \Lambda(n)$.

Let $\mathfrak{g} = \mathfrak{vect}(0|n)$; set further

$$W = \Lambda(n), \quad V = \Lambda(n)/\mathbb{C}\cdot 1, \quad V_0 = \{\varphi \in \Lambda(n) \mid \varphi(0) = 0\}.$$

The projection $p\colon W \longrightarrow V$ establishes a natural isomorphism between V and V_0. Let $i\colon V_0 \longrightarrow W$ be the "inverse" embedding.

Denote by $\mathrm{mult}\colon W \otimes W \longrightarrow W$ the multiplication on W given by the tensor mult of valency $(2,1)$. Since V_0 is an ideal in the associative supercommutative superalgebra W, the image $\mathrm{mult}\,|_{V_0\otimes V_0}$ is contained in V_0. Denote by mult' the tensor which coincides with mult on $V_0 \otimes V_0$ and vanishes on $\mathbb{C}\cdot 1 \otimes W \bigoplus W \otimes \mathbb{C}\cdot 1$. By means of the projection p and the embedding i we can $\mathfrak{g}$-invariantly transport mult' to V. The tensor obtained will be also denoted by mult'.

For any monomial $\varphi \in W$, denote by φ^* the dual functional; let $B(W)$ denote the monomial basis in W. Then

$$\mathrm{mult} = \mathrm{mult}' + \sum_{\varphi\in B(W)} 1^* \otimes \varphi^* \otimes \varphi = \mathrm{mult}' + 1^* \otimes \sum_{\varphi\in B(W)} \varphi^* \otimes \varphi.$$

By definition of $\mathfrak{g}$, it preserves mult, i.e., $L_D(\mathrm{mult}) = 0$ for any $D \in \mathfrak{g}$. Hence,

$$L_D(\mathrm{mult}') = -L_D(1^*) \otimes \sum_{\varphi\in B(W)} \varphi^* \otimes \varphi - (1^*) \otimes L_D\left(\sum_{\varphi\in B(W)} \varphi^* \otimes \varphi\right).$$

Under the restriction onto $V_0 \otimes V_0$ the second summand vanishes. Observe that

$$\sum_{\varphi\in B(W)} \varphi^* \otimes \varphi = \mathrm{id}\,|_W.$$

Thus,

$$L_D(\mathrm{mult}'|_{V_0\otimes V_0}) = -L_D(1^*)\otimes \mathrm{id}|_{V_0}.$$

The lift of the identity operator $\mathrm{id}|_{V_0}$ to W reads as follows:

$$L_D(\mathrm{mult}) = \alpha(D)\otimes \mathrm{id}|_W \text{ for a 1-form } \alpha \text{ on } W.$$

Thus, all the structures preserved by $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ are clear, except for those preserved by several of the exceptional algebras. Namely, these structures are:

(1) the tensor products $B\otimes$mult of a bilinear or a volume form B preserved (perhaps, conformally, up to multiplication by a scalar) in the fiber of a vector bundle over a $0|r$-dimensional supermanifold on which the structure governed by mult is preserved,

(2) mult$'$, or mult twisted by λ-density. Observe that the volume element B may be not just $\mathrm{vol}(\xi)$ but $(1+\alpha\xi_1\dots\xi_n)\,\mathrm{vol}(\xi)$ as well.

The structures of other types, namely certain pseudodifferential forms preserved by $\mathfrak{b}_\lambda(n)$, are already described.

2.4. Summary: The list of simple $\mathbb{Z}$-graded Lie superalgebras of polynomial vector fields

2.4.0.1. Notation. Comments. We will describe all the Lie superalgebras from the tables in our Main Theorem in detail in due course, but to simplify grasping the general picture from the displayed formulas of the following theorem, let us immediately inform the reader that the prime example, the general vectorial algebra, $\mathfrak{vect}(m|n;r)$, is the Lie superalgebra of vector fields whose coefficients are polynomials (or formal power series, depending on the setting) in m commuting and n anti-commuting indeterminates with the filtration (and grading) determined by equating the degrees of r $(0\le r\le m)$ of odd indeterminates to 0, the degrees of all (even and odd) of the remaining indeterminates being equal to 1. The regradings of other series are determined similarly, but in a more complicated way, see below. Usually, we do not indicate parameter r if $r=0$.

$\mathfrak{k}$ and $\mathfrak{h}$ are the straightforward analogs of the contact and hamiltonian series, respectively; $\mathfrak{po}(2n|m)$, the central extension of $\mathfrak{h}(2n|m)$, is the Poisson Lie superalgebra.

$\mathfrak{m}$ and $\mathfrak{le}$ are the "odd" analogs of the contact and hamiltonian series, respectively. The bracket in $\mathfrak{b}$, the central extension of $\mathfrak{le}$, is the classical Schouten bracket, more popular now under the name antibracket. We say "odd" in quotation marks: the series $\mathfrak{m}$ preserves the Pfaff equation with an *even* contact form, and all these "odd" analogs have both even and odd parts.

The antibracket can be deformed, the corresponding deforms within ZGLAPG class are $\mathfrak{b}_\lambda$, where $\lambda\in\mathbb{C}P^1$.

The algebras $\mathfrak{sm}$, $\mathfrak{sle}$, and $\mathfrak{sb}$ are divergence free subalgebras in respective algebras.

$\widetilde{\mathfrak{sb}}$ is a simple algebra, it is a deformation of a non-simple Lie superalgebra $\mathfrak{sb}(n|n+1;n)$.

Observe **an outstanding property of the bracket in $\mathfrak{h}_\lambda(2|2)\simeq\mathfrak{b}_\lambda(2;2)$: it can be considered as a deformation of an even bracket as well as a deformation of an odd one**.

The *standard* (a.k.a. *identity* or natural) representation of a matrix Lie superalgebra $\mathfrak{g}$ or its subalgebra $\mathfrak{g}\subset\mathfrak{gl}(V)$ in V, and sometimes **the module V itself** are denoted by id or, for clarity, $\mathrm{id}_{\mathfrak{g}}$. The contents prevents confusion of these notations with that of the identity (scalar) operator id_V on the space V, as in the next paragraph:

For $\mathfrak{g}=\oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$, the trivial representation of $\mathfrak{g}_0$ is denoted by $\mathbb{C}$ (if $\mathfrak{g}_0$ is simple) whereas $\mathbb{C}[k]$ denotes a representation of $\mathfrak{g}_0$ trivial on its semi-simple part and such that k is the value of the central element z from $\mathfrak{g}_0$, where z is chosen so that $z|_{\mathfrak{g}_i}=i\cdot\mathrm{id}_{\mathfrak{g}_i}$.

Further elucidations. In formulas (2.93)–(2.96) below: in parentheses after the "family name" of the algebra there stands the superdimension of the superspace of indeterminates and — after semicolon — a shorthand description of the regrading $r\neq 0$.

Passing from one regrading to another one, we take a "minimal" realization (i.e., with a minimal $\dim\mathcal{L}/\mathcal{L}_0$) as the point of reference. For the exceptional Lie superalgebras, another point of reference — the compatible (with parity) grading — is often more convenient, it is denoted by K.

The regradings of the series are governed by a parameter r described in sec. 2.3.6. All regradings are given in (hopefully) suggestive notations, e.g., $\mathfrak{kas}^\xi(1|6;3\eta)$ means that, having taken $\mathfrak{kas}^\xi$ as the point of reference, we set $\deg\eta=0$ for each of the three η's (certainly, this imposes some conditions on the degrees of the other indeterminates).

The exceptional grading $\mathrm{Reg}_{\mathfrak{h}}$ of $\mathfrak{h}_\lambda(2|2)$ is described in passing in the list of occasional isomorphisms (2.97); it is described in detail for another incarnation of this algebra (2.79).

The finite dimensional Lie superalgebra $\mathfrak{h}(0|n)$ of hamiltonian vector fields is not simple: it contains a simple ideal $\mathfrak{h}'(0|n)$.

On drop-outs. Several algebras are "drop-outs" from the series. Examples: the algebras $\mathfrak{svect}'(1|n;r)$ are "drop-outs" from the series $\mathfrak{svect}(m|n;r)$ since the latter are not simple for $m=1$ but contain the simple ideal $\mathfrak{svect}'(1|n;r)$.

Similarly, $\mathfrak{le}(n|n;r)$, $\mathfrak{b}'_1(n|n+1;r)$ and $\mathfrak{b}'_\infty(n|n+1;r)$ are "drop-outs" from the series $\mathfrak{b}_\lambda(n|n+1;r)$ corresponding to $\lambda=0$, 1 and ∞, respectively, having either a simple ideal of codimension 1 or a center the quotient modulo which is simple. (For $n=2$, there are more exceptional values of λ.)

Though $\mathfrak{sm}$ is not a drop-out due to the above reason, it is singled out by its divergence free property, hence deserves a separate line.

One should not treat the drop-outs lightly, be it the case of characteristic p, or the super-case. People justly consider the Lie algebras of Hamiltonian vector fields and its central extension, Poisson algebra (or loop algebra and its affine Kac-Moody relative) as totally distinct algebras. Likewise, the difference between $\mathfrak{sl}$ and $\mathfrak{psl}$ in characteristic p and in the super-case is enormous: For example, the irreducible representation of least dimension of $\mathfrak{psl}$ is the adjoint one.

Thus, all drop-outs, without exception, are quite separate items on the list.

2.4.1. Theorem (Solution to Problem A)**.** *The simple W-graded vectorial Lie superalgebras $\mathfrak{g}$ constitute the following series* (2.93)–(2.95) *and five exceptional families of fifteen individual algebras* (2.96)*. They are pair-wise non-isomorphic, as graded and filtered superalgebras, bar occasional isomorphisms* (2.97)*.*

For completeness, our tables include finite dimensional "degenerations" labeled by "FD" instead of the number.

All these algebras are the results of the (generalized) Cartan prolongation (described below), and therefore are determined by the terms $\mathfrak{g}_i$ with $i \leq 0$ (or $i \leq 1$ in some cases).

N	the family and conditions for its simplicity
1	$\mathfrak{vect}(m\vert n;r)$ for $m \geq 1$ and $0 \leq r \leq n$
FD	$\mathfrak{vect}(0\vert n;r)$ for $n > 1$ and $0 \leq r \leq n$
2	$\mathfrak{svect}(m\vert n;r)$ for $m > 1$, $0 \leq r \leq n$
FD	$\mathfrak{svect}(0\vert n;r)$ for $n > 2$ and $0 \leq r \leq n$
3	$\mathfrak{svect}'(1\vert n;r)$ for $n > 1$, $0 \leq r \leq n$
FD	$\widetilde{\mathfrak{svect}}(0\vert n)$ for $n > 2$

(2.93)

N	the family and conditions for its simplicity
4	$\mathfrak{k}(2m+1\vert n;r)$ for $0 \leq r \leq [\frac{n}{2}]$ unless $(m\vert n) = (0\vert 2k)$ $\mathfrak{k}(1\vert 2k;r)$ for $0 \leq r \leq k$ except $r = k-1$
5	$\mathfrak{h}(2m\vert n;r)$ for $m > 0$ and $0 \leq r \leq [\frac{n}{2}]$ $\mathfrak{h}_\lambda(2\vert 2;r)$ for $\lambda \neq -2, -1, 0, 1, \infty$, and $r = 0$, 1 and $\mathrm{Reg}_{\mathfrak{h}}$ (see (2.97) and sec. 1.3.1)
FD	$\mathfrak{h}'(0\vert n)$ for $n > 3$

(2.94)

N	the family and conditions for its simplicity
6	$\mathfrak{m}(n\vert n+1;r)$ for $0 \leq r \leq n$ except $r = n-1$
7	$\mathfrak{sm}(n\vert n+1;r)$ for $n > 1$ but $n \neq 3$ and $0 \leq r \leq n$ except $r = n-1$
8	$\mathfrak{b}_\lambda(n\vert n+1;r)$ for $n > 1$, where $\lambda \neq 0, 1, \infty$ and $0 \leq r \leq n$ except $r = n-1$ $\mathfrak{b}_\lambda(2\vert 3;r)$, for $\lambda \neq -2, -1, 0, 1, \infty$, and $r = 0$, 1 and $\mathrm{Reg}_{\mathfrak{b}}$ (see (2.97) and sec. 1.3.1)
9	$\mathfrak{b}'_1(n\vert n+1;r)$ for $n > 1$ and $0 \leq r \leq n$ except $r = n-1$
10	$\mathfrak{b}'_\infty(n\vert n+1;r)$ for $n > 1$ and $0 \leq r \leq n$ except $r = n-1$
11	$\mathfrak{le}(n\vert n;r)$ for $n > 1$ and $0 \leq r \leq n$ except $r = n-1$
12	$\mathfrak{sle}'(n\vert n;r)$ for $n > 2$ and $0 \leq r \leq n$ except $r = n-1$
13	$\widetilde{\mathfrak{sb}}_\mu(2^{n-1}-1\vert 2^{n-1})$ for $\mu \neq 0$ and $n > 2$

(2.95)

More notations. Hereafter we abbreviate algebras $\mathfrak{m}(n|n+1;r)$, $\mathfrak{b}_\lambda(n|n+1;r)$, $\mathfrak{le}(n|n;r)$ and so on from (2.95) to $\mathfrak{m}(n;r)$, $\mathfrak{b}_\lambda(n;r)$, $\mathfrak{le}(n;r)$, and so on, respectively.

Let $\lambda = \dfrac{2a}{n(a-b)} \in \mathbb{C} \cup \{\infty\}$. Then, as we will see, $\mathfrak{b}_{a,b}(n;r)$ is a more natural notation of $\mathfrak{b}_\lambda(n;r)$. The notations $\mathfrak{svect}_{a,b}(0|n)$ and $\mathfrak{spe}_{a,b}(n)$ though look similar to $\mathfrak{b}_{a,b}(n;r)$ mean something different: they are shorthand for $\mathfrak{g} \Subset \mathbb{C}(az+bd)$, where $\mathfrak{g}$ stands for $\mathfrak{svect}(0|n)$ or $\mathfrak{spe}(n)$, respectively, d is the operator that determines the standard $\mathbb{Z}$-grading of $\mathfrak{g}$, and z is the trivial center.

Exceptional simple algebras. We concede that our notations of exceptional simple vectorial superalgebras, though reflect the way they are constructed and the geometry preserved ($\mathfrak{ksle}$ reflects that it is a subalgebra of $\mathfrak{k}$ related to $\mathfrak{sle}$, and so on), are rather long. But to write just $\mathfrak{e}$(sdim) is to create confusion: the superdimensions of the superspaces $(\mathcal{L}/\mathcal{L})^*$ on which the algebra $\mathcal{L}$ is realized may coincide for different regradings. So, in accordance with table (2.67), we set (by K we denote *compatible* gradings, CK is an exceptional grading found by Cheng and Kac):

Lie superalgebra	its regradings (shorthand)
$\mathfrak{vle}(4\vert 3;r)$, $r = 0, 1, K$	$\mathfrak{vle}(4\vert 3)$, $\mathfrak{vle}(5\vert 4)$, and $\mathfrak{vle}(3\vert 6)$
$\mathfrak{vas}(4\vert 4)$	$\mathfrak{vas}(4\vert 4)$
$\mathfrak{kas}(1\vert 6;r)$, $r = 0, 1\xi, 3\xi, 3\eta$	$\mathfrak{kas}(1\vert 6)$, $\mathfrak{kas}(5\vert 5)$, $\mathfrak{kas}(4\vert 4)$, and $\mathfrak{kas}(4\vert 3)$
$\mathfrak{mb}(4\vert 5;r)$, $r = 0, 1, K$	$\mathfrak{mb}(4\vert 5)$, $\mathfrak{mb}(5\vert 6)$, and $\mathfrak{mb}(3\vert 8)$
$\mathfrak{ksle}(9\vert 6;r)$, $r = 0, 2, K, CK$	$\mathfrak{ksle}(9\vert 6)$, $\mathfrak{ksle}(11\vert 9)$, $\mathfrak{ksle}(5\vert 10)$, and $\mathfrak{ksle}(9\vert 11)$

(2.96)

2.4.1.1. Occasional isomorphisms. We have:

$$\begin{aligned}
&\mathfrak{vect}(1|1) \cong \mathfrak{vect}(1|1;1);\\
&\mathfrak{svect}(2|1) \cong \mathfrak{le}(2;2); \mathfrak{svect}(2|1;1) \cong \mathfrak{le}(2)\\
&\mathfrak{sm}(n) \cong \mathfrak{b}_{2/(n-1)}(n);\ \text{in particular, } \mathfrak{sm}(2) \cong \mathfrak{b}_2(2), \text{ and } \mathfrak{sm}(3) \cong \mathfrak{b}_1(3),\\
&\text{hence } \mathfrak{sm}(3) \text{ is not simple;}\\
&\mathfrak{sle}'(3) \cong \mathfrak{sle}'(3;3);\\
&\mathfrak{b}_{1/2}(2;2) \cong \mathfrak{h}_{1/2}(2|2) = \mathfrak{h}(2|2); \mathfrak{h}_\lambda(2|2) \cong \mathfrak{b}_\lambda(2;2); \mathfrak{h}_\lambda(2|2;1) \cong \mathfrak{b}_\lambda(2);\\
&\mathfrak{b}'_1(2;\mathrm{Reg}_{\mathfrak{b}}) \cong \mathfrak{le}(2),\ \mathfrak{le}(2;\mathrm{Reg}_{\mathfrak{b}}) \cong \mathfrak{b}'_1(2) \text{ and } \mathfrak{b}'_\infty(2;\mathrm{Reg}_{\mathfrak{b}}) \cong \mathfrak{b}'_\infty(2);\\
&\mathfrak{b}_\lambda(2;\mathrm{Reg}_{\mathfrak{b}}) \cong \mathfrak{b}_{-1-\lambda}(2) \text{ and } \mathfrak{h}_\lambda(2|2) \cong \mathfrak{h}_{-1-\lambda}(2|2),\ \text{hence}\\
&\text{the fundamental domain is } \operatorname{Re}\lambda \geq -\frac{1}{2} \text{ for } \lambda \neq 0,1,\infty\\
&\qquad\qquad\qquad\qquad \text{or } \operatorname{Re}\lambda \leq -\frac{1}{2} \text{ for } \lambda \neq -2,-1;\\
&\widetilde{\mathfrak{sb}}_\mu(2^{n-1}-1|2^{n-1}) \cong \widetilde{\mathfrak{sb}}_\nu(2^{n-1}-1|2^{n-1}) \text{ for } \mu\nu \neq 0.
\end{aligned} \tag{2.97}$$

The isomorphism $\mathfrak{h}_\lambda(2|2) \cong \mathfrak{b}_\lambda(2;2)$ and the exceptional regrading $\mathrm{Reg}_{\mathfrak{b}}$ of $\mathfrak{b}_\lambda(2)$ and the determine an exceptional grading $\mathrm{Reg}_{\mathfrak{h}}$ of $\mathfrak{h}_\lambda(2|2)$. These regradings ($\mathrm{Reg}_{\mathfrak{b}}$ and $\mathrm{Reg}_{\mathfrak{h}}$) are not, however, of any interest from the classification point of view, thanks to isomorphisms (2.97). On the other hand, these regradings contribute to the group of automorphisms of $\mathfrak{h}_\lambda(2|2) \cong \mathfrak{b}_\lambda(2;2)$.

2.4.1.2. Remarks. 1) Though $\mathfrak{b}_\lambda(2;r)$ and $\mathfrak{h}_\lambda(2|2;r)$ are isomorphic, we consider them separately because they are deformations of very distinct structures: the odd and the even bracket, respectively.

Actually, $\mathfrak{h}_\lambda(2|2) \cong \mathfrak{b}_\lambda(2;2)$ should be considered as an exceptional family.

2) **Warning**. Isomorphic abstract Lie superalgebras might be quite distinct as filtered or graded: e.g., regradings provide us with isomorphisms ([ALSh])

$$\mathfrak{k}(1|2) \cong \mathfrak{vect}(1|1) \cong \mathfrak{m}(1) \text{ as abstract algebras.} \tag{2.98}$$

Observe that only one of the above three non-isomorphic graded algebras is W-graded.

3) The excluded regrading $\mathfrak{k}(1|2)$ is often considered in physical papers, despite the fact that this grading (or filtration) is not a Weisfeiler one: For it, the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is reducible. Over $\mathbb{R}$, however, $\mathfrak{k}(1|2k;k-1)$ is a W-grading, at least, for some real forms, and it is these forms that physicists consider.

4) $\widetilde{\mathfrak{svect}}(0|n)$, as well as $\widetilde{\mathfrak{sb}}_\mu(2^{n-1}-1|2^{n-1})$, depend on an odd parameter if n is odd.

5) The above Lie superalgebras sometimes admit deformations that do not possess Weisfeiler filtrations.

Chapter 3

Invariant differential operators: solving O. Veblen's problem (P. Grozman, D. Leites, I. Shchepochkina)

3.1. Introduction

This chapter is a version of the paper [GLS2]. In particular, we have eliminated references (which can be found in [GLS2]) to inaccessible papers by Kochetkov, whose results we redo, anyway.

Setting of the problem. The problem we address — calculation of invariant differential operators acting in the (super)spaces of tensor fields on (super)manifolds with various structures — was a part of the agenda of our Seminar on Supersymmetries since mid-1970's. Here we use Grozman's code `SuperLie` [Gr] to verify and correct earlier results and obtain new ones, especially when bare hands are inadequate. The awful-looking lists of singular vectors we give in this chapter are to be interpreted in reasonable terms of invariant operators acting in the spaces of tensor fields; this is an interesting and important open problem. We review the whole field with its open problems and recall interesting Kirillov's results and problems buried in the VINITI collection [Ki] which is not very accessible. Another nice (and accessible) review we can recommend in addition to [Ki] is [KMS].

3.1.1. Veblen's problem. The topology of differentiable manifolds has always been related with various geometric objects on them and, in particular, with operators invariant with respect to the group of diffeomorphisms of the manifold, operators which act in the spaces of sections of "natural bundles" ([KMS]) whose sections are tensor fields, or connections, and so on. For example, an important invariant of the manifold, its *cohomology*, stems from the de Rham complex whose neighboring terms are bridged by an invariant differential operator — the exterior differential.

The role of invariance had been appreciated already in XIX century in relation with physics; indeed, differential operators invariant with respect to the group of diffeomorphisms preserving a geometric structure are essential both in formulation of Maxwell's laws of electricity and magnetism and in Einstein–Hilbert's formulation of relativity.

Simultaneously, invariance became a topic of conscious interest for mathematicians: the representation theory flourished in works of F. Klein, followed by Lie and É. Cartan, to name the most important contributors; it provided with the language and technique adequate in the study of geometric structures. Still, it was not until O. Veblen's talk in 1928 at the Mathematical Congress in Bologna ([Veb]) that invariant operators (such as, say, Lie derivative, the exterior differential, or integral) became the primary object of the study. In what follows we rule out the integral and other non-local operators; except in Kirillov's example, we only consider local operators.

Schouten and Nijenhuis tackled Veblen's problem: they reformulated it in terms of modern differential geometry and found several new bilinear invariant differential operators. Schouten conjectured that there is essentially one **un**ary invariant differential operator: the exterior differential of differential forms. This conjecture had been proved in particular cases by a number of people, and in full generality in 1977–78 by A. A. Kirillov and, independently, C. L. Terng ([Ki], [Ter]).

Thanks to the usual clarity and an enthusiastic way of Kirillov's presentation he drew new attention to this problem, at least, in Russia. Under the light of this attention it became clear (to J. Bernstein) that in 1973 A. Rudakov [R1] also proved this conjecture (or, rather, an equivalent to it dual problem) by a *simple algebraic* method which reduces Veblen's problem for differential operators to a "computerizable" one.

Thus, a tough analytic problem was reduced to a problem formally understandable by any first year undergraduate: a series of systems of linear equations in small dimensions plus (easy) induction on dimension. The only snag is the volume of calculations: to list all unary operators in the key cases requires a half page; for binary operators one needs about 30 pages and the induction becomes rather nontrivial, see [G]; for r-nary operators with $r > 2$ only some cases seem to be feasible (and of interest), for example, anti-symmetric operators described by Feigin and Fuchs on the line.

Later Rudakov for the Lie algebras $\mathfrak{g}$ of divergence-free and Hamiltonian (and his student I. Kostrikin for the contact series) classified unary $\mathfrak{g}$-invariant differential operators $D : T(V) \longrightarrow T(W)$, where $T(U)$ is the space of formal tensor fields with fiber U on the manifold with the $\mathfrak{g}_0$-structure [1)] (resp. with the contact structure). Or, rather, they listed the corresponding *singular vectors*, i.e., the vacuum vectors in the dual spaces $I(U)$ corresponding to the maps $D^* : I(W^* \otimes \mathrm{str}) \longrightarrow I(V^* \otimes \mathrm{str})$.

In passing, the definition of the tensor field was generalized and **primitive forms came to foreground**.

Speculations on modern physical applications. Broadhurst and Kac observed [Ka1] that some of the exceptional Lie superalgebras (listed in [Sh], [CK]) might pertain to a SUSY GUTs (Supersymmetric Grand Unified Theories) or the Standard Model, their linear parts being isomorphic to $\mathfrak{sl}(5)$ or $\mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{gl}(1)$. Kac demonstrated [Ka2] that, for the Standard Model with $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ as the gauge group, a certain remarkable relation between $\mathfrak{vle}(3|6)$ and some of the known elementary particles does take place; it seems that, for $\mathfrak{mb}(3|8)$, there is an even better correspondence.

[1] We advise the reader to refresh the definition of the G-structure on a manifold, see Index.

The total lack of enthusiasm from the physicists' community concerning these correspondences is occasioned, perhaps, by the fact that no real form of any of the simple Lie superalgebras of vector fields with polynomial coefficients has a unitary Lie algebra as its linear part.

Undeterred by this, Kac and Rudakov calculated [KR1] some $\mathfrak{vle}(3|6)$-invariant differential operators $D : T(V) \longrightarrow T(W)$, where $T(U)$ is the space of formal tensor fields with irreducible over $(\mathfrak{vle}(3|6))_0$ (in the standard $\mathbb{Z}$-grading) and **finite dimensional** fiber U. More precisely, they have calculated *singular vectors* in the dual spaces $I(U) := (T(U^* \otimes \mathrm{str}))^*$. To interpret the answer obtained in terms of singular vectors of the spaces $I(U)$ in geometric terms of tensor fields is an *open problem*.

The restriction on finite dimension of fibers makes calculations a sight easier but strikes out many operators. Nevertheless, the amount of calculations for $\mathfrak{mb}(3|8)$ is too high to be performed by hands without mistakes.

Main results of this Chapter. (1) We observe that the linear parts of two of the W-regradings of $\mathfrak{mb}$ are Lie superalgebras containing $\mathfrak{sl}(3)\oplus\mathfrak{sl}(2)\oplus\mathfrak{gl}(1)$. They (or certain of their real forms) are natural candidates for the algebras of The *would be* Standard Models since the modern "no-go" theorems do not preclude them.

(2) We list (degeneracy conditions for) all invariant differential operators, or rather the corresponding to them so-called *singular vectors*, of degrees 1, and 2 and, in some cases, of all possible degrees (which often are ≤ 2), where invariance is considered with respect to (separately, not simultaneously) each of the exceptional simple Lie superalgebras of polynomial vector fields. When degeneracy conditions are violated (absence of singular vectors) the corresponding induced and coinduced modules are irreducible. For some of these exceptional simple Lie superalgebras, EVERY module $I(V)$ has a singular vector. This is a totally new feature never encountered before in the study of singular vectors in modules over simple serial vectorial Lie superalgebras (of series $\mathfrak{vect}$, $\mathfrak{svect}$, $\mathfrak{h}$ and $\mathfrak{k}$ in their standard grading).

3.2. How to solve Veblen's problem

3.2.1. Rudakov's breakthrough (following Bernstein [BL2]). Hereafter the ground field $\mathbb{K}$ is $\mathbb{C}$ or $\mathbb{R}$. Without going into details which will be given later, observe that the spaces in which invariant operators act fall into two major cases: spaces of tensor fields (transformations depend on the 1-jet of diffeomorphism) and spaces depending on higher jets, called *HJ-tensors* for short. We will only study tensors here, not HJ-tensors.

1) Instead of considering $\mathrm{Diff}(U)$-invariant operators, where U is a local chart, let us consider $\mathfrak{vect}(U)$-invariant operators, where $\mathfrak{vect}(U)$ is the Lie algebra of vector fields on U with polynomial coefficients, or its formal completion. (*A posteriori* one proves that the global and the local problems are equivalent, cf. [BL2]). Accordingly, instead of tensor fields with smooth coefficients, we consider their formal version: $T(V) = V \otimes \mathbb{K}[[x]]$, where $x = (x_1, \dots, x_n)$ and $n = \dim U$.

2) We assume here that V is an *irreducible* $\mathfrak{gl}(n)$*-module with lowest weight*. Observe that while the requirement of lowest weight seems to be "obviously" reasonable, that on irreducibility is not, unless we confine ourselves to finite dimensional modules V. In super setting we are forced, in the absence of complete reducibility, to consider *indecomposable* representations even for finite dimensional modules. Irreducible modules is just the simplest first step.

3) Instead of the coinduced module, $T(V)$, consider the dual induced module, $I(V^*) = \mathbb{K}[\partial] \otimes V^*$, where $\partial_i = \dfrac{\partial}{\partial x_i}$. The reason: formulas for $\mathfrak{vect}(U)$-action are simpler for $I(V^*)$ than for $T(V)$. (The results, contrariwise, are more graphic in terms of tensor fields.)
Observe that each induced module is a "highest weight one" with respect to the whole $\mathfrak{g} = \mathfrak{vect}(U)$, i.e., the vector of the most highest weight with respect to the linear vector fields from $\mathfrak{g}_0 = \mathfrak{gl}(n)$ is annihilated by $\mathfrak{g}_+$, the subalgebra of $\mathfrak{g}$ consisting of all operators of degree > 0 relative the standard grading ($\deg x_i = 1$ for all i).
In what follows the vectors annihilated by $\mathfrak{g}_+$ will be called *singular* ones.

4) To every r-nary operator $D\colon T(V_1) \otimes \dots \otimes T(V_r) \longrightarrow T(V)$ the dual operator corresponds

$$D^*\colon I(W) \longrightarrow I(W_1) \otimes \dots \otimes I(W_r), \quad \text{where} \quad W = V^*,\ W_i = W_i^*,$$

and, since (for details, see [R1]) each induced module is a highest weight one,

$$\begin{aligned}&\textit{to list all the } D\textit{'s, is the same as}\\ &\textit{to list all the } \mathfrak{g}_0\textit{-highest singular vectors } D^* \in I(W_1) \otimes \dots \otimes I(W_r).\end{aligned} \tag{3.1}$$

In what follows r is called the *arity* of D.

5) In super setting, as well as for non-super but infinite dimensional fibers, the above statement (3.1) is not true: The submodule generated by a singular vector does not have to be a maximal one; it could have another singular vector of the same degree due to the lack of complete reducibility.
For unary operators invariant with respect to Lie algebras over ground fields of characteristic 0, this nuisance does not happen; this was one of the (unreasonable) psychological motivations to stick to the finite dimensional case even for Lie superalgebras, compare [BL2] and [Sm4].

6) Rudakov's paper [R1] contains two results:

(A) description of $\mathfrak{vect}(n)$-singular vectors which in terms of invariant differential operators in tensor fields reads (thanks to the Poincaré lemma): only the exterior differential exists;

(B) proof of the fact that between the spaces of HJ-tensors there are no unary invariant operators.

3.2.2. Problems. 1) Describe r-nary invariant operators in the spaces of HJ-tensors for $r > 1$.

2) Describe r-nary invariant operators in the *super*spaces of HJ-tensors.

The dual operators. Kirillov noticed Ki that by means of the invariant pairing (we consider tensor fields with compact support on a manifold M and tensoring is performed over the space $\mathcal{F}$ of functions on M)

$$B\colon T(V) \times (T(V^*) \otimes_{\mathcal{F}} \mathrm{Vol}(M)) \longrightarrow \mathbb{R}, \quad (t, t^* \otimes \mathrm{vol}) \mapsto \int (t, t^*)\,\mathrm{vol}$$

one can define the duals of the known invariant operators. For the fields with formal coefficients we consider there is, of course, no pairing, but we consider a *would be* pairing induced by smooth fields with compact support. So the *formal dual* of $T(V)$ is not $T(V^*)$ because the pairing returns a function instead of a volume form to be integrated to get a number, and not $T(V)^*$ because $T(V)^*$ is a highest weight module while we need a lowest weight one. Answer: the formal dual of $T(V)$ is $T(V^*) \otimes_{\mathcal{F}} \mathrm{Vol}(M)$.

Possibility to dualize, steeply diminishes the number of cases to consider in computations and helps to check the results. Indeed, with every invariant operator $D\colon T(V) \longrightarrow T(W)$ the dual operator

$$D^*\colon T(W^*) \otimes_{\mathcal{F}} \mathrm{Vol}(M) \longrightarrow T(V^*) \otimes_{\mathcal{F}} \mathrm{Vol}(M)$$

is also invariant. For example, what is the dual of $d\colon \Omega^k \longrightarrow \Omega^{k+1}$? Clearly, it is the same d but in another incarnation: $d\colon \Omega^{n-k-1} \longrightarrow \Omega^{n-k}$. Though, roughly speaking, we only have one operator, d (even its expression in coordinates $d = \sum dx_i \partial_i$ does not depend on its domain of definition, on the degree of the space of exterior forms it acts on), the shape of singular vectors corresponding to d differs with k and having found several "new" singular vectors we must verify that the corresponding operators are indeed distinct. This might be not easy.

Observe that the dualization arguments do not work when we allow infinite dimensional fibers (dualization sends the highest module into a lowest weight one, so it is unclear if a highest weight module with a singular vector always correspond to this lowest weight one). Sometimes, being tired of calculations, or when the computer gave up, we formulated the description of singular vectors "up to dualization"; sometimes even the computer became "tired". We will mention such cases extra carefully.

3.2.2.1. Problem. To reconsider these cases on a more powerful computer is an open **Problem**.

3.2.3. Further ramifications of Veblen's problem. Rudakov's arguments [R1] show that the fibers of HJ-tensors have to be of infinite dimension; the same holds for Lie superalgebras, though arguments are different.

Traditionally, fibers of tensor bundles were only considered to be of finite dimension, though even in his first paper on the subject Rudakov [R1] digressed from traditions.

1°. In the study of invariant operators, one of the "reasons" for confining to tensors, moreover, the ones corresponding to finite dimensional fibers, is provided by two of Rudakov's results:

(1) there are no invariant operators between HJ-tensors,

(2) starting with any highest weight modules $I(V)$, Rudakov unearthed singular vectors only for fundamental (hence, finite dimensional) representations.

Though (1) only applies to *unary* operators, researchers were somewhat discouraged to consider HJ-tensors even speaking about operators of higher arity.

3.2.3.1. Problem. Consider invariant operators of arity > 1 between the spaces of HJ-tensors. Is it true that in this case there are no invariant operators either?

2°. Kirillov proved that (having fixed the dimension of the manifold and arity) *the degree of invariant (with respect to $\mathfrak{vect}(n)$) differential operators is bounded, even* dim *of the space of invariant operators is bounded.*

There seemed to be no doubt that a similar statement holds on supermanifolds ... but Kochetkov's examples reproduced below and our own ones show that these expectations are false in some cases.

3.2.3.2. Problem. Figure out the conditions under which the dimension of the space of invariant operators is bounded. (We conjecture that this is true for all the serial simple vectorial Lie superalgebras in their standard gradings.)

3°. On the line, all tensors are λ-densities and every r-linear differential operator is of the form

$$L\colon (f_1 dx^{\lambda_1}, \dots, f_r dx^{\lambda_r}) \longrightarrow P_L(f_1, \dots, f_r) dx^{\lambda}.$$

Kirillov showed (with ease and elegance) that invariance of L is equivalent to the system

$$\sum_{s=1}^{r} \left[t_s \frac{\partial^{j+1}}{\partial t_s^{j+1}} + (j+1)\lambda_s \frac{\partial^j}{\partial t_s^j} \right] P_L(t) = \begin{cases} \lambda P_L(t) & \text{for } j = 0 \\ 0 & \text{for } j > 0. \end{cases} \tag{3.2}$$

Clearly, differential operators correspond to polynomial solutions $P_L(t)$ and in this case $\lambda = \sum\limits_{s=1}^{r} \lambda_s - \deg P_L$. Kirillov demonstrated that nonpolynomial solutions of (3.2) do exist: for $r = 2$ and $\lambda_1 = \lambda_2 = 0$ the function

$$P_L(t) = \frac{t_1 - t_2}{t_1 + t_2}$$

satisfies (3.2) for $\lambda = 0$.

3.2.3.3. Problem. What invariant operator corresponds to this solution? Describe all (any) of the nonpolynomial solutions of (3.2) and the corresponding operators.

4°. To select a reasonable type of r-nary operators is a good problem. Symmetric and anti-symmetric operators, as well as operators on λ-densities are the first choices but even in such simple cases there are few results. These results, though scanty, are rather interesting: quite unexpectedly, some of them are related to calculation of the N. Shapovalov determinant for the Virasoro algebra, cf. [FF1, FF2].

5°. Since the real forms of simple vectorial Lie algebras are only trivial ones (in the natural polynomial basis replace all complex coefficients with reals), the results for $\mathbb{R}$ and $\mathbb{C}$ are identical. In super cases for nontrivial real forms some new operators might appear.

3.2.3.4. Problem. To consider these cases is an open **Problem**.

3.2.4. Arity > 1. Grozman added a new dimension to Rudakov's solution of Veblen's problem: In 1978, he described all **bi**nary invariant differential operators ([G]). It turned out that there are plenty of them but not too many: modulo dualizations and permutations of arguments there are eight series of first order operators and several second and third order operators all of which are compositions of first order operators with one exception: the 3rd order irreducible Grozman operator on the line. There are no invariant bilinear operators of order > 3.

Miraculously, the 1st order differential operators determine, bar a few exceptions, a Lie superalgebra structure on their domain. (Here Lie superalgebras timidly indicated their usefulness in a seemingly non-super problem. Other examples, such as Quillen's proof of the index theorem, and several remarkable Witten's super observations followed soon.)

Limits of applicability of Rudakov's method. Though fans of Rudakov's method, [2] let us point out that its application to simple finite dimensional subalgebras of the algebras of vector fields is extremely voluminous computational job; therefore, it is ill applicable, say, to isometries of a Riemannian manifold or the group preserving the Laplace operator.

Fortunately, usually if Rudakov's method fails, one can apply other methods (Laplace-Casimir operators, the Shapovalov determinant, etc.).

3.2.5. Generalized tensors and primitive forms. In [R2], Rudakov considered differential operators invariant with respect to the Lie algebra of Hamiltonian vector fields on the symplectic manifold (M^{2n}, ω). Thanks to non-degeneracy of ω we can identify Ω^i with Ω^{n-i}. So the operator $d\colon \Omega^{n-i-1} \longrightarrow \Omega^{n-i}$ dual to the exterior differential $d\colon \Omega^i \longrightarrow \Omega^{i+1}$, and

[2] Interplay between restriction and induction functors goes back to Frobenius, but discovery of each instance deserves an acknowledgement, we presume.

which, as we already know, is another incarnation of d, looks like a completely new operator, $\delta\colon \Omega^{i+1} \longrightarrow \Omega^i$, the co-differential. There are also (proportional to each other) compositions $\delta\circ\omega\circ\delta$ and $d\circ\omega^{-1}\circ d$, where ω^{-1} is the convolution with the bivector dual to ω.

A novel feature is provided by the fact that "tensors" now are sections of the representation of $\mathfrak{sp}(V)$, not $\mathfrak{gl}(V)$. Since various representations of $\mathfrak{sp}(V)$ can not be extended to representations of $\mathfrak{gl}(V)$ these "tensors" are, strictly speaking, new notions.

Another novel feature we encounter considering subalgebras $\mathfrak{g}$ of $\mathfrak{vect}$ are *primitive forms*. If the $\mathfrak{vect}$-module $I(V)$ contains a singular vector with respect to $\mathfrak{vect}$, then it also contains a singular vector with respect to $\mathfrak{g}$. But the irreducible $\mathfrak{vect}_0$-module V does not have to remain irreducible with respect to submodule of the $\mathfrak{g}_0$. The $\mathfrak{g}_0$-irreducible component with the biggest highest weight in V is called the $\mathfrak{g}_0$-*primitive* (usually, briefly called just primitive) component. Examples: the primitive components appeared in symplectic geometry (we encounter their counterparts in finite dimensional purely odd picture as spherical harmonics, [Shap2, LSH1]) contact analogues of primitive forms are described in [Le3]. Spaces of primitive differential and integrable forms are just restrictions of the "usual" domain of the exterior differential; but for the other types of Lie superalgebras $\mathfrak{g}$ we may encounter new types of primitive tensors as domains of totally new invariant operators.

Further examples. A. Shapovalov and Shmelev considered the Lie *super*algebras of Hamiltonian vector fields and (following Bernstein who considered the non-super case) their central extension, the Poisson Lie superlagebra, see [Shap1], [Shap2] and [Sm2]–[Sm4]. Shmelev also considered the operators invariant with respect to the funny exceptional deformation $\mathfrak{h}_\lambda(2|2)$ of the Lie superalgebra $\mathfrak{h}(2|2)$ of Hamiltonian vector fields, cf. [Sm1]. For further description of $\mathfrak{h}_\lambda(2|2)$, see also [LSh].

By that time I. Kostrikin described singular vectors for the contact Lie algebras and found a "new" 2nd order invariant operator. This operator was actually well-known in differential geometry as an *Euler operator* (for its description, see [Ly]; here we just briefly observe that it is not $\sum x_i\partial_i$, this is another Euler operator); it is needed for invariant formulation of Monge-Ampére equations, cf. [LRC]. Leites [Le3] generalized I. Kostrikin's calculations to contact Lie *super*algebras and found out that there seem to be no analogue of Euler's operator in the super setting. This makes one contemplate on the following:

3.2.5.1. Problem. What are superanalogs of Monge-Ampére equations, if any?

In 1977, "odd" analogs of the hamiltonian and contact series ($\mathfrak{le}$, $\mathfrak{m}$, their divergence free subalgebras and their deformations) were discovered [Le2, ALSh]. Batalin and Vilkovisky rediscovered the antibracket related to these series and showed its importance, cf. [GPS]. In a series of papers, Kochetkov undertook the task of calculating the singular vectors in the modules $I(V)$ over the Lie superalgebras of series $\mathfrak{le}$, $\mathfrak{m}$, $\mathfrak{sle}$, $\mathfrak{sm}$ which constituted his Ph.D. thesis. He also considered two of the three known at that time Shchepochkina's exceptions (and named Ш$_1$ and Ш$_2$ after her using the first Cyrillic letter of her name), one of these exceptional Lie superalgebras was recently reconsidered in another realization in [KR1]. We have found out more singular vectors (= invariant operators) than Kochetkov did, even with finite dimensional fibers; `SuperLie` is indeed indispensable. We have also considered Ш$_2$; we reproduce Kochetkov's result for comparison. For the lack of resources we left out some possible cases of singular vectors, but we are sure they are improbable. Though for $\mathfrak{vle}(4|3)$ and $\mathfrak{mb}(4|5)$, all degrees can occur, we are sure induction à la Kochetkov (complete list of singular vectors) can be performed. Various W-regradings of $\mathfrak{ksle}$ seem to be a tougher problem.

3.2.6. Superization leads to new developments. The study of invariant differential operators on supermanifolds began in 1976 as a byproduct of attempts to construct an integration theory on supermanifolds similar to the integration theory of differential forms on manifolds. Bernstein and Leites became interested in Veblen's problem when they tried to construct an integration theory for supermanifolds containing an analog of the Stokes formula ([LSoS]). At that time there were only known the differential forms which are impossible to integrate and the volume forms of the highest degree. Unlike the situation on manifolds, no volume form coincides with any differential form, and there was known no analogs of volume forms of lesser degrees.

Having discovered *integrable* forms [LSoS] (i.e., the forms that can be integrated; Deligne calls them *integral* forms [D]) Bernstein and Leites wanted to be sure that there were no other tensor objects that can be integrated. Observe several points of this delicate question.

(1) The conventional Stokes formula on a manifold exists due to the fact that there is an invariant operator on the space of differential forms. The uniqueness of the integration theory with Stokes formula follows then from the above result by Rudakov and its superization due to Bernstein and Leites.

Since there are several superanalogs of the determinant, it follows that on supermanifolds, there are, perhaps, several analogs of integration theory, see [LInt], some of them without Stokes formula. Still, if we wish to construct an integration theory for supermanifolds containing an analog of the Stokes formula, and, moreover, coinciding with it when the supermanifold degenerates to a manifold, we have to describe all differential operators in tensor fields on supermanifolds.

(2) Bernstein and Leites confined themselves to the study of spaces of tensor fields $T(V)$ with finite dimensional modules V owing to the tradition which says that a tensor field is a section of a vector bundle with a finite dimensional fiber V on which the general linear group or its Lie algebra $\mathfrak{gl}(n)$ acts. Even sticking to the "traditional" definition, Bernstein and Leites had to digress somewhat from the conventions and consider, since it

was natural, ALL finite dimensional irreducible modules V over the general linear Lie superalgebra. Some of such representations can not be integrated to a representation of the general linear supergroup.

Inspired by Duflo, and courageous physicists who incorporated infinite dimensional representations in their household long ago, Leites used calculations of [BL2] to describe invariant differential operators acting in the superspaces of tensor fields with *infinite* dimensional fibers, see [LKV]. These operators of order > 1 are totally new, though similar to fiber-wise integration along the odd coordinates. The operators of order 1 are also not bad: though they are, actually, the good old exterior differential d, the new domain is that of semi-infinite forms, certain class of pseudodifferential forms. Observe that quite criminally (using V. I. Arnold's terms) no example of the corresponding new type homology is calculated yet (and this is an **open problem**), except some preliminary (but important) results of Shander, and Voronov and Zorich (see [LSoS]).

(3) Even under all the restrictions Bernstein and Leites imposed, to say that "the only invariant differential operator is just the exterior differential" would be to disregard how drastically they expanded its domain (even though they ignored semi-infinite possibilities). It acts in the superspace of differential forms and in the space of *integrable forms*, which is natural, since the space of integrable forms is just the dual space to the superspace of differential forms. Though Bernstein and Leites did not find any new invariant differential operator (*this proves that an integration theory on supermanifolds containing an analog of the Stokes formula can only be constructed with integrable forms*), they enlarged the domain of the exterior differential to the superspace of pseudodifferential and pseudointegrable forms. These superspaces are not *tensor fields* on $\mathcal{M}^{m,n}$ unless $n = 1$, but they are always tensor fields on the supermanifold $\widehat{\mathcal{M}}$ whose structure sheaf $\mathcal{O}_{\widehat{\mathcal{M}}}$ is a completion of the sheaf of differential forms on $\mathcal{M}$; namely, the sections of $\mathcal{O}_{\widehat{\mathcal{M}}}$ are arbitrary functions of differentials, not only polynomial ones.

(4) Bernstein and Leites did not consider indecomposable representations ρ which are more natural in both the supersetting and for infinite dimensional fibers. The first to consider indecomposable cases was Shmelev; [Sm4] his result was, however, "not interesting": there are no totally new operators, just compositions of the known ones with projections. For a review of indecomposable representations of simple Lie superalgebras, see [LInd].

3.2.6.1. Integration and invariant differential operators for infinite dimensional fibers. There are new operators invariant with respect to the already considered (super)groups of diffeomorphisms or, equivalently, their Lie superalgebras, if we let them act in the superspaces of sections of vector bundles with infinite dimensional fibers. These operators of high order have no counterparts on manifolds and are versions of the Berezin integral applied fiber-wise. A year after the talk with these results was delivered (see [LKV]), I. Penkov and V. Serganova interpreted some of these new operators

as acting in the superspaces of certain tensor fields on "curved" superflag and supergrassmann supervarieties, see [PS].

We hope to relate with some of these operators new topological invariants (or perhaps old, like cobordisms, but from a new viewpoint). Recall that since the de Rham cohomology of a supermanifold are the same as those of its underlying manifold, the "old type" operators are inadequate to study "topological" invariants of supermanifolds. The operators described here and related to vector bundles of infinite rank lead to new (co)homology theories (we prefix them with a "pseudo"). This pseudocohomology provide us with invariants different from de Rham cohomology; regrettably, never computed yet.

The approach adopted here for the operators in the natural bundles with *infinite dimensional* fibers on supermanifolds prompts us to start looking for same on manifolds. From the explicit calculations in Grozman's thesis,[G] it is clear that there are some new bilinear operators acting in the spaces of sections of tensor fields with infinite dimensional fibers.

3.2.7. An infinitesimal version of Veblen's problem. Let $\mathcal{F} = \mathbb{K}[[x]]$, where $x = (u_1, \dots, u_n, \xi_1, \dots, \xi_m)$ so that $p(u_i) = \bar{0}$ and $p(\xi_j) = \bar{1}$. Denote by (x) the maximal ideal in $\mathcal{F}$ generated by the x_i. Define a topology in $\mathcal{F}$ so that the ideals $(x)^r$, $r = 0, 1, 2, \dots$ are neighborhoods of zero, i.e., two series are *r-close* if they coincide up to order r. We see that $\mathcal{F}$ is complete with respect to this topology.

Denote by $\mathfrak{vect}(n|m)$ the Lie superalgebra of formal vector fields, i.e., of continuous derivations of $\mathbb{K}[[x]]$. By abuse of notations we denote $\mathfrak{der}\mathbb{K}[x]$, the Lie superalgebra of *polynomial* vector fields, also by $\mathfrak{vect}(n|m)$.

Define partial derivatives $\partial_i = \dfrac{\partial}{\partial x_i} \in \mathfrak{vect}(n|m)$ by setting $\partial_i(x_j) = \delta_{ij}$ with super-Leibniz rule. Clearly, $p(\partial_i) = p(x_i)$ and $[\partial_i, \partial_j] = 0$. Any element $D \in \mathfrak{vect}(n|m)$ is of the form $D = \sum f_i \partial_i$, where $f_i = D(x_i) \in \mathcal{F}$. We will denote $\mathfrak{vect}(n|m)$ by $\mathcal{L}$. In $\mathcal{L}$, define a filtration of the form $\mathcal{L} = \mathcal{L}_{-1} \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \dots$ setting

$$\mathcal{L}_r = \{D \in \mathfrak{vect}(n|m) \mid D(\mathcal{F}) \subset (x)^{r+1}\}.$$

This filtration defines a topology on $\mathcal{L}$, the superspaces $\mathcal{L}_r$ being the base of the topology, open neighborhoods of zero.

Denote by $L = \oplus L_r$, where $L_r = \mathcal{L}_r/\mathcal{L}_{r+1}$, the associated graded Lie superalgebra. Clearly, $L_0 \simeq \mathfrak{gl}(n|m)$ with $E_{ij} \longleftrightarrow x_j \partial_i$.

Let ρ be an irreducible representation of the Lie superalgebra $L_0 = \mathfrak{gl}(n|m)$ with lowest weight in a superspace V. Define a $\mathfrak{vect}(n|m)$-module $T(\rho)$ also denoted by $T(V)$ by setting $T(V) = \mathcal{F} \otimes_{\mathbb{K}} V$. The superspace $T(V)$ evidently inherits the topology of $\mathcal{F}$. To any vector field D, assign the operator $L_D\colon T(V) \longrightarrow T(V)$ — the Lie derivative — such that for $f \in \mathcal{F}$ and $v \in V$

$$L_D(fv) = D(f)v + (-1)^{p(D)p(f)} \sum D^{ij} \rho(E_{ij})(v), \tag{3.3}$$

where $D^{ij} = (-1)^{p(x_i)(p(f)+1)}\partial_i f_j$. We will usually write just D instead of L_D.

The elements $t \in T(V)$ will be called *tensor fields of type* V. The modules $T(V)$ are topological; their duals are spaces with discrete topology.

Observe that even if V is finite dimensional, the elements of $T(V)$ are generalized tensors as compared with the classical notion: the space V might not be realized in the tensor product of co- and contra-variant tensors, only as a subquotient of such; e.g., unlike the determinant (or trace, speaking on the Lie algebra level), the supertrace is not realized in tensors and we have to introduce new type of "tensors" — the λ-densities.

For any L_0-module V with highest weight and any L_0-module W with lowest weight set

$$I(V) = U(\mathcal{L}) \otimes_{U(\mathcal{L}_0)} V; \quad T(W) = \mathrm{Hom}_{U(\mathcal{L}_0)}(U(\mathcal{L}), \tag{3.4}$$

where we have extended the action of L_0 to a $U(\mathcal{L}_0)$-action by setting $\mathcal{L}_1 V = 0$ and $\mathcal{L}_1 W = 0$. Clearly,

a) $I(V)$ is an $\mathcal{L}$-module with discrete topology;

b) $(I(V))^* \cong T(V^*)$; observe that this — correct — answer differs from our "formally dual" answer $(T(V))^* := T(V^* \otimes \mathrm{str})$ which we had to invent wishing to confine dualization within the class of tensor fields with lowest weight vectors (analogously, "formally dual" answer in the dual category is $(I(W))^* := I(W^* \otimes \mathrm{str})$ for the highest weight modules);

c) definition of the tensor fields with $\mathcal{L}$-action (3.3) is equivalent to the one given by (3.4).

Thus, instead of studying invariant maps $T(W_1) \longrightarrow T(W_2)$ (or $T(W_1) \otimes T(W_2) \longrightarrow T(W_3)$, etc.) we may study submodules — or, equivalently, singular vectors — of $I(V)$ (resp. of $I(V_1) \otimes I(V_2)$, etc.). They are much easier to describe.

Further generalization of tensors. Let

$$\mathcal{L} = \mathcal{L}_{-d} \supset \dots \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \dots$$

be a Lie superalgebra of vector fields with formal or polynomial coefficients and endowed with a Weisfeiler filtration/grading described in what follows (for the time being consider a "most natural" grading, like that in $\mathfrak{vect}$ above). We define the space of *generalized* tensor fields and its dual by the same formula (3.4) as for the usual tensor fields given any L_0-module V with highest weight and any L_0-module W with lowest weight such that $\mathcal{L}_1 V = 0$ and $\mathcal{L}_1 W = 0$.

Observe that for the Lie algebra of divergence-free vector fields the spaces $T(W)$ are the same as for $\mathfrak{vect}$. For some other Lie superalgebras the notion of tensors we give is different because there are representations of L_0 distinct from tensor powers of the identity one. For example, for the Lie superalgebra $\mathcal{L}$ of Hamiltonian vector fields $\mathfrak{h}(2n|m)$ such is the spinor representation (for $n = 0$); if we consider infinite dimensional fibers such is the oscillator representation (for $m = 0$), and in the general case such is the spinor/oscillator representation, cf. [LSH1].

Thus, the first step in the study of $\mathcal{L}$-invariant operators is a description of irreducible L_0-modules, at least in terms of the highest/lowest weight. For the majority of the L_0's this is not a big deal, but the catch is that for some L_0's there is no (easy to formulate, or even none at all) highest/lowest weight theorem, even for finite dimensional modules. We will encounter this phenomenon with $\mathfrak{as}$, the linear part of $\mathfrak{vas}$.

An aside remark: being interested not only in representations of vectorial algebras (with polynomial coefficients) but in their stringy analogs (with Laurent coefficients), too, observe that vacuum over L_0 can be degenerate.

3.2.7.1. Problem. For all Weisfeiler gradings of simple vectorial superalgebras $\mathcal{L}$, describe conditions for the highest (lowest) weight under which the irreducible quotient of the Verma module over L_0 is finite dimensional and describe the corresponding module (say, in terms of a character formula, cf. [PS]).

Examples of generalized tensor fields. Clearly, for $\mathcal{L} = \mathfrak{vect}(n|m)$ we have $\mathcal{L} \equiv T(\mathrm{id})$, where $\mathrm{id} = \mathrm{Span}(\partial_i \mid 1 \leq i \leq n+m)$ is the (space V of the) identity representation of $L_0 = \mathfrak{gl}(V) = \mathfrak{gl}(n|m)$. The spaces $T(E^i(\mathrm{id}^*))$ are denoted by Ω^i; their elements are called *differential* i*-forms* and the right dual elements to ∂_i are denoted by $\widehat{x_i} = dx_i$, where $p(\widehat{x_i}) = p(x_i) + \bar{1}$. In particular, let $\mathcal{F} = \Omega^0$ be the algebra of functions.

The algebra $\widehat{\Omega}$ of arbitrary, not only polynomial, functions in $\widehat{x_i} = dx_i$ is called the algebra of *pseudodifferential forms*. An important, as Shander showed in [LSoS], subspace $\widehat{\Omega}^{(\lambda)}$ of homogeneous *pseudodifferential forms of homogeneity degree* $\lambda \in \mathbb{K}$ is naturally defined as functions of homogeneity degree λ with respect to the hatted indeterminates.

Define the space of volume forms Vol to be $T(\mathrm{str})$; denote the volume element by $\mathrm{vol}(x)$ or $\mathrm{vol}(u|\theta)$. (Observe again that it is a bad habit to denote, as many people still do, vol by $d^n u d^m \theta$: their transformation rules are totally different, see, e.g., [BL2], [D].)

The space of *integrable* i-forms is $\Sigma_i = \mathrm{Hom}_{\mathcal{F}}(\Omega^i, \mathrm{Vol})$. In other words, integrable forms are Vol-valued polyvector fields. Pseudointegrable forms are defined as elements of $\widehat{\Sigma} = \mathrm{Hom}_{\mathcal{F}}(\widehat{\Omega}, \mathrm{Vol})$; the subspace $\widehat{\Sigma}_{(\lambda)} = \mathrm{Hom}_{\mathcal{F}}(\widehat{\Omega}^{(\lambda)}, \mathrm{Vol})$ of homogeneous forms is also important.

Particular cases: a) $m = 0$. We see that $\Omega^i = 0$ for $i > n$ and $\Sigma_i = 0$ for $i < 0$. In addition, the mapping $\mathrm{vol} \mapsto \hat{x}_1 \cdots \hat{x}_n$ defines an isomorphism of Ω^i with Σ_i preserving all structures.

b) $n = 0$. In this case there is an even $\mathcal{L}$-module morphism $\int : \Sigma_{-m} \longrightarrow \mathbb{K}$ called the *Berezin integral*. It is defined by the formula

$$\int \xi_1 \cdots \xi_m \ \mathrm{vol} = 1, \text{and} \int \xi_1^{\nu_1} \cdots \xi_m^{\nu_m} \ \mathrm{vol} = 0 \quad \text{if} \quad \prod \nu_i = 0.$$

We will also denote by $\int$ the composition $\int : \Sigma_{-m} \to \mathbb{K} \hookrightarrow \Omega^0$ of the Berezin integral and the natural embedding.

c) $m=1$. We generalize Ω^i and Σ_j to the spaces Φ^λ of pseudointegrodifferential forms containing Ω^i and Σ_j, where $\lambda\in\mathbb{K}$. Let $x=(u_1,\cdots,u_n,\xi)$. Consider a $\mathbb{K}$-graded Ω-module $\Phi=\oplus\Phi^\lambda$ (we assume that $\deg\hat{x}_i=1\in\mathbb{K}$) generated by $\hat{\xi}^\lambda$, where $\deg\hat{\xi}^\lambda=\lambda$ and $p(\hat{\xi}^\lambda)=\bar{0}$, with relations $\hat{\xi}\cdot\hat{\xi}^\lambda=\hat{\xi}^{\lambda+1}$. Define the action of partial derivatives ∂_i and $\hat{\partial}_j$ for $1\leq i,j\leq n+1$ via $\hat{\partial}_j(x_i)=0$, $\partial_i(\hat{\xi}^\lambda)=0$, $\partial_{\hat{u}_i}(\hat{\xi}^\lambda)=0$ and $\partial_{\hat{\xi}}(\hat{\xi}^\lambda)=\lambda\hat{\xi}^{\lambda-1}$.

On Φ, the derivations d, $i_{\mathfrak{D}}$ and $L_{\mathfrak{D}}$ consistent with the exterior derivation d, the inner product $i_{\mathfrak{D}}$ and the Lie derivative $L_{\mathfrak{D}}$ on Ω are naturally defined.

It is easy to see that $\Phi=\oplus\Phi^\lambda$ is a supercommutative superalgebra.

Clearly, Φ is a superspace of tensor fields and for $\Phi^{\mathbb{Z}}=\oplus_{r\in\mathbb{Z}}\Phi^r$ we have a sequence

$$0\longrightarrow\Omega\xrightarrow{\alpha}\Phi^{\mathbb{Z}}\xrightarrow{\beta}\Sigma\longrightarrow 0 \tag{3.5}$$

where the maps α and β are defined by

$$\alpha(\omega)=\omega\hat{\xi}^0,\quad \beta(\hat{u}_1\cdots\hat{u}_n\hat{\xi}^{-1})=\mathrm{vol}.$$

Clearly, the homomorphisms α and β are consistent with the Ω-module structure and the operators d, $i_{\mathfrak{D}}$ and $L_{\mathfrak{D}}$. The explicit form of the $\mathcal{F}$-basis in Ω, Σ and Φ easily implies that (3.5) is exact.

3.2.8. Operators invariant with respect to nonstandard realizations. At the moment the $\mathcal{L}$-invariant differential operators are described for all but one series of simple vectorial Lie superalgebras in the standard realization. Contrariwise, about operators invariant with respect to same in nonstandard realizations almost nothing is known, except for $\mathfrak{vect}(m|n;1)$, see [LKV].

For series, the standard realization is the one for which $\dim\mathcal{L}/\mathcal{L}_0$ is minimal; for exceptional algebras the notion of the standard realization is more elusive, and since there are 1 to 4 realizations, it is reasonable and feasible to consider all of them. It is also natural to consider $\mathfrak{h}_\lambda(2|2)$ and $\mathfrak{h}_\lambda(2|2;1)$ as exceptional algebras, especially at exceptional values of λ.

3.2.9. On description of irreducible $\mathcal{L}$-modules. Having described $\mathfrak{vect}(n|m)$-invariant differential operators in tensor fields with finite dimensional fibers (answer: only d, and $\int$ if $n=0$, $m\neq 0$), we consider the quotients of $T(V)$ modulo the image of the invariant operator. It could be that the quotient also contains a submodule. In the general case there are no such submodules (Poincaré lemma), in other cases anything can happen.

Observe that to describe irreducible $\mathcal{L}$-modules, it does not always suffice to consider only one realization of $\mathcal{L}$. It is like considering generalized Verma modules induced or co-induced from distinct parabolic subalgebras. Similarly, the description of invariant operators must be performed from scratch in each realization.

Here we do not specifically consider the irreducible $\mathcal{L}$-modules; so far, the answers are known for tensors with finite dimensional fibers and in two cases only: [KR2] ($\mathfrak{vle}(3|6)$) and [Le3] ($\mathfrak{k}(1|n)$; weights of singular vectors are corrected below).

In what follows in this Chapter, m_1 is a nonzero highest weight vector in the irreducible $\mathfrak{g}_0$-module V.

3.3. Singular vectors for $\mathfrak{g}=\mathfrak{vle}(3|6)$

We denote the indeterminates by x (even) and ξ (odd); the corresponding partial derivatives by ∂ and δ. The Cartan subalgebra is spanned by

$$\begin{aligned}
h_1&=-2\,x_4\otimes\partial_4-\xi_1\otimes\delta_1-\xi_2\otimes\delta_2-\xi_3\otimes\delta_3,\\
h_2&=-x_2\otimes\partial_2-x_3\otimes\partial_3-\xi_1\otimes\delta_1,\\
h_3&=-x_1\otimes\partial_1-x_3\otimes\partial_3-\xi_2\otimes\delta_2,\\
h_4&=-x_1\otimes\partial_1-x_2\otimes\partial_2-\xi_3\otimes\delta_3
\end{aligned}$$

We consider the following negative operators from $\mathfrak{g}_0$:

$$\begin{aligned}
&a_1=\partial_4,\\
&a_{12}=-{x_4}^2\partial_4-x_4\xi_1\delta_1-x_4\xi_2\delta_2-x_4\xi_3\delta_3+\xi_1\xi_2\partial_3-\xi_1\xi_3\partial_2+\xi_2\xi_3\partial_1
\end{aligned}$$

$$\begin{array}{ll}
a_2=-x_1\partial_1-x_2\partial_2-x_3\partial_3+x_4\partial_4, & a_3=x_2\partial_2+x_3\partial_3+\xi_1\delta_1\\
a_4=-x_2\partial_1+\xi_1\delta_2, & a_5=-x_3\partial_1+\xi_1\delta_3\\
a_6=-x_1\partial_2+\xi_2\delta_1, & a_7=x_1\partial_1+x_3\partial_3+\xi_2\delta_2\\
a_8=-x_3\partial_2+\xi_2\delta_3, & a_9=-x_1\partial_3+\xi_3\delta_1\\
a_{10}=-x_2\partial_3+\xi_3\delta_2, & a_{11}=x_1\partial_1+x_2\partial_2+\xi_3\delta_3
\end{array}$$

and the operators from $\mathfrak{g}_-$:

$$\begin{array}{lll}
n_1=\delta_1, & n_4=x_4\delta_3-\xi_1\partial_2+\xi_2\partial_1, & n_7=\partial_1\\
n_2=\delta_2, & n_5=\ x_4\delta_2-\xi_1\partial_3+\xi_3\partial_1, & n_8=\partial_2\\
n_3=\delta_3, & n_6=x_4\delta_1-\xi_2\partial_3+\xi_3\partial_2, & n_9=\partial_3
\end{array}$$

The m_i are the following elements of the irreducible $\mathfrak{g}_0$-module V:

$$\begin{array}{ll}
m_1 \text{ is the highest weight vector} & m_9=a_4\cdot a_8\cdot m_1\\
m_2=a_{12}\cdot m_1 & m_{10}=a_8\cdot a_8\cdot m_1\\
m_3=a_4\cdot m_1 & m_{11}=a_5\cdot m_1\\
m_4=a_8\cdot m_1 & m_{12}=a_{12}\cdot a_{12}\cdot a_{12}\cdot m_1\\
m_5=a_{12}\cdot a_{12}\cdot m_1 & m_{13}=a_{12}\cdot a_{12}\cdot a_4\cdot m_1\\
m_6=a_{12}\cdot a_4\cdot m_1 & m_{14}=a_{12}\cdot a_{12}\cdot a_8\cdot m_1\\
m_7=a_{12}\cdot a_8\cdot m_1 & m_{15}=a_{12}\cdot a_4\cdot a_4\cdot m_1\\
m_8=a_4\cdot a_4\cdot m_1 & m_{16}=a_{12}\cdot a_4\cdot a_8\cdot m_1
\end{array}$$

Theorem. *In $I(V)$, there are only the following singular vectors of degree d):*

1a) $(k,k,l,l)\longrightarrow(k+1,k+1,l,l)$*:* $n_1\otimes m_1$*;*

1b) $(k,k,k-1,k-1)\longrightarrow(k+1,k,k,k-1)$*:* $n_2\otimes m_1+n_1\otimes m_3$*;*

1c) $(k-3,k,k,k-1)\longrightarrow(k-2,k,k,k)$: $n_3\otimes m_1+n_2\otimes m_4+n_1\otimes m_9$;

1d) $(-k,k-2,l,1)\longrightarrow(-k-1,k-1,l,l)$, *where* $k\neq 0$

$$k\,n_6\otimes m_1+n_1\otimes m_2;$$

1e) $(-k,k-2,k-1,k-1)\longrightarrow(-k-1,k-2,k,k)$, *where* $k\neq 0,-1$

$$k\,(n_5\otimes m_1+n_6\otimes m_3)-n_2\otimes m_2+n_1\otimes m_6;$$

1f) $(-k,k+1,k+1,k)\longrightarrow(-k-1,k+1,k+1,k+1)$, *where* $k\neq 0$

$$k\,(n_4\otimes m_1-n_5\otimes m_4+n_6\otimes m_9)+n_3\otimes m_2+n_2\otimes m_7+n_1\otimes m_{16};$$

2a) $(0,-2,k,k)\longrightarrow(0,0,k-1,k-1)$: $(n_6{\cdot}n_1)\otimes m_1$;

2b) $(0,-2,0,0)\longrightarrow(0,-1,1,0)$:

$$(n_5{\cdot}n_1-n_6{\cdot}n_2)\otimes m_1+(n_6{\cdot}n_1)\otimes m_3$$

2c) $(1,0,0,-1)\longrightarrow(1,1,0,0)$:

$$-(n_8+n_4{\cdot}n_1)\otimes m_1+(n_3{\cdot}n_1)\otimes m_2+(n_9+n_5{\cdot}n_1)\otimes m_4+(n_2{\cdot}n_1)\otimes m_7-(n_6{\cdot}n_1)\otimes m_9$$

2d) $(-3,0,0,-1)\longrightarrow(-3,1,0,0)$

$$\begin{aligned}&(n_8+n_4{\cdot}n_1-2\,(n_6{\cdot}n_3))\otimes m_1+(n_3{\cdot}n_1)\otimes m_2\\&-(n_9+n_5{\cdot}n_1+2\,(n_6{\cdot}n_2))\otimes m_4+(n_2{\cdot}n_1)\otimes m_7-(n_6{\cdot}n_1)\otimes m_9\end{aligned}$$

2e) $(0,k,-1,k+1)\longrightarrow(0,k,1,k+1)$ *(The dual cases were not calculated.)*

$$-k\,(1+k)\,(n_5{\cdot}n_2)\otimes m_1-k\,(n_5{\cdot}n_1-k\,n_6{\cdot}n_2)\otimes m_3+(n_6{\cdot}n_1)\otimes m_8$$

3a) $(k-2,k,k,k)\longrightarrow(k+1,k+1,k+1,k+1)$: $(n_3{\cdot}n_2{\cdot}n_1)\otimes m_1$;

3b) $(-3,-1,-1,-1)\longrightarrow(-2,0,0,0)$ *(The dual cases were not calculated.)*

$$(n_7{\cdot}n_1+n_8{\cdot}n_2+n_9{\cdot}n_3-n_4{\cdot}n_2{\cdot}n_1-n_5{\cdot}n_3{\cdot}n_1-n_6{\cdot}n_3{\cdot}n_2)\otimes m_1-(n_3{\cdot}n_2{\cdot}n_1)\otimes m_2$$

3.4. Singular vectors for $\mathfrak{g}=\mathfrak{vle}(4|3)$

Here $\mathfrak{g}=\mathfrak{vle}(4|3)$, former $\mathfrak{m}_1$. In $\mathfrak{g}_0=\mathfrak{c}(\mathfrak{vect}(0|3))$ considered in the standard grading, we take the usual basis of Cartan subalgebra, $\xi_i\dfrac{\partial}{\partial_i}$, and z we identify the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ with $\Pi(\Lambda(\xi)/\mathbb{C}\cdot 1)$, by setting

$$\begin{aligned}&\partial_i=\Pi(\xi_i);\quad \partial_0=\Pi(\xi_1\xi_2\xi_3),\\&\delta_i=\mathrm{sign}(ijk)\Pi(\xi_j\xi_k)\text{ for }(i,j,k)\in S_3.\end{aligned}$$

We consider the following negative operators from $\mathfrak{g}_0$:

$$\begin{array}{ll}y_1=-x_1\partial_2+\xi_2\delta_1 & y_2=-x_2\partial_3+\xi_3\delta_2\\ y_3=-x_0\delta_3-\xi_1\partial_2+\xi_2\partial_1 & y_4=x_1\partial_3-\xi_3\delta_1\\ y_5=x_0\delta_2-\xi_1\partial_3+\xi_3\partial_1 & y_6=x_0\delta_1+\xi_2\partial_3-\xi_3\partial_2\end{array}$$

and the basis of Cartan subalgebra:

$$\begin{aligned}h_0&=x_1\partial_1+x_2\partial_2+x_3\partial_3+x_0\partial_0+\xi_1\delta_1+\xi_2\delta_2+\xi_3\delta_3\\h_1&=x_2\partial_2+x_3\partial_3+\xi_1\delta_1\\h_2&=x_1\partial_1+x_3\partial_3+\xi_2\delta_2\\h_3&=x_1\partial_1+x_2\partial_2+\xi_3\delta_3\end{aligned}$$

The m_i are the following elements of the irreducible $\mathfrak{g}_0$-module V:

$$\begin{array}{lll}m_2=y_1\,m_1 & m_{17}=y_5\,m_1 & m_{54}=y_2\,y_3\,y_4\,m_1\\ m_3=y_2\,m_1 & m_{26}=y_1\,y_5\,m_1 & m_{57}=y_3\,y_5\,m_1\\ m_7=y_3\,m_1 & m_{31}=y_3\,y_4\,m_1 & m_{82}=y_1\,y_3\,y_5\,m_1\\ m_8=y_4\,m_1 & m_{32}=(y_4)^2\,m_1 & m_{94}=y_3\,y_6\,m_1\\ m_{12}=y_1\,y_3\,m_1 & m_{33}=y_6\,m_1 & m_{148}=y_3\,y_4\,y_5\,m_1\\ m_{15}=y_2\,y_3\,m_1 & m_{46}=y_1\,y_2\,y_5\,m_1 & m_{150}=y_5\,y_6\,m_1\end{array}$$

and $m_{320}=y_3\,y_5\,y_6\,m_1$. Observe that our choice of ordering obscures the fact that the vectors m_{129}, m_{148}, and m_{150} are proportional.

Theorem. *In $I(V)$ in degrees indicated, there are only the following singular vectors:*

1a) $\lambda\longrightarrow\lambda+(-1,1,1,1)$, *where* $2\lambda_1=\lambda_2+\lambda_3+\lambda_4$: $\partial_0\,m_1$

1b) $(k,l,l,k-l+1)\longrightarrow(k-1,l+1,l+1,k-l+1)$: $\partial_0\,m_7+(k-l)\delta_3)\,m_1$

1c) $(k,k-1,1,k)\longrightarrow(k-1,k,1,k+1)$:

$$\partial_0\,(m_{15}+(k-2)m_{17})-(k-1)\delta_2\,m_1+\delta_3\,m_3$$

1d) $(k,l,k-l,k-l)\longrightarrow(k-1,l,k-l+1,k-l+1)$:

$$\partial_0\,(m_{26}+m_{31}+(1-k+2l)m_{33})+((k-2l)(1+l))\delta_1\,m_1-(l+1)\delta_2\,m_2+(l+1)\delta_3\,m_8$$

1e) $(1,1,0,0)\longrightarrow(0,1,1,0)$:

$$-2\partial_1\,m_2+2\partial_2\,m_1+\partial_0\,(m_{82}+2m_{94})+2\delta_1\,m_7+2\delta_2\,m_{12}+\delta_3\,(m_{26}-m_{31}+2m_{33})$$

1f) $(0,0,0,-1)\longrightarrow(-1,0,0,0)$:

$$\begin{aligned}&-\partial_1\,m_8+\partial_2\,m_3-\partial_3\,m_1\\&+\partial_0\,(m_{129}+m_{148})+\delta_1\,(2m_{15}-m_{17})+\delta_2\,(2m_{31}-m_{33})+\delta_3\,(m_{46}+m_{54})\end{aligned}$$

2a) $\lambda \longrightarrow \lambda + (-2,2,2,2)$, *where* $\lambda_1 = \lambda_2 + \lambda_3 + \lambda_4 + 1$: $\partial_0^2\, m_1$
2b) $(k, k-2, k-2, 3) \longrightarrow (k-2, k, k, 4)$: $\partial_0^2\, m_7 + 2\partial_0\, \delta_3\, m_1$
2c) $(k, k-2, 2, k-1) \longrightarrow (k-2, k, 3, k+1)$:

$$\partial_0^2\,(m_{15} + (k-4)m_{17}) - 2(k-3)\partial_0\,\delta_2\,m_1 + 2\partial_0\,\delta_3\,m_3$$

2d) $(k, 1, k-1, k-1) \longrightarrow (k-2, 2, k+1, k+1)$:

$$\partial_0^2\,(m_{26} + m_{31} + (3-k)m_{33}) + 2(k-2)\partial_0\,\delta_1\,m_1 - 2\partial_0\,\delta_2\,m_2 + 2\partial_0\,\delta_3\,m_8$$

2e) $(2,1,1,1) \longrightarrow (0,2,2,2)$:

$$\begin{aligned}&\partial_0^2\, m_{320} + 2\partial_1\,\partial_0\,(m_{26}+m_{31}) + 2\partial_1\,\delta_1\,m_1 - 2\partial_1\,\delta_2\,m_2\\ &+2\partial_1\,\delta_3\,m_8 - 2\partial_2\,\partial_0\,m_{15} + 2\partial_2\,\delta_2\,m_1 - 2\partial_2\,\delta_3\,m_3\\ &+2\partial_3\,\partial_0\,m_7 + 2\partial_3\,\delta_3\,m_1 - 2\partial_0\,\delta_1\,m_{57} + \partial_0\,\delta_2\,(-m_{82}+m_{94})\\ &+\partial_0\,\delta_3\,(m_{148}+2m_{150}) - 2\delta_1\,\delta_2\,m_7 - 2\delta_1\,\delta_3\,m_{17} + 2\delta_2\,\delta_3\,m_{33}\end{aligned}$$

3a) $\lambda \longrightarrow \lambda + (-3,3,3,3)$, *where* $\lambda_1 = \lambda_2 + \lambda_3 + \lambda_4 + 2$: $\partial_0^3\, m_1$
3b) $(k, k-3, k-3, 4) \longrightarrow (k-3, k, k, 6)$: $\partial_0^3\, m_7 + 3\partial_0^2\,\delta_3\,m_1$
3c) $(k, k-3, 3, k-2) \longrightarrow (k-3, k, 5, k+1)$:

$$\partial_0^3\,(m_{15} + (k-6)m_{17}) - 3(k-5)\partial_0^2\,\delta_2\,m_1 + 3\partial_0^2\,\delta_3\,m_3$$

3d) $(k, 2, k-2, k-2) \longrightarrow (k-3, 4, k+1, k+1)$:

$$\partial_0^3\,(m_{26} + m_{31} + (5-k)m_{33}) + 3(k-4)\partial_0^2\,\delta_1\,m_1 - 3\partial_0^2\,\delta_2\,m_2 + 3\partial_0^2\,\delta_3\,m_8$$

3e) $(2k, k, k, k) \longrightarrow (2k-3, k+2, k+2, k+2)$:

$$\begin{aligned}&\partial_0^3\,m_{320} + (k+1)\partial_1\,\partial_0^2\,(m_{26}+m_{31}) + k(k+1)\partial_1\,\partial_0\,\delta_1\,m_1\\ &-k(k+1)\partial_1\,\partial_0\,\delta_2\,m_2 + k(k+1)\partial_1\,\partial_0\,\delta_3\,m_8\\ &-(k+1)\partial_2\,\partial_0^2\,m_{15} + k(k+1)\partial_2\,\partial_0\,\delta_2\,m_1\\ &-k(k+1)\partial_2\,\partial_0\,\delta_3\,m_3 + (k+1)\partial_3\,\partial_0^2\,m_7\\ &+k(k+1)\partial_3\,\partial_0\,\delta_3\,m_1 - (k+1)\partial_0^2\,\delta_1\,m_{57}\\ &+\partial_0^2\,\delta_2\,(-m_{82}+km_{94}) + \partial_0\,\delta_3\,(m_{148}+(k+1)m_{150})\\ &-k(k+1)\partial_0\,\delta_1\,\delta_2\,m_7 - k(k+1)\partial_0\,\delta_1\,\delta_3\,m_{17}\\ &+k(k+1)\partial_0\,\delta_2\,\delta_3\,m_{33} - (k-1)k(k+1)\delta_1\,\delta_2\,\delta_3\,m_1\end{aligned}$$

Remarks. Cases a) have an obvious generalization to any degree. Some expressions can be shortened by an appropriate ordering of the elements of the enveloping algebra, in other words, some vectors represent zero, e.g., m_2, m_3, m_8, m_{15} in cases 2e) and 3e). We did not always perform such renormalization; the cases 2e) and 3e) are left as they are to entertain the reader. The following case — $\mathfrak{g} = \mathfrak{mb}(4|5)$ — is strikingly similar.

3.5. Singular vectors for $\mathfrak{g} = \mathfrak{mb}(4|5)$ (after Kochetkov)

Here $\mathfrak{g} = \mathfrak{mb}(4|5)$, former Ш$_2$. Recall that in terms of generating functions we identify the $\mathfrak{g}_0$-module $\mathfrak{g}_{-2}$ with $\Pi(\mathbb{C}\cdot 1)$; we denote by $\mathbf{1} \in \mathfrak{m}(4)$ the image of $\Pi(1)$; so $f\mathbf{1}$ denotes $M_f M_1$. We identify $\mathfrak{g}_{-1}$ with $\Pi(\Lambda(\xi))$ by setting

$$\begin{aligned}&x_0 = \Pi(1), \quad x_i = \operatorname{sign}(ijk)\Pi(\xi_j\xi_k) \text{ for } (i,j,k) \in S_3,\\ &\eta_0 = \Pi(\xi_1\xi_2\xi_3), \quad \eta_i = \Pi(\xi_i).\end{aligned}$$

Let V be an irreducible finite dimensional $\mathfrak{g}_0$-module with highest weight Λ, and v_Λ the corresponding vector; let $f \in I(V)$ be a nonzero singular vector.

3.5.1. Theorem. *In $I(V)$, there are only the following singular vectors:*

1) $\Lambda = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 3)$, *and* $m = 1$ *or* 3

$$\eta_0^m \otimes v + \sum_{1\le i\le 3} \eta_0^{m-1} x_i \otimes v_i + \sum_{1\le i\le 3} \eta_0^{m-1}\eta_i \otimes w_i + \eta_0^{m-1} x_0 \otimes v_\Lambda$$

2) $\Lambda = (0,0,0;2)$ *and* $m = 3$

$$\eta_0(\sum_{0\le i\le 3} \eta_i x_i) \otimes v_\Lambda + 2\eta_0\mathbf{1} \otimes v_\Lambda;$$

3) $\Lambda = (0,0,0;a)$: $\partial_0^2 \otimes v_\Lambda$
4) $\Lambda = (0,0,-1;0)$ *for* $m = 1$ *or* 3

$$\partial_0^m \otimes v_\Lambda + \sum_{1\le i\le 3} \partial_0^{m-1}\delta_i \otimes w_i + \sum_{1\le i\le 3} \partial_0^{m-1}\partial_j \otimes v_j, \text{ where } v_3 = v_\Lambda.$$

3.5.2. Remark. In Kochetkov's paper no description of the v_j and w_j is given; now we can compare his result with our latest result:

We give the weights with respect to the following basis of Cartan subalgebra:

$$\begin{aligned}&h_1 = \tau; && h_2 = -q_0\,\xi_0 + q_1\,\xi_1,\\ &h_3 = -q_0\,\xi_0 + q_2\,\xi_2, && h_4 = -q_0\,\xi_0 + q_3\,\xi_3.\end{aligned}$$

For the negative elements of $\mathfrak{g}_0$ we take

$$\begin{aligned}&y_1 = q_2\,\xi_1, && y_2 = q_3\,\xi_2, && y_3 = -q_0\,q_1 + \xi_2\,\xi_3,\\ &y_4 = -q_3\,\xi_1, && y_5 = -q_0\,q_2 - \xi_1\,\xi_3, && y_6 = -q_0\,q_3 + \xi_1\,\xi_2.\end{aligned}$$

The m_i are the following elements of the irreducible $\mathfrak{g}_0$-module V:

$$\begin{array}{lll} m_2 = y_1\, m_1, & m_{17} = y_5\, m_1 & m_{57} = y_3\, y_5\, m_1 \\ m_3 = y_2\, m_1, & m_{24} = y_1\, y_2\, y_3\, m_1 & m_{91} = y_2\, y_3\, y_5\, m_1 \\ m_5 = y_1\, y_2\, m_1 & m_{30} = y_2\, y_5\, m_1 & m_{94} = y_3\, y_6\, m_1 \\ m_7 = y_3\, m_1, & m_{31} = y_3\, y_4\, m_1 & m_{148} = y_3\, y_4\, y_5\, m_1 \\ m_8 = y_4\, m_1, & m_{33} = y_6\, m_1 & m_{150} = y_5\, y_6\, m_1 \\ m_{12} = y_1\, y_3\, m_1 & m_{320} = y_3\, y_5\, y_6\, m_1 & \end{array}$$

Observe that our choice of ordering obscures the fact that some of the vectors either are proportional or represent zero.

3.5.3. Theorem. *In $I(V)$, there are only the following singular vectors:*

1a) $\lambda \longrightarrow \lambda + (-1,1,1,1)$, *where* $\lambda_1 = \lambda_2 + \lambda_3 + \lambda_4$*:* $\xi_0\, m_1$.

1b) $(-k+2l-2, k, l, l) \longrightarrow (-k+2l-3, k+1, l, l)$: $-kq_1\, m_1 + \xi_0\, m_7$

1c) $(k, k, k+1, k+1) \longrightarrow (k-1, k, k+2, k+1)$:

$$-kq_1\, m_2 - kq_2\, m_1 + \xi_0\, m_{12}$$

1d) $(2k+1, k, 0, k+1) \longrightarrow (2k, k, 1, k+1)$

$$q_1\, m_2 - kq_2\, m_1 + \xi_0\, (m_{12} - (k+1)m_{17})$$

1e) $(k+3, k, k, k-1) \longrightarrow (k+2, k, k, k)$

$$-(k-3)q_1(m_5+m_8)+2(k-3)q_2m_3-2(k-3)q_3m_1+\xi_0(m_{24}-3m_{30}+m_{31}+5m_{33})$$

1f) $(2k+1, k, k, 1) \longrightarrow (2k, k, k, 2)$

$$\begin{aligned} &q_1\, (m_5 + (k-1)m_8) - kq_2\, m_3 + k(k-1)q_3\, m_1 \\ &+\xi_0\, (m_{24} - (k+1)m_{30} + (k-1)m_{31} + (k^2+1)m_{33}) \end{aligned}$$

1g) $(3, 1, 1, 1) \longrightarrow (2, 0, 0, 0)$

$$\begin{aligned} &q_0\, m_1 - q_1\, (m_{132} + m_{148}) + q_2\, m_{91} - q_3\, m_{57} - \xi_0\, m_{320} \\ &+\xi_1\, m_7 + \xi_2\, m_{12} - \xi_3\, (-m_{30} + m_{31}) \end{aligned}$$

2a) $\lambda \longrightarrow \lambda + (-2,2,2,2)$, *where* $\lambda_1 = \lambda_2 + \lambda_3 + \lambda_4 + 2$*:* $\xi_0^2\, m_1$.

2b) $(2k, -2, k, k) \longrightarrow (2k-2, 0, k+1, k+1)$*:* $\xi_0^2\, m_7 + 2q_1\, \xi_0\, m_1$.

2c) $(2k, k-1, -1, k) \longrightarrow (2k-2, k, 1, k+1)$

$$\xi_0^2\, (m_{12} - (k+1)m_{17}) + 2q_1\, \xi_0\, m_2 - 2kq_2\, \xi_0\, m_1$$

2d) $(2k+2, k, k, 0) \longrightarrow (2k, k+1, k+1, 2)$

$$\begin{aligned} &\xi_0^2\, (-m_{30} + m_{31} + (k+1)m_{33}) + q_1\, \xi_0\, (m_5 + m_8) \\ &-2q_2\, \xi_0\, m_3 + 2kq_3\, \xi_0\, m_1 \end{aligned}$$

2e) $(2k, 0, 0, 0) \longrightarrow (2k-2, 0, 0, 0)$

$$((3-k) + q_0\, \xi_0 + q_1\, \xi_1) + q_2\, \xi_2 + q_3\, \xi_3)\, m_1$$

3a) $\lambda \longrightarrow \lambda + (-3,3,3,3)$, *where* $\lambda_1 = \lambda_2 + \lambda_3 + \lambda_4 + 4$*:* $\xi_0\, m_1$.

3b) $(2k+1, -3, k, k) \longrightarrow (2k-2, 0, k+2, k+2)$*:* $\xi_0^3\, m_7 + 3q_1\, \xi_0^2\, m_1$.

3c) $(2k+1, k+1, -2, k) \longrightarrow (2k-2, k+4, 1, k+2)$

$$\xi_0^3\, (m_{12} - (k+2)m_{17}) + 3q_1\, \xi_0^2\, m_2 - 3(k+1)q_2\, \xi_0^2\, m_1$$

3d) $(2k+3, k, k, -1) \longrightarrow (2k, k+2, k+2, 2)$

$$\begin{aligned} &\xi_0^3\, (m_{24} - 2m_{30} + (3+k)m_{33}) + q_1\, \xi_0^2\, ((2-k)m_5 + (1+k)m_8) \\ &-3q_2\, \xi_0^2\, m_3 + 3(1+k)q_3\, \xi_0^2\, m_1 \end{aligned}$$

3e) $(2, 0, 0, 0) \longrightarrow (-1, 1, 1, 1)$

$$(q_0\, \xi_0^2 + 3\xi_0^3 + q_1\, \xi_0\, \xi_1 + q_2\, \xi_0\, \xi_2 + q_3\, \xi_0\, \xi_3)\, m_1$$

3f) $(k+2, k, k, k) \longrightarrow (k-1, k+1, k+1, k+1)$

$$\begin{aligned} &\xi_0^3\, m_{320} + (-2+k)q_0\, \xi_0^2\, m_1 + q_1\, \xi_0^2\, (m_{132} + m_{148} + (1-k)m_{150}) \\ &+q_2\, \xi_0^2\, (-m_{91} + (-1+k)m_{94}) + (2-k)q_3\, \xi_0^2\, m_{57} \\ &-(-3+k)(-2+k)\xi_0\, m_1 + (-2+k)\xi_0^2\, \xi_1\, m_7) + (-2+k)\xi_0^2\, \xi_2\, m_{12} \\ &+\xi_0^2\, \xi_3\, ((-2+k)m_{30} + (2-k)m_{31}) - (-2+k)(-1+k)kq_1\, q_2\, q_3\, m_1 \\ &+q_1\, q_2\, \xi_0\, (m_{24} - km_{30} + (-2+k)m_{31} + (2-2k+k^2)m_{33}) \\ &+q_1\, q_3\, \xi_0\, ((-2+k)m_{12} - (-2+k)km_{17}) - (-2+k)(-1+k)q_1\, \xi_0\, \xi_1\, m_1 \\ &-(-2+k)kq_1\, \xi_0\, \xi_2\, m_2 + q_1\, \xi_0\, \xi_3\, ((2-k)m_5 + (-2+k)km_8) \\ &+(-2+k)(-1+k)(q_2\, q_3\, \xi_0\, m_7 - (-2+k)(-1+k)q_2\, \xi_0\, \xi_2\, m_1 \\ &-(-2+k)(-1+k)q_2\, \xi_0\, \xi_3\, m_3 - (-2+k)(-1+k)q_3\, \xi_0\, \xi_3\, m_1 \end{aligned}$$

3.6. Singular vectors for $\mathfrak{g} = \mathfrak{mb}(3|8)$

We give the weights with respect to the following basis of Cartan subalgebra:

$$\begin{aligned} &H_1 = \frac{1}{2}\tau + \frac{3}{2}q_1\xi_1 - \frac{1}{2}q_2\xi_2 - \frac{1}{2}q_3\xi_3 - \frac{1}{2}q_4\xi_4; \\ &H_2 = -q_1\xi_1 + q_2\xi_2, \quad H_3 = -q_1\xi_1 + q_3\xi_3, \quad H_3 = -q_1\xi_1 + q_4\xi_4. \end{aligned}$$

The basis elements of $\mathfrak{g}_-$ are denoted by

$$\begin{array}{llll} q_0 & q_1 & q_2 & q_3, \\ I & \xi_1 & \xi_2 & \xi_3, \end{array}$$

and

$$A = -q_0 q_1 + \xi_2\xi_3, \quad B = -q_0 q_2 - \xi_1\xi_3, \quad C = -q_0 q_3 + \xi_1\xi_2.$$

3.6.1. Theorem. *In $I(V)$, there are only the following singular vectors calculated up to dualization (though some dual vectors are also given):*

1a) $(k, 0, l, l) \longrightarrow (k+1, 0, l-1, l-1)$: $A \otimes m_1$;

1b) $(k, l, -k, l+1) \longrightarrow (k-1, l, -k+1, l+1)$, *where* $k \neq 0, -l$

$$k\, q_1 \otimes (q_2\xi_1 \cdot m_1) - (k+l)\, B \otimes (\xi_0 \cdot m_1) - k\ (k+l)\ q_2 \otimes m_1 + A \otimes (\xi_0 \cdot q_2\xi_1 \cdot m_1).$$

1c) $(k, l, l, 2) \longrightarrow (k+1, l-1, l-1, 2)$, *where* $l \neq 2$

$$A \otimes (q_2\xi_1 \cdot q_3\xi_2 \cdot m_1) - B \otimes (q_3\xi_2 \cdot m_1) - (2-l)\, C \otimes m_1.$$

1d) $(k, l, l, 1-k) \longrightarrow (k-1, l, l, 2-k)$, *where* $k + l \neq 1$

$$-k\, q_1 \otimes (q_3\xi_1 \cdot m_1) - B \otimes (\xi_0 \cdot q_3\xi_2 \cdot m_1) - k\ (1-k-l)\ q_3 \otimes m_1$$
$$-(1-k-l)\, C \otimes (\xi_0 \cdot m_1) - k\, q_2 \otimes (q_3\xi_2 \cdot m_1) - A \otimes (\xi_0 \cdot q_3\xi_1 \cdot m_1).$$

1e) $(k, l, 1, l+1) \longrightarrow (k+1, l-1, 1, l-1)$, *where* $l \neq 1$

$$(1-l)\, B \otimes m_1 + A \otimes (q_2\xi_1 \cdot m_1).$$

1f) $(k, -k-1, l, l) \longrightarrow (k-1, -k+1, l, l)$, *where* $k \neq 0$

$$k\, q_1 \otimes m_1 + A \otimes (\xi_0 \cdot m_1).$$

2a) $(0, -1, l, l) \longrightarrow (0, 0, l-1, l-1)$: $(q_1 \cdot A) \otimes m_1$;

2b) $(0, -1, 1, 1) \longrightarrow (0, -1, 1, 0)$

$$(q_2 \cdot A + q_1 \cdot B) \otimes m_1 + (q_1 \cdot A) \otimes (q_2\xi_1 \cdot m_1).$$

2c) $(2, 0, 0, -1) \longrightarrow (2, 0, -1, -1)$

$$2\,\xi_2 \otimes m_1 + 2\,\xi_3 \otimes (q_3\xi_2 \cdot m_1) - 2\ (q_1 \cdot A) \otimes (q_2\xi_1 \cdot q_3\xi_2 \cdot m_1) + 2\ (q_2 \cdot A) \otimes (q_3\xi_2 \cdot m_1)$$
$$-2\ (q_3 \cdot A) \otimes m_1 + (B \cdot A) \otimes (\xi_0 \cdot q_3\xi_2 \cdot m_1) - (C \cdot A) \otimes (\xi_0 \cdot m_1)$$

2d) $(-4, 3, 3, 2) \longrightarrow (-4, 3, 2, 2)$

$$-2\ (q_1 \cdot A) \otimes (q_2\xi_1 \cdot q_3\xi_2 \cdot m_1) - \ \xi_2 \otimes m_1 - \ \xi_3 \otimes (q_3\xi_2 \cdot m_1) - \ (C \cdot A) \otimes (\xi_0 \cdot m_1)$$
$$-3\,(q_1 \cdot C) \otimes m_1 + \ (q_3 \cdot A) \otimes m_1 + 3\ (q_1 \cdot B) \otimes (q_3\xi_2 \cdot m_1) - (q_2 \cdot A) \otimes (q_3\xi_2 \cdot m_1)$$
$$+ (B \cdot A) \otimes (\xi_0 \cdot q_3\xi_2 \cdot m_1).$$

2e) $(0, k, 0, k+1) \longrightarrow (0, k-1, 1, k)$

$$(q_1 \cdot A) \otimes \left((q_2\xi_1)^2 \cdot m_1\right) + (1-k)\ (q_1 \cdot B) \otimes (q_2\xi_1 \cdot m_1)$$
$$+(1-k)\ (q_2 \cdot A) \otimes (q_2\xi_1 \cdot m_1) + (-1+k)\ k\ (q_2 \cdot B) \otimes m_1.$$

2f) $(0, 1, 0, 2) \longrightarrow (0, 0, 0, 2)$

$$(q_1 \cdot A) \otimes \left((q_2\xi_1)^2 \cdot q_3\xi_2 \cdot m_1\right) - (q_1 \cdot B) \otimes (q_2\xi_1 \cdot q_3\xi_2 \cdot m_1)$$
$$-2\ (q_1 \cdot C) \otimes (q_2\xi_1 \cdot m_1) - (q_2 \cdot A) \otimes (q_2\xi_1 \cdot q_3\xi_2 \cdot m_1)$$
$$+2\ (q_2 \cdot B) \otimes (q_3\xi_2 \cdot m_1) + 2\ (q_2 \cdot C) \otimes m_1 - 2\ (q_3 \cdot A) \otimes (q_2\xi_1 \cdot m_1) + 2\ (q_3 \cdot B) \otimes m_1.$$

3a) $(-3, 2, 2, 2) \longrightarrow (-2, 1, 1, 1)$

$$-\,q_0 \otimes m_1 - (\xi_1 \cdot A) \otimes m_1 - (\xi_2 \cdot B) \otimes m_1 - (\xi_3 \cdot C) \otimes m_1 - (q_1 \cdot C \cdot B) \otimes m_1$$
$$+ (q_2 \cdot C \cdot A) \otimes m_1 - (q_3 \cdot B \cdot A) \otimes m_1 + (C \cdot B \cdot A) \otimes (\xi_0 \cdot m_1).$$

3a*) $(-2, 2, 2, 2) \longrightarrow (-3, 2, 2, 2)$

$$-4\, I \otimes m_1 - 2\, q_0 \otimes (\xi_0 \cdot m_1) + 2\ (\xi_1 \cdot q_1) \otimes m_1 - 2\ (\xi_1 \cdot A) \otimes (\xi_0 \cdot m_1)$$
$$+2\ (\xi_2 \cdot q_2) \otimes m_1 - 2\ (\xi_2 \cdot B) \otimes (\xi_0 \cdot m_1) + 2\ (\xi_3 \cdot q_3) \otimes m_1 - 2\ (\xi_3 \cdot C) \otimes (\xi_0 \cdot m_1)$$
$$-2\ (q_1 \cdot C \cdot B) \otimes (\xi_0 \cdot m_1) + 2\ (q_2 \cdot q_1 \cdot C) \otimes m_1 + 2\ (q_2 \cdot C \cdot A) \otimes (\xi_0 \cdot m_1)$$
$$-2\ (q_3 \cdot q_1 \cdot B) \otimes m_1 + 2\ (q_3 \cdot q_2 \cdot A) \otimes m_1 - 2\ (q_3 \cdot B \cdot A) \otimes (\xi_0 \cdot m_1) + C \cdot B \cdot A \otimes \left(\xi_0^2 \cdot m_1\right)$$

3b) $(1-k, k, k, k) \longrightarrow (-2-k, k+1, k+1, k+1)$, *where* $k \neq -1,\ 0,\ 1$

$$-2\, k\ (1+k)\ I \otimes (\xi_0 \cdot m_1) - (1+k)\, q_0 \otimes \left({\xi_0}^2 \cdot m_1\right) + k\ (1+k)\, (\xi_1 \cdot q_1) \otimes (\xi_0 \cdot m_1)$$
$$-\,(1+k)\, (\xi_1 \cdot A) \otimes \left({\xi_0}^2 \cdot m_1\right) + k\ (1+k)\, (\xi_2 \cdot q_2) \otimes (\xi_0 \cdot m_1)$$
$$-\,(1+k)\, (\xi_2 \cdot B) \otimes \left({\xi_0}^2 \cdot m_1\right) + k\ (1+k)\, (\xi_3 \cdot q_3) \otimes (\xi_0 \cdot m_1)$$
$$-\,(1+k)\, (\xi_3 \cdot C) \otimes \left({\xi_0}^2 \cdot m_1\right) - (1+k)\, (q_1 \cdot C \cdot B) \otimes \left({\xi_0}^2 \cdot m_1\right)$$
$$+k\ (1+k)\, (q_2 \cdot q_1 \cdot C) \otimes (\xi_0 \cdot m_1) + (1+k)\, (q_2 \cdot C \cdot A) \otimes \left({\xi_0}^2 \cdot m_1\right)$$
$$-k\ (1+k)\, (q_3 \cdot q_1 \cdot B) \otimes (\xi_0 \cdot m_1)$$
$$-\,(-1+k)\ k\ (1+k)\, (q_3 \cdot q_2 \cdot q_1) \otimes m_1$$
$$+k\ (1+k)\, (q_3 \cdot q_2 \cdot A) \otimes (\xi_0 \cdot m_1) - (1+k)\, (q_3 \cdot B \cdot A) \otimes \left({\xi_0}^2 \cdot m_1\right)$$
$$+\,(C \cdot B \cdot A) \otimes \left({\xi_0}^3 \cdot m_1\right)$$

3b*) $(k, 2, 2, 2) \longrightarrow (k+3, 0, 0, 0)$: $C \cdot B \cdot A \otimes m_1$.

4a) $(0, 2, 2, 1) \longrightarrow (2, 0, 0, 0)$

$$(q_0 \cdot A) \otimes (q_2\xi_1 \cdot q_3\xi_2 \cdot m_1) - (q_0 \cdot B) \otimes (q_3\xi_2 \cdot m_1) + (q_0 \cdot C) \otimes m_1$$
$$+\,(\xi_1 \cdot B \cdot A) \otimes (q_3\xi_2 \cdot m_1) - (\xi_1 \cdot C \cdot A) \otimes m_1 + (\xi_2 \cdot B \cdot A) \otimes (q_2\xi_1 \cdot q_3\xi_2 \cdot m_1)$$
$$+\,(\xi_2 \cdot C \cdot A) \otimes (q_2\xi_1 \cdot m_1) - (\xi_2 \cdot C \cdot B) \otimes m_1 + (\xi_3 \cdot C \cdot A) \otimes (q_2\xi_1 \cdot q_3\xi_2 \cdot m_1)$$
$$-\,(\xi_3 \cdot C \cdot B) \otimes (q_3\xi_2 \cdot m_1) + (q_1 \cdot C \cdot B \cdot A) \otimes (q_2\xi_1 \cdot q_3\xi_2 \cdot m_1)$$
$$-\,(q_2 \cdot C \cdot B \cdot A) \otimes (q_3\xi_2 \cdot m_1) + (q_3 \cdot C \cdot B \cdot A) \otimes m_1$$

3.7. Singular vectors for $\mathfrak{g} = \mathfrak{ksle}(5|10)$

We set: $\delta_{ij} = \dfrac{\partial}{\partial\theta_{ij}} + \sum\limits_{\text{even permutations } (ijklm)} \theta_{kl}\partial_m$; e.g.,

$$\delta_{12} = \frac{\partial}{\partial\theta_{12}} + \theta_{34}\partial_5 + \theta_{45}\partial_3 - \theta_{35}\partial_4,$$
$$\delta_{13} = \frac{\partial}{\partial\theta_{13}} + \theta_{25}\partial_4 - \theta_{24}\partial_5 - \theta_{45}\partial_2,$$
$$\delta_{14} = \frac{\partial}{\partial\theta_{14}} + \theta_{23}\partial_5 + \theta_{35}\partial_2 - \theta_{25}\partial_3, \text{ etc.}$$

The x-part of the elements of $\mathfrak{g}_0 = \mathfrak{sl}(5)$ is obvious. The negative elements are:

$$y_{ij} = x_i\partial_j + \sum_k \theta_{jk}\delta_{ki} \quad \text{for } i < j$$

and the basis of Cartan subalgebra is $h_i = y_{ii} - y_{i+1,i+1}$.

Let us estimate the possible degree of invariant operators. Since the $\mathbb{Z}$-grading is compatible (with parity) and $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{svect}(5|0)$, we see that the degree of the singular vector can not exceed $2 \times 2 + 10 = 14$: each element from $\mathfrak{g}_{-1}$ can only contribute once and the degree of singular vector of the $\mathfrak{svect}(5|0)$ modules can not exceed 2; each counted with weight 2. In reality, the degree of singular vectors is much lower, even with infinite dimensional fibers. To compute the singular vectors directly is possible on modern computers, but hardly on a workstation; the in-built `Mathematica`'s restrictions aggravate the problem.

Still, even simple-minded direct calculations provide us with several first and second order operators. The only "known" operator, the exterior differential, is inhomogeneous in the consistent grading and consists of parts of degree 1 and parts of degree 2. To match these parts with our operators is a problem.

The m_i are the following elements of the irreducible $\mathfrak{g}_0$-module V:

$$\begin{array}{ll}
m_2 = y_{21}\cdot m_1 & m_{18} = -\,y_{53}\cdot m_1\\
m_3 = y_{32}\cdot m_1 & m_{24} = y_{21}\cdot y_{32}\cdot y_{43}\cdot m_1\\
m_4 = y_{43}\cdot m_1 & m_{25} = y_{21}\cdot y_{32}\cdot y_{54}\cdot m_1\\
m_5 = y_{54}\cdot m_1 & m_{30} = -\,y_{21}\cdot y_{42}\cdot m_1\\
m_7 = y_{21}\cdot y_{32}\cdot m_1 & m_{31} = -\,y_{21}\cdot y_{53}\cdot m_1\\
m_8 = y_{21}\cdot y_{43}\cdot m_1 & m_{36} = y_{32}\cdot y_{43}\cdot y_{54}\cdot m_1\\
m_9 = y_{21}\cdot y_{54}\cdot m_1 & m_{40} = -\,y_{32}\cdot y_{53}\cdot m_1\\
m_{11} = y_{32}\cdot y_{43}\cdot m_1 & m_{44} = -\,y_{43}\cdot y_{31}\cdot m_1\\
m_{12} = y_{32}\cdot y_{54}\cdot m_1 & m_{48} = -\,y_{54}\cdot y_{31}\cdot m_1\\
m_{14} = y_{43}\cdot y_{54}\cdot m_1 & m_{49} = -\,y_{54}\cdot y_{42}\cdot m_1\\
m_{16} = -\,y_{31}\cdot m_1 & m_{51} = -\,y_{41}\cdot m_1\\
m_{17} = -\,y_{42}\cdot m_1 & m_{52} = -\,y_{52}\cdot m_1
\end{array}$$

$$\begin{array}{ll}
m_{70} = y_{21}\cdot y_{32}\cdot y_{43}\cdot y_{54}\cdot m_1 & m_{181} = -\,y_{21}\cdot y_{32}\cdot y_{43}\cdot y_{53}\cdot m_1\\
m_{73} = -\,y_{21}\cdot y_{32}\cdot y_{42}\cdot m_1 & m_{184} = -\,y_{21}\cdot y_{32}\cdot y_{54}\cdot y_{42}\cdot m_1\\
m_{74} = -\,y_{21}\cdot y_{32}\cdot y_{53}\cdot m_1 & m_{187} = -\,y_{21}\cdot y_{32}\cdot y_{52}\cdot m_1\\
m_{83} = -\,y_{21}\cdot y_{54}\cdot y_{42}\cdot m_1 & m_{241} = -\,y_{32}\cdot y_{43}\cdot y_{54}\cdot y_{31}\cdot m_1\\
m_{86} = -\,y_{21}\cdot y_{52}\cdot m_1 & m_{250} = -\,y_{32}\cdot y_{54}\cdot y_{41}\cdot m_1\\
m_{101} = -\,y_{32}\cdot y_{43}\cdot y_{53}\cdot m_1 & m_{254} = y_{32}\cdot y_{31}\cdot y_{53}\cdot m_1\\
m_{115} = -\,y_{43}\cdot y_{54}\cdot y_{31}\cdot m_1 & m_{258} = -\,y_{32}\cdot y_{51}\cdot m_1\\
m_{124} = -\,y_{54}\cdot y_{41}\cdot m_1 & m_{279} = y_{43}\cdot y_{31}\cdot y_{53}\cdot m_1\\
m_{127} = y_{31}\cdot y_{42}\cdot m_1 & m_{291} = y_{54}\cdot y_{31}\cdot y_{42}\cdot m_1\\
m_{128} = y_{31}\cdot y_{53}\cdot m_1 & m_{298} = y_{31}\cdot y_{52}\cdot m_1\\
m_{130} = y_{42}\cdot y_{53}\cdot m_1 & m_{301} = y_{53}\cdot y_{41}\cdot m_1\\
m_{132} = -\,y_{51}\cdot m_1 & m_{397} = -\,y_{21}\cdot y_{32}\cdot y_{32}\cdot y_{43}\cdot y_{53}\cdot m_1\\
m_{171} = y_{21}\cdot y_{32}\cdot y_{32}\cdot y_{43}\cdot y_{54}\cdot m_1 & m_{539} = y_{32}\cdot y_{43}\cdot y_{31}\cdot y_{53}\cdot m_1\\
m_{175} = -\,y_{21}\cdot y_{32}\cdot y_{32}\cdot y_{53}\cdot m_1 &
\end{array}$$

The Cartan subalgebra is spanned by

$$h_1 = x_1\partial_1 - \theta_{12}\delta_{12} - \theta_{13}\delta_{13} - \theta_{14}\delta_{14} - \theta_{15}\delta_{15} - x_2\partial_2 - \theta_{12}\delta_{12} - \theta_{23}\delta_{23} - \theta_{24}\delta_{24} - \theta_{25}\delta_{25}$$
$$h_2 = x_2\partial_2 - \theta_{12}\delta_{12} - \theta_{23}\delta_{23} - \theta_{24}\delta_{24} - \theta_{25}\delta_{25} - x_3\partial_3 - \theta_{13}\delta_{13} - \theta_{23}\delta_{23} - \theta_{34}\delta_{34} - \theta_{35}\delta_{35}$$
$$h_3 = x_3\partial_3 - \theta_{13}\delta_{13} - \theta_{23}\delta_{23} - \theta_{34}\delta_{34} - \theta_{35}\delta_{35} - x_4\partial_4 - \theta_{14}\delta_{14} - \theta_{24}\delta_{24} - \theta_{34}\delta_{34} - \theta_{45}\delta_{45}$$
$$h_4 = x_4\partial_4 - \theta_{14}\delta_{14} - \theta_{24}\delta_{24} - \theta_{34}\delta_{34} - \theta_{45}\delta_{45} - \xi_1\partial_5 - \theta_{15}\delta_{15} - \theta_{25}\delta_{25} - \theta_{35}\delta_{35} - \theta_{45}\delta_{45}$$

3.7.1. Theorem. *In $I(V)$ in degree d), there are only the following singular vectors (computed for degree 2 up to dualization):*

1a) $(k, l, 0, 0) \longrightarrow (k, l+1, 0, 0)$: $\delta_{12} \otimes m_1$;

1a*) $(0, 0, k, l) \longrightarrow (0, 0, k-1, l)$, *where* $k \neq 0$ *and* $k + l + 1 \neq 0$;[3]

$$\begin{aligned}
&2k\,(1+k+l)\,\delta_{45}\otimes m_1 - 2\,(1+k+l)\,\delta_{35}\otimes m_4\\
&+2\delta_{25}\otimes((1+k-l)\;m_{11} + 2\,l\,m_{17})\\
&+2\delta_{15}\otimes(3\;(-1+k-l)\;m_{24} - 2\;(-1+2\,k-2\,l)\;m_{44} + 2\;(-1+2\,k-l)\;m_{51})\\
&+2\delta_{34}\otimes(m_{14} - (1+k)\;m_{18}) + 2\delta_{24}\otimes(m_{36} + (1+k)\;m_{40} - 2\,m_{49} + 2\,m_{52})\\
&-2\delta_{23}\otimes(m_{101} + 2\,m_{130})\\
&+2\delta_{14}\otimes(3\,m_{70} + 3\,(-1+k)\,m_{74} - 4\,m_{115} + 2\,m_{124} - 2\,(-1+2\,k)\,m_{128} - 2\,m_{132})\\
&-2\delta_{13}\otimes(3\,m_{181} - 4\,m_{279} + 2\,m_{301}) + \delta_{12}\otimes(m_{397} - 4\,m_{539})
\end{aligned}$$

1b) $(k, l, -k-1, 0) \longrightarrow (k+1, l-1, -k, 0)$, *where* $l \neq 0$

$$-l\,\delta_{13}\otimes m_1 + \delta_{12}\otimes m_3$$

[3] Hereafter in similar statements the reader can check our restrictions: the coefficient of $\otimes m_1$ must not vanish.

1b*) $(0,k,l,-k-1) \longrightarrow (0,k-1,l+1,-k)$, *where* $k \neq 0,-1$ *and* $l \neq -1$

$$\begin{aligned}
&k(1+k)(1+l)\delta_{35}\otimes m_1-(1+k)(1+l)\delta_{25}\otimes m_3\\
&+\delta_{15}\otimes((1+k)\ (1+k-l)\ m_7-(1+k)\ (k-2\,l)\ m_{16})\\
&+k(1+l)\delta_{34}\otimes m_5-(1+l)\delta_{24}\otimes m_{12}\\
&+\delta_{23}\otimes(m_{36}-(1+k)\ m_{40}-(1+k)\ m_{49}-k\ (1+k)\ m_{52})\\
&+\delta_{14}\otimes((1+k-l)\ m_{25}-(k-2\,l)\ m_{48})\\
&+\delta_{13}\otimes(m_{70}-m_{74}+(1+k)\ m_{83}+k\,m_{86}-2\,m_{115}+2\,m_{124}+(2+k)\ m_{128}\\
&-\left(2+k^2\right)\ m_{132})-\delta_{12}\otimes(m_{184}-k\,m_{258}-2\,m_{291}-k\,m_{298})
\end{aligned}$$

1c) $(k,0,l,-l-1) \longrightarrow (k+1,0,l-1,-l)$, *where* $l \neq 0$

$$l\,\delta_{14}\otimes m_1-\delta_{13}\otimes m_4+\delta_{12}\otimes m_{17}$$

1c*) $(k,-k-1,0,l) \longrightarrow (k-1,-k,0,l-1)$, *where* $k \neq 0$ *and* $l \neq 0$

$$\begin{aligned}
&-k\,l\delta_{25}\otimes m_1+l\,\delta_{15}\otimes m_2+k\,\delta_{24}\otimes m_5-k\,\delta_{23}\otimes m_{18}-\delta_{14}\otimes m_9\\
&+\delta_{13}\otimes m_{31}+\delta_{12}\otimes(m_{86}+(1+k)\ m_{132})
\end{aligned}$$

1d) $(k,l,-k-l-2,0) \longrightarrow (k-1,l,-k-l-1,0)$, *where* $k \neq 0$ *and* $k+l+1 \neq 0$

$$\begin{aligned}
&k(1+k+l)\delta_{23}\otimes m_1-(1+k+l)\delta_{13}\otimes m_2\\
&+\delta_{12}\otimes(m_7+(-1-k)\ m_{16})
\end{aligned}$$

1d*) $(0,k,l,-k-l-2) \longrightarrow (0,k-1,l,-k-l-1)$, *where* $k \neq 0$ *and* $k+l+1 \neq 0$

$$\begin{aligned}
&k(1+k+l)\delta_{34}\otimes m_1-(1+k+l)\delta_{24}\otimes m_3+\delta_{23}\otimes(m_{11}-(1+k)\,m_{17})\\
&+\delta_{14}\otimes((1+k-l)\,m_7+2\,l\,m_{16})+\delta_{13}\otimes(m_{24}+(1+k)\,m_{30}-2\,m_{44}+2\,m_{51})\\
&-\delta_{12}\otimes(m_{73}-2\,m_{127})
\end{aligned}$$

1e) $(k,0,0,l) \longrightarrow (k+1,0,0,l-1)$, *where* $l \neq 0$

$$l\,\delta_{15}\otimes m_1-\delta_{14}\otimes m_5+\delta_{13}\otimes m_{18}+\delta_{12}\otimes m_{52}$$

1f) $(k,-k-1,l,-l-1) \longrightarrow (k-1,-k,l-1,-l)$, *where* $k \neq 0$ *and* $l \neq 0,-1$

$$\begin{aligned}
&(k\,l\ -1-l)\delta_{24}\otimes m_1+k(1+l)\delta_{23}\otimes m_4+l(1+l)\delta_{14}\otimes m_2\\
&-(1+l)\,\delta_{13}\otimes m_8+\delta_{12}\otimes(m_{24}+l\,m_{30}-(1+k)\ m_{44}+(1+k)\ (1+l)\ m_{51})
\end{aligned}$$

2a) $(k,0,0,1) \longrightarrow (k+1,1,0,0)$

$$\delta_{15}\delta_{12}m_1-\delta_{14}\delta_{12}m_5+\delta_{13}\delta_{12}m_{18}$$

2b) $(k,-k-1,0,1) \longrightarrow (k-1,-k+1,0,0)$, *where* $k \neq 0$

$$-k\,\delta_{25}\delta_{12}m_1+\delta_{15}\delta_{12}m_2+k\,\delta_{24}\delta_{12}m_5-\delta_{14}\delta_{12}m_9-k\,\delta_{23}\delta_{12}m_{18}+\delta_{13}\delta_{12}m_{31}$$

3.8. Singular vectors for $\mathfrak{g}=\mathfrak{ksle}(9|11)$

Consider the following negative operators from $\mathfrak{g}_0$:

$$\begin{aligned}
&y_1=x_2\partial_1-\theta_{13}\delta_{23}-\theta_{14}\delta_{24}-\theta_{15}\delta_{25} && y_5=-x_3\partial_1-\theta_{12}\delta_{23}+\theta_{14}\delta_{34}+\theta_{15}\delta_{35}\\
&y_2=x_3\partial_2-\theta_{12}\delta_{13}-\theta_{24}\delta_{34}-\theta_{25}\delta_{35} && y_6=-\delta_{13}-\theta_{25}\partial_4+\theta_{24}\partial_5+\theta_{45}\partial_2\\
&y_3=\delta_{12}+\theta_{34}\partial_5+\theta_{45}\partial_3-\theta_{35}\partial_4 && y_7=-\delta_{23}-\theta_{14}\partial_5-\theta_{45}\partial_1+\theta_{15}\partial_4\\
&y_4=x_5\partial_4-\theta_{14}\delta_{15}-\theta_{24}\delta_{25}-\theta_{34}\delta_{35}
\end{aligned}$$

and the operators from $\mathfrak{g}_-$:

$$\begin{aligned}
&n_1=\partial_4 && n_{10}=\xi_1\partial_1+\theta_{12}\delta_{25}+\theta_{13}\delta_{35}+\theta_{14}\delta_{45}\\
&n_2=\partial_5 && n_{11}=x_4\partial_2-\theta_{12}\delta_{14}+\theta_{23}\delta_{34}-\theta_{25}\delta_{45}\\
&n_3=\delta_{14}+\theta_{23}\partial_5+\theta_{35}\partial_2-\theta_{25}\partial_3 && n_{12}=\xi_1\partial_2-\theta_{12}\delta_{15}+\theta_{23}\delta_{35}+\theta_{24}\delta_{45}\\
&n_4=\delta_{15}+\theta_{24}\partial_3-\theta_{23}\partial_4-\theta_{34}\partial_2 && n_{13}=x_4\partial_3-\theta_{13}\delta_{14}-\theta_{23}\delta_{24}-\theta_{35}\delta_{45}\\
&n_5=\delta_{24}+\theta_{15}\partial_3-\theta_{13}\partial_5-\theta_{35}\partial_1 && n_{14}=\xi_1\partial_3-\theta_{13}\delta_{15}-\theta_{23}\delta_{25}+\theta_{34}\delta_{45}\\
&n_6=\delta_{25}+\theta_{13}\partial_4+\theta_{34}\partial_1-\theta_{14}\partial_3 && n_{17}=\partial_1\\
&n_7=\delta_{34}+\theta_{12}\partial_5+\theta_{25}\partial_1-\theta_{15}\partial_2 && n_{18}=\partial_2\\
&n_8=\delta_{35}+\theta_{14}\partial_2-\theta_{12}\partial_4-\theta_{24}\partial_1 && n_{19}=\partial_3\\
&n_9=x_4\partial_1+\theta_{12}\delta_{24}+\theta_{13}\delta_{34}-\theta_{15}\delta_{45} && n_{20}=\delta_{45}+\theta_{12}\partial_3+\theta_{23}\partial_1-\theta_{13}\partial_2
\end{aligned}$$

$$\begin{aligned}
n_{15}=&-\xi_1\theta_{12}\partial_3+\xi_1\theta_{13}\partial_2-\xi_1\theta_{23}\partial_1+2\,\theta_{12}\theta_{13}\theta_{23}\partial_4-\theta_{12}\theta_{13}\theta_{24}\partial_3+\\
&\theta_{12}\theta_{13}\theta_{34}\partial_2-\theta_{12}\theta_{14}\theta_{23}\partial_3-\theta_{12}\theta_{23}\theta_{34}\partial_1+\theta_{13}\theta_{14}\theta_{23}\partial_2+\theta_{13}\theta_{23}\theta_{24}\partial_1-\xi_1\delta_{45}+\\
&2\,\theta_{12}\theta_{13}\delta_{15}+2\,\theta_{12}\theta_{23}\delta_{25}-\theta_{12}\theta_{34}\delta_{45}+2\,\theta_{13}\theta_{23}\delta_{35}+\theta_{13}\theta_{24}\delta_{45}+\theta_{14}\theta_{23}\delta_{45}
\end{aligned}$$

$$\begin{aligned}
n_{16}=&-x_4\theta_{12}\partial_3+x_4\theta_{13}\partial_2-x_4\theta_{23}\partial_1+2\,\theta_{12}\theta_{13}\delta_{14}+2\,\theta_{12}\theta_{23}\delta_{24}+\theta_{12}\theta_{35}\delta_{45}+\\
&2\,\theta_{13}\theta_{23}\delta_{34}-\theta_{13}\theta_{25}\delta_{45}-\theta_{15}\theta_{23}\delta_{45}-2\,\theta_{12}\theta_{13}\theta_{23}\partial_5+\theta_{12}\theta_{13}\theta_{25}\partial_3-\theta_{12}\theta_{13}\theta_{35}\partial_2+\\
&\theta_{12}\theta_{15}\theta_{23}\partial_3+\theta_{12}\theta_{23}\theta_{35}\partial_1-\theta_{13}\theta_{15}\theta_{23}\partial_2-\theta_{13}\theta_{23}\theta_{25}\partial_1-x_4\delta_{45}
\end{aligned}$$

The m_i are the following elements of the irreducible $\mathfrak{g}_0$-module V:

m_1 is the highest weight vector	$m_{15}=y_5\cdot m_1$	$m_{88}=y_4\cdot y_7\cdot m_1$
$m_2=y_1\cdot m_1$	$m_{21}=y_1\cdot y_2\cdot y_2\cdot m_1$	$m_{91}=y_1\cdot y_1\cdot y_1\cdot y_1\cdot y_1\cdot m_1$
$m_3=y_2\cdot m_1$	$m_{22}=y_1\cdot y_2\cdot y_3\cdot m_1$	$m_{92}=y_1\cdot y_1\cdot y_1\cdot y_1\cdot y_2\cdot m_1$
$m_4=y_3\cdot m_1$	$m_{27}=y_1\cdot y_6\cdot m_1$	$m_{93}=y_1\cdot y_1\cdot y_1\cdot y_1\cdot y_3\cdot m_1$
$m_5=y_4\cdot m_1$	$m_{31}=y_2\cdot y_3\cdot y_4\cdot m_1$	$m_{94}=y_1\cdot y_1\cdot y_1\cdot y_1\cdot y_4\cdot m_1$
$m_7=y_1\cdot y_2\cdot m_1$	$m_{36}=y_3\cdot y_5\cdot m_1$	$m_{95}=y_1\cdot y_1\cdot y_1\cdot y_2\cdot y_2\cdot m_1$
$m_9=y_1\cdot y_4\cdot m_1$	$m_{39}=y_4\cdot y_5\cdot m_1$	$m_{96}=y_1\cdot y_1\cdot y_1\cdot y_2\cdot y_3\cdot m_1$
$m_{10}=y_2\cdot y_2\cdot m_1$	$m_{41}=y_7\cdot m_1$	$m_{97}=y_1\cdot y_1\cdot y_1\cdot y_2\cdot y_4\cdot m_1$
$m_{11}=y_2\cdot y_3\cdot m_1$	$m_{56}=y_1\cdot y_2\cdot y_3\cdot y_4\cdot m_1$	$m_{98}=y_1\cdot y_1\cdot y_1\cdot y_3\cdot y_4\cdot m_1$
$m_{12}=y_2\cdot y_4\cdot m_1$	$m_{65}=y_1\cdot y_4\cdot y_6\cdot m_1$	$m_{99}=y_1\cdot y_1\cdot y_1\cdot y_4\cdot y_4\cdot m_1$
$m_{13}=y_3\cdot y_4\cdot m_1$	$m_{82}=y_3\cdot y_4\cdot y_5\cdot m_1$	$m_{100}=y_1\cdot y_1\cdot y_1\cdot y_5\cdot m_1$

3.8.1. Theorem. *In $I(V)$ in degree 1) (higher degrees were not considered), there are only the following singular vectors:*

1a) $\lambda \longrightarrow \lambda+(0,0,-2,1)$: $n_{16}\otimes m_1$ *for ANY* λ;

1b) $\lambda \longrightarrow \lambda+(0,0,-1,-1)$: $-\lambda_4 n_{15}\otimes m_1+n_{16}\otimes m_5$ *for ANY* λ;

1c) $(k,l,1,1) \longrightarrow (k,l+1,0,0)$

$$-n_{15}\otimes m_4+2\,n_{14}\otimes m_1+n_{16}\otimes m_{13}-2\,n_{13}\otimes m_5$$

1d) $(k,l,2,0) \longrightarrow (k,l+1,0,1)$: $n_{16}\otimes m_4-2\,n_{13}\otimes m_1$;

1e) $(k,-1,1,1) \longrightarrow (k,0,1,0)$

$$-n_{15}\otimes m_{11}-2\,n_{12}\otimes m_1+2\,n_{14}\otimes m_3+n_{16}\otimes m_{31}+2\,n_{11}\otimes m_5-2\,n_{13}\otimes m_{12}$$

1f) $(-1,k,k+2,1) \longrightarrow (-2,k,k+2,0)$, *where* $k\neq 0$

$$\begin{aligned}&-n_{15}\otimes(m_{22}-(1+k)\,m_{27})-2\,k\,n_{10}\otimes m_1+2\,k\,n_{12}\otimes m_2+2\,n_{14}\otimes m_7\\&+n_{16}\otimes(m_{56}-(1+k)\,m_{65})+2\,k\,n_9\otimes m_5-2\,k\,n_{11}\otimes m_9-2\,n_{13}\otimes m_{23}\end{aligned}$$

1g) $(k,0,-k-1,1) \longrightarrow (k-1,0,-k-1,0)$, *where* $k\neq 0,-1$

$$\begin{aligned}&-n_{15}\otimes(m_{22}-(k+2)\,m_{27}-(k+1)\,m_{36}-(k+1)^2\,m_{41})\\&+2\,k\,(k+1)\,n_{10}\otimes m_1+2\,(k+1)\,n_{12}\otimes m_2\\&+n_{14}\otimes(-2(3+2\,k)\,m_7-2\,(1+k)\,m_{15})\\&+n_{16}\otimes(m_{56}-(k+2)\,m_{65}-(k+1)\,m_{82}-(k+1)^2\,m_{88})\\&-2\,k\,(1+k)\,n_9\otimes m_5-2\,(1+k)\,n_{11}\otimes m_9+2\,(1+k)\,n_{13}\otimes(m_{23}+m_{39})\end{aligned}$$

1h) $(k,-k-2,1,1) \longrightarrow (k-1,-k-2,1,0)$, *where* $k\neq 0$

$$\begin{aligned}&-n_{15}\otimes(m_{22}-(1+k)\,m_{36}+(1+k)\,m_{41})-2\,k\,n_{10}\otimes m_1-2\,n_{12}\otimes m_2\\&+2\,n_{14}\otimes(m_7-(1+k)\,m_{15})+n_{16}\otimes(m_{56}-(1+k)\,m_{82}+(1+k)\,m_{88})\\&+2\,k\,n_9\otimes m_5+2\,n_{11}\otimes m_9-2\,n_{13}\otimes(m_{23}-(1+k)\,m_{39})\end{aligned}$$

3.9. Singular vectors for $\mathfrak{g}=\mathfrak{ksle}(11|9)$

Here we realize the elements of $\mathfrak{g}$, as in [KR1], as divergence-free vector fields and closed 2-forms with shifted parity. We consider the following negative operators from $\mathfrak{g}_0$:

$$\begin{array}{lll} y_1=x_2\partial_1 & y_2=x_4\partial_3 & y_3=x_5\partial_4\\ y_4=\pi dx_1dx_2 & y_5=-x_5\partial_3 & \end{array}$$

and the elements of Cartan subalgebra

$$\begin{array}{ll} h_1=x_1\partial_1-x_2\partial_2 & h_2=-\frac{1}{2}(x_1\partial_1+x_2\partial_2)+x_3\,\partial_3,\\ h_3=-\frac{1}{2}(x_1\partial_1+x_2\partial_2)+x_4\,\partial_4, & h_4=-\frac{1}{2}(x_1\partial_1+x_2\partial_2)+x_5\,\partial_5\end{array}$$

The m_i are the following elements of the irreducible $\mathfrak{g}_0$-module V:

$$\begin{array}{lll}
m_2=y_1\,m_1 & m_{16}=y_1\,y_2\,y_3\,m_1 & m_{42}=(y_2)^2\,(y_3)^2\,m_1\\
m_3=y_2\,m_1 & m_{18}=y_1\,y_5\,m_1 & m_{45}=y_2\,y_3\,y_5\,m_1\\
m_4=y_3\,m_1 & m_{20}=(y_2)^2\,y_3\,m_1 & m_{49}=y_3\,y_4\,m_1\\
m_6=y_1\,y_2\,m_1 & m_{21}=y_2\,(y_3)^2\,m_1 & m_{50}=(y_5)^2\,m_1\\
m_7=y_1\,y_3\,m_1 & m_{22}=y_2\,y_5\,m_1 & m_{74}=y_1\,y_3\,y_4\,m_1\\
m_8=(y_2)^2\,m_1 & m_{24}=y_3\,y_5\,m_1 & m_{85}=y_2\,y_3\,y_4\,m_1\\
m_9=y_2\,y_3\,m_1 & m_{25}=y_4\,m_1 & m_{89}=(y_3)^2\,y_4\,m_1\\
m_{10}=(y_3)^2\,m_1 & m_{34}=y_1\,(y_2)^2\,y_3\,m_1 & m_{126}=y_1\,y_2\,y_3\,y_4\,m_1\\
m_{11}=y_5\,m_1 & m_{36}=y_1\,y_2\,y_5\,m_1 & m_{146}=y_2\,(y_3)^2\,y_4\,m_1\\
m_{15}=y_1\,(y_2)^2\,m_1 & m_{39}=y_1\,y_4\,m_1 & m_{231}=(y_2)^2\,(y_3)^2\,y_4\,m_1
\end{array}$$

3.9.1. Theorem. *In $I(V)$ in degrees d), there are only the following singular vectors (where $x_i\partial_jx_k\partial_lm_r$ means $(x_i\partial_j)(x_k\partial_l)m_r$):*

1a) $(2k,-k,l,m) \longrightarrow (2k+1,-k+\frac{3}{2},l+\frac{1}{2},m+\frac{1}{2})$: $x_3\,\partial_2\,m_1$;

1b) $(2k,l,1-k,m) \longrightarrow (2k+1,l+\frac{1}{2},-k+\frac{5}{2},m+\frac{1}{2})$:

$$(x_3\,\partial_2)\,m_3+(1-k-l)(x_4\,\partial_2)\,m_1;$$

1c) $(2k,l,m,2-k) \longrightarrow (2k+1,l+\frac{1}{2},m+\frac{1}{2},-k+\frac{7}{2})$

$$(x_3\partial_2)(m_{11}+(-2+k+m)m_{15})+(1-k-l)(x_4\partial_2)m_4+(-1+k+l)(-2+k+m)(x_5\partial_2)m_1$$

1d) $(2k,3-k,3-k,2-k) \longrightarrow (2k+1,\frac{5}{2}-k,\frac{5}{2}-k,\frac{5}{2}-k)$

$$(\pi dx_1dx_3)m_{11}-(\pi dx_1dx_4)m_4+(\pi dx_1dx_5)m_1-(x_3\partial_2)m_{85}+(x_4\partial_2)m_{49}-(x_5\partial_2)m_{25}$$

1e) $(2k,k+1,l,m) \longrightarrow (2k-1,k+\frac{5}{2},l+\frac{1}{2},m+\frac{1}{2})$: $2k(x_3\,\partial_1)\,m_1+(x_3\,\partial_2)\,m_2$

1f) $(2k,l,2+k,m) \longrightarrow (2k-1,l+\frac{1}{2},k+\frac{7}{2},m+\frac{1}{2})$

$$2k(x_3\,\partial_1)\,m_3+(x_3\,\partial_2)\,m_6+2k(2+k-l)(x_4\,\partial_1)\,m_1+(2+k-l)(x_4\,\partial_2)\,m_2$$

1g) $(2k,l,m,3+k) \longrightarrow (2k-1,l+\frac{1}{2},m+\frac{1}{2},k+\frac{9}{2})$

$$\begin{aligned}&(x_3\,\partial_1)\,(2km_9-2k(3+k-m)m_{11})+(x_3\,\partial_2)\,(m_{16}+(-3-k+m)m_{18})\\&+2k(2+k-l)(x_4\,\partial_1)\,m_4+(2+k-l)(x_4\,\partial_2)\,m_7\\&+2k(2+k-l)(3+k-m)(x_5\,\partial_1)\,m_1+(2+k-l)(3+k-m)(x_5\,\partial_2)\,m_2\end{aligned}$$

1h) $(2k,4+k,4+k,3+k) \longrightarrow (2k-1,\frac{7}{2}-k,\frac{7}{2}-k,\frac{7}{2}-k)$

$(\pi\, dx_1dx_3)\, m_{18} - (\pi\, dx_1dx_4)\, m_7(\pi\, dx_1dx_5)\, m_2 - 2k(\pi\, dx_2dx_3)\, m_{11}2k(\pi\, dx_2dx_4)\, m_4$
$-2k(\pi\, dx_2dx_5)\, m_1 - 2k(x_3\,\partial_1)\, m_{85} - (x_3\,\partial_2)\, m_{126} + 2k(x_4\,\partial_1)\, m_{49} + (x_4\,\partial_2)\, m_{74})$
$-2k(x_5\,\partial_1)\, m_{25} - (x_5\,\partial_2)\, m_{39}$

2a) $(2k, -k-1, l, m) \longrightarrow (2k+2, -k+2, l+1, m+1)$: $(x_3\,\partial_2)^2\, m_1$

2b) $(2k, -k-1, -k+1, l) \longrightarrow (2k+2, -k+1, -k+3, l+1)$:

$$(x_3\,\partial_2)^2\, m_3 + 2x_3\partial_2\, x_4\partial_2\, m_1$$

2c) $(2k, -k-1, l, -k+2) \longrightarrow (2k+2, -k+1, l+1, -k+4)$

$$(x_3\partial_2)^2\,(m_9 + (-2+k+l)m_{11}) + 2x_3\partial_2\, x_4\partial_2\, m_4 - 2(-2+k+l)x_3\partial_2\, x_5\partial_2\, m_1$$

2d) $(2k, l, -k, m) \longrightarrow (2k+2, l+1, -k+3, m+1)$

$$(x_3\partial_2)^2\, m_8 + (-1+k+l)(k+l)(x_4\partial_2)^2\, m_1 - 2(-1+k+l)x_3\partial_2\, x_4\partial_2\, m_3$$

2e) $(2k, l, -k, -k+2) \longrightarrow (2k+2, l+1, -k+2, -k+4)$

$$(x_3\partial_2)^2\,(m_{20} - (2m_{22})) + (-1+k+l)(k+l)(x_4\partial_2)^2\, m_4$$
$$-2(-1+k+l)x_3\partial_2\, x_4\,\partial_2\,(m_9 - m_{11}) - 2(-1+k+l)x_3\,\partial_2\, x_5\partial_2\, m_3$$
$$+2(-1+k+l)(k+l)x_4\partial_2\, x_5\partial_2\, m_1$$

In particular,

2ea) $l = 1-k$:

$$(x_3\partial_2)^2\, m_{22} + (x_4\partial_2)^2\, m_4 - x_3\partial_2\, x_4\partial_2\, m_9 - 2x_3\partial_2\, x_5\partial_2\, m_3 + 2x_4\partial_2\, x_5\partial_2\, m_1$$

2f) $(2k, l, m, -k+1) \longrightarrow (2k+2, l+1, m+1, -k+4)$

$(x_3\partial_2)^2(m_{42}+2(-2+k+m)m_{45}+(-2+k+m)(-1+k+m)m_{50})$
$+(-1+k+l)(k+l)(x_4\partial_2)^2m_{10}+(-1+k+l)(k+l)(-2+k+m)(-1+k+m)(x_5\partial_2)^2m_1$
$-2(-1+k+l)x_3\partial_2x_4\partial_2(m_{21}+(-2+k+m)m_{24})$
$+2(-1+k+l)(-2+k+m)x_3\partial_2x_5\partial_2(m_9+(-1+k+m)m_{11})-$
$2(-1+k+l)(k+l)(-2+k+m)x_4\partial_2x_5\partial_2m_4$

In particular,

2fa) $(2k, -k+1, -k, -k+1) \longrightarrow (2k+2, -k+2, -k+1, -k+4)$:

$$(x_3\partial_2)^2\,(m_{42} - 4m_{50}) + 2(x_4\partial_2)^2\, m_{10} + 4(x_5\partial_2)^2\, m_1$$
$$+4x_3\partial_2\, x_4\partial_2\,(-m_{21} + 2m_{24}) + 8x_3\partial_2\, x_5\partial_2)\,(-m_9 + m_{11}) + 8x_4\partial_2\, x_5\partial_2\, m_4$$

2fb) $(2k, -k, -k, -k+1) \longrightarrow (2k+2, -k+1, -k+1, -k+4)$:

$(x_3\partial_2)^2m_{45}+(x_4\partial_2)^2m_{10}+2(x_5\partial_2)^2m_1-x_3\partial_2x_4\partial_2m_{21}-4x_3\partial_2x_5\partial_2m_9+4x_4\partial_2x_5\partial_2m_4$

2g) $(2k, 3-k, 3-k, 1-k) \longrightarrow (2k+2, 3-k, 3-k, 3-k)$

$$(x_3\partial_2)^2\, m_{231} + 2(x_4\partial_2)^2\, m_{89} + 4(x_5\,\partial_2)^2\, m_{25}$$
$$-4\pi dx_1dx_3\, x_3\partial_2\, m_{45} + 4\pi dx_1dx_3\, x_4\partial_2\, m_{24} - 4\pi dx_1dx_3\, x_5\partial_2\, m_{11}$$
$$+4\pi dx_1dx_4\, x_3\partial_2\,(m_{21} - m_{24}) - 4\pi dx_1dx_4\, x_4\partial_2\, m_{10} + 4\pi dx_1dx_4\, x_5\partial_2\, m_4$$
$$+4\pi dx_1dx_5\, x_3\partial_2\,(-3m_9 + 2m_{11}) + 4\pi dx_1dx_5\, x_4\partial_2\, m_4 - 8\pi dx_1dx_5\, x_5\partial_2\, m_1$$
$$-2x_3\partial_2\, x_4\partial_2\, m_{146} + 4x_3\partial_2\, x_5\partial_2\, m_{85}$$
$$-4x_4\partial_2\, x_5\partial_2\, m_{49}$$

2h) $(-2, 0, k, l) \longrightarrow (-2, 3, k+1, l+1)$: $(x_3\partial_2)^2\, m_2 - 2x_3\partial_1\, x_3\partial_2\, m_1$

2i) $(2k, -k, l, m) \longrightarrow (2k, -k+2, l+2, m+1)$

$$(x_3\partial_2)^2\, m_6 + 2kx_3\partial_1\, x_3\partial_2\, m_3 + 4k(1+k)x_3\partial_2\, x_4\partial_1\, m_1 + 2(1+k)x_3\partial_2\, x_4\partial_2\, m_2$$

2j) $(2k, k+1, l, m) \longrightarrow (2k, k+3, l+2, m+1)$

$(x_3\partial_2)^2\, m_6 + 2kx_3\partial_1\, x_3\partial_2\, m_3 - 2k(1+2k)x_3\partial_1\, x_4\partial_2\, m_1$
$+2kx_3\partial_2\, x_4\partial_1\, m_1 - 2kx_3\partial_2\, x_4\partial_2\, m_2 - 2(-1+k+l)(k+l)(-2+k+m)x_4\partial_2\, x_5\partial_2\, m_4$

In particular,

2ja) $(0, 1, 1, m) \longrightarrow (0, 3, 3, m+1)$:

$$-x_3\partial_1\, x_4\partial_2\, m_1 + x_3\partial_2\, x_4\partial_1\, m_1$$

2k) $(2k, -k, l, k+3) \longrightarrow (2k, -k+2, l+1, k+4)$

$$(x_3\partial_2)^2\,(m_{16} + (-3-k+l)m_{18}) + 2kx_3\partial_1\, x_3\partial_2\,(m_9 - (3+k-l)m_{11})$$
$$+4k(1+k)x_3\partial_2\, x_4\partial_1\, m_4 + 2(1+k)x_3\partial_2\, x_4\partial_2\, m_7$$
$$+4k(1+k)(3+k-l)x_3\partial_2\, x_5\partial_1\, m_1 + 2(1+k)(3+k-l)x_3\partial_2\, x_5\partial_2\, m_2$$

2l) $(2k, k+1, l, 2-k) \longrightarrow (2k, k+3, l+1, 4-k)$

$$(x_3\partial_2)^2\,(m_{16} + (-2+k+l)m_{18}) + 2kx_3\partial_1\, x_3\partial_2\,(m_9 + (-2+k+l)m_{11})$$
$$-2k(1+2k)x_3\partial_1\, x_4\partial_2\, m_4 + 2k(1+2k)(-2+k+l)x_3\partial_1\, x_5\partial_2\, m_1$$
$$+2kx_3\partial_2\, x_4\partial_1\, m_4 - 2kx_3\partial_2\, x_4\partial_2\, m_7$$
$$-2k(-2+k+l)x_3\partial_2\, x_5\partial_1\, m_1 + 2k(-2+k+l)x_3\partial_2\, x_5\partial_2\, m_2$$

In particular,

2la) $(0, 1, l, 2) \longrightarrow (0, 3, l+1, 4)$:

$$x_3\partial_1\, x_4\partial_2\, m_4 - (l-2)x_3\partial_1\, x_5\partial_2\, m_1 - x_3\partial_2\, x_4\partial_1\, m_4 + (l-2)x_3\partial_2\, x_5\partial_1\, m_1$$

2m) $(-2, k, 1, l) \longrightarrow (-2, k+1, 4, l+1)$

$(x_3\partial_2)^2\, m_{15} + (-2+k)(-1+k)(x_4\partial_2)^2\, m_2 - 2x_3\partial_1\, x_3\partial_2\, m_8 + 2(-2+k)x_3\partial_1\, x_4\partial_2\, m_3$
$+2(-2+k)x_3\partial_2\, x_4\partial_1\, m_3 - 2(-2+k)x_3\partial_2\, x_4\partial_2\, m_6 - 2(-2+k)(-1+k)x_4\partial_1\, x_4\partial_2\, m_1$

2n) $(2k, l, -k+1, k+3) \longrightarrow (2k, l+1, -k+3, k+5)$

$$(x_3\partial_2)^2(m_{34}-2(1+k)m_{36})-(2+k-l)(-1+k+l)(x_4\partial_2)^2m_7+ 2kx_3\partial_1x_3\partial_2(m_{20}-2(1+k)m_{22})+2k(-1+k+l)x_3\partial_1x_4\partial_2(-m_9 +2(1+k)m_{11})+2kx_3\partial_2x_4\partial_1((3+k-l)m_9-2(1+k)m_{11})+ 2x_3\partial_2x_4\partial_2(-(-2+l)m_{16}+(1+k)(-2+k+l)m_{18})+4k(1+k)(2+k-l)x_3\partial_2x_5\partial_1m_3 +2(1+k)(2+k-l)x_3\partial_2x_5\partial_2m_6-2k(2+k-l)(-1+k+l)x_4\partial_1x_4\partial_2m_4- 4k(1+k)(2+k-l)(-1+k+l)x_4\partial_2x_5\partial_1m_1-2(1+k)(2+k-l)(-1+k+l)x_4\partial_2x_5\partial_2m_2$$

2o) $(2k, l, k+2, -k+2) \longrightarrow (2k, l+1, k+4, -k+4)$

$$(x_3\partial_2)^2(m_{34}+(2k)m_{36})-(2+k-l)(-1+k+l)(x_4\partial_2)^2m_7+2kx_3\partial_1x_3\partial_2(m_{20}+2km_{22}) +2k(-1+k+l)x_3\partial_1x_4\partial_2(-m_9+m_{11})+2k(1+2k)(-1+k+l)x_3\partial_1x_5\partial_2m_3 +2kx_3\partial_2x_4\partial_1((3+k-l)m_9+(1+5k+2k^2-l-2kl)m_{11}) +2x_3\partial_2x_4\partial_2(-(-2+l)m_{16}+k(3+k-l)m_{18})-2k(-1+k+l)x_3\partial_2x_5\partial_1m_3 +2k(-1+k+l)x_3\partial_2x_5\partial_2m_6-2k(2+k-l)(-1+k+l)x_4\partial_1x_4\partial_2m_4 +2k(1+2k)(2+k-l)(-1+k+l)x_4\partial_1x_5\partial_2m_1-2k(2+k-l)(-1+k+l)x_4\partial_2x_5\partial_1m_1 +2k(2+k-l)(-1+k+l)x_4\partial_2x_5\partial_2m_2$$

3.10. Singular vectors for $\mathfrak{g} = \mathfrak{kas}$ and $\mathfrak{g} = \mathfrak{k}(1|n)$

The coordinates of the weights are given with respect to the following basis of $\mathfrak{g}_0$:

$$(K_t, K_{\xi_1\eta_1}, \dots, K_{\xi_s\eta_s}), \quad s = \left[\frac{n}{2}\right],$$

and, for brevity, hereafter in this Chapter we write just f instead of K_f, so $f \cdot g := K_f K_g$ whereas $fg := K_{fg}$.

3.10.1. Theorem. *In $I(V)$, there are only the following singular for $\mathfrak{k}(1|3)$ vectors:*

1a) $(k, -k) \longrightarrow (k-1, -k+1)$: $\xi_1 \otimes m_1$;

1a*) $(k+1, k) \longrightarrow (k, k-1)$

$$\xi_1 \otimes \left((\eta_1\theta_1)^2 \cdot m_1\right) - k\ (-1+2\,k)\ \eta_1 \otimes m_1 + (1-2\,k)\,\theta_1 \otimes (\eta_1\theta_1 \cdot m_1)$$

1b) $(1, -1) \longrightarrow (0, -1)$: $\xi_1 \otimes (\eta_1\theta_1 \cdot m_1) + \theta_1 \otimes m_1$;

2a) $\frac{1}{2}(3, 1) \longrightarrow \frac{1}{2}(-1, 1)$

$$I \otimes m_1 - 2\ (\xi_1 \cdot \theta_1) \otimes (\eta_1\theta_1 \cdot m_1) + (\eta_1 \cdot \xi_1) \otimes m_1$$

3.10.2. Theorem. *In $I(V)$, there are only the following singular for $\mathfrak{k}(1|4)$ vectors:*

1a) $(k, -k, 0) \longrightarrow (k-1, -k+1, 0)$: $\xi_1 \otimes m_1$;

1a*) $(k+1, -1, k) \longrightarrow (k, -1, k-1)$, *where* $k \neq 1$

$$\xi_1 \otimes (\eta_1\eta_2 \cdot m_1) + (1-k)\,\eta_2 \otimes m_1$$

1b) $(k, -1, 1-k) \longrightarrow (k-1, -1, 2-k)$, *where* $k \neq 2$

$$(-2+k)\,\xi_2 \otimes m_1 + \xi_1 \otimes (\xi_2\eta_1 \cdot m_1)$$

1c) $(k+2, k, 0) \longrightarrow (k+1, k-1, 0)$, *where* $k \neq 0$

$$\xi_1 \otimes (\eta_1\eta_2 \cdot m_1) - k\,\eta_2 \otimes m_1$$

2a) $(1, -1, -1) \longrightarrow (-1, 0, 0)$: $(\xi_2 \cdot \xi_1) \otimes m_1$;

2b) $(2, -1, 1) \longrightarrow (0, 0, 0)$: $(\eta_2 \cdot \xi_1) \otimes m_1$;

2c) $(2, -1, -1) \longrightarrow (0, -1, -1)$

$$I \otimes m_1 - (\xi_2 \cdot \xi_1) \otimes (\eta_1\eta_2 \cdot m_1) + (\eta_1 \cdot \xi_1) \otimes m_1 + (\eta_2 \cdot \xi_2) \otimes m_1$$

3.10.3. Theorem. *In $I(V)$, there are only the following singular for $\mathfrak{k}(1|6)$ and $\mathfrak{kas}$ vectors:*

1a) $\lambda = (k, -k, l, l) \longrightarrow \lambda + (-1, 1, 0, 0)$; $\mathfrak{kas}$ *and* $\mathfrak{k}(1|6)$: $\xi_1 \otimes m_1$ *(for $\mathfrak{k}(1|6)$ only if $l = 0$; for $\mathfrak{kas}$ without restrictions)*;

1a*) $\lambda = (k, l, 1-k, l+1) \longrightarrow \lambda + (-1, 0, 1, 0)$

$\mathfrak{kas}$: $(-1+k+l)\,\xi_2 \otimes m_1 + \xi_1 \otimes (\xi_2\eta_1 \cdot m_1)$

$\mathfrak{k}(1|6)$: *the above for* $l = -1$

1b) $\lambda = (k, l, l, 2-k) \longrightarrow \lambda + (-1, 0, 0, 1)$, *where* $l + k \neq 2$

$\mathfrak{kas}$: $\xi_1 \otimes (\xi_2\eta_1 \cdot \xi_3\eta_2 \cdot m_1) + \xi_2 \otimes (\xi_3\eta_2 \cdot m_1) + (-2+k+l)\,\xi_3 \otimes m_1$

$\mathfrak{k}(1|6)$: *the above for* $l = -1$

1c) $\lambda = (k, l, -l-2, k-2) \longrightarrow \lambda + (-1, 0, 0, -1)$, *where* $l + k \neq 1$ *and* $k - l \neq 4$

$\mathfrak{kas}$: $\xi_1 \otimes (\xi_2\eta_1 \cdot \eta_2\eta_3 \cdot m_1) + (-4+k-l)\,\xi_1 \otimes (\eta_1\eta_3 \cdot m_1)$
$+(-1+k+l)\,\xi_2 \otimes (\eta_2\eta_3 \cdot m_1) - (-4+k-l)\,(-1+k+l)\,\eta_3 \otimes m_1$

$\mathfrak{k}(1|6)$: *the above for* $l = -1$

The singular vectors of degree 2 for $\mathfrak{kas}$ and $\mathfrak{k}(1|6)$ are the same:

2a) $(3,-1,-1,-1) \longrightarrow (1,-1,-1,-1)$

$$\begin{aligned}&-2\,I\otimes m_1+(\xi_2{\cdot}\xi_1)\otimes(\xi_3\eta_2{\cdot}\eta_1\eta_3{\cdot}m_1)+(\xi_3{\cdot}\xi_1)\otimes(\eta_1\eta_3{\cdot}m_1)\\&+(\xi_3{\cdot}\xi_2)\otimes(\eta_2\eta_3{\cdot}m_1)-(\eta_1{\cdot}\xi_1)\otimes m_1-(\eta_2{\cdot}\xi_1)\otimes(\xi_2\eta_1{\cdot}m_1)\\&-(\eta_2{\cdot}\xi_2)\otimes m_1+(\eta_3{\cdot}\xi_1)\otimes(\xi_2\eta_1{\cdot}\xi_3\eta_2{\cdot}m_1)\\&-(\eta_3{\cdot}\xi_1)\otimes(\xi_3\eta_1{\cdot}m_1)-(\eta_3{\cdot}\xi_3)\otimes m_1\end{aligned}$$

2a*) $(3,-1,0,0) \longrightarrow (1,-1,0,0)$

$$\begin{aligned}&-I\otimes m_1+(\xi_2{\cdot}\xi_1)\otimes(\xi_3\eta_2{\cdot}\eta_1\eta_3{\cdot}m_1)+(\xi_3{\cdot}\xi_1)\otimes(\eta_1\eta_3{\cdot}m_1)\\&+(\xi_3{\cdot}\xi_2)\otimes(\eta_2\eta_3{\cdot}m_1)-(\eta_1{\cdot}\xi_1)\otimes m_1-(\eta_2{\cdot}\xi_1)\otimes(\xi_2\eta_1{\cdot}m_1)\\&+(\eta_3{\cdot}\xi_1)\otimes(\xi_2\eta_1{\cdot}\xi_3\eta_2{\cdot}m_1)-(\eta_3{\cdot}\xi_1)\otimes(\xi_3\eta_1{\cdot}m_1)-(\eta_3{\cdot}\xi_2)\otimes(\xi_3\eta_2{\cdot}m_1)\end{aligned}$$

2b) $(3,-1,-1,1) \longrightarrow (1,-1,-1,1)$

$$\begin{aligned}&-I\otimes m_1+(\xi_2{\cdot}\xi_1)\otimes(\eta_2\eta_3{\cdot}\xi_3\eta_1{\cdot}m_1)-(\xi_3{\cdot}\xi_1)\otimes(\xi_2\eta_1{\cdot}\eta_2\eta_3{\cdot}m_1)\\&+(\xi_3{\cdot}\xi_1)\otimes(\eta_1\eta_3{\cdot}m_1)-(\eta_1{\cdot}\xi_1)\otimes m_1-(\eta_2{\cdot}\xi_1)\otimes(\xi_2\eta_1{\cdot}m_1)\\&-(\eta_2{\cdot}\xi_2)\otimes m_1-(\eta_3{\cdot}\xi_1)\otimes(\xi_3\eta_1{\cdot}m_1)\\&-(\eta_3{\cdot}\xi_2)\otimes(\xi_3\eta_2{\cdot}m_1)+(\eta_3{\cdot}\xi_3)\otimes m_1\end{aligned}$$

3.11. Singular vectors for $\mathfrak{g}=\mathfrak{kas}(1|6;3\xi)$

Set

$$\begin{array}{ll} m_2=\xi_1\eta_2\,m_1 & m_{15}=\xi_1\eta_2\,(\xi_3\eta_1)^2\,m_1\\ m_3=\xi_3\eta_1\,m_1 & m_{16}=\xi_1\eta_2\,\xi_3\eta_1\,\eta_3\,m_1\\ m_4=\eta_3\,m_1 & m_{18}=\xi_1\eta_2\,\eta_1\,m_1\\ m_6=\xi_1\eta_2\,\xi_3\eta_1\,m_1 & m_{23}=\eta_3\,\xi_3\eta_2\,m_1\\ m_7=\xi_1\eta_2\,\eta_3\,m_1 & m_{25}=-(\eta_2\,m_1)\\ m_9=\xi_3\eta_1\,\eta_3\,m_1 & m_{34}=\xi_1\eta_2\,(\xi_3\eta_1)^2\,\eta_3\,m_1\\ m_{10}=\xi_3\eta_2\,m_1 & m_{36}=\xi_1\eta_2\,\xi_3\eta_1\,\eta_1\,m_1\\ m_{11}=\eta_1\,m_1 & m_{46}=-(\xi_3\eta_1\,\eta_2\,m_1)\\ & m_{49}=\xi_3\eta_2\,\eta_1\,m_1\end{array}$$

3.11.1. Theorem. *In $I(V)$ in degrees d), there are only the following singular vectors:*

1a) $\lambda \longrightarrow \lambda+(1,1,1,1)$ *for ANY* λ: $\xi_1\xi_2\xi_3\,m_1$;

1b) $(k,l-1,-l-1,k) \longrightarrow (k,l,-l,k)$: $\xi_1\xi_2\,m_1+\xi_1\xi_2\xi_3\,m_4$;

1c) $(k,l,k-1,-l-1) \longrightarrow (k,l,k,-l)$, *where* $k+l\neq 0$

$$\xi_1\xi_2\,m_3-(k+l)\xi_2\xi_3\,m_1+\xi_1\xi_2\xi_3\,(m_9-(k+l+1)m_{11})$$

1d) $(k+1,k-1,l,-l) \longrightarrow (k+1,k-1,l+1,-l+1)$, *where* $k+l\neq 0$, $k\neq l+1$

$$\begin{aligned}&\xi_1\xi_2\,(m_6+(-k+l)m_{10})-(-1+k-l)(k+l)\xi_1\xi_3\,m_1\\&-(k+l)\xi_2\xi_3\,m_2+\xi_1\xi_2\xi_3\,(m_{16}-(k+l+1)m_{18}+(-k+l)m_{23}-(k-l)(k+l)m_{25})\end{aligned}$$

1e) $(k+1,k-1,0,0) \longrightarrow (k+1,k-1,1,1)$, *where* $k\neq 1$ *(new for* $k=0$ *only)*

$$\begin{aligned}&\xi_1\xi_2\,(m_6-2m_{10})+2(1-k)\xi_1\xi_3\,m_1\\&-2\xi_2\xi_3\,m_2-2\xi_1\xi_2\xi_3\,(m_{18}+m_{23}+km_{25})\end{aligned}$$

2a) $\lambda \longrightarrow \lambda+(1,2,2,1)$, *where* $\lambda_4=-2-\lambda_3$: $\xi_1\xi_2\cdot\xi_1\xi_2\xi_3\,m_1$;

2b) $\lambda \longrightarrow \lambda+(1,2,1,2)$, *where* $\lambda_4=-1-\lambda_2$, $\lambda_2+\lambda_3\neq -1$

$$\xi_1\xi_2\cdot\xi_1\xi_2\xi_3\,m_3+(-1-\lambda_2-\lambda_3)(\xi_2\xi_3\cdot\xi_1\xi_2\xi_3\,m_1)$$

2c) $\lambda \longrightarrow \lambda+(1,1,2,2)$, *where* $\lambda_4=-\lambda_3$, $\lambda_2\neq\lambda_3$, $\lambda_2+\lambda_3\neq -1$

$$\begin{aligned}&(\xi_1\xi_2\cdot\xi_1\xi_2\xi_3)\,(m_6+(-1-\lambda_2+\lambda_3)m_{10})-(\lambda_2-\lambda_3)(1+\lambda_2+\lambda_3)\\&\times(\xi_1\xi_3\cdot\xi_1\xi_2\xi_3)\,m_1+(-1-\lambda_2-\lambda_3)(\xi_2\xi_3\cdot\xi_1\xi_2\xi_3)\,m_2\end{aligned}$$

2d) $(k,l,0,0) \longrightarrow (k+1,l+1,2,2)$, *where* $l\neq 0$ *(new for* $l=-1$ *only)*

$$(\xi_1\xi_2\cdot\xi_1\xi_2\xi_3)\,m_{10}+l(\xi_1\xi_3\cdot\xi_1\xi_2\xi_3)\,m_1+(\xi_2\xi_3\cdot\xi_1\xi_2\xi_3)\,m_2$$

2e) $(k,-k,k-3,k-1) \longrightarrow (k,-k+2,k-2,k)$

$$(\xi_1\xi_2)^2\,m_3-2\xi_2\cdot\xi_1\xi_2\xi_3\,m_1+2\xi_1\xi_2\cdot\xi_2\xi_3\,m_1+\xi_1\xi_2\cdot\xi_1\xi_2\xi_3\,(m_9-m_{11})+2\xi_2\xi_3\cdot\xi_1\xi_2\xi_3\,m_4$$

2f) $(k,-k,l,l) \longrightarrow (k,-k+2,l+1,l+1)$, *where* $k\neq l$

$$(\xi_1\xi_2)^2\,m_3-4(k-l)\xi_2\cdot\xi_1\xi_2\xi_3\,m_1-4\xi_1\xi_2\cdot\xi_1\xi_2\xi_3\,m_{11}+4\xi_2\xi_3\cdot\xi_1\xi_2\xi_3\,m_4$$

2g) $(1+k,l,-k,1+l) \longrightarrow (1+k,l+1,2-k,2+l)$

$$\begin{aligned}&(k-l)(1+k-l)(k+l)\xi_1\cdot\xi_1\xi_2\xi_3\,m_1+(k-l)(1+k-l)\xi_2\cdot\xi_1\xi_2\xi_3\,m_2\\&+\xi_1\xi_2\cdot\xi_1\xi_2\xi_3\,(m_{16}+(k-l)m_{18}+(-1-k-l)m_{23}+(1+k-l)(1+k+l)m_{25})\\&-(1+k-l)(k+l)\xi_1\xi_3\cdot\xi_1\xi_2\xi_3\,m_4+(-1-k+l)\xi_2\xi_3\cdot\xi_1\xi_2\xi_3\,m_7\end{aligned}$$

2h) $(k+2,k-2,k,-k) \longrightarrow (k+2,k-1,k+1,-k+2)$

$$\begin{aligned}&(\xi_1\xi_2)^2\,m_{15}+2k(-1+2k)(\xi_2\xi_3)^2\,m_2-2(-1+2k)\xi_1\cdot\xi_1\xi_2\xi_3\,m_3\\&-2\xi_2\cdot\xi_1\xi_2\xi_3\,(m_6+2km_{10})-4k(-1+2k)\xi_3\cdot\xi_1\xi_2\xi_3\,m_1+2(-1+2k)\xi_1\xi_2\cdot\xi_1\xi_3\,m_3\\&-2(-1+2k)\xi_1\xi_2\cdot\xi_2\xi_3\,m_6+\xi_1\xi_2\cdot\xi_1\xi_2\xi_3\,(m_{34}-2(1+k)m_{36}-2km_{46}-2km_{49})\\&-4k(-1+2k)\xi_1\xi_3\cdot\xi_2\xi_3\,m_1+2(-1+2k)\xi_1\xi_3\cdot\xi_1\xi_2\xi_3\,(m_9-(1+2k)m_{11})\\&+2\xi_2\xi_3\cdot\xi_1\xi_2\xi_3\,(-(-1+k)m_{16}+(-1+2k^2)m_{18}+km_{23}+2(-1+2k)m_{25})\end{aligned}$$

2i) $(-l+1,k,k,l+1)\longrightarrow(-l+2,k+2,k+2,l+2)$

$$\begin{aligned}&(k-l)(\xi_2\xi_3)^2\,m_2-2(k+l)\xi_1\cdot\xi_1\xi_2\xi_3\,m_3+2\xi_2\cdot\xi_1\xi_2\xi_3\,(-m_6+(1-k-l)m_{10})\\&-2(-1+k-l)(k+l)\xi_3\cdot\xi_1\xi_2\xi_3\,m_1+2\xi_1\xi_2\cdot\xi_2\xi_3\,(-m_6+m_{10})\\&+2\xi_1\xi_2\cdot\xi_1\xi_2\xi_3\,(m_{46}+m_{49})+2\xi_1\xi_3\cdot\xi_1\xi_2\xi_3\,(-m_9+(k-l)m_{11})\\&+2\xi_2\xi_3\cdot\xi_1\xi_2\xi_3\,(m_{18}-m_{23}+(2-k+l)m_{25})\end{aligned}$$

3.12. Singular vectors for $\mathfrak{g}=\mathfrak{kas}(1|6;3\eta)$

The m_i are the following elements of the irreducible $\mathfrak{g}_0$-module V:

$m_2=\xi_2\,m_1$	$m_{13}=\xi_2\,\xi_1\eta_2\,\xi_3\eta_1\,m_1$	$m_{30}=\xi_2\,\xi_1\eta_2\,\xi_3\eta_2\,m_1$
$m_3=\xi_1\eta_2\,m_1$	$m_{16}=\xi_2\,\xi_3\eta_2\,m_1$	$m_{34}=-(\xi_2\,\xi_3\,m_1)$
$m_4=\xi_3\eta_1\,m_1$	$m_{18}=(\xi_1\eta_2)^2\,\xi_3\eta_1\,m_1$	$m_{37}=(\xi_1\eta_2)^2\,(\xi_3\eta_1)^2\,m_1$
$m_5=\xi_2\,\xi_1\eta_2\,m_1$	$m_{19}=\xi_1\eta_2\,(\xi_3\eta_1)^2\,m_1$	$m_{41}=\xi_1\eta_2\,\xi_3\eta_1\,\xi_1\,m_1$
$m_6=\xi_2\,\xi_3\eta_1\,m_1$	$m_{21}=\xi_1\eta_2\,\xi_3\eta_2\,m_1$	$m_{42}=\xi_1\eta_2\,\xi_3\eta_1\,\xi_3\eta_2\,m_1$
$m_7=(\xi_1\eta_2)^2\,m_1$	$m_{23}=\xi_3\eta_1\,\xi_1\,m_1$	$m_{43}=-(\xi_1\eta_2\,\xi_3\,m_1)$
$m_8=\xi_1\eta_2\,\xi_3\eta_1\,m_1$	$m_{24}=\xi_3\eta_1\,\xi_3\eta_2\,m_1$	$m_{48}=\xi_1\,\xi_3\eta_2\,m_1$
$m_9=(\xi_3\eta_1)^2\,m_1$	$m_{25}=-(\xi_3\,m_1)$	$m_{49}=(\xi_3\eta_2)^2\,m_1$
$m_{10}=\xi_1\,m_1$	$m_{27}=\xi_2\,(\xi_1\eta_2)^2\,\xi_3\eta_1\,m_1$	$m_{86}=-(\xi_1\,\xi_3\,m_1)$
$m_{11}=\xi_3\eta_2\,m_1$		

3.12.1. Theorem. *In $I(V)$ in degrees d), there are only the following singular vectors:*

1a) $(k,l,m,-m)\longrightarrow(l,l,m-1,-m-1)$: $\eta_1\eta_3\,m_1$;

1b) $(k,l,m,-l-1)\longrightarrow(k,l-1,m,-l-2)$: $\eta_1\eta_3\,m_3+(-l+m)\eta_2\eta_3\,m_1$;

1c) $(k,l,-l-2,m)\longrightarrow(k,l-1,-l-3,m)$

$$(1+l-m)(2+l+m)\eta_1\eta_2\,m_1+\eta_1\eta_3\,(m_8+(-2-l-m)m_{11})+(-1-l+m)\eta_2\eta_3\,m_4$$

1d) $(k+3,-k-2,k,k-1)\longrightarrow(k+2,-k-2,k-1,k-1)$

$$2k\eta_1\,m_1-2k\eta_3\,m_4+2k\eta_1\eta_2\,m_2+\eta_1\eta_3\,(m_{13}+m_{16})+2k\eta_2\eta_3\,m_6$$

1e) $(k+3,k-1,-k-1,k-1)\longrightarrow(k+2,k-2,-k-1,k-1)$

$$\begin{aligned}&2k\eta_1\,m_3-4k^2\eta_2\,m_1+2k\eta_3\,(-m_8+(1+2k)m_{11})\\&+2k\eta_1\eta_2\,(m_5+2km_{10})+\eta_1\eta_3\,(m_{27}-2km_{30}+2km_{41}-2km_{43}-4k^2m_{48})\\&+2k\eta_2\eta_3\,(-m_{16}-m_{23}+m_{25})\end{aligned}$$

1f) $(4,0,-1,-1)\longrightarrow(3,-1,-1,-1)$

$$\begin{aligned}&-2\eta_1\,m_3)+2\eta_2\,m_1+\eta_3\,(m_8-2m_{11})-\eta_1\eta_2\,(m_5+m_{10})\\&+\eta_1\eta_3\,(m_{27}+m_{43}+m_{48})-\eta_2\eta_3\,(2m_{13}-m_{16}+m_{23})\end{aligned}$$

1g) $(4,0,0,0)\longrightarrow(2,0,0,0)$

$$6\,m_1+\eta_1\,(m_5+3m_{10})+3\eta_2\,m_2-\eta_3\,(m_{16}+m_{23})+\eta_1\eta_2\,m_{15}+\eta_1\eta_3\,m_{86}+\eta_2\eta_3\,m_{34}$$

2a) $(k,l,m,2-m)\longrightarrow(k,l,m-2,-m)$: $(\eta_1\eta_3)^2\,m_1$;

2b) $(k,l,l+2,-l-1)\longrightarrow(0,l-1,l+1,-l-3)$

$$(\eta_1\eta_3)^2\,m_3+2\eta_1\eta_3\cdot\eta_2\eta_3\,m_1$$

2c) $(k,l-2,-l,l+1)\longrightarrow(k,l-3,-l-2,l)$, *where* $l\neq-\frac{1}{2}$

$$(\eta_1\eta_3)^2\,(m_8-(1+2l)m_{11})-2(1+2l)\eta_1\eta_2\cdot\eta_1\eta_3\,m_1+2\eta_1\eta_3\cdot\eta_2\eta_3\,m_4$$

2d) $(k,l,m,-l)\longrightarrow(k,l-2,m,-l-2)$

$$(\eta_1\eta_3)^2\,m_7+(-1+l-m)(l-m)(\eta_2\eta_3)^2\,m_1-2(-1+l-m)\eta_1\eta_3\cdot\eta_2\eta_3\,m_3$$

2e) $(k,l-1,-l-1,-l+1)\longrightarrow(k,l-3,-1-2,-l)$

$$\begin{aligned}&(\eta_1\eta_3)^2\,(m_{18}-2m_{21})+2l(-1+2l)(\eta_2\eta_3)^2\,m_4+2(-1+2l)\eta_1\eta_2\cdot\eta_1\eta_3\,m_3\\&-4l(-1+2l)\eta_1\eta_2\cdot\eta_2\eta_3\,m_1+2(-1+2l)\eta_1\eta_3\cdot\eta_2\eta_3\,(-2m_8+m_{11})\end{aligned}$$

2ea) *Particular solution for* $l=\frac{1}{2}$*:*

$$(\eta_1\eta_3)^2\,m_{21}+(\eta_2\eta_3)^2\,m_4+2\eta_1\eta_2\cdot\eta_1\eta_3\,m_3-2\eta_1\eta_2\cdot\eta_2\eta_3\,m_1-\eta_1\eta_3\cdot\eta_2\eta_3\,m_8$$

2f) $(k,l,-l-1,m)\longrightarrow(k,l-2,-l-3,m)$, *where* $m\neq l,l+1,-l-1,-l-2$

$$\begin{aligned}&(l-m)(1+l-m)(1+l+m)(2+l+m)(\eta_1\eta_2)^2\,m_1\\&+(\eta_1\eta_3)^2\,(m_{37}-2(2+l+m)m_{42}+(1+l+m)(2+l+m)m_{49})\\&+(l-m)(1+l-m)(\eta_2\eta_3)^2\,m_9\\&+2(l-m)(2+l+m)\eta_1\eta_2\cdot\eta_1\eta_3\,(m_8-(1+l+m)m_{11})\\&-2(l-m)(1+l-m)(2+l+m)\eta_1\eta_2\cdot\eta_2\eta_3\,m_4\\&+2(l-m)\eta_1\eta_3\cdot\eta_2\eta_3\,(-m_{19}+(2+l+m)m_{24})\end{aligned}$$

2fa) *Particular solution for* $l=m=0$*:*

$$\begin{aligned}&4(\eta_1\eta_2)^2\,m_1+(\eta_1\eta_3)^2\,(m_{37}-4m_{49})+2(\eta_2\eta_3)^2\,m_9\\&+8\eta_1\eta_2\cdot\eta_1\eta_3\,(m_8-m_{11})-8\eta_1\eta_2\cdot\eta_2\eta_3\,m_4+4\eta_1\eta_3\cdot\eta_2\eta_3\,(-m_{19}+2m_{24})\end{aligned}$$

2fb) *Particular solution for* $l=-\frac{1}{2}$, $m=\frac{1}{2}$*:*

$$\begin{aligned}&2(\eta_1\eta_2)^2\,m_1+(\eta_1\eta_3)^2\,m_{42}+(\eta_2\eta_3)^2\,m_9+4\eta_1\eta_2\cdot\eta_1\eta_3\,m_8\\&-4\eta_1\eta_2\cdot\eta_2\eta_3\,m_4-\eta_1\eta_3\cdot\eta_2\eta_3\,m_{19}\end{aligned}$$

2g) $(4,-3,1,0)\longrightarrow(3,-3,-1,-1)$

$$(\eta_1\eta_3)^2\,(m_{13}+m_{16})+2\eta_1\cdot\eta_1\eta_3\,m_1-2\eta_3\cdot\eta_1\eta_3\,m_4+2\eta_1\eta_2\cdot\eta_1\eta_3\,m_2+2\eta_1\eta_3\cdot\eta_2\eta_3\,m_6$$

3.13. Singular vectors for $\mathfrak{g} = \mathfrak{vas}(4|4)$

We consider the following negative operators from $\mathfrak{g}_0$:

$$\begin{array}{ll}
a_5 = x_2\delta_3 + x_3\delta_2 & a_{14} = -x_4\partial_1 + \xi_1\delta_4 \\
a_6 = x_3\delta_3 & a_{15} = -2x_3\delta_4 - \xi_1\partial_2 + \xi_2\partial_1 \\
a_8 = x_2\delta_4 + x_4\delta_2 & a_{18} = -x_3\partial_2 + \xi_2\delta_3 \\
a_9 = x_3\delta_4 + x_4\delta_3 & a_{19} = -x_4\partial_2 + \xi_2\delta_4 \\
a_{10} = x_4\delta_4 & a_{20} = 2x_2\delta_4 - \xi_1\partial_3 + \xi_3\partial_1 \\
a_{12} = -x_2\partial_1 + \xi_1\delta_2 & a_{25} = -x_4\partial_3 + \xi_3\delta_4 \\
a_{13} = -x_3\partial_1 + \xi_1\delta_3 & a_{26} = -2x_2\delta_3 - \xi_1\partial_4 + \xi_4\partial_1
\end{array}$$

For the basis of Cartan subalgebra we take

$$\begin{aligned}
a_{11} &= -\frac{1}{2}x_1\partial_1 + \frac{1}{2}x_2\partial_2 + \frac{1}{2}x_3\partial_3 + \frac{1}{2}x_4\partial_4 + \xi_1\delta_1) \\
a_{17} &= \frac{1}{2}x_1\partial_1 - \frac{1}{2}x_2\partial_2 + \frac{1}{2}x_3\partial_3 + \frac{1}{2}x_4\partial_4 + \xi_2\delta_2 \\
a_{24} &= \frac{1}{2}x_1\partial_1 + \frac{1}{2}x_2\partial_2 - \frac{1}{2}x_3\partial_3\frac{1}{2}x_4\partial_4 + \xi_3\delta_3 \\
a_{32} &= \frac{1}{2}x_1\partial_1 + \frac{1}{2}x_2\partial_2 + \frac{1}{2}x_3\partial_3 - \frac{1}{2}x_4\partial_4 + \xi_4\delta_4)
\end{aligned}$$

The m_i are the following elements of the irreducible $\mathfrak{g}_0$-module V:

$$\begin{array}{ll}
m_1 \text{ is the highest weight vector} & m_{10} = a_8 \cdot m_1 \\
m_2 = a_5 \cdot m_1 & m_{11} = a_{20} \cdot m_1 \\
m_3 = a_{25} \cdot m_1 & m_{24} = a_{26} \cdot a_{20} \cdot m_1 \\
m_4 = a_{26} \cdot m_1 & m_{27} = -a_{12} \cdot m_1 \\
m_8 = a_{25} \cdot a_{26} \cdot m_1 &
\end{array}$$

3.13.1. Theorem. *In $I(V)$, there are only the following singular vectors:*

1a) $(k,l,l,l) \longrightarrow (k+1,l,l,l)$: $\delta_1 \otimes m_1$;

1b) $(-1,0,0,0) \longrightarrow \frac{1}{2}(-1,1,1,-1)$: $\partial_4 m_1 + \delta_1 m_4$

1c) $\frac{1}{2}(-1,1,1,-1) \longrightarrow (0,1,0,0)$: $\partial_3 m_1 - \partial_4 m_3 + \delta_1 m_{11}$

1d) $(l,k+l,l,l) \longrightarrow (l,k+l+1,l,l)$; *two particular cases:*

1da) $l \neq 0 \Longrightarrow k \neq -1$:

$$-\partial_3\,(4\,l\,m_2 + m_4) - \partial_4\,(-m_8 + 4\,l\,m_{10}) + \delta_1\,(m_{24} - 4\,l\,m_{27}) - 4\,(1+k)\,l\,\delta_2 m_1$$

1db) $l = 0 \Longrightarrow k \neq 0$: $\partial_3 m_2 + \partial_4 m_{10} + \delta_1 m_{27} + k\,\delta_2 m_1$

Chapter 4

The Lie superalgebra $\mathfrak{osp}(1|2)$, connections over symplectic manifolds and representations of Poisson algebras (J. Bernstein)

4.0. Introduction

Here, I describe irreducible representations of Poisson algebras. For this, I study actions of the central extension of the group of symplectomorphisms on tensor fields with values in the line bundle (L, ∇) endowed with a “maximally non-flat” connection ∇. The language of Lie superalgebras enables us to divide the space of primitive forms with values in this bundle into tinier parts.

This decomposition is a particular case of manifestation of the Howe duality. A similar construction takes place in the hyperkälerian case. Generalization of this construction to supermanifolds is also possible, in, at least, two ways, see [Shap2] and [LSH1, LSh3].

We show that each continuous (with respect to a natural topology) irreducible representation of the Poisson algebra $\mathfrak{po}(2n)$ is realized in the space of sections of “twisted” tensor fields over the symplectic manifold. The Lie superalgebra $\mathfrak{osp}(1|2)$ appears in this description of irreducible representations of $\mathfrak{po}(2n)$ in the same way as $\mathfrak{sl}(2)$ appears to introduce the primitive forms on the symplectic manifold. Thus with the help of $\mathfrak{osp}(1|2)$ we construct a “square root” of the Hodge–Lepage[1)] decomposition.

I am thankful to D. Leites who took notes of my 1977 talk at his “Seminar on Supersymmetries”, and the Department of Mathematics of Stockholm University for financing a preprint of these notes.

4.1. Primitive forms, invariant differential operators and irreducible representations of Lie algebras of Hamiltonian vector fields

4.1.1. Let (M, ω) be a $2n$-dimensional (real) symplectic manifold, G the group of its diffeomorphisms preserving symplectic structure, $\Omega = \oplus\Omega^i$ the

[1] For history and more, see [KLR].

algebra of differential forms on M. The following result is well known, see [Bou], [Weil].

In Ω, let X_+ be the operator of (left exterior) multiplication by ω, let X_- be the operator of the inner multiplication by (i.e., the convolution with) the bivector dual to the form ω and let $H := [X_+, X_-]$. The following statement is an obvious corollary of definitions, but very important.

4.1.1.1. Theorem. *On Ω, the operators X_+, X_- and H define an $\mathfrak{sl}(2)$-action commuting with the G-action.*

The elements of the space $P^i = \operatorname{Ker} X_- \cap \Omega^i$ are said to be *primitive forms* of degree i. The $\mathfrak{sl}(2)$-action in Ω is completely reducible; it is the sum of an infinite number of finite dimensional $\mathfrak{sl}(2)$-modules with lowest weights $-n, -n+1, \dots, 0$. The primitive forms are, in fact, the lowest weight vectors of the irreducible components.

4.1.2. Let ρ be a representation of the group $\mathrm{Sp}(2n;\mathbb{R})$ in a finite dimensional space V. Let us give two equivalent definitions of tensor fields of type (ρ, V) on the symplectic manifold (M,ω). The space of tensor fields of type (ρ, V) will be denoted by $T(\rho, V)$ or just $T(V)$.

a) The group G naturally acts on the space $T(\rho)$: in any coordinate system x the object $t \in T(\rho)$ is defined by the V-valued vector function $t(x)$ such that

$$t(y(x)) = \rho\left(\frac{\partial y}{\partial x}\right) t(x), \quad \text{where } \frac{\partial y}{\partial x} \text{ is the Jacobi matrix.}$$

b) Let $G_p \subset G$ be the stabilizer of point $p \in M$. Clearly, $G_p \cong \mathrm{Sp}(2n;\mathbb{R})\cdot N$, where N is a normal subgroup. Let $E \longrightarrow M$ be a vector bundle with fibres isomorphic to V and let the G-action on E be such that G_p acts in the fibre over point p via ρ. Then the space of sections of the bundle E and the space $T(\rho)$ from definition a) coincide.

In this subsection we will describe all G-invariant differential operators $c\colon T(\rho_1) \longrightarrow T(\rho_2)$. The problem of their description is, in fact, equivalent to the following formal problem raised and solved by A. N. Rudakov. For its various generalizations, see [GLS2].

Let $\mathbb{K}$ be an algebraically closed field, $\operatorname{Char}\mathbb{K} = 0$. The Lie algebra of Hamiltonian vector fields $\mathfrak{h}(2n)$ consists of fields that preserve the form ω, i.e., $\mathfrak{h}(2n) = \{D \mid L_D(\omega) = 0\}$. In other words, $\mathfrak{h}(2n)$ consists of the vector fields of the form

$$H_f = \sum \left(\frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i}\right) \quad \text{for any } f \in \mathbb{K}[[q,p]].$$

The filtration of $\mathbb{K}[[q,p]]$ defined by the powers of the maximal ideal (q,p) induces on $\mathfrak{h}(2n) = \mathcal{L}$ the filtration

$$\mathcal{L} = \mathcal{L}_{-1} \supset \mathcal{L}_0 \supset \mathcal{L}_i \supset \dots,$$

where $\mathcal{L}_i = \{H_f \mid f \in (q,p)^{i+2}\}$. Set $L_i = \mathcal{L}_i/\mathcal{L}_{i+1}$. Note that $L_0 \cong \mathfrak{sp}(2n)$.

Let V be an L_0-module relative a representation ρ. Let us extend this representation to $\mathcal{L}_0$-module such that $\mathcal{L}_1 V = 0$. The elements of the space

$$T(\rho, V) := \operatorname{Hom}_{U(\mathcal{L}_0)}(U(\mathcal{L}), V) \simeq \mathbb{K}[[q,p]] \otimes V$$

are said to be (formal) *tensor fields of type* (ρ, V) because of the isomorphism above which follows from the Poincaré-Birkhoff-Witt theorem: Indeed,

$$U(\mathcal{L}) \simeq U(\mathcal{L}_0 \oplus L_{-1}) \simeq U(\mathcal{L}_0) \otimes U(L_{-1}) \text{ and } U(L_{-1}) \simeq \mathbb{K}[\partial_q, \partial_p], \text{ so}$$
$$U(L_{-1})^* \simeq \mathbb{K}[[q,p]].$$

4.1.2.1. Theorem. *Let V_1 and V_2 be irreducible $\mathfrak{sp}(2n)$-modules with lowest weight vector and $c\colon T(V_1) \longrightarrow T(V_2)$ a non-zero $\mathfrak{h}(2n)$-invariant differential operator. Then one and only one of the following cases may occur:*

1) *$V_1 \cong V_2$ and c is a scalar operator extending this isomorphism;*

2) *$T(V_1)$ and $T(V_2)$ are neighboring terms in the sequence of primitive forms induced by the de Rham complex*

$$P^0 \xrightarrow{d_p} P^1 \xrightarrow{d_p} \dots \xrightarrow{d_p} P^n \xrightarrow{d_p^*} P^{n-1} \xrightarrow{d_p^*} \dots \xrightarrow{d_p^*} P^0 \tag{4.1}$$

and c is a multiple of d_p, where d_p is the composition of the exterior differential d and the projection $\Omega^i \longrightarrow P^i$.

3) *$T(V_1) = T(V_2) = P^i$ and c is a multiple of $d_p X_- d$.*

This theorem and Appendix to [BL2] imply that G-invariant differential operators are the same as in the formal case. Actually, Appendix to [BL2] implies that $\mathcal{L}$-invariant differential operators are also invariant with respect to the Lie algebra of smooth Hamiltonian vector fields. Since $\mathcal{L}$-invariant operators are also invariant with respect to reflections, i.e., with respect to elements of G/G_0, where G_0 is the connected component of the unit, they are also G-invariant.

The operator $d_p^*\colon P^i \longrightarrow P^{i-1}$ is often called the *codifferential* and denoted by δ.

4.1.3. The description of $\mathfrak{h}(2n)$-modules. Sec. 4.1.2 and the following analogue of the Poincaré lemma imply the description of irreducible topological representations of the Lie algebra $\mathfrak{h}(2n)$.

4.1.3.1. Theorem. *The sequence* (4.1) *is exact in all terms except P^0; we have $P^0 \cap \operatorname{Ker} d_p = \mathbb{K}$.*

The proof of this theorem is analogous to that of Theorem 1 of [BL2].

4.1.3.2. Theorem. ([R2]) *Let V be an irreducible $\mathfrak{sp}(2n)$-module. Let*

$$\operatorname{irr} T(V) = \begin{cases} T(V) & \text{if } T(V) \not\simeq P^i, \\ \operatorname{Im} d_p X_- d \cap P^i & \text{if } T(V) \simeq P^i \text{ for } i > 0, \\ \mathbb{K} & \text{if } T(V) \simeq P^0. \end{cases}$$

All irreducible topological $\mathfrak{h}(2n)$*-modules are submodules of the modules of tensor fields* $T(V)$ *and are of the form* irr $T(V)$ *for some* V.

In other words, if V is an irreducible $\mathfrak{sp}(2n)$-module and $T(V) \neq P^i$, then the module $T(V)$ is irreducible, whereas if $T(V) = P^i$, then the structure of its submodules, though not extremely complicated, requires a moment of consideration: there are two submodules: kernels of d and δ, respectively, and the quotient modulo their union is isomorphic to their intersection.

4.2. The Lie superalgebra $\mathfrak{osp}(1|2)$ and forms with values in the bundle with the most curved connection on (M, ω)

4.2.1. The Lie superalgebra $\mathfrak{osp}(1|2)$ and its representations. The Lie superalgebra $\mathfrak{osp}(1|2)$ is the next of kin of the Lie algebra $\mathfrak{sl}(2)$. Most of their properties are identical. Here is the standard matrix realization of the Lie superalgebra $\mathfrak{osp}(1|2)$, where only basis elements are written. By definition, X_+, X_- and H are even, while D_+ and D_- are odd:

$$X_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$D_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The defining relations are

$$[D_+, D_-] = H, \quad [H, D_\pm] = \pm D_\pm, \tag{4.2}$$

where we take simultaneously either all the upper or all the lower signs $+$ or $-$. Other relations with the non-zero right hand side are listed below for the convenience of possible computations

$$\begin{aligned} &[H, X_\pm] = \pm X_\pm, \quad [X_+, X_-] = H, \\ &[X_\pm, D_\mp] = D_\pm, \quad [D_\pm, D_\pm] = \mp 2X_\pm, \end{aligned} \tag{4.3}$$

Define the $\mathfrak{osp}(1|2)$-module L^m: Let $\{l_{-m}, l_{-m+1}, \dots, l_m\}$ be a homogeneous basis of L^m, with a parity defined by the eq. $p(l_i) \equiv i \mod 2$ and the $\mathfrak{osp}(1|2)$-action given by

$$\begin{aligned} D_- l_k &= l_{k-1}, \\ D_+ l_m &= 0, \\ H l_m &= m l_m. \end{aligned} \tag{4.4}$$

The relations (4.2) easily imply that

$$\begin{aligned} &H l_k = k l_k \text{ for any } k, \\ &D_+ l_k = c(k) l_{k+1} \text{ for an easy to find coefficient } c(k). \end{aligned} \tag{4.5}$$

Exercise. Find $c(k)$.

A non-zero vector $v \in L$, where L is an $\mathfrak{osp}(1|2)$-module, is said to be the *highest* (*lowest*) weight vector if $D_+ v = 0$ (resp., $D_- v = 0$) and $Hv = \lambda v$ for some $\lambda \in \mathbb{K}$. This λ is called the *highest* (*lowest*) weight.

4.2.1.1. Theorem. 1) *The element*

$$\begin{aligned} \triangle = 2(X_+X_- + X_-X_+ + \frac{1}{2}(D_+D_- + D_-D_+ + H^2) = \\ 4X_-X_+ - 2D_-D_+ + H^2 + 3H \end{aligned}$$

belongs to the center of the universal enveloping algebra of the Lie superalgebra $\mathfrak{osp}(1|2)$.

2) *If* $l \in L^m$, *then* $\triangle(l) = m(m+3)l$.

3) *Each finite dimensional* $\mathfrak{osp}(1|2)$*-module contains a submodule isomorphic to either* L^m *or* $\Pi(L^m)$ *for some* m.

4) *Finite dimensional representations of* $\mathfrak{osp}(1|2)$ *are completely reducible.*

5) L^m *is an irreducible* $\mathfrak{osp}(1|2)$*-module.*

6) *Modules* L^m *and* $\Pi(L^m)$ *exhaust the collection of finite dimensional irreducible* $\mathfrak{osp}(1|2)$*-modules up to isomorphism.*

Proof of this theorem is similar to that of corresponding statements for $\mathfrak{sl}(2)$, see [Bou]. Note that Theorem 4.2.1.1 implies that any finite dimensional irreducible $\mathfrak{osp}(1|2)$-module is uniquely defined not just by its highest (lowest) weight, as for simple Lie algebras over $\mathbb{C}$, but by the pair: (the highest (lowest) weight, and the parity of the highest (lowest) weight vector).

4.2.2. Let L be the space of a (complex) line bundle over a connected symplectic manifold (M, ω) with a connection ∇ such that the curvature form of ∇ is equal to $\hbar\omega$ for some $\hbar \in \mathbb{C}$. Denote by $\widehat{G}$ the group of ∇-preserving automorphisms of the bundle $L \longrightarrow M$.

The $\widehat{G}$-action on M defines a homomorphism $\widehat{G} \longrightarrow G$. It is known (see [Bou]) that this homomorphism is epimorphic in a neighborhood of the unit of the group G and its kernel is isomorphic to $\mathbb{C}^\times$.

It is clear that $\widehat{G}$ acts in the space of tensor fields on M with values in the bundle L so that $1 \in \mathbb{C}^\times$ acts on these fields as multiplication by $\hbar \in \mathbb{C}$. This $\hbar$ will be called a *weight*; the space of tensor fields of type (ρ, V) and of weight $\hbar$ will be denoted by $T_\hbar(\rho, V)$.

4.2.3. Let us naturally extend the action of X_+, X_- and H from Ω to $\Omega_\hbar$ using the isomorphism of *spaces*

$$T_\hbar(\rho, V) \simeq T(\rho, V) \otimes \Omega_\hbar,$$

i.e., $X_+ \mapsto X_+ \otimes 1$, etc. Let D_+ be the connection ∇ itself and $D_- = [X_-, D_+]$. On $\Omega_\hbar$, we introduce a superspace structure by setting

$$p(\varphi \otimes s) = \deg \varphi \pmod 2, \text{ for any } \varphi \in \Omega \text{ and } s \in \Omega^0_\hbar.$$

Theorem. *On $\Omega_\hbar$, the operators X_+, X_-, H and D_+, D_- define an $\mathfrak{osp}(1|2)$-action commuting with the $\hat{G}$-action.*

Proof is a straightforward verification of relations (4.2). $\square$

The elements of the space $\sqrt{P}^i_\hbar = \operatorname{Ker} D_- \cap P^i_\hbar$ will be called ∇-*primitive forms* of degree i (and weight $\hbar$). Clearly, the $\mathfrak{osp}(1|2)$-action on $\Omega_\hbar$ is completely reducible. The module $\Omega_\hbar$ decomposes into the direct sum of infinitely many finite dimensional $\mathfrak{osp}(1|2)$-modules with lowest weights $-n, -n+1, \cdots, 0$, so that ∇-primitive forms are the lowest weight vectors of irreducible components.

4.2.4. Coordinate expressions of certain operators in terms of the supermanifold $\widehat{M} = (M, \Omega)$. Though in this paper we can do without supermanifolds, the formulas of this section will be clearer in terms of supermanifolds. For the definition of general supermanifolds as ringed spaces whose structure sheaf is locally isomorphic to the sheaf of sections of the exterior algebra of a vector bundle, see Ch. 1.

Consider Ω as the superalgebra of functions on a supermanifold, denoted in what follows $\widehat{M} = (M, \Omega)$; the coordinates on $\widehat{M}$ are the even x's and the odd $\hat{x}$'s, where, speaking informally, $\widehat{x}_i = dx_i$.

Let $D = \sum f_i \partial_i$ be a vector field on (M, ω). Then, in the coordinates $x, \hat{x}$, where $x = (q, p)$, $\partial_i = \dfrac{\partial}{\partial x_i}$, we have the following expressions for the Lie derivative and several other operators on $\Omega(M)$:

$$\begin{aligned}
&L_D = D + \sum \widehat{x}_j \frac{\partial f_i}{\partial x_j} \partial_i;\\
&d = \sum \widehat{x}_i \partial_i;\\
&\delta = \sum \frac{\partial}{\partial \widehat{p}_i} \frac{\partial}{\partial q_i} - \frac{\partial}{\partial \widehat{q}_i} \frac{\partial}{\partial p_i};\\
&X_+ = \sum \widehat{q}_i \widehat{p}_i; \quad X_- = -\sum \frac{\partial}{\partial \widehat{p}_i} \frac{\partial}{\partial \widehat{q}_i}; \quad H = \sum \widehat{x}_i \frac{\partial}{\partial \widehat{x}_i} - n.
\end{aligned}$$

Let

$$\alpha = dt + \frac{\hbar}{2} \sum (p_i dq_i - q_i dp_i)$$

be the form of the connection ∇; here t is the coordinate in the fiber of the line bundle. Then the operators D_+ and D_- are of the form

$$\begin{aligned}
D_+ &= \sum \widehat{x}_i \frac{\partial}{\partial \widehat{x}_i} - \frac{\hbar}{2} \sum (\widehat{q}_i p_i - \widehat{p}_i q_i),\\
D_- &= \sum \left(\frac{\partial}{\partial \widehat{p}_i} \frac{\partial}{\partial q_i} - \frac{\partial}{\partial \widehat{q}_i} \frac{\partial}{\partial p_i} \right) - \frac{\hbar}{2} \sum \left(p_i \frac{\partial}{\partial \widehat{p}_i} + q_i \frac{\partial}{\partial \widehat{q}_i} \right).
\end{aligned}$$

4.3. Irreducible representations of Poisson algebras

4.3.1. The space of the *Poisson algebra* is the space of functions on a symplectic manifold; the Lie algebra structure is locally defined by the *Poisson bracket*.

Denote by $\mathfrak{po}(2n)$ the formal analogue of this Lie algebra. Explicitly, the space of $\mathfrak{po}(2n)$ coincides with $\mathbb{K}[[q, p]]$, where $q = (q_1, \ldots, q_n)$, $p = (p_1, \ldots, p_n)$, and the Poisson bracket is given by the formula

$$\{f, g\} = \sum \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

The powers of the maximal ideal $x = (q, p)$ define a filtration in $\mathfrak{po}(2n)$ of the form

$$\widehat{\mathcal{L}} = \widehat{\mathcal{L}}_{-2} \subset \widehat{\mathcal{L}}_{-1} \subset \widehat{\mathcal{L}}_0 \subset \widehat{\mathcal{L}}_1 \subset \ldots,$$

where $\widehat{\mathcal{L}}_i = \{f \mid f \in (x)^{i+2}\}$. Set $\widehat{\mathcal{L}} = \mathfrak{po}(2n)$ and let $\hat{L}_i = \widehat{\mathcal{L}}_i / \widehat{\mathcal{L}}_{i+1}$. Clearly, $\hat{L}_0 \cong \mathfrak{sp}(2n)$.

To any function $f \in \mathfrak{po}(2n)$ we assign the *Hamiltonian vector field*

$$H_f = \sum \left(\frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} \right).$$

Sometimes we have to distinguish between the functions f from $\mathbb{C}[[q, p]]$ and the elements of $\mathfrak{po}(2n)$ these functions generate; we denote the latter by K_f (for reasons, see Ch. 2). Obviously, the sequence

$$0 \longrightarrow \mathbb{K}K_1 \longrightarrow \mathfrak{po}(2n) \longrightarrow \mathfrak{h}(2n) \longrightarrow 0$$

is exact.

Let V be an $\widehat{\mathcal{L}}_0$-module such that $\widehat{\mathcal{L}}_1 V = 0$. The elements of the space

$$T_\hbar(V) = \operatorname{Hom}_{U(\widehat{\mathcal{L}}_0)}(U(\widehat{\mathcal{L}})/(K_1 - \hbar), V)$$

are called (formal) *tensor fields of type V and of weight $\hbar$*.

The operator $K_f \in \widehat{\mathcal{L}}$ acts in $T_\hbar(V)$ by the formula

$$K_f(s) = L_{H_f}(s) + (\hbar f + \alpha(H_f))s,$$

where $s \in T_\hbar(V)$ and α is the form of the connection ∇. This formula implies a geometric interpretation of the Lie algebra $\widehat{\mathcal{L}} = \mathfrak{po}(2n)$ as t h e L i e a l g e b r a p r e s e r v i n g t h e c o n n e c t i o n w i t h f o r m α, cf. [Ko].

4.3.2. In this subsection we will assume that $\hbar \neq 0$. The case $\hbar = 0$ is completely considered in Theorem 4.1.2.1.

Theorem. A) $P^i_\hbar = \sqrt{P}^i_\hbar \oplus D_+ \sqrt{P}^{i-1}_\hbar$.

B) *Let* V_1 *and* V_2 *be irreducible* $\mathfrak{sp}(2n)$*-modules with lowest weight vector and* $c\colon T_{\hbar_1}(V_1) \longrightarrow T_{\hbar_2}(V_2)$ *a non-zero* $\mathfrak{po}(2n)$*-invariant differential operator. Then* $\hbar_1 = \hbar_2 = \hbar$ *and only the following possibilities may occur:*

1) $V_1 \cong V_2$ *and* c *is either the scalar operator extending this isomorphism or* $T_\hbar(V_1) = T_\hbar(V_2) = P^i_\hbar$ *and* c *is a multiple of* D_+D_-*;*

2) $T_\hbar(V_1)$ *and* $T_\hbar(V_2)$ *are neighboring terms in the sequence*

$$P^0_\hbar \stackrel{D_+\circ pr}{\longrightarrow} P^1_\hbar \stackrel{D_+\circ pr}{\longrightarrow} \dots \stackrel{D_+\circ pr}{\longrightarrow} P^n_\hbar \stackrel{D_-}{\longrightarrow} P^{n-1}_\hbar \stackrel{D_-}{\longrightarrow} \dots \stackrel{D_-}{\longrightarrow} P^0_\hbar \tag{4.6}$$

and c *is a multiple of the respective* D_- *or* D_+*.*

4.3.3. Proof of Theorem 4.3.2. This proof is a direct generalizes the method due to A. N. Rudakov [R1], [R2] (for an account, see [BL2, GLS2]). In this subsection we use the results and notations from [R2], [BL2]. Set $\widehat{\mathcal{L}} = \mathfrak{po}(2n)$ and

$$I_\hbar(V) = U(\widehat{\mathcal{L}})/(K_1 - \hbar) \otimes_{U(\widehat{\mathcal{L}}_0)} V.$$

4.3.3.1. Lemma. $I_\hbar(V)^* = T_{-\hbar}(V^*)$.

The $\widehat{\mathcal{L}}$-module $I_\hbar$, where $\hbar$ is the value of the scalar by which the operator $K_1 \in \widehat{\mathcal{L}}$ multiplies it, is called *discrete* whenever $\dim\ U(\widehat{\mathcal{L}}_0)v < \infty$ and $\widehat{\mathcal{L}}_1^{r(v)}v = 0$ for some $r(v) \in \mathbb{N}$ for any vector $v \in I_\hbar$. The modules $I_\hbar(V)$ are examples of discrete $\widehat{\mathcal{L}}$-modules.

Clearly, $\mathrm{Hom}_{\widehat{\mathcal{L}}}(I_{\hbar_1}, I_{\hbar_2}) = 0$ if $\hbar_1 \neq \hbar_2$. The elements of $I_\hbar^{\widehat{\mathcal{L}}_1}$ are called *singular* vectors, see [R2].

4.3.3.2. Lemma. $\mathrm{Hom}_{\widehat{\mathcal{L}}}(I_\hbar(V), I_\hbar) \cong \mathrm{Hom}_{\widehat{\mathcal{L}}_0}(V, I_\hbar^{\widehat{\mathcal{L}}_1})$.

This lemma reduces the problem of description of $\widehat{\mathcal{L}}$-invariant operators $c\colon I_\hbar(V_1) \longrightarrow I_\hbar(V_2)$ to that of $\hat{L}_0$-homomorphisms $c_0\colon V_1 \longrightarrow I_\hbar(V_2)^{\widehat{\mathcal{L}}_1}$. If V_1 is irreducible, then c_0 is defined by the highest weight singular vector; therefore, the description of different homomorphisms c is reduced to that of highest weight singular vectors in $I_\hbar(V)$. The identification

$$U(\widehat{\mathcal{L}})/(K_1 - \hbar) \cong \mathbb{K}[\widehat{\partial}_q, \widehat{\partial}_p] \otimes U(\widehat{\mathcal{L}}_0)$$

implies that $I_\hbar(V) \cong \mathbb{K}[\widehat{\partial}_q, \widehat{\partial}_p] \otimes V$, where $\widehat{\partial}_{q_i}$ and $\widehat{\partial}_{p_i}$ do not commute but satisfy the relation

$$[\widehat{\partial}_{q_i}, \widehat{\partial}_{p_j}] = \delta_{ij}\hbar \tag{4.7}$$

The singular vectors in the $\mathfrak{po}(2n)$-modules $I_\hbar(V)$ are the same as in the $\mathfrak{h}(2n)$-modules $I(V)$ (found in [R1]), but because of relations (4.7) the fact that the vector is singular does not yet guarantee that this vector spans a proper submodule. However, singular vectors may only occur in the dual of $P^i_\hbar$; hence, the proof of headings 1)–3) of Theorem 4.3.2 is completed.

4.3.3.3. Corollary. *Let* V *be an irreducible* $\mathfrak{sp}(2n)$*-module. Let*

$$\begin{aligned} &\mathrm{irr}\ T_\hbar(V) := T_\hbar(V) \quad \text{if } T(V) \neq P^i \\ &\mathrm{irr}\ P^i_\hbar = \sqrt{P}^i_\hbar. \end{aligned}$$

The modules irr $T_\hbar(V)$ *exhaust all irreducible continuous* $\mathfrak{po}(2n)$*-modules.*

Chapter 5

Poisson superalgebras as analogs of the general linear Lie algebra. The spinor and oscillator representations (D. Leites, I. Shchepochkina)

5.1. Introduction

Traditionally, mathematicians learn about the Lie algebra $\mathfrak{gl}(n)$ rather early, in courses of Linear Algebra and, in accordance with modern overspecialization and divorce of physics from mathematics in the standard curricula, may graduate from the university without taking any course where the Poisson algebra is mentioned.

Unlike mathematicians, physics majors do learn about Poisson algebra, perhaps, even earlier than Linear Algebra, but the revelation of the crucial fact that the Poisson algebra $\mathfrak{po}(2n)$ is, basically, a "quasiclassical limit" of the "quantum object" — $\mathfrak{gl}(\infty)$ (in one of many incarnations of the general linear algebra acting on an infinite dimensional space) is seldom made. Students discover this later, in the postgraduate school.

And, until now, no student ever hears that the there is a one-parameter family of simple Lie superalgebras isomorphic to $\mathfrak{gl}$, or its "queer" version $\mathfrak{q}$, at all values of the parameter but one, and the spinor representations — traditionally looked at as weird second cousins in the family of tensors — are just the identity representations of the Lie superalgebras from the mentioned parametric family at the generic value of the parameter.

The super point of view makes the notions involved simple and transparent.

The importance of the *spinor* representation became clear rather early, more than a century ago. One of the reasons is the following. As is known from any textbook on representation theory[1)], the *fundamental representations* (the ones whose highest weights constitute a basis of the lattice of weights)

$$R(\varphi_1) = W, \quad R(\varphi_2) = \Lambda^2(W), \quad \dots, \quad R(\varphi_{n-1}) = \Lambda^{n-1}(W) \tag{5.1}$$

[1] Among the best, currently, are [FH], [OV].

of $\mathfrak{sl}(W)$, where $\dim W = n$ and φ_i is the highest weight of $\Lambda^i(W)$, are irreducible. Any finite dimensional irreducible $\mathfrak{sl}(n)$-module L^λ is completely determined by its highest weight $\lambda = \sum \lambda_i\varphi_i$ with $\lambda_i \in \mathbb{Z}_+$. The module L^λ can be realized as a submodule (or quotient) of $\underset{i}{\otimes}\left(R(\varphi_i)^{\otimes\lambda_i}\right)$.

Similarly, every irreducible $\mathfrak{gl}(n)$-module L^λ, where $\lambda = (\lambda_1, \dots \lambda_{n-1}; c)$ and c is the eigenvalue of the unit matrix, is realized in the space of tensors, perhaps, twisted with the help of c-densities, namely in the space $\underset{i}{\otimes}\left(R(\varphi_i)^{\otimes\lambda_i}\right) \otimes \mathrm{tr}^c$, where tr^c is the 1-dimensional representation — the Lie algebraic (infinitesimal) version of the representation given by the cth power of the determinant, i.e., infinitesimally, by c-multiple of trace — given, for any $c \in \mathbb{C}$, by the formula

$$X \mapsto c\cdot \mathrm{tr}(X) \quad \text{for any matrix } X \in \mathfrak{gl}(W). \tag{5.2}$$

Thus, all the irreducible finite dimensional representations of $\mathfrak{sl}(W)$ are naturally realized in the space of tensors, i.e., in the subspaces or quotient spaces (in view of complete reducibility both are true) of the space

$$T^p_q = \underbrace{W \otimes \dots \otimes W}_{p} \underbrace{\otimes W^* \otimes \dots \otimes W^*}_{q}, \tag{5.3}$$

where W is the space of the identity representation.

For $\mathfrak{gl}(W)$, we have to consider the spaces $T^p_q \otimes \mathrm{tr}^c$.

For $\mathfrak{sp}(W)$, the construction of irreducible finite dimensional modules is similar to that for $\mathfrak{sl}(W)$, except that the fundamental module $R(\varphi_i)$ is now not the whole module $\Lambda^i(\mathrm{id})$ but a *part* of it consisting of the *primitive forms*.

For $\mathfrak{o}(W)$, the situation is totally different: not all fundamental representations can be realized as (parts of) the modules $\Lambda^i(\mathrm{id})$. The exceptional one (or two, for $\mathfrak{o}(2n)$) of them is called the *spinor representation*; for $\mathfrak{o}(W)$, where $\dim W = 2n$, it is realized in the Grassmann algebra $E^\bullet(V)$ of a "half" of W, where $W = V \oplus V^*$ is a decomposition into the direct sum of subspaces isotropic with respect to the form preserved by $\mathfrak{o}(W)$. For $\dim W = 2n+1$, it is realized in the Grassmann algebra $E^\bullet(V \oplus W_0)$, where $W = V \oplus V^* \oplus W_0$ and W_0 is the 1-dimensional space on which the orthogonal form is nondegenerate.

The quantization of the harmonic oscillator leads to an infinite dimensional analog of the spinor representation which, following Howe, we call the *oscillator* representation of $\mathfrak{sp}(W)$. It is realized in $S^\bullet(V)$, where as above, V is a maximal isotropic subspace of W (with respect to the skew form preserved by $\mathfrak{sp}(W)$). The remarkable likeness of the spinor and oscillator representations was explained and generalized in a theory of *dual Howe's pairs*, [H1], [H2].

The importance of spinor-oscillator representations is different for distinct classes of Lie algebras and their representations. In the description of irreducible finite dimensional representations of the classical matrix Lie algebras $\mathfrak{gl}(n)$, $\mathfrak{sl}(n)$ and $\mathfrak{sp}(2n)$ we can do without either spinor or oscillator representations. We can not do without spinor representation for $\mathfrak{o}(n)$, but a pessimist might say that spinor representation constitutes only $\frac{1}{n}$th of the building bricks. Our, optimistic, point of view identifies the spinor representations as one of the two possible types of the building bricks.

For the Witt algebra $\mathfrak{witt}$ and its central extension, the Virasoro algebra $\mathfrak{vir}$, **every** irreducible highest weight module is realized as a quotient of a spinor or, equivalently, oscillator representation, see [FF]. This miraculous equivalence is known in physics under the name of *bose-fermi correspondence*, see [GSW], [K3].

For the list of generalizations of $\mathfrak{witt}$ and $\mathfrak{vir}$, i.e., simple (or close to simple) stringy Lie superalgebras (in other words, Lie superalgebras of vector fields on N-extended supercircles), often called by an unfortunate (as explained in [GLS1]) name "superconformal algebras", see [GLS1]. The importance of spinor-oscillator representations diminishes as N grows, but for the most interesting — *distinguished* ([GLS1]) — stringy superalgebras it is as high as for $\mathfrak{vir}$, cf. [FST].

5.2. The Poisson superalgebra $\mathfrak{g} = \mathfrak{po}(2n|m)$

5.2.1. Certain $\mathbb{Z}$-gradings of $\mathfrak{g}$. Recall that $\mathfrak{g}$ is the Lie superalgebra whose superspace is $\mathbb{C}[q, p, \Theta]$, where $q = (q_1, \dots, q_n)$, $p = (p_1, \dots, p_n)$, $\Theta = (\Theta_1, \dots, \Theta_m)$, and the bracket is the *Poisson bracket* $\{\cdot,\cdot\}_{P.b.}$ (in the realization with the form ω_0) given by the formula

$$\begin{gathered}\{f, g\}_{P.b.} = \sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right) - \\ (-1)^{p(f)}\sum_{j\leq m}\frac{\partial f}{\partial \Theta_j}\frac{\partial g}{\partial \Theta_j} \quad \text{for any } \ f, g \in \mathbb{C}[p, q, \Theta].\end{gathered} \tag{5.4}$$

It is often more convenient to redenote the Θ's and set (over $\mathbb{R}$ such a transformation is impossible)

$$\begin{cases}\xi_j = \frac{1}{\sqrt{2}}(\Theta_j - i\Theta_{r+j}) \\ \eta_j = \frac{1}{\sqrt{2}}(\Theta_j + i\Theta_{r+j})\end{cases} \quad \text{for } \ j \leq r = \left[\frac{m}{2}\right] \ (\text{ here } \ i^2 = -1), \tag{5.5}$$
$$\theta = \Theta_{2r+1}$$

and accordingly modify the bracket (if $m = 2r$, there is no term with θ):

$$\begin{gathered}\{f, g\}_{P.b.} = \sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right) - \\ (-1)^{p(f)}\left(\sum_{j\leq m}\left(\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j}\frac{\partial g}{\partial \xi_j}\right) + \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \theta}\right).\end{gathered} \tag{5.6}$$

Setting $\deg p_i = \deg q_i = \deg \theta_j$ for all i, j, and further setting

$$\deg_{Lie} f = \deg f - 2 \text{ for any monomial } f \in \mathbb{C}[p, q, \theta], \tag{5.7}$$

we obtain the *standard* $\mathbb{Z}$-grading of $\mathfrak{g}$ (indicated are elements that span $\mathfrak{g}_k$):

degree	-2	-1	0	1	...
elements	1	$p,\ q,\ \theta$	$f \mid \deg f = 2$	$f \mid \deg f = 3$	...

Clearly, $\mathfrak{g} = \bigoplus\limits_{i \geq -2} \mathfrak{g}_i$.

Exercise. $\mathfrak{g}_0 \simeq \mathfrak{osp}(m|2n)$.

Consider now another, "*rough*", grading of $\mathfrak{g}$. To this end, introduce: $Q = (q, \xi)$, $P = (p, \eta)$ and set

$$\deg Q_i = 0,\ \deg \theta = 1,\ \deg P_i = \begin{cases} 1 \text{ if } m = 2k \\ 2 \text{ if } m = 2k+1. \end{cases} \tag{5.8}$$

Remark. Physicists prefer to use half-integer values of the degree for m odd by setting $\deg \theta = \dfrac{1}{2}$ and $\deg P_i = 1$ at all times. Like other mathematicians, we prefer to deal with integers rather than with fractions.

The above grading (5.8) of the polynomial algebra induces the following *rough grading* of the Lie superalgebra $\mathfrak{g}$:

$m = 2k$:

degree	...	-1	0	1
elements	...	$\mathbb{C}[Q]P^2$	$\mathbb{C}[Q]P$	$\mathbb{C}[Q]$

(5.9)

$m = 2k+1$:

degree	...	-2	-1	0	1	2
elements	...	$\mathbb{C}[Q]P^2$	$\mathbb{C}[Q]P\theta$	$\mathbb{C}[Q]P$	$\mathbb{C}[Q]\theta$	$\mathbb{C}[Q]$

5.2.2. Quantization. The nontrivial deformation $\mathcal{Q}$ of the Lie algebra $\mathfrak{g}$ is called *quantization*. There are many ways to quantize $\mathfrak{g}$, but all of them are equivalent (for an overview of the setting of the problem and formulation of results, see [LSh3]). Recall that we only consider $\mathfrak{g}$ whose elements are represented by polynomials; for functions of other types (say, Laurent polynomials) the uniqueness of quantization may be violated, cf. [Dzh], [KST].

Consider the following quantization, so-called QP-quantization, given on linear terms by the formulas:

$$\mathcal{Q} : Q \mapsto \hat{Q}, \quad P \mapsto \hat{P} := \hbar \frac{\partial}{\partial Q}, \tag{5.10}$$

where $\hat{Q}$ is the operator of left multiplication by Q. Given an arbitrary monomial, rearrange it first so that the Q-terms precede all P-terms and then apply (5.10) term-wise, assuming $\mathcal{Q}$ to be linear over the ground field.

This quantization $\mathcal{Q}$ is a deformation of the Lie superalgebra structure (for every value of parameter $\hbar$ the supercommutator of the images $[\mathcal{Q}(f), \mathcal{Q}(g)]$ determines a Lie superalgebra structure on $\mathcal{Q}(\mathfrak{po}(2n|m))$.

The deformed Lie superalgebra $\mathcal{Q}(\mathfrak{po}(2n|2k))$ is, clearly, the Lie superalgebra $\mathfrak{diff}(n|k)$ of differential operators with polynomial coefficients on $\mathbb{C}^{n|k}$.

Actually, $\mathfrak{diff}(n|k)$ is an analog of $\mathfrak{gl}(V)$. This is most clearly seen for $n = 0$, $m = 2k$ when $\mathfrak{diff}(n|k)$ *IS* $\mathfrak{gl}(V)$ for $V = \Lambda^{\bullet}(\xi)$. Indeed,

$$\mathcal{Q}(\mathfrak{po}(0|2k)) = \mathfrak{gl}(\Lambda^{\bullet}(\xi)) = \mathfrak{gl}(2^{k-1}|2^{k-1}). \tag{5.11}$$

For $n \neq 0$,

$$\mathcal{Q}(\mathfrak{po}(n|2k)) = \text{“}\mathfrak{gl}\text{”}(\mathcal{F}(Q)) = \mathfrak{diff}(\mathbb{R}^{n|k}), \tag{5.12}$$

We put $\mathfrak{gl}$ in quotation marks because, for $\dim V = \infty$, there are several versions of $\mathfrak{gl}(V)$, see, e.g., [E].

For $m = 2k-1$, we consider $\mathfrak{po}(0|2k-1)$ as a subalgebra of $\mathfrak{po}(0|2k)$.

Exercise. The quantization $\mathcal{Q}$ sends $\mathfrak{po}(0|2k)$ into $\mathfrak{q}(2^{k-1})$.

For $n \neq 0$, the image of $\mathcal{Q}$ is an infinite dimensional analog of $\mathfrak{q}(2^k)$, namely,

$$\begin{gathered}\mathcal{Q}(\mathfrak{po}(n|2k-1)) = \mathfrak{qdiff}(\mathbb{C}^{n|k}) = \\ \left\{ D \in \mathfrak{diff}(\mathbb{C}^{n|k}) \mid \begin{cases} [D, J] = 0 & \text{for } J = i(\theta + \frac{\partial}{\partial \theta}), \\ [D, \Pi] = 0 & \text{for } \Pi = (\theta + \frac{\partial}{\partial \theta}). \end{cases} \right\}\end{gathered} \tag{5.13}$$

5.2.3. Fock spaces and spinor-oscillator representations. The Lie superalgebras $\mathfrak{diff}(\mathbb{C}^{n|k})$ and $\mathfrak{qdiff}(\mathbb{C}^{n|k})$ have indescribably many irreducible representations even for $n = 0$. But one of them, the identity one, in the superspace of functions on $\mathbb{C}^{n|k}$, is the "smallest" one.

This smallest representation can be singled out by its other property related with the fact that on the same space $\mathfrak{diff}(\mathbb{C}^{n|k})$ or $\mathfrak{qdiff}(\mathbb{C}^{n|k})$ there is not only a Lie superalgebra structure but also and a structure of an associative algebra.

To distinguish these Lie and associative structures, we denote the respective associative algebra by $\mathrm{Diff}(\mathbb{C}^{n|k})$ or $\mathrm{QDiff}(\mathbb{C}^{n|k})$. Each of these *associative* superalgebras has only *one* irreducible representation — the same "smallest" identity one. This representation is called *the Fock space*.

As is known, over $\mathbb{C}$, the Lie superalgebras $\mathfrak{osp}(m|2n)$ are rigid for $(m, 2n) \neq (4, 2)$. Therefore the through map

$$\mathfrak{h} \longrightarrow \mathfrak{g}_0 = \mathfrak{osp}(m|2n) \subset \mathfrak{g} = \mathfrak{po}(2n|m) \xrightarrow{\mathcal{Q}} \begin{cases} \mathfrak{diff}(\mathbb{C}^{n|k}) & \text{for } m \text{ even} \\ \mathfrak{qdiff}(\mathbb{C}^{n|k}) & \text{for } m \text{ odd} \end{cases} \tag{5.14}$$

sends any subsuperalgebra $\mathfrak{h}$ of $\mathfrak{osp}(m|2n)$ (for $(m, 2n) \neq (4, 2)$) into its isomorphic image.

5.2.3.1. Problem. What are the spinor/oscillator representations of $\mathfrak{osp}(4|2;\lambda)$ or any other non-rigid Lie (super)algebra?

As module over $\mathfrak{h}$, the Fock space splits into several submodules. The irreducible $\mathfrak{h}$-submodule which contains the constants is called the *spinor-oscillator representation* of $\mathfrak{h}$.

In particular cases, for $n=0$ and $\mathfrak{o}(m)$, or $m=0$ and $\mathfrak{sp}(2n)$, this spinor-oscillator representation turns into the usual *spinor* representation of $\mathfrak{o}(m)$ or, respectively, *oscillator representation* of $\mathfrak{sp}(2n)$. We have just given a unified description of them.

Recall that on the space of functions, even pure even ones, there is a parity function $\tilde{p}$ given by the property

$$\tilde{p}(f)=\begin{cases}\tilde{\bar{0}} & \text{if } f(-x)=f(x),\\ \tilde{\bar{1}} & \text{if } f(-x)=-f(x).\end{cases} \tag{5.15}$$

To distinguish this parity from the one induced by the Θ's, we denoted it by $\tilde{p}$ and will call it *t-parity.* The Fock space splits, clearly, into the direct sum of two invariant components: the superspaces of t-even and t-odd functions with respect to the t-parity; let $\widetilde{\Pi}$ be the functor of t-parity change.

Statement. *As $\mathfrak{o}(m)$- or $\mathfrak{sp}(2n)$-modules, the spaces of t-even and t-odd functions with respect to the t-parity are irreducible and differ by $\widetilde{\Pi}$.*

Exercise. 1) As $\mathfrak{osp}(m|2n)$-modules, the spaces of t-even and t-odd functions with respect to the t-parity are irreducible and isomorphic if $n=0$; they differ by $\widetilde{\Pi}$ if $n\neq 0$.

1) What is the relation between $\widetilde{\Pi}$ and Π?

5.2.4. Primitive alias harmonic elements. The elements of $\mathfrak{osp}(m|2n)$ (or its subalgebra $\mathfrak{h}$) act in the space of the spinor-oscillator representation by inhomogeneous differential operators of order ≤ 2 (this order is just the filtration associated with the "rough" grading):

$m=2k:$

degree	-1	0	1
elements	$\hat{P}^2$	$\hat{P}\hat{Q}$	$\hat{Q}^2$

(5.16)

$m=2k+1:$

degree	-2	-1	0	1	2
elements	$\hat{P}^2$	$\hat{P}\hat{\theta}$	$\hat{P}\hat{Q}$	$\hat{Q}\hat{\theta}$	$\hat{Q}^2$

Recall that $M^{\mathfrak{g}}:=\{m\in M\mid gm=0 \text{ for any } g\in\mathfrak{g} \text{ and } m\in M\}$ is a standard notation of the set of $\mathfrak{g}$-invariant elements of the $\mathfrak{g}$-module M.

The elements of $(\mathbb{C}[Q])^{\hat{P}^2}$ for $m=2k$ and $(\mathbb{C}[Q,\theta])^{\hat{P}\hat{\theta}}$ for $m=2k+1$ are called *primitive* or *harmonic* ones.

More generally, let $\mathfrak{h}\subset\mathfrak{osp}(m|2n)$ be a $\mathbb{Z}$-graded Lie superalgebra embedded consistently with the rough grading of $\mathfrak{osp}(m|2n)$. Then the elements of $(\mathbb{C}[Q])^{\mathfrak{h}_{-1}}$ for $m=2k$ or $(\mathbb{C}[Q,\theta])^{\mathfrak{h}_{-1}}$ for $m=2k+1$ are said to be $\mathfrak{h}$-*primitive* or $\mathfrak{h}$-*harmonic*.

5.2.4.1. Nonstandard $\mathbb{Z}$-gradings of $\mathfrak{g}$. It is well known that one simple Lie superalgebra can have several nonequivalent Cartan matrices and systems of Chevalley generators. Accordingly, the divisions into *positive* and *negative* root vectors are distinct.

Problem. How the passage to nonstandard gradings affects the highest weight of the spinor-oscillator representation defined in sec. 5.2.3 (Cf. [NH])

5.2.5. Examples of dual pairs. Two subalgebras Γ,Γ' of $\mathfrak{g}_0=\mathfrak{osp}(m|2n)$ will be called a *dual pair* if one of them is the centralizer of the other in $\mathfrak{g}_0$.

If $\Gamma\oplus\Gamma'$ is a maximal subalgebra in $\mathfrak{g}_0$, then, clearly, Γ,Γ' is a dual pair. For a number of such examples see [ShM]. Let us consider several of these examples in detail.

5.2.5.1. Let $\Gamma=\mathfrak{sp}(2n)=\mathfrak{sp}(W)$ and $\Gamma'=\mathfrak{sp}(2)=\mathfrak{sl}(2)=\mathfrak{sp}(V\oplus V^*)$. Clearly, $\mathfrak{h}=\Gamma\oplus\Gamma'$ is a maximal subalgebra in $\mathfrak{o}(W\otimes(V\oplus V^*))$. The Fock space is just $\Lambda^{\bullet}(W)$.

The following classical theorem and its analog 5.2.5.2 illustrate the importance of the above notions and constructions.

Theorem. *The Γ'-primitive elements of $\Lambda^{\bullet}(W)$ of each degree i constitute an irreducible Γ-module P^i, $0\leq i\leq n$.*

This action of Γ' in the space of differential forms on any symplectic manifold is well known: Γ' is generated (as a Lie algebra) by operators X_+ of left multiplication by the symplectic form ω and X_-, the convolution with the dual to ω.

5.2.5.2. $\Gamma=\mathfrak{o}(2n)=\mathfrak{sp}(W)$ and $\Gamma'=\mathfrak{sp}(2)=\mathfrak{sl}(2)=\mathfrak{sp}(V\oplus V^*)$. Clearly, $\mathfrak{h}=\Gamma\oplus\Gamma'$ is a maximal subalgebra in $\mathfrak{sp}(W\otimes(V\oplus V^*))$. The Fock space is just $S^{\bullet}(W)$.

Theorem. *The Γ'-primitive elements of $S^{\bullet}(W)$ of each degree i constitute an irreducible Γ-module P^i, where $i=0,1,\dots$*

This action of Γ' in the space of polynomial functions on any Riemann manifold is also well known: Γ' is generated (as a Lie algebra) by operators X_+ of left multiplication by the quadratic polynomial representing the metric g and X_- is the corresponding Laplace operator.

Clearly, a mixture of Examples 5.2.5.1 and 5.2.5.2 corresponding to symmetric or skew-symmetric forms on a supermanifold is also possible: *the space of Γ'-primitive elements of $S^{\bullet}(W)$ of each degree i is an irreducible Γ-module*, cf. [NH], [Ser1], [Ser2].

In [H1], [H2] the dual pairs had to satisfy one more condition: the through action of both Γ and Γ' on the identity $\mathfrak{g}_0$-module should be completely reducible. (This is why dual pairs were sometimes referred to as *reductive* pairs.) However, the very first example from [H1], [H2] in which superalgebras appear not idly — the proof of the Poincaré lemma — fails to satisfy this requirement and indeed it is seldom needed, cf. [Ser1, Ser2].

5.2.5.3. (Bernstein's square root of the Lefschetz decomposition) Let L be the space of a (complex) line bundle over a connected symplectic manifold (M^{2n}, ω) with connection ∇ such that the curvature form of ∇ is equal to $\hbar\omega$ for some $\hbar \in \mathbb{C}$. This $\hbar$ will be called a *twist*; the space of tensor fields of type ρ (here $\rho\colon \mathfrak{sp}(2n) \longrightarrow \mathfrak{gl}(U)$ is a representation which defines the space $\Gamma(M, U)$ of tensor fields with values in U), and twist $\hbar$ will be denoted by $T_\hbar(\rho)$. Let us naturally extend the action of X_+, X_- from the space Ω of differential forms on M onto the space $\Omega_\hbar$ of twisted differential forms using the isomorphism of *spaces* $T_\hbar(\rho) \simeq T(\rho) \otimes \Gamma(L)$, where $\Gamma(L) = \Omega^0_\hbar$ is the space of sections of the line bundle L, i.e., the space of twisted functions.

Namely, set $X_+ \mapsto X_+ \otimes 1$, and so on. Let $D_+ = d + \alpha$ be the connection ∇ itself and
$$D_- = [X_-, D_+]. \tag{5.17}$$
On $\Omega_\hbar$, introduce a superspace structure setting $p(\varphi \otimes s) = \deg \varphi \pmod 2$, for any $\varphi \in \Omega$ and $s \in \Omega^0_\hbar$.

Theorem ([Ber])**.** *On $\Omega_\hbar$, the operators D_+ and D_- generate an action of the Lie superalgebra $\mathfrak{osp}(1|2)$ commuting with the action of the group $\hat{G}$ of ∇-preserving automorphisms of the bundle L.*

Bernstein studied the $\hat{G}$-action, more exactly, the action of the Lie algebra $\mathfrak{po}(2n|0)$ corresponding to $\hat{G}$; we are interested in the part of this action only: in $\mathfrak{po}(2n|0)_0$-action.

In Example 5.2.5.1 the space P^i consisted of differential forms with constant coefficients. Denote by $\mathcal{P}^i = P^i \otimes S^\bullet(V)$ the space of primitive forms with polynomial coefficients. The elements of the space $\sqrt{\mathcal{P}}^i_\hbar = \ker D_- \cap \mathcal{P}^i_\hbar$ will be called ∇-*primitive forms* of degree i (and twist h).

Bernstein showed that $\sqrt{\mathcal{P}}^i_\hbar$ is an irreducible $\mathfrak{g} = \mathfrak{po}(2n|0)$-module and Shapovalov and Shmelev literally generalized his result for supermanifolds, see review [Le3]. It could be that over subalgebra $\mathfrak{g}_0$ this module will be reducible but the general theorem of Howe (which is true for $\mathfrak{osp}(1|2n)$) states that this is not the case, it remains irreducible.

5.2.5.4. Inspired by Bernstein's construction, let us similarly define a "square root" of the hyper-Kähler structure. Namely, on any hyper-Kählerean manifold (M, ω_1, ω_2), consider a line bundle L with two connections: ∇_1 and ∇_2, whose curvature forms are equal, respectively, to $\hbar_1\omega_1$ and $\hbar_2\omega_2$ for some $\hbar_1, \hbar_2 \in \mathbb{C}$. The pair $\hbar = (\hbar_1, \hbar_2)$ will be called a *twist*; the space of tensor fields of type ρ and twist $\hbar$ will be denoted by $T_\hbar(\rho)$. Verbitsky [Ver] defined the action of $\mathfrak{sp}(4)$ in the space Ω of differential forms on M. Let us naturally extend the action of the generators $X^\pm_j$ for $j = 1, 2$ of of $\mathfrak{sp}(4)$ from Ω onto the space $\Omega_\hbar$ of twisted differential forms using the isomorphism $T_\hbar(\rho) \simeq T(\rho) \otimes \Gamma(L)$, where $\Gamma(L) = \Omega^0_\hbar$ is the space of sections of the line bundle L.

Define the space of primitive i-forms (with constant coefficients) on the hyper-Kählerean manifold (M, ω_1, ω_2) by setting
$$P^i = \ker X^-_1 \cap \ker X^-_2 \cap \Omega^i. \tag{5.18}$$
According to the general theorem of Howe [H2], this space should be an irreducible $\mathfrak{sp}(2n; \mathbb{H})$-module.

The promised square root of this decomposition is the space
$$\mathcal{P}^i_\hbar = \ker D^-_1 \cap \ker D^-_2 \cap \Omega^i_\hbar, \tag{5.19}$$
where the operators D^+_i are the above connections. Together with the operators $D^-_i = [X^-_i, D^+_i]$ they generate $\mathfrak{osp}(1|4)$.

5.2.6. Examples of dual pairs. The following subalgebras $\mathfrak{g}_1(V_1) \oplus \mathfrak{g}_2(V_2)$ are maximal in $\mathfrak{g}(V_1 \otimes V_2)$:

$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}$	
$\mathfrak{osp}(n_1\vert 2m_1)$	$\mathfrak{osp}(n_2\vert 2m_2)$	$\mathfrak{osp}(n_1n_2 + 4m_1m_2\vert 2n_1m_2 + 2n_2m_1)$	
$\mathfrak{o}(n)$	$\mathfrak{osp}(n_2\vert 2m_2)$	$\mathfrak{osp}(nn_2\vert 2nm_2), n \neq 2, 4$	
$\mathfrak{sp}(2n)$	$\mathfrak{osp}(n_2\vert 2m_2)$	$\mathfrak{osp}(2mn_2\vert 4nm_2)$	
$\mathfrak{pe}(n_1)$	$\mathfrak{pe}(n_2)$	$\mathfrak{osp}(2n_1n_2\vert 2n_1n_2), n_1, n_2 > 2$	(5.20)
$\mathfrak{osp}(n_1\vert 2m_1)$	$\mathfrak{pe}(n_2)$	$\mathfrak{pe}(n_1n_2 + 2m_1n_2)$ if $n_1 \neq 2m_1$	
		$\mathfrak{spe}(n_1n_2 + 2m_1n_2)$ if $n_1 = 2m_1$	
$\mathfrak{o}(n)$	$\mathfrak{pe}(m)$	$\mathfrak{pe}(nm)$	
$\mathfrak{sp}(2n)$	$\mathfrak{pe}(m)$	$\mathfrak{pe}(2nm)$	

In particular, on the superspace of polyvector fields, there is a natural $\mathfrak{pe}(n)$-module structure, and its dual $\mathfrak{pe}(1)$ in $\mathfrak{osp}(2n|2n)$ is spanned by the divergence operator Δ ("odd Laplacian"), called the *BRST (in honor of Becchi, Rouet, Stora, and (independently) Tyutin) operator* ([BT]), the even operator being $\deg_x - \deg_\theta$, where $\theta_i = \dfrac{\partial}{\partial x_i}$.

For further examples of maximal subalgebras in $\mathfrak{gl}$ and $\mathfrak{q}$, see [ShM]. These subalgebras give rise to other new examples of Howe dual pairs. For the decomposition of the tensor algebra corresponding to some of these examples see [Ser1, Ser2], some of the latter are further elucidated in [CW].

Chapter 6

Irreducible representations of solvable Lie superalgebras (A. Sergeev)

6.0. Introduction

Hereafter the ground field is $\mathbb{C}$ and all the modules and superalgebras are finite dimensional; $\mathbb{Z}/2 = \{\bar{0}, \bar{1}\}$ and $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a solvable Lie superalgebra.

The description of irreducible representations of solvable Lie superalgebras given in [K2] (Theorem 7) contains an error. In reality, to give such a description one has to imitate the description of infinite dimensional solvable Lie algebras [Di], i.e., we must consider *twisted* induced representations. In what follows I give a correct description of irreducible representations of solvable Lie superalgebras. I also show where a mistake crept into [K2] (this is a subtle point) and give a counterexample to Theorem 7 from [K2].

The proof given in what follows was delivered at Leites' *Seminar on Supersymmetries* in 1983 and is preprinted in [LSoS] #22 in a form considerably edited by I. Shchepochkina and D. Leites. My acknowledgements are due to them and also to the Department of Mathematics of Stockholm University that financed publication of the preprint, see also `arXiv:math/9810109` and Represent. Theory 3 (1999), 435–443.

6.1. Main result

6.1.1. Polarizations. Set $L = \{\lambda \in \mathfrak{g}^* \mid \lambda(\mathfrak{g}_{\bar{1}}) = 0 \text{ and } \lambda([\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]) = 0\}$. Recall that a *superspace* is a $\mathbb{Z}/2$-graded space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ and its *superdimension* is the pair $(\dim V_{\bar{0}}, \dim V_{\bar{1}})$. By the usual abuse of language $\lambda \in L$ denotes a character and also the (1, 0)-dimensional representation of the Lie algebra $\mathfrak{g}_{\bar{0}}$ determined by the character λ. Every functional $\lambda \in L$ determines a symmetric form f_λ on $\mathfrak{g}_{\bar{1}}$ by the formula $f_\lambda(\xi_1, \xi_2) = \lambda([\xi_1, \xi_2])$.

A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is said to be a *polarization* for $\lambda \in L$ if $\lambda([\mathfrak{h}, \mathfrak{h}]) = 0$, $\mathfrak{h} \supset \mathfrak{g}_{\bar{0}}$ and $\mathfrak{h}_{\bar{1}}$ is a maximal fully isotropic subspace for f_λ.

6.1.2. Lemma. *For every $\lambda \in L$, there exists a polarization $\mathfrak{h}$.*

Proof follows from Lemma 6.2.4.

6.1.3. Twisted representations. If $\mathfrak{h}$ is a polarization for $\lambda \in L$, then, clearly, λ determines a $(1,0)$-dimensional representation of $\mathfrak{h}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subsuperalgebra that contains $\mathfrak{g}_{\bar 0}$. Define a functional $\theta_{\mathfrak{h}} \in L$ by setting

$$\theta_{\mathfrak{h}}(g) = \begin{cases} -\frac{1}{2}\operatorname{tr}_{\mathfrak{g}/\mathfrak{h}}(\operatorname{ad}_g) & \text{for } g \in \mathfrak{g}_{\bar 0} \\ 0 & \text{for } g \in \mathfrak{g}_{\bar 1}. \end{cases}$$

Note that $\theta_{\mathfrak{h}}([\mathfrak{h},\mathfrak{h}]) = 0$. Therefore, $\theta_{\mathfrak{h}}$ is a character of a $(1,0)$-dimensional representation of $\mathfrak{h}$.

Let $\mathfrak{h}$ be a polarization for $\lambda \in L$. Define the *twisted* (by the character $\theta_{\mathfrak{h}}$) induced and coinduced representations by setting

$$I^{\mathfrak{g}}_{\mathfrak{h}}(\lambda) = \operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{h}}(\lambda + \theta_{\mathfrak{h}}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} (\lambda + \theta_{\mathfrak{h}});$$
$$CI^{\mathfrak{g}}_{\mathfrak{h}}(\lambda) = \operatorname{Coind}^{\mathfrak{g}}_{\mathfrak{h}}(\lambda - \theta_{\mathfrak{h}}) = \operatorname{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), \lambda - \theta_{\mathfrak{h}}).$$

6.1.4. Lemma. 1) $I^{\mathfrak{g}}_{\mathfrak{h}}(\lambda)$ *is finite dimensional and irreducible.*

2) $I^{\mathfrak{g}}_{\mathfrak{h}}(\lambda)$ *does not depend on the choice of a polarization* $\mathfrak{h}$*; therefore, notation* $I(\lambda)$ *(*$= I^{\mathfrak{g}}_{\mathfrak{h}}(\lambda)$ *for some* $\mathfrak{h}$*) is well-defined.*

3) $CI(\lambda) \cong I(\lambda)$.

For the proof, see Corollaries 6.3.3 and 6.4.3.

6.1.5. Main Theorem. Let $Z = \{(\lambda, \mathfrak{h}) \mid \lambda \in L$ and let $\mathfrak{h}$ be a polarization for $\lambda\}$. Define an equivalence relation on Z by setting

$$(\lambda, \mathfrak{h}) \sim (\mu, \mathfrak{t}) \Longleftrightarrow \lambda - \theta_{\mathfrak{h}} = \mu - \theta_{\mathfrak{t}}$$

Clearly, this relation is well-defined.

Recall that the representation of a Lie superalgebra $\mathfrak{g}$ is called *irreducible of G-type* if it has no invariant subspaces; it is called *irreducible of Q-type* if it has no invariant sub*super*spaces. Recall also that to every superspace $V = V_{\bar 0} \oplus V_{\bar 1}$ the change of parity functor Π assigns the superspace $\Pi(V)$ such that $\Pi(V)_{\bar i} = V_{\bar i + \bar 1}$. Observe that the modules $I(\lambda)$ and $\Pi(I(\lambda))$ are not isomorphic as $\mathfrak{g}$-modules (unless they are of Q-type); they are always isomorphic, however, as $\mathfrak{g}_{\bar 0}$-modules.

6.1.6. Theorem. 1) *Every irreducible finite dimensional representation of* $\mathfrak{g}$ *is isomorphic up to application of the change of parity functor* Π *to a representation of the form* $I(\lambda)$ *for some* λ.

2) *The map* $\lambda \mapsto I(\lambda)$ *is (up to* Π*) a 1-1 correspondence between elements of* L *and the irreducible finite dimensional representations of* $\mathfrak{g}$.

3) *Let* $(\lambda, \mathfrak{h}), (\mu, \mathfrak{t}) \in Z$. *Then* $\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{h}}(\lambda) \cong \operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{t}}(\mu)$ *if and only if* $(\lambda, \mathfrak{h}) \sim (\mu, \mathfrak{t})$.

4) *If* $\operatorname{rk} f_\lambda$ *is even, then* $I(\lambda)$ *is a* G*-type representation; if* $\operatorname{rk} f_\lambda$ *is odd, then* $I(\lambda)$ *is a* Q*-type representation.*

For the proof, see sec. 6.3.3, 6.3.5, 6.4.2 and 6.4.3.

6.1.6.1. Remark. For examples of irreducible representations of dimension > 1 of solvable Lie superalgebras (and interesting examples of the latter), see [ShM1].

6.2. Prerequisites for the proof of Main theorem

Let $\mathfrak{k} \subset \mathfrak{g}$ be a subsuperalgebra, $\operatorname{codim}\ \mathfrak{k} = (0,1)$, and μ the character of the representation of $\mathfrak{g}_{\bar 0}$ in $\mathfrak{g}/\mathfrak{k}$.

6.2.1. Lemma. μ *is a character of* $\mathfrak{g}$.

Proof. Let $\xi \in \mathfrak{g}$ and $\xi \notin \mathfrak{k}$. Since in $\mathfrak{g}/\mathfrak{k}$ there is a $\mathfrak{k}$-action, it suffices to prove that $\mu([\mathfrak{k}, \xi]) = \mu([\xi, \xi]) = 0$. By the Jacobi identity $[[\xi, \xi], \xi] = 0$ which proves that $\mu([\xi, \xi]) = 0$. Let $\eta \in \mathfrak{k}_{\bar 1}$. Then $[[\eta, \xi], \xi] = \frac{1}{2}[\eta, [\xi, \xi]] \in \mathfrak{k}$, and therefore $\mu([\mathfrak{k}_{\bar 1}, \xi]) = 0$. □

6.2.2. Corollary. *Let* $\mathfrak{k} \subset \mathfrak{k}_1$ *be subalgebras in* $\mathfrak{g}$ *such that both containing* $\mathfrak{g}_{\bar 0}$ *and such that* $\dim \mathfrak{k}_1/\mathfrak{k} = (0,1)$. *Let* λ *be the character of an irreducible factor of* $\mathfrak{g}/\mathfrak{k}$ *considered as the* $\mathfrak{g}_{\bar 0}$*-module. Then* λ *is a character of* $\mathfrak{k}_1$.

Proof. Let $\dim \mathfrak{g}/\mathfrak{k} = (0, l)$; we will induct on l. If $l = 1$, the statement of Corollary holds thanks to Lemma 6.2.1.

Let $l > 1$ and $\mathfrak{k}_2$ a subalgebra of $\mathfrak{g}$ such that $\dim \mathfrak{g}/\mathfrak{k}_2 = (0,1)$ and $\mathfrak{k}_2 \supset \mathfrak{k}_1$. Any irreducible factor of $\mathfrak{g}/\mathfrak{k}$ is a factor of either $\mathfrak{k}_2/\mathfrak{k}$ or $\mathfrak{g}/\mathfrak{k}_2$. In the first case Corollary holds by the inductive hypothesis. In the second case let λ be the character of an irreducible factor of $\mathfrak{g}/\mathfrak{k}_2$. Then λ is a character of $\mathfrak{k}_2$; hence, a character of $\mathfrak{k}_1$. □

6.2.3. Corollary ([K2], Prop. 1.3.3, p. 25). *A Lie superalgebra* $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ *is solvable if and only if so is* $\mathfrak{g}_{\bar 0}$.

Proof. Here is an independent proof. We induct on $l = \dim \mathfrak{g}_{\bar 1}$. If $l = 0$, the statement is obvious. Let $l > 0$. Set

$$\tilde{\mathfrak{g}} = [\mathfrak{g}_{\bar 0}, \mathfrak{g}_{\bar 0}] \oplus [\mathfrak{g}_{\bar 1}, \mathfrak{g}_{\bar 1}] \oplus \mathfrak{g}_{\bar 1}.$$

Since $[\mathfrak{g}, \mathfrak{g}] \subset \tilde{\mathfrak{g}}$, it suffices to demonstrate that $\tilde{\mathfrak{g}}$ is solvable. Let $\mathfrak{h} \subset \mathfrak{g}$ and $\dim \mathfrak{g}/\mathfrak{h} = (0,1)$. By Lemma 6.2.1 we see that $[\tilde{\mathfrak{g}}_{\bar 0}, \mathfrak{g}_{\bar 1}] \subset \mathfrak{h}_{\bar 1}$. Hence, $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] \subset \tilde{\mathfrak{g}}_{\bar 0} \oplus \mathfrak{h}_{\bar 1}$. By the inductive hypothesis, $\tilde{\mathfrak{g}}_{\bar 0} \oplus \mathfrak{h}_{\bar 1}$ is a solvable Lie superalgebra, hence, so is $\mathfrak{g}$.

The converse statement is obvious. □

6.2.4. Lemma. *Let* W *be a finite dimensional* $\mathfrak{g}$*-module,* f *a symmetric* $\mathfrak{g}$*-invariant form on* W *and* V *a* $\mathfrak{g}$*-invariant fully isotropic subspace. Then there exists a maximal* $\mathfrak{g}$*-invariant* f*-isotropic subspace in* W *containing* V.

Proof. Without loss of generality we may assume that f is nondegenerate. Let $\operatorname{rk}\ f = \dim W = 2l$.

i) Let us prove first that W contains a nonzero isotropic $\mathfrak{g}$-invariant one-dimensional subspace. Since $\mathfrak{g}$ is solvable, there exists a $w \in W$ such that $xw = \lambda(x)w$ for any $x \in \mathfrak{g}$. If $f(w, w) = 0$, we are done. If $f(w, w) \neq 0$, then the invariance implies that

$$0 = f(xw, w) + f(w, xw) = 2\lambda(x)f(w, w).$$

Therefore, $\lambda(x) = 0$ and w is a $\mathfrak{g}$-invariant. Furthermore, $W = \operatorname{Span}(w) \oplus W_1$, where $W_1 = \operatorname{Span}(w)^{\perp}$.

In W_1, select a one-dimensional $\mathfrak{g}$-invariant subspace $\operatorname{Span}(w_1)$. If $f(w_1, w_1) = 0$, we are done. If $f(w_1, w_1) \neq 0$, the above arguments show that w_1 is a $\mathfrak{g}$-invariant. Then $w_2 = w + \alpha w_1$ is an isotropic and $\mathfrak{g}$-invariant vector for $\alpha = \sqrt{-\dfrac{f(w, w)}{f(w_1, w_1)}}$.

ii) Now let us induct on l. If $l = 1$, let us apply step i). If $l > 1$ we may assume, thanks to i), that $V \neq 0$. If $V = V^{\perp}$, we are done. But if $V \neq V^{\perp}$, then $V \subset V^{\perp}$, since V is fully isotropic; moreover, the restriction of f onto $V^{\perp}/V$ is nondegenerate. The equality $\dim V + \dim V^{\perp} = 2l$ implies that $\dim V^{\perp}/V$ is even. Therefore, by the induction we prove that $V^{\perp}/V$ contains a maximal $\mathfrak{g}$-invariant fully isotropic subspace $\bar{U}$. But then its preimage U in $V^{\perp}$ is a maximal fully isotropic $\mathfrak{g}$-invariant subspace of W containing V.

The case $\operatorname{rk} f = \dim W = 2l + 1$ is treated similarly. $\square$

6.2.5. Corollary. *If $\mathfrak{h}$ is a polarization for $\lambda \in L$, $\mathfrak{n}$ the kernel of f_λ and $\lambda_1, \dots, \lambda_l$ are characters of irreducible subfactors of $\mathfrak{g}_{\bar{1}}/\mathfrak{n}$ concidered as a $\mathfrak{g}_{\bar{0}}$-module, then $\mathfrak{h}$ is also polarization for $\mu = \lambda + \alpha_1\lambda_1 + \dots + \alpha_l\lambda_l$ for any $\alpha_1, \dots, \alpha_l \in \mathbb{C}^*$.*

Proof. Since f_λ determines a nondegenerate $\mathfrak{g}_{\bar{0}}$-invariant pairing

$$\mathfrak{h}^{\perp}/\mathfrak{n} \times \mathfrak{g}_{\bar{1}}/\mathfrak{h} \longrightarrow \mathbb{C},$$

the characters $\lambda_1, \dots, \lambda_l$ coincide, up to a sign, with characters of irreducible factors of $\mathfrak{g}_{\bar{1}}/\mathfrak{h}$. But the latter space is a $\mathfrak{h}$-module, so $\lambda_i([\mathfrak{h}_{\bar{1}}, \mathfrak{h}_{\bar{1}}]) = 0$, and therefore $\mathfrak{h}_{\bar{1}}$ is fully isotropic for f_μ.

If $\mathfrak{h}_{\bar{1}}$ is a *maximal* fully isotropic subspace for f_μ, we are done. Otherwise, i.e., if $\mathfrak{h}_{\bar{1}}$ is not a maximal fully isotropic subspace for f_μ, select a $\mathfrak{g}_{\bar{0}}$-invariant subspace $\mathfrak{b}_{\bar{1}}$ of $\mathfrak{g}_{\bar{1}}$ distinct from $\mathfrak{h}_{\bar{1}}$ containing $\mathfrak{h}_{\bar{1}}$ and isotropic with respect to f_μ.

Next, in the module $\mathfrak{b}_{\bar{1}}/\mathfrak{h}_{\bar{1}}$ select a one-dimensional $\mathfrak{g}_{\bar{0}}$-invariant subspace $\operatorname{Span}(\bar{\xi})$, where $\xi \in \mathfrak{b}_{\bar{1}}$. Then $\mathfrak{k}_{\bar{1}} = \mathfrak{h}_{\bar{1}} \oplus \operatorname{Span}(\xi)$ is fully isotropic with respect to f_μ and $\mathfrak{k} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ is a subalgebra of $\mathfrak{g}$. Then $\lambda_i([\mathfrak{k}_{\bar{1}}, \mathfrak{k}_{\bar{1}}]) = 0$ by Corollary 6.2.2. But $\lambda = \mu - \alpha_1\lambda_1 - \dots - \alpha_l\lambda_l$; hence, $\lambda([\mathfrak{k}_{\bar{1}}, \mathfrak{k}_{\bar{1}}]) = 0$. In other words, $\mathfrak{k}_{\bar{1}}$ is fully isotropic with respect to f_λ. But this contradicts to the maximality of $\mathfrak{h}_{\bar{1}}$. $\square$

The following two statements are standard, so their proofs are omitted.

6.2.6. Lemma. *Let $\mathfrak{k} \subset \mathfrak{g}$ be a Lie subsuperalgebra, $\dim \mathfrak{g}/\mathfrak{k} = (0, 1)$. If (V, ρ) is an irreducible representation of $\mathfrak{k}$ in a superspace V, then $W = \operatorname{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(V)$ is reducible if and only if V admits a $\mathfrak{g}$-module structure that extends ρ.*

6.2.7. Lemma (see [K2], Lemma 5.2.2 b))**.** *Let $\mathfrak{k} \subset \mathfrak{g}$ be a Lie subsuperalgebra, $\dim \mathfrak{g}/\mathfrak{k} = (0, 1)$. If W is an irreducible $\mathfrak{g}$-module and $V \subset W$ is an irreducible proper $\mathfrak{k}$-submodule, then $W = \operatorname{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(V)$.*

6.3. Description of irreducible modules

6.3.1. Proposition. *Let $\lambda \in L$, let $\mathfrak{p} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}}$ be a polarization for λ, let $\mathfrak{n}$ be the kernel of f_λ and $F \subset \mathfrak{p}_{\bar{1}}$ a subspace such that $\mathfrak{p}_{\bar{1}} = F \oplus \mathfrak{n}$. Define ξ_0 as follows: if $\operatorname{rk}\ f_\lambda$ is even, then we set $\xi_0 = 0$ and let ξ_0 be from $\mathfrak{p}_{\bar{1}}^{\perp}$ but so as $\xi_0 \notin \mathfrak{p}_{\bar{1}}$ if $\operatorname{rk}\ f_\lambda$ is odd. Let $xv = \lambda(x)v$ be a one-dimensional representation of $\mathfrak{p}$ in $V = \operatorname{Span}(v)$. Denote: $I_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) = \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(V)$*

If $u \in I_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ and $Fu = 0$, then $u \in \operatorname{Span}(v, \xi_0 v)$.

Proof. Induction on $\operatorname{rk}\ f_\lambda$. If $\operatorname{rk}\ f_\lambda = 0$, then $F = 0$ and the statement is obvious.

Let $\operatorname{rk}\ f_\lambda > 0$. Select a subalgebra $\mathfrak{h} \subset \mathfrak{p}$ such that $\dim \mathfrak{g}_{\bar{1}}/\mathfrak{h}_{\bar{1}} = 1$. The two cases are possible: $\mathfrak{h}_{\bar{1}}^{\perp} \not\subset \mathfrak{h}_{\bar{1}}$ and $\mathfrak{h}_{\bar{1}}^{\perp} \subset \mathfrak{h}_{\bar{1}}$.

i) $\mathfrak{h}_{\bar{1}}^{\perp} \not\subset \mathfrak{h}_{\bar{1}}$. Then $\mathfrak{g}_{\bar{1}} = \mathfrak{h}_{\bar{1}} \oplus \operatorname{Span}(\xi)$, where $\xi \perp \mathfrak{h}_{\bar{1}}$. Hence, $\xi \perp \mathfrak{p}_{\bar{1}}$ and $\xi \notin \mathfrak{p}_{\bar{1}}$. Therefore, we may assume that $\xi = \xi_0$ and $\operatorname{rk}\ f_\lambda$ is an odd number. Clearly, $\mathfrak{p}$ is a polarization for the restriction f_λ onto $\mathfrak{h}_{\bar{1}}$ and $\operatorname{rk}\ f_\lambda$ is an even number. Further on,

$$\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) = \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{h}}(\lambda) \oplus \xi_0 \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda).$$

Let $u = u_0 + \xi_0 u_1 \in \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ and $pu = 0$ for any $p \in F$. Then

$$0 = pu = pu_0 + [p, \xi_0]u_1 + \xi_0 pu_1,$$

therefore, $pu_1 = 0$. By the induction, $u_1 \in \operatorname{Span}(v)$, where v is the generator of $\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$. Since $\xi_0 \perp \mathfrak{p}_{\bar{1}}$, it follows that $[p, \xi_0]u_1 = f_\lambda(p, \xi_0)u_1 = 0$. Therefore, $pu_0 = 0$ and $u_0 \in \operatorname{Span}(v)$. Hence, $u \in \operatorname{Span}(v, \xi_0 v)$.

ii) Let us show now that the weight of $\xi_0 v$ with respect to $\mathfrak{g}_{\bar{0}}$ is also equal to λ. If $\xi_0 = 0$, all is clear. So let $\xi_0 \neq 0$. Since $[x, \xi_0] \perp \mathfrak{p}_{\bar{1}}$ for any $x \in \mathfrak{g}_{\bar{0}}$, it follows that $[x, \xi_0] = \mu(x)\xi_0 + p$ for some $p \in \mathfrak{p}_{\bar{1}}$. Furthermore, $[x, [\xi_0, \xi_0] = 2[[x, \xi_0], \xi_0]$; hence,

$$\begin{aligned} 0 = \lambda([x, [\xi_0, \xi_0]) &= 2\lambda([\mu(x)\xi_0 + p, \xi_0]) = \\ 2\mu(x)\lambda([\xi_0, \xi_0]) &+ 2\lambda([p, \xi_0]) = 2\mu(x)\lambda([\xi_0, \xi_0]). \end{aligned}$$

But $\lambda([\xi_0, \xi_0]) \neq 0$, so, $\mu(x) = 0$ and the weight of $\xi_0 v$ is equal to λ.

iii) $\mathfrak{h}_{\bar{1}}^{\perp} \subset \mathfrak{h}_{\bar{1}}$. Then the *restriction* of the form f_λ onto $\mathfrak{h}_{\bar{1}}$ is of rank by 2 less than that of f_λ itself.

Select $\xi \notin \mathfrak{h}_{\bar{1}}$ and set $F_1 = F \cap \mathrm{Span}(\xi)^{\perp}$. Let $\mathfrak{n}$ be the kernel of f_λ. Then

$$\dim \mathfrak{h}_{\bar{1}} + \dim \mathfrak{h}_{\bar{1}}^{\perp} = \dim \mathfrak{g}_{\bar{1}} + \dim \mathfrak{n},$$

so $\dim \mathfrak{h}_{\bar{1}}^{\perp} = \dim \mathfrak{n} + 1$. Therefore, there exists an element $\eta \in \mathfrak{h}_{\bar{1}}^{\perp} \cap F$ and such that $\eta \notin \mathfrak{n}$. Clearly, $f_\lambda(\xi, \eta) \neq 0$ and $\mathfrak{p}_{\bar{1}}^{\perp} \subset \mathfrak{h}_{\bar{1}}$. Therefore,

$$F = F_1 \oplus \mathrm{Span}(\eta), \quad \xi \perp F_1 \text{ and } f_\lambda(\xi, \eta) \neq 0.$$

Let

$$u = u_0 + \xi u_1, \text{ where } u \in \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda),\ u_0, u_1 \in \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{h}}(\lambda) \text{ and } pu = 0 \text{ for any } p \in F.$$

Then

$$0 = pu = pu_0 + [p, \xi]u_1 + \xi p u_1,$$

hence, $pu_1 = 0$ and by the induction $u_1 \in \mathrm{Span}(v, \xi_0 v)$. Thanks to ii) $[p, \xi]u_1 = f_\lambda(p, \xi)u_1$ and if $p \in F_1$, then $f_\lambda(p, \xi)u_1 = 0$; hence, $pu_0 = 0$ for any $p \in F_1$. By the induction we deduce that $u_0 \in \mathrm{Span}(v, \xi_0 v)$. Further on,

$$0 = \eta u = \eta u_0 + \eta \xi u_1 = [\eta, \xi]u_1 = f_\lambda(\eta, \xi)u_1$$

and since $f_\lambda(\eta, \xi) \neq 0$, then $u_1 = 0$ and $u = u_0 \in \mathrm{Span}(v, \xi_0 v)$. □

6.3.2. Corollary. *If $\mathfrak{h} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ is a polarization for λ, then $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$ is an irreducible module.*

Proof. Observe that irreducibility is equivalent to the absence of vectors ahhihilated by $\mathfrak{b}_{\bar{1}}$ that do not lie in $\mathrm{Span}(v, \xi_0 v)$). □

6.3.3. Corollary. *Heading* 1) *of Lemma* 6.1.3 *and heading* 4) *of the Main Theorem* 6.1.5 *hold.*

6.3.4. Corollary. *Let U be an irreducible finite dimensional $\mathfrak{g}$-module. Then $U = \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$ for some $\lambda \in L$ and a polarization $\mathfrak{h}$.*

Proof. Induction on $\dim \mathfrak{g}_{\bar{1}}$. If $\mathfrak{g} = \mathfrak{g}_{\bar{0}}$, then this is Lies theorem. Let $\mathfrak{k} \subset \mathfrak{g}$ and $\dim \mathfrak{g}_{\bar{1}}/\mathfrak{k}_{\bar{1}} = 1$.

Let U be irreducible as a $\mathfrak{k}$-module. Then there exist $\lambda \in L$ and a polarization $\mathfrak{h} \subset \mathfrak{k}$ for $\lambda \in L$ such that $U = \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{k}}(\lambda)$. If $\mathfrak{h}$ were a polarization for λ in $\mathfrak{g}$, too, then by Corollary 6.3.2 the representation

$$\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda) = \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{k}}(\lambda))$$

would have been irreducible contradicting Lemma 6.2.6.

Let $\hat{\mathfrak{h}} \supset \mathfrak{h}$ be a polarization for λ in $\mathfrak{g}$ and $\xi \in \hat{\mathfrak{h}}$ so that $\xi \notin \mathfrak{h}$. If v is an element of $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{k}}(\lambda)$ as the one described in 6.3.1 and $p \in \mathfrak{h}_{\bar{1}}$, then

$$p\xi v = [p, \xi]v = f_\lambda(p, \xi)v = 0,$$
$$\xi\xi v = \frac{1}{2}[\xi, \xi]v = \frac{1}{2}f_\lambda(\xi, \xi)v = 0.$$

Therefore, there exists a non-zero $\mathfrak{g}$-module homomorphism

$$\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}} \lambda \longrightarrow \mathrm{Ind}_{\mathfrak{h}}^{\hat{\mathfrak{k}}}(\lambda) = U$$

and since both modules are irreducible, this is an "odd isomorphism", i.e., the composition of an isomorpism with the change of parity.

Now let U be reducible as a $\mathfrak{k}$-module. Then by Lemma 6.2.7 $U = \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}} V$, and, by the induction, $V = \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{k}}(\lambda)$ for a polarization $\mathfrak{h} \subset \mathfrak{k}$ and $\lambda \in L$. If $\mathfrak{h}$ is not a polarization for λ in $\mathfrak{g}$, then let $\hat{\mathfrak{h}} \supset \mathfrak{h}$ be a polarization. We have a non-zero $\mathfrak{g}$-module homomorphism $U = \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda) \longrightarrow \mathrm{Ind}_{\hat{\mathfrak{h}}}^{\mathfrak{g}}(\lambda)$ and since both modules are irreducible, this is an isomorphism which is impossible because $\dim \mathrm{Ind}_{\hat{\mathfrak{h}}}^{\mathfrak{g}}(\lambda) < \dim \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$. Therefore, $\mathfrak{h}$ is a polarization for λ in $\mathfrak{g}$ and $U = \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$. □

6.3.5. Corollary. *Heading* 1) *of Theorem holds.*

6.3.6. A subsuperalgebra subordinate for $\boldsymbol{\lambda \in L}$. Recall, see [K2] p. 79, that if

$$\mathfrak{g}_\lambda = \{g \in \mathfrak{g} \mid \lambda([g, g_1]) = 0 \text{ for all } g_1 \in \mathfrak{g}\},$$

then a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is said to be *subordinate to* λ if $\lambda([\mathfrak{p}, \mathfrak{p}]) = 0$ and $\mathfrak{p} \supset \mathfrak{g}_\lambda$.

6.3.6.1. Corollary. *Let $\lambda \in L$, $\mathfrak{p}$ a subalgebra subordinate to λ. Then $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is irreducible if and only if $\mathfrak{p}$ is a polarization for λ.*

6.4. Classification of modules $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\boldsymbol{\lambda})$

6.4.1. Lemma. *If $(\lambda, \mathfrak{h}) \sim (\mu, \mathfrak{t})$, then $\mathfrak{h}$ is a polarization for μ.*

Proof. By 6.2.5 $\mathfrak{h}$ is a polarization for $\lambda - \theta_{\mathfrak{h}}$. Since $\lambda - \theta_{\mathfrak{h}} = \mu - \theta_{\mathfrak{t}}$, then $\mathfrak{t}$ is also a polarization for $\lambda - \theta_{\mathfrak{h}}$. Let $\mathfrak{n}$ be the kernel of $f_{\lambda - \theta_{\mathfrak{h}}}$, then $\mathfrak{t} \supset \mathfrak{n}$. Hence, $\mathfrak{g}_{\bar{1}}/\mathfrak{t}$ is a subquotient of $\mathfrak{g}_{\bar{1}}/\mathfrak{n}$. Therefore, by Lemma 6.2.5 we see that $\mathfrak{h}$ is a polarization for $\mu = (\lambda - \theta_{\mathfrak{h}}) + \theta_{\mathfrak{t}}$. □

6.4.2. Proof of heading 3) of Theorem. Let $(\lambda, \mathfrak{h}) \sim (\mu, \mathfrak{t})$. We will carry the proof out by the induction on $k = \dim \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{t})$. If $k = 0$ the statement is obvious. Let $k = 1$, then, obviously, $\dim \mathfrak{t}/(\mathfrak{h} \cap \mathfrak{t}) = 1$. Consider the space $\mathfrak{h} + \mathfrak{t}$. By Lemma 6.4.1 $\mathfrak{t}$ is a polarization for λ, and therefore the kernel of f_λ on the subspace $\mathfrak{h} + \mathfrak{t}$ is equal to $\mathfrak{h} \cap \mathfrak{t}$.

Let $\xi \in \mathfrak{h}$ and $\eta \in \mathfrak{t}$ be such that $\bar{\xi} \in \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{t}), \bar{\xi} \neq 0$ and $\bar{\eta} \in \mathfrak{t}/(\mathfrak{h} \cap \mathfrak{t}), \bar{\eta} \neq 0$. We may assume that $f_\lambda(\xi, \eta) = 1$.

Let $v \in \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$ be as in Prop. 6.3.1. Then for $r \in \mathfrak{h} \cap \mathfrak{t}$, we have

$$r\eta v = [r, \eta]v = \lambda([r, \eta])v = 0, \quad \eta\eta v = \frac{1}{2}[\eta, \eta]v = \frac{1}{2}\lambda([\eta, \eta])v = 0,$$

i.e., $\mathfrak{t}_{\bar{1}}(\eta v) = 0$, and therefore there exists a non-zero homomorphism $\mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(\tilde{\mu}) \longrightarrow \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$, where $\tilde{\mu}$ is the weight of ηv.

Since $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$ is irreducible and $\dim \mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(\mu) = \dim \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$, this homomorphism is an isomorphism. Let $g \in \mathfrak{g}_{\bar{0}}$. Then

$$g(\eta v) = \eta(gv) + [g, \eta]v = [\lambda + \mathrm{tr}_{\mathfrak{t}/(\mathfrak{t}\cap\mathfrak{h})} \mathrm{ad}_g]\eta v,$$

i.e., $\tilde{\mu} = \lambda + \mathrm{tr}_{\mathfrak{t}/(\mathfrak{t}\cap\mathfrak{h})} \mathrm{ad}_g$.

Since $\lambda \in L$, it follows that

$$\begin{aligned} 0 = \lambda([g,[\xi,\eta]]) &= \lambda([[g,\xi],\eta]) + \lambda([\xi,[g,\eta]]) = \\ &= (\mathrm{tr}_{\mathfrak{t}/\mathfrak{t}\cap\mathfrak{h}} \mathrm{ad}_g + \mathrm{tr}_{\mathfrak{h}/\mathfrak{t}\cap\mathfrak{h}} \mathrm{ad}_g)\lambda([\xi,\eta]). \end{aligned}$$

Since $\lambda([\xi,\eta] = 1$, it follows that $\mathrm{tr}_{\mathfrak{t}/\mathfrak{t}\cap\mathfrak{h}} \mathrm{ad}_g = -\mathrm{tr}_{\mathfrak{h}/\mathfrak{t}\cap\mathfrak{h}} \mathrm{ad}_g$, and

$$\mu = \lambda - \theta_{\mathfrak{p}} - \theta_{\mathfrak{t}} = \lambda + \mathrm{tr}_{\mathfrak{t}/\mathfrak{t}\cap\mathfrak{h}} = \tilde{\mu},$$

i.e., $\mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(\mu) \cong \pi(\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda))$.

Let $k > 1$. On $\mathfrak{g}_{\bar{1}}$, consider the form f_λ. Let $\mathfrak{h} = \mathfrak{g}_{\bar{0}} + \mathfrak{h}_{\bar{1}}$ and $\mathfrak{t} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{t}_{\bar{1}}$. Select F so that $\mathfrak{h}_{\bar{1}} \cap \mathfrak{t}_{\bar{1}} \subset F \subset \mathfrak{h}_{\bar{1}}$, $F \neq \mathfrak{h}_{\bar{1}}$ and $F \neq \mathfrak{h}_{\bar{1}} \cap \mathfrak{t}_{\bar{1}}$, where F is a $\mathfrak{g}_{\bar{0}}$-submodule in $\mathfrak{g}_{\bar{1}}$. Set $\mathfrak{r}_{\bar{1}} = F + (F^\perp \cap \mathfrak{t}_{\bar{1}})$. It is not difficult to verify that $\mathfrak{r} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{r}_{\bar{1}}$ is a polarization for λ. Set

$$\nu(x) = \lambda(x) - \mathrm{tr}_{\mathfrak{h}_{\bar{1}}/(\mathfrak{h}_{\bar{1}}\cap\mathfrak{r}_{\bar{1}})}(\mathrm{ad}_x).$$

Since $\mathfrak{h}_{\bar{1}}/(\mathfrak{h}_{\bar{1}} \cap \mathfrak{r}_{\bar{1}})$ is a subfactor in $\mathfrak{g}_{\bar{1}}/\mathfrak{n}$, where $\mathfrak{n}$ is the kernel of f_λ, it follows from Lemma 6.2.5 that $\mathfrak{n}$ is a polarization for ν.

Since $\mathfrak{h}_{\bar{1}} \cap \mathfrak{r}_{\bar{1}} \supset F \supset \mathfrak{h}_{\bar{1}} \cap \mathfrak{t}_{\bar{1}}$, then $\dim \mathfrak{r}_{\bar{1}}/(\mathfrak{h}_{\bar{1}} \cap \mathfrak{r}_{\bar{1}}) < \dim \mathfrak{h}_{\bar{1}}/(\mathfrak{h}_{\bar{1}} \cap \mathfrak{t}_{\bar{1}})$.

Further, the diagram of inclusions

$$\begin{array}{ccc} \mathfrak{h}_{\bar{1}} \cap \mathfrak{r}_{\bar{1}} & \longrightarrow & \mathfrak{h}_{\bar{1}} \\ \downarrow & & \downarrow \\ \mathfrak{r}_{\bar{1}} & \longrightarrow & \mathfrak{g}_{\bar{1}} \end{array}$$

shows that

$$2\theta_{\mathfrak{h}}(x) - \mathrm{tr}_{\mathfrak{h}_{\bar{1}}/(\mathfrak{h}_{\bar{1}} \cap \mathfrak{r}_{\bar{1}})} \mathrm{ad}_x = 2\theta_{\mathfrak{r}}(x) - \mathrm{tr}_{\mathfrak{r}_{\bar{1}}/(\mathfrak{r}_{\bar{1}} \cap \mathfrak{h}_{\bar{1}})} \mathrm{ad}_x .$$

By duality, there exists a nondegenerate pairing

$$(\mathfrak{h}_{\bar{1}}/(\mathfrak{h}_{\bar{1}} \cap \mathfrak{r}_{\bar{1}})) \times (\mathfrak{r}_{\bar{1}}/(\mathfrak{h}_{\bar{1}} \cap \mathfrak{r}_{\bar{1}})) \longrightarrow \mathbb{C}$$

and since

$$\mathrm{tr}_{\mathfrak{h}_{\bar{1}}/(\mathfrak{h}_{\bar{1}} \cap \mathfrak{r}_{\bar{1}})} \mathrm{ad}_x = -\mathrm{tr}_{\mathfrak{r}_{\bar{1}}/(\mathfrak{h}_{\bar{1}} \cap \mathfrak{r}_{\bar{1}})} \mathrm{ad}_x,$$

then

$$\mathrm{tr}_{\mathfrak{r}_{\bar{1}}/(\mathfrak{h}_{\bar{1}} \cap \mathfrak{r}_{\bar{1}})} \mathrm{ad}_x = -\theta_{\mathfrak{h}}(x) + \theta_{\mathfrak{r}}(x).$$

Thus,

$$\nu(x) - \theta_{\mathfrak{r}}(x) = \lambda(x) + \mathrm{tr}_{\mathfrak{r}_{\bar{1}}/(\mathfrak{h}_{\bar{1}} \cap \mathfrak{r}_{\bar{1}})} \mathrm{ad}_x - \theta_{\mathfrak{r}}(x) = \lambda(x) - \theta_{\mathfrak{h}}(x),$$

i.e., $(\lambda, \mathfrak{h}) \sim (\nu, \mathfrak{r})$ and, by the induction, $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda) = \mathrm{Ind}_{\mathfrak{r}}^{\mathfrak{g}}(\nu)$. Besides, $\nu - \theta_{\mathfrak{r}} = \lambda - \theta_{\mathfrak{h}} = \mu - \theta_{\mathfrak{t}}$ and $\mathfrak{t}_{\bar{1}} \cap \mathfrak{r}_{\bar{1}} \supset F^\perp \cap \mathfrak{t}_{\bar{1}} \supset \mathfrak{t}_{\bar{1}} \cap \mathfrak{h}_{\bar{1}}$, where the latter inclusion is a strict one because $F \neq \mathfrak{h}_{\bar{1}}$; therefore,

$$\dim \mathfrak{t}_{\bar{1}}/\mathfrak{t}_{\bar{1}} \cap \mathfrak{r}_{\bar{1}} < \dim \mathfrak{t}_{\bar{1}}/(\mathfrak{t}_{\bar{1}} \cap \mathfrak{h}_{\bar{1}}).$$

By the induction, $\mathrm{Ind}_{\mathfrak{r}}^{\mathfrak{g}}(\nu) \cong \mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(\mu)$, therefore, $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda) = \mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(\mu)$.

Conversely, let $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda) \cong \mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(\mu)$. Then $\lambda = \mu + \lambda_1 + \dots + \lambda_k$, where the λ_i are the weights of $\mathfrak{g}_{\bar{1}}/\mathfrak{t}_{\bar{1}}$. Therefore, by Lemma 6.2.5 $\mathfrak{t}$ is a polarization for λ and, thanks to sec. 6.4.1, for $\tilde{\mu} = \lambda - \theta_{\mathfrak{h}} + \theta_{\mathfrak{t}}$, too. Since $\tilde{\mu} - \theta_{\mathfrak{t}} = \lambda - \theta_{\mathfrak{h}}$, then by the above $\mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(\tilde{\mu}) = \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda) = \mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(\mu)$. Let $\tilde{v} \in \mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(\tilde{\mu})$ be as in Prop. 6.3.1 and $\mathfrak{t}_{\bar{1}}\tilde{v} = 0$. By 6.4.1 $\tilde{v} \in \mathrm{Span}(v, \xi_0 v)$, where $v \in \mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(\mu)$ be as in Prop. 6.3.1; therefore, $\tilde{\mu} = \mu$ and $(\mu, \mathfrak{t}) \sim (\lambda, \mathfrak{h})$. □

6.4.3. Corollary. *Heading* 2) *of Theorem and heading* 2) *of Lemma* 6.1.3 *hold.*

Proof. Due to sec. 6.2.5 it is clear that $\mathfrak{h}$ is a polarization for $\lambda + \theta_{\mathfrak{h}}$, and therefore $I_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$ is irreducible. If $\mathfrak{t}$ is another polarization for λ, then by sect. 6.4.2

$$I_{\mathfrak{t}}^{\mathfrak{g}}(\lambda) = \mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(\lambda + \theta_{\mathfrak{t}}) = \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda + \theta_{\mathfrak{h}}) = I_{\mathfrak{h}}^{\mathfrak{g}}(\lambda).$$

If U is irreducible, then by sect. 6.3.4 $U \cong I_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$ for some λ and $\mathfrak{h}$.

If $I(\lambda) = I(\mu)$, then $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda + \theta_{\mathfrak{h}}) \cong \mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(\mu + \theta_{\mathfrak{t}})$ and by sect. 6.4.2

$$\lambda = \lambda + \theta_{\mathfrak{h}} - \theta_{\mathfrak{h}} = \mu + \theta_{\mathfrak{t}} - \theta_{\mathfrak{t}} = \mu. \quad \square$$

Proof of heading 3) of Lemma 6.1.3. Let us prove that

$$I_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) \cong CI_{\mathfrak{p}}^{\mathfrak{g}}(\lambda).$$

For this, we use the isomorphisms

$$(I_{\mathfrak{p}}^{\mathfrak{g}}(\lambda))^* \cong CI_{\mathfrak{p}}^{\mathfrak{g}}(-\lambda) \quad \text{and} \quad (I_{\mathfrak{p}}^{\mathfrak{g}}(\lambda))^* \cong I_{\mathfrak{p}}^{\mathfrak{g}}(-\lambda + 2\theta_{\mathfrak{p}}).$$

The first of these isomorphisms follows from the definitions of the induced and coinduced modules.

Let us prove the other one. Select a basis $\xi_1, \dots, \xi_n$ in the complement to $\mathfrak{p}_{\bar{1}}$ in $\mathfrak{g}_{\bar{1}}$ and consider the following filtration of $I_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$:

$$I_0 = \mathrm{Span}(v) \subset I_1 = \mathrm{Span}(v, \xi_1 v, \xi_2 v, \dots, \xi_n v) \subset \dots \subset I_n = I_{\mathfrak{p}}^{\mathfrak{g}}(\lambda),$$

where v is as in Prop. 6.3.1. It is clear that the elements $\xi_1, \xi_2, \dots, \xi_n$ can be chosen so that each I_k is a $\mathfrak{g}_{\bar{0}}$-module. Let $l \in (I_{\mathfrak{p}}^{\mathfrak{g}}(\lambda))^*$ be such that $l(I_n) \neq 0$ while $l(I_{n-1}) = 0$. Then it is easy to verify that $\mathfrak{p}_{\bar{1}} l = 0$ and the weight l with respect to $\mathfrak{g}_{\bar{0}}$ is equal to $-\lambda + 2\theta_{\mathfrak{p}}$. Therefore, there exists a nonzero homomorphism $\varphi\colon I_{\mathfrak{p}}^{\mathfrak{g}}(-\lambda + 2\theta_{\mathfrak{p}}) \longrightarrow (I_{\mathfrak{p}}^{\mathfrak{g}}(\lambda))^*$.

Since the dimensions of these modules are equal and the first of them is irreducible, φ is an isomorphism. Hence,

$$CI(\lambda) = CI_{\mathfrak{p}}^{\mathfrak{g}}(\lambda - \theta_{\mathfrak{p}}) \cong (I_{\mathfrak{p}}^{\mathfrak{g}}(-\lambda + \theta_{\mathfrak{p}}))^* = I_{\mathfrak{p}}^{\mathfrak{g}}(\lambda - \theta_{\mathfrak{p}} + 2\theta_{\mathfrak{p}}) = I(\lambda). \quad \square$$

6.5. An example

Let $\Lambda(2) = \mathbb{C}[\xi_1, \xi_2]$ be the Grassmann superalgebra on two indeterminates with the natural $\mathbb{Z}/2$-grading (parity). In $\mathfrak{gl}(\Lambda(2))$, consider the linear hull $\mathfrak{g}$ of the operators

$$\begin{aligned}
&x = \xi_1 \frac{\partial}{\partial \xi_1}, \quad y = \xi_2 \frac{\partial}{\partial \xi_1}, \quad z = \xi_1 \xi_2, \quad u = 1;\\
&\eta_1 = \frac{\partial}{\partial \xi_1}, \quad \eta_2 = \frac{\partial}{\partial \xi_2} - \xi_1 \xi_2 \frac{\partial}{\partial \xi_1},\\
&\eta_{-1} = \xi_1, \quad \eta_{-2} = \xi_2,
\end{aligned}$$

where $f \in \Lambda(2)$ is identified with the operator of left multiplication by f. It is not difficult to verify that $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ is a Lie superalgebra. It is solvable since so is $\mathfrak{g}_{\bar 0}$. Moreover, $[\mathfrak{g}_{\bar 0}, \mathfrak{g}_{\bar 0}] = \mathrm{Span}(y, z)$ and $[\mathfrak{g}_{\bar 1}, \mathfrak{g}_{\bar 1}] = \mathfrak{g}_{\bar 0}$.

Let u^*, y^*, z^*, x^* be the basis of $\mathfrak{g}_{\bar 0}^*$ left dual u, y, z, x, respectively, and let $\lambda = u^*$. Then $\mathfrak{h} = \mathfrak{g}_{\bar 0} \oplus \mathrm{Span}(\eta_{-1}, \eta_{-2})$ and $\mathfrak{t} = \mathfrak{g}_{\bar 0} \oplus \mathrm{Span}(\eta_1, \eta_2)$ are polarizations for λ.

As is easy to verify, the characters of the irreducible factors of the $\mathfrak{g}_{\bar 0}$-module $I_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$ are λ and $\lambda - x^*$ whereas the characters of the irreducible factors of the $\mathfrak{g}_{\bar 0}$-module $I_{\mathfrak{t}}^{\mathfrak{g}}(\lambda)$ are λ and $\lambda + x^*$. Hence, $I_{\mathfrak{h}}^{\mathfrak{g}}(\lambda) \not\simeq I_{\mathfrak{t}}^{\mathfrak{g}}(\lambda)$.

Moreover, $\lambda - (\lambda - x^*) = x^*$ but $x^*([\mathfrak{g}_{\bar 1}, \mathfrak{g}_{\bar 1}]) \neq 0$ contradicting the statement of Theorem 7 of [K2].

The error in the proof of Theorem 7 of [K2] is not easy to find: it is an incorrect induction in the proof of heading a) on p. 80 (of [K2]). Namely, if, in notations of [K2], the subalgebra H is of codimension (0, 1), then the irreducible factors of W considered as G_0-modules belong by the inductive hypothesis to one class from L/L_0^H, where

$$L_0^H = \{\lambda \in \mathfrak{g}^* \mid \lambda([H, H]) = 0\},$$

NOT to one class from L/L_0^G as stated on p. 80, line 13 from below.

Chapter 7

How to realize Lie algebras by vector fields (I. Shchepochkina)

7.1. Introduction

This chapter is a version of the paper [Shch].

Here I offer an algorithm which explicitly describes how to embed any $\mathbb{Z}$-graded Lie algebra (or Lie superalgebra) $\mathfrak{n} := \bigoplus\limits_{k \geq -d} \mathfrak{n}_k$ such that

$$\mathfrak{n}_{-1} \text{ generates } \mathfrak{n}_- := \bigoplus_{k<0} \mathfrak{n}_k \text{ and } \dim \mathfrak{n}_- < \infty \tag{7.1}$$

into a Lie algebra (resp., Lie superalgebra) of polynomial vector fields over $\mathbb{R}$ or $\mathbb{C}$ or over a field $\mathbb{K}$ of characteristic $p > 0$. [1)]

For almost a decade, whenever asked, I described the algorithm I propose here but was reluctant to publish it as a research paper: the algorithm is straightforward and was, actually, used more than a century ago by Cartan [C], and recently by Yamaguchi [Y].

Grozman and Leites convinced me, however, that the algorithm, and its usefulness, were never expressed explicitly. Most convincingly, they used the algorithm not only for interpreting known, but mysterious, simple Lie algebras, and Lie superalgebras, especially in characteristic $p > 0$, but in order to get new examples in the absence of classification ([GL3]). So here it is. Grozman already implemented it in his `SuperLie` package [Gr].

Having started to write, I added something new as compared with [C]: a description by means of differential equations of *partial* prolongs — subalgebras of the Lie algebras of polynomial vector fields embedded "projective-like". Such description is particularly important if $p > 0$, and for some Lie superalgebras.

At the last moment, I learned that, for $p > 0$, Fei and Shen [FSh] proved existence of embeddings I consider and illustrated it with a description of the

[1] Although p denotes the characteristic of the ground field, parity, and is also used as an index, the context is always clear.

simple Lie algebras of contact vector fields for $p = 2$. They also formulate questions this chapter answers.

For reviews of related to our result realizations of Lie (super)algebras by differential operators (not necessarily first order homogeneous ones), see [BGLS, VM].

Problem formulation, facts known, and our reasons. Let $\mathfrak{n} := \mathop{\oplus}\limits_{k=-d}^{-1} \mathfrak{n}_k$, be an n-dimensional $\mathbb{Z}$-graded Lie algebra of depth $d > 1$ satisfying (7.1). Let $f : \mathfrak{n} \longrightarrow \mathfrak{vect}(n) = \mathfrak{der}\mathbb{K}[x_1, \dots, x_n]$ be an embedding. The image $f(\mathfrak{n})$ is a subspace in the space of vector fields; every vector field can be evaluated at any point; let $f(\mathfrak{n})(0)$ be the span of these evaluations at 0.

Problem 1. *Embed* $\mathfrak{n}$ *into* $\mathfrak{vect}(n)$ *so that the* $\dim f(\mathfrak{n})(0) = n$.

Comment. Roughly speaking, we wish the image of $\mathfrak{n}$ be spanned by all partial derivatives modulo vector fields that vanish at the origin.

Such an embedding determines a non-standard[2)] grading of depth d on $\mathfrak{vect}(n)$. We will denote $\mathfrak{vect}(n)$ considered with this non-standard grading by $\mathfrak{v} = \mathop{\oplus}\limits_{k=-d}^{\infty} \mathfrak{v}_k$. Let $\mathfrak{g}_-$ be the image of $\mathfrak{n}$ in $\mathfrak{v}$, i.e., $\mathfrak{g}_- \subset \mathfrak{v}_- := \mathop{\oplus}\limits_{k<0} \mathfrak{v}_k$.

Problem 2. *Compute the* **complete algebraic prolong** *of* $\mathfrak{g}_-$, *i.e., the maximal subalgebra* $(\mathfrak{g}_-)_* = \mathop{\oplus}\limits_{k\geq -d} \mathfrak{g}_k \subset \mathfrak{v}$ *with the given* **negative** *part.*

Problem 3. *Single out* **partial prolongs** *of* $\mathfrak{g}_-$ *in* $(\mathfrak{g}_-)_*$. *In particular, given not only* $\mathfrak{n}$, *but* $\mathfrak{n}_0 \subset \mathfrak{der}_0\mathfrak{n}$, *where the subscript* 0 *singles out derivations that preserve the* $\mathbb{Z}$*-grading, we should automatically have an embedding* $\mathfrak{n}_0 \subset \mathfrak{g}_0$.

If the inclusion $\mathfrak{n}_0 \subset \mathfrak{g}_0$ *is a strict one, we wish to be able to single out* $\mathfrak{n}_0$ *in* $\mathfrak{g}_0$ *as well as to single out the algebraic prolong* $(\mathfrak{g}_-, \mathfrak{n}_0)_*$ *— the maximal subalgebra of* $\mathfrak{v}$ *with a given* **non-positive** *part — in* $(\mathfrak{g}_-)_*$.

If $\mathfrak{n}_0 = \mathfrak{g}_0$ *but the component* $\mathfrak{g}_1$ *forms a reducible* $\mathfrak{g}_0$*-module with a submodule* $\mathfrak{h}_1$, *how to singe out the maximal subalgebra* $(\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{h}_1)_* \subset \mathfrak{v}$ *with a given "beginning part"(components of grading* ≤ 1*)?*

In utmost generality, single out in $\mathfrak{v}$ *the maximal subalgebra* $\mathfrak{h}_* = \mathop{\oplus}\limits_{k\geq -d} \mathfrak{h}_k$ *with a given* beginning part $\mathfrak{h} = \mathfrak{g}_- \oplus \left(\mathop{\oplus}\limits_{0\leq k\leq K} \mathfrak{h}_k\right)$. *Naturally, the beginning part* $\mathfrak{h}$ *should be compatible with the bracket, i.e.,* $[\mathfrak{h}_i, \mathfrak{h}_j] \subset \mathfrak{h}_{i+j}$ *for all* i, j *such that* $i + j \leq K$.

The components $\mathfrak{h}_k$ with $k > K$ are defined recurrently:

$$\mathfrak{h}_k = \{X \in \mathfrak{g}_k \mid [X, \mathfrak{g}_{-1}] \subset \mathfrak{h}_{k-1}\}. \tag{7.2}$$

[2] The grading $\deg x_i = 1$ for all i associated with the (x)-adic filtration is said to be *standard*; any other grading is *non-standard*.

We prove the inclusion $[\mathfrak{h}_k, \mathfrak{h}_l] \subset \mathfrak{h}_{k+l}$ for all indices by induction on $k+l$ with an appeal to (1); it guarantees that $\mathfrak{h}_* := \mathfrak{g}_- \oplus \left(\mathop{\oplus}\limits_{0\leq k} \mathfrak{h}_k\right)$ is a subalgebra of $\mathfrak{v}$. The Lie algebra $\mathfrak{h}_* = \mathop{\oplus}\limits_{k\geq -d} \mathfrak{h}_k$ is a generalization of Cartan prolong.

Remark. *De facto*, for simple Lie algebras over $\mathbb{R}$ and $\mathbb{C}$, the number K is always ≤ 1, but if $\operatorname{Char}\mathbb{K} > 0$, and for superalgebras, then $K > 1$ is possible.

Discussion. If a Lie group N with a Lie algebra $\mathfrak{n}$ is given explicitly, i.e., if we know explicit expressions for the product of the group elements in some coordinates, then there is no problem to describe an embedding $\mathfrak{n} \subset \mathfrak{vect}(n)$: the Lie algebras of left- and right-invariant vector fields on N are isomorphic to $\mathfrak{n}$. (This is, actually, one of the definitions of the Lie algebra of N.) If the group N is not explicitly given, then to describe an embedding $\mathfrak{n} \subset \mathfrak{vect}(n)$ is a part of the problem of recovering the Lie group from its Lie algebra (in the cases where one can speak about Lie groups). Of course, the Campbell-Hausdorff formula gives a solution to this problem. Unfortunately, despite its importance in theoretical discussions, the Campbell-Hausdorff formula is not convenient in actual calculations.

For $\mathbb{K} = \mathbb{R}$ and $\mathbb{C}$, another method of constructing an embedding $\mathfrak{n} \subset \mathfrak{vect}(n)$ and recovering a Lie group from its Lie algebra is integration of the Maurer-Cartan equations, cf. [DFN]. Although the algorithm I offer does not use a Lie group of $\mathfrak{n}$ and is applicable even for the cases where no analog of a Lie groups can be offered, it is viewing a given Lie algebra as the Lie algebra of left-invariant vector fields on a Lie group that gives us a key lead.

For other algorithms for embedding $\mathfrak{n} \subset \mathfrak{vect}(n)$, based on explicit descriptions of the $\mathfrak{n}$-action in $U(\mathfrak{n})$, see [BGLS, VM]. Now, let me list reasons that lead to the algorithm.

Reason 1. Let $X_1, \dots, X_n$ be vector fields linearly independent at each point of an n-dimensional (super)domain, and

$$[X_i, X_j] = \sum_k c_{ij}^k X_k, \quad c_{ij}^k \in \mathbb{K}. \tag{7.3}$$

Let $\omega^1, \dots \omega^n$ be the dual basis of differential 1-forms ($\omega^i(X_j) = \delta_j^i$). Then (a standard exercise)

$$d\omega^k = -\frac{1}{2}\sum_{ij} c_{ij}^k \omega^i \wedge \omega^j = -\sum_{i<j} c_{ij}^k \omega^i \wedge \omega^j, \tag{7.4}$$

and *vise versa*: if the 1-forms $\omega^1, \dots \omega^n$ satisfy (7.4) then the dual vector fields $X_1, \dots, X_n$ satisfy (7.3).

Observe that, although in the super setting the expression for $d\omega$, i.e.,

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]), \tag{7.5}$$

acquires some signs, eq. (7.4) is valid for superalgebras as well: the extra signs in eq. (7.5) appearing due to super nature of its constituents do not affect (7.4).

Recall that if the fields X_i form a basis of left-invariant vector fields on the group N, eqs. (7.4) are called the *Maurer-Cartan equations*; in this case, c_{ij}^k are the structure constants of the Lie algebra $\mathfrak{n}$.

If $\omega^i = \sum_k V_k^i(x)dx^k$, then eqs. (7.4) can be expressed as equations for the functions V_k^i:

$$\partial_j V_i^k - \partial_i V_j^k = \sum_{p,q} c_{pq}^k V_i^p V_j^q. \tag{7.6}$$

Reason 2. In the real or complex situation, eqs. (7.6) are easy to integrate in "nice"coordinates for any Lie algebra $\mathfrak{n}$ (not only nilpotent). Namely, introduce functions

$$W_j^i(t,x) = tV_j^i(\exp(tx)), \quad \text{where } t \in \mathbb{R},\ x \in \mathfrak{n}^*.$$

(In other words, we should integrate eqs. (7.6) along one-parameter subgroups.) As is easy to check, the functions W satisfy ODE

$$\frac{dW_j^i}{dt} = \delta_j^i + \sum_{p,q} c_{pq}^i W_j^p x^q \tag{7.7}$$

with the initial condition $W_j^i(0,x) = 0$.

Actually, since $\mathfrak{n}$ is $\mathbb{Z}$-graded nilpotent, the system (7.6) is so simple that one can integrate it directly, without appealing to auxiliary functions W, and over any ground field. This direct solution of (7.6) allows us to construct an embedding $\mathfrak{n} \longrightarrow \mathfrak{vect}(n)$ most suitable for our purposes[3)], and find all possible embeddings.

Namely, select a basis $B = \{e_1, \dots, e_n\}$ of $\mathfrak{n}$ compatible with the grading. This means that its first n_1 elements form a basis of $\mathfrak{n}_{-1}$, the next n_2 elements form a basis of $\mathfrak{n}_{-2}$, and so on. Let I_s be the set of indices corresponding to $\mathfrak{n}_{-s}$, and $I = \cup I_s$. Let c_{ij}^k be the structure constants in this basis:

$$[e_i, e_j] = \sum_k c_{ij}^k e_k, \tag{7.8}$$

and $x_1, \dots, x_n$ be the determined by B coordinates of $\mathfrak{n}^*$, the dual space to $\mathfrak{n}$. The nonstandard $\mathbb{Z}$-grading $\mathfrak{v} = \underset{k\geq -d}{\oplus} \mathfrak{v}_k$ of $\mathfrak{vect}(n)$ compatible with the $\mathbb{Z}$-grading of $\mathfrak{n}$ is determined by setting

[3] For example, why even the authors of [BGLS] were reluctant to use any of the three algorithms presented in [BGLS]? I tested all of their three algorithms: they work, although some clarifications are needed. The only explanation I can deduce from the questions Grozman and Leites asked me, is the fact that the formulas in [BGLS] are *fixed*, and some of them involve divisions. And what to do if, say, one wants to avoid division (by 2 or 3) in coefficients?! Whereas I give the customer a possibility to select the embedding to taste. *I. Shch.*

$$\deg x_i = s \quad \text{for any } i \in I_s. \tag{7.9}$$

Let $X_i \in \mathfrak{v}$ be the image of e_i under our embedding. Then the value of X_i at 0 is equal to $\partial_{x_i} := \partial_i$ and the value of the dual form ω^i at 0 is equal to dx^i. If $i \in I_s$, then the field X_i and the form ω^i are homogeneous of degree $-s$ and s respectively. We have

$$\begin{aligned}
&\omega^i = dx^i && \text{for } i \in I_1;\\
&\omega^i = dx^i + \sum_{j,k\in I_1} a_{jk}^i x^j\, dx^k && \text{for } i \in I_2;\\
&\omega^i = dx^i + \sum_{j\in I_1, k\in I_2} a_{jk}^i x^j\, dx^k + \\
&\quad \sum_{k\in I_1}\Big(\sum_{s,t\in I_1} a_{stk}^i x^s x^t + \sum_{s\in I_2} a_{sk}^i x^s\Big)\, dx^k && \text{for } i \in I_3;\\
&\dots\ \dots\dots\dots\dots\dots && \dots\dots
\end{aligned}$$

The grading guarantees automatic fulfillment of a part of conditions (7.6): For example, for $k \in I_1$, all the functions V_i^k are known: $V_i^k = \delta_i^k$; for $k \in I_2$, the rhs of (7.6) only contains the known functions (V_i^k with $k \in I_1$), and so on.

The system for a_{jk}^i is highly undetermined but if we are interested in getting some embedding only, we do not need all the solutions; any solution (the simpler looking, the better) will do. Then we proceed in the same way with V_i^k for $k \in I_3$, and so on. The Jacobi identity guarantees the compatibility of the system.

Reason 3. How the complete prolongations are singled out. Over $\mathbb{R}$, the connected simply connected Lie group N with Lie algebra $\mathfrak{n}$, left-invariant forms ω^i, where $i \in I$, and the structure constants c_{ij}^k given by (7.4) possess a universal property ([St2]):

Let M be a smooth manifold with a collection of linearly independent at each point differential 1-forms α^i, satisfying (7.4) with the same constants c_{ij}^k. Then, for every point $x \in M$, there exists its neighborhood U and a diffeomorphism $f : U \longrightarrow N$ such that

$$\alpha^i = f^*(\omega^i).$$

Any two such diffeomorphisms differ by a translation.

Hence, as soon as we have found forms ω^i satisfying (7.4), we can think of them as of left-invariant forms of the group N and of the dual vector fields X_i as of left-invariant vector fields, $\mathfrak{g}_- = Span\{X_1, \dots, X_n\} \subset \mathfrak{vect}(N)$.

Let $Y_1, \dots, Y_n$ be the right-invariant vector fields, such that

$$X_i(e) = Y_i(e),$$

and $\theta^1, \dots, \theta^n$ be the dual right-invariant 1-forms.

Clearly, both $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ span Lie subalgebras of $\mathfrak{v}_-$.

Let us define a right-invariant distribution $\mathcal{D}$ on N such that $D(e) = \mathfrak{n}_{-1}$. Clearly, $\mathcal{D}$ is singled out by the system of equations for $X \in \mathfrak{vect}(n)$:

$$\theta^i(X) = 0 \text{ for any } i \in I_2 \cup I_3 \cup \cdots \cup I_d. \tag{7.10}$$

Since left- and right-invariant vector fields on a Lie group always commute with each other, each X_j preserves $\mathcal{D}$ and hence the Lie algebra $\mathfrak{g}_-$ preserves $\mathcal{D}$. Moreover, since $\mathfrak{n}$ is $\mathbb{Z}$-graded, it follows that the fact "$X \in \mathfrak{v}_-$ preserves $\mathcal{D}$" is equivalent to the fact "X commutes with all Y_i, where $i \in I_1$", and hence with all Y_i, where $i \in I$, since $\mathfrak{n}_{-1}$ generates $\mathfrak{n}$.

Thus, $\mathfrak{g}_-$ is characterized as the maximal subalgebra of $\mathfrak{v}_-$ preserving $\mathcal{D}$. But then the complete prolongation of $\mathfrak{g}_-$ is the maximal subalgebra of $\mathfrak{vect}(n)$ preserving $\mathcal{D}$.

Of course we can reformulate all this without appealing to N. All we need is $\mathfrak{cent}_{\mathfrak{v}_-}(\mathfrak{g}_-)$, the centralizer of $\mathfrak{g}_-$ in $\mathfrak{v}_-$. It is also clear that, having represented $Y \in \mathfrak{v}_{-s}$ as a sum of homogeneous components in the standard grading ($\deg x^i = 1$):

$$Y = \sum_{p=-1}^{d-s-1} Y_{(p)},$$

we see that for the fields that vanish at the origin (for them, $Y_{(-1)} = 0$), the lowest component of $[X_i, Y]$ coincides with the bracket of the lowest component of Y with ∂_i, and therefore is nonzero.

The other way round, for any Y such that $Y_{(-1)} \neq 0$ the equations $[X_i, Y] = 0$, where $i = 1, \ldots, n$, enable us to uniquely recover, consecutively, all the components $Y_{(p)}$ for $p \geq 0$ starting with $Y_{(-1)}$ using the recurrence:

$$[\partial_i, Y_{(p)}] = -\sum_{s=-1}^{p-1} [(X_i)_{(p-1-s)}, Y_{(s)}] \quad \text{for } i = 1, \ldots, n. \tag{7.11}$$

Let $Y_i \in \mathfrak{cent}_{\mathfrak{v}_-}(\mathfrak{g}_-)$ be such that $(Y_i)_{(-1)} = \partial_i$. Then

$$\begin{gathered}[Y_i, Y_j]_{(-1)} = [\partial_i, (Y_j)_{(0)}] + [(Y_i)_{(0)}, \partial_j] = -[(X_i)_{(0)}, \partial_j] - [\partial_i, (X_j)_{(0)}] = \\ -[X_i, X_j]_{(-1)} = -\textstyle\sum_k c_{ij}^k (X_k)_{(-1)} = -\sum_k c_{ij}^k (Y_k)_{(-1)},\end{gathered} \tag{7.12}$$

and, since the fields from $\mathfrak{cent}_{\mathfrak{v}_-}(\mathfrak{g}_-)$ are uniquely determined by their (-1)st components, we get:

$$[Y_i, Y_j] = -\sum_k c_{ij}^k Y_k,$$

i.e., $\mathfrak{cent}_{\mathfrak{v}_-}(\mathfrak{g}_-)$ is isomorphic to $\mathfrak{n}$.

Let the θ^i constitute a basis of 1-forms dual to the $\{Y_i\}_{i \in I}$ (i.e., $\theta^i(Y_j) = \delta^i_j$). Then any vector field $X \in \mathfrak{vect}(n)$ is of the form

$$X = \sum_i \theta^i(X) Y_i.$$

Since $[X_i, Y_j] = 0$ for any $i, j = 1, \ldots, n$, we have

$$\theta^i([X_j, X]) = X_j(\theta^i(X)). \tag{7.13}$$

Now let us consider the distribution $\mathcal{D}$ defined by (7.10). As we have already observed, $\mathfrak{g}_-$ is characterized as the maximal subalgebra of $\mathfrak{v}_-$ preserving $\mathcal{D}$.

Observe, first of all, that although "any X_i preserves any form θ^j", the condition in quotation marks does not survive the operation (7.2) of complete prolongation whereas the condition "preserve $\mathcal{D}$" is not so strong and survives it.

Indeed, a field $X \in \mathfrak{v}$ preserves $\mathcal{D}$ if and only if

$$\theta^k([X, Y_i]) = 0 \text{ for any } i = 1, \ldots, n_1, \text{ and any } k > n_1. \tag{7.14}$$

Let (7.14) be valid for any $X \in \mathfrak{g}_{s-1}$. Then, due to (7.2), $X \in \mathfrak{g}_s$ if and only if

$$\theta^k([[X_j, X], Y_i]) = X_j \theta^k([X, Y_i]) = 0 \ \text{ for any } i, j = 1, \ldots, n_1, \text{ and } k > n_1. \tag{7.15}$$

(We have taken (7.13) into account.)

Finally, since $\mathfrak{n}_{-1}$ generates the algebra $\mathfrak{n}$, (7.15) is equivalent to

$$\partial_j(\theta^k([X, Y_i]) = 0 \ \text{ for all } j = 1, \ldots, n. \tag{7.16}$$

But if $k \in I_l$ ($l \geq 2$), then $\theta^k([X, Y_i])$ is a homogeneous (in our nonstandard grading) polynomial of degree $s - 1 + l \geq s + 1 \geq 1$, and hence (7.16) is equivalent to

$$\theta^k([X, Y_i]) = 0 \ \text{ for any } i = 1, \ldots, n_1, \text{ and } k > n_1, \tag{7.17}$$

and hence X preserves $\mathcal{D}$.

Let us rewrite the system (7.14) for coordinates of X more explicitly:

$$\begin{gathered} Y_i(\theta^k(X)) - \textstyle\sum_j (-1)^{p(Y_i)p(\theta^j(X))} c_{ij}^k \theta^j(X) = 0 \\ \text{for any } i = 1, \ldots, n_1, \text{ and } k = n_1 + 1, \ldots, n. \end{gathered} \tag{7.18}$$

Since $\mathfrak{g}_-$ is $\mathbb{Z}$-graded, eqs. (7.18) are of a particular form. Let

$$F_i = \theta^{n - n_d + i}(X), \text{ where } i = 1, \ldots, n_d,$$

be the coordinates of a vector field X lying in the component $\mathfrak{g}_{-d}$ of maximal depth.

If the functions F_i are given, then eqs. (7.18), where $k \in I_{d-1}$, constitute a system of linear (not differential) equations for the coordinates $\theta^j(X)$ corresponding to the component $\mathfrak{g}_{-d+1}$, and if this component does not contain central elements of the whole algebra $\mathfrak{g}_-$, then all the coordinates of the level

$-d+1$ enter the system. After all these coordinates are determined, eqs. (7.18), where now $k \in I_{d-2}$, become a system of linear equations for coordinates on the next level, $-d+2$, and so on.

Therefore, the F_i are generating functions for X. In the general case, one should take for generating functions the functions corresponding to all central basis elements of $\mathfrak{g}_-$.

Now, we are able to formulate the algorithm for the first two of our problems.

7.2. The algorithm: Solving Problems 1 and 2

• In $\mathfrak{n}$, take a basis B compatible with the grading and compute the corresponding structure constants c_{ij}^k.

• Seek the basis of 1-forms $\{\omega^i\}_{i\in I}$ satisfying (7.4), i.e., solve system (7.6) upwards, i.e., starting with degree 1 and proceeding up to degree d.

• Seek the dual basis of vector fields $\{X_i\}_{i\in I}$ upwards, i.e., starting with degree $-d$ and proceeding up to degree -1. The fields $\{X_i\}_{i\in I}$ determine an embedding of $\mathfrak{n}$ into $\mathfrak{vect}(n)$.

Problem 1 is solved.

• Seek a basis $\{Y_i\}_{i\in I}$ of $\mathfrak{cent}_{\mathfrak{v}_-}(\mathfrak{g}_-)$ in $\mathfrak{v}_-$ by means of (7.11) and the dual basis of 1-forms $\{\theta^i\}_{i\in I}$.

• To find the component $\mathfrak{g}_s$ of the complete prolongation of $\mathfrak{g}_-$, we seek the field $X \in \mathfrak{g}_s$ in the form $X = \sum \theta^i(X)Y_i$. For this, we express each of the n_d generating functions $F^i = \theta^{n-n_d+i}(X)$ as a sum of monomials of degree $d+s$ (in the nonstandard grading) with undetermined coefficients and solve the system (7.18) of linear homogeneous equations for these coefficients.

For debugging, we compare, for $s<0$, the fields thus obtained with the X_i.

Problem 2 is solved.

7.2.1. Example. Consider the exceptional Lie algebra $\mathfrak{g}(2)$[4)] in its $\mathbb{Z}$-grading of depth 3, as in [C, Y]. In what follows,

$$x^{(k)} \text{ denotes } \begin{cases} \dfrac{x^k}{k!} & \text{over } \mathbb{R} \text{ or } \mathbb{C} \\ u^{(k)} (\text{the divided power}) & \text{in characteristic } p>0. \end{cases}$$

Then (recall that $\mathfrak{n}$ is the given abstract algebra whose image in the Lie algebra of vector fields is designated by $\mathfrak{g}$)

$$\mathfrak{n} = \mathfrak{n}_{-3} \oplus \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}, \quad \text{where } \dim \mathfrak{n}_{-1} = 2, \dim \mathfrak{n}_{-2} = 1, \dim \mathfrak{n}_{-3} = 2.$$

[4] We denote the exceptional Lie algebras in the same way as the serial ones, like $\mathfrak{sl}(n)$; we thus avoid confusing $\mathfrak{g}(2)$ with the second component $\mathfrak{g}_2$ of a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$. *I. Shch.*

Let us see how the algorithm works for the embedding $f(\mathfrak{n}) = \mathfrak{g}_- \subset \mathfrak{vect}(5)$.

1) A basis compatible with the $\mathbb{Z}$-grading and structure constants are of the form:

$$\begin{aligned}
&\mathfrak{n}_{-1} = \mathrm{Span}(e_1, e_2), \quad \mathfrak{n}_{-2} = \mathrm{Span}(e_3), \quad \mathfrak{n}_{-3} = \mathrm{Span}(e_4, e_5);\\
&[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4, \quad [e_2, e_3] = e_5;\\
&c_{ij}^k = 0 \text{ for } k = 1, 2; \quad c_{12}^3 = -c_{21}^3 = 1, c_{ij}^3 = 0 \text{ otherwise };\\
&c_{13}^4 = -c_{31}^4 = 1, \qquad c_{ij}^4 = 0 \text{ otherwise};\\
&c_{23}^5 = -c_{32}^5 = 1, \qquad c_{ij}^5 = 0 \text{ otherwise}.
\end{aligned}$$

2) We have

$$\begin{aligned}
&\omega^1 = dx^1, \quad \omega^2 = dx^2 \Longrightarrow V_i^k = \delta_i^k \text{ for } k = 1, 2\\
&\omega^3 = dx^3 + \sum_{i,j=1}^{2} a_{ij}^3 x^i dx^j \Longrightarrow V_4^3 = V_5^3 = 0; \quad V_3^3 = 1;\\
&V_1^3 = a_{11}^3 x^1 + a_{21}^3 x^2; \quad V_2^3 = a_{12}^3 x^1 + a_{22}^3 x^2.
\end{aligned}$$

Eqs. (7.6) give one non-trivial relation on the V_i^3:

$$\partial_2 V_1^3 - \partial_1 V_2^3 = V_1^1 V_2^2 - V_1^2 V_2^1 = 1,$$

or, equivalently,

$$a_{21}^3 - a_{12}^3 = 1. \tag{7.19}$$

Select a solution which seems to be a simplest one:

$$a_{11}^3 = a_{22}^3 = a_{12}^3 = 0, \quad a_{21}^3 = 1.$$

(In canonical coordinates of first kind, $a_{11}^3 = a_{22}^3 = 0$, $a_{12}^3 = a_{21}^3 = \frac{1}{2}$. These are most symmetric coordinates. We wish, however, to evade division if possible.) Thus, $V_1^3 = x^2$, $V_2^3 = 0$, and hence

$$\omega^3 = dx^3 + x^2 dx^1.$$

Further, for $k = 4, 5$,

$$\omega^k = dx^k + \sum_{j=1}^{3} V_j^k dx^j,$$

where

$$\begin{aligned}
&V_3^k = a_1^k x^1 + a_2^k x^2,\\
&V_j^k = \alpha_j^k (x^1)^{(2)} + \beta_j^k x^1 x^2 + \gamma_j^k (x^2)^{(2)} + \varepsilon_j^k x^3, \quad j = 1, 2.
\end{aligned}$$

Eqs. (7.6) give three nontrivial relation for each function V_j^k, where $k = 4, 5$, $j = 1, 2, 3$:

$$\partial_2 V_1^4 - \partial_1 V_2^4 = 0; \quad \partial_3 V_1^4 - \partial_1 V_3^4 = 1; \quad \partial_3 V_2^4 - \partial_2 V_3^4 = 0;$$
$$\partial_2 V_1^5 - \partial_1 V_2^5 = -x^2; \quad \partial_3 V_1^5 - \partial_1 V_3^5 = 0; \quad \partial_3 V_2^5 - \partial_2 V_3^5 = 1,$$

or, in terms of coefficients:

$$\beta_1^4 x^1 + \gamma_1^4 x^2 = \alpha_2^4 x^1 + \beta_2^4 x^2; \quad \varepsilon_1^4 - a_1^4 = 1; \quad \varepsilon_2^4 - a_2^4 = 0;$$
$$\beta_1^5 x^1 - \gamma_1^5 x^2 - \alpha_2^5 x^1 - \beta_2^5 x^2 = -x^2; \quad \varepsilon_1^5 - a_1^5 = 1; \quad \varepsilon_2^5 - a_2^5 = 1.$$

Select a simpler looking solution:

$$\omega^4 = dx^4 - x^1 dx^3, \quad \omega^5 = dx^5 - x^2 dx^3 - (x^2)^{(2)} dx^1.$$

Finally,

$$\begin{aligned}
&\omega^1 = dx^1, \quad \omega^2 = dx^2,\\
&\omega^3 = dx^3 + x^2 dx^1,\\
&\omega^4 = dx^4 - x^1 dx^3,\\
&\omega^5 = dx^5 - x^2 dx^3 - (x^2)^{(2)} dx^1.
\end{aligned}$$

3) Now seek the dual fields X_i:

$$\begin{aligned}
&X_5 = \partial_5, \qquad X_4 = \partial_4,\\
&X_3 = \partial_3 + x^1\partial_4 + x^2\partial_5,\\
&X_2 = \partial_2, \qquad X_1 = \partial_1 - x^2\partial_3 - x^1x^2\partial_4 - (x^2)^{(2)}\partial_5.
\end{aligned}$$

We get $\mathfrak{g}_- = \operatorname{Span}\{X_1, \dots, X_5\}$.

4) Now we seek homogeneous fields $Y_i = \partial_i + \dots$, commuting with all the X_j. Since the brackets with X_2, X_4, X_5 vanish, the coordinates of the Y_i can only depend on x^1 and x^3. Therefore

$$\begin{aligned}
&Y_4 = \partial_4, \quad Y_5 = \partial_5;\\
&Y_3 = \partial_3 + ax^1\partial_4 + bx^1\partial_5,
\end{aligned}$$

and $[X_1, Y_3] = 0$ implies that $Y_3 = \partial_3$.

Finally, for $i = 1, 2$, we have

$$Y_i = \partial_i + \alpha_i x^1 \partial_3 + \sum_{j=4}^{5} (\beta_i^j (x^1)^{(2)} + \gamma_i^j x^3)\partial_j.$$

Bracketing Y_i with X_1 and X_3, we get

$$\begin{aligned}
&Y_1 = \partial_1 + x^3\partial_4,\\
&Y_2 = \partial_2 - x^1\partial_3 - (x^1)^{(2)}\partial_4 + x^3\partial_5.
\end{aligned} \tag{7.20}$$

It only remains to find the forms θ^i left-dual to Y_i. The routine computations yield: $\theta^i = dx^i$ for $i = 1, 2$ and

$$\begin{aligned}
&\theta^3 = dx^3 + x^1 dx^2;\\
&\theta^4 = dx^4 - x^3 dx^1 + (x^1)^{(2)} dx^2;\\
&\theta^5 = dx^5 - x^3 dx^2.
\end{aligned} \tag{7.21}$$

5) Now, we seek all the vector fields X preserving $\mathcal{D} = \operatorname{Span}\{Y_1, Y_2\}$, or, which is the same, all the fields that belong to the complete prolong of $\mathfrak{g}_-$. Let $X = \sum f^i Y_i$, where $f^i = \theta^i(X)$. To find the f^i, we solve eqs. (7.18). In our case they are:

$$Y_1(f^4) = f^3, \ Y_1(f^5) = 0, \ Y_2(f^4) = 0, \ Y_2(f^5) = f^3, \ Y_1(f^3) = f^2, \ Y_2(f^3) = -f^1.$$

We see that X is completely determined by the functions f^4 and f^5 which must satisfy the three relations:

$$Y_1(f^5) = 0, \ Y_2(f^4) = 0, \ Y_1(f^4) = Y_2(f^5). \tag{7.22}$$

For control, let us look what are the corresponding fields in the component $\mathfrak{v}_{-2}$. In this case, both f^4 and f^5 should be of degree 1 in our grading, i.e., must be of the form $f^i = a^i x^1 + b^i x^2$ for $i = 4, 5$. Then $Y_1(f^i) = a^i$, $Y_2(f^i) = b^i$, and hence eqs. (7.22) mean that

$$f^4 = ax^1, \quad f^5 = ax^2 \Longrightarrow f^3 = a, f^1 = f^2 = 0.$$

Therefore, any field preserving $\mathcal{D}$ and lying in $\mathfrak{v}_{-2}$ is proportional to

$$X = Y_3 + x^1 Y_4 + x^2 Y_5 = \partial_3 + x^1\partial_4 + x^2\partial_5 = X_3,$$

as should be.

We similarly check that, in $\mathfrak{v}_{-1}$, our equations single out precisely the subspace spanned by X_1 and X_2.

Now, let us compute $\mathfrak{g}_0$. Its generating functions must be of degree 3 in our grading, i.e., of the form (for $i = 4, 5$)

$$\begin{aligned}
f^i = {}& a_1^i (x^1)^{(3)} + a_2^i (x^1)^{(2)} x^2 + a_3^i x^1 (x^2)^{(2)} + a_4^i (x^2)^{(3)} +\\
& b_1^i x^1 x^3 + b_2^i x^2 x^3 + c_1^i x^4 + c_2^i x^5.
\end{aligned}$$

Then

$$\begin{aligned}
Y_1(f^i) &= a_1^i (x^1)^{(2)} + a_2^i x^1 x^2 + a_3^i (x^2)^{(2)} + (b_1^i + c_1^i) x^3;\\
Y_2(f^i) &= (a_2^i - 2b_1^i - c_1^i)(x^1)^{(2)} + (a_3^i - b_2^i) x^1 x^2 + a_4^i (x^2)^{(2)} + (b_2^i + c_2^i) x^3.
\end{aligned}$$

In this case, eqs. (7.22) take the following form:

$$\begin{cases} a_1^5 = a_2^5 = a_3^5 = 0, \\ b_1^5 + c_1^5 = 0, \\ a_4^4 = 0, \\ b_2^4 + c_2^4 = 0, \\ a_3^4 - b_2^4 = 0, \\ a_2^4 - 2b_1^4 + c_1^4 = 0, \\ a_1^4 = a_2^5 - 2b_1^5 - c_1^5, \\ a_2^4 = a_3^5 - b_2^5, \\ a_3^4 = a_4^5, \\ b_1^4 + c_1^4 = b_2^5 + c_2^5. \end{cases}$$

The solution to this system is:

$$\begin{aligned} &a_1^4 = -b_1^5 = c_1^5 = \alpha, \\ &a_2^4 = -b_2^5 = \beta, \\ &a_3^4 = a_4^5 = b_2^4 = -c_2^4 = \gamma, \\ &a_4^4 = a_1^5 = a_2^5 = a_3^5 = 0, \\ &b_1^4 = \delta, \\ &c_1^4 = \beta - 2\delta, \\ &c_2^5 = 2\beta - \delta. \end{aligned}$$

Hence

$$\begin{aligned} &f^4 = \alpha(x^1)^{(3)} + \beta(x^1)^{(2)}x^2 + \gamma x^1(x^2)^{(2)} + \delta x^1x^3 + \gamma x^2x^3 + (\beta - 2\delta)x^4 - \gamma x^5, \\ &f^5 = \gamma(x^2)^3 - \alpha x^1x^3 - \beta x^2x^3 + \alpha x^4 + (2\beta - \delta)x^5, \\ &f^3 = \alpha(x^1)^{(2)} + \beta x^1x^2 + \gamma(x^2)^{(2)} + (\beta - \delta)x^3, \\ &f^2 = Y_1(f^3) = \alpha x^1 + \beta x^2, \\ &f^1 = -Y_2(f^3) = -\delta x^1 - \gamma x^2. \end{aligned}$$

For a basis of $\mathfrak{g}_0$ we take the vectors $X_\alpha, X_\beta, X_\gamma, X_\delta$ corresponding to the only one non-zero parameter (for example, X_α corresponds to $\alpha = 1, \beta = \gamma = \delta = 0$ and so on):

$$\begin{aligned} X_\alpha &= x^1Y_2 + (x^1)^{(2)}Y_3 + (x^1)^{(3)}Y_4 + (-x^1x^3 + x^4)Y_5 = \\ &\quad x^1\partial_2 + x^4\partial_5 - (x^1)^{(2)}\partial_3 - 2(x^1)^{(2)}\partial_4, \\ X_\beta &= x^2\partial_2 + x^3\partial_3 + x^4\partial_4 + 2x^5\partial_5, \\ X_\gamma &= -x^2\partial_1 - x^5\partial_4 + (x^2)^{(2)}\partial_3 + x^1(x^2)^{(2)}\partial_4 + (x^2)^{(3)}\partial_5, \\ X_\delta &= -x^1\partial_1 - x^3\partial_3 - 2x^4\partial_4 - x^5\partial_5. \end{aligned}$$

For $\alpha = \gamma = 0$, $\delta = -\beta = 1$, we get the grading operator

$$X = -x^1\partial_1 - x^2\partial_2 - 2x^3\partial_3 - 3x^4\partial_4 - 3x^5\partial_5.$$

The higher components can be calculated in a similar way.

Interpretation. There are three realizations of $\mathfrak{g} = \mathfrak{g}(2)$ as a Lie algebra that preserves a non-integrable distribution on $\mathfrak{g}_-$ related with the three (incompressible) $\mathbb{Z}$-gradings of $\mathfrak{g}$: with one or both coroots of degree 1. Above we considered the grading $(1, 0)$; Cartan used it to give the first interpretation of $\mathfrak{g}(2)$, then recently discovered by Killing, see [C]. [5)]

In this realization (by fields X_i) $\mathfrak{g} = \mathfrak{g}(2)$ preserves the distribution in the tangent bundle on $\mathfrak{g}_-$ given by the system of Pfaff equations for vector fields X

$$\theta^3(X) = 0; \qquad \theta^4(X) = 0; \qquad \theta^5(X) = 0.$$

Equivalently, but a bit more economically, we can describe $\mathfrak{g} = \mathfrak{g}(2)$ as preserving the codistribution in the cotangent bundle on $\mathfrak{g}_-$ given by the vectors (7.20), i.e., as the following system of equations for 1-forms α:

$$\alpha(Y_1) = 0; \qquad \alpha(Y_2) = 0.$$

Obviously, description in terms of codistributions is sometimes shorter: any distribution of codimension r requires for its description r Pfaff equations, whereas the dual codistribution requires $n - r$ equations.

One can similarly describe the remaining realizations of $\mathfrak{g}(2)$ corresponding to the other $\mathbb{Z}$-gradings, various realizations of $\mathfrak{f}(4)$ and $\mathfrak{e}(6) - \mathfrak{e}(8)$ and of exceptional Lie superalgebras, as well as Lie algebras over fields of characteristic p.

There s e e m e d to be no need to consider nonintegrable distributions associated with various $\mathbb{Z}$-gradings of *non-exceptional* Lie algebras (their usual description as preserving volume or a nondegenerate form seems to be sufficiently clear); Cartan himself, though understood importance of description of Lie algebras in terms of distributions, only considered one or two $\mathbb{Z}$-gradings and related distributions of exceptional Lie algebras and none for non-exceptional. If, however, we apply the algorithm presented here to $\mathfrak{g}(2)$, $\mathfrak{o}(7)$, $\mathfrak{sp}(4)$ and $\mathfrak{sp}(10)$ in characteristic $p = 2, 3$ or 5, we elucidate the meaning of some of the simple Lie algebras specific to $p = 2, 3, 5$ and, with luck and in the absence of classification, distinguish new examples, as in [GL3].

Other gradings of other algebras are now being under consideration.

[5] Cartan also considered the grading $(1, 1)$, see [Y]; Cartan used it to study Hilbert's equation $f' = (g'')^2$. *I. Shch.*

7.3. How to single out partial prolongs: Solving Problem 3

Thus, we have described the **complete** prolong of the Lie (super)algebra $\mathfrak{g}_-$, i.e., as we have already observed, the maximal subalgebra $\mathfrak{g} = (\mathfrak{g}_-)_* \subset \mathfrak{v}$ with a given negative part. Let us consider now a subspace

$$\mathfrak{h} = \mathfrak{g}_- \oplus \left(\bigoplus_{0\leq k\leq K} \mathfrak{h}_k\right) \subset \mathfrak{g},$$

closed with respect to the bracket within limits of its degrees, i.e., such that $[\mathfrak{h}_i, \mathfrak{h}_j] \subset \mathfrak{h}_{i+j}$ whenever $i + j \leq K$. Let us describe the **partial** prolong $\mathfrak{h}_* = \bigoplus\limits_{k\geq -d} \mathfrak{h}_k \subset \mathfrak{g}$ of the subspace $\mathfrak{h}$, i.e., the maximal subalgebra of $\mathfrak{g}$ with the given *beginning part* $\mathfrak{h}$. The components $\mathfrak{h}_k$ with $k > K$ are singled out by condition (7.2). Here by description we mean a way to single out $\mathfrak{h}$ in $\mathfrak{g}$ by a system of differential equations.

7.3.1. Remark. Observe that the Cartan prolong $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ (where $\mathfrak{g}_-$ is commutative, the depth is $d = 1$, and $\mathfrak{g}_0 \subset \mathfrak{gl}(n)$) is a particular case of the above construction with $\mathfrak{g} = \mathfrak{vect}(n)$, and $\mathfrak{h} = \mathfrak{g}_- \oplus \mathfrak{g}_0$. For examples of descriptions of Cartan prolongations by means of differential equations, see[Sh], and [ShP].

The homogeneous component $\mathfrak{h}_m$ of $\mathfrak{h}$ is said to be *defining*, if $\mathfrak{h}_k = \mathfrak{g}_k$ for all $k < m$ but $\mathfrak{h}_m \neq \mathfrak{g}_m$. Let us consider an algorithm of description of $\mathfrak{h}_*$ in the case where the defining component is of the maximal degree — $\mathfrak{h}_K$. The case of defining component of smaller degree $m < K$ can be reduced to our case; indeed, we first describe the partial prolong

$$\mathfrak{h}_*^m = \left(\bigoplus_{0\leq k\leq m} \mathfrak{h}_k\right)_*,$$

compare the components $\mathfrak{h}_k$, where $m < k \leq K$ with the corresponding components of this prolong $\mathfrak{h}_*^m$, find out the new defining component, if any, and so on.

Thus, let $Z_1, \dots, Z_{\dim \mathfrak{h}_K}$ be a basis of the defining component $\mathfrak{h}_K \subset \mathfrak{g}_K$.

The first thing to do is to single out the subspace $\mathfrak{h}_K$ in $\mathfrak{g}_K$ by means of a system of linear (algebraic) equations (i.e., find out a basis of the annihilator of $\mathfrak{h}_K$ in $(\mathfrak{g}_K)^*$, or, equivalently, find out the fundamental system of solutions $\alpha^1, \dots, \alpha^r$ of the system of equations for an unknown 1-form $\alpha \in (\mathfrak{g}_K)^*$:

$$\alpha(Z_i) = 0 \text{ for all } i = 1, \dots, \dim \mathfrak{h}_K. \tag{7.23}$$

The subspace $\mathfrak{h}_K$ is then singled out by a system of homogeneous linear equations for an unknown vector field $X \in \mathfrak{g}_K$:

$$\alpha^i(X) = 0 \text{ for all } i = 1, \dots, r. \tag{7.24}$$

Observe now that in $\mathfrak{g}_K$ there is a convenient for us basis consisting of the fields of the form fY_j, where f is a monomial of degree $K + s$ if $j \in I_s$. Accordingly, the dual basis consists of the elements of degree $-K$ and of the form

$$A^j_{i_1,\dots,i_t} = S(Y_{i_1} \dots Y_{i_t})\theta^j,$$

and the forms α^l can be expressed in this basis as

$$\alpha^l = \sum a_j^{l;i_1,\dots,i_t} A^j_{i_1,\dots,i_t}. \tag{7.25}$$

Substituting (7.25) into (7.24) we get a system of homogeneous linear differential equations with constant coefficients for the coordinates of the vector field $X = \theta^i(X)Y_i \in \mathfrak{h}_K$:

$$\sum a_j^{l;i_1,\dots,i_t} S(Y_{i_1} \dots Y_{i_t})\theta^j(X) \text{ for all } l = 1, \dots, r. \tag{7.26}$$

Observe now that, for Lie algebras, equation (7.26) survive prolongation procedure (7.2). Indeed, for $k > K$, by the induction hypothesis $X \in \mathfrak{h}_k$ if and only if the brackets $[X_i, X]$ satisfy (7.26) for any $i = 1, \dots, n_1$. Set

$$f^l = \sum a_j^{l;i_1,\dots,i_t} S(Y_{i_1} \dots Y_{i_t})\theta^j(X).$$

Since all the X_i commute with all the Y_j, the system (7.26) for the brackets $[X_i, X]$ is equivalent to the system

$$X_i(f^l) = 0 \text{ for all } i = 1, \dots, n_1 \text{ and } l = 1, \dots, r, \tag{7.27}$$

which thanks to (7.1) is, in its turn, equivalent to the system

$$\partial_i(f^l) = 0 \text{ for all } i = 1, \dots, n \text{ and } l = 1, \dots, r.$$

This implies that $f^l = \text{const}$ for all $l = 1, \dots, r$. Since the functions f^l are homogeneous polynomials of degree $k - K > 0$, it follows that $f^l = 0$. Hence, $X \in \mathfrak{h}_k$ if and only if X satisfies system (7.26).

In super case the fields X_i and Y_j supercommute, not commute, and this does not allow us, generally speaking, break out the X_i and pass from system (7.26) for the brackets to the system (7.27). There is, however, a simple and well-know consideration that saves us. Recall that p denotes the parity function and Pty is the parity operator, i.e.,

$$\mathrm{Pty}(x) = (-1)^{p(x)}x.$$

7.3.2. Lemma. *Let $X, Y \in \mathrm{End}\, V$ supercommute and $p(Y) = \bar{1}$. Then X and $\hat{Y} = Y\,\mathrm{Pty}$ commute (in the usual sense), i.e., $X\hat{Y} = \hat{Y}X$.*

Indeed,

$$X\hat{Y}(v) = XY\,\mathrm{Pty}(v) = (-1)^{p(v)}XY(v) =$$
$$(-1)^{p(v)}(-1)^{p(X)p(Y)}YX(v) = (-1)^{p(v)+p(X)}YX(v) = \hat{Y}X(v).$$

Therefore, in the super case, the system (7.26) should be written with operators $\hat{Y}_i$ instead of Y_i (if $p(Y_i) = \bar{0}$, we set $\hat{Y}_i = Y_i$).

Finally, if $d > 1$, then any field $X \in \mathfrak{g}$ is completely determined by its generating functions F^i. Therefore, it suffices to write equations (7.26) for the generating functions only.

Examples: Depth 1. Let $\mathfrak{g}_- = \mathfrak{g}_{-1}$ be commutative, hence

$$\mathfrak{g} = (\mathfrak{g}_-)_* = \mathfrak{vect}(n)$$

in the standard $\mathbb{Z}$-grading (the degree of each indeterminate is equal to 1). Let $\mathfrak{g}_0 = \mathfrak{gl}(n) = \mathfrak{vect}(n)_0$. The degree 1 component $\mathfrak{vect}(n)_1$ consists, as is well-known, of the two irreducible $\mathfrak{gl}(n)$-modules. Let

$$X = \sum a^k_{ij}x^ix^j\partial_k := \sum f^k(x)\partial_k. \tag{7.28}$$

Then these submodules are:

$$\mathfrak{h}_{1(1)} := \mathrm{Span}\{x^i\sum_j x^j\partial_j \mid i = 1,\dots,n\} \tag{7.29}$$

and

$$\mathfrak{h}_{1(2)} := \mathrm{Span}\{X = \sum_{j=1}^n d^k_{ij}x^ix^j\partial_k \mid d^i_{ii} + \sum d^j_{ij} = 0 \ \text{ for } \ i = 1,\dots,n.\} \tag{7.30}$$

Let us single out the partial prolongs $\mathfrak{h}_{*(i)} = (\mathfrak{g}_{-1} \oplus \mathfrak{gl}(n) \oplus \mathfrak{h}_{1(i)})_*$, where $i = 1, 2$ in $\mathfrak{vect}(n)$ by means of differential equations on the functions $f^k(x)$, see (7.28). In this case $X_i = Y_i = \partial_i$.

The conditions on $\mathfrak{h}_{1(2)}$ can be immediately expressed as

$$\sum_j \partial_i\partial_j(f^j) = 0 \ \text{ for all } \ i = 1,\dots,n$$

or, equivalently, as

$$\partial_i\left(\sum_j \frac{\partial f^j}{\partial x^j}\right) = 0 \ \text{ for all } \ i = 1,\dots,n. \tag{7.31}$$

This is exactly the system (7.26) for $\mathfrak{h}_{*(2)}$ which can be rewritten in a well-known way:

$$\sum_j \frac{\partial f^j}{\partial x^j} = \mathrm{div}\,X = \mathrm{const}\,.$$

Hence, as is well-known,

$$\mathfrak{h}_{*(2)} = \mathfrak{dsvect}(n) := \mathfrak{svect}(n) \Subset \mathbb{K}E, \ \text{ where } E = \sum x^i\partial_i.$$

Now let us consider $\mathfrak{h}_{*(1)}$ (which is, of course, $\mathfrak{sl}(n+1)$ embedded into $\mathfrak{vect}(n)$). Having expressed $X \in \mathfrak{h}_{1(1)}$ as

$$(c_1(x^1)^2 + c_2x^1x^2 + \dots + c_nx^1x^n)\partial_1 + \dots + (c_1x^1x^n + c_2x^2x^n + \dots + c_n(x^n)^2)\partial_n$$

we immediately see that $d^k_{ij} = 0$ if $i \neq k$ and $j \neq k$, and $d^k_{kk} = d^i_{ki}$ for any $i \neq k$. The corresponding system of differential equations is

$$\begin{aligned} &\frac{\partial^2 f^k}{\partial x^i\partial x^j} = 0 \ \text{ for } i, j \neq k;\\ &\frac{1}{2}\frac{\partial^2 f^k}{(\partial x^k)^2} = \frac{\partial^2 f^i}{\partial x^i\partial x^k} \ \text{ for } i \neq k. \end{aligned} \tag{7.32}$$

Superization. For superalgebras, as we have seen, one should take compositions of $Y_i = \partial_i$ with the parity operators, i.e., instead of the ∂_i we should take operators

$$\nabla_i(f) := (-1)^{p(f)p(\partial_i)}\partial_i(f).$$

These ∇_i commute (not *super*commute) with any operator $X_j = \partial_j$ from $\mathfrak{g}_{-1}$. The system (7.31) will take form

$$\nabla_i\left(\sum_j \nabla_j(f^j)\right) = 0 \text{ for all } i = 1,\dots,n,$$

which yields, nevertheless, the same condition $\mathrm{div}\,X = \mathrm{const}$. (This is one more way to see why the coordinate expression of divergence in the super case must contain some signs: eqs. (7.31) do not survive the prolongation procedure (7.2).)

Having in mind that

$$d^k_{ij} = -(-1)^{p(f^k)}\frac{\partial^2 f^k}{\partial x^i\partial x^j}; \ \text{ so } d^k_{kk} = 0 \text{ for } x^k \text{ odd} \tag{7.33}$$

we deduce that the second line in (7.32) takes the following form

$$\begin{aligned} &\frac{1}{2}\frac{\partial^2 f^k}{(\partial x^k)^2} = (-1)^{p(x^i)p(f^i)+1}\frac{\partial^2 f^i}{\partial x^i\partial x^k} \ \text{ for } p(x^i) = \bar{0} \text{ and } i \neq k;\\ &(-1)^{p(x^j)p(f^j)}\frac{\partial^2 f^i}{\partial x^j\partial x^k} = (-1)^{p(x^i)p(f^i)}\frac{\partial^2 f^i}{\partial x^i\partial x^k} \ \text{ for } p(x^i) = \bar{1} \text{ and } i, j \neq k. \end{aligned} \tag{7.34}$$

Depth > 1. We consider several more-or-less well-known examples and a new one ($\mathfrak{kas}$).

Let $\mathfrak{n} = \mathfrak{n}_{-1} \oplus \mathfrak{n}_{-2}$ be the Heisenberg Lie algebra: $\dim \mathfrak{n}_{-1} = 2n$, $\dim \mathfrak{n}_{-2} = 1$. The complete prolong of $\mathfrak{n}$ is the Lie algebra $\mathfrak{k}(2n+1)$ of contact vector fields. Having embedded $\mathfrak{n}$ into $\mathfrak{vect}(p_1, \dots, p_n, q_1, \dots, q_n; t)$ with the grading $\deg p_i = \deg q_i = 1$ for all i and $\deg t = 2$ we can take for the X-vectors, for example,

$$X_{q_i} = \partial_{q_i} + p_i \partial_t, \quad X_{p_i} = \partial_{p_i} - q_i \partial_t; \quad X_t = \partial_t.$$

Hence $\mathfrak{g}_- = \operatorname{Span}\{X_{p_1}, \dots, X_{p_n}, X_{q_1}, \dots, X_{q_n}, X_t\}$ and the contact vector fields in consideration preserve the distribution $\mathcal{D}$ given by the Pfaff equation $\alpha(X) = 0$ for vector fields X, where $\alpha = dt + \sum_i (p_i dq_i - q_i dp_i)$.

The Y-vectors in this case are of the form

$$Y_{q_i} = \partial_{q_i} - p_i \partial_t, \quad Y_{p_i} = \partial_{p_i} + q_i \partial_t; \quad Y_t = \partial_t.$$

In this particular example, a contact vector field K is determined by only one generating function F which is exactly the coefficient of Y_t in the decomposition of K with respect to the Y-basis and there are no restrictions on the function F. Denoting $F = 2f$ and solving eqs. (7.18), we get the formula for any contact vector field K_f:

$$\begin{aligned} K_f = 2fY_t + \textstyle\sum_i (-Y_{q_i}(f) Y_{p_i} + Y_{p_i}(f) Y_{q_i}) = \\ (2-E)(f)\partial_t + \frac{\partial f}{\partial t} E + \textstyle\sum_i \left(\frac{\partial f}{\partial_{p_i}} \partial_{q_i} - \frac{\partial f}{\partial_{q_i}} \partial_{p_i} \right), \end{aligned} \tag{7.35}$$

where $E = \sum_i (p_i \partial_{p_i} + q_i \partial_{q_i})$. Of course, this is exactly the standard formula of the contact vector field with the generating function f. Further we use the realization of $\mathfrak{k}(2n+1)$ in generation functions f.

If $\mathfrak{h}_0 \neq \mathfrak{k}_0$, then $\mathfrak{h}_0$ is the defining component. The component $\mathfrak{k}_0$ is generated by 2nd order homogeneous polynomials in p, q, t. Thus, for a basis Z in $\mathfrak{k}_0$ we can take monomials $p_i p_j$, $q_i q_j$, $p_i q_j$; t and for a basis Z^* of the dual space we then take operators

$$Y_{p_i} Y_{p_j}, \;\; Y_{q_i} Y_{q_j}, \;\; Y_{p_i} Y_{q_j} \text{ for } i \neq j; \;\; \frac{1}{2}(Y_{p_i} Y_{q_i} + Y_{q_i} Y_{p_i}); \quad Y_t.$$

To describe the complete prolongation of $\mathfrak{g}_- \oplus \mathfrak{h}_0$, one should first single out $\mathfrak{h}_0$ in $\mathfrak{k}_0$ in terms of equations for functions generating $\mathfrak{h}_0$ in basis Z, then rewrite the equations in terms of Z^*.

1) If $\mathfrak{h}_0 = \mathfrak{sp}(2n)$, the generating functions do not depend on t, which means that

$$Y_t(f) = 0.$$

This equation singles out the Poisson subalgebra $\mathfrak{po}(2n)$ in $\mathfrak{k}(2n+1)$.

2) If $\mathfrak{h}_0 \simeq \mathbb{C}$ id, the generating function is t, which means that

$$\begin{aligned} &Y_{p_i} Y_{p_j}(f) = 0, \;\; Y_{q_i} Y_{q_j}(f) = 0, \;\; Y_{p_i} Y_{q_j}(f) = 0 \text{ for } i \neq j; \\ &(Y_{p_i} Y_{q_i} + Y_{q_i} Y_{p_i})(f) = 0. \end{aligned} \tag{7.36}$$

For $i \neq j$, these equations imply

$$Y_{q_i} Y_{p_i} Y_{p_j}(f) - Y_{p_i} Y_{q_i} Y_{p_j}(f) = Y_t Y_{p_j}(f) = Y_{p_j}(Y_t(f)) = 0.$$

Analogously, $Y_{q_j}(Y_t(f)) = 0$, and hence $Y_t(Y_t(f)) = 0$, i.e., $Y_t(f) = const$ and $f = ct + f_0$ while eqs. (7.36) imply $\deg_{p,q} f_0 \leq 1$. Hence the prolong of $\mathfrak{g}_- \oplus \mathfrak{h}_0$ coincides with $\mathfrak{g}_- \oplus \mathfrak{h}_0$.

Let $\mathfrak{h}_0 = \mathfrak{k}_0$, and $\mathfrak{h}_1 \subset \mathfrak{k}_1$. As a $\mathfrak{k}_0$-module, $\mathfrak{k}_1$ decomposes into the direct sum of two (over $\mathbb{C}$; for $\operatorname{Char} \mathbb{K} = 3$ and in super setting, even over $\mathbb{C}$, the situation is more involved) irreducible submodules, W_1 spanned by cubic monomials in p and q, and W_2 spanned by tp_i and tq_i. The dual bases of W_1 and W_2 are given by order 3 symmetric polynomials in the Y_{p_i}, Y_{q_i}, and, respectively, spanned by $Y_{p_j} Y_t$ and $Y_{q_j} Y_t$.

Hence the subspace W_1 is singled out by conditions

$$Y_{p_j} Y_t(f) = Y_{q_j} Y_t(f) = 0 \Longrightarrow Y_t(f) = const.$$

In $\mathfrak{k}(2n+1)$, this equation singles out $\mathfrak{der}(\mathfrak{po}(2n)) = \mathfrak{po}(2n) \oplus \mathbb{C}K_t$, the derivation algebra of the Poisson algebra.

To single out W_2, we have the system

$$\begin{aligned} &Y_{p_i} Y_{p_j} Y_{p_k}(f) = 0, \;\; Y_{q_i} Y_{q_j} Y_{q_k}(f) = 0, \\ &Y_{p_i} Y_{p_j} Y_{q_k}(f) = 0, \;\; Y_{p_k} Y_{q_i} Y_{q_j}(f) = 0, \;\; \text{for } k \neq i, j; \\ &Y_{p_i}\left(Y_{p_j} Y_{q_j} + Y_{q_j} Y_{p_j}\right)(f) = 0, \; Y_{q_i}\left(Y_{p_j} Y_{q_j} + Y_{q_j} Y_{p_j}\right)(f) = 0 \text{ for } i \neq j; \\ &\left(Y_{p_i}^2 Y_{q_i} + Y_{p_i} Y_{q_i} Y_{p_i} + Y_{q_i} Y_{p_i}^2\right)(f) = 0, \\ &\left(Y_{q_i}^2 Y_{p_i} + Y_{q_i} Y_{p_i} Y_{q_i} + Y_{p_i} Y_{q_i}^2\right)(f) = 0. \end{aligned} \tag{7.37}$$

which implies that $Y_t(f)$ satisfies eqs. (7.36), and hence

$$\frac{\partial f}{\partial t} \in \operatorname{Span}\{1;\; p_1, \dots, p_n,\; q_1, \dots, q_n,;\; t\},$$

whereas

$$f \in \operatorname{Span}\{1;\; p_1, \dots, p_n,\; q_1, \dots, q_n;\; t,\; tp_1, \dots, tp_n,\; tq_1, \dots, tq_n;\; t^2\}.$$

Hence the complete prolongation $(\mathfrak{k}_- \oplus \mathfrak{k}_0 \oplus W_2)_*$ is isomorphic to $\mathfrak{sp}(2n+2)$.

$\mathfrak{kas} \subset \mathfrak{k}(1|6)$. Let $\mathfrak{n} = \mathfrak{n}_{-1} \oplus \mathfrak{n}_{-2}$ be the Heisenberg Lie superalgebra $\mathfrak{hei}(1|6)$: $\dim \mathfrak{n}_{-1} = 0|6$, $\dim \mathfrak{n}_{-2} = 1|0$. The complete prolong of $\mathfrak{n}$ is $\mathfrak{g} = \mathfrak{k}(1|6)$ with $\mathfrak{g}_0 = \mathfrak{co}(6)$. The component $\mathfrak{g}_1$ consists of three irreducible $\mathfrak{g}_0$-modules.

If we consider $\mathfrak{k}(1|6)$ in realization by generating functions in $t, \theta_1, \dots, \theta_6$, i.e., when

$$K_f = (2-E)(f)\partial_t + \frac{\partial f}{\partial t}E - (-1)^{p(f)}\sum_i \frac{\partial f}{\partial \theta_i}\partial_{\theta_i}, \quad f \in \mathbb{C}[t, \theta_1, \dots, \theta_6],$$

where $E = \sum_i \theta_i \partial_{\theta_i}$ and $\{\theta_i, \theta_j\}_{k.b.} = \delta_{ij}$, then

$$\mathfrak{g}_1 \simeq t\Lambda(\theta) \oplus \Lambda^3(\theta) = t\Lambda(\theta) \oplus \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-$$

with $\mathfrak{g}_1^\pm \subset \Lambda^3(\theta)$ singled out with the help of the Hodge star *:

$$\mathfrak{g}_1^\pm = \{f \in \Lambda^3(\theta) \mid f^* = \pm\sqrt{-1}f\}. \tag{7.38}$$

Recall that the Hodge star * is just the Fourier transformation in odd indeterminates whereas t is considered a parameter:

$$* : f(\xi, t) \mapsto f^*(\eta, t) = \int \exp(\sum \eta_i \xi_i) f(\xi, t)\,\mathrm{vol}(\xi).$$

The exceptional simple Lie superalgebra $\mathfrak{kas}$ is defined as a partial prolong of $\mathfrak{h} = \mathop{\oplus}\limits_{k=-2}^{1} \mathfrak{h}_k$, where $\mathfrak{h}_k = \mathfrak{k}(1|6)_k$ for $-2 \leq k \leq 0$ and where $\mathfrak{h}_1 = t\Lambda(\theta) \oplus \mathfrak{g}_1^+$. Hence $\mathfrak{h}_1$ is the defining component.

Then

$$X_i = \partial_{\theta_i} + \theta_i \partial_t \text{ for } i = 1, \dots 6, \quad X_7 = \partial_t,$$

and

$$Y_i = \partial_{\theta_i} - \theta_i \partial_t \text{ for } i = 1, \dots 6, \quad Y_7 = X_7.$$

Let $I = \{i_1, i_2, i_3\} \subset \{1, \dots, 6\}$ be an ordered subset of indices, and $I^* = \{j_1, j_2, j_3\}$ the dual subset of indices (i.e., $\{I, I^*\}$ is an even permutation of $\{1, \dots, 6\}$). Set:

$$Y_I = Y_{i_1} Y_{i_2} Y_{i_3}, \quad Y_{I^*} = Y_{j_1} Y_{j_2} Y_{j_3},$$

and define $\Delta_{Y_I} : \mathbb{C}[t, \theta] \longrightarrow \mathbb{C}[t, \theta]$ by the eqs.

$$\Delta_{Y_I}(f) = (-1)^{p(f)} Y_I(f).$$

Observe that $\Delta_{Y_I}(t\theta_s) = 0$ for any $s = 1, \dots, 6$. Therefore $\mathfrak{h}_1$ can be singled out in $\mathfrak{k}(1|6)_1$ by the following 10 equations parameterized by partitions (I, I^*) of $(1, \dots, 6)$ constituting even permutations:

$$(\Delta_{Y_I} - \sqrt{-1}\Delta_{Y_I^*})(f) = 0. \tag{7.39}$$

Clearly, (7.39) is equivalent to

$$(Y_I - \sqrt{-1}Y_{I^*})(f) = 0.$$

The solutions of this system span the following subspace of the space of generating functions:

$$\begin{aligned}
&f(t) - \sqrt{-1}f'''(t)1^*,\\
&f_j(t)\theta_j - \sqrt{-1}f_j''(t)\theta_j^*,\\
&f_{jk}(t)\theta_j\theta_k - \sqrt{-1}f_{jk}'(t)(\theta_j\theta_k)^*,\\
&f_{jkl}(t)\left(\theta_j\theta_k\theta_l - \sqrt{-1}(\theta_j\theta_k\theta_l)^*\right).
\end{aligned} \tag{7.40}$$

In (7.40), j, k, l are distinct indices 1 to 6.

Chapter 8

The analogs of Riemann and Penrose tensors on supermanifolds (E. Poletaeva)

This chapter is an edited version of [Po]. It expounds the results that appeared mainly in not very accessible papers listed in [Po] and complements [LPS]. Meanwhile there appeared Grozman's package `SuperLie` that confirmed our — not very easy — calculations; for some of Grozman's independently obtained results (the cases of exceptional algebras), see [LPS].

Our results also make it clear why one can not just "superize" metric in order to get Einstein-Hilbert's equations. Their true superizations — various SUGRAs — correspond to $\mathbb{Z}$-graded Lie superalgebras of depth $d > 1$. Such Lie superalgebras were discussed by Yu. Manin [MaG] but the corresponding structure functions were not calculated yet except in the cases considered by Grozman and Leites in [GL11].

There is, however, a paper ([GL22]) where approach similar to the one described in what follows is applied to $\mathbb{Z}$-graded Lie superalgebras of depth $d = 1$ and the results are interpreted as supergravity equations since the tensor, obtained after deleting everything that depends on odd parameters, is exactly the standard Riemannian tensor.

8.0. Introduction

8.0.1. Structure functions. The main object of the study of Riemannian geometry is the properties of the Riemann tensor, which in turn splits into the Weyl tensor, the traceless Ricci tensor, and the scalar curvature. All these tensors are obstructions to the possibility of "flattening" the manifold on which they are considered. The word "splits" above means that at every point of the Riemannian manifold M^n for $n \neq 4$ the space of values of the Riemann tensor constitutes an $O(n)$-module which splits into the sum of three irreducible components (for $n = 4$ there are four of them, because the Weyl tensor splits additionally in this case) [ALV, Kob].

More generally, let $G \subset \mathrm{GL}(n)$ be any Lie group, not necessarily $O(n)$. A reduction of the principal $\mathrm{GL}(n)$-bundle on M to the principal G-bundle is called a *G-structure* on M.

Recall that on a manifold with a G-structure there is a canonical connection. For a Riemannian manifold this is the *Levi-Civita connection*. The so-called *structure functions* (SFs) constitute the complete set of obstructions to integrability of the canonical connection or, in other words, to the possibility of local flattening of a manifold with G-structure. The Riemann tensor is an example of a SF. Among the most known other examples of SFs are the following ones:

- a *conformal structure*, $G = O(n) \times \mathbb{R}^*$, SFs are called the *Weyl tensor*;
- Penrose's *twistor* theory, $G = S(U(2) \times U(2)) \times \mathbb{C}^*$, SFs-Penrose's tensors — split into two components called the "*α-forms*" and "*β-forms*";
- an almost *complex structure*, $G = GL(n;\mathbb{C}) \subset GL(2n;\mathbb{R})$, SFs are called the *Nijenhuis tensor*;
- an almost *symplectic structure*, $G = \mathrm{Sp}(2n)$, no accepted name for SFs.

8.0.2. Spencer cohomology groups. Recall necessary definitions [St2, Gu].

The simplest G-structure is the *flat* G-structure defined as follows. Let V be $\mathbb{K}^n$ with a fixed frame. Consider the bundle over V whose fiber over $v \in V$ consists of all frames obtained from the fixed one under the G-action, V being identified with T_vV.

Obstructions to identification of the $(k+1)$-st infinitesimal neighborhood of a point $m \in M$ on a manifold M with G-structure and that of a point of the flat manifold V with the above G-structure are called *structure functions of order k*. The identification is performed inductively and is possible provided the obstructions of lesser orders vanish. At each point of a manifold M SFs take values in certain cohomology groups, called *Spencer cohomology groups*. The corresponding complex is defined as follows. Let S^iV denote the i-th symmetric power of a vector space V and $Lie(G)$ denote the Lie algebra of the Lie group G. Set $\mathfrak{g}_{-1} = T_mM, \mathfrak{g}_0 = \mathfrak{g} = Lie(G)$ and for $i > 0$ set:

$$\begin{aligned}\mathfrak{g}_i &= \{X \in \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{i-1}) \mid X(v)(w) = X(w)(v) \text{ for any } v, w \in \mathfrak{g}_{-1}\}\\ &= (\mathfrak{g}_0 \otimes S^i(\mathfrak{g}_{-1})^*) \cap (\mathfrak{g}_{-1} \otimes S^{i+1}(\mathfrak{g}_{-1})^*).\end{aligned}$$

Now set $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = \underset{i\geq -1}{\oplus}\mathfrak{g}_i$. Suppose that the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is faithful.

Then $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) \subset \mathfrak{vect}(n) = \mathfrak{der}\mathbb{K}[[x_1, \dots, x_n]]$, where $n = \dim \mathfrak{g}_{-1}$. It can be verified that the Lie algebra structure on $\mathfrak{vect}(n)$ induces such a structure on $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$. The Lie algebra $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$, usually abbreviated $\mathfrak{g}_*$, will be called the *Cartan prolong* of the pair $(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$.

Let E^iV be the i-th exterior power of a vector space V. Set

$$C^{k,s}_{\mathfrak{g}_0} = \mathfrak{g}_{k-s} \otimes E^s(\mathfrak{g}^*_{-1}).$$

Define the differentials $\partial^{k,s}_{\mathfrak{g}_0} : C^{k,s}_{\mathfrak{g}_0} \longrightarrow C^{k,s+1}_{\mathfrak{g}_0}$ as follows: for any $g_1, \dots, g_{s+1} \in \mathfrak{g}_{-1}$

$$(\partial^{k,s}_{\mathfrak{g}_0} f)(g_1, \dots, g_{s+1}) = \sum_i (-1)^i [f(g_1, \dots, \widehat{g_{s+1-i}}, \dots, g_{s+1}),\ g_{s+1-i}] \quad (8.1)$$

As expected, $\partial^{k,s}_{\mathfrak{g}_0}\partial^{k,s+1}_{\mathfrak{g}_0} = 0$. The cohomology of bidegree $(k,\ s)$ of this complex is called the $(k,\ s)$-th *Spencer cohomology group* $H^{k,s}_{\mathfrak{g}_0}$. It turns out that structure functions of order k on a manifold M with G-structure are sections of certain vector bundles over M with fiber over a point $m \in M$ isomorphic to $H^{k,2}_{\mathfrak{g}}(T_mM)$, where $\mathfrak{g} = Lie(G)$.

8.0.3. Generalized conformal structures. A generalization of the notion of conformal structure is a G-structure of type X, where X is a classical space, i.e., an irreducible compact Hermitian symmetric space (CHSS). These G-structures were introduced and intensively studied by A. Goncharov, who calculated the corresponding structure functions [Go]. In his examples G is the reductive part of the stabilizer of a point of X. The usual conformal structure is the one that corresponds to $X = Q_n$, a quadric in the projective space. The complex grassmannian $X = Gr^4_2$ corresponds to Penrose's *twistors*.

Recall that Penrose's idea is to embed the Minkowski space M^4 into the complex Grassmann manifold Gr^4_2 of planes in $\mathbb{C}^4$ (or straight lines in $\mathbb{CP}^3$) and to express the conformal structure on M^4 in terms of the incidence relation of the straight lines in $\mathbb{CP}^3$ [Pen].

The conformal structure on M^4 is given by a field of quadratic cones in the tangent spaces to the points of M^4. In Penrose's case these cones possess two families of two-dimensional flat generators, the so-called "α-planes" and "β-planes." The geometry of these families is vital for Penrose's considerations. In particular, the Weyl tensor gets a lucid description in terms of these families.

It is interesting to include 4-dimensional Penrose theory into a more general theory of geometric structures. A. Goncharov has shown that there is an analogous field of quadratic cones for any irreducible compact Hermitian symmetric space X of rank greater than one [Go].

Let S be a simple complex Lie group, P its parabolic subgroup with the Levi decomposition $P = GN$, i.e., G is reductive and N is the radical of P. As is known (see [He]), N is Abelian if and only if $X = S/P$ is a CHSS, and in this case $G = G_0 \times \mathbb{C}^*$, where G_0 is semisimple.

Let $P_x = G_xN_x$ be the Levi decomposition of the stabilizer of $x \in X$ in S. Denote by C_x the cone of highest weight vectors in the G_x-module T_xX, i.e., each element in C_x is highest with respect to some Borel subgroup in G_x. Since $s \in S$ transforms C_x to C_{sx}, then with X there is associated the cone $C(X) \subset T_{\bar{e}}X$, where $\bar{e}$ is the image of the unit $e \in S$ in X.

Let $\mathrm{rk}(X) > 1$, i.e., $X \neq \mathbb{CP}^n$. Then on a manifold M said to be a *generalized conformal structure* of type X is given if M is endowed with a family of cones C_m and $\mathbb{C}$-linear isomorphisms $A_m\colon T_{\bar{e}}X \longrightarrow T_mM$ such that $A_m(C(x)) = C_m$.

Goncharov has shown (see [Go]) that a manifold M with generalized conformal structure of type X is a manifold with a $\tilde{G}$-structure, where $\tilde{G}$ is a group of linear automorphisms of the cone $C(X)$ and the connected component of the identity of this group is precisely G.

8.0.4. The case of a simple Lie algebra $\mathfrak{g}_*$ over $\mathbb{C}$. The following remarkable fact, though known to experts, is seldom formulated explicitly [LRC, KN].

8.0.5. Proposition. *Let $\mathbb{K}=\mathbb{C}$, $\mathfrak{g}_*=\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$ be simple. Then only the following cases are possible:*

1) $\mathfrak{g}_2\neq 0$, then $\mathfrak{g}_*$ is either $\mathfrak{vect}(n)$ or its special subalgebra $\mathfrak{svect}(n)$ of divergence-free vector fields, or its subalgebra $\mathfrak{h}(2n)$ of Hamiltonian vector fields.

2) $\mathfrak{g}_2=0$, $\mathfrak{g}_1\neq 0$, then $\mathfrak{g}_*$ is the Lie algebra of the complex Lie group of automorphisms of a CHSS (see sect. 8.0.3).

Let $R(\sum\limits_i a_i\pi_i)$ be the irreducible $\mathfrak{g}_0$-module with the highest weight $\sum\limits_i a_i\pi_i$, where π_i is the i-th fundamental weight.

8.0.6. Theorem (Serre [St2])**.** *In case 1) of Proposition* 8.0.5*, SFs can only be of order* 1*. More precisely: for $\mathfrak{g}_*=\mathfrak{vect}(n)$ and $\mathfrak{svect}(n)$, SFs vanish, for $\mathfrak{g}_*=\mathfrak{h}(2n)$, the nonzero SFs constitute $R(\pi_1)$ for $n=2$, and $R(\pi_1)\oplus R(\pi_3)$ for $n>2$.*

When $\mathfrak{g}_*$ is a simple finite dimensional Lie algebra over $\mathbb{C}$ computation of SFs becomes an easy corollary of the Bott-Borel-Weil (BBW) theorem in a form due to W. Shmid [Shm], cf. [Go]. Indeed, by definition,

$$\bigoplus_k H^{k,2}_{\mathfrak{g}_0}=H^2(\mathfrak{g}_{-1},\ \mathfrak{g}_*).$$

The BBW theorem implies that as a $\mathfrak{g}_0$-module, $H^2(\mathfrak{g}_{-1},\ \mathfrak{g}_*)$ has as many components as $H^2(\mathfrak{g}_{-1})$. Thanks to commutativity of $\mathfrak{g}_{-1}$ one has $H^2(\mathfrak{g}_{-1})=E^2\mathfrak{g}^*_{-1}$, which facilitates the count of components. The BBW theorem also gives the formula for the highest weights of these components.

8.0.7. Reduced structures. Let $X=S/P$, where

$$Lie(S)=\mathfrak{g}_*=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1,\quad Lie(P)=\mathfrak{g}_0\oplus\mathfrak{g}_1,$$

be a CHSS. Let $\hat{\mathfrak{g}_0}$ be the semisimple part of $\mathfrak{g}_0=Lie(G)$. A $\hat{G}$-structure, where $Lie(\hat{G})=\hat{\mathfrak{g}_0}$, will be referred to as a *Riemannian structure* of type X. To reduce the structure group G to its semisimple part $\hat{G}$ is an action similar to distinguishing a metric from a conformal class on a conformal manifold.

The structure functions of the $\hat{G}$-structures form an analogue of the Riemann tensor for the metric. They include the structure functions of the G-structure and several other irreducible components, some of which are analogues of the traceless Ricci tensor or the scalar curvature.

More precisely, the structure functions of the G-structure are defined as the part of the structure functions of the $\hat{G}$-structure obtained by a reduction of the G-structure that does not depend on the choice of reduction. In other words, this is a generalized conformally invariant part of the structure functions of the $\hat{G}$-structure.

In the case of the Riemannian structure we have

$$\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0})=\mathfrak{g}_{-1}\oplus\hat{\mathfrak{g}_0},$$

so there only exist SFs of orders 1 and 2. Though the BBW theorem doesn't work in this case, SFs are describable thanks to the following proposition:

8.0.8. Proposition ([Go])**.** 1) $H^{1,2}_{\hat{\mathfrak{g}_0}}=H^{1,2}_{\mathfrak{g}_0}$;

2) $H^{2,2}_{\hat{\mathfrak{g}_0}}=H^{2,2}_{\mathfrak{g}_0}\oplus S^2(\mathfrak{g}^*_{-1})$.

8.0.9. The Riemannian structure in the classical case of Riemannian geometry. Einstein equations. Let $G=O(n)$. In this case $\mathfrak{g}_1=\mathfrak{g}_{-1}$ and a 1-dimensional subspace is distinguished in $S^2(\mathfrak{g}_{-1})^*$. The sections through this subspace constitute a Riemannian metric g on M. The usual way to determine a metric on M is to define a matrix-valued function, but actually this function with values in symmetric matrices depends only on one functional parameter. The values of the Riemann tensor at a point of M constitute an $O(n)$-module $H^2(\mathfrak{g}_{-1},\ \mathfrak{g}_*)$, which contains a trivial component. Let a section through it be denoted by R. This trivial component is naturally realized as a submodule in a module isomorphic to $S^2(\mathfrak{g}_{-1})^*$.

Thus, there exist two matrix-valued functions: g and R, both preserved by $O(n)$. Now let R correspond to the Levi-Civita connection. The process of restoring R from g involves differentiations and in this way one gets a nonlinear PDE, which constitutes one of the two conditions called *Einstein equations* [Le4, LSV, LPS]:[1] **[Ol: a kak zdes'? Ostavit' tak?]** 1

$$R=\lambda g,\ \text{where } \lambda\in\mathbb{R}. \qquad (EE_0)$$

The other condition is that the other component belonging to $S^2(\mathfrak{g}_{-1})^*$, the traceless Ricci tensor Ric, vanishes:

$$\mathrm{Ric}\ =0. \qquad (EE_{ric})$$

There is a close relation between G-structures and so-called F-structures, which are also of interest, in particular, because of their application to Penrose's geometry. This relation will now be explained.

8.0.10. F-structures and their structure functions. Recall that the notion of F-structure is a generalization of the notion of distribution, i.e., a subbundle in TM and the SFs of an F-structure generalize the notion of the Frobenius form [Go].

Let $V=T_mM$, $F\subset Gr_k(V)$ be a manifold with a transitive action of a subgroup $G_F\subset GL(V)$, $\mathbb{F}(M)$ be a subbundle of $Gr_k(TM)$, where the fiber of

$Gr_k(TM)$ is $Gr_k(T_mM)$. The bundle $\mathbb{F}(M) \longrightarrow M$ is called an *F-structure* on M if, for any point m of M, there is a linear isomorphism $I_m : V \longrightarrow T_mM$, which induces a diffeomorphism $I_m(F) = \mathbb{F}(m)$. A submanifold $Z \subset M$ of dimension k such that $T_zZ \subset \mathbb{F}(z)$ for any $z \in Z$ is called an integral submanifold. An F-structure is integrable if for any $z \in Z$ and for any subspace $V(z) \subset \mathbb{F}(z)$ there is an integral manifold Z with $T_zZ = V(z)$.

SFs of an F-structure are defined as follows. For $f \in F$ let $V_f \subset V$ be the subspace corresponding to f. Set

$$(T_fF)_{-1} = V/V_f, \quad (T_fF)_0 = T_fF.$$

Define

$$(T_fF)_s = ((T_fF)_{s-1} \otimes V_f^*) \cap ((T_fF)_{s-2} \otimes S^2V_f^*)$$

for $s > 0$, and

$$C^{k,s}_{T_fF} = (T_fF)_{k-s} \otimes E^sV_f^*$$

Define the differentials as in (8.1). Then the cohomology groups $H^{k,s}_{T_fF}$ are naturally defined. It turns out that the obstruction to integrability of order k of an F-structure on a manifold M is a section of a certain vector bundle over $\mathbb{F}(M)$ with fiber over a point $\psi \in \mathbb{F}(m)$ isomorphic to $H^{k,2}_{T_{I_m^{-1}(\psi)}F}$. Moreover, there exists a map $H^{k,s}_{\mathfrak{g}_F} \longrightarrow H^{k,s}_{T_fF}$, where $\mathfrak{g}_F = Lie(G_F)$ (see [Go]).

The relation between SFs of a G_F-structure and the obstructions to integrability of an F-structure generalizes a theorem of Penrose, which states that the anti-selfdual part of the Weyl tensor on a 4-dimensional manifold with a conformal structure vanishes if and only if α-surfaces exist, in other words, the metric is α-integrable [AHS, Gi].

More precisely, for a generalized conformal structure of type X, where $X = Gr^{m+n}_m(\mathbb{C})$, there exist two families of m and n-dimensional flat generators–analogues of Penrose's α-planes and β-planes. When neither m nor n is equal to 1, i.e., the grassmannian is not a projective space, SFs decompose into the direct sum of two components, which are analogues of the self-dual and anti-self-dual parts of the Weyl tensor on a 4-dimensional manifold with a conformal structure. The integrability of each of two families of generators is equivalent to the vanishing of the corresponding component of the SFs.

8.0.11. Structure functions on supermanifolds. The necessary background on Lie superalgebras and supermanifolds is gathered in [LSoS].

The classical superspaces (homogeneous compact Hermitian symmetric superspaces), which are the super analogues of CHSS, considered by Goncharov, are listed in [LPS].

The above definitions of SFs are generalized to Lie superalgebras via the Sign Rule. However, in the super case new phenomena appear, which have no analogues in the classical case:

- Cartan prolongations of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ and of $(\Pi\mathfrak{g}_{-1}, \mathfrak{g}_0)$ are essentially different;
- faithfulness of the $\mathfrak{g}_0$-action on $\mathfrak{g}_{-1}$ is violated in natural examples of supergrassmannians of subsuperspaces in an (n, n)-dimensional superspace when the center z of $\mathfrak{g}_0$ acts trivially.
- the formulation of Serre's theorem and the BBW theorem fail to be literally true for Lie superalgebras.

8.0.12. Description of results. In §1 I compute the SFs for the odd analogue of the metric on the supermanifolds and for several related G-structures (see section 8.1.2). In this case $\mathfrak{g}_0 = Lie(G)$ is the periplectic Lie superalgebra, the special periplectic Lie superalgebra, or their central extensions. It turns out that unlike the classical case of Riemannian geometry, the $\mathfrak{g}_0$-module $H^{k,2}_{\mathfrak{g}_0}$ is not completely reducible, and I describe the Jordan-Hölder series for this module. Thus, my computations show that there is no analogue of (EE_0) for the odd metric.

In §2 and §3 I obtain an explicit description of the Spencer cohomology groups $H^{k,2}_{\mathfrak{g}_0}$ for simple finite-dimensional complex classical Lie superalgebras endowed with $\mathbb{Z}$-grading of depth 1: $\mathfrak{g} = \underset{i\geq -1}{\oplus} \mathfrak{g}_i$, where $\mathfrak{g}_0$ is the zero-th part of the grading.

It is known [K1, Se2] that all such $\mathbb{Z}$-gradings are of the form

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

except for the case where $\mathfrak{g}$ is the special periplectic superalgebra considered in §1. Thus, the cohomology groups $H^{k,2}_{\mathfrak{g}_0}$ constitute the space of values of SFs of G-structures corresponding to homogeneous compact Hermitian symmetric superspaces, where G is a reductive complex Lie supergroup of classical type and $\mathfrak{g}_0 = Lie(G)$. The groups $H^{k,2}_{\mathfrak{g}_0}$ correspond to structures of Riemannian type.

An important particular case is $\mathfrak{g} = \mathfrak{sl}(m|n)$, where $m \neq n$, corresponding to general supergrassmannians.

In §2 I consider a $\mathbb{Z}$-grading of $\mathfrak{g}$ for which $\mathfrak{g}_0$ is a reductive Lie algebra. Thus, the $\mathfrak{g}_0$-module $H^{k,2}_{\mathfrak{g}_0}$ is completely reducible, and for $m, n > 2$ decomposes into the direct sum of two irreducible components — super analogues of Penrose's tensors for the usual complex grassmannians (see Theorem 8.2.3.1 [2] **[Ol: 8.2.3.1 or 8.2.5.1?]**).

The case $\mathfrak{g} = \mathfrak{sl}(n|n)$ is also interesting, because I discovered a phenomenon which has no an analogue in the classical case. Indeed, the center z of $\mathfrak{g}_0$ acts trivially on $\mathfrak{g}_{-1}$. If one retains the same definition of the Cartan prolongation, then it has the form of the semidirect sum $S^*(\mathfrak{g}^*_{-1}) \Subset \mathfrak{g}_*(\mathfrak{g}_{-1}, \mathfrak{g}_0/z)$ (the ideal is $S^*(\mathfrak{g}^*_{-1})$) with the natural $\mathbb{Z}$-grading and Lie superalgebra structure, but this Lie superalgebra is not a subsuperalgebra of $\mathfrak{vect}(\dim \mathfrak{g}_{-1})$ anymore (see Theorem 8.2.1.1).

In §3 I describe the Spencer cohomology groups for the other $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{sl}(m|n)$ and $\mathfrak{psl}(n|n)$ (see Theorem 8.3.1.4 and Theorem

8.3.1.5). These theorems show that the superspace of SFs can be not completely reducible, and I get the answer in terms of non-split exact sequences of $\mathfrak{g}_0$-modules.

Finally, in the cases when m or n are equal to 1, I get SFs of the Lie superalgebra of vector fields $\mathfrak{vect}(m|n)$ or of divergence-free vector fields $\mathfrak{svect}(m|n)$ (see Theorem 8.2.3.1 and Theorem 8.3.1.3).

Theorem 8.3.2.3 shows that the SFs for queer grassmannians constitute a module looking exactly the same as that for grassmannians of generic dimensions.

The case $\mathfrak{g} = \mathfrak{osp}(m|2n)$ is similar to Riemannian geometry. The SFs constitute an irreducible $\mathfrak{g}_0$-module, which is an analogue of the Weyl tensor, and the superspace of the SFs for the reduced structure decomposes into the direct sum of three components–the super analogues of the Weyl tensor, the traceless Ricci tensor, and the scalar curvature. I find the highest weights of these components (see Theorem 8.3.3.3).

Finally, I describe the Spencer cohomology groups for exceptional Lie superalgebras $\mathfrak{osp}(4|2;\alpha)$ and $\mathfrak{ab}(3)$ (see Theorem 8.3.4.3 and Theorem 8.3.5.3, respectively).

8.0.13. Algebraic methods. As in the classical case (Lie theory), computation of Spencer cohomology groups reduces to certain problems of representation theory. However, in the super case computations become much more complicated, because of the absence of complete reducibility. I could not directly apply the usual tools for computing (co)homology (spectral sequences and the BBW theorem) to superalgebras and had to retreat a step and apply these tools to the even parts of the considered Lie superalgebras. Then, using certain necessary conditions, I verified whether two modules over a Lie superalgebra that could be glued into an indecomposable module were glued or not.

My method of computing the structure functions is based on the Hochshild-Serre spectral sequence [Fu]. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and M be a $\mathfrak{g}$-module. On the superspace of k-dimensional cochains $C^k = C^k(\mathfrak{g}, M)$ define a filtration:

$$F^0C^k = C^k \supset F^1C^k \supset F^2C^k \supset \dots F^jC^k \supset \dots \supset F^{k+1}C^k = 0,$$

where for $0 \leq j \leq k+1$, we have

$$F^jC^k = \{c \in C^k \mid c(g_1, \dots, g_i, \dots, g_k) = 0 \text{ if } k-j+1 \text{ arguments lie in } \mathfrak{g}_0\}.$$

Using this filtration define the usual corresponding spectral sequence $E_r^{p,q}$ [GM]. Thus,

$$H^2(\mathfrak{g}, M) = \underset{p+q=2}{\oplus} E_\infty^{p,q} = E_3^{2,0} \oplus E_3^{1,1} \oplus E_4^{0,2}.$$

In particular, $E_1^{p,q} = H^q(\mathfrak{g}_0,\ M \otimes S^p\mathfrak{g}_1^*)$ [Fu]. Since, in the case of Spencer cohomology, $\mathfrak{g} = \mathfrak{g}_{-1} = (\mathfrak{g}_{-1})_0 \oplus (\mathfrak{g}_{-1})_1$ is a commutative Lie superalgebra, we have

$$E_1^{p,q} = H^q((\mathfrak{g}_{-1})_0,\ \mathfrak{g}_* \otimes S^p(\mathfrak{g}_{-1})_1^*) = H^q((\mathfrak{g}_{-1})_0,\ \mathfrak{g}_*) \otimes S^p(\mathfrak{g}_{-1})_1^*.$$

Then in special cases I use the BBW theorem to compute $H^q((\mathfrak{g}_{-1})_0,\ \mathfrak{g}_*)$ as a module over $(\mathfrak{g}_0)_0$.

Open problems

It is interesting to compute the Spencer cohomology groups $H^{k,s}_{\mathfrak{g}_0}$ for simple finite-dimensional Lie superalgebras of vector fields with nonstandard $\mathbb{Z}$-gradings of depth 1. For example, for $\mathfrak{g} = \mathfrak{vect}(0|n)$ and $\mathfrak{h}'(0|n)$.

It is much more difficult to compute the Spencer cohomology groups for vectorial Lie superalgebras than for "matrix" ones, because in this case the number of the irreducible quotient modules in the Jordan-Hölder series does depend on n. Moreover, even for small n computations seem to be very complicated, because of the absence of complete reducibility with respect to $\mathfrak{g}_0$, cf. [LPS].

8.1. The analogues of the Riemannian tensors for the odd metric on supermanifolds

8.1.1. Periplectic superalgebras and their Cartan prolongations. Let $z = 1_{2n}$ be the unit matrix and $\tau = \mathrm{diag}(1_n, \ -1_n)$.

Let P be a nondegenerate supersymmetric odd bilinear form on a superspace V. Clearly, $\dim V = (n, \ n)$. Define the odd analogue of the symplectic Lie algebra, the periplectic Lie superalgebra $\mathfrak{pe}(n)$, and its special subsuperalgebra $\mathfrak{spe}(n)$, setting

$$\begin{aligned}
\mathfrak{pe}(n) &= \{X \in \mathfrak{gl}(n|n) \mid X^{st}P + (-1)^{p(X)}PX = 0\},\\
\mathfrak{spe}(n) &= \mathfrak{pe}(n) \cap \mathfrak{sl}(n|n).
\end{aligned}$$

Thus,

$$\mathfrak{pe}(n) = \mathfrak{spe}(n) \Subset \langle \tau \rangle.$$

Denote by $\varepsilon_1, \dots, \varepsilon_n$ the standard basis of the space dual to the space of diagonal matrices in $\mathfrak{gl}(n) \subset \mathfrak{pe}(n)$. Denote by V_λ the irreducible $\mathfrak{gl}(n)$-module with highest weight λ and highest vector v_λ and by X_λ the irreducible $\mathfrak{pe}(n)$-module with highest weight λ and an even highest vector.

Let $V = V_0 \oplus V_1$ be the standard (identity) $\mathfrak{pe}(n)$-module, $e_1, \dots, e_n$ be a basis of V_0, and $f_1, \dots, f_n$ be a basis of V_1 with respect to which the form P on V takes the form $P = \mathrm{antidiag}(1_n, 1_n)$. With respect to this basis the elements $X \in \mathfrak{pe}(n)$ are represented by matrices of the standard format $(n, \ n)$:

$$X = \mathrm{diag}(A, -A^t) + \mathrm{antidiag}(B, C), \text{ where } A \in \mathfrak{gl}(n), \ B^t = B, \ C^t = -C.$$

In what follows we will often use a natural abbreviation: e.g., $B_{1,n}$ stands for the matrix X whose components A and C are zero and all the entries of B are also zero except for $(1, \ n)$-th and $(n, \ 1)$-st.

Denote by $\tilde{e}_1, \dots, \tilde{e}_n$ and $\tilde{f}_1, \dots, \tilde{f}_n$ the basis of V^* dual to the above basis of V, i.e., $\tilde{f}_i(e_j) = \tilde{e}_i(f_j) = \delta_{ij}$. Since the form P preserved by $\mathfrak{pe}(n)$ is odd, then V^* and $\Pi(V)$ are isomorphic as $\mathfrak{pe}(n)$-modules. Notice that as $\mathfrak{pe}(n)$-modules,

$$\mathfrak{pe}(n) \cong \Pi(E^2V^*).$$

8.1.1.1. Lemma. *a) There exists a $\mathbb{Z}$-grading of the Lie superalgebra $\mathfrak{pe}(n+1)$ of the form*

$$\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where

$$\begin{aligned}
\mathfrak{g}_{-1} &= V,\\
\mathfrak{g}_0 &= \mathfrak{cpe}(n),\\
\mathfrak{g}_1 &= V^* = \Pi(V),\\
\mathfrak{g}_2 &= \Pi(\langle 1 \rangle).
\end{aligned}$$

b) There exists a $\mathbb{Z}$-grading of the Lie superalgebra $\mathfrak{spe}(n+1)$ of the form

$$\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where

$$\begin{aligned}
\mathfrak{g}_{-1} &= V,\\
\mathfrak{g}_0 &= \mathfrak{spe}(n) \Subset \langle \tau + nz \rangle,\\
\mathfrak{g}_1 &= V^* = \Pi(V),\\
\mathfrak{g}_2 &= \Pi(\langle 1 \rangle).
\end{aligned}$$

Proof. Let $W = W_0 \oplus W_1$ be the standard (identity) $\mathfrak{pe}(n+1)$-module, $e_1, \dots, e_{n+1}$ be a basis of W_0, and $f_1, \dots, f_{n+1}$ be a basis of W_1 with respect to which the form P on W takes the form $P = \mathrm{antidiag}(1_{n+1}, \ 1_{n+1})$. Denote by $\tilde{e}_1, \dots, \tilde{e}_{n+1}$ and $\tilde{f}_1 \dots, \tilde{f}_{n+1}$ the basis of W^* dual to the above basis of W, e.g., $\tilde{f}_i(e_j) = \tilde{e}_i(f_j) = \delta_{ij}$.

Note that

$$\mathfrak{pe}(n+1) = \Pi(E^2W^*) = \Pi(E^2W_1 \oplus W_0 \wedge W_1 \oplus S^2W_0).$$

Thus,

$$\mathfrak{pe}(n+1) = \langle e_i\tilde{e}_j, e_i \wedge \tilde{f}_j, f_i \wedge \tilde{f}_j \rangle (1 \leq i, \ j \leq n+1),$$

where

$$\begin{aligned}
e_i\tilde{e}_j &= \frac{1}{2}(e_i \otimes \tilde{e}_j + e_j \otimes \tilde{e}_i),\\
e_i \wedge \tilde{f}_j &= \frac{1}{2}(e_i \otimes \tilde{f}_j - f_j \otimes \tilde{e}_i),\\
f_i \wedge \tilde{f}_j &= \frac{1}{2}(f_i \otimes \tilde{f}_j - f_j \otimes \tilde{f}_i).
\end{aligned} \tag{8.2}$$

Note that the commutator in $\mathfrak{pe}(n+1)$ is defined as follows:

$$[w_i \wedge \tilde{w}_j, \ w_s \wedge \tilde{w}_t] = \frac{1}{2}(\tilde{w}_j(w_s)(w_i \wedge \tilde{w}_t) - (-1)^{p(w_s)}\tilde{w}_t(w_i)(w_s \wedge \tilde{w}_j)), \tag{8.3}$$

where $w_l \in \{e_1, \dots, e_{n+1}; f_1, \dots, f_{n+1}\}$ for $1 \leq l \leq n+1$.

Let $V = V_0 \oplus V_1 = \langle e_1, \dots, e_n; f_1, \dots, f_n\rangle$ be $(n,\ n)$-dimensional subsuperspace in W. Then

$$\begin{aligned}
&\Pi(E^2W_1) = \Pi(E^2V_1) \oplus V_1 \wedge \langle \tilde{f}_{n+1}\rangle,\\
&\Pi(W_0 \wedge W_1) = \Pi(V_0 \wedge V_1) \oplus V_0 \wedge \langle \tilde{f}_{n+1}\rangle \oplus V_1 \wedge \langle \tilde{e}_{n+1}\rangle \oplus \langle e_{n+1} \wedge \tilde{f}_{n+1}\rangle,\\
&\Pi(S^2W_0) = \Pi(S^2V_0) \oplus V_0 \wedge \langle \tilde{e}_{n+1}\rangle \oplus \langle e_{n+1}\tilde{e}_{n+1}\rangle.
\end{aligned}$$

Set

$$\begin{aligned}
&\mathfrak{g}_{-1} = V_0 \wedge \langle \tilde{f}_{n+1}\rangle \oplus V_1 \wedge \langle \tilde{f}_{n+1}\rangle,\\
&\mathfrak{g}_0 = \Pi(E^2V_1 \oplus V_0 \wedge V_1 \oplus S^2V_0) \oplus \langle e_{n+1} \wedge \tilde{f}_{n+1}\rangle,\\
&\mathfrak{g}_1 = V_0 \wedge \langle \tilde{e}_{n+1}\rangle \oplus V_1 \wedge \langle \tilde{e}_{n+1}\rangle,\\
&\mathfrak{g}_2 = \langle e_{n+1}\tilde{e}_{n+1}\rangle.
\end{aligned} \tag{8.4}$$

According to (8.3), formulas (8.4) indeed define a $\mathbb{Z}$-grading of $\mathfrak{pe}(n+1)$, described in Lemma 8.1.1.1.

In order to define a $\mathbb{Z}$-grading of $\mathfrak{spe}(n+1)$ we set

$$\mathfrak{g}_0 = \Pi(E^2V_1 \oplus S^2V_0) \oplus \Big\langle \sum_{i,j=1,\dots,n} a_{ij}e_i \wedge \tilde{f}_j \mid \sum_{i=1,\dots,n} a_{ii} = 0\Big\rangle \oplus \Big\langle \Big(\sum_{i=1,\dots,n} e_i \wedge \tilde{f}_i\Big) - ne_{n+1} \wedge \tilde{f}_{n+1}\Big\rangle.$$

Note that by (8.3) we have $2[e_{n+1} \wedge \tilde{f}_{n+1},\ \mathfrak{g}_i] = i\mathfrak{g}_i$ for $-1 \le i \le 2$. Hence, $-2e_{n+1} \wedge \tilde{f}_{n+1} = z$. Since by (8.2) $2\sum\limits_{i=1,\dots,n} e_i \wedge \tilde{f}_i = \tau$, then

$$2\Big(\sum_{i=1,\dots,n} e_i \wedge \tilde{f}_i - ne_{n+1} \wedge \tilde{f}_{n+1}\Big) = \tau + nz.$$

Thus, $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle \tau + nz\rangle$. This proves Lemma 8.1.1.1. □

8.1.1.2. Theorem. *Let $\mathfrak{g}_{-1} = V$. Then*

a) If $\mathfrak{g}_0 = \mathfrak{spe}(n)$, $\mathfrak{pe}(n)$, $\mathfrak{cspe}(n)$ or $\mathfrak{spe}(n) \Subset \langle a\tau + bz\rangle$, where $a, b \in \mathbb{C}$ are such that $a, b \ne 0$ and $b/a \ne n$, then $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$.

b) If $\mathfrak{g}_0 = \mathfrak{cpe}(n)$ or $\mathfrak{spe}(n) \Subset \langle \tau + nz\rangle$, then $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$ is either $\mathfrak{pe}(n+1)$ or $\mathfrak{spe}(n+1)$, respectively, in the $\mathbb{Z}$-grading described in Lemma 8.1.1.1.

Proof. Let us consider the case where $\mathfrak{g}_0 = \mathfrak{cpe}(n)$. By Lemma 8.1.1.1 we have

$$\mathfrak{pe}(n+1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where

$$\begin{aligned}
&\mathfrak{g}_{-1} = V,\\
&\mathfrak{g}_0 = \mathfrak{cpe}(n),\\
&\mathfrak{g}_1 = V^* = \Pi(V),\\
&\mathfrak{g}_2 = \Pi(\langle 1\rangle).
\end{aligned}$$

Therefore,

$$\mathfrak{pe}(n+1) \subset \mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0). \tag{8.5}$$

In fact, since $\mathfrak{spe}(n+1)$ is a simple Lie superalgebra, then it is transitive, (i.e., if there exists $g \in \mathfrak{g}_i (i \ge 0)$ such that $[\mathfrak{g}_{-1},\ g] = 0$, then $g = 0$). It follows that $\mathfrak{g}_i \subset \mathfrak{g}_{i-1} \otimes \mathfrak{g}^*_{-1}$. The Jacobi identity implies $\mathfrak{g}_i \subset \mathfrak{g}_{i-2} \otimes S^2\mathfrak{g}^*_{-1}$. □

Let us find $\mathfrak{g}_1$.

8.1.1.3. Lemma. *As a $\mathfrak{gl}(n)$-module, $\mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1}$ is the direct sum of irreducible $\mathfrak{gl}(n)$-submodules whose highest weights and highest vectors are listed in Table 1.*

Convention. Let v, w be elements of a vector space. Set

$$vw = \frac{1}{2}(v \otimes w + (-1)^{p(v)p(w)} w \otimes v),\ \ v \wedge w = \frac{1}{2}(v \otimes w - (-1)^{p(v)p(w)} w \otimes v).$$

Proof of Lemma 8.1.1.3 consists of:

a) a verification of the fact that vectors v from Table 1 are indeed highest with respect to $\mathfrak{gl}(n)$, i.e., $A_{i,j}v = 0$ for $i < j$,

b) a calculation of dimensions of the corresponding irreducible submodules by the formula from Appendix. □

Let us show with the help of Table 1 that if $\lambda \ne \varepsilon_1,\ -\varepsilon_n$, then $v_\lambda \notin \mathfrak{g}_{-1} \otimes S^2\mathfrak{g}^*_{-1}$. Indeed, if $\lambda = -\varepsilon_{n-1} - 2\varepsilon_n$, then

$$v_\lambda(e_n)(e_{n-1}) = -\frac{1}{2}f_n, v_\lambda(e_{n-1})(e_n) = 0;$$

if $\lambda = -\varepsilon_{n-2} - \varepsilon_{n-1} - \varepsilon_n$, then

$$v_\lambda(e_n)(e_{n-1}) = \frac{1}{2}f_{n-2}, v_\lambda(e_{n-1})(e_n) = -\frac{1}{2}f_{n-2};$$

if $\lambda = \varepsilon_1 - 2\varepsilon_n$, then

$$v_\lambda(e_n)(f_1) = \frac{1}{2}f_n, v_\lambda(f_1)(e_n) = 0;$$

if $\lambda = \varepsilon_1 + \varepsilon_2 - \varepsilon_n$, then

$$v_\lambda(f_2)(e_n) = -\frac{1}{2}e_1, v_\lambda(e_n)(f_2) = 0;$$

if $\lambda = 3\varepsilon_1$, then

$$v_\lambda(f_1)(f_1) = e_1 \ne 0;$$

if $\lambda = 2\varepsilon_1 + \varepsilon_2$, then

$$v_\lambda(f_1)(f_1) = \frac{1}{2}e_2 \ne 0.$$

Let $\lambda = \varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$. According to Table 1, $\mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1}$ contains two highest vectors of weight λ. Let

$v_\lambda = k_1 f_{n-1} \wedge \tilde{f}_n \otimes \tilde{e}_1 + k_2(f_{n-1} \wedge \tilde{e}_1 \otimes \tilde{f}_n - f_n \wedge \tilde{e}_1 \otimes \tilde{f}_{n-1}),$ where $k_1,\ k_2 \in \mathbb{C}$,

be a linear combination of these vectors. The condition

$$v_\lambda(e_n)(e_{n-1}) = v_\lambda(e_{n-1})(e_n)$$

implies $k_2 = 0$. Then the condition

$$v_\lambda(e_n)(f_1) = v_\lambda(f_1)(e_n)$$

implies $k_1 = 0$.

Let $\lambda = 2\varepsilon_1 - \varepsilon_n$. Let

$$v_\lambda = k_1 f_n \wedge \tilde{e}_1 \otimes \tilde{e}_1 + k_2 e_1 \tilde{e}_1 \otimes \tilde{f}_n, \text{ where } k_1,\ k_2 \in \mathbb{C},$$

be a linear combination of the highest vectors of weight λ which belong to $\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*$. The condition $v_\lambda(f_1)(f_1) = 0$ implies $k_1 = 0$. Then the condition $v_\lambda(e_n)(f_1) = v_\lambda(f_1)(e_n)$ implies $k_2 = 0$. Therefore, if $\lambda \neq \varepsilon_1,\ -\varepsilon_n$, then $v_\lambda \notin \mathfrak{g}_{-1} \otimes S^2\mathfrak{g}_{-1}^*$, hence $v_\lambda \notin \mathfrak{g}_1$.

Let $\lambda = \varepsilon_1$. According to Table 1, $\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*$ has four highest vectors of weight λ. Let

$$v_\lambda = k_1 \sum_{i=1}^n f_i \wedge \tilde{e}_i \otimes \tilde{e}_1 + k_2 \sum_{i=1}^n f_i \wedge \tilde{e}_1 \otimes \tilde{e}_i + k_3 \sum_{i=1}^n e_1 \tilde{e}_i \otimes \tilde{f}_i + k_4 \sum_{i=1}^n e_i \tilde{f}_i \otimes \tilde{e}_1,$$

where $k_1,\ k_2,\ k_3,\ k_4 \in \mathbb{C}$, be their linear combination. Note that $v_\lambda \in \mathfrak{g}_{-1} \otimes S^2\mathfrak{g}_{-1}^*$ if and only if the following conditions are satisfied:

$$\begin{aligned}
&v_\lambda(f_1)(f_1) = 0,\\
&v_\lambda(f_1)(f_i) = -v_\lambda(f_i)(f_1) \text{ for } i \neq 1,\\
&v_\lambda(f_1)(e_1) = v_\lambda(e_1)(f_1),\\
&v_\lambda(f_1)(e_i) = v_\lambda(e_i)(f_1) \text{ for } i \neq 1,\\
&v_\lambda(f_i)(e_i) = v_\lambda(e_i)(f_i) \text{ for } i \neq 1,
\end{aligned}$$

which determine, respectively, the following system of linear equations:

$$\begin{aligned}
&k_1 + k_2 + k_4 = 0,\\
&k_1 + k_4 = -k_2,\\
&\frac{1}{2}(-k_1 - k_2 + k_4) = k_3,\\
&-k_1 + k_4 = k_3,\\
&-k_2 = k_3.
\end{aligned}$$

The solution of this system is

$$k_1 = 0,\ k_3 = -k_2 = k_4.$$

Therefore,

$$v_{\varepsilon_1} = -\sum_{i=1}^n f_i \wedge \tilde{e}_1 \otimes \tilde{e}_i + \sum_{i=1}^n e_1 \tilde{e}_i \otimes \tilde{f}_i + \sum_{i=1}^n e_i \tilde{f}_i \otimes \tilde{e}_1 \in \mathfrak{g}_1. \tag{8.6}$$

Since $\mathfrak{g}_1$ is a $\mathfrak{pe}(n)$-module and v_{ε_1} is an odd vector, we have $\mathfrak{g}_1 = V^*$.

Let us find $\mathfrak{g}_2$.

8.1.1.4. Lemma. *There exist the following nonsplit sequences of $\mathfrak{pe}(n)$-modules:*

$$0 \longrightarrow X_{2\varepsilon_1} \longrightarrow E^2V^* \longrightarrow \Pi(\langle 1 \rangle) \longrightarrow 0, \tag{8.7}$$
$$0 \longrightarrow \Pi(\langle 1 \rangle) \longrightarrow S^2V^* \longrightarrow X_{\varepsilon_1+\varepsilon_2} \longrightarrow 0. \tag{8.8}$$

Proof. First of all recall that $\Pi(E^2V^*)$ and $\mathfrak{pe}(n)$ itself are isomorphic $\mathfrak{pe}(n)$-modules and there exists the following nonsplit sequence of $\mathfrak{pe}(n)$-modules

$$0 \longrightarrow \mathfrak{spe}(n) \longrightarrow \mathfrak{pe}(n) \longrightarrow \langle \tau \rangle \longrightarrow 0. \tag{8.9}$$

Note that as a $\mathfrak{gl}(n)$-module, E^2V^* is isomorphic to

$$E^2V_0^* \oplus V_0^* \wedge V_0 \oplus S^2V_0, \text{ where } S^2V_0 = V_{2\varepsilon_1}, v_{2\varepsilon_1} = \tilde{e}_1^2. \tag{8.10}$$

Since $B_{i,j}v_{2\varepsilon_1} = 0$, then $v_{2\varepsilon_1}$ is a $\mathfrak{pe}(n)$-highest vector. Since $\mathfrak{spe}(n)$ is simple, we get eq. (8.7) after the change of parity.

Let us prove eq. (8.8). Notice that, as $\mathfrak{gl}(n)$-modules,

$$S^2V^* = S^2V_0^* \oplus V_0^* \cdot V_0 \oplus E^2V_0, \text{ where}$$

$$\begin{aligned}
&S^2V_0^* = V_{-2\varepsilon_n}, v_{-2\varepsilon_n} = \tilde{f}_n^2,\\
&V_0^* \cdot V_0 = V_{\varepsilon_1-\varepsilon_n} \oplus \langle v_0 \rangle,\ v_{\varepsilon_1-\varepsilon_n} = \tilde{f}_n \tilde{e}_1,\ v_0 = \sum_{i=1}^n \tilde{f}_i \tilde{e}_i,\\
&E^2V_0 = V_{\varepsilon_1+\varepsilon_2},\ v_{\varepsilon_1+\varepsilon_2} = \tilde{e}_1 \wedge \tilde{e}_2.
\end{aligned} \tag{8.11}$$

Note that $\langle v_0 \rangle$ is the trivial 1-dimensional $\mathfrak{pe}(n)$-module. Indeed,

$$B_{i,j}(v_0) = -\tilde{e}_j \wedge \tilde{e}_i + -\tilde{e}_i \wedge \tilde{e}_j = 0,\ C_{i,j}(v_0) = \tilde{f}_i \tilde{f}_j + \tilde{f}_j(-\tilde{f}_i) = 0.$$

Since $B_{i,j}v_{\varepsilon_1+\varepsilon_2} = 0$, then $v_{\varepsilon_1+\varepsilon_2}$ is a $\mathfrak{pe}(n)$-highest vector.

Let us prove that as $\mathfrak{gl}(n)$-modules,

$$X_{\varepsilon_1+\varepsilon_2} \cong V_{\varepsilon_1+\varepsilon_2} \oplus V_{\varepsilon_1-\varepsilon_n} \oplus V_{-2\varepsilon_n}.$$

Indeed,

$$\begin{aligned}
&C_{2,n}(v_{\varepsilon_1+\varepsilon_2}) = v_{\varepsilon_1-\varepsilon_n},\ -B_{2,n}(v_{\varepsilon_1-\varepsilon_n}) = v_{\varepsilon_1+\varepsilon_2},\\
&C_{1,n}(v_{\varepsilon_1-\varepsilon_n}) = v_{-2\varepsilon_n},\ B_{1,n}(v_{-2\varepsilon_n}) = v_{\varepsilon_1-\varepsilon_n}.
\end{aligned}$$

Finally, we get

$$v_0 = \frac{1}{2}\sum_{i=1}^{n-1} B_{i,n}A_{n,i} + nB_{n,n}v_{-2\varepsilon_n}.$$

This proves (8.8) and Lemma 8.1.1.4. □

Let us prove that if v_λ from $\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}^*$ is a $\mathfrak{pe}(n)$-highest vector of weight either $\lambda = 2\varepsilon_1$ or $\varepsilon_1 + \varepsilon_2$, then $v_\lambda \notin \mathfrak{g}_0 \otimes S^2\mathfrak{g}_{-1}^*$. In fact, if $\lambda = 2\varepsilon_1$, then by (8.6) and (8.10) we have

$$v_\lambda = \Big(-\sum_{i=1}^n f_i \wedge \tilde{e}_1 \otimes \tilde{e}_i + \sum_{i=1}^n e_1\tilde{e}_i \otimes \tilde{f}_i + \sum_{i=1}^n e_i\tilde{f}_i \otimes \tilde{e}_1\Big) \otimes \tilde{e}_1.$$

Then

$$v_\lambda(f_1)(f_1) = -f_1 \wedge \tilde{e}_1 + \sum_{i=1}^n e_i\tilde{f}_i \neq 0.$$

Therefore, $v_\lambda \notin \mathfrak{g}_0 \otimes S^2\mathfrak{g}_{-1}^*$.

If $\lambda = \varepsilon_1 + \varepsilon_2$, then by (8.6) and (8.11)

$$v_\lambda = \Big(-\sum_{i=1}^n f_i \wedge \tilde{e}_1 \otimes \tilde{e}_i + \sum_{i=1}^n e_1\tilde{e}_i \otimes \tilde{f}_i + \sum_{i=1}^n e_i\tilde{f}_i \otimes \tilde{e}_1\Big) \otimes \tilde{e}_2$$
$$- \Big(-\sum_{i=1}^n f_i \wedge \tilde{e}_2 \otimes \tilde{e}_i + \sum_{i=1}^n e_2\tilde{e}_i \otimes \tilde{f}_i + \sum_{i=1}^n e_i\tilde{f}_i \otimes \tilde{e}_2\Big) \otimes \tilde{e}_1.$$

Thus,

$$v_\lambda(f_2)(e_i) = e_1\tilde{e}_i \neq 0 \text{ and } v_\lambda(e_i)(f_2) = 0.$$

Hence, $v_\lambda \notin \mathfrak{g}_0 \otimes S^2\mathfrak{g}_{-1}^*$.

Let $\lambda = 0$. According to Lemma 8.1.1.4, the Jordan-Hölder series of the $\mathfrak{pe}(n)$-module $\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}^*$ contains two $\mathfrak{pe}(n)$-modules with highest weight 0. By Lemma 8.1.1.4 the sequence (8.7) is nonsplit and we have already proved that $\mathfrak{g}_2$ has no irreducible $\mathfrak{pe}(n)$-module with highest weight $2\varepsilon_1$. Therefore, either $\mathfrak{g}_2$ consists of one trivial $\mathfrak{pe}(n)$-module or $\mathfrak{g}_2 = 0$. But by (8.5)

$$\mathfrak{pe}(n+1) \subset \mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0).$$

Hence, $\mathfrak{g}_2 = \Pi(\langle 1\rangle)$.

Finally, let us show that $\mathfrak{g}_3 = 0$. By definition

$$\mathfrak{g}_3 = (\mathfrak{g}_2 \otimes \mathfrak{g}_{-1}^*) \cap (\mathfrak{g}_1 \otimes S^2\mathfrak{g}_{-1}^*).$$

Note that

$$\mathfrak{g}_2 \otimes \mathfrak{g}_{-1}^* = \Pi(\langle 1\rangle) \otimes V^* \cong V,$$

as $\mathfrak{pe}(n)$-modules. By (8.4) the $\mathfrak{pe}(n)$-highest vector in $\mathfrak{g}_2 \otimes \mathfrak{g}_{-1}^*$ is $v = e_{n+1}\tilde{e}_{n+1} \otimes \tilde{e}_1$.

By the explicit formula (8.3) of multiplication in $\mathfrak{pe}(n+1)$ we have

$$v(f_1)(e_1) = [e_{n+1}\tilde{e}_{n+1}, e_1 \wedge \tilde{f}_{n+1}] = e_1\tilde{e}_{n+1} \in \mathfrak{g}_1.$$

On the other hand, $v(e_1)(f_1) = 0$. Therefore, $v \notin \mathfrak{g}_1 \otimes S^2\mathfrak{g}_{-1}^*$. Hence, $\mathfrak{g}_3 = 0$.

Thus, Theorem 8.1.1.2 is proved for $\mathfrak{g}_0 = \mathfrak{cpe}(n)$. This result and part b) of Lemma 8.1.1.1 imply the statement of Theorem 8.1.1.2 for $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle \tau + nz\rangle$.

Let us prove that $\mathfrak{g}_1 = 0$ for $\mathfrak{g}_0 = \mathfrak{spe}(n)$, $\mathfrak{pe}(n)$, $\mathfrak{cspe}(n)$, or $\mathfrak{spe}(n) \Subset \langle a\tau + bz\rangle$, where $a,\ b \in \mathbb{C}$, $a,\ b \neq 0$, and $b/a \neq n$.

Indeed, as has been shown, $\mathfrak{g}_1 = \Pi(V)$ for $\mathfrak{g}_0 = \mathfrak{cpe}(n)$ or $\mathfrak{spe}(n) \Subset \langle \tau + nz\rangle$, and by (8.6) the corresponding $\mathfrak{spe}(n)$-highest vector is

$$v_{\varepsilon_1} = -\sum_{i=1}^n f_i \wedge \tilde{e}_1 \otimes \tilde{e}_i + \sum_{i=1}^n e_1\tilde{e}_i \otimes \tilde{f}_i + \sum_{i=1}^n e_i\tilde{f}_i \otimes \tilde{e}_1.$$

Then

$$v_{\varepsilon_1}(f_1) = -f_1 \wedge \tilde{e}_1 + \sum_{i=1}^n e_i\tilde{f}_i \in \mathfrak{spe}(n) \Subset \langle \tau + nz\rangle.$$

Note that

$$-f_1 \wedge \tilde{e}_1 + \sum_{i=1}^n e_i\tilde{f}_i \notin \mathfrak{spe}(n).$$

Hence $v_{\varepsilon_1}(f_1) \notin \mathfrak{g}_0$ for $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle a\tau + bz\rangle$, where $a,\ b \in \mathbb{C}$, $a \neq 0$ and $b/a \neq n$, or $a = 0$. Therefore,

$$\mathfrak{g}_1 = 0 \text{ for } \mathfrak{g}_0 = \mathfrak{spe}(n),\ \mathfrak{pe}(n),\ \mathfrak{cspe}(n),\ \text{or } \mathfrak{spe}(n) \Subset \langle a\tau + bz\rangle,$$

where $a,\ b \in \mathbb{C}$, $a,\ b \neq 0$ and $b/a \neq n$. Thus, in these cases

$$\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0.$$

8.1.2. The main theorem. The following theorem describes SFs for the odd analogues of Riemannian metric and various conformal versions.

8.1.2.1. Theorem. *For the G-structures with the following Lie $(G) = \mathfrak{g}_0$ the nonzero SFs are of orders not exceeding 2 and as follows:*

order 1: *if $\mathfrak{g}_0 = \mathfrak{spe}(n)$ or $\mathfrak{spe}(n) \Subset \langle \tau + nz\rangle$ we have V^*;*

order 2: *if $\mathfrak{g}_0 = \mathfrak{spe}(n)$, where $n > 3$, we have the following non-split exact sequence of $\mathfrak{spe}(n)$-modules:*

$$0 \longrightarrow X_{\varepsilon_1+\varepsilon_2} \longrightarrow H^{2,2}_{\mathfrak{g}_0} \longrightarrow \Pi(X_{2\varepsilon_1+2\varepsilon_2}) \longrightarrow 0;$$

if $\mathfrak{g}_0 = \mathfrak{spe}(3)$, then another space is added to the SFs: we have the following nonsplit exact sequence of $\mathfrak{spe}(3)$-modules:

$$0 \longrightarrow X \longrightarrow H^{2,2}_{\mathfrak{g}_0} \longrightarrow \Pi(X_{3\varepsilon_1}) \longrightarrow 0,$$

where X is determined from the following non-split exact sequence of $\mathfrak{spe}(3)$-modules:

$$0 \longrightarrow X_{\varepsilon_1+\varepsilon_2} \longrightarrow X \longrightarrow \Pi(X_{2\varepsilon_1+2\varepsilon_2}) \longrightarrow 0;$$

if $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle a\tau + bz\rangle$, where $a,\ b \in \mathbb{C}$ are such that $a = 0$, $b \neq 0$ or $a \neq 0$, $b/a \neq n$, then for $n > 2$, we have the following non-split exact sequence of $\mathfrak{spe}(n)$-modules:

$$0 \longrightarrow H^{2,2}_{\mathfrak{spe}(n)} \longrightarrow H^{2,2}_{\mathfrak{g}_0} \longrightarrow X_{2\varepsilon_1} \longrightarrow 0;$$

if $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle\tau + nz\rangle$, *then for* $n > 3$, $H^{2,2}_{\mathfrak{g}_0} = \Pi(X_{2\varepsilon_1+2\varepsilon_2})$ *is an irreducible* $\mathfrak{spe}(n)$*-module, whereas for* $n = 3$, *we have the following non-split exact sequence of* $\mathfrak{spe}(3)$*-modules:*

$$0 \longrightarrow \Pi(X_{2\varepsilon_1+2\varepsilon_2}) \longrightarrow H^{2,2}_{\mathfrak{g}_0} \longrightarrow \Pi(X_{3\varepsilon_1}) \longrightarrow 0;$$

if $\mathfrak{g}_0 = \mathfrak{cpe}(n)$, $n > 2$, *then*

$$H^{2,2}_{\mathfrak{cpe}(n)} = \Pi(S^2(E^2V/\Pi(\langle 1\rangle))/E^4V),$$

more precisely, we have the following non-split exact sequence of $\mathfrak{spe}(n)$*-modules:*

$$0 \longrightarrow H^{2,2}_{\mathfrak{spe}(n)\Subset\langle\tau+nz\rangle} \longrightarrow H^{2,2}_{\mathfrak{g}_0} \longrightarrow X_{2\varepsilon_1} \longrightarrow 0.$$

8.1.3. Proof of the main theorem.

8.1.3.1. Calculation of SFs of order 1. Recall that the bidegree of the differentials in the Spencer complex is $(-1,\ 1)$. We will often refer to the following

Lemma. *Let* $(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$ *be an arbitrary pair, where* $\mathfrak{g}_{-1}$ *is a faithful module*
3 *over a Lie superalgebra* $\mathfrak{g}_0$, *and let* [3] **[Ol: m.b. nizhe** $(k \geq 1)$ **pridvinut' k f-le, chtob ne vyglyadelo, kak nomer?]**

$$\mathfrak{g}_{k-1} \otimes \mathfrak{g}^*_{-1} \xrightarrow{\partial^{k+1,1}_{\mathfrak{g}_0}} \mathfrak{g}_{k-2} \otimes E^2\mathfrak{g}^*_{-1} \xrightarrow{\partial^{k,2}_{\mathfrak{g}_0}} \mathfrak{g}_{k-3} \otimes E^3\mathfrak{g}^*_{-1} \qquad (k \geq 1)$$

be the corresponding Spencer cochain sequence. Then

$$\operatorname{Im} \partial^{k+1,1}_{\mathfrak{g}_0} \cong (\mathfrak{g}_{k-1} \otimes \mathfrak{g}^*_{-1})/\mathfrak{g}_k. \qquad (8.12)$$

Proof. By the definition of the Cartan prolongation

$$\mathfrak{g}_k = (\mathfrak{g}_{k-1} \otimes \mathfrak{g}^*_{-1}) \cap (\mathfrak{g}_{k-2} \otimes S^2\mathfrak{g}^*_{-1}).$$

Let $c \in \mathfrak{g}_{k-1} \otimes \mathfrak{g}^*_{-1}$. Then $c \in \mathfrak{g}_k$ if and only if $c(g_1)g_2 = (-1)^{p(g_1)p(g_2)}c(g_2)g_1$ for any (homogeneous) $g_1, g_2 \in \mathfrak{g}_{-1}$. On the other hand

$$\partial^{k+1,1}_{\mathfrak{g}_0} c(g_1, g_2) = -(-1)^{p(g_1)p(g_2)}c(g_2)g_1 + c(g_1)g_2.$$

Hence, $\operatorname{Ker} \partial^{k+1,1}_{\mathfrak{g}_0} = \mathfrak{g}_k$. This proves the Lemma. □

In particular, to define $H^{1,2}_{\mathfrak{g}_0}$, we have the following Spencer cochain sequence:

$$\mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1} \xrightarrow{\partial^{2,1}_{\mathfrak{g}_0}} \mathfrak{g}_{-1} \otimes E^2\mathfrak{g}^*_{-1} \xrightarrow{\partial^{1,2}_{\mathfrak{g}_0}} 0, \text{ where } \operatorname{Ker} \partial^{2,1}_{\mathfrak{g}_0} = \mathfrak{g}_1.$$

Let us prove that

$$H^{1,2}_{\mathfrak{g}_0} = 0 \text{ if either } \mathfrak{g}_0 = \mathfrak{cpe}(n) \text{ or } \mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle a\tau + bz\rangle,$$

where $a = 0,\ b \neq 0$ or $a \neq 0,\ b/a \neq n$.

Let $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle\tau\rangle = \mathfrak{pe}(n)$. Since $\mathfrak{pe}(n)$ and $\Pi(E^2V^*)$ are isomorphic $\mathfrak{g}_0$-modules, we see that the $\mathfrak{pe}(n)$-module $\mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1}$ is isomorphic to

$$\Pi(E^2V^*) \otimes V^* \cong E^2V^* \otimes V \cong \mathfrak{g}_{-1} \otimes E^2\mathfrak{g}^*_{-1} = \operatorname{Ker} \partial^{1,2}_{\mathfrak{g}_0}.$$

By part a) of Theorem 8.1.1.2 we have $\mathfrak{g}_1 = 0$. Therefore, by (8.12)

$$\operatorname{Im} \partial^{2,1}_{\mathfrak{g}_0} \cong \mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1} \cong \operatorname{Ker} \partial^{1,2}_{\mathfrak{g}_0},$$

i.e., $H^{1,2}_{\mathfrak{pe}(n)} = 0$.

Let $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle a\tau + bz\rangle$, where $a = 0,\ b \neq 0$ or $a \neq 0,\ b/a \neq n$. By Theorem 8.1.1.2, $\mathfrak{g}_1 = 0$ for such $\mathfrak{g}_0$. Note that $\dim \mathfrak{g}_0 = \dim \mathfrak{pe}(n)$. Therefore,

$$\dim \operatorname{Im} \partial^{2,1}_{\mathfrak{g}_0} = \dim \mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1} = \dim \mathfrak{pe}(n) \otimes V^* = \dim E^2V^* \otimes V = \dim \operatorname{Ker} \partial^{1,2}_{\mathfrak{g}_0}.$$

Hence, $H^{1,2}_{\mathfrak{g}_0} = 0$.

Let $\mathfrak{g}_0 = \mathfrak{cpe}(n)$. Then by part b) of Theorem 8.1.1.2 $\mathfrak{g}_1 = V^*$. Note that

$$\mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1} = (\mathfrak{pe}(n) \oplus \langle z\rangle) \otimes V^* = \mathfrak{pe}(n) \otimes V^* \oplus \langle z\rangle \otimes V^* \cong V \otimes E^2V^* \oplus V^*.$$

Therefore,

$$\operatorname{Im} \partial^{2,1}_{\mathfrak{g}_0} = V \otimes E^2V^* = \operatorname{Ker} \partial^{1,2}_{\mathfrak{g}_0}.$$

Hence, $H^{1,2}_{\mathfrak{cpe}(n)} = 0$.

Let us prove that

$$H^{1,2}_{\mathfrak{g}_0} = V^* \text{ if } \mathfrak{g}_0 = \mathfrak{spe}(n) \text{ or } \mathfrak{spe}(n) \Subset \langle\tau + nz\rangle.$$

Let $\mathfrak{g}_0 = \mathfrak{spe}(n)$. By part a) of Theorem 8.1.1.2 $\mathfrak{g}_1 = 0$. Therefore,

$$\operatorname{Im} \partial^{2,1}_{\mathfrak{g}_0} \cong \mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1} = \mathfrak{spe}(n) \otimes V^*.$$

As has been shown for the case $\mathfrak{g}_0 = \mathfrak{cspe}(n) = \mathfrak{spe}(n) \oplus \langle z\rangle$, we have

$$\mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1} = (\mathfrak{spe}(n) \oplus \langle z\rangle) \otimes V^* \cong V \otimes E^2V^*.$$

Therefore,

$$\operatorname{Ker} \partial^{1,2}_{\mathfrak{spe}(n)} / \operatorname{Im} \partial^{2,1}_{\mathfrak{spe}(n)} \cong V^*.$$

Hence $H^{1,2}_{\mathfrak{spe}(n)} = V^*$.

Finally, let $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle\tau + nz\rangle$. By part b) of Theorem 8.1.1.2 $\mathfrak{g}_1 = V^*$. Since

$$\mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1} = (\mathfrak{spe}(n) \Subset \langle\tau + nz\rangle) \otimes V^*,$$

we see that the Jordan-Hölder series for the $\mathfrak{spe}(n)$-module $\operatorname{Im} \partial^{2,1}_{\mathfrak{g}_0}$ contains the same irreducible quotient modules as that for the $\mathfrak{spe}(n)$-module $\mathfrak{spe}(n) \otimes V^*$. Since

$$\operatorname{Ker} \partial^{1,2}_{\mathfrak{g}_0} = V \otimes E^2V^* \cong (\mathfrak{spe}(n) \oplus \langle z\rangle) \otimes V^*,$$

then $H^{1,2}_{\mathfrak{g}_0} = V^*$. This proves Theorem 8.1.2.1 in the case of the SFs of the first order.

8.1.3.2. Hochschild-Serre spectral sequence. The continuation of the proof of Theorem 8.1.2.1 is based on the Hochschild-Serre spectral sequence. Let us recall the corresponding formulations in a form convenient for us, since the case of Lie superalgebras is hardly reflected in the literature (one might think that the union of [Fu] and [GM] should suffice, but the sign rule applied to the Lie algebra case does not completely solve the problem).

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and M be a $\mathfrak{g}$-module. On the superspace of k-dimensional cochains $C^k = C^k(\mathfrak{g}, M)$ define a filtration:

$$F^0C^k = C^k \supset F^1C^k \supset F^2C^k \supset \ldots \supset F^jC^k \supset \ldots \supset F^{k+1}C^k = 0,$$

where

$$F^jC^k = \{c \in C^k \mid c(g_1, \ldots, g_i, \ldots, g_k) = 0 \text{ if } k-j+1 \text{ arguments lie } \mathfrak{g}_0\},$$

$0 \le j \le k+1$. Set

$$Z_r^{p,q} = \{c \in F^pC^{p+q} \mid dc \in F^{p+r}C^{p+q+1}\}. \tag{8.13}$$

Finally, set

$$E_r^{p,q} = Z_r^{p,q}/(Z_{r-1}^{p+1,q-1} + dZ_{r-1}^{p-r+1,q+r-2}). \tag{8.14}$$

Notice that the differential d induces the differentials

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1} \tag{8.15}$$

and $E_{r+1}^{p,q} = H^{p,q}(E_r)$ [GM].

Since $d(F^jC^k) \subset F^jC^{k+1}$, we get the induced filtration on $H^k = H^k(\mathfrak{g}, M)$ such that $F^pH^k/F^{p+1}H^k = E_\infty^{p,q}$, where $p+q=k$.

We want to compute the group $H^2(V, \mathfrak{g}_*)$, where $V = V_0 \oplus V_1$ is the standard $\mathfrak{pe}(n)$-module, $\mathfrak{g}_* = \mathfrak{g}_*(V, \mathfrak{cpe}(n))$. The Hochschild-Serre spectral sequence corresponding to the subalgebra V_0 converges to $H^2(V, \mathfrak{g}_*)$. Thus, $H^2(V, \mathfrak{g}_*) = \oplus_{p+q=2}E_\infty^{p,q}$ and in order to compute the limit terms of the spectral sequence $E_\infty^{p,q}$ we have to consider three cases:

1) $p=2$, $q=0$. Then by formula (8.15) we have

$$\begin{gathered} E_1^{1,0} \xrightarrow{d_1^{1,0}} E_1^{2,0} \xrightarrow{d_1^{2,0}} E_1^{3,0}, \\ E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \xrightarrow{d_2^{2,0}} 0, \\ 0 \xrightarrow{d_3^{-1,2}} E_3^{2,0} \xrightarrow{d_3^{2,0}} 0. \end{gathered} \tag{8.16}$$

Therefore, $E_\infty^{2,0} = E_3^{2,0}$.

2) $p=1$, $q=1$. Then by formula (8.15) we have

$$\begin{gathered} E_1^{0,1} \xrightarrow{d_1^{0,1}} E_1^{1,1} \xrightarrow{d_1^{1,1}} E_1^{2,1}, \\ 0 \xrightarrow{d_2^{-1,2}} E_2^{1,1} \xrightarrow{d_2^{1,1}} E_2^{3,0}, \\ 0 \xrightarrow{d_3^{-2,3}} E_3^{1,1} \xrightarrow{d_3^{1,1}} 0. \end{gathered} \tag{8.17}$$

Therefore, $E_\infty^{1,1} = E_3^{1,1}$.

3) $p=0, q=2$. Then by formula (8.15) we have

$$\begin{gathered} 0 \xrightarrow{d_1^{-1,2}} E_1^{0,2} \xrightarrow{d_1^{0,2}} E_1^{1,2}, \\ 0 \xrightarrow{d_2^{-2,3}} E_2^{0,2} \xrightarrow{d_2^{0,2}} E_2^{2,1}, \\ 0 \xrightarrow{d_3^{-3,4}} E_3^{0,2} \xrightarrow{d_3^{0,2}} E_3^{3,0}, \\ 0 \xrightarrow{d_4^{-4,5}} E_4^{0,2} \xrightarrow{d_4^{0,2}} 0. \end{gathered} \tag{8.18}$$

Therefore, $E_\infty^{0,2} = E_4^{0,2}$.

8.1.3.3. Continuation of the proof. Notice that by [Fu]

$$E_1^{p,q} = H^q(\mathfrak{g}_0,\ M \otimes S^p\mathfrak{g}_1^*).$$

Since in our case $V = V_0 \oplus V_1$ is a commutative Lie superalgebra, then

$$E_1^{p,q} = H^q(V_0,\ \mathfrak{g}_* \otimes S^pV_1^*) = H^q(V_0,\ \mathfrak{g}_*) \otimes S^pV_1^*. \tag{8.19}$$

Let us calculate $H^q(V_0,\ \mathfrak{g}_*)$ for $q = 0, 1, 2$. By Lemma 8.1.1.1

$$\begin{aligned} &\mathfrak{g}_* = \mathfrak{pe}(n+1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \text{ where} \\ &\mathfrak{g}_{-1} = V_0 \oplus V_1, \mathfrak{g}_0 = \mathfrak{cpe}(n) = \mathfrak{sl}(n) \oplus \Pi(S^2V_0) \oplus \Pi(E^2V_1) \oplus \langle\tau\rangle \oplus \langle z\rangle, \\ &\mathfrak{g}_1 = \Pi(V_1) \oplus \Pi(V_0), \\ &\mathfrak{g}_2 = \Pi(\langle 1\rangle). \end{aligned} \tag{8.20}$$

Recall that as an $\mathfrak{sl}(n+1)$-module,

$$\mathfrak{pe}(n+1) \cong \mathfrak{sl}(n+1) \oplus E^2W_0^* \oplus S^2W_0 \oplus \langle\ \mathrm{d}\ \rangle,$$

where d is $\mathrm{diag}(1_{n+1},\ -1_{n+1})$ and W_0 is the standard $\mathfrak{sl}(n+1)$-module. Clearly,

$$\begin{aligned} &E^2W_0^* = \Pi(E^2V_1) \oplus V_1, \\ &S^2W_0 = \Pi(S^2V_0) \oplus \Pi(V_0) \oplus \mathfrak{g}_2, \\ &\mathfrak{sl}(n+1) = \mathfrak{sl}(n) \oplus \langle\tau + nz\rangle \oplus V_0 \oplus \Pi(V_1), \\ &\mathrm{d}\ = \tau - z. \end{aligned}$$

Let $\varepsilon_1, \ldots, \varepsilon_{n+1}$ be the standard basis of the dual space to the space of the diagonal matrices in $\mathfrak{gl}(n+1)$ and the ordering is performed so that

$$\Delta_+ = \{\varepsilon_i - \varepsilon_j, i < j\},\ \Delta_- = \{\varepsilon_i - \varepsilon_j, i > j\}.$$

Let $E_{\varepsilon_i - \varepsilon_j} (i \ne j)$ be the corresponding root vectors. Then V_0 is the subspace of $\mathfrak{sl}(n+1)$ generated by

$$E_{\varepsilon_1 - \varepsilon_{n+1}},\ E_{\varepsilon_2 - \varepsilon_{n+1}}, \ldots,\ E_{\varepsilon_n - \varepsilon_{n+1}}.$$

Let V_λ be the irreducible $\mathfrak{sl}(n+1)$-module with highest weight λ. The BBW theorem says [Kos] that there exists a 1-1 correspondence between the irreducible components of $H^q(V_0,\ V_\lambda)$, considered as $\mathfrak{gl}(n)$-module, and elements $w \in W(\mathfrak{sl}(n+1))$ of length q from the Weyl group of the Lie algebra $\mathfrak{sl}(n+1)$ such that

$$w(\Delta_-) \cap \Delta_+ \subset \{\varepsilon_1 - \varepsilon_{n+1},\ \varepsilon_2 - \varepsilon_{n+1}, \dots, \varepsilon_n - \varepsilon_{n+1}\}. \tag{8.21}$$

Moreover, the highest weight of the $\mathfrak{gl}(n)$-module corresponding to w is equal to $w(\lambda+\rho)-\rho$, where $\rho = \frac{1}{2}(\sum\limits_{\alpha\in\Delta_+}\alpha)$.

Notice that $E^2W_0^*$, S^2W_0, $\mathfrak{sl}(n+1)$, and $\langle\tau - z\rangle$ are all irreducible $\mathfrak{sl}(n+1)$-modules with highest weights, respectively,

$$\varepsilon_1 + \ldots + \varepsilon_{n-1},\ 2\varepsilon_1,\ 2\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n,\ 0.$$

Let us find the highest weights of irreducible $\mathfrak{gl}(n)$-submodules of $H^q(V_0,\ V_\lambda)$ for each of the indicated λ.

1) $q = 0$. The only element of the Weyl group of length 0 is the unit. Hence $w(\lambda+\rho)-\rho = \lambda$.

2) $q = 1$. Let $\langle\alpha_1, \dots, \alpha_n\rangle$, where

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \dots,\ \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \dots,\ \alpha_n = \varepsilon_n - \varepsilon_{n+1}$$

be the system of simple roots. The elements of the Weyl group of length 1 are reflections corresponding to the simple roots:

$$r_{\alpha_i} : \alpha \longrightarrow \alpha - \frac{2(\alpha_i, \alpha)}{(\alpha_i, \alpha_i)}\alpha_i.$$

Since the only element r_{α_i} satisfying (8.21) is r_{α_n}, we have

$$w(\lambda+\rho)-\rho = r_{\alpha_n}(\lambda) - \alpha_n.$$

For

$$\lambda = \varepsilon_1 + \ldots + \varepsilon_{n-1},\ 2\varepsilon_1,\ 2\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n,\ 0$$

this expression is equal to, respectively:

$$\begin{aligned}&-2\varepsilon_n,\ \varepsilon_1 - \varepsilon_2 - \ldots - \varepsilon_{n-1} - 2\varepsilon_n,\\ &-\varepsilon_2 - \varepsilon_3 - \ldots - \varepsilon_{n-1} - 3\varepsilon_n,\ -\varepsilon_1 - \varepsilon_2 - \ldots - \varepsilon_{n-1} - 2\varepsilon_n.\end{aligned}$$

3) $q = 2$. The elements of length 2 are of the form $r_{\alpha_i}r_{\alpha_j}$. The only such element satisfying (8.21) is $r_{\alpha_n}r_{\alpha_{n-1}}$. Then

$$w(\lambda+\rho)-\rho = r_{\alpha_n}r_{\alpha_{n-1}}(\lambda) - \alpha_{n-1} - 2\alpha_n.$$

For $\lambda = \varepsilon_1 + \ldots + \varepsilon_{n-1}$, $2\varepsilon_1$, $2\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n$, 0 this expression is equal to, respectively:

$$\begin{aligned}&-2\varepsilon_1 - 2\varepsilon_2 - \ldots - 2\varepsilon_{n-2} - 4\varepsilon_{n-1} - 4\varepsilon_n,\\ &-3\varepsilon_2 - 3\varepsilon_3 \text{ (if } n=3) \text{ or } -2\varepsilon_2 - \ldots - 2\varepsilon_{n-2} - 3\varepsilon_{n-1} - 3\varepsilon_n \text{ (if } n>3),\\ &-\varepsilon_1 - 3\varepsilon_2 - 4\varepsilon_3 \text{ (if } n=3) \text{ or } -\varepsilon_1 - 2\varepsilon_2 - \ldots - 2\varepsilon_{n-2} - 3\varepsilon_{n-1} - 4\varepsilon_n \text{ (if } n>3),\\ &-2\varepsilon_1 - \ldots - 2\varepsilon_{n-2} - 3\varepsilon_{n-1} - 3\varepsilon_n.\end{aligned}$$

Remark. We have obtained the weights with respect to $\mathfrak{gl}(n) = \mathfrak{sl}(n)\oplus\langle\tau+nz\rangle$ embedded into $\mathfrak{sl}(n+1)$. Now it is not difficult to rewrite these weights as the highest ones with respect to $\mathfrak{pe}(n)_0 = \mathfrak{sl}(n)\oplus\langle\tau\rangle$. We collect all our results in the following

8.1.3.4. Lemma. $\mathfrak{gl}(n) = \mathfrak{pe}(n)_0$ *module* $H^q(V_0,\ \mathfrak{g}_*)$ $(q = 0,\ 1,\ 2)$ *is the direct sum of irreducible submodules with the highest weights and highest vectors listed in Table 2.*

8.1.3.5. Lemma. *Let V_λ be an irreducible $\mathfrak{gl}(n)$-module with highest weight λ. Then $E_1^{p,0}(p = 1,\ 2,\ 3)$, $E_1^{p,1}(p = 0,\ 1,\ 2)$, $E_1^{p,2}$ $(p = 0,\ 1)$ are the direct sums of irreducible $\mathfrak{gl}(n)$-submodules with highest weights given in the corresponding columns of Tables 3, 4, and 5, respectively.*

Proof. By formula (8.19) the following $\mathfrak{gl}(n)$-modules are isomorphic: $E_1^{p,q} \cong H^q(V_0,\ \mathfrak{g}_*)\otimes S^pV_0$. Making use of the description of $H^q(V_0,\ \mathfrak{g}_*)$ as a $\mathfrak{gl}(n)$-module given in Lemma 8.1.3.4 we find the decomposition of the indicated tensor product into irreducible components described in Tables 3, 4, and 5. □

8.1.3.6. Lemma. $E_0^{p,0}(p = 1,\ 2,\ 3)$ *are the direct sums of the irreducible $\mathfrak{gl}(n)$-modules with highest weights described in the corresponding columns of Table 6.*

Proof. By formula (8.14) we have

$$E_0^{p,0} = Z_0^{p,0} = \mathfrak{g}_*\otimes S^pV_1^* = \mathfrak{g}_*\otimes S^pV_0.$$

Making use of the description of $\mathfrak{g}_*$ as $\mathfrak{gl}(n)$-module given in (8.20) we find the decomposition of the indicated tensor products into direct sum of the irreducible components described in Table 6. □

8.1.3.7. Lemma. $E_\infty^{2,0}$ *is an irreducible $\mathfrak{gl}(n)$-module with highest weight* $2\varepsilon_1 + 2\varepsilon_2$.

Proof. First, recall that $H^{1,2}_{\mathfrak{cpe}(n)} = 0$ by the already proved part of Theorem 8.1.2.1 for the case of SFs of order 1. Making use of Lemma 8.1.3.4, we note that if $E_\infty^{2,0}$ had contained a $\mathfrak{gl}(n)$-submodule belonging to either $V_{-\varepsilon_n}\otimes S^2V_0$ or $V_{\varepsilon_1}\otimes S^2V_0$, then this submodule would have belonged to $H^{1,2}_{\mathfrak{cpe}(n)}$. □

Therefore, with the help of Table 3 we deduce that $E_\infty^{2,0}$ has no $\mathfrak{gl}(n)$-submodules with highest weights $2\varepsilon_1-\varepsilon_n$, ε_1, $3\varepsilon_1$, $2\varepsilon_1+\varepsilon_2$.

Let us show that $E_\infty^{2,0}$ has no irreducible $\mathfrak{gl}(n)$-submodules with highest weights $4\varepsilon_1$, $3\varepsilon_1+\varepsilon_2$, and $2\varepsilon_1$ either. More precisely, let us show that even $E_2^{2,0}$ does not have them.

Recall that the corresponding differentials act as follows:

$$E_1^{1,0}\xrightarrow{d_1^{1,0}}E_1^{2,0}\xrightarrow{d_1^{2,0}}E_1^{3,0}.$$

Note that according to (8.14), $E_1^{p,0}=Z_1^{p,0}$ for $p=1,2,3$. Let us show that $\operatorname{Ker} d_1^{2,0}$ has no components with weights $4\varepsilon_1$, $3\varepsilon_1+\varepsilon_2$, and $2\varepsilon_1$. It follows from Tables 2 and 3 that the corresponding highest vectors in $E_1^{2,0}$ are

$$v_{4\varepsilon_1}=e_1\tilde e_1\otimes\tilde e_1^2,\ v_{3\varepsilon_1+\varepsilon_2}=e_1\tilde e_2\otimes\tilde e_1^2-e_1\tilde e_1\otimes\tilde e_1\tilde e_2,\ v_{2\varepsilon_1}=(\tau-z)\otimes\tilde e_1^2.$$

We remind the reader that the differentials d in our case are the same as the differentials $\partial_{\mathfrak g_0}^{k,s}$. Notice that if $c\in E_1^{2,0}$ then

$$d_1^{2,0}c(f_1,f_1,f_1)=dc(f_1,f_1,f_1)=-3c(f_1,f_1)(f_1).$$

Therefore,

$$d_1^{2,0}v_{4\varepsilon_1}(f_1,f_1)=3e_1,\ d_1^{2,0}v_{3\varepsilon_1+\varepsilon_2}(f_1,f_1)=\frac{3}{2}e_2,\ d_1^{2,0}v_{2\varepsilon_1}(f_1,f_1)=-6f_1. \tag{8.22}$$

Hence,

$$v_{4\varepsilon_1},\ v_{3\varepsilon_1+\varepsilon_2},\ v_{2\varepsilon_1}\notin\operatorname{Ker} d_1^{2,0}.$$

Finally, let us prove that in $E_\infty^{2,0}$ there is an irreducible $\mathfrak{gl}(n)$-submodule of highest weight $2\varepsilon_1+2\varepsilon_2$. Notice that $E_2^{2,0}$ has submodule of highest weight $2\varepsilon_1+2\varepsilon_2$, since according to Table 3, this module is contained in $E_1^{2,0}$ and is not contained in either $E_1^{1,0}$ or $E_1^{3,0}$. Recall that the corresponding differentials act as follows:

$$E_2^{0,1}\xrightarrow{d_2^{0,1}}E_2^{2,0}\xrightarrow{d_2^{2,0}}0.$$

Therefore, $\operatorname{Ker} d_2^{2,0}=E_2^{2,0}$ has a component of weight $2\varepsilon_1+2\varepsilon_2$. By Table 4 $E_1^{0,1}$ has no components of weight $2\varepsilon_1+2\varepsilon_2$. Hence, neither $E_2^{0,1}$ nor $\operatorname{Im} d_2^{0,1}$ has such a component. Therefore, it must be in $E_3^{2,0}=E_\infty^{2,0}$.

8.1.3.8. Lemma. a) *As a $\mathfrak{gl}(n)$-module, $E_\infty^{1,1}$ can only have the irreducible submodules with the following highest weights, each of multiplicity not greater then 1:*

$$2\varepsilon_1,\ 2\varepsilon_1+\varepsilon_2-\varepsilon_n,\ \varepsilon_1+\varepsilon_2-2\varepsilon_n,\ \textit{and}\ \varepsilon_1-\varepsilon_n;$$

b) *$E_\infty^{1,1}$ has an irreducible $\mathfrak{gl}(n)$-submodule with highest weight $2\varepsilon_1$.*

Proof. By Theorem 8.1.2.1 for the case of SFs of order 1 and Tables 2 and 4 we see that $E_\infty^{1,1}$ has no irreducible $\mathfrak{gl}(n)$-submodules of highest weight $\varepsilon_1-2\varepsilon_n$ and $-\varepsilon_n$, since they would have corresponded to SFs of order 1.

Let us show that there are no components of weight $2\varepsilon_1-2\varepsilon_n$ or 0 in $E_\infty^{1,1}$, more precisely, that even $E_2^{1,1}$ does not have them. Recall that the corresponding differentials act as follows:

$$E_1^{0,1}\xrightarrow{d_1^{0,1}}E_1^{1,1}\xrightarrow{d_1^{1,1}}E_1^{2,1}.$$

By (8.14) we have

$$\begin{aligned}E_1^{0,1}&=Z_1^{0,1}/(Z_0^{1,0}+dZ_0^{0,0}),\\ E_1^{1,1}&=Z_1^{1,1}/(Z_0^{2,0}+dZ_0^{1,0}),\\ E_1^{2,1}&=Z_1^{2,1}/(Z_0^{3,0}+dZ_0^{2,0}).\end{aligned}\tag{8.23}$$

By Tables 2 and 4 the highest vectors of weights $2\varepsilon_1-2\varepsilon_n$ and 0 in $E_1^{1,1}$ are, respectively,

$$v_{2\varepsilon_1-2\varepsilon_n}=(e_1\wedge\tilde f_n)\otimes\tilde e_1\wedge\tilde f_n \text{ and } v_0=\sum_{i=1}^n(\tau-z)\otimes\tilde f_i\wedge\tilde e_i.$$

We see that

$$\begin{aligned}&dv_{2\varepsilon_1-2\varepsilon_n}(e_n,f_1,f_1)=-v_{2\varepsilon_1-2\varepsilon_n}(f_1,f_1)(e_n)-2v_{2\varepsilon_1-2\varepsilon_n}(e_n,f_1)(f_1)=-\frac{1}{2}f_n\neq0,\\ &dv_0(e_1,f_1,f_1)=-v_0(f_1,f_1)(e_1)-2v_0(e_1,f_1)(f_1)=2f_1\neq0.\end{aligned}$$

Suppose that

$$dv_{2\varepsilon_1-2\varepsilon_n}\in Z_0^{3,0}+dZ_0^{2,0}.$$

Then there exist highest $\mathfrak{gl}(n)$-vectors $v'_{2\varepsilon_1-2\varepsilon_n}\in Z_0^{3,0}$ and $v''_{2\varepsilon_1-2\varepsilon_n}\in dZ_0^{2,0}$ of weight $2\varepsilon_1-2\varepsilon_n$ such that

$$dv_{2\varepsilon_1-2\varepsilon_n}=v'_{2\varepsilon_1-2\varepsilon_n}+v''_{2\varepsilon_1-2\varepsilon_n}.$$

Since $e_n\in V_0$, it follows that $v'_{2\varepsilon_1-2\varepsilon_n}(e_n,f_1,f_1)=0$. Hence $v''_{2\varepsilon_1-2\varepsilon_n}\neq0$ and therefore, $dZ_0^{2,0}$ has an irreducible $\mathfrak{gl}(n)$-submodule of highest weight $2\varepsilon_1-2\varepsilon_n$.

Similarly, having assumed that $dv_0\in Z_0^{3,0}+dZ_0^{2,0}$, we deduce that $dZ_0^{2,0}$ has an irreducible $\mathfrak{gl}(n)$-submodule of weight 0. Note that according to Table 6, $E_0^{2,0}$ has no submodules of highest weight $2\varepsilon_1-2\varepsilon_n$ or 0. Since $E_0^{2,0}=Z_0^{2,0}$, then $Z_0^{2,0}$ and $dZ_0^{2,0}$ have no such components, either. Therefore, $dv_{2\varepsilon_1-2\varepsilon_n}$ and dv_0 do not lie in $Z_0^{3,0}+dZ_0^{2,0}$. Thanks to eq. (8.23) this implies that

$$d_1^{1,1}v_{2\varepsilon_1-2\varepsilon_n}\neq0 \text{ and } d_1^{1,1}v_0\neq0.$$

Hence, $\operatorname{Ker} d_1^{1,1}$, and therefore $E_2^{1,1}$, have no irreducible $\mathfrak{gl}(n)$-submodules of highest weight $2\varepsilon_1-2\varepsilon_n$ and 0.

Let us prove now that $E_\infty^{1,1}=E_3^{1,1}$ has no irreducible $\mathfrak{gl}(n)$-submodule of highest weight $3\varepsilon_1-\varepsilon_n$. Notice that $E_2^{1,1}$ has such a submodule, since thanks to Table 4 it is contained in $E_1^{1,1}$ and is not contained in either $E_1^{0,1}$ or $E_1^{2,1}$.

Tables 2 and 4 imply that the $\mathfrak{gl}(n)$-highest vector in $E_1^{1,1}$ of weight $3\varepsilon_1 - \varepsilon_n$ is $v_{3\varepsilon_1-\varepsilon_n} = e_1\tilde{e}_1 \otimes \tilde{f}_n \wedge \tilde{e}_1$.

Recall that the corresponding differentials act as follows:

$$0 \xrightarrow{d_2^{-1,2}} E_2^{1,1} \xrightarrow{d_2^{1,1}} E_2^{3,0}.$$

By formula (8.14) we have

$$\begin{aligned} E_2^{1,1} &= Z_2^{1,1}/(Z_1^{2,0} + dZ_1^{0,1}),\\ E_2^{3,0} &= Z_2^{3,0}/dZ_1^{2,0}. \end{aligned} \tag{8.24}$$

Thanks to formulas (8.23) we see that the $\mathfrak{gl}(n)$-highest vector in $E_2^{1,1}$ of weight $3\varepsilon_1 - \varepsilon_n$ is

$$w_{3\varepsilon_1-\varepsilon_n} = v_{3\varepsilon_1-\varepsilon_n} + v'_{3\varepsilon_1-\varepsilon_n} + v''_{3\varepsilon_1-\varepsilon_n},$$

where $v'_{3\varepsilon_1-\varepsilon_n}$ and $v''_{3\varepsilon_1-\varepsilon_n}$ are $\mathfrak{gl}(n)$-highest vectors in $Z_0^{2,0}$ and $dZ_0^{1,0}$, respectively.

$\boxed{4}$? Since by eq. (8.13)[4] $dw_{3\varepsilon_1-\varepsilon_n} \in Z_2^{3,0}$, then $dw_{3\varepsilon_1-\varepsilon_n}(e_n, f_1, f_1) = 0$. Since $v''_{3\varepsilon_1-\varepsilon_n} \in dZ_0^{1,0}$, then $dv''_{3\varepsilon_1-\varepsilon_n} = 0$. We have

$$dv_{3\varepsilon_1-\varepsilon_n}(e_n, f_1, f_1) = -2v_{3\varepsilon_1-\varepsilon_n}(e_n, f_1)(f_1) = -e_1 \neq 0.$$

Therefore, $v'_{3\varepsilon_1-\varepsilon_n} \neq 0$. Looking at Table 6 we see that the unique highest vector of weight $3\varepsilon_1-\varepsilon_n$ in $Z_0^{2,0}$ is $e_1\wedge\tilde{f}_n\otimes\tilde{e}_1^2$. Hence, $v'_{3\varepsilon_1-\varepsilon_n} = ke_1\wedge\tilde{f}_n\otimes\tilde{e}_1^2$, where $k \in \mathbb{C}^*$. Note that since $v_{3\varepsilon_1-\varepsilon_n}(f_1, f_1) = 0$, then $dv_{3\varepsilon_1-\varepsilon_n}(f_1, f_1, f_1) = 0$. Therefore,

$$\begin{aligned} &dw_{3\varepsilon_1-\varepsilon_n}(f_1, f_1, f_1) = dv'_{3\varepsilon_1-\varepsilon_n}(f_1, f_1, f_1) =\\ &-3v'_{3\varepsilon_1-\varepsilon_n}(f_1, f_1)(f_1) = -\frac{3}{2}kf_n \neq 0. \end{aligned}$$

Note that $dw_{3\varepsilon_1-\varepsilon_n} \notin dZ_1^{2,0}$. In fact, by Table 3 $E_1^{2,0}$ has no irreducible $\mathfrak{gl}(n)$-component with highest weight $3\varepsilon_1 - \varepsilon_n$. Since $E_1^{2,0} = Z_1^{2,0}$, then $Z_1^{2,0}$ and $dZ_1^{2,0}$ have no such component, either. Therefore, by (8.24) we have $d_2^{1,1}w_{3\varepsilon_1-\varepsilon_n} \neq 0$. Hence, $\operatorname{Ker} d_2^{1,1} = E_3^{1,1}$ has no components with highest weight $3\varepsilon_1 - \varepsilon_n$.

Let us prove that the irreducible component with highest weight $\varepsilon_1 - \varepsilon_n$ cannot be contained in $E_\infty^{1,1}$ with multiplicity greater than 1. Note that $E_1^{1,1}$ has two components of weight $\varepsilon_1 - \varepsilon_n$. According to Tables 2 and 4, one of the $\mathfrak{gl}(n)$-highest vectors of weight $\varepsilon_1 - \varepsilon_n$ in $E_1^{1,1}$ is $v_{\varepsilon_1-\varepsilon_n} = (\tau - z) \otimes \tilde{f}_n \wedge \tilde{e}_1$. We see that

$$dv_{\varepsilon_1-\varepsilon_n}(e_n, f_1, f_1) = -2v_{\varepsilon_1-\varepsilon_n}(e_n, f_1)(f_1) = -(\tau - z)(f_1) = 2f_1 \neq 0.$$

Suppose that $dv_{\varepsilon_1-\varepsilon_n} \in Z_0^{3,0} + dZ_0^{2,0}$. Then there exist $\mathfrak{gl}(n)$-highest vectors $v'_{\varepsilon_1-\varepsilon_n} \in Z_0^{3,0}$ and $v''_{\varepsilon_1-\varepsilon_n} \in dZ_0^{2,0}$ of weight $\varepsilon_1 - \varepsilon_n$ such that

$$dv_{\varepsilon_1-\varepsilon_n} = v'_{\varepsilon_1-\varepsilon_n} + v''_{\varepsilon_1-\varepsilon_n}.$$

Since $e_n \in V_0$, then $v'_{\varepsilon_1-\varepsilon_n}(e_n, f_1, f_1) = 0$. Therefore, $v''_{\varepsilon_1-\varepsilon_n} \neq 0$. Now note that

$$dv_{\varepsilon_1-\varepsilon_n}(e_1, f_1, f_1) = -v_{\varepsilon_1-\varepsilon_n}(f_1, f_1)(e_1) - 2v_{\varepsilon_1-\varepsilon_n}(e_1, f_1)(f_1) = 0.$$

Since $e_1 \in V_0$, then $v'_{\varepsilon_1-\varepsilon_n}(e_1, f_1, f_1) = 0$. Therefore,

$$v''_{\varepsilon_1-\varepsilon_n}(e_1, f_1, f_1) = 0. \tag{8.25}$$

By Table 6 $E_0^{2,0} = Z_0^{2,0}$ contains a unique highest vector of weight $\varepsilon_1 - \varepsilon_n$, namely,

$$\sum_{i=1}^n f_n \wedge \tilde{f}_i \otimes \tilde{e}_i\tilde{e}_1.$$

Then

$$v''_{\varepsilon_1-\varepsilon_n} = kd(\sum_{i=1}^n f_n \wedge \tilde{f}_i \otimes \tilde{e}_i\tilde{e}_1), \text{ where } k \in \mathbb{C}^*.$$

Note that in this case

$$\begin{aligned} &v''_{\varepsilon_1-\varepsilon_n}(e_1, f_1, f_1) = kd(\sum_{i=1}^n f_n \wedge \tilde{f}_i \otimes \tilde{e}_i\tilde{e}_1)(e_1, f_1, f_1) =\\ &-k(\sum_{i=1}^n f_n \wedge \tilde{f}_i \otimes \tilde{e}_i\tilde{e}_1)(f_1, f_1)(e_1) = \frac{k}{2}f_n \neq 0, \end{aligned}$$

which contradicts (8.25). Thus, $dv_{3\varepsilon_1-\varepsilon_n} \notin Z_0^{3,0} + dZ_0^{2,0}$. Then (8.23) yields $d_1^{1,1}v_{\varepsilon_1-\varepsilon_n} \neq 0$. Therefore, the component of highest weight $\varepsilon_1 - \varepsilon_n$ can not be contained in $\operatorname{Ker} d_1^{1,1}$ and hence, in $E_\infty^{1,1}$, with multiplicity exceeding 1. So part a) of Lemma 8.1.3.8 is proved.

Let us prove that $E_\infty^{1,1} = E_3^{1,1}$ contains an irreducible $\mathfrak{gl}(n)$-submodule of highest weight $2\varepsilon_1$. Note that $E_2^{1,1}$ does contain such a submodule, since by Table 4 it is contained in $E_1^{1,1}$ and is not contained in either $E_1^{0,1}$ or $E_1^{2,1}$. Let $u_{2\varepsilon_1}$ be the $\mathfrak{gl}(n)$-highest vector of weight $2\varepsilon_1$ in $E_1^{1,1}$. By (8.23) $u_{2\varepsilon_1}$ can be chosen so that

$$u_{2\varepsilon_1}(v_1, v_2) = 0 \text{ for any } v_1, v_2 \in V_1.$$

According to (8.23), the $\mathfrak{gl}(n)$-highest vector of weight $2\varepsilon_1$ in $E_2^{1,1}$ is

$$w_{2\varepsilon_1} = u_{2\varepsilon_1} + t_{2\varepsilon_1} + s_{2\varepsilon_1}, \tag{8.26}$$

where $t_{2\varepsilon_1} \in Z_0^{2,0}$ and $s_{2\varepsilon_1} \in dZ_0^{1,0}$ are $\mathfrak{gl}(n)$-highest vectors. If $dw_{2\varepsilon_1} = 0$, then

$$w_{2\varepsilon_1} \in \operatorname{Ker} d_2^{1,1} = E_3^{1,1}$$

and therefore, $E^{1,1}_\infty$ has an irreducible $\mathfrak{gl}(n)$-submodule with highest weight $2\varepsilon_1$.

Suppose that $dw_{2\varepsilon_1} \neq 0$. Let us prove that then

$$dw_{2\varepsilon_1} \in dZ_1^{2,0}. \tag{8.27}$$

Recall that $Z_1^{2,0} = E_1^{2,0}$ and by Table 3 $E_1^{2,0}$ has one highest vector of weight $2\varepsilon_1$, namely, $v_{2\varepsilon_1}$. Let us show that

$$dw_{2\varepsilon_1} = kdv_{2\varepsilon_1}, \text{ where } k \in \mathbb{C}^*. \tag{8.28}$$

Note that by (8.13) $dw_{2\varepsilon_1} \in Z_2^{3,0}$ and $dZ_1^{2,0} \subset Z_0^{3,0}$. Therefore, in order to prove (8.28), it suffices to show that

$$dw_{2\varepsilon_1}(v_1, v_2, v_3) = kdv_{2\varepsilon_1}(v_1, v_2, v_3), \text{ where } k \in \mathbb{C}^*, \text{ for any } v_1, v_2, v_3 \in V_1.$$

We have

$$du_{2\varepsilon_1}(v_1, v_2, v_3) = 0 \text{ for any } v_1, v_2, v_3 \in V_1.$$

Since $s_{2\varepsilon_1} \in dZ_0^{1,0}$, then $ds_{2\varepsilon_1} = 0$. Therefore,

$$dw_{2\varepsilon_1}(v_1, v_2, v_3) = dt_{2\varepsilon_1}(v_1, v_2, v_3) \text{ for any } v_1, v_2, v_3 \in V_1. \tag{8.29}$$

Since $t_{2\varepsilon_1} \in Z_0^{2,0}$, then $dt_{2\varepsilon_1} \in Z_0^{2,1}$. Hence $dt_{2\varepsilon_1} = t'_{2\varepsilon_1} + t''_{2\varepsilon_1}$,where $t'_{2\varepsilon_1}, t''_{2\varepsilon_1}$ are $\mathfrak{gl}(n)$-highest vectors from $Z_0^{2,1}$ such that

$$t'_{2\varepsilon_1}(v_1, v_2, v_3) = 0 \text{ for all } v_1, v_2, v_3 \in V_1 \text{ and } t''_{2\varepsilon_1} \in Z_0^{3,0}.$$

Since by hypothesis $dw_{2\varepsilon_1} \neq 0$, then $t''_{2\varepsilon_1} \neq 0$. Since $v_{2\varepsilon_1} \in Z_1^{2,0}$, then $dv_{2\varepsilon_1} \in Z_0^{3,0}$.

In Lemma 8.1.3.7 we have proved that $dv_{2\varepsilon_1} \neq 0$ (see (8.22)). By Table 6 $Z_0^{3,0} = E_0^{3,0}$ has only one irreducible $\mathfrak{gl}(n)$-submodule with highest weight $2\varepsilon_1$. Hence $t''_{2\varepsilon_1} = kdv_{2\varepsilon_1}$, where $k \in \mathbb{C}^*$. Therefore,

$$dt_{2\varepsilon_1}(v_1, v_2, v_3) = kdv_{2\varepsilon_1}(v_1, v_2, v_3) \text{ for any } v_1, v_2, v_3 \in V_1.$$

Thus, by (8.29)

$$dw_{2\varepsilon_1}(v_1, v_2, v_3) = kdv_{2\varepsilon_1}(v_1, v_2, v_3) \text{ for any } v_1, v_2, v_3 \in V_1$$

and formula (8.28) is proved. Then by (8.24) $d_2^{1,1} w_{2\varepsilon_1} = 0$. Therefore, $w_{2\varepsilon_1} \in \operatorname{Ker} d_2^{1,1} = E_3^{1,1}$. Thus, $E^{1,1}_\infty$ contains an irreducible $\mathfrak{gl}(n)$-submodule with highest weight $2\varepsilon_1$. This proves part b) of Lemma 8.1.3.8. $\square$

8.1.3.9. Lemma. *Only the following highest weights of irreducible $\mathfrak{gl}(n)$-submodules can be encountered among those in $E^{0,2}_\infty$:*

$$-2\varepsilon_{n-1} - 2\varepsilon_n,\ 2\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n,\ \varepsilon_1 - \varepsilon_{n-1} - 2\varepsilon_n,\ \ -\varepsilon_{n-1} - \varepsilon_n.$$

Proof. By Table 5 $E_1^{0,2}$ is a direct sum of irreducible $\mathfrak{gl}(n)$-components with the indicated highest weights.

Note that due to Table 2 $H^2(V,\ \mathfrak{g}_*)$ can only possess first and second order SFs. By the statement of Theorem 8.1.2.1 for SFs of order 1, $H^{1,2}_{\mathfrak{cpe}(n)} = 0$. Therefore, by Lemmas 8.1.3.7–8.1.3.9 $H^{2,2}_{\mathfrak{cpe}(n)}$ can only contain irreducible $\mathfrak{gl}(n)$-submodules with the following highest weights:

$$\begin{gathered}2\varepsilon_1 + 2\varepsilon_2,\ 2\varepsilon_1,\ 2\varepsilon_1 + \varepsilon_2 - \varepsilon_n,\ \varepsilon_1 + \varepsilon_2 - 2\varepsilon_n,\ \varepsilon_1 - \varepsilon_n,\\ -2\varepsilon_{n-1} - 2\varepsilon_n,\ 2\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n,\ \varepsilon_1 - \varepsilon_{n-1} - 2\varepsilon_n,\ \ -\varepsilon_{n-1} - \varepsilon_n\end{gathered}$$

each with multiplicity not greater than one, and the components with highest weights $2\varepsilon_1 + 2\varepsilon_2$ and $2\varepsilon_1$ are contained in $H^{2,2}_{\mathfrak{cpe}(n)}$ with multiplicity one each.

Recall that by definition $H^{2,2}_{\mathfrak{g}_0}$ is determined by the sequence

$$\mathfrak{g}_1 \otimes \mathfrak{g}^*_{-1} \xrightarrow{\partial^{3,1}_{\mathfrak{g}_0}} \mathfrak{g}_0 \otimes E^2\mathfrak{g}^*_{-1} \xrightarrow{\partial^{2,2}_{\mathfrak{g}_0}} \mathfrak{g}_{-1} \otimes E^3\mathfrak{g}^*_{-1}.$$

Thus, $\mathfrak{g}_0 \otimes E^2\mathfrak{g}^*_{-1}$ for

$$\mathfrak{g}_0 = \mathfrak{cpe}(n) \text{ or } \mathfrak{spe}(n) \mathbin{\in\!\!\!\!+} \langle \tau + nz\rangle$$

is equal to

$$\mathfrak{cpe}(n) \otimes E^2\mathfrak{g}^*_{-1} \text{ or } (\mathfrak{spe}(n) \mathbin{\in\!\!\!\!+} \langle \tau + nz\rangle) \otimes E^2\mathfrak{g}^*_{-1},$$

respectively. Note that $\operatorname{Im} \partial^{3,1}_{\mathfrak{cpe}(n)} \simeq \operatorname{Im} \partial^{3,1}_{\mathfrak{spe}(n) \mathbin{\in\!\!\!\!+} \langle \tau + nz\rangle}$ as $\mathfrak{spe}(n)$-modules by part b) of Theorem 8.1.1.2 and eq. (8.12).[5] $\boxed{5}$?

Note also that by Lemma 8.1.1.4 the Jordan-Hölder series for E^2V^* contains $\mathfrak{spe}(n)$-modules with highest weights $2\varepsilon_1$ and 0. Therefore, the Jordan-Hölder series of the $\mathfrak{spe}(n)$-module $H^{2,2}_{\mathfrak{cpe}(n)}$, as compared with that of $H^{2,2}_{\mathfrak{spe}(n) \mathbin{\in\!\!\!\!+} \langle \tau + nz\rangle}$, can additionally contain only the irreducible $\mathfrak{spe}(n)$-modules with highest weights $2\varepsilon_1$ and 0. But we have shown that $H^{2,2}_{\mathfrak{cpe}(n)}$ has no trivial $\mathfrak{gl}(n)$-submodule and therefore, $H^{2,2}_{\mathfrak{cpe}(n)}$ and $H^{2,2}_{\mathfrak{spe}(n) \mathbin{\in\!\!\!\!+} \langle \tau + nz\rangle}$ can only differ by an irreducible $\mathfrak{spe}(n)$-submodule with highest weight $2\varepsilon_1$, which, being considered as $\mathfrak{sl}(n)$-module, is the sum of irreducible $\mathfrak{sl}(n)$-submodules with highest weights

$$2\varepsilon_1,\ \varepsilon_1 - \varepsilon_n, \text{ and } -\varepsilon_{n-1} - \varepsilon_n \text{ (see (1.4.4))}.$$

$\square$

8.1.3.10. Lemma. *$H^{2,2}_{\mathfrak{spe}(n) \mathbin{\in\!\!\!\!+} \langle \tau + nz\rangle}$ has no irreducible $\mathfrak{sl}(n)$-submodule with highest weight $2\varepsilon_1$.*

Proof. Making use of Lemma 8.1.3.8, let us prove that $E_\infty^{1,1}$ considered as an $\mathfrak{sl}(n)$-module has no irreducible component with highest weight $2\varepsilon_1$ in the case where $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle \tau + nz \rangle$.

Indeed, in Lemma 8.1.3.8 we have shown that if $w_{2\varepsilon_1}$ is the $\mathfrak{sl}(n)$-highest vector of weight $2\varepsilon_1$ in $E_2^{1,1}$ such that $dw_{2\varepsilon_1} \neq 0$ then $dw_{2\varepsilon_1} \in dZ_1^{2,0}$ (see (8.27)).

According to Table 2, $H^0(V_0, \mathfrak{g}_*(\mathfrak{g}_{-1}, \mathfrak{g}_0))$ has no irreducible $\mathfrak{sl}(n)$-submodule with highest weight 0 when $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle \tau + nz \rangle$ Therefore, by Table 3 $E_1^{2,0}$ has no irreducible $\mathfrak{sl}(n)$-submodule with highest weight $2\varepsilon_1$. Since $Z_1^{2,0} = E_1^{2,0}$, then $Z_1^{2,0}$ and $dZ_1^{2,0}$ have no such component either. Thus, by (8.24) $d_2^{1,1} w_{2\varepsilon_1} \neq 0$ and therefore, $w_{2\varepsilon_1} \notin E_3^{1,1} = E_\infty^{1,1}$. It remains to show that $dw_{2\varepsilon_1} \neq 0$. Recall that $w_{2\varepsilon_1} = u_{2\varepsilon_1} + t_{2\varepsilon_1} + s_{2\varepsilon_1}$, where $u_{2\varepsilon_1} \in E_1^{1,1}, t_{2\varepsilon_1} \in Z_0^{2,0}$ and $s_{2\varepsilon_1} \in dZ_0^{1,0}$ are $\mathfrak{sl}(n)$-highest vectors (see (8.26)). By Tables 2 and 4

$$u_{2\varepsilon_1} = 2\sum_{i=1}^n e_1\tilde{e}_i \otimes \tilde{f}_i \wedge \tilde{e}_1 - (n+1)\sum_{i=1}^n e_1\tilde{e}_1 \otimes \tilde{f}_i \wedge \tilde{e}_i. \tag{8.30}$$

Therefore,

$$\begin{aligned} &du_{2\varepsilon_1}(e_1, f_1, f_1) = -u_{2\varepsilon_1}(f_1, f_1)(e_1) - 2u_{2\varepsilon_1}(e_1, f_1)(f_1) = \\ &- 2(e_1\tilde{e}_1 - \frac{n+1}{2}e_1\tilde{e}_1)(f_1) = (n-1)e_1 \neq 0. \end{aligned}$$

Since $s_{2\varepsilon_1} \in dZ_0^{1,0}$, then $ds_{2\varepsilon_1} = 0$. Hence, if $dw_{2\varepsilon_1} = 0$, then $t_{2\varepsilon_1} \neq 0$ and since $u_{2\varepsilon_1}$ is an even vector, then the vector $t_{2\varepsilon_1}$ must be even.

By Table 6 the space $Z_0^{2,0} = E_0^{2,0}$ has 4 irreducible $\mathfrak{sl}(n)$-submodules with highest weight $2\varepsilon_1$. The corresponding highest vectors are

$$\begin{aligned} &(\sum_{i=2}^n e_i \wedge \tilde{f}_i - (n-1)e_1 \wedge \tilde{f}_1) \otimes \tilde{e}_1^2 - n\sum_{i=2}^n e_1 \wedge \tilde{f}_i \otimes \tilde{e}_1\tilde{e}_i, \\ &\tau \otimes \tilde{e}_1^2, z \otimes \tilde{e}_1^2, \text{ and } \mathfrak{g}_2 \otimes \tilde{e}_1^2. \end{aligned}.$$

Only the first three of these vectors are even. Therefore, if we confine ourselves to the case $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle \tau + nz \rangle$, we see that there should be two $\mathfrak{sl}(n)$-highest vectors of weight $2\varepsilon_1$ in $E_0^{2,0}$. Let

$$\begin{aligned} t_{2\varepsilon_1} = {} & k_1((\sum_{i=2}^n e_i \wedge \tilde{f}_i - (n-1)e_1 \wedge \tilde{f}_1) \otimes \tilde{e}_1^2 - n\sum_{i=2}^n e_1 \wedge \tilde{f}_i \otimes \tilde{e}_1\tilde{e}_i) \\ &+ k_2((\tau + nz) \otimes \tilde{e}_1^2), \text{ where } k_1, \ k_2 \in \mathbb{C}, \end{aligned}$$

be a linear combination of these vectors. Note that

$$\begin{aligned} &du_{2\varepsilon_1}(e_2, f_2, f_1) = -u_{2\varepsilon_1}(f_2, f_1)(e_2) - u_{2\varepsilon_1}(e_2, f_1)(f_2) - u_{2\varepsilon_1}(e_2, f_2)(f_1) = \\ &- 2(\frac{1}{2}e_1\tilde{e}_2)(f_2) + (n+1)(\frac{1}{2}e_1\tilde{e}_1)(f_1) = -\frac{1}{2}e_1 + \frac{n+1}{2}e_1 = \frac{n}{2}e_1, \end{aligned}$$

$$\begin{aligned} &dt_{2\varepsilon_1}(e_2, f_2, f_1) = -t_{2\varepsilon_1}(f_2, f_1)(e_2) - t_{2\varepsilon_1}(e_2, f_1)(f_2) - t_{2\varepsilon_1}(e_2, f_2)(f_1) = \\ &- \frac{1}{2}k_1 n e_1 \wedge \tilde{f}_2(e_2) = -\frac{1}{4}k_1 n e_1. \end{aligned}$$

Therefore, if $dw_{2\varepsilon_1} = 0$, then $k_1 = 2$. Observe that

$$\begin{aligned} &dt_{2\varepsilon_1}(e_1, f_1, f_1) = -t_{2\varepsilon_1}(f_1, f_1)(e_1) - 2t_{2\varepsilon_1}(e_1, f_1)(f_1) = \\ &- k_1(n-1)(e_1 \wedge \tilde{f}_1)(e_1) + k_2(\tau + nz)(e_1) = (-k_1\frac{n-1}{2} + k_2(n+1))e_1. \end{aligned}$$

Since $du_{2\varepsilon_1}(e_1, f_1, f_1) = (n-1)e_1$, then

$$-(n-1) + k_2(n+1) + (n-1) = 0.$$

Hence, $k_2 = 0$. But then

$$\begin{aligned} &dw_{2\varepsilon_1}(f_1, f_1, f_1) = dt_{2\varepsilon_1}(f_1, f_1, f_1) = -3t_{2\varepsilon_1}(f_1, f_1)(f_1) = \\ &- 3k_1(n-1)e_1 \wedge \tilde{f}_1(f_1) = \frac{3}{2}k_1(n-1)f_1 = 3(n-1)f_1 \neq 0. \end{aligned}$$

This proves Lemma 8.1.3.10. □

Lemma 8.1.3.10 implies that $H^{2,2}_{\mathfrak{cpe}(n)}$ and $H^{2,2}_{\mathfrak{spe}(n) \Subset \langle \tau + nz \rangle}$ differ by an irreducible $\mathfrak{spe}(n)$-module with highest weight $2\varepsilon_1$. Thus, $H^{2,2}_{\mathfrak{spe}(n) \Subset \langle \tau + nz \rangle}$ can only contain irreducible $\mathfrak{sl}(n)$-submodules with highest weights

$$\begin{aligned} &2\varepsilon_1 + 2\varepsilon_2, \ 2\varepsilon_1 + \varepsilon_2 - \varepsilon_n, \ \varepsilon_1 + \varepsilon_2 - 2\varepsilon_n, \\ &- 2\varepsilon_{n-1} - 2\varepsilon_n, \ 2\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n, \ \varepsilon_1 - \varepsilon_{n-1} - 2\varepsilon_n \end{aligned}$$

each with multiplicity not greater than 1, and the multiplicity of the submodule with highest weight $2\varepsilon_1 + 2\varepsilon_2$ is precisely 1.

8.1.3.11. Lemma. *The irreducible $\mathfrak{pe}(n)$-module with highest weight $2\varepsilon_1+2\varepsilon_2$ is the direct sum of irreducible $\mathfrak{gl}(n)$-modules with the following highest weights:*

a) *for $n > 3$:*

$$\begin{aligned} &2\varepsilon_1 + 2\varepsilon_2, \ 2\varepsilon_1 + \varepsilon_2 - \varepsilon_n, \ \varepsilon_1 + \varepsilon_2 - 2\varepsilon_n, \\ &- 2\varepsilon_{n-1} - 2\varepsilon_n, \ 2\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n, \ \varepsilon_1 - \varepsilon_{n-1} - 2\varepsilon_n; \end{aligned}$$

b) *for $n = 3$:*

$$2\varepsilon_1 + 2\varepsilon_2, \ 2\varepsilon_1 + \varepsilon_2 - \varepsilon_3, \ \varepsilon_1 + \varepsilon_2 - 2\varepsilon_3.$$

Proof. Let us consider the $\mathfrak{pe}(n)$-module $S^2(S^2V)$. Note that

$$v_{2\varepsilon_1+2\varepsilon_2} = (e_1e_2)^2 - (e_1^2)(e_2^2)$$

is a $\mathfrak{pe}(n)$-highest vector. Indeed, $B_{i,j}v_{2\varepsilon_1+2\varepsilon_2} = 0$ for any i and j. Set

$$
\begin{aligned}
v_{2\varepsilon_1+\varepsilon_2-\varepsilon_n} &= (e_1e_2)(e_1f_n) - (e_1^2)(e_2f_n),\\
v_{\varepsilon_1+\varepsilon_2-2\varepsilon_n} &= (e_1f_n)\wedge(e_2f_n),\\
v_{-2\varepsilon_{n-1}-2\varepsilon_n} &= (f_{n-1}\wedge f_n)(f_{n-1}\wedge f_n),\\
v_{2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n} &= (e_1f_{n-1})\wedge(e_1f_n) - (e_1^2)(f_{n-1}\wedge f_n),\\
v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n} &= (f_{n-1}\wedge f_n)(e_1f_n).
\end{aligned}
$$

Notice that these vectors are the $\mathfrak{gl}(n)$-highest ones. Moreover,

$$C_{2,n}v_{2\varepsilon_1+2\varepsilon_2} = 2v_{2\varepsilon_1+\varepsilon_2-\varepsilon_n}, \tag{8.31}$$
$$B_{2,n}v_{2\varepsilon_1+\varepsilon_2-\varepsilon_n} = v_{2\varepsilon_1+2\varepsilon_2}, \tag{8.32}$$
$$C_{1,n}v_{2\varepsilon_1+\varepsilon_2-\varepsilon_n} = -3v_{\varepsilon_1+\varepsilon_2-2\varepsilon_n}, \tag{8.33}$$
$$B_{1,n}v_{\varepsilon_1+\varepsilon_2-2\varepsilon_n} = -v_{2\varepsilon_1+\varepsilon_2-\varepsilon_n}. \tag{8.34}$$

If $n > 3$, then additionally

$$
\begin{aligned}
C_{2,n-1}v_{2\varepsilon_1+\varepsilon_2-\varepsilon_n} &= v_{2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n},\\
B_{2,n-1}v_{2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n} &= v_{2\varepsilon_1+\varepsilon_2-\varepsilon_n},\\
C_{1,n}v_{2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n} &= -3v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n},\\
B_{1,n}v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n} &= -v_{2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n},\\
C_{1,n-1}v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n} &= v_{-2\varepsilon_{n-1}-2\varepsilon_n},\\
B_{1,n-1}v_{-2\varepsilon_{n-1}-2\varepsilon_n} &= 2v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n}.
\end{aligned}
$$

Therefore, if $n > 3$, then the irreducible $\mathfrak{pe}(n)$-module with highest weight $2\varepsilon_1+2\varepsilon_2$ contains irreducible $\mathfrak{gl}(n)$-modules with highest weights

$$
\begin{aligned}
&2\varepsilon_1+2\varepsilon_2,\ 2\varepsilon_1+\varepsilon_2-\varepsilon_n,\ \varepsilon_1+\varepsilon_2-2\varepsilon_n,\\
&-2\varepsilon_{n-1}-2\varepsilon_n,\ 2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n,\ \varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n.
\end{aligned}
$$

We have already shown that the $\mathfrak{spe}(n)$-module $H^{2,2}_{\mathfrak{spe}(n)\subset\!\!\!\!+\langle\tau+nz\rangle}$ does contain irreducible $\mathfrak{sl}(n)$-submodules with these highest weights exactly, their multiplicities are not greater than 1, and the multiplicity of $\mathfrak{sl}(n)$-submodule with highest weight $2\varepsilon_1+2\varepsilon_2$ is precisely one.

From Tables 2 and 3 we see that the corresponding $\mathfrak{sl}(n)$-highest vector is

$$v_{2\varepsilon_1+2\varepsilon_2} = (e_1\tilde{e}_1)\otimes(\tilde{e}_2\tilde{e}_2) + (e_2\tilde{e}_2)\otimes(\tilde{e}_1\tilde{e}_1) - 2(e_1\tilde{e}_2)\otimes(\tilde{e}_1\tilde{e}_2). \tag{8.35}$$

Hence, $v_{2\varepsilon_1+2\varepsilon_2}$ is an odd $\mathfrak{spe}(n)$-highest vector. So part a) of Lemma 8.1.3.11 and Theorem 8.1.2.1 for the case where $\mathfrak{g}_0 = \mathfrak{spe}(n)\subset\!\!\!\!+\langle\tau+nz\rangle, n > 3$, are proved.

Let $n = 3$. Consider the $\mathfrak{spe}(3)$-module E^3V. As an $\mathfrak{sl}(3)$-module, this module is isomorphic to

$$E^3(V_0\oplus V_0^*) = E^3V_0 \oplus (E^2V_0)(V_0^*) \oplus V_0(S^2V_0^*) \oplus S^3V_0^*.$$

Therefore, E^3V is the direct sum of irreducible $\mathfrak{sl}(3)$-modules with the highest weights and highest vectors listed in Table 7. Note that the vectors of weights 0 and ε_1 are the $\mathfrak{spe}(3)$-highest ones. Therefore, the Jordan-Hölder series of the $\mathfrak{spe}(3)$-module E^3V contains as quotient modules the trivial and the standard ones. Notice that the vector of weight $-2\varepsilon_3$ is the $\mathfrak{spe}(3)$-highest one in the corresponding quotient module which can only contain $\mathfrak{sl}(3)$-submodules with highest weights $-2\varepsilon_3,\ \varepsilon_1-2\varepsilon_3,$ and $-3\varepsilon_3$.

Since $v_{2\varepsilon_1+2\varepsilon_2}$ is the $\mathfrak{sl}(3)$-highest vector of weight $-2\varepsilon_3$, then the relations (8.31)–(8.34) imply part b) of Lemma 8.1.3.11. □

8.1.3.12. Lemma. *For $n = 3$ we have the following non-split exact sequence of $\mathfrak{spe}(3)$-modules*

$$0 \longrightarrow \Pi(X_{2\varepsilon_1+2\varepsilon_2}) \longrightarrow H^{2,2}_{\mathfrak{spe}(3)\subset\!\!\!\!+\langle\tau+3z\rangle} \longrightarrow \Pi(X_{3\varepsilon_1}) \longrightarrow 0. \tag{8.36}$$

Proof. By part b) of Lemma 8.1.3.11 and (8.35) we see that $H^{2,2}_{\mathfrak{spe}(3)\subset\!\!\!\!+\langle\tau+3z\rangle}$ contains an irreducible $\mathfrak{spe}(3)$-module with highest weight $-2\varepsilon_3$, which being considered as an $\mathfrak{sl}(3)$-module, is the sum of irreducible $\mathfrak{sl}(3)$-components with highest weights $-2\varepsilon_3,\ \varepsilon_1-2\varepsilon_3,$ and $-3\varepsilon_3$.

In addition to these $\mathfrak{sl}(3)$-components, $H^{2,2}_{\mathfrak{spe}(3)\subset\!\!\!\!+\langle\tau+3z\rangle}$ can only contain irreducible $\mathfrak{sl}(3)$-components with the following highest weights:

$$3\varepsilon_1,\ 2\varepsilon_1,\ \text{and}\ 2\varepsilon_1-\varepsilon_3.$$

Let us show that these components are indeed contained in $H^{2,2}_{\mathfrak{spe}(3)\subset\!\!\!\!+\langle\tau+3z\rangle}$, and that their sum is an irreducible $\mathfrak{spe}(3)$-quotient module with highest weight $3\varepsilon_1$.

First, note that $E^{0,2}_\infty = E^{0,2}_4$ has an irreducible $\mathfrak{gl}(n)$-module with highest weight $2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n$. In fact, by Table 5 such a submodule is contained in $E^{0,2}_1$ but is not contained in $E^{1,2}_1$. Therefore, by (8.18) it is contained in $E^{0,2}_2$.

According to Table 4, in $E^{2,1}_1$ there is no submodule with highest weight $2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n$, hence such a submodule is not contained in $E^{2,1}_2$ either. Therefore, by (8.18) the submodule with this highest weight is contained in $E^{0,2}_3$.

According to Table 6, $E^{3,0}_0$ has no submodule with highest weight $2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n$, hence it is not contained in $E^{3,0}_3$ either. Therefore, by (8.18) it is contained in $E^{0,2}_\infty = E^{0,2}_4$.

By Tables 2 and 5 the $\mathfrak{gl}(n)$-highest vectors in $E^{0,2}_1$ of weights

$$-2\varepsilon_{n-1}-2\varepsilon_n,\ 2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n,\ \text{and}\ \varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n$$

are, respectively,

$$
\begin{aligned}
v_{-2\varepsilon_{n-1}-2\varepsilon_n} &= (f_{n-1}\wedge\tilde{f}_n)\otimes(\tilde{f}_{n-1}\wedge\tilde{f}_n),\\
v_{2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n} &= (e_1\tilde{e}_1)\otimes(\tilde{f}_{n-1}\wedge\tilde{f}_n),\ \text{and}\\
v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n} &= (e_1\wedge\tilde{f}_n)\otimes(\tilde{f}_{n-1}\wedge\tilde{f}_n).
\end{aligned}
$$

Note that

$$C_{1,n}v_{2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n} = -2v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n},$$
$$B_{1,n}v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n} = -v_{2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n} - (e_1\wedge\tilde f_n)\otimes(\tilde f_{n-1}\wedge\tilde e_1),$$

$$C_{1,n-1}v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n} = v_{-2\varepsilon_{n-1}-2\varepsilon_n},$$
$$B_{1,n-1}v_{-2\varepsilon_{n-1}-2\varepsilon_n} = v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n} + (f_{n-1}\wedge\tilde f_n)\otimes(\tilde e_1\wedge\tilde f_n).$$

Therefore, for $n=3$, the components with highest weights $3\varepsilon_1$, $2\varepsilon_1$, and $2\varepsilon_1-\varepsilon_3$ constitute an irreducible quotient module with highest weight $3\varepsilon_1$.

From Tables 2 and 4 we see that the $\mathfrak{gl}(n)$-highest vector in $E_1^{1,1}$ of weight $2\varepsilon_1+\varepsilon_2-\varepsilon_n$ is

$$v_{2\varepsilon_1+\varepsilon_2-\varepsilon_n} = (e_1\tilde e_2)\otimes(\tilde e_1\wedge\tilde f_n) - (e_1\tilde e_1)\otimes(\tilde e_2\wedge\tilde f_n).$$

Observe that

$$(B_{1,2}A_{2,1} - \frac{1}{2}B_{2,2})(v_{2\varepsilon_1-\varepsilon_2-\varepsilon_3}) = 2v_{2\varepsilon_1+\varepsilon_2-\varepsilon_3}.$$

Therefore, the sequence (8.36) is non-split. This proves Lemma 8.1.3.12, and Theorem 8.1.2.1 in the case where $\mathfrak{g}_0 = \mathfrak{spe}(3) \Subset \langle\tau+3z\rangle$. □

Recall that the Jordan-Hölder series of $\mathfrak{spe}(n)$-module $H^{2,2}_{\mathfrak{cpe}(n)}$, as compared to that of $H^{2,2}_{\mathfrak{spe}(n)\Subset\langle\tau+nz\rangle}$, contains in addition the $\mathfrak{spe}(n)$-component with highest weight $2\varepsilon_1$. Recall that by (8.30) the highest $\mathfrak{gl}(n)$-vector with weight $2\varepsilon_1$ in $E_1^{1,1}$ is

$$u_{2\varepsilon_1} = 2\sum_{i=1}^{n} e_1\tilde e_i\otimes\tilde f_i\wedge\tilde e_1 - (n+1)\sum_{i=1}^{n} e_1\tilde e_1\otimes\tilde f_i\wedge\tilde e_i.$$

Note that

$$C_{n-1,n}(u_{2\varepsilon_1}) = -2(n+1)v_{2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n} + 2(e_1\wedge\tilde f_{n-1}\otimes\tilde f_n\wedge\tilde e_1 - e_1\wedge\tilde f_n\otimes\tilde f_{n-1}\wedge\tilde e_1).$$

This proves Theorem 8.1.2.1 in the case where $\mathfrak{g}_0 = \mathfrak{cpe}(n)$.

The proof of Theorem 8.1.2.1 in the case where $\mathfrak{g}_0 = \mathfrak{spe}(n)$ follows from the fact that the Jordan-Hölder series of $\mathfrak{spe}(n)$-module $H^{2,2}_{\mathfrak{spe}(n)}$, as compared to that of $H^{2,2}_{\mathfrak{spe}(n)\Subset\langle\tau+nz\rangle}$, contains in addition the $\mathfrak{spe}(n)$-component with highest weight $\varepsilon_1+\varepsilon_2$.

Finally, the proof of Theorem 8.1.2.1 in the case where $\mathfrak{g}_0 = \mathfrak{spe}(n)\Subset\langle a\tau+bz\rangle$, where a, $b\in\mathbb{C}$ are such that $a=0$, $b\neq 0$, or $a\neq 0$, $b/a\neq n$, follows from the fact that the Jordan-Hölder series of $\mathfrak{spe}(n)$-module $\operatorname{Ker}\partial^{2,2}_{\mathfrak{spe}(n)\Subset\langle a\tau+bz\rangle}$, as compared to that of $\operatorname{Ker}\partial^{2,2}_{\mathfrak{spe}(n)}$, contains in addition the $\mathfrak{spe}(n)$-component with highest weight $2\varepsilon_1$.

8.2. The analogues of Penrose's tensors

8.2.1. Standard $\mathbb{Z}$-grading of $\mathfrak{sl}(m|n)$ and the corresponding Cartan prolongations. Let $V=V(m|0)$ and $U=U(0|n)$ be the standard (identity) $\mathfrak{gl}(m)$- and $\mathfrak{gl}(n)$-modules. (Hereafter $\mathfrak{gl}(m)=\mathfrak{gl}(m|0)$, $\mathfrak{gl}(n)=\mathfrak{gl}(0|n)$, etc.)

In what follows we will consider the standard (compatible) $\mathbb{Z}$-grading of $\mathfrak{g}=\mathfrak{sl}(m|n)$ with $m\leq n$ and let the degrees of all even roots be zero. This yields the $\mathbb{Z}$-grading of the form:

$$\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1,\ \text{where } \mathfrak{g}_0=\mathfrak{sl}(m)\oplus\mathfrak{sl}(n)\oplus\mathbb{C},\ \mathfrak{g}_{-1}=\mathfrak{g}_1^*=U\otimes V^*.$$

Let $\hat{\mathfrak{g}}_0$ be the Levi subalgebra of $\mathfrak{g}_0$, i.e., $\hat{\mathfrak{g}}_0=\mathfrak{sl}(m)\oplus\mathfrak{sl}(n)$. The weights are given with respect to the bases $\varepsilon_1,\dots,\varepsilon_m$ and $\delta_1,\dots,\delta_n$ of the dual spaces to the maximal tori of $\mathfrak{gl}(m|n)$. Let $e_1,\dots,e_m$ be the weight basis of V and $f_1,\dots,f_n$ be the weight basis of U. Let $\tilde e_1,\dots,\tilde e_m$ and $\tilde f_1,\dots,\tilde f_n$ be the bases of the dual spaces to V and U, respectively, normed so that $\tilde e_i(e_j)=\tilde f_i(f_j)=\delta_{ij}$. If $\bigoplus\limits_\lambda k_\lambda V_\lambda$ is a direct sum of irreducible $\mathfrak{g}_0$-modules (here k_λ is the multiplicity of V_λ) with highest weight λ, denote by v^i_λ the highest weight vectors of the corresponding components: $i=1,\dots,k_\lambda$. We will often represent the elements of $\mathfrak{gl}(m|n)$ by the matrices

$$X=\operatorname{diag}(A,\ D)+\operatorname{antidiag}(B,\ C)$$

where the dimensions of the matrices A, B, C, and D are $m\times m$, $m\times n$, $n\times m$ and $n\times n$, respectively. Denote by $A_{i,j}$ the matrix X whose components B, C, and D are zero and all the entries of A are also zero except for the $(i,\ j)$-th. The matrices $B_{i,j}$, $C_{i,j}$, and $D_{i,j}$ are defined similarly.

8.2.1.1. Theorem. a) *If $m=1$, $n>1$, then $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)=\mathfrak{vect}(0|n)$, $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}}_0)=\mathfrak{svect}(0|n)$;*

b) *if $m,\ n>1$ and $m\neq n$, then $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)=\mathfrak{g}$, $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}}_0)=\mathfrak{g}_{-1}\oplus\hat{\mathfrak{g}}_0$;*

c) *if $m=n=2$, then*

$$\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}}_0)=\mathfrak{h}(0|4)\quad\text{and } \mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)=S^*(\mathfrak{g}_{-1}^*)\Subset\mathfrak{h}(0|4);$$

d) *if $m=n>2$, then*

$$\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}}_0)=\mathfrak{psl}(n|n)\quad\text{and } \mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)=S^*(\mathfrak{g}_{-1}^*)\Subset\mathfrak{psl}(n|n).$$

Proof. Consider all cases mentioned in Theorem 8.2.1.1.

8.2.1.2. $m=1$, $n\geq 2$. Then $\mathfrak{g}_0=\mathfrak{sl}(n)\oplus\mathbb{C}=\mathfrak{gl}(n)$ and $\hat{\mathfrak{g}}_0=\mathfrak{sl}(n)$, where $\mathfrak{g}_{-1}$ is the standard $\mathfrak{g}_0$ (or $\hat{\mathfrak{g}}_0$) module. Therefore, $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)=\mathfrak{vect}(0|n)$, $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}}_0)=\mathfrak{svect}(0|n)$.

Notice that if $m\neq n$, then

$$\mathfrak{sl}(m|n)\subset\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)\tag{8.37}$$

and if $m = n$, then
$$\mathfrak{psl}(n|n) \subset \mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0}). \tag{8.38}$$
Indeed, the Lie superalgebras $\mathfrak{sl}(m|n)$, where $m \neq n$, and $\mathfrak{psl}(n|n)$ are simple and therefore, they are transitive (i.e., if there exists $g \in \mathfrak{g}_i$ $(i \geq 0)$ such that $[\mathfrak{g}_{-1},\ g] = 0$, then $g = 0$. It follows that $\mathfrak{g}_1$ is embedded into $\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*$ (or $\hat{\mathfrak{g}_0} \otimes \mathfrak{g}_{-1}^*$). The Jacobi identity implies $\mathfrak{g}_1 \subset \mathfrak{g}_{-1} \otimes S^2\mathfrak{g}_{-1}^*$.

8.2.1.3. Calculation of the first term of the Cartan prolongation for $m,\ n \geq 2,\ m \neq n$. Let $\mathfrak{g}_1'$ be the first term of the Cartan prolongation of the pair $(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$. Let us show that $\mathfrak{g}_1' = \mathfrak{g}_1$. By definition,

$$\begin{aligned}&\mathfrak{g}_1' = (\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*) \cap (\mathfrak{g}_{-1} \otimes S^2\mathfrak{g}_{-1}^*), \text{ where, as } \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)\text{-modules },\\ &\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^* \cong [(V \otimes V^*)/\mathbb{C} \oplus (U \otimes U^*)/\mathbb{C} \oplus \mathbb{C}] \otimes (U^* \otimes V).\end{aligned}$$

Note that if $g \in \mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*$, then

$$g \in \mathfrak{g}_{-1} \otimes S^2\mathfrak{g}_{-1}^* \text{ if and only if } g(g_1)(g_2) = -g(g_2)(g_1) \text{ for any } g_1, g_2 \in \mathfrak{g}_{-1},$$

since $\mathfrak{g}_{-1}$ is purely odd.

8.2.2. Lemma. *The $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-module $\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*$ is the direct sum of irreducible submodules whose highest weights and respective vectors are listed in Table 8.*

Proof. The proof of the Lemma consists of: *a)* a verification of the fact that vectors v_λ from Table 8 are indeed highest with respect to $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$, i.e. $A_{i,j}v_\lambda = D_{i,j}v_\lambda = 0$ for $i < j$;

b) a calculation of dimension of $\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*$ and of dimensions of the irreducible submodules of $\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*$ by the formula from Appendix.

Let us show that if

$$\lambda = 2\varepsilon_1 - \varepsilon_m - \delta_n,\ \varepsilon_1 + \delta_1 - 2\delta_n,\ \varepsilon_1 + \varepsilon_2 - \varepsilon_m - \delta_n \text{ (if } m \geq 3),$$

or
$$\lambda = \varepsilon_1 + \delta_1 - \delta_{n-1} - \delta_n \text{ (if } n \geq 3),$$
then $v_\lambda \notin \mathfrak{g}_1'$. For this it suffices to indicate $g_1, g_2 \in \mathfrak{g}_{-1}$ such that

$$v_\lambda(g_1)(g_2) \neq -v_\lambda(g_2)(g_1) \tag{8.39}$$

or, perhaps, there exists just one $g \in \mathfrak{g}_{-1}$ such that

$$v_\lambda(g)(g) \neq 0. \tag{8.40}$$

Let $\lambda = 2\varepsilon_1 - \varepsilon_m - \delta_n$. Then

$$v_\lambda(f_n \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_1) = A_{1,m}(f_n \otimes \tilde{e}_1) = -f_n \otimes \tilde{e}_m \neq 0.$$

If $\lambda = \varepsilon_1 + \delta_1 - 2\delta_n$, then

$$v_\lambda(f_n \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_1) = D_{1,n}(f_n \otimes \tilde{e}_1) = f_1 \otimes \tilde{e}_1 \neq 0.$$

If $\lambda = \varepsilon_1 + \varepsilon_2 - \varepsilon_m - \delta_n$ (for $m \geq 3$), then

$$\begin{aligned}&v_\lambda(f_n \otimes \tilde{e}_2)(f_{n-1} \otimes \tilde{e}_1) = A_{1,m}(f_{n-1} \otimes \tilde{e}_1) = -f_{n-1} \otimes \tilde{e}_m,\\ &\text{but } v_\lambda(f_{n-1} \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_2) = 0.\end{aligned}$$

Finally, if $\lambda = \varepsilon_1 + \delta_1 - \delta_{n-1} - \delta_n$ (for $n \geq 3$), then

$$\begin{aligned}&v_\lambda(f_n \otimes \tilde{e}_1)(f_{n-1} \otimes \tilde{e}_2) = D_{1,n-1}(f_{n-1} \otimes \tilde{e}_2) = f_1 \otimes \tilde{e}_2,\\ &\text{but } v_\lambda(f_{n-1} \otimes \tilde{e}_2)(f_n \otimes \tilde{e}_1) = 0.\end{aligned}$$

Now, let us show that if $\lambda = \varepsilon_1 - \delta_n$, then $\mathfrak{g}_1'$ contains precisely one irreducible $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-module with highest weight λ. Notice that by (8.37) $\mathfrak{g}_1'$ contains at least one such module. Let

$$v_\lambda = k_1 v_\lambda^1 + k_2 v_\lambda^2 + k_3 v_\lambda^3, \text{ where } k_1, k_2, k_3 \in \mathbb{C},$$

be a linear combination of highest vectors of weight λ. Then the condition

$$v_\lambda(f_n \otimes \tilde{e}_2)(f_{n-1} \otimes \tilde{e}_1) = -v_\lambda(f_{n-1} \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_2)$$

implies
$$mk_1 = nk_2, \tag{8.41}$$
whereas the condition

$$v_\lambda(f_n \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_1) = 0$$

implies
$$k_1(m-1) + k_2(1-n) + k_3(m-n) = 0.$$
Hence,
$$k_2 = mk_1/n \text{ and } k_3 = -k_1/n. \tag{8.42}$$
Thus, $\mathfrak{g}_1' = V_{\varepsilon_1 - \delta_n}$ and $\mathfrak{g}_1' = \mathfrak{g}_1$. $\square$

8.2.2.1. Calculation of the second term of the Cartan prolongation for $m,\ n \geq 2,\ m \neq n$. Let $\mathfrak{g}_2$ be the second term of the Cartan prolongation of $(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$.

Let us show that $\mathfrak{g}_2 = 0$. Indeed, $\mathfrak{g}_2 := (\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}^*) \cap (\mathfrak{g}_0 \otimes S^2\mathfrak{g}_{-1}^*)$. Notice that, as $\mathfrak{g}_0$-module,

$$\begin{aligned}&\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}^* \cong (U^* \otimes V) \otimes (U^* \otimes V) =\\ &S^2U^* \otimes S^2V \oplus \Lambda^2U^* \otimes \Lambda^2V \oplus \Lambda^2U^* \otimes S^2V \oplus S^2U^* \otimes \Lambda^2V.\end{aligned}$$

This decomposition and Table 5 of [OV] imply the following

Lemma. *The $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-module $(U^* \otimes V) \otimes (U^* \otimes V)$ is the direct sum of irreducible submodules whose highest weights and the corresponding highest vectors are listed in Table 9.*

Let us show that $v_\lambda \notin \mathfrak{g}_2$, where v_λ is any of the highest vectors listed in Table 9. Let us indicate $g_1, g_2 \in \mathfrak{g}_{-1}$ for which either (8.6) or (8.40) holds.

Let $\lambda = 2\varepsilon_1 - 2\delta_n$. Then

$$v_\lambda(f_n \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_2) = B_{1,n}(f_n \otimes \tilde{e}_2) = e_1 \otimes \tilde{e}_2, \text{ but } v_\lambda(f_n \otimes \tilde{e}_2)(f_n \otimes \tilde{e}_1) = 0.$$

If $\lambda = \varepsilon_1 + \varepsilon_2 - 2\delta_n$, then

$$v_\lambda(f_n \otimes \tilde{e}_2)(f_n \otimes \tilde{e}_2) = B_{1,n}(f_n \otimes \tilde{e}_2) = e_1 \otimes \tilde{e}_2 \neq 0.$$

If $\lambda = 2\varepsilon_1 - \delta_{n-1} - \delta_n$, then

$$v_\lambda(f_{n-1} \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_2) = -B_{1,n}(f_n \otimes \tilde{e}_2) = -e_1 \otimes \tilde{e}_2,$$

but $v_\lambda(f_n \otimes \tilde{e}_2)(f_{n-1} \otimes \tilde{e}_1) = 0$. Let $\lambda = \varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$. Then if $n > 2$, we have

$$v_\lambda(f_{n-1} \otimes \tilde{e}_2)(f_1 \otimes \tilde{e}_1) = B_{1,n}(f_1 \otimes \tilde{e}_1) = f_1 \otimes \tilde{f}_n,$$

but $v_\lambda(f_1 \otimes \tilde{e}_1)(f_{n-1} \otimes \tilde{e}_2) = 0$. If $m > 2$, then

$$v_\lambda(f_{n-1} \otimes \tilde{e}_2)(f_n \otimes \tilde{e}_m) = B_{1,n}(f_n \otimes \tilde{e}_m) = e_1 \otimes \tilde{e}_m,$$

but $v_\lambda(f_n \otimes \tilde{e}_m)(f_{n-1} \otimes \tilde{e}_2) = 0$. Therefore, $\mathfrak{g}_2 = 0$ and $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = \mathfrak{g}$. Note that by (8.42) we have $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0}) = \mathfrak{g}_{-1} \oplus \hat{\mathfrak{g}_0}$. This proves part b) of Theorem 8.2.1.1.

8.2.2.2. $\boldsymbol{m = n}$. Let $m = n = 2$. Since $\hat{\mathfrak{g}_0} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) = \mathfrak{o}(4)$ and $\mathfrak{g}_{-1}$ is the standard $\mathfrak{o}(4)$-module (considered as purely odd superspace), then $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0}) = \mathfrak{h}(0|4)$.

Let $m = n > 2$ and $\mathfrak{g}_1'$ be the first term of the Cartan prolongation of the pair $(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0})$. Let us show that $\mathfrak{g}_1' = \mathfrak{g}_1$. Indeed, by (8.38) $\mathfrak{g}_1 \subset \mathfrak{g}_1'$. By the results of 8.2.1.3 and Table 8 we see that the only highest weights of $\mathfrak{g}_1'$ are all equal to $\varepsilon_1 - \delta_n$. Then formula (8.41) implies that the highest vector of such weight in $\mathfrak{g}_1'$ is precisely one and therefore, $\mathfrak{g}_1' = \mathfrak{g}_1$. By the results of 8.2.2.1 the second term of the Cartan prolongation of the pair $(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0})$ is zero. Hence, for $m = n > 2$, we have $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0}) = \mathfrak{psl}(n|n)$.

Let $m = n > 1$ and let

$$\mathfrak{g}_k = (\mathfrak{g}_0 \otimes S^k\mathfrak{g}_{-1}^*) \cap (\mathfrak{g}_{-1} \otimes S^{k+1}\mathfrak{g}_{-1}^*) \quad \text{for } k \geq 1$$

be the k-th term of the Cartan prolongation of the pair $(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$. Observe that

$$\mathfrak{g}_0 \otimes S^k\mathfrak{g}_{-1}^* = (\hat{\mathfrak{g}_0} \oplus \langle z\rangle) \otimes S^k\mathfrak{g}_{-1}^*, \text{ where } z = 1_{2n} \text{ is the center of } \mathfrak{sl}(n|n).$$

Note that

$$\langle z\rangle \otimes S^k\mathfrak{g}_{-1}^* \subset \mathfrak{g}_{-1} \otimes S^{k+1}\mathfrak{g}_{-1}^*.$$

Therefore,

$$\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = S^*(\mathfrak{g}_{-1}^*) \Subset \mathfrak{g}_*(\mathfrak{g}_{-1}, \hat{\mathfrak{g}_0}).$$

□

8.2.3. Structure functions of Lie superalgebras $\mathfrak{vect}(0|n)$ and $\mathfrak{svect}(0|n)$. Let U be the purely odd standard $\mathfrak{gl}(n)$-module.

8.2.3.1. Theorem. *If $m = 1, n > 1$, then*

a) $H^{k,2}_{\mathfrak{g}_0} = 0$ for any $k > 0$;

b) $H^{k,2}_{\hat{\mathfrak{g}_0}} = \Pi^n(\mathbb{C})\delta_{kn}$.

8.2.3.2. Proof of part a). Since $\mathfrak{g}_*(\mathfrak{g}_{-1}, \mathfrak{g}_0) = \mathfrak{vect}(0|n) = \mathop{\oplus}\limits_{i=-1}^{n-1} \mathfrak{g}_i$, where $\mathfrak{g}_0 = \mathfrak{gl}(n)$, and the $\mathfrak{gl}(n)$-module $\mathfrak{g}_i$ is isomorphic to $U \otimes S^{i+1}U^*$, then for $k \geq n+2$, we have $H^{k,2}_{\mathfrak{g}_0} = 0$, and for $k \leq n+1$, there exist the following Spencer cochain sequences:

$$\begin{array}{lll}
\mathfrak{gl}(n) \otimes U^* \xrightarrow{\partial^{2,1}_{\mathfrak{gl}(n)}} U \otimes \Lambda^2U^* \xrightarrow{\partial^{1,2}_{\mathfrak{gl}(n)}} 0 & (k=1), \\
C^{k+1,1}_{\mathfrak{gl}(n)} \xrightarrow{\partial^{k+1,1}_{\mathfrak{gl}(n)}} C^{k,2}_{\mathfrak{gl}(n)} \xrightarrow{\partial^{k,2}_{\mathfrak{gl}(n)}} C^{k-1,3}_{\mathfrak{gl}(n)} & (2 \leq k \leq n), \\
0 \xrightarrow{\partial^{n+2,1}_{\mathfrak{gl}(n)}} C^{n+1,2}_{\mathfrak{gl}(n)} \xrightarrow{\partial^{n+1,2}_{\mathfrak{gl}(n)}} C^{n,3}_{\mathfrak{gl}(n)} & (k = n+1),
\end{array}$$

where

$$\begin{aligned}
C^{k+1,1}_{\mathfrak{gl}(n)} &= \mathfrak{g}_{k-1} \otimes \mathfrak{g}_{-1}^* \cong (U \otimes S^kU^*) \otimes U^*, \\
C^{k,2}_{\mathfrak{gl}(n)} &= \mathfrak{g}_{k-2} \otimes \Lambda^2\mathfrak{g}_{-1}^* \cong (U \otimes S^{k-1}U^*) \otimes \Lambda^2U^*, \\
C^{k-1,3}_{\mathfrak{gl}(n)} &= \mathfrak{g}_{k-3} \otimes \Lambda^3\mathfrak{g}_{-1}^* \cong (U \otimes S^{k-2}U^*) \otimes \Lambda^3U^*.
\end{aligned}$$

Recall (see section 8.1.3.1) that if $\mathfrak{g}_{-1}$ is a faithful $\mathfrak{g}_0$-module and [6] **[Ol: analogichno]** 6

$$\mathfrak{g}_{k-1} \otimes \mathfrak{g}_{-1}^* \xrightarrow{\partial^{k+1,1}_{\mathfrak{g}_0}} \mathfrak{g}_{k-2} \otimes \Lambda^2\mathfrak{g}_{-1}^* \xrightarrow{\partial^{k,2}_{\mathfrak{g}_0}} \mathfrak{g}_{k-3} \otimes \Lambda^3\mathfrak{g}_{-1}^* \qquad (k \geq 1)$$

is the Spencer cochain sequence corresponding to the pair $(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$, then

$$\operatorname{Im} \partial^{k+1,1}_{\mathfrak{g}_0} \cong (\mathfrak{g}_{k-1} \otimes \mathfrak{g}_{-1}^*)/\mathfrak{g}_k. \qquad (8.43)$$

Let us show that $H^{1,2}_{\mathfrak{g}_0} = 0$. Indeed, by (8.43)

$$\operatorname{Im} \partial^{2,1}_{\mathfrak{g}_0} \cong (\mathfrak{gl}(n) \otimes U^*)/(U \otimes S^2U^*) \cong U \otimes \Lambda^2U^* = \operatorname{Ker} \partial^{1,2}_{\mathfrak{g}_0}.$$

We will prove that for $2 \leq k \leq n+1$, we have $H^{k,2}_{\mathfrak{gl}(n)} = 0$, using the following

8.2.4. Lemma. *As $\mathfrak{gl}(n)$-modules $C^{k,2}_{\mathfrak{gl}(n)}$, $C^{k+1,1}_{\mathfrak{gl}(n)}$, and $\mathfrak{g}_k$, where $2 \leq k \leq n+1$, are the direct sums of the irreducible submodules whose highest weights and highest vectors are listed in Tables 10, 11, and 12, respectively.* (Here r, s, and t denote the cyclic permutations of $(n-k, \dots, n)$, $(n-k+1, \dots, n)$, and $(n-k+2, \dots, n)$, respectively.)

Proof. The proof of the Lemma consists of

a) a verification of the fact that vectors $v_{\lambda_l}, v_{\beta_l}, v_{\gamma_l}$ from Tables 10, 11, and 12 are indeed highest with respect to $\mathfrak{gl}(n)$, i.e.,

$$D_{i,j}v_{\lambda_l} = D_{i,j}v_{\beta_l} = D_{i,j}v_{\gamma_l} = 0 \quad \text{for } i < j;$$

b) a calculation of dimensions of given modules and dimensions of their irreducible submodules by the formula from Appendix.

Notice that $D_{i,j}f_p = \delta_{jp}f_i,\ D_{i,j}\tilde{f}_p = -\delta_{ip}\tilde{f}_j$. According to Table 10,

$$\lambda_1 = \delta_1 - \delta_{n-k+2} - \ldots - \delta_{n-1} - 3\delta_n,\ v_{\lambda_1} = (f_1 \otimes \tilde{f}_{n-k+2} \wedge \tilde{f}_{n-k+3} \wedge \ldots \wedge \tilde{f}_n) \otimes \tilde{f}_n^{\ 2}.$$

Then for $i < n-k+2$, we have $D_{i,j}v_{\lambda_1} = 0$. For $j > i \geq n-k+2$, we have

$$D_{i,j}(v_{\lambda_1}) = D_{i,j}(f_1 \otimes \tilde{f}_{n-k+2} \wedge \ldots \wedge \tilde{f}_i \wedge \ldots \wedge \tilde{f}_j \wedge \ldots \wedge \tilde{f}_n) \otimes \tilde{f}_n^{\ 2} =$$
$$(f_1 \otimes \tilde{f}_{n-k+2} \wedge \ldots \wedge (-\tilde{f}_j) \wedge \ldots \wedge \tilde{f}_j \wedge \ldots \wedge \tilde{f}_n) \otimes \tilde{f}_n^{\ 2} = 0.$$

Thus, v_{λ_1} is a highest vector. The proof of the fact that the other vectors from Tables 10, 11, and 12 are highest with respect to $\mathfrak{gl}(n)$ is similar.

Using the formula from Appendix we find the dimensions of the $\mathfrak{gl}(n)$-modules given in Table 10:

$$\begin{aligned}
&\dim V_{\lambda_1} = \tfrac{n(n+3)(n+1)!}{2(n-k+2)(k+1)(k-2)!(n-k)!} \text{ if } 2 \leq k \leq n,\\
&\dim V_{\lambda_1} = \tfrac{(n-1)n(n+2)}{2} \text{ if } 3 \leq k = n+1,\\
&\dim V_{\lambda_2} = \tfrac{(n+1)!}{(k-2)!((n-k+1)!k} \text{ if } 2 \leq k \leq n+1,\\
&\dim V_{\lambda_3} = \dim V_{\lambda_2} \text{ if } k = 2 \leq n,\\
&\dim V_{\lambda_3} = \tfrac{(n+2)!}{2(k-3)!(n-k+2)!k} \text{ if } 3 \leq k \leq n,\\
&\dim V_{\lambda_4} = n^2 - 1 \text{ if } 3 \leq k = n,\\
&\dim V_{\lambda_4} = \tfrac{n(n+2)n!}{(n-k+1)(k-1)!(n-k-1)!(k+1)} \text{ if } 2 \leq k \leq n-1,\\
&\dim V_{\lambda_5} = \tfrac{n!}{k!(n-k)!} \text{ if } 2 \leq k \leq n,\\
&\dim V_{\lambda_6} = \dim V_{\lambda_2} \text{ if } 3 \leq k \leq n-1.
\end{aligned} \tag{8.44}$$

Therefore, if $2 = k = n$, then

$$\dim V_{\lambda_1} + 2\dim V_{\lambda_2} + \dim V_{\lambda_5} = \frac{n^3(n+1)}{2} = \dim(U \otimes U^*) \otimes \Lambda^2 U^*,$$

$$\begin{aligned}
&\sum_{l=1}^{5} \dim V_{\lambda_l} = \tfrac{n^3(n+1)}{2} = \dim(U \otimes U^*) \otimes \Lambda^2 U^*, \text{ if } 2 = k \leq n-1,\\
&\sum_{l=1}^{2} \dim V_{\lambda_l} = \tfrac{n^2(n+1)}{2} = \dim U \otimes \Lambda^2 U^* \text{ if } 3 \leq k = n+1,\\
&\sum_{l=1}^{5} \dim V_{\lambda_l} = \tfrac{n^3(n+1)}{2} = \dim(U \otimes S^{n-1}U^*) \otimes \Lambda^2 U^* \text{ if } 3 \leq k = n;
\end{aligned}$$

if $3 \leq k \leq n-1$, then

$$\sum_{l=1}^{6} \dim V_{\lambda_l} = \frac{n^2(n+1)!}{2(n-k+1)!(k-1)!} = \dim(U \otimes S^{k-1}U^*) \otimes \Lambda^2 U^*.$$

In order to find the dimensions of the $\mathfrak{gl}(n)$-modules given in Table 11, note that if $k \geq 2$, then $\beta_1 = \lambda_4, \beta_2 = \beta_4 = \lambda_5, \beta_3 = \lambda_2$. Using the formula from Appendix we get

$$\dim V_{\beta_5} = \frac{(n+1)!}{(n-k)(k+1)!(n-k-2)!} \quad \text{for } 2 \leq k \leq n-2. \tag{8.45}$$

Therefore

$$\begin{aligned}
&\sum_{l=1}^{2} \dim V_{\beta_l} = n^2 = \dim U \otimes U^* \text{ if } 2 \leq k = n,\\
&\sum_{l=1}^{4} \dim V_{\beta_l} = n^3 = \dim(U \otimes S^{n-1}U^*) \otimes U^* \text{ if } 2 \leq k = n-1,\\
&\sum_{l=1}^{5} \dim V_{\beta_l} = \tfrac{n^2 n!}{(n-k)!k!} = \dim(U \otimes S^k U^*) \otimes U^* \text{ if } 2 \leq k \leq n-2.
\end{aligned}$$

Finally, in order to find the dimensions of the $\mathfrak{gl}(n)$-modules given in Table 12, note that if $2 \leq k \leq n-2$, then $\gamma_1 = \beta_5$, and $\gamma_2 = \lambda_5$. Therefore, if $2 \leq k \leq n-2$, then

$$\sum_{l=1}^{2} \dim V_{\gamma_l} = \frac{nn!}{(n-k-1)!(k+1)!} = \dim(U \otimes S^{k+1}U^*) = \dim \mathfrak{g}_k,$$

and if $k = n-1$, then $\dim V_{\gamma_1} = n = \dim U = \dim \mathfrak{g}_k$. □

Let $k \geq 2$ and $\lambda = \lambda_1,\ \lambda_2,\ \lambda_3$. Then

$$v_\lambda \notin \operatorname{Ker} \partial^{k,2}_{\mathfrak{gl}(n)}. \tag{8.46}$$

Indeed,

$$\partial^{k,2}_{\mathfrak{gl}(n)} v_\lambda(f_n, f_n, f_n) = -3v_\lambda(f_n, f_n)(f_n),$$

and according to Table 10, $v_\lambda(f_n, f_n)(f_n) \neq 0$. Note that if $2 \leq k = n$, then

$$\operatorname{Im} \partial^{k+1,1}_{\mathfrak{gl}(n)} = V_{\lambda_2} \oplus V_{\lambda_5}. \tag{8.47}$$

In fact, according to Table 11,

$$C^{k+1,1}_{\mathfrak{gl}(n)} = V_{\lambda_2} \oplus V_{\lambda_5},$$

and we get (8.47) by (8.43), since $\mathfrak{g}_k = 0$. Note that if $2 \leq k \leq n-1$, then

$$\operatorname{Im} \partial^{k+1,1}_{\mathfrak{gl}(n)} = V_{\lambda_2} \oplus V_{\lambda_4} \oplus V_{\lambda_5}. \tag{8.48}$$

Indeed, according to Table 11, if $2 \le k = n-1$, then

$$C^{k+1,1}_{\mathfrak{gl}(n)} = V_{\lambda_2} \oplus V_{\lambda_4} \oplus 2V_{\lambda_5}, \tag{8.49}$$

and if $2 \le k \le n-2$, then

$$C^{k+1,1}_{\mathfrak{gl}(n)} = V_{\lambda_2} \oplus V_{\lambda_4} \oplus 2V_{\lambda_5} \oplus V_{\beta_5}. \tag{8.50}$$

Since by Table 12

$$\begin{aligned} \mathfrak{g}_k &= V_{\lambda_5} \text{ for } 2 \le k = n-1, \\ \mathfrak{g}_k &= V_{\lambda_5} \oplus V_{\beta_5} \text{ for } 2 \le k \le n-2, \end{aligned}$$

we get (8.48) by (8.43).

We will show now that for $2 \le k \le n+1$, we have

$$\operatorname{Ker} \partial^{k,2}_{\mathfrak{gl}(n)} = \operatorname{Im} \partial^{k+1,1}_{\mathfrak{gl}(n)}. \tag{8.51}$$

Let

$$\operatorname{Ker} \partial^{k,2}_{\mathfrak{gl}(n)} = \bigoplus_{\lambda} k_\lambda V_\lambda. \tag{8.52}$$

Let $k=2$. According to Table 10, if $n=2$, then

$$C^{k,2}_{\mathfrak{gl}(n)} = V_{\lambda_1} \oplus 2V_{\lambda_2} \oplus V_{\lambda_5}, \tag{8.53}$$

and if $n \ge 3$, then

$$C^{k,2}_{\mathfrak{gl}(n)} = V_{\lambda_1} \oplus 2V_{\lambda_2} \oplus V_{\lambda_4} \oplus V_{\lambda_5}.$$

Therefore, in (8.52) $k_{\lambda_5} \le 1$, and by (8.46) $k_{\lambda_1} = 0, k_{\lambda_2} \le 1$. Note that if $n=2$, then $k_{\lambda_4} = 0$, and if $n \ge 3$, then $k_{\lambda_4} \le 1$. Thus, by (8.47) and (8.48) we get (8.51).

Let $k \ge 3$. Then according to Table 10, if $k = n+1$, then

$$C^{k,2}_{\mathfrak{gl}(n)} = V_{\lambda_1} \oplus V_{\lambda_2}.$$

Hence, by (8.46) $\operatorname{Ker} \partial^{k,2}_{\mathfrak{gl}(n)} = 0$. If $k=n$, then

$$C^{k,2}_{\mathfrak{gl}(n)} = V_{\lambda_1} \oplus 2V_{\lambda_2} \oplus V_{\lambda_3} \oplus V_{\lambda_5}. \tag{8.54}$$

Therefore, in (8.52) $k_{\lambda_5} \le 1$ and by (8.46) $k_{\lambda_1} = k_{\lambda_3} = 0, k_{\lambda_2} \le 1$. So from (8.47) we get (8.51).

Finally, if $k \le n-1$, then

$$C^{k,2}_{\mathfrak{gl}(n)} = V_{\lambda_1} \oplus 2V_{\lambda_2} \oplus V_{\lambda_3} \oplus V_{\lambda_4} \oplus V_{\lambda_5}.$$

Therefore, in (8.52) $k_{\lambda_4} \le 1$, $k_{\lambda_5} \le 1$ and by (8.46) $k_{\lambda_1} = k_{\lambda_3} = 0, k_{\lambda_2} \le 1$. Thus, by (8.48) we get (8.51). This proves part a) of Theorem 8.2.3.1.

8.2.4.1. Proof of part b) of Theorem 8.2.3.1. Note that

$$\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}}_0) = \mathfrak{svect}(0|n) = \bigoplus_{i=-1}^{n-2} \mathfrak{g}_i,$$

where $\hat{\mathfrak{g}}_0 = \mathfrak{sl}(n)$, and the $\mathfrak{sl}(n)$-module $\mathfrak{g}_i$ is isomorphic to U if $i=-1$ and to $V_{\delta_1 - \delta_{n-k} - \delta_{n-k+1} - \ldots - \delta_n}$ if $0 \le k \le n-2$.

Hence, for $k \ge n+1$, we have $H^{k,2}_{\mathfrak{sl}(n)} = 0$ and for $1 \le k \le n$, there exist the following Spencer cochain sequences:

$$\begin{array}{ll} \mathfrak{sl}(n) \otimes U^* \xrightarrow{\partial^{2,1}_{\mathfrak{sl}(n)}} U \otimes \Lambda^2 U^* \xrightarrow{\partial^{1,2}_{\mathfrak{sl}(n)}} 0 & (k=1), \\ C^{k+1,1}_{\mathfrak{sl}(n)} \xrightarrow{\partial^{k+1,1}_{\mathfrak{sl}(n)}} C^{k,2}_{\mathfrak{sl}(n)} \xrightarrow{\partial^{k,2}_{\mathfrak{sl}(n)}} C^{k-1,3}_{\mathfrak{sl}(n)} & (2 \le k \le n-1), \\ 0 \xrightarrow{\partial^{n+1,1}_{\mathfrak{sl}(n)}} C^{n,2}_{\mathfrak{sl}(n)} \xrightarrow{\partial^{n,2}_{\mathfrak{sl}(n)}} C^{n-1,3}_{\mathfrak{sl}(n)} & (k=n). \end{array}$$

First, we will show that $H^{1,2}_{\mathfrak{sl}(n)} = 0$. In fact, since $\mathfrak{g}_1 \cong V_{\delta_1 - \delta_{n-1} - \delta_n}$, then by (8.43)[7] 7 ?

$$\operatorname{Im} \partial^{2,1}_{\mathfrak{sl}(n)} \cong (\mathfrak{sl}(n) \otimes U^*)/V_{\delta_1 - \delta_{n-1} - \delta_n}.$$

Since $\dim \mathfrak{sl}(n) \otimes U^* = (n^2-1)n$ and by the formula from Appendix

$$\dim V_{\delta_1 - \delta_{n-1} - \delta_n} = \frac{n(n+1)(n-2)}{2},$$

then

$$\begin{aligned} &\dim \operatorname{Im} \partial^{2,1}_{\mathfrak{sl}(n)} = (n^2-1)n - \frac{n(n+1)(n-2)}{2} = \frac{n^3+n^2}{2} = \\ &\dim U \otimes \Lambda^2 U^* = \dim \operatorname{Ker} \partial^{1,2}_{\mathfrak{sl}(n)}. \end{aligned}$$

Next, we will prove that for $2 \le k \le n-1$, we have $H^{k,2}_{\mathfrak{sl}(n)} = 0$.

Lemma. *If $2 \le k = n-1$, then*

$$C^{k+1,1}_{\mathfrak{sl}(n)} = V_{\lambda_4} \oplus V_{\lambda_5}. \tag{8.55}$$

If $2 \le k \le n-2$, then

$$C^{k+1,1}_{\mathfrak{sl}(n)} = V_{\lambda_4} \oplus V_{\lambda_5} \oplus V_{\beta_5}. \tag{8.56}$$

Proof. Let $c \in C^{k+1,1}_{\mathfrak{gl}(n)}$. Then $c \in C^{k+1,1}_{\mathfrak{sl}(n)}$ if and only if

$$c \in (\mathfrak{sl}(n) \otimes S^{k-1}U^*) \otimes U^*. \tag{8.57}$$

Let

$$C^{k+1,1}_{\mathfrak{sl}(n)} = \bigoplus_{\beta} k_\beta V_\beta. \tag{8.58}$$

Using the decomposition of $C^{k+1,1}_{\mathfrak{gl}(n)}$ into direct sum of irreducible $\mathfrak{gl}(n)$-modules given in (8.49) and (8.50), we check condition (8.57) for the corresponding highest vectors: let $\beta = \lambda_4 = \beta_1$, then up to a complex constant,

$$v_{\beta_1}(f_n)(f_{j_1} \wedge f_{j_2} \wedge \ldots \wedge f_{j_{k-1}}) = f_1 \otimes \tilde{f}_{j_k}, \text{ where } 2 \leq j_k \leq n.$$

Since $f_1 \otimes \tilde{f}_{j_k} \in \mathfrak{sl}(n)$, then in (8.58) $k_{\lambda_4} = 1$. Let $\beta = \lambda_2 = \beta_3$, then up to a nonzero constant,

$$v_{\beta_3}(f_n)(f_{n-k+2} \wedge f_{n-k+3} \wedge \ldots \wedge f_n) = \sum_{j=1}^{n} f_j \otimes \tilde{f}_j.$$

Since $\sum\limits_{j=1}^{n} f_j \otimes \tilde{f}_j \notin \mathfrak{sl}(n)$, then $k_{\lambda_2} = 0$. Let $\beta = \beta_5$. Then up to a constant,

$$v_{\beta_5}(f_{j_1})(f_{j_2} \wedge \ldots \wedge f_{j_k}) = f_1 \otimes \tilde{f}_{j_{k+1}}, \text{ where } 2 \leq j_{k+1} \leq n.$$

Since $f_1 \otimes \tilde{f}_{j_{k+1}} \in \mathfrak{sl}(n)$, then $k_{\beta_5} = 1$. Let us show that $k_{\lambda_5} = 1$. Indeed, by the formula from Appendix

$$\dim V_{\delta_1 - \delta n-k+1-\ldots-\delta_n} = \frac{(n+1)!}{(n-k+1)k!(n-k-1)!}.$$

Therefore,

$$\dim C^{k+1,1}_{\mathfrak{sl}(n)} = \dim(V_{\delta_1 - \delta n-k+1-\ldots-\delta_n} \otimes U^*) = \frac{n(n+1)!}{(n-k+1)k!(n-k-1)!}.$$

8 Then by (8.44) and (**??**)[8] **Ol: kakaya-to f-la mezhdu** (8.44) **i** (8.45) – **nomer propushchen!**] if $2 \leq k = n-1$, then

$$\dim C^{k+1,1}_{\mathfrak{sl}(n)} = \dim V_{\lambda_4} + \dim V_{\lambda_5},$$

9 if $2 \leq k \leq n-2$, then by (8.44), (**??**)[9] **Ol: f-la mezhdu** (8.44) **i** (8.45)], and (8.45)

$$\dim C^{k+1,1}_{\mathfrak{sl}(n)} = \dim V_{\lambda_4} + \dim V_{\lambda_5} + \dim V_{\beta_5},$$

Thus, $k_{\lambda_5} = 1$. This proves the Lemma. □

8.2.4.2. Lemma. Ker $\partial^{k,2}_{\mathfrak{sl}(n)}$ *does not contain irreducible* $\mathfrak{sl}(n)$*-submodules with highest weight* λ_2.

Proof. Let $c \in C^{k,2}_{\mathfrak{gl}(n)}$. Then $c \in C^{k,2}_{\mathfrak{sl}(n)}$ if and only if

$$c \in (\mathfrak{sl}(n) \otimes S^{k-2}U^*) \otimes \lambda^2 U^*.$$

According to Table 10, each highest vector with weight λ_2 in $C^{k,2}_{\mathfrak{gl}(n)}$ is

$$v = av_{\lambda_2} + bv_{\lambda_3}, \text{ if } k = 2 \leq n,$$
$$v = av_{\lambda_2} + bv_{\lambda_6}, \text{ if } 3 \leq k \leq n-1, \text{ where } a,\ b \in \mathbb{C}.$$

Therefore, up to a nonzero constant,

$$v(f_n, f_n)(f_{n-k+2} \wedge \ldots \wedge f_{n-1}) = (-1)^k a f_n \otimes \tilde{f}_n + b \sum_{j=1}^{n} f_j \otimes \tilde{f}_j.$$

Note that

$$(-1)^k a f_n \otimes \tilde{f}_n + b \sum_{j=1}^{n} f_j \otimes \tilde{f}_j \in \mathfrak{sl}(n)$$

if and only if $(-1)^k a + nb = 0$. But in this case $v \notin \operatorname{Ker}^{k,2}_{\mathfrak{sl}(n)}$. In fact,

$$\partial^{k,2}_{\mathfrak{sl}(n)}(f_n, f_n, f_n) = -3v(f_n, f_n)(f_n) \neq 0.$$

This proves Lemma 8.2.4.2. □

In order to prove that for $2 \leq k \leq n-1$

$$\operatorname{Im} \partial^{k+1,1}_{\mathfrak{sl}(n)} = \operatorname{Ker} \partial^{k,2}_{\mathfrak{sl}(n)}, \tag{8.59}$$

observe that if $2 \leq k = n-1$, then $\mathfrak{g}_k = 0$ and if $2 \leq k \leq n-2$, then $\mathfrak{g}_k = V_{\beta_5}$.
Thus, by (8.43),[10] (8.55), and (8.56) 10 ?

$$\operatorname{Im} \partial^{k+1,1}_{\mathfrak{sl}(n)} = V_{\lambda_4} \oplus V_{\lambda_5}.$$

Therefore, by (8.48), (8.51), and Lemma 8.2.4.2, we get (8.59).

Finally, let $2 \leq k = n$. Notice that by (8.47) and (8.51)

$$\operatorname{Ker} \partial^{n,2}_{\mathfrak{gl}(n)} = V_{\lambda_2} \oplus V_{\lambda_5}, \tag{8.60}$$

where V_{λ_5} is a trivial $\mathfrak{sl}(n)$-module. By (8.53) and (8.54) the multiplicity of λ_5 in $C^{n,2}_{\mathfrak{gl}(n)}$ is 1. Moreover, this trivial submodule is contained in $C^{n,2}_{\mathfrak{sl}(n)}$, because the $\mathfrak{sl}(n)$-module $C^{n,2}_{\mathfrak{sl}(n)}$ is isomorphic to $\Lambda^2 U \otimes \Lambda^2 U^*$, which contains a trivial $\mathfrak{sl}(n)$-submodule (generated by $\sum\limits_{i,j} f_i f_j \otimes \tilde{f}_i \tilde{f}_j$). Thus, Ker $\partial^{n,2}_{\mathfrak{sl}(n)}$ must contain a trivial submodule. According to (8.60) and Lemma 8.2.4.2, Ker $\partial^{n,2}_{\mathfrak{sl}(n)}$ is a trivial $\mathfrak{sl}(n)$-submodule, generated by

$$v_{\lambda_5} = \sum_{j=0}^{n-1} (-1)^{(n-1)j} \sum_{i=1}^{n} f_i \otimes \tilde{f}_{s^j(1)} \wedge \tilde{f}_{s^j(2)} \wedge \ldots \wedge \tilde{f}_{s^j(n-1)} \otimes \tilde{f}_{s^j(n)} \tilde{f}_i,$$

where s is a cyclic permutation of $(1,\ 2, \ldots,\ n)$. This proves Theorem 8.2.3.1.

8.2.5. Penrose's tensors.

8.2.5.1. Theorem. *If $m, n > 1$, then $H^{k,2}_{\mathfrak{g}_0} = 0$ for $k > 2$ and the $\mathfrak{g}_0$-modules $H^{1,2}_{\mathfrak{g}_0}$ and $H^{2,2}_{\mathfrak{g}_0}$ are the direct sums of irreducible submodules whose highest weights are given in Table 13.*

If $m = n$, then $H^{k,2}_{\hat{\mathfrak{g}_0}} = H^{k,2}_{\mathfrak{g}_0}$ for any k and if $m \neq n$, then $H^{1,2}_{\hat{\mathfrak{g}_0}} = H^{1,2}_{\mathfrak{g}_0}$ whereas

$$H^{2,2}_{\hat{\mathfrak{g}_0}} = H^{2,2}_{\mathfrak{g}_0} \oplus V_{\varepsilon_1+\varepsilon_2-2\delta_n} \oplus V_{2\varepsilon_1-\delta_{n-1}-\delta_n}$$

if either $m = 2$ or $n = 2$;

$$H^{2,2}_{\hat{\mathfrak{g}_0}} = H^{2,2}_{\mathfrak{g}_0} \oplus V_{\varepsilon_1+\varepsilon_2-2\delta_n} \oplus V_{2\varepsilon_1-\delta_{n-1}-\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2-\delta_{n-1}-\delta_n} \text{ if } m,\ n > 2.$$

8.2.5.2. Calculation of $H^{1,2}_{\mathfrak{g}_0}$ and $H^{1,2}_{\hat{\mathfrak{g}_0}}$ for $m,\ n \geq 2,\ m \neq n$. For $k = 1$ the Spencer cochain sequence is of the form

$$\mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1} \xrightarrow{\partial^{2,1}_{\mathfrak{g}_0}} \mathfrak{g}_{-1} \otimes \Lambda^2 \mathfrak{g}^*_{-1} \xrightarrow{\partial^{1,2}_{\mathfrak{g}_0}} 0.$$

Observe that

$$\begin{aligned}
&\mathfrak{g}_{-1} \otimes \Lambda^2 \mathfrak{g}^*_{-1} = (U \otimes V^*) \otimes \Lambda^2(U^* \otimes V) \cong\\
&(U \otimes V^*) \otimes (\Lambda^2 U^* \otimes S^2 V \oplus S^2 U^* \otimes \Lambda^2 V) \cong\\
&(\Lambda^2 U^* \otimes U) \otimes (S^2 V \otimes V^*) \oplus (S^2 U^* \otimes U) \otimes (\Lambda^2 V \otimes V^*),\\
&\mathfrak{g}_0 \otimes \mathfrak{g}^*_{-1} = (V \otimes V^*/\mathbb{C} \oplus U \otimes U^*/\mathbb{C} \oplus \mathbb{C}) \otimes (U^* \otimes V),\\
&\mathfrak{g}_1 = U^* \otimes V.
\end{aligned}$$

Therefore, as $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-modules,

$$\operatorname{Im} \partial^{2,1}_{\mathfrak{g}_0} \cong (V \otimes V^*/\mathbb{C} \oplus U \otimes U^*/\mathbb{C}) \otimes (U^* \otimes V) \tag{8.61}$$

and $H^{1,2}_{\mathfrak{g}_0} \cong$

$$(\Lambda^2 U^* \otimes U/U^*) \otimes (S^2 V \otimes V^*/V) \oplus (S^2 U^* \otimes U/U^*) \otimes (\Lambda^2 V \otimes V^*/V).$$

Note that

$$\begin{aligned}
&S^2 V \otimes V^*/V = V_{2\varepsilon_1-\varepsilon_m},\\
&\Lambda^2 V \otimes V^*/V = V_{\varepsilon_1+\varepsilon_2-\varepsilon_m} \text{ for } m > 2,\\
&\Lambda^2 V \otimes V^*/V = 0 \text{ for } m = 2.
\end{aligned}$$

Since U is purely odd, we deduce with the help of Table 5 of [OV] that

$$\begin{aligned}
&\Lambda^2 U^* \otimes U/U^* = V_{\delta_1-2\delta_n},\\
&S^2 U^* \otimes U/U^* = V_{\delta_1-\delta_{n-1}-\delta_n} \text{ for } n > 2,\\
&S^2 U^* \otimes U/U^* = 0 \text{ for } n = 2.
\end{aligned}$$

Therefore, we have

$$H^{1,2}_{\mathfrak{g}_0} = V_{2\varepsilon_1-\varepsilon_m+\delta_1-2\delta_n} \text{ if } m = 2,\ n > 2$$

and

$$H^{1,2}_{\mathfrak{g}_0} = V_{2\varepsilon_1-\varepsilon_m+\delta_1-2\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_m+\delta_1-\delta_{n-1}-\delta_n} \text{ if } m,\ n > 2.$$

By part b) of Theorem 8.2.1.1 $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0}) = \mathfrak{g}_{-1} \oplus \hat{\mathfrak{g}_0}$. Therefore, by (8.43),[11] we have

$$\operatorname{Im} \partial^{2,1}_{\hat{\mathfrak{g}_0}} = \hat{\mathfrak{g}_0} \otimes \mathfrak{g}^*_{-1} = \operatorname{Im} \partial^{2,1}_{\mathfrak{g}_0}.$$

Hence, $H^{1,2}_{\hat{\mathfrak{g}_0}} = H^{1,2}_{\mathfrak{g}_0}$.

8.2.5.3. Calculation of $H^{1,2}_{\hat{\mathfrak{g}_0}}$ for $m = n > 1$. Since by parts c) and d) of Theorem 8.2.1.1 the first term of the Cartan prolongation $\mathfrak{g}_*(\mathfrak{g}_{-1}, \hat{\mathfrak{g}_0})$ is $U^* \otimes V$, then by (8.43)[12] $\boxed{12}$?

$$\operatorname{Im} \partial^{2,1}_{\hat{\mathfrak{g}_0}} = [(V \otimes V^*/\mathbb{C} \oplus U \otimes U^*/\mathbb{C}) \otimes (U^* \otimes V)]/(U^* \otimes V).$$

Therefore, by (8.61) and (8.2.5.2),

$$H^{1,2}_{\hat{\mathfrak{g}_0}} \cong (\Lambda^2 U^* \otimes U/U^*) \otimes (S^2 V \otimes V^*/V) \oplus (S^2 U^* \otimes U/U^*) \otimes (\Lambda^2 V \otimes V^*/V) \oplus (U^* \otimes V).$$

Hence,

$$H^{1,2}_{\hat{\mathfrak{g}_0}} = V_{2\varepsilon_1-\varepsilon_2+\delta_1-2\delta_2} \oplus V_{\varepsilon_1-\delta_2} \text{ for } n = 2$$

and

$$H^{1,2}_{\hat{\mathfrak{g}_0}} = V_{2\varepsilon_1-\varepsilon_n+\delta_1-2\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_n+\delta_1-\delta_{n-1}-\delta_n} \oplus V_{\varepsilon_1-\delta_n} \text{ for } n > 2.$$

8.2.5.4. Calculation of $H^{2,2}_{\mathfrak{g}_0}$ for $m,\ n > 1,\ m \neq n$. For $k = 2$ the Spencer cochain sequence is of the form

$$\mathfrak{g}_1 \otimes \mathfrak{g}^*_{-1} \xrightarrow{\partial^{3,1}_{\mathfrak{g}_0}} \mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1} \xrightarrow{\partial^{2,2}_{\mathfrak{g}_0}} \mathfrak{g}_{-1} \otimes \Lambda^3 \mathfrak{g}^*_{-1}.$$

Observe that

$$\begin{aligned}
&\mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1} = (V \otimes V^*/\mathbb{C} \oplus U \otimes U^*/\mathbb{C} \oplus \mathbb{C}) \otimes (\Lambda^2 U^* \otimes S^2 V \oplus S^2 U^* \otimes \Lambda^2 V),\\
&\mathfrak{g}_1 \otimes \mathfrak{g}^*_{-1} = (U^* \otimes V) \otimes (U^* \otimes V),\\
&\mathfrak{g}_2 = 0.
\end{aligned}$$

Lemma. *As $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-module, $\mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1}$ is the direct sum of the irreducible submodules whose highest weights and highest vectors are listed in Table 14.* (Here s and t denote the cyclic permutations of $(1,\ 2,\ 3)$ and $(n-2,\ n-1,\ n)$, respectively.)

The proof follows from the formula given in Appendix.

Let us show that if

$$\lambda = 3\varepsilon_1 - \varepsilon_m - 2\delta_n,\ 2\varepsilon_1 + \varepsilon_2 - \varepsilon_m - 2\delta_n\ (m > 2),\ 2\varepsilon_1 + \varepsilon_2 - \varepsilon_m - \delta_{n-1} - \delta_n,$$
$$2\varepsilon_1 + \delta_1 - 3\delta_n,\ 2\varepsilon_1 + \delta_1 - \delta_{n-1} - 2\delta_n\ (n > 2),\ \text{or } \varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-1} - 2\delta_n,$$

then $v_\lambda \notin \operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$. Recall that if $v \in \mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1}$, then

$$\partial^{2,2}_{\mathfrak{g}_0} v(g_1, g_2, g_3) = -v(g_1, g_2)g_3 - v(g_1, g_3)g_2 - v(g_2, g_3)g_1 \tag{8.62}$$

for any $g_1, g_2, g_3 \in \mathfrak{g}_{-1}$.

Let $\lambda = 3\varepsilon_1 - \varepsilon_m - 2\delta_n$. Then

$$\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1) = -3v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_1) = \\ = 3A_{1,m}(f_n \otimes \tilde{e}_1) = -3f_n \otimes \tilde{e}_m \neq 0.$$

Let $\lambda = 2\varepsilon_1 + \varepsilon_2 - \varepsilon_m - 2\delta_n$ $(m > 2)$. Then

$$\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_{n-1} \otimes \tilde{e}_2) = -v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1)(f_{n-1} \otimes \tilde{e}_2) = \\ = -A_{2,m}(f_{n-1} \otimes \tilde{e}_2) = f_{n-1} \otimes \tilde{e}_m \neq 0.$$

Let $\lambda = 2\varepsilon_1 + \varepsilon_2 - \varepsilon_m - \delta_{n-1} - \delta_n$. Then

$$\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_{n-1} \otimes \tilde{e}_2) = -2v_\lambda(f_n \otimes \tilde{e}_1, f_{n-1} \otimes \tilde{e}_2)(f_n \otimes \tilde{e}_1) = \\ = -A_{1,m}(f_n \otimes \tilde{e}_1) = f_n \otimes \tilde{e}_m \neq 0.$$

The proof of the fact that $v_\lambda \notin \operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$ for

$$\lambda = 2\varepsilon_1 + \delta_1 - 3\delta_n,\ 2\varepsilon_1 + \delta_1 - \delta_{n-1} - 2\delta_n\ (n > 2),\ \text{and}\ \varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-1} - 2\delta_n$$

is similar.

Let $\lambda = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_m - \delta_{n-1} - \delta_n$ $(m > 3)$. Let us show that if $n = 2$, then $v_\lambda \in \operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$ and if $n > 2$, then $v_\lambda \notin \operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$. Indeed, if $n = 2$, then for $j = 0,\ 1,\ 2$ we have

$$\begin{aligned} &\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_1 \otimes \tilde{e}_{s^j(2)}, f_2 \otimes \tilde{e}_{s^j(3)}, f_1 \otimes \tilde{e}_{s^j(1)}) = \\ &- v_\lambda(f_1 \otimes \tilde{e}_{s^j(2)}, f_2 \otimes \tilde{e}_{s^j(3)})(f_1 \otimes \tilde{e}_{s^j(1)}) \\ &- v_\lambda(f_2 \otimes \tilde{e}_{s^j(3)}, f_1 \otimes \tilde{e}_{s^j(1)})(f_1 \otimes \tilde{e}_{s^j(2)}) = \\ &A_{s^j(1),m}\frac{1}{2}(f_1 \otimes \tilde{e}_{s^j(1)}) - A_{s^j(2),m}\frac{1}{2}(f_1 \otimes \tilde{e}_{s^j(2)}) = \\ &-\frac{1}{2}f_1 \otimes \tilde{e}_m + \frac{1}{2}f_1 \otimes \tilde{e}_m = 0, \end{aligned}$$

$$\begin{aligned} &\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_1 \otimes \tilde{e}_{s^j(2)}, f_2 \otimes \tilde{e}_{s^j(3)}, f_2 \otimes \tilde{e}_{s^j(1)}) = \\ &- v_\lambda(f_1 \otimes \tilde{e}_{s^j(2)}, f_2 \otimes \tilde{e}_{s^j(3)})(f_2 \otimes \tilde{e}_{s^j(1)}) - \\ &- v_\lambda(f_2 \otimes \tilde{e}_{s^j(1)}, f_1 \otimes \tilde{e}_{s^j(2)})(f_2 \otimes \tilde{e}_{s^j(3)}) = \\ &A_{s^j(1),m}\frac{1}{2}(f_2 \otimes \tilde{e}_{s^j(1)}) - A_{s^j(3),m}\frac{1}{2}(f_2 \otimes \tilde{e}_{s^j(3)}) = \\ &-\frac{1}{2}f_2 \otimes \tilde{e}_m + \frac{1}{2}f_2 \otimes \tilde{e}_m = 0. \end{aligned}$$

Therefore, $v_\lambda \in \operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$. If $n > 2$, then

$$\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_{n-1} \otimes \tilde{e}_2, f_n \otimes \tilde{e}_3, f_1 \otimes \tilde{e}_1) = -v_\lambda(f_{n-1} \otimes \tilde{e}_2, f_n \otimes \tilde{e}_3)(f_1 \otimes \tilde{e}_1) = \\ = A_{1,m}(f_1 \otimes \tilde{e}_1)/2 = -f_1 \otimes \tilde{e}_m/2 \neq 0.$$

The proof of the fact that if $\lambda = \varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-2} - \delta_{n-1} - \delta_n$ $(n \geq 4)$, then $v_\lambda \in \operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$ for $m = 2$ and $v_\lambda \notin \operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$ for $m > 2$ is similar.

Finally, let us show that if

$$\lambda = 2\varepsilon_1 - 2\delta_n,\ \varepsilon_1 + \varepsilon_2 - 2\delta_n,\ 2\varepsilon_1 - \delta_{n-1} - \delta_n,\ \text{or}\ \varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$$

and $v_\lambda \in \operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$, then $v_\lambda \in \operatorname{Im} \partial^{3,1}_{\mathfrak{g}_0}$. Note that since $\mathfrak{g}_2 = 0$, then, as $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-modules,

$$\operatorname{Im} \partial^{3,1}_{\mathfrak{g}_0} \cong \mathfrak{g}_1 \otimes \mathfrak{g}^*_{-1} = (U^* \otimes V) \otimes (U^* \otimes V).$$

Therefore, by Table 9,

$$\operatorname{Im} \partial^{3,1}_{\mathfrak{g}_0} = V_{2\varepsilon_1 - 2\delta_n} \oplus V_{\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n} \oplus V_{\varepsilon_1 + \varepsilon_2 - 2\delta_n} \oplus V_{2\varepsilon_1 - \delta_{n-1} - \delta_n}.$$

Let $\lambda = \varepsilon_1 + \varepsilon_2 - 2\delta_n$. By Table 14 $\mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1}$ contains two irreducible components with the indicated highest weight, and one of the corresponding highest vectors is v^1_λ. Observe that

$$\partial^{2,2}_{\mathfrak{g}_0} v^1_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_{n-1} \otimes \tilde{e}_2) = -v^1_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1)(f_{n-1} \otimes \tilde{e}_2) = \\ = -A_{2,1}(f_{n-1} \otimes \tilde{e}_2) = f_{n-1} \otimes \tilde{e}_1 \neq 0.$$

Therefore, $\operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$ contains precisely one irreducible submodule with highest weight $\varepsilon_1 + \varepsilon_2 - 2\delta_n$ and this submodule belongs to $\operatorname{Im} \partial^{3,1}_{\mathfrak{g}_0}$. Similarly, $\mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1}$ contains two irreducible submodules with highest weight $2\varepsilon_1 - \delta_{n-1} - \delta_n$, one of which belongs to $\operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$, and therefore to $\operatorname{Im} \partial^{3,1}_{\mathfrak{g}_0}$.

Let $\lambda = 2\varepsilon_1 - 2\delta_n$. Then by Table 14 any $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-highest vector of weight λ, which belongs to $\mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1}$, is

$$v_\lambda = k_1 v^1_\lambda + k_2 v^2_\lambda + k_3 v^3_\lambda,\ \text{where}\ k_1, k_2, k_3 \in \mathbb{C}.$$

If $v_\lambda \in \operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$, then the condition $\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1) = 0$ implies

$$k_1(m-1) - k_2(n-1) + k_3(m-n) = 0, \tag{8.63}$$

and the condition $\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_n \otimes \tilde{e}_2, f_n \otimes \tilde{e}_1, f_1 \otimes \tilde{e}_1) = 0$ implies that

$$k_1 m - k_2 n = 0. \tag{8.64}$$

Thus, for $m \neq n$ we have

$$k_2 = mk_1/n, k_3 = -k_1/n. \tag{8.65}$$

Therefore, $\operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$ contains precisely one irreducible submodule with highest weight $2\varepsilon_1 - 2\delta_n$ and this submodule belongs to $\operatorname{Im} \partial^{3,1}_{\mathfrak{g}_0}$.

Finally, let $\lambda = \varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$. Then by Table 14 any highest vector with weight λ, which belongs to $\mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}_{-1}^*$, is

$$v_\lambda = k_1 v_\lambda^1 + k_2 v_\lambda^2 + k_3 v_\lambda^3, \text{ where } k_1, k_2, k_3 \in \mathbb{C},$$

and if $m = 2$, then $k_1 = 0$. Note that if $v_\lambda \in \operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$, then

$$\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_{n-1} \otimes \tilde{e}_1, f_n \otimes \tilde{e}_2, f_n \otimes \tilde{e}_1) = 0$$

implies that

$$k_1 + k_2 + k_3(n - m) = 0. \tag{8.66}$$

Thus, if $m = 2$, then

$$k_2 = (2 - n)k_3. \tag{8.67}$$

If $m,\ n > 2$, then the condition

$$\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_{n-1} \otimes \tilde{e}_m, f_n \otimes \tilde{e}_2, f_1 \otimes \tilde{e}_1) = 0 \tag{8.68}$$

implies that $k_1 + k_2 = 0$. Hence,

$$k_2 = -k_1,\ k_3 = 0. \tag{8.69}$$

Therefore, $\operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$ contains precisely one highest vector of weight

$$\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$$

which belongs to $\operatorname{Im} \partial^{3,1}_{\mathfrak{g}_0}$. Thus, we have the description of $H^{2,2}_{\mathfrak{g}_0}$ given in Table 13.

8.2.5.5. Calculation of $H^{2,2}_{\hat{\mathfrak{g}}_0}$ for $m,\ n > 1,\ m \neq n$. By part b) of Theorem 8.2.1.1

$$\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}}_0) = \mathfrak{g}_{-1} \oplus \hat{\mathfrak{g}}_0.$$

Therefore, the Spencer cochain sequence for $k = 2$ takes the form

$$0 \xrightarrow{\partial^{3,1}_{\hat{\mathfrak{g}}_0}} \hat{\mathfrak{g}}_0 \otimes \Lambda^2 \mathfrak{g}_{-1}^* \xrightarrow{\partial^{2,2}_{\hat{\mathfrak{g}}_0}} \mathfrak{g}_{-1} \otimes \Lambda^3 \mathfrak{g}_{-1}^*.$$

Note that since $\mathfrak{g}_0 = \hat{\mathfrak{g}}_0 \oplus \mathbb{C}$, then

$$\mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}_{-1}^* = \hat{\mathfrak{g}}_0 \otimes \Lambda^2 \mathfrak{g}_{-1}^* \oplus V_{2\varepsilon_1 - 2\delta_n} \oplus V_{\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n}. \tag{8.70}$$

13 ? As we have shown in sec. 8.2.5.4[13] if λ is one of the weights from Table 14, then an irreducible module with highest weight λ is contained in the decomposition of $\operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$ into irreducible $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-modules if and only if

$$\begin{aligned} &\lambda = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_m - \delta_{n-1} - \delta_n\ (m > 3),\\ &\varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-2} - \delta_{n-1} - \delta_n\ (n > 3),\\ &\varepsilon_1 + \varepsilon_2 - 2\delta_n,\ 2\varepsilon_1 - \delta_{n-1} - \delta_n,\ 2\varepsilon_1 - 2\delta_n \text{ or } \varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n \end{aligned} \tag{8.71}$$

and its multiplicity is 1. Therefore, by (8.70), if

$$\begin{aligned} &\lambda = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_m - \delta_{n-1} - \delta_n\ (m > 3),\\ &\varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-2} - \delta_{n-1} - \delta_n\ (n > 3),\\ &\varepsilon_1 + \varepsilon_2 - 2\delta_n,\ \text{ or } 2\varepsilon_1 - \delta_{n-1} - \delta_n, \end{aligned}$$

then the corresponding submodule is contained in $\operatorname{Ker} \partial^{2,2}_{\hat{\mathfrak{g}}_0}$ as well.

Let $\lambda = 2\varepsilon_1 - 2\delta_n$ and $v_\lambda \in \operatorname{Ker} \partial^{2,2}_{\hat{\mathfrak{g}}_0}$. Then (8.65) where $k_3 = 0$, implies that $v_\lambda \notin \operatorname{Ker} \partial^{2,2}_{\hat{\mathfrak{g}}_0}$.

Let $\lambda = \varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n,\ v_\lambda \in \operatorname{Ker} \partial^{2,2}_{\mathfrak{g}_0}$. Then (8.67) implies that $v_\lambda \notin \operatorname{Ker} \partial^{2,2}_{\hat{\mathfrak{g}}_0}$ for $m = 2$, and (8.69) implies that $v_\lambda \in \operatorname{Ker} \partial^{2,2}_{\hat{\mathfrak{g}}_0}$ for $m,\ n > 2$. Thus, we have

$$H^{2,2}_{\hat{\mathfrak{g}}_0} = H^{2,2}_{\mathfrak{g}_0} \oplus V_{\varepsilon_1 + \varepsilon_2 - 2\delta_n} \oplus V_{2\varepsilon_1 - \delta_{n-1} - \delta_n} \text{ if either } m = 2 \text{ or } n = 2$$

and

$$H^{2,2}_{\hat{\mathfrak{g}}_0} = H^{2,2}_{\mathfrak{g}_0} \oplus V_{\varepsilon_1 + \varepsilon_2 - 2\delta_n} \oplus V_{2\varepsilon_1 - \delta_{n-1} - \delta_n} \oplus V_{\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n} \text{ if } m,\ n > 2.$$

8.2.5.6. Calculation of $H^{2,2}_{\hat{\mathfrak{g}}_0}$ for $m = n > 1$. By parts c) and d) of Theorem 8.2.1.1 the first term of $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}}_0)$ is isomorphic to $U^* \otimes V$ and the second one to $\mathbb{C}$ for $n = 2$ and is 0 for $n > 2$. By formula (8.43)[14] we have 14 ?

$$\operatorname{Im} \partial^{3,1}_{\hat{\mathfrak{g}}_0} = (U^* \otimes V) \otimes (U^* \otimes V) \text{ for } n > 2 \tag{8.72}$$

and

$$\operatorname{Im} \partial^{3,1}_{\hat{\mathfrak{g}}_0} = (U^* \otimes V) \otimes (U^* \otimes V)/\mathbb{C} \text{ for } n = 2. \tag{8.73}$$

Therefore, by Table 9,

$$\operatorname{Im} \partial^{3,1}_{\hat{\mathfrak{g}}_0} = V_{2\varepsilon_1 - 2\delta_n} \oplus V_{2\varepsilon_1 - \delta_{n-1} - \delta_n} \oplus V_{\varepsilon_1 + \varepsilon_2 - 2\delta_n} \text{ for } n = 2$$

and

$$\operatorname{Im} \partial^{3,1}_{\hat{\mathfrak{g}}_0} = V_{2\varepsilon_1 - 2\delta_n} \oplus V_{2\varepsilon_1 - \delta_{n-1} - \delta_n} \oplus V_{\varepsilon_1 + \varepsilon_2 - 2\delta_n} \oplus V_{\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n} \text{ for } n > 2.$$

Therefore, by (8.67) and (8.71),

$$H^{2,2}_{\hat{\mathfrak{g}}_0} = 0 \text{ for } n = 2,\ 3$$

and

$$H^{2,2}_{\hat{\mathfrak{g}}_0} = V_{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_n - \delta_{n-1} - \delta_n} \oplus V_{\varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-2} - \delta_{n-1} - \delta_n} \text{ for } n > 3.$$

8.2.5.7. Computation of $H^{3,2}_{\mathfrak{g}_0}$ for $m,\ n>1,\ m\neq n$. For $k=3$, the Spencer cochain sequence is of the form

$$\mathfrak{g}_2\otimes\mathfrak{g}_{-1}^*\xrightarrow{\partial^{4,1}_{\mathfrak{g}_0}}\mathfrak{g}_1\otimes\Lambda^2\mathfrak{g}_{-1}^*\xrightarrow{\partial^{3,2}_{\mathfrak{g}_0}}\mathfrak{g}_0\otimes\Lambda^3\mathfrak{g}_{-1}^*.$$

Observe that

$$\begin{aligned}&\mathfrak{g}_1\otimes\Lambda^2\mathfrak{g}_{-1}^*=(U^*\otimes V)\otimes\Lambda^2(U^*\otimes V)\cong\\&(\Lambda^2U^*\otimes U^*)\otimes(S^2V\otimes V)\oplus(S^2U^*\otimes U^*)\otimes(\Lambda^2V\otimes V),\\&\mathfrak{g}_2=0.\end{aligned}$$

By Table 5 from [OV]

$$\begin{aligned}&S^2V\otimes V=V_{3\varepsilon_1}\oplus V_{2\varepsilon_1+\varepsilon_2},\\&\Lambda^2V\otimes V=V_{2\varepsilon_1+\varepsilon_2}\oplus V_{\varepsilon_1+\varepsilon_2+\varepsilon_3}\text{ for }m>2,\\&\Lambda^2V\otimes V=V_{2\varepsilon_1+\varepsilon_2}\text{ for }m=2.\end{aligned}$$

Since U is purely odd,

$$\begin{aligned}&\Lambda^2U^*\otimes U^*=V_{-3\delta_n}\oplus V_{-\delta_{n-1}-2\delta_n},\\&S^2U^*\otimes U^*=V_{-\delta_{n-1}-2\delta_n}\oplus V_{-\delta_{n-2}-\delta_{n-1}-\delta_n}\text{ for }n>2,\\&S^2U^*\otimes U^*=V_{-\delta_{n-1}-2\delta_n}\text{ for }n=2.\end{aligned}$$

The above decompositions imply the following

Lemma. *The $\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)$-module $\mathfrak{g}_1\otimes\Lambda^2\mathfrak{g}_{-1}^*$ is the direct sum of irreducible submodules whose highest weights and highest vectors are listed in Table 15.* (Here s and t denote the cyclic permutations of $(1,\ 2,\ 3)$ and $(n-2,\ n-1,\ n)$, respectively.)

Let us show that $\operatorname{Ker}\partial^{3,2}_{\mathfrak{g}_0}=0$. Let $\lambda=3\varepsilon_1-3\delta_n$. Then

$$\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_n\otimes\tilde e_1,f_n\otimes\tilde e_1,f_n\otimes\tilde e_1)=3B_{1,n}(f_n\otimes\tilde e_1)=3(e_1\otimes\tilde e_1+f_n\otimes\tilde f_n)\neq0.$$

Let $\lambda=2\varepsilon_1+\varepsilon_2-3\delta_n$. Then

$$\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_n\otimes\tilde e_1,f_n\otimes\tilde e_1,f_n\otimes\tilde e_1)=-3B_{2,n}(f_n\otimes\tilde e_1)=-3e_2\otimes\tilde e_1\neq0.$$

Let $\lambda=3\varepsilon_1-\delta_{n-1}-2\delta_n$. Then

$$\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_n\otimes\tilde e_1,f_n\otimes\tilde e_1,f_n\otimes\tilde e_1)=3B_{1,n-1}(f_n\otimes\tilde e_1)=3f_n\otimes\tilde f_{n-1}\neq0.$$

Let $\lambda=2\varepsilon_1+\varepsilon_2-\delta_{n-1}-2\delta_n$. Since by Table 15 $\mathfrak{g}_1\otimes\Lambda^2\mathfrak{g}_{-1}^*$ contains two irreducible submodules with highest weight λ, then any highest vector of weight λ in $\mathfrak{g}_1\otimes\Lambda^2\mathfrak{g}_{-1}^*$ is of the form

$$v_\lambda=k_1v^1_\lambda+k_2v^2_\lambda,\ \text{where }k_1,k_2\in\mathbb{C}.$$

Let $v_\lambda\in\operatorname{Ker}\partial^{3,2}_{\mathfrak{g}_0}$. If $m>2$, then

$$\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_{n-1}\otimes\tilde e_2,f_n\otimes\tilde e_1,f_n\otimes\tilde e_m)=-\frac12k_2B_{1,n}(f_n\otimes\tilde e_m)=-\frac12k_2e_1\otimes\tilde e_m=0.$$

Therefore, $k_2=0$. Moreover,

$$\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_n\otimes\tilde e_1,f_n\otimes\tilde e_1,f_{n-1}\otimes\tilde e_m)=k_1B_{2,n-1}(f_{n-1}\otimes\tilde e_m)=k_1e_2\otimes\tilde e_m=0.$$

Hence, $k_1=0$. If $n>2$, then

$$\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_{n-1}\otimes\tilde e_2,f_n\otimes\tilde e_1,f_1\otimes\tilde e_1)=-\frac12k_2B_{1,n}(f_1\otimes\tilde e_1)=-\frac12k_2f_1\otimes\tilde f_n=0.$$

Therefore, $k_2=0$. Moreover,

$$\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_n\otimes\tilde e_1,f_n\otimes\tilde e_1,f_1\otimes\tilde e_2)=k_1B_{2,n-1}(f_1\otimes\tilde e_2)=k_1f_1\otimes\tilde f_{n-1}=0.$$

Hence, $k_1=0$.

Let $\lambda=\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_{n-1}-2\delta_n$. Then

$$\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_{n-1}\otimes\tilde e_2,f_n\otimes\tilde e_3,f_n\otimes\tilde e_2)=\frac12B_{1,n}(f_n\otimes\tilde e_2)=\frac12e_1\otimes\tilde e_2\neq0.$$

Let $\lambda=2\varepsilon_1+\varepsilon_2-\delta_{n-2}-\delta_{n-1}-\delta_n$. Then

$$\begin{aligned}&\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_{n-1}\otimes\tilde e_2,f_n\otimes\tilde e_1,f_{n-1}\otimes\tilde e_1)=\\&\frac12B_{1,n-2}(f_{n-1}\otimes\tilde e_1)=\frac12f_{n-1}\otimes\tilde f_{n-2}\neq0.\end{aligned}$$

Finally, let $\lambda=\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_{n-2}-\delta_{n-1}-\delta_n$. Then

$$\begin{aligned}&\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_{n-1}\otimes\tilde e_2,f_n\otimes\tilde e_3,f_{n-2}\otimes\tilde e_1)=\\&\frac12(B_{1,n-2}(f_{n-2}\otimes\tilde e_1)+B_{3,n}(f_n\otimes\tilde e_3)+B_{2,n-1}(f_{n-1}\otimes\tilde e_2))=\\&\frac12(e_1\otimes\tilde e_1+f_{n-2}\otimes\tilde f_{n-2}+e_3\otimes\tilde e_3+f_n\otimes\tilde f_n+e_2\otimes\tilde e_2+f_{n-1}\otimes\tilde f_{n-1})\neq0.\end{aligned}$$

Thus, $H^{3,2}_{\mathfrak{g}_0}=0$.

8.2.5.8. Calculation of $H^{3,2}_{\hat{\mathfrak{g}}_0}$ for $m=n>1$. By part d) of Theorem 8.2.1.1 for $n>2$ the first term of the Cartan prolongation of the pair $(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}}_0)$ is $U^*\otimes V$ and the second one is zero. Therefore, by arguments similar to those from section 8.2.5.7 we get $H^{3,2}_{\hat{\mathfrak{g}}_0}=0$.

If $n=2$, then by part c) of Theorem 8.2.1.1 the first term of $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}}_0)$ is $U^*\otimes V$, the second one is the 1-dimensional $\mathfrak{gl}(2)\oplus\mathfrak{gl}(2)$-module with highest weight $\varepsilon_1+\varepsilon_2-\delta_1-\delta_2$, and the third one is zero. Thus, by (8.43),[15] 15

$$\operatorname{Im}\partial^{4,1}_{\hat{\mathfrak{g}}_0}=V_{2\varepsilon_1+\varepsilon_2-\delta_1-2\delta_2}.$$

By Table 15 $(U^*\otimes V)\otimes\Lambda^2(U^*\otimes V)$ contains two irreducible $\mathfrak{gl}(2)\oplus\mathfrak{gl}(2)$-modules with highest weight $\lambda=2\varepsilon_1+\varepsilon_2-\delta_1-2\delta_2$ and one of the corresponding highest vectors is v^2_λ. Since

$$\partial^{3,2}_{\hat{\mathfrak{g}}_0}v^2_\lambda(f_1\otimes\tilde e_1,f_2\otimes\tilde e_2,f_2\otimes\tilde e_2)=B_{1,2}(f_2\otimes\tilde e_2)=e_1\otimes\tilde e_2\neq0,$$

then $\operatorname{Ker}\partial^{3,2}_{\hat{\mathfrak{g}}_0}=\operatorname{Im}\partial^{4,1}_{\hat{\mathfrak{g}}_0}$. Thus, $H^{3,2}_{\hat{\mathfrak{g}}_0}=0$.

8.2.5.9. Calculation of $H^{4,2}_{\hat{\mathfrak{g}_0}}$ for $m = n = 2$. For $k = 4$, the Spencer cochain sequence is of the form

$$\mathfrak{g}_3 \otimes \mathfrak{g}^*_{-1} \xrightarrow{\partial^{5,1}_{\hat{\mathfrak{g}_0}}} \mathfrak{g}_2 \otimes \Lambda^2\mathfrak{g}^*_{-1} \xrightarrow{\partial^{4,2}_{\hat{\mathfrak{g}_0}}} \mathfrak{g}_1 \otimes \Lambda^3\mathfrak{g}^*_{-1}.$$

By part c) of Theorem 8.2.1.1 the second term of $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0})$ is

$$\mathfrak{g}_2 = V_{\varepsilon_1+\varepsilon_2-\delta_1-\delta_2} = \langle g\rangle,$$

the 1-dimensional $\mathfrak{gl}(2) \oplus \mathfrak{gl}(2)$-module, and the third term is zero. Since by Table 9

$$\Lambda^2\mathfrak{g}^*_{-1} = V_{2\varepsilon_1-2\delta_2} \oplus V_{\varepsilon_1+\varepsilon_2-\delta_1-\delta_2},$$

then

$$\mathfrak{g}_2 \otimes \Lambda^2\mathfrak{g}^*_{-1} = V_{3\varepsilon_1+\varepsilon_2-\delta_1-3\delta_2} \oplus V_{2\varepsilon_1+2\varepsilon_2-2\delta_1-2\delta_2}.$$

Let $\lambda = 3\varepsilon_1 + \varepsilon_2 - \delta_1 - 3\delta_2$. Then by Table 9 $v_\lambda = g \otimes (\tilde{f}_2 \otimes e_1) \wedge (\tilde{f}_2 \otimes e_1)$. Let v be an element from the basis of $\mathfrak{g}_{-1}$ such that $g(v) \neq 0$. If $v = f_2 \otimes \tilde{e}_1$, then

$$\partial^{4,2}_{\hat{\mathfrak{g}_0}}(f_2 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_1, v) = -3v_\lambda(f_2 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_1)(v) = 3g(v) \neq 0,$$

and if $v \neq f_2 \otimes \tilde{e}_1$, then

$$\partial^{4,2}_{\hat{\mathfrak{g}_0}}v_\lambda(f_2 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_1, v) = -v_\lambda(f_2 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_1)(v) = g(v) \neq 0.$$

Let $\lambda = 2\varepsilon_1 + 2\varepsilon_2 - 2\delta_1 - 2\delta_2$. Then by Table 9

$$\begin{aligned} v_\lambda = g \otimes ((\tilde{f}_2 \otimes e_1) \wedge (\tilde{f}_1 \otimes e_2) - (\tilde{f}_2 \otimes e_2) \wedge (\tilde{f}_1 \otimes e_1)-\\ - (\tilde{f}_1 \otimes e_1) \wedge (\tilde{f}_2 \otimes e_2) + (\tilde{f}_1 \otimes e_2) \wedge (\tilde{f}_2 \otimes e_1)).\end{aligned}$$

Let v be an element of the basis of $\mathfrak{g}_{-1}$ such that $g(v) \neq 0$. Then

$$\begin{aligned}&\partial^{4,2}_{\hat{\mathfrak{g}_0}}v_\lambda(f_2 \otimes \tilde{e}_1, f_1 \otimes \tilde{e}_2, v) = -2v_\lambda(f_2 \otimes \tilde{e}_1, f_1 \otimes \tilde{e}_2)(v) = g(v) \neq 0\\ &\text{if either } v = f_2 \otimes \tilde{e}_1 \text{ or } v = f_1 \otimes \tilde{e}_2,\end{aligned}$$

and

$$\begin{aligned}&\partial^{4,2}_{\hat{\mathfrak{g}_0}}v_\lambda(f_1 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_2, v) = -2v_\lambda(f_1 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_2)(v) = -g(v) \neq 0\\ &\text{if either } v = f_2 \otimes \tilde{e}_2 \text{ or } v = f_1 \otimes \tilde{e}_1.\end{aligned}$$

Therefore, $H^{4,2}_{\hat{\mathfrak{g}_0}} = 0$.

8.2.5.10. Calculation of $H^{k,2}_{\mathfrak{g}_0}$ for $m = n > 1,\ k > 0$.

Lemma. $H^{k,2}_{\mathfrak{g}_0} = H^{k,2}_{\hat{\mathfrak{g}_0}}$.

Proof. Note that if $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0}) = \mathfrak{g}_{-1} \oplus (\bigoplus\limits_{k\geq 0} \hat{\mathfrak{g}}_k)$ is the Cartan prolongation of the pair $(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0})$, then, since $\mathfrak{g}_k = \hat{\mathfrak{g}_k} \oplus S^k(\mathfrak{g}^*_{-1})$ $(k \geq 0)$, the Spencer cochain sequence is of the form

$$\begin{aligned}&(\hat{\mathfrak{g}_0} \oplus \mathbb{C}) \otimes \mathfrak{g}^*_{-1} \xrightarrow{\partial^{2,1}_{\mathfrak{g}_0}} \mathfrak{g}_{-1} \otimes \Lambda^2\mathfrak{g}^*_{-1} \xrightarrow{\partial^{1,2}_{\mathfrak{g}_0}} 0 &&\text{for } k = 1,\\ &(\hat{\mathfrak{g}}_{k-1} \oplus S^{k-1}(\mathfrak{g}^*_{-1})) \otimes \mathfrak{g}^*_{-1} \xrightarrow{\partial^{k+1,1}_{\mathfrak{g}_0}} (\hat{\mathfrak{g}}_{k-2} \oplus S^{k-2}(\mathfrak{g}^*_{-1})) \otimes \Lambda^2\mathfrak{g}^*_{-1} \xrightarrow{\partial^{k,2}_{\mathfrak{g}_0}}\\ &(\hat{\mathfrak{g}}_{k-3} \oplus S^{k-3}(\mathfrak{g}^*_{-1})) \otimes \Lambda^3\mathfrak{g}^*_{-1} &&\text{for } k > 1.\end{aligned}$$

Note that since $\mathfrak{g}_*(\mathfrak{g}_{-1}, \mathfrak{g}_0) = S^*(\mathfrak{g}^*_{-1}) \Subset \mathfrak{g}_*(\mathfrak{g}_{-1}, \hat{\mathfrak{g}_0})$, then the sequence

$$S^{k-1}(\mathfrak{g}^*_{-1}) \otimes \mathfrak{g}^*_{-1} \xrightarrow{\bar{\partial}^{k+1,1}_{\mathfrak{g}_0}} S^{k-2}(\mathfrak{g}^*_{-1}) \otimes \Lambda^2\mathfrak{g}^*_{-1} \xrightarrow{\bar{\partial}^{k,2}_{\mathfrak{g}_0}} S^{k-3}(\mathfrak{g}^*_{-1}) \otimes \Lambda^3\mathfrak{g}^*_{-1} \text{ for } k \geq 1,$$

where $\bar{\partial}^{k+1,1}_{\mathfrak{g}_0}$ and $\bar{\partial}^{k,2}_{\mathfrak{g}_0}$ are the restrictions of the operators $\partial^{k+1,1}_{\mathfrak{g}_0}$ and $\partial^{k,2}_{\mathfrak{g}_0}$ to $S^{k-1}(\mathfrak{g}^*_{-1}) \otimes \mathfrak{g}^*_{-1}$ and $S^{k-2}(\mathfrak{g}^*_{-1}) \otimes \Lambda^2\mathfrak{g}^*_{-1}$, respectively, and $S^k(\mathfrak{g}^*_{-1}) = 0$ for $k < 0$, is well-defined. Hence the corresponding cohomology groups

$$\bar{H}^{k,2}_{\mathfrak{g}_0} = \operatorname{Ker} \bar{\partial}^{k,2}_{\mathfrak{g}_0} / \operatorname{Im} \bar{\partial}^{k+1,1}_{\mathfrak{g}_0}$$

are well-defined and $H^{k,2}_{\mathfrak{g}_0} = H^{k,2}_{\hat{\mathfrak{g}_0}} \oplus \bar{H}^{k,2}_{\mathfrak{g}_0}$.

Let us show that $\bar{H}^{k,2}_{\mathfrak{g}_0} = 0$ for $k > 0$. For $k = 1$ this is obvious. Let $k = 2$. Since $S^{k-2}(\mathfrak{g}^*_{-1}) \otimes \Lambda^2\mathfrak{g}^*_{-1} = \langle z\rangle \otimes \Lambda^2\mathfrak{g}^*_{-1}$, where z is a generator of the center of $\mathfrak{gl}(n|n)$, then

$$\operatorname{Ker} \bar{\partial}^{k,2}_{\mathfrak{g}_0} \cong \Lambda^2\mathfrak{g}^*_{-1}.$$

By formula (8.43)[16] 16

$$\operatorname{Im} \bar{\partial}^{k+1,1}_{\mathfrak{g}_0} \cong \mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1}/S^2\mathfrak{g}^*_{-1} = \Lambda^2\mathfrak{g}^*_{-1}.$$

Therefore, $\bar{H}^{2,2}_{\mathfrak{g}_0} = 0$. Let $k = 3$. Observe that

$$S^2(U^* \otimes V) \otimes (U^* \otimes V) = (S^2U^* \otimes U^*) \otimes (S^2V \otimes V) \oplus (\Lambda^2U^* \otimes U^*) \otimes (\Lambda^2V \otimes V).$$

By Table 5 from [OV] we get:

$$\begin{aligned}&S^2V \otimes V = V_{3\varepsilon_1} \oplus V_{2\varepsilon_1+\varepsilon_2},\ \Lambda^2V \otimes V = V_{2\varepsilon_1+\varepsilon_2} \text{ if } n = 2,\\ &\Lambda^2V \otimes V = V_{2\varepsilon_1+\varepsilon_2} \oplus V_{\varepsilon_1+\varepsilon_2+\varepsilon_3} \text{ if } n > 2.\end{aligned}$$

Since U is odd,

$$\begin{aligned}&\Lambda^2U^* \otimes U^* = V_{-3\delta_n} \oplus V_{-\delta_{n-1}-2\delta_n},\\ &S^2U^* \otimes U^* = V_{-\delta_{n-1}-2\delta_n} \text{ if } n = 2,\\ &S^2U^* \otimes U^* = V_{-\delta_{n-1}-2\delta_n} \oplus V_{-\delta_{n-2}-\delta_{n-1}-\delta_n} \text{ if } n > 2.\end{aligned}$$

Therefore,

$$S^2(U^*\otimes V)\otimes(U^*\otimes V)=$$
$$V_{3\varepsilon_1-\delta_{n-1}-2\delta_n}\oplus V_{2\varepsilon_1+\varepsilon_2-3\delta_n}\oplus 2V_{2\varepsilon_1+\varepsilon_2-\delta_{n-1}-2\delta_n}\ \text{ if } n=2$$

and

$$\begin{aligned}&S^2(U^*\otimes V)\otimes(U^*\otimes V)=\\&V_{3\varepsilon_1-\delta_{n-1}-2\delta_n}\oplus V_{2\varepsilon_1+\varepsilon_2-3\delta_n}\oplus 2V_{2\varepsilon_1+\varepsilon_2-\delta_{n-1}-2\delta_n}\oplus V_{3\varepsilon_1-\delta_{n-2}-\delta_{n-1}-\delta_n}\oplus\\&V_{\varepsilon_1+\varepsilon_2+\varepsilon_3-3\delta_n}\oplus V_{2\varepsilon_1+\varepsilon_2-\delta_{n-2}-\delta_{n-1}-\delta_n}\oplus V_{\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_{n-1}-2\delta_n}\qquad\text{if } n>2.\end{aligned}$$

Moreover, $S^3(U^*\otimes V)=$

$$\begin{cases}V_{2\varepsilon_1+\varepsilon_2-\delta_1-2\delta_2} & \text{if } n=2\\ V_{\varepsilon_1+\varepsilon_2+\varepsilon_3-3\delta_n}\oplus V_{3\varepsilon_1-\delta_{n-2}-\delta_{n-1}-\delta_n}\oplus V_{2\varepsilon_1+\varepsilon_2-\delta_{n-1}-2\delta_n} & \text{if } n>2.\end{cases}$$

17 Thus, by (8.43)[17]

$$\begin{aligned}&\operatorname{Im}\bar\partial^{4,1}_{\mathfrak g_0}=V_{3\varepsilon_1-\delta_{n-1}-2\delta_n}\oplus V_{2\varepsilon_1+\varepsilon_2-3\delta_n}\oplus V_{2\varepsilon_1+\varepsilon_2-\delta_{n-1}-2\delta_n} && \text{if } n=2,\\&\operatorname{Im}\bar\partial^{4,1}_{\mathfrak g_0}=V_{3\varepsilon_1-\delta_{n-1}-2\delta_n}\oplus V_{2\varepsilon_1+\varepsilon_2-3\delta_n}\oplus V_{2\varepsilon_1+\varepsilon_2-\delta_{n-1}-2\delta_n}\oplus\\&\oplus V_{2\varepsilon_1+\varepsilon_2-\delta_{n-2}-\delta_{n-1}-\delta_n}\oplus V_{\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_{n-1}-2\delta_n} && \text{if } n>2.\end{aligned}$$

Finally, the decomposition of the $\mathfrak{gl}(n)\oplus\mathfrak{gl}(n)$-module $\mathfrak g^*_{-1}\otimes\Lambda^2\mathfrak g^*_{-1}$ into the direct sum of irreducible components is given in Table 15. Checking the action of $\bar\partial^{3,2}_{\mathfrak g_0}$ on the highest vectors we get:

$$\operatorname{Im}\bar\partial^{4,1}_{\mathfrak g_0}=\operatorname{Ker}\bar\partial^{3,2}_{\mathfrak g_0}.$$

Note that for $k>3$ the cohomology groups $\bar H^{k,2}_{\mathfrak g_0}$ coincide with the Spencer cohomology groups $H^{k-2,2}_{\mathfrak o(n^2)}$ corresponding to the Cartan prolongation $\mathfrak g_*(V(0|n^2),\ \mathfrak o(n^2))=\mathfrak h(0|n^2)$, where $V(0|n^2)$ is the standard (odd) $\mathfrak o(n^2)$-module. These groups vanish for $k>3$ (see Theorem 8.3.1.3).

□

8.3. The analogues of the Riemann–Weyl tensors for classical superspaces

Recall that $\mathbb Z$-grading of depth 1 of a Lie (super)algebra $\mathfrak g$ is the $\mathbb Z$-grading of the form $\mathfrak g=\underset{i\geq -1}{\oplus}\mathfrak g_i$. All such $\mathbb Z$-gradings of simple finite-dimensional complex Lie superalgebras are listed in [S2]. Denote by V_λ the irreducible module over a Lie superalgebra with highest weight λ and an even highest vector.

8.3.1. Spencer cohomology of $\mathfrak{sl}(m|n)$ and $\mathfrak{psl}(n|n)$.

8.3.1.1. Description of the $\mathbb Z$-gradings of depth 1. Let $V(m-p|q)$ and $U(p|n-q)$ be the standard $\mathfrak{sl}(m-p|q)$ and $\mathfrak{sl}(p|n-q)$- modules, respectively.

All $\mathbb Z$-gradings of depth 1 of $\mathfrak g=\mathfrak{sl}(m|n)$ and $\mathfrak{psl}(n|n)$ are of the form $\mathfrak g=\mathfrak g_{-1}\oplus\mathfrak g_0\oplus\mathfrak g_1$, where $\mathfrak g_1=\mathfrak g^*_{-1}=V(m-p|q)\otimes U(p|n-q)^*$.

A) For $\mathfrak{sl}(m|n)$, where $m\neq n$, there are the following possible values of $\mathfrak g_0$ for the $\mathbb Z$-gradings of depth 1:

a) $\mathfrak c(\mathfrak{sl}(m)\oplus\mathfrak{sl}(n))$;

b) $\mathfrak c(\mathfrak{sl}(m|q)\oplus\mathfrak{sl}(n-q))$, if $p=0,q\neq 0,n-q\neq 0$;

c) $\mathfrak c(\mathfrak{sl}(m-p)\oplus\mathfrak{sl}(p|n))$, if $q=0,p\neq 0,m-p\neq 0$;

d) $\mathfrak c(\mathfrak{sl}(m-p|q)\oplus\mathfrak{sl}(p|n-q))$, if $p\neq 0,q\neq 0$.

B) For $\mathfrak{sl}(n|n)$, there are the following possible values of $\mathfrak g_0$ for the $\mathbb Z$-gradings of depth 1:

a) $\mathfrak c(\mathfrak{sl}(n)\oplus\mathfrak{sl}(n))$;

b) $\mathfrak c(\mathfrak{sl}(n|q)\oplus\mathfrak{sl}(n-q))$, if $p=0,q\neq 0,n-q\neq 0$;

c) $\mathfrak c(\mathfrak{sl}(n-p)\oplus\mathfrak{sl}(p|n))$, if $q=0,p\neq 0,n-p\neq 0$;

d) $\mathfrak c(\mathfrak{sl}(n-p|q)\oplus\mathfrak{sl}(p|n-q))$, if $p\neq 0,q\neq 0$.

C) The $\mathbb Z$-gradings of $\mathfrak{psl}(n|n)$ are similar to those of $\mathfrak{sl}(n|n)$, only $\mathfrak g_0$ is centerless.

8.3.1.2. Theorem (Cartan prolongs)**.** *For the cases of section 8.3.1.1, we have:*

A) $\mathfrak g=\mathfrak{sl}(m|n)$, where $m\neq n$. Then $\mathfrak g_*(\mathfrak g_{-1},\ \mathfrak g_0)=\mathfrak g$, except for the following cases:

a) if $n=1$, then $\mathfrak g_*=\mathfrak{vect}(0|m)$, if $m=1$, then $\mathfrak g_*=\mathfrak{vect}(0|n)$;

b) if $n-q=1$, then $\mathfrak g_*=\mathfrak{vect}(q|m)$, if $m=0,q=1$, then $\mathfrak g_*=\mathfrak{vect}(n-1|0)$;

c) if $m-p=1$, then $\mathfrak g_*=\mathfrak{vect}(p|n)$, if $n=0,p=1$, then $\mathfrak g_*=\mathfrak{vect}(m-1|0)$;

d) if $n-q=0,p=1$, then $\mathfrak g_*=\mathfrak{vect}(m-1|n)$, if $m-p=0,q=1$, then $\mathfrak g_*=\mathfrak{vect}(n-1|m)$.

B) $\mathfrak g=\mathfrak{sl}(n|n)$. Then $\mathfrak g_*(\mathfrak g_{-1},\ \mathfrak g_0)=S^*(\mathfrak g^*_{-1})\Subset\mathfrak{psl}(n|n)$, except for the following cases:

a) if $n=2$, then $\mathfrak g_*=S^*(\mathfrak g^*_{-1})\Subset\mathfrak h(0|4)$;

b) if $n-q=1$, then $\mathfrak g_*=\mathfrak{vect}(q|n)$;

c) if $n-p=1$, then $\mathfrak g_*=\mathfrak{vect}(p|n)$;

d) if $n-q=0,p=1$ or $n-p=0,q=1$ then $\mathfrak g_*=\mathfrak{vect}(n-1|n)$.

C) $\mathfrak g=\mathfrak{psl}(n|n)$. Then $\mathfrak g_*(\mathfrak g_{-1},\ \mathfrak g_0)=\mathfrak g$, except for the following cases:

a) if $n=2$, then $\mathfrak g_*=\mathfrak h(0|4)$;

b) if $n-q=1$, then $\mathfrak g_*=\mathfrak{svect}(q|n)$;

c) if $n-p=1$, then $\mathfrak g_*=\mathfrak{svect}(p|n)$;

d) if $n-q=0,p=1$ or $n-p=0,q=1$, then $\mathfrak g_*=\mathfrak{svect}(n-1|n)$.

Let $\mathfrak g=\mathfrak{sl}(m|n)$, where $m\neq n$, or $\mathfrak{psl}(n|n)$. We will describe the Spencer cohomology groups for all $\mathbb Z$-gradings of depth 1 listed in section 8.3.1.1.

First consider the cases easiest to formulate. Let $\langle\pi_i\rangle$ be the irreducible module whose highest weight is the i-th fundamental weight of $\mathfrak g_0$.

8.3.1.3. Theorem. 1) *For* $\mathfrak{g}_* = \mathfrak{vect}(m|n)$, $\mathfrak{svect}(m|n)$, *SFs vanish except for* $\mathfrak{svect}(0|n)$, *when SFs are of order* n *and constitute the* $\mathfrak{g}_0$*-module* $\Pi^n(\langle 1\rangle)$.

2) *For* $\mathfrak{g}_*$ *of series* $\mathfrak{h}(0|n)$, *the nonzero SFs are of order 1. For* $n>3$, *SFs constitute* $\mathfrak{g}_0$*-module* $\Pi(V_{3\pi_1}\oplus V_{\pi_1})$.

3) *For* $\mathfrak{g}_* = \mathfrak{sh}(0|n)$, *the nonzero SFs are the same as for* $\mathfrak{h}(0|n)$ *and additionally* $\Pi^{n-1}(V_{\pi_1})$ *of order* $n-1$.

Consider the $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{g}=\mathfrak{sl}(m|n)$ $(m\neq n)$ and $\mathfrak{psl}(n|n)$ listed in section 8.3.1.1 for which $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)=\mathfrak{g}$. Describe the corresponding SFs.

Case a) was discussed in §2. Consider case b).

8.3.1.4. Theorem. *The nonzero SFs are of orders 1 and 2. The* $\mathfrak{g}_0$*-module* $H^{2,2}_{\mathfrak{g}_0}$ *splits into the direct sum of irreducible components whose weights are given in Table 16. Table 16 also contains the highest weights (with respect to the bases* $\varepsilon_1,\dots,\varepsilon_{m+q}$ *and* $\delta_1,\dots,\delta_{n-q}$ *of the dual spaces to the maximal tori of* $\mathfrak{sl}(m|q)$ *and* $\mathfrak{sl}(n-q)$*, respectively) of irreducible components of* $H^{1,2}_{\mathfrak{g}_0}$ *for the cases when* $H^{1,2}_{\mathfrak{g}_0}$ *does split into the direct sum of irreducible* $\mathfrak{g}_0$*-modules.*

Exceptional cases are as follows: if $m=q-1, m>1, n-q\geq 3$, then

$$H^{1,2}_{\mathfrak{g}_0} = V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m+q}+\delta_1-\delta_{n-q-1}-\delta_{n-q}}\oplus X,$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$0\longrightarrow V_{2\varepsilon_1-\varepsilon_{m+q}+\delta_1-2\delta n-q}\longrightarrow X\longrightarrow \Pi(V_{\varepsilon_1+\delta_1-2\delta_{n-q}})\longrightarrow 0; \tag{8.74}$$

if $m=q-1, m>1, n-q=2$, then $H^{1,2}_{\mathfrak{g}_0}=X$, where X is given by (8.74);

if $m=1, q=2, n-q\geq 3$, then

$$H^{1,2}_{\mathfrak{g}_0} = \Pi(V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-\delta_{n-q-1}-\delta_{n-q}})\oplus X,$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$\begin{aligned}&0\longrightarrow V_{2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_{n-q}}\oplus \Pi(V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta_1-2\delta_{n-q}})\\ &\longrightarrow X\longrightarrow \Pi(V_{\varepsilon_1+\delta_1-2\delta_{n-q}})\longrightarrow 0;\end{aligned} \tag{8.75}$$

if $m=1,\ q=2,\ n-q=2$, then $H^{1,2}_{\mathfrak{g}_0}=X$, where X is given by (8.75); if $m=q+1,\ n-q\geq 3$, then

$$H^{1,2}_{\mathfrak{g}_0} = V_{2\varepsilon_1-\varepsilon_{m+q}+\delta_1-2\delta_{n-q}}\oplus X,$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$\begin{aligned}&0\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m+q}+\delta_1-\delta_{n-q-1}-\delta_{n-q}}\longrightarrow X\\ &\longrightarrow \Pi(V_{\varepsilon_1+\delta_1-\delta_{n-q-1}-\delta_{n-q}})\longrightarrow 0 \qquad (q\geq 2),\\ &0\longrightarrow V_{\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta_1-\delta_{n-2}-\delta_{n-1}}\oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-\delta_{n-2}-\delta_{n-1}}\longrightarrow X\\ &\longrightarrow \Pi(V_{\varepsilon_1+\delta_1-\delta_{n-2}-\delta_{n-1}})\longrightarrow 0 \qquad (q=1).\end{aligned}$$

Case c) is similar to case b). Consider case d).

8.3.1.5. Theorem. *The nonzero SFs are of orders 1 and 2. The* $\mathfrak{g}_0$*-module* $H^{2,2}_{\mathfrak{g}_0}$ *splits into the direct sum of irreducible components whose weights are given in Table 17. Table 17 also contains the highest weights (with respect to the bases* $\varepsilon_1,\dots,\varepsilon_{m-p+q}$ *and* $\delta_1,\dots,\delta_{p+n-q}$ *of the dual spaces to the maximal tori of* $\mathfrak{sl}(m-p|q)$ *and* $\mathfrak{sl}(p|n-q)$*, respectively) of irreducible components of* $H^{1,2}_{\mathfrak{g}_0}$ *for the cases when* $H^{1,2}_{\mathfrak{g}_0}$ *does split into the direct sum of irreducible* $\mathfrak{g}_0$*-modules.*

Exceptional cases are $m=p+q\pm 1$ and $n=p+q\pm 1$. More precisely: if $m=p+q+1,\ n\neq p+q\pm 1,\ q$, then

$$H^{1,2}_{\mathfrak{g}_0} = V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}}\oplus Y,$$

where Y is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$\begin{aligned}&0\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}}\oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}}\longrightarrow Y\\ &\longrightarrow V_{\varepsilon_1+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}}\longrightarrow 0 \qquad (q=1),\\ &0\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}}\longrightarrow Y\\ &\longrightarrow V_{\varepsilon_1+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}}\longrightarrow 0 \qquad (q\geq 2);\end{aligned}$$

if $m=p+q+1,\ n=p+q-1$, then $H^{1,2}_{\mathfrak{g}_0}=X\oplus Y$, where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$\begin{aligned}&0\longrightarrow V_{2\varepsilon_1-\varepsilon_{m-2+q}+\delta_1-2\delta_3}\oplus V_{2\varepsilon_1-\varepsilon_{m-2+q}-\delta_1-\delta_2+\delta_3}\longrightarrow\\ &X\longrightarrow V_{2\varepsilon_1-\varepsilon_{m-2+q}-\delta_3}\longrightarrow 0 \qquad (p=2),\\ &0\longrightarrow V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}}\longrightarrow X\\ &\longrightarrow V_{2\varepsilon_1-\varepsilon_{m-p+q}-\delta_{p+n-q}}\longrightarrow 0 \qquad (p\geq 3),\end{aligned}$$

and Y is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$\begin{aligned}&0\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}}\oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}}\longrightarrow\\ &Y\longrightarrow V_{\varepsilon_1+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}}\longrightarrow 0 \qquad (q=1),\\ &0\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}}\longrightarrow Y\\ &\longrightarrow V_{\varepsilon_1+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}}\longrightarrow 0 \qquad (q\geq 2);\end{aligned}$$

if $m=p+q+1,\ n=q$, then

$$H^{1,2}_{\mathfrak{g}_0} = V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-\delta_{p-1}-\delta_p}\oplus Y \quad (p\geq 3) \quad \text{or} \quad H^{1,2}_{\mathfrak{g}_0}=Y \quad (p=2),$$

where Y is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$\begin{aligned}&0\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-2\delta_p}\oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta_1-2\delta_p}\longrightarrow Y\longrightarrow V_{\varepsilon_1+\delta_1-2\delta_p}\longrightarrow 0 \quad (q=1),\\ &0\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-2\delta_p}\longrightarrow Y\longrightarrow V_{\varepsilon_1+\delta_1-2\delta_p}\longrightarrow 0 \qquad (q\geq 2);\end{aligned}$$

if $n=p+q+1,\ m\neq p+q\pm 1,\ p$, then

$$H^{1,2}_{\mathfrak{g}_0} = V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}} \oplus Y,$$

where

$$\begin{aligned}
&0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}+\delta_1-\delta_2-\delta_3} \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}-\delta_1+\delta_2-\delta_3} \longrightarrow \\
&Y \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}-\delta_3} \longrightarrow 0 \qquad (p=1),\\
&0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \longrightarrow Y \\
&\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}-\delta_{p+n-q}} \longrightarrow 0 \qquad (p\geq 2);
\end{aligned}$$

if $n=p+q+1$, $m=p+q-1$, then $H^{1,2}_{\mathfrak{g}_0} = X \oplus Y$, where

$$\begin{aligned}
&0 \longrightarrow V_{2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_{p+n-2}} \oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta_1-2\delta_{p+n-2}} \longrightarrow X \\
&\longrightarrow V_{\varepsilon_1+\delta_1-2\delta_{p+n-2}} \longrightarrow 0 \qquad (q=2),\\
&0 \longrightarrow V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}} \longrightarrow X \longrightarrow V_{\varepsilon_1+\delta_1-2\delta_{p+n-q}} \longrightarrow 0 \quad (q\geq 3),
\end{aligned}$$

and Y is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$\begin{aligned}
&0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}+\delta_1-\delta_2-\delta_3} \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}-\delta_1+\delta_2-\delta_3} \\
&\longrightarrow Y \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}-\delta_3} \longrightarrow 0 \qquad (p=1),\\
&0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \longrightarrow Y \\
&\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}-\delta_{p+n-q}} \longrightarrow 0 \qquad (p\geq 2);
\end{aligned}$$

if $n=p+q+1$, $m=p$, then

$$H^{1,2}_{\mathfrak{g}_0} = V_{\varepsilon_1+\varepsilon_2-\varepsilon_q+\delta_1-2\delta_{p+n-q}} \oplus Y \quad (q\geq 3), \quad \text{and} \quad H^{1,2}_{\mathfrak{g}_0} = Y \quad (q=2),$$

where

$$\begin{aligned}
&0 \longrightarrow V_{2\varepsilon_1-\varepsilon_q+\delta_1-\delta_2-\delta_3} \oplus V_{2\varepsilon_1-\varepsilon_q-\delta_1+\delta_2-\delta_3} \longrightarrow Y \longrightarrow V_{2\varepsilon_1-\varepsilon_q-\delta_3} \longrightarrow 0 \quad (p=1),\\
&0 \longrightarrow V_{2\varepsilon_1-\varepsilon_q+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \longrightarrow Y \longrightarrow V_{2\varepsilon_1-\varepsilon_q-\delta_{p+n-q}} \longrightarrow 0 \qquad (p\geq 2);
\end{aligned}$$

if $m=p+q-1$, $n\neq p+q\pm 1$, q, then

$$H^{1,2}_{\mathfrak{g}_0} = X \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}},$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$\begin{aligned}
&0 \longrightarrow V_{2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_{p+n-2}} \oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta_1-2\delta_{p+n-2}} \longrightarrow \\
&X \longrightarrow V_{\varepsilon_1+\delta_1-2\delta_{p+n-2}} \longrightarrow 0 \qquad (q=2),\\
&0 \longrightarrow V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}} \longrightarrow X \longrightarrow V_{\varepsilon_1+\delta_1-2\delta_{p+n-q}} \longrightarrow 0 \quad (q\geq 3);
\end{aligned}$$

if $m=p+q-1$, $n=q$, then

$$H^{1,2}_{\mathfrak{g}_0} = X \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-2\delta_p} \quad (p\geq 3),$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$\begin{aligned}
&0 \longrightarrow V_{2\varepsilon_1-\varepsilon_3+\delta_1-\delta_{p-1}-\delta_p} \oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta_1-\delta_{p-1}-\delta_p} \longrightarrow \\
&X \longrightarrow V_{\varepsilon_1+\delta_1-\delta_{p-1}-\delta_p} \longrightarrow 0 \qquad (q=2),\\
&0 \longrightarrow V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-\delta_{p-1}-\delta_p} \longrightarrow X \longrightarrow V_{\varepsilon_1+\delta_1-\delta_{p-1}-\delta_p} \longrightarrow 0 \quad (q\geq 3);
\end{aligned}$$

if $n=p+q-1$, $m\neq p+q\pm 1$, p, then

$$H^{1,2}_{\mathfrak{g}_0} = X \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}},$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$\begin{aligned}
&0 \longrightarrow V_{2\varepsilon_1-\varepsilon_{m-2+q}+\delta_1-2\delta_3} \oplus V_{2\varepsilon_1-\varepsilon_{m-2+q}-\delta_1-\delta_2+\delta_3} \longrightarrow \\
&X \longrightarrow V_{2\varepsilon_1-\varepsilon_{m-2+q}-\delta_3} \longrightarrow 0 \qquad (p=2),\\
&0 \longrightarrow V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}} \longrightarrow X \longrightarrow V_{2\varepsilon_1-\varepsilon_{m-p+q}-\delta_{p+n-q}} \longrightarrow 0 \quad (p\geq 3);
\end{aligned}$$

if $n=p+q-1$, $m=p$, then

$$H^{1,2}_{\mathfrak{g}_0} = X \oplus V_{2\varepsilon_1-\varepsilon_q+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \quad (q\geq 3),$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$\begin{aligned}
&0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_q+\delta_1-2\delta_3} \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_q-\delta_1-\delta_2+\delta_3} \longrightarrow \\
&X \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_q-\delta_3} \longrightarrow 0 \qquad (p=2),\\
&0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_q+\delta_1-2\delta_{p+n-q}} \longrightarrow X \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_q-\delta_{p+n-q}} \longrightarrow 0 \quad (p\geq 3);
\end{aligned}$$

if $m=n=p+q+1$, then

$$H^{1,2}_{\mathfrak{g}_0} = X \oplus V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}} \oplus V_{\varepsilon_1-\delta_{p+n-q}},$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$0 \longrightarrow Y \longrightarrow X \longrightarrow V_{\varepsilon_1-\delta_{p+n-q}} \longrightarrow 0,$$

and Y is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$\begin{aligned}
&0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta p+n-q} \longrightarrow Y \\
&\longrightarrow V_{\varepsilon_1+\delta_1-\delta_{p+n-q-1}-\delta p+n-q} \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}-\delta p+n-q} \longrightarrow 0 \qquad (p\geq 2,\ q\geq 2),\\
&0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}+\delta_1-\delta_2-\delta_3} \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}-\delta_1+\delta_2-\delta_3} \longrightarrow Y \\
&\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}-\delta_3} \oplus V_{\varepsilon_1+\delta_1-\delta_2-\delta_3} \oplus V_{\varepsilon_1-\delta_1+\delta_2-\delta_3} \longrightarrow 0 \qquad (p=1,\ q\geq 2),\\
&0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}} \oplus \\
&V_{\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}} \longrightarrow Y \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_3-\delta_{p+n-1}} \oplus \\
&V_{\varepsilon_1+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}} \oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3-\delta_{p+n-1}} \longrightarrow 0 \qquad (p\geq 2,\ q=1),\\
&0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-\delta_2-\delta_3} \oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta_1-\delta_2-\delta_3} \oplus \\
&V_{\varepsilon_1+\varepsilon_2-\varepsilon_3-\delta_1+\delta_2-\delta_3} \oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3-\delta_1+\delta_2-\delta_3} \longrightarrow Y \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_3-\delta_3} \oplus \\
&V_{\varepsilon_1-\varepsilon_2+\varepsilon_3-\delta_3} \oplus V_{\varepsilon_1+\delta_1-\delta_2-\delta_3} \oplus V_{\varepsilon_1-\delta_1+\delta_2-\delta_3} \longrightarrow 0 \qquad (p=1,\ q=1)
\end{aligned}$$

if $m=n=p+q-1$, then

$$H^{1,2}_{\mathfrak{g}_0}=X\oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}}\oplus V_{\varepsilon_1-\delta_{p+n-q}},$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$0\longrightarrow Y\longrightarrow X\longrightarrow V_{\varepsilon_1-\delta_{p+n-q}}\longrightarrow 0,$$

and Y is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$\begin{aligned}&0\longrightarrow V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}}\longrightarrow Y\\&\longrightarrow V_{2\varepsilon_1-\varepsilon_{m-p+q}-\delta_{p+n-q}}\oplus V_{\varepsilon_1+\delta_1-2\delta_{p+n-q}}\longrightarrow 0\qquad (p\geq 3, q\geq 3),\end{aligned}$$

$$\begin{aligned}&0\longrightarrow V_{2\varepsilon_1-\varepsilon_{m-2+q}+\delta_1-2\delta_3}\oplus V_{2\varepsilon_1-\varepsilon_{m-2+q}-\delta_1-\delta_2+\delta_3}\longrightarrow Y\\&\longrightarrow V_{2\varepsilon_1-\varepsilon_{m-2+q}-\delta_3}\oplus V_{\varepsilon_1+\delta_1-2\delta_3}\oplus V_{\varepsilon_1-\delta_1-\delta_2+\delta_3}\longrightarrow 0\qquad (p=2,\ q\geq 3),\end{aligned}$$

$$\begin{aligned}&0\longrightarrow V_{2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_{p+n-2}}\oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta_1-2\delta_{p+n-2}}\longrightarrow Y\\&\longrightarrow V_{\varepsilon_1+\delta_1-2\delta_{p+n-2}}\oplus V_{2\varepsilon_1-\varepsilon_3-\delta_{p+n-2}}\oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_{p+n-2}}\longrightarrow 0\quad (p\geq 3,\ q=2),\end{aligned}$$

$$\begin{aligned}&0\longrightarrow V_{2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_3}\oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta_1-2\delta_3}\oplus V_{2\varepsilon_1-\varepsilon_3-\delta_1-\delta_2+\delta_3}\oplus\\&V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_1-\delta_2+\delta_3}\longrightarrow Y\longrightarrow V_{2\varepsilon_1-\varepsilon_3-\delta_3}\oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_3}\oplus\\&V_{\varepsilon_1+\delta_1-2\delta_3}\oplus V_{\varepsilon_1-\delta_1-\delta_2+\delta_3}\longrightarrow 0\qquad (p=2,\ q=2).\end{aligned}$$

Remark. The irreducible $\mathfrak{g}_0$-modules in the above listed non-split exact sequences are given regardless of their parity, which can be easily recovered from the corresponding highest weights.

8.3.2. Spencer cohomology of $\mathfrak{psq}(n)$.

8.3.2.1. Definitions. Set

$$\mathfrak{s}(\mathfrak{q}(p)\oplus\mathfrak{q}(n-p))=\{X\in\mathfrak{q}(p)\oplus\mathfrak{q}(n-p)\ \mid \operatorname{qtr}\mathfrak{q}(p)+\operatorname{qtr}\mathfrak{q}(n-p)=0\},$$

$$\mathfrak{ps}(\mathfrak{q}(p)\oplus\mathfrak{q}(n-p))=\mathfrak{s}(\mathfrak{q}(p)\oplus\mathfrak{q}(n-p))/\langle 1_p+1_{n-p}\rangle.$$

8.3.2.2. $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{psq}(n)$. Let $V(n\ \mid n)$ be the standard $\mathfrak{q}(n)$-module. All $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{g}=\mathfrak{psq}(n)$ are of the form $\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$, where $\mathfrak{g}_0=\mathfrak{ps}(\mathfrak{q}(p)\oplus\mathfrak{q}(n-p))$, $p>0$, and as $\mathfrak{g}_0$-modules $\mathfrak{g}_1\cong\mathfrak{g}_{-1}^*$, where $\mathfrak{g}_{-1}$ is either one of the two irreducible $\mathfrak{g}_0$-modules in $V(p\ \mid p)^*\otimes V(n-p\ \mid n-p)$. Explicitly:

$$\mathfrak{g}_{-1}=\langle(x\pm\Pi(x))\otimes(y\pm\Pi(y))\rangle,\ \text{where } x\in V(p\mid p)^*,\quad y\in V(n-p\mid n-p).$$

Let $\varepsilon_1,\dots,\varepsilon_p$ and $\delta_1,\dots,\delta_{n-p}$ be the standard bases of the dual spaces to the spaces of diagonal matrices in $\mathfrak{q}(p)$ and $\mathfrak{q}(n-p)$, respectively.

8.3.2.3. Theorem. 1) $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)=\mathfrak{g}$,

2) *all SFs are of order 1 and split into the direct sum of two irreducible $\mathfrak{g}_0$-submodules with highest weights $2\varepsilon_1-\varepsilon_p+\delta_1-2\delta_{n-p}$ and $\varepsilon_1-\delta_{n-p}$.*

8.3.3. Spencer cohomology of $\mathfrak{osp}(m|2n)$.

8.3.3.1. Definition of $\mathfrak{osp}(m|2n)$. Let $\mathfrak{osp}(m|2n)$ be the Lie superalgebra, which preserves a nondegenerate supersymmetric even bilinear form on a superspace V of $\operatorname{sdim}V=(m|2n)$.

8.3.3.2. Consider the $\mathbb{Z}$-grading of depth 1 of $\mathfrak{g}=\mathfrak{osp}(m|2n)$, which is defined as follows: $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$, where $\mathfrak{g}_0=\mathfrak{cosp}(m-2|2n)$ is the central extension of $\hat{\mathfrak{g}_0}=\mathfrak{osp}(m-2|2n)$, $\mathfrak{g}_1\cong\mathfrak{g}_{-1}$ is the standard $\mathfrak{g}_0$-module.

Let $m=2r+2$ or $m=2r+3$, $n>0$. Let $\varepsilon_1,\dots,\varepsilon_r$ and $\delta_1,\dots,\delta_n$ be the standard bases of the dual spaces to the spaces of diagonal matrices in $\mathfrak{o}(m-2)$ and $\mathfrak{sp}(n)$, respectively.

8.3.3.3. Theorem. 1) $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)=\mathfrak{g}$, $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0})=\mathfrak{g}_{-1}\oplus\hat{\mathfrak{g}_0}$.

2) *If $k\neq 2$, then $H^{k,2}_{\mathfrak{g}_0}=H^{k,2}_{\hat{\mathfrak{g}_0}}=0$. As a $\hat{\mathfrak{g}_0}$-module $H^{2,2}_{\hat{\mathfrak{g}_0}}$ is isomorphic to $S^2(E^2(\mathfrak{g}_{-1}))/E^4(\mathfrak{g}_{-1})$ and splits into the direct sum of three irreducible components (analogues of the Weyl tensor, the traceless Ricci tensor, and the scalar curvature). The highest weights of these components are listed in Table 18. As $\hat{\mathfrak{g}_0}$-modules, $H^{2,2}_{\hat{\mathfrak{g}_0}}\cong H^{2,2}_{\mathfrak{g}_0}\oplus S^2(\mathfrak{g}_{-1})$. The $\mathfrak{g}_0$-module $H^{2,2}_{\mathfrak{g}_0}$ is irreducible.*

8.3.4. Spencer cohomology of $\mathfrak{osp}(4|2;\alpha)$.

8.3.4.1. Definition of $\mathfrak{osp}(4|2;\alpha)$. $\mathfrak{osp}(4|2;\alpha)$, where $\alpha\in\mathbb{C}\backslash\{0,-1\}$, is a one-parameter family consisting of all simple Lie superalgebras for which $\mathfrak{osp}(4|2;\alpha)_0=\mathfrak{sl}(2)_1\oplus\mathfrak{sl}(2)_2\oplus\mathfrak{sl}(2)_3$ and its representation on $\mathfrak{osp}(4|2;\alpha)_1$ is $V_1\otimes V_2\otimes V_3$, where V_i is the standard $\mathfrak{sl}(2)_i$-module.

8.3.4.2. $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{osp}(4|2;\alpha)$. Let $\varepsilon_1,\varepsilon_2,\varepsilon_3$ be the standard basis of the dual space to the space of diagonal matrices in $\mathfrak{gl}(1|2)$, V_λ be the irreducible $\mathfrak{sl}(1|2)$-module with highest weight λ and an even highest weight vector. All $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{osp}(4|2;\alpha)$ are of the form $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$, where $\mathfrak{g}_0\cong\mathfrak{gl}(1|2)$. There are the following possible values of $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ for the $\mathbb{Z}$-gradings of depth 1:

a) $\mathfrak{g}_1=V_{(1+\alpha)\varepsilon_1}$, $\mathfrak{g}_{-1}=V_{-\alpha\varepsilon_1}$,
b) $\mathfrak{g}_1=V_{(\frac{1+\alpha}{\alpha})\varepsilon_1}$, $\mathfrak{g}_{-1}=V_{-\frac{1}{\alpha}\varepsilon_1}$,
c) $\mathfrak{g}_1=V_{(\frac{\alpha}{1+\alpha})\varepsilon_1}$, $\mathfrak{g}_{-1}=V_{\frac{1}{1+\alpha}\varepsilon_1}$.

More explicitly, let e_1^i, e_2^i be the basis of the standard $\mathfrak{sl}(2)_i$-module V_i. Then the $\mathbb{Z}$-grading in case a) can be described as follows: $\mathfrak{g}_0=(\mathfrak{g}_0)_0\oplus(\mathfrak{g}_0)_1$, where

$$(\mathfrak{g}_0)_0=\mathfrak{sl}(2)_1\oplus\langle\begin{pmatrix}1&0\\0&-1\end{pmatrix}_2\rangle\oplus\langle\begin{pmatrix}1&0\\0&-1\end{pmatrix}_3\rangle,(\mathfrak{g}_0)_1=V_1\otimes e_1^2\otimes e_2^3\oplus V_1\otimes e_2^2\otimes e_1^3;$$

$\mathfrak{g}_1=(\mathfrak{g}_1)_0\oplus(\mathfrak{g}_1)_1$, where

$$(\mathfrak{g}_1)_0=\langle\begin{pmatrix}0&1\\0&0\end{pmatrix}_2\rangle\oplus\langle\begin{pmatrix}0&1\\0&0\end{pmatrix}_3\rangle,(\mathfrak{g}_1)_1=V_1\otimes e_1^2\otimes e_1^3;$$

$\mathfrak{g}_{-1}=(\mathfrak{g}_{-1})_0\oplus(\mathfrak{g}_{-1})_1$, where

$$(\mathfrak{g}_{-1})_0 = \langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_2 \rangle \oplus \langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_3 \rangle, (\mathfrak{g}_{-1})_1 = V_1 \otimes e_2^2 \otimes e_2^3.$$

The $\mathbb{Z}$-gradings in cases b) and c) can be described similarly.

8.3.4.3. Theorem. *For all $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{g} = \mathfrak{osp}(4|2;\alpha)$ we have*

1) $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = \mathfrak{g}$.

2) *The nonzero SFs are of order 2, and for the cases considered in section 8.3.4.2, the $\mathfrak{g}_0$-module $H^{2,2}_{\mathfrak{g}_0}$ is isomorphic to*

a) $\Pi(V_{(2\alpha+1)\varepsilon_1+\varepsilon_2})$,

b) $\Pi(V_{(\frac{\alpha+2}{\alpha})\varepsilon_1+\varepsilon_2})$,

c) $\Pi(V_{(\frac{\alpha-1}{\alpha+1})\varepsilon_1+\varepsilon_2})$, *respectively.*

8.3.5. Spencer cohomology of $\mathfrak{ab}(3)$.

8.3.5.1. Definition of $\mathfrak{ab}(3)$. $\mathfrak{ab}(3)$ is a simple Lie superalgebra for which $(\mathfrak{ab}(3))_0 = \mathfrak{sl}(2) \oplus \mathfrak{o}(7)$ and its representation on $(\mathfrak{ab}(3))_1$ is $V \otimes spin_7$.

8.3.5.2. $\mathbb{Z}$-grading of depth 1 of $\mathfrak{ab}(3)$. Let $\varepsilon_1, \delta_1, \delta_2$ be the standard basis of the dual space to the space of diagonal matrices in $\mathfrak{osp}(2|4)$, V_λ be an irreducible $\mathfrak{osp}(2|4)$-module with highest weight λ and an even highest vector.

There is only one $\mathbb{Z}$-grading of depth 1 in $\mathfrak{g} = \mathfrak{ab}(3)$, namely, $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_{-1} = V_{-\varepsilon_1+\delta_1+\delta_2}$, $\mathfrak{g}_0 = \mathfrak{cosp}(2|4)$, $\mathfrak{g}_1 = V_{3\varepsilon_1}$.

Note that $\mathfrak{o}(7) = V_1 \oplus \mathfrak{o}(5) \oplus \mathbb{C} \oplus V_2$, where V_1, V_2 are standard $\mathfrak{o}(5)$-modules. The space of the representation $spin_7$ after restriction of $\mathfrak{o}(7)$ to $\mathfrak{o}(5)$ decomposes into the direct sum of two irreducible subspaces, which we denote by U_1 and U_2. Let e_1, e_2 be the basis of the standard $\mathfrak{sl}(2)$-module V. Then $\mathfrak{g}_0 = (\mathfrak{g}_0)_0 \oplus (\mathfrak{g}_0)_1$, where

$(\mathfrak{g}_0)_0 = \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle \oplus \mathfrak{o}(5) \oplus \mathbb{C}, (\mathfrak{g}_0)_1 = e_1 \otimes U_1 \oplus e_2 \otimes U_2$;

$\mathfrak{g}_1 = (\mathfrak{g}_1)_0 \oplus (\mathfrak{g}_1)_1$, where $(\mathfrak{g}_1)_0 = \langle \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \rangle \oplus V_2, (\mathfrak{g}_1)_1 = e_1 \otimes U_2$;

$\mathfrak{g}_{-1} = (\mathfrak{g}_{-1})_0 \oplus (\mathfrak{g}_{-1})_1$, where $(\mathfrak{g}_{-1})_0 = \langle \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \rangle \oplus V_1, (\mathfrak{g}_{-1})_1 = e_2 \otimes U_1$.

8.3.5.3. Theorem. 1) $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = \mathfrak{g}$.

2) *The nonzero SFs are of order 1. The $\mathfrak{g}_0$- module $H^{1,2}_{\mathfrak{g}_0}$ is given by the non-split exact sequence of $\mathfrak{g}_0$- modules*

$$0 \longrightarrow X \longrightarrow H^{1,2}_{\mathfrak{g}_0} \longrightarrow V_{\varepsilon_1+2\delta_1} \longrightarrow 0,$$

where X is given by the non-split exact sequence of $\mathfrak{g}_0$-modules

$$0 \longrightarrow \Pi(V_{4\varepsilon_1+2\delta_1+\delta_2}) \longrightarrow X \longrightarrow V_{3\varepsilon_1+2\delta_1} \longrightarrow 0.$$

Appendix. The dimension formula for irreducible $\mathfrak{sl}(n)$-modules

Let $\varepsilon_1, \ldots, \varepsilon_n$ be the standard basis of the dual space to the space of diagonal matrices in $\mathfrak{gl}(n), V_\lambda$ be the irreducible $\mathfrak{sl}(n)$-module with highest weight $\lambda = k_1\varepsilon_1 + k_2\varepsilon_2 + \ldots + k_n\varepsilon_n$, where $k_i \in \mathbb{Z}$. Then

$$\dim V_\lambda = \prod_{i=1}^{n-1} \prod_{j=1}^{n-i} \left(1 + \frac{k_i - k_{i+j}}{j}\right).$$

Proof. A weight λ is the highest weight of an irreducible $\mathfrak{sl}(n)$-module if and only if λ is a dominant integer form, i.e., if

$$2\frac{(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}_+.$$

It is known [GG1] that the inner products of the weights ε_i and of weight ρ, where $\rho = \frac{1}{2} \sum\limits_{\beta \in \Delta_+} \beta$, with fundamental weights α_j are:

$$\begin{array}{llll}
(\varepsilon_1, \alpha_1) = \frac{1}{2n}, & (\varepsilon_1, \alpha_j) = 0 & & \text{for } 2 \le j \le n-1; \\
(\varepsilon_i, \alpha_{i-1}) = -\frac{1}{2n}, & (\varepsilon_i, \alpha_i) = \frac{1}{2n}, & (\varepsilon_i, \alpha_j) = 0 & \text{for } 2 \le i \le n-1,\ j \ne i-1, i; \\
(\varepsilon_n, \alpha_{n-1}) = -\frac{1}{2n} & (\varepsilon_n, \alpha_j) = 0 & & \text{for } 1 \le j \le n-2; \\
(\rho, \alpha_i) = \frac{1}{2n} & & & \text{for } 1 \le i \le n-1.
\end{array}$$

Thus,

$$(\lambda, \alpha_i) = \frac{k_i - k_{i+1}}{2n} \text{ and } k_i \ge k_{i+1}.$$

By Weyl's character formula [GG1]

$$\dim V_\lambda = \prod_{\beta \in \Delta_+} \left(1 + \frac{(\lambda, \beta)}{(\rho, \beta)}\right).$$

For $\mathfrak{sl}(n)$,we have $\Delta_+ = \{\alpha_i + \alpha_{i+1} + \ldots + \alpha_j \mid 1 \le i \le n-1, j \ge i\}$. Since

$$(\lambda, \alpha_i + \ldots + \alpha_j) = \frac{1}{2n}((k_i - k_{i+1}) + (k_{i+1} - k_{i+2}) + \ldots + (k_j - k_{j+1})) = \frac{k_i - k_{j+1}}{2n},$$

we have

$$\prod_{\beta \in \Delta_+} \left(1 + \frac{(\lambda, \beta)}{(\rho, \beta)}\right) = \prod_{i=1}^{n-1} \prod_{j=i}^{n-1} \left(1 + \frac{\frac{k_i - k_{j+1}}{2n}}{\frac{j-i+1}{2n}}\right) = \prod_{i=1}^{n-1} \prod_{j=1}^{n-i} \left(1 + \frac{k_i - k_{i+j}}{j}\right). \quad \square$$

Tables

Table 1. Irreducible $\mathfrak{gl}(n)$-submodules of $\mathfrak{cpc}(n) \otimes V^*$.

$\mathfrak{gl}(n)$-submodule	Highest weight	Highest weight vector
$E^2V_0^* \otimes V_0$	$\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$ $-\varepsilon_n$	$f_{n-1} \wedge \tilde{f}_n \otimes \tilde{e}_1$ $\sum_{i=1}^{n} f_n \wedge \tilde{f}_i \otimes \tilde{e}_i$
$E^2V_0^* \otimes V_0^*$	$-\varepsilon_{n-1} - 2\varepsilon_n$ $-\varepsilon_{n-2} - \varepsilon_{n-1} - \varepsilon_n$	$f_{n-1} \wedge \tilde{f}_n \otimes \tilde{f}_n$ $f_{n-2} \wedge \tilde{f}_{n-1} \otimes \tilde{f}_n + f_{n-1} \wedge \tilde{f}_n \otimes \tilde{f}_{n-2}$ $+f_n \wedge \tilde{f}_{n-2} \otimes \tilde{f}_{n-1}$
$V_0^* \wedge V_0 \otimes V_0$	$2\varepsilon_1 - \varepsilon_n$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_n$ ε_1 ε_1	$f_n \wedge \tilde{e}_1 \otimes \tilde{e}_1$ $f_n \wedge \tilde{e}_1 \otimes \tilde{e}_2 - f_n \wedge \tilde{e}_2 \otimes e_1$ $\sum_{i=1}^{n} f_i \wedge \tilde{e}_i \otimes \tilde{e}_1$ $\sum_{i=1}^{n} f_i \wedge \tilde{e}_1 \otimes \tilde{e}_i$
$V_0^* \wedge V_0 \otimes V_0^*$	$\varepsilon_1 - 2\varepsilon_n$ $\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$ $-\varepsilon_n$ $-\varepsilon_n$	$f_n \wedge \tilde{e}_1 \otimes \tilde{f}_n$ $f_{n-1} \wedge \tilde{e}_1 \otimes \tilde{f}_n - f_n \wedge \tilde{e}_1 \otimes \tilde{f}_{n-1}$ $\sum_{i=1}^{n} f_i \wedge \tilde{e}_i \otimes \tilde{f}_n$ $\sum_{i=1}^{n} f_n \wedge \tilde{e}_i \otimes \tilde{f}_i$
$S^2V_0 \otimes V_0$	$3\varepsilon_1$ $2\varepsilon_1 + \varepsilon_2$	$e_1\tilde{e}_1 \otimes \tilde{e}_1$ $e_1\tilde{e}_2 \otimes \tilde{e}_1 - e_1\tilde{e}_1 \otimes \tilde{e}_2$
$S^2V_0 \otimes V_0^*$	$2\varepsilon_1 - \varepsilon_n$ ε_1	$e_1\tilde{e}_1 \otimes \tilde{f}_n$ $\sum_{i=1}^{n} e_1\tilde{e}_i \otimes \tilde{f}_i$
$z \otimes V_0$	ε_1	$\sum_{i=1}^{n} e_i\tilde{f}_i \otimes \tilde{e}_1$
$z \otimes V_0^*$	$-\varepsilon_n$	$\sum_{i=1}^{n} e_i\tilde{f}_i \otimes \tilde{f}_n$

Table 2. Irreducible $\mathfrak{gl}(n)$-submodules of $H^q(V_0, \mathfrak{g}_*)$.

q	Highest weight	Highest weight vector
0	$-\varepsilon_n$ $2\varepsilon_1$ ε_1 0	f_n $e_1\tilde{e}_1$ e_1 $\tau - z$
1	$-2\varepsilon_n$ $2\varepsilon_1 - \varepsilon_n$ $\varepsilon_1 - 2\varepsilon_n$ $-\varepsilon_n$	$f_n \otimes \tilde{f}_n$ $(e_1\tilde{e}_1) \otimes \tilde{f}_n$ $(e_1 \wedge \tilde{f}_n) \otimes \tilde{f}_n$ $(\tau - z) \otimes \tilde{f}_n$
2	$-2\varepsilon_{n-1} - 2\varepsilon_n$ $2\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$ $\varepsilon_1 - \varepsilon_{n-1} - 2\varepsilon_n$ $-\varepsilon_{n-1} - \varepsilon_n$	$(f_{n-1} \wedge \tilde{f}_n) \otimes \tilde{f}_{n-1} \wedge \tilde{f}_n$ $e_1\tilde{e}_1 \otimes \tilde{f}_{n-1} \wedge \tilde{f}_n$ $(e_1 \wedge \tilde{f}_n) \otimes \tilde{f}_{n-1} \wedge \tilde{f}_n$ $(\tau - z) \otimes \tilde{f}_{n-1} \wedge \tilde{f}_n$

Table 3. Irreducible $\mathfrak{gl}(n)$-submodules of $E_1^{p,0}$.

weight			
λ	$\mathbf{E_1^{1,0}} = \bigoplus_\lambda \mathbf{V}_\lambda \otimes \mathbf{V_0}$	$\mathbf{E_1^{2,0}} = \bigoplus_\lambda \mathbf{V}_\lambda \otimes \mathbf{S^2V_0}$	$\mathbf{E_1^{3,0}} = \bigoplus_\lambda \mathbf{V}_\lambda \otimes \mathbf{S^3V_0}$
$-\varepsilon_n$	$\varepsilon_1 - \varepsilon_n$ 0	$2\varepsilon_1 - \varepsilon_n$ ε_1	$3\varepsilon_1 - \varepsilon_n$ $2\varepsilon_1$
$2\varepsilon_1$	$3\varepsilon_1$ $2\varepsilon_1 + \varepsilon_2$	$4\varepsilon_1$ $2\varepsilon_1 + 2\varepsilon_2$ $3\varepsilon_1 + \varepsilon_2$	$5\varepsilon_1$ $4\varepsilon_1 + \varepsilon_2$ $3\varepsilon_1 + 2\varepsilon_2$
ε_1	$2\varepsilon_1$ $\varepsilon_1 + \varepsilon_2$	$3\varepsilon_1$ $2\varepsilon_1 + \varepsilon_2$	$4\varepsilon_1$ $3\varepsilon_1 + \varepsilon_2$
0	ε_1	$2\varepsilon_1$	$3\varepsilon_1$

Table 4. Irreducible $\mathfrak{gl}(n)$-submodules of $E_1^{p,1}$.

weight			
λ	$\mathbf{E_1^{0,1}} = \bigoplus_\lambda \mathbf{V}_\lambda$	$\mathbf{E_1^{1,1}} = \bigoplus_\lambda \mathbf{V}_\lambda \otimes \mathbf{V_0}$	$\mathbf{E_1^{2,1}} = \bigoplus_\lambda \mathbf{V}_\lambda \otimes \mathbf{S^2V_0}$
$-2\varepsilon_n$	$-2\varepsilon_n$	$\varepsilon_1 - 2\varepsilon_n$ $-\varepsilon_n$	$2\varepsilon_1 - 2\varepsilon_n$ $\varepsilon_1 - \varepsilon_n$ 0
$2\varepsilon_1 - \varepsilon_n$	$2\varepsilon_1 - \varepsilon_n$	$3\varepsilon_1 - \varepsilon_n$ $2\varepsilon_1$ $2\varepsilon_1 + \varepsilon_2 - \varepsilon_n$	$4\varepsilon_1 - \varepsilon_n$ $3\varepsilon_1$ $3\varepsilon_1 + \varepsilon_2 - \varepsilon_n$ $2\varepsilon_1 + 2\varepsilon_2 - \varepsilon_n$ $2\varepsilon_1 + \varepsilon_2$
$\varepsilon_1 - 2\varepsilon_n$	$\varepsilon_1 - 2\varepsilon_n$	$2\varepsilon_1 - 2\varepsilon_n$ $\varepsilon_1 + \varepsilon_2 - 2\varepsilon_n$ $\varepsilon_1 - \varepsilon_n$	$3\varepsilon_1 - 2\varepsilon_n$ $2\varepsilon_1 - \varepsilon_n$ ε_1 $2\varepsilon_1 + \varepsilon_2 - 2\varepsilon_n$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_n$
$-\varepsilon_n$	$-\varepsilon_n$	$\varepsilon_1 - \varepsilon_n$ 0	$2\varepsilon_1 - \varepsilon_n$ ε_1

Table 5. Irreducible $\mathfrak{gl}(n)$-submodules of $E_1^{p,2}$.

weight		
λ	$\mathbf{E_1^{0,2}} = \bigoplus_\lambda \mathbf{V}_\lambda$	$\mathbf{E_1^{1,2}} = \bigoplus_\lambda \mathbf{V}_\lambda \otimes \mathbf{V_0}$
$-2\varepsilon_{n-1} - 2\varepsilon_n$	$-2\varepsilon_{n-1} - 2\varepsilon_n$	$\varepsilon_1 - 2\varepsilon_{n-1} - 2\varepsilon_n$ $-\varepsilon_{n-1} - 2\varepsilon_n$
$2\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$	$2\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$	$3\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$ $2\varepsilon_1 - \varepsilon_n$
$\varepsilon_1 - \varepsilon_{n-1} - 2\varepsilon_n$	$\varepsilon_1 - \varepsilon_{n-1} - 2\varepsilon_n$	$2\varepsilon_1 - \varepsilon_{n-1} - 2\varepsilon_n$ $\varepsilon_1 - 2\varepsilon_n$ $\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$
$-\varepsilon_{n-1} - \varepsilon_n$	$-\varepsilon_{n-1} - \varepsilon_n$	$\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$ $-\varepsilon_n$

Table 6. Irreducible $\mathfrak{gl}(n)$-submodules of $E_0^{p,0}$.

$\mathbf{U}$	$\mathbf{E_0^{1,0} = \bigoplus_U U \otimes V_0}$	$\mathbf{E_0^{2,0} = \bigoplus_U U \otimes S^2V_0}$	$\mathbf{E_0^{3,0} = \bigoplus_U U \otimes S^3V_0}$
$V_0(\text{mult } 2)$	$2\varepsilon_1$ $\varepsilon_1+\varepsilon_2$	$3\varepsilon_1$ $2\varepsilon_1+\varepsilon_2$	$4\varepsilon_1$ $3\varepsilon_1+\varepsilon_2$
$V_0^*(\text{mult } 2)$	$\varepsilon_1-\varepsilon_n$ 0	$2\varepsilon_1-\varepsilon_n$ ε_1	$3\varepsilon_1-\varepsilon_n$ $2\varepsilon_1$
S^2V_0	$3\varepsilon_1$ $2\varepsilon_1+\varepsilon_2$	$4\varepsilon_1$ $3\varepsilon_1+\varepsilon_2$ $2\varepsilon_1+2\varepsilon_2$	$5\varepsilon_1$ $4\varepsilon_1+\varepsilon_2$ $3\varepsilon_1+2\varepsilon_2$
$E^2V_0^*$	$\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n$ $-\varepsilon_n$	$2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n$ $\varepsilon_1-\varepsilon_n$	$3\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n$ $2\varepsilon_1-\varepsilon_n$
$\mathfrak{sl}(n)$	$2\varepsilon_1-\varepsilon_n$ ε_1 $\varepsilon_1+\varepsilon_2-\varepsilon_n$	$3\varepsilon_1-\varepsilon_n$ $2\varepsilon_1$ $2\varepsilon_1+\varepsilon_2-\varepsilon_n$ $\varepsilon_1+\varepsilon_2$	$4\varepsilon_1-\varepsilon_n$ $3\varepsilon_1+\varepsilon_2-\varepsilon_n$ $3\varepsilon_1$ $2\varepsilon_1+\varepsilon_2$
$\mathbb{C}(\text{mult } 3)$	ε_1	$2\varepsilon_1$	$3\varepsilon_1$

Table 7. Irreducible $\mathfrak{gl}(n)$-submodules of E^3V.

Space	**Highest weight vectors**	**Highest weights**
E^3V_0	$e_1\wedge e_2\wedge e_3$	0
$(E^2V_0)(V_0^*)$	$(e_1\wedge e_2)f_3$ $\sum_{i=1}^{3}(e_1\wedge e_i)f_i$	$-2\varepsilon_3$ ε_1
$V_0(S^2V_0^*)$	$e_1f_3^2$ $\sum_{i=1}^{3} e_if_if_3$	$\varepsilon_1-2\varepsilon_3$ $-\varepsilon_3$
$S^3V_0^*$	f_3^3	$-3\varepsilon_3$

Table 8. Irreducible $\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)$-submodules of $\mathfrak{g}_0\otimes\mathfrak{g}_{-1}^*$.

$\mathfrak{gl}(\mathbf{m})\oplus\mathfrak{gl}(\mathbf{n})$**-module**	**Highest weight**	**Highest weight vector**
$(V\otimes V^*)/\mathbb{C}\otimes(U^*\otimes V)$	$2\varepsilon_1-\varepsilon_m-\delta_n$ $\varepsilon_1-\delta_n$ $\varepsilon_1+\varepsilon_2-\varepsilon_m-\delta_n$ (if $m\geq 3$)	$(e_1\otimes\tilde{e}_m)\otimes(\tilde{f}_n\otimes e_1)$ $v_\lambda^1=\sum_{i=1}^{m}(e_i\otimes\tilde{e}_i)\otimes(\tilde{f}_n\otimes e_1)-$ $-m\sum_{i=1}^{m}(e_1\otimes\tilde{e}_i)\otimes(\tilde{f}_n\otimes e_i)$ $(e_1\otimes\tilde{e}_m)\otimes(\tilde{f}_n\otimes e_2)-$ $-(e_2\otimes\tilde{e}_m)\otimes(\tilde{f}_n\otimes e_1)$
$(U\otimes U^*)/\mathbb{C}\otimes(U^*\otimes V)$	$\varepsilon_1+\delta_1-2\delta_n$ $\varepsilon_1-\delta_n$ $\varepsilon_1+\delta_1-\delta_{n-1}-\delta_n$ (if $n\geq 3$)	$(f_1\otimes\tilde{f}_n)\otimes(\tilde{f}_n\otimes e_1)$ $v_\lambda^2=\sum_{i=1}^{n}(f_i\otimes\tilde{f}_i)\otimes(\tilde{f}_n\otimes e_1)-$ $-n\sum_{i=1}^{n}(f_i\otimes\tilde{f}_n)\otimes(\tilde{f}_i\otimes e_1)$ $(f_1\otimes\tilde{f}_{n-1})\otimes(\tilde{f}_n\otimes e_1)-$ $-(f_1\otimes\tilde{f}_n)\otimes(\tilde{f}_{n-1}\otimes e_1)$
$\mathbb{C}\otimes(U^*\otimes V)$	$\varepsilon_1-\delta_n$	$v_\lambda^3=(n\sum_{i=1}^{m}e_i\otimes\tilde{e}_i+$ $+m\sum_{i=1}^{n}f_i\otimes\tilde{f}_i)\otimes(\tilde{f}_n\otimes e_1)$

Table 9. Irreducible $\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)$-submodules of $(U^*\otimes V)\otimes(U^*\otimes V)$.

$\mathfrak{gl}(\mathbf{m})\oplus\mathfrak{gl}(\mathbf{n})$-module	**Highest weight**	**Highest weight vector**
$\Lambda^2U^*\otimes S^2V$	$2\varepsilon_1-2\delta_n$	$(\tilde{f}_n\otimes e_1)\otimes(\tilde{f}_n\otimes e_1)$
$S^2U^*\otimes\Lambda^2V$	$\varepsilon_1+\varepsilon_2-\delta_{n-1}-\delta_n$	$(\tilde{f}_n\otimes e_1)\otimes(\tilde{f}_{n-1}\otimes e_2)-$ $-(\tilde{f}_n\otimes e_2)\otimes(\tilde{f}_{n-1}\otimes e_1)-$ $-(\tilde{f}_{n-1}\otimes e_1)\otimes(\tilde{f}_n\otimes e_2)+$ $+(\tilde{f}_{n-1}\otimes e_2)\otimes(\tilde{f}_n\otimes e_1)$
$\Lambda^2U^*\otimes\Lambda^2V$	$\varepsilon_1+\varepsilon_2-2\delta_n$	$(\tilde{f}_n\otimes e_1)\otimes(\tilde{f}_n\otimes e_2)-$ $-(\tilde{f}_n\otimes e_2)\otimes(\tilde{f}_n\otimes e_1)$
$S^2U^*\otimes S^2V$	$2\varepsilon_1-\delta_{n-1}-\delta_n$	$(\tilde{f}_{n-1}\otimes e_1)\otimes(\tilde{f}_n\otimes e_1)-$ $-(\tilde{f}_n\otimes e_1)\otimes(\tilde{f}_{n-1}\otimes e_1)$

Table 10. Irreducible $\mathfrak{gl}(n)$-submodules of $C^{k,2}_{\mathfrak{gl}(n)}$.

k	Highest weight	Highest weight vector
2	$\lambda_1 = \delta_1 - 3\delta_n$	$v_{\lambda_1} = (f_1 \otimes \tilde{f}_n) \otimes \tilde{f}_n^2$
	$\lambda_2 = -2\delta_n$	$v_{\lambda_2} = \sum_{j=1}^{n} (f_j \otimes \tilde{f}_n) \otimes \tilde{f}_j \tilde{f}_n$
	$\lambda_3 = -2\delta_n$	$v_{\lambda_3} = \sum_{j=1}^{n} (f_j \otimes \tilde{f}_j) \otimes \tilde{f}_n^2$
	$\lambda_4 = \delta_1 - \delta_{n-1} - 2\delta_n (\text{ if } n \geq 3)$	$v_{\lambda_4} = (f_1 \otimes \tilde{f}_{n-1}) \otimes \tilde{f}_n^2 - (f_1 \otimes \tilde{f}_n) \otimes \tilde{f}_{n-1}\tilde{f}_n$
	$\lambda_5 = -\delta_{n-1} - \delta_n$	$v_{\lambda_5} = \sum_{j=1}^{n} ((f_j \otimes \tilde{f}_{n-1}) \otimes \tilde{f}_j \tilde{f}_n - (f_j \otimes \tilde{f}_n) \otimes \tilde{f}_j \tilde{f}_{n-1})$
$3 \leq k \leq n+1$	$\lambda_1 = \delta_1 - \delta_{n-k+2} - \ldots - \delta_{n-1} - 3\delta_n$	$v_{\lambda_1} = (f_1 \otimes \tilde{f}_{n-k+2} \wedge \ldots \wedge \tilde{f}_n) \otimes \tilde{f}_n^2$
	$\lambda_2 = -\delta_{n-k+2} - \ldots - \delta_{n-1} - 2\delta_n$	$v_{\lambda_2} = \sum_{j=1}^{n} (f_j \otimes \tilde{f}_{n-k+2} \wedge \ldots \wedge \tilde{f}_n) \otimes \tilde{f}_j \tilde{f}_n$
$3 \leq k \leq n$	$\lambda_3 = -\delta_{n-k+3} - \ldots - \delta_n - 2\delta_n$	$v_{\lambda_3} = \sum_{j=1}^{n} (f_j \otimes \tilde{f}_j \wedge \tilde{f}_{n-k+3} \wedge \ldots \wedge \tilde{f}_n) \otimes \tilde{f}_n^2$
	$\lambda_4 = \delta_1 - \delta_{n-k+1} - \ldots - \delta_{n-1} - 2\delta_n$	$v_{\lambda_4} = \sum_{j=0}^{k-1} (-1)^{(k-1)j} (f_1 \otimes \tilde{f}_{s^j(n-k+1)} \wedge \ldots$ $\wedge \tilde{f}_{s^j(n-1)}) \otimes \tilde{f}_{s^j(n)} \tilde{f}_n$
	$\lambda_5 = -\delta_{n-k+1} - \delta_{n-k+2} - \ldots - \delta_n$	$v_{\lambda_5} = \sum_{j=0}^{k-1} (-1)^{(k-1)j} \sum_{i=1}^{n} (f_i \otimes \tilde{f}_{s^j(n-k+1)} \wedge \ldots$ $\wedge \tilde{f}_{s^j(n-1)}) \otimes \tilde{f}_i \tilde{f}_{s^j(n)}$
$3 \leq k \leq n-1$	$\lambda_6 = -\delta_{n-k+2} - \ldots - \delta_{n-1} - 2\delta_n$	$v_{\lambda_6} = \sum_{j=0}^{k-2} (-1)^{(k-2)j} \sum_{i=1}^{n} (f_i \otimes \tilde{f}_i \wedge \tilde{f}_{t^j(n-k+2)} \wedge \ldots$ $\wedge \tilde{f}_{t^j(n-1)}) \otimes \tilde{f}_{t^j(n)} \tilde{f}_n$

Table 11. Irreducible $\mathfrak{gl}(n)$-submodules of $C^{k+1,1}_{\mathfrak{gl}(n)}$.

k	Highest weight	Highest weight vector
$2 \leq k \leq n$	$\beta_1 = \delta_1 - \delta_{n-k+1} - \ldots - \delta_{n-1} - 2\delta_n$	$v_{\beta_1} = (f_1 \otimes \tilde{f}_{n-k+1} \wedge \ldots \wedge \tilde{f}_{n-1} \wedge \tilde{f}_n) \otimes \tilde{f}_n$
	$\beta_2 = -\delta_{n-k+1} - \ldots - \delta_{n-1} - \delta_n$	$v_{\beta_2} = \sum_{j=1}^{n} f_j \otimes \tilde{f}_{n-k+1} \wedge \ldots \wedge \tilde{f}_{n-1} \wedge \tilde{f}_n) \otimes \tilde{f}_j$
$2 \leq k \leq n-1$	$\beta_3 = -\delta_{n-k+2} - \ldots - \delta_{n-1} - 2\delta_n$	$v_{\beta_3} = \sum_{j=6}^{n} (f_j \otimes \tilde{f}_j \wedge \tilde{f}_{n-k+2} \wedge \ldots \wedge \tilde{f}_{n-1} \wedge \tilde{f}_n) \otimes \tilde{f}_n$
	$\beta_4 = -\delta_{n-k+1} - \ldots - \delta_{n-1} - \delta_n$	$v_{\beta_4} = \sum_{j=0}^{k-1} (-1)^{(k-3)j} \sum_{i=1}^{n} (f_i \otimes \tilde{f}_i \wedge \tilde{f}_{s^j(n-k+1)} \wedge \ldots$ $\wedge f_{s^j(n-1)}) \otimes \tilde{f}_{s^j(n)}$
$2 \leq k \leq n-2$	$\beta_5 = \delta_1 - \delta_{n-k} - \ldots - \delta_{n-1} - \delta_n$	$v_{\beta_5} = \sum_{j=0}^{k} (-1)^{kj} (f_1 \otimes \tilde{f}_{r^j(n-k)} \wedge \ldots$ $\wedge \tilde{f}_{r^j(n-1)}) \otimes \tilde{f}_{r^j(n)}$

Table 12. Irreducible $\mathfrak{gl}(n)$-submodules of $\mathfrak{g}_k$.

k	Highest weight	Highest weight vector
$2 \leq k \leq n-1$	$\gamma_1 = \delta_1 - \delta_{n-k} - \delta_{n-k+1} - \ldots - \delta_n$	$v_{\gamma_1} = f_1 \otimes \tilde{f}_{n-k} \wedge \ldots \wedge \tilde{f}_n$
$2 \leq k \leq n-2$	$\gamma_2 = -\delta_{n-k+1} - \delta_{n-k+2} - \ldots - \delta_n$	$v_{\gamma_2} = \sum_{j=1}^{n} f_j \otimes \tilde{f}_j \wedge \tilde{f}_{n-k+1} \wedge \ldots \wedge \tilde{f}_n$

Table 13. Structure functions of $\mathfrak{sl}(m|n)$ endowed with the standard $\mathbb{Z}$-grading.

m	n	$H^{1,2}_{\mathfrak{g}_0}$	$H^{2,2}_{\mathfrak{g}_0}$
3	2	$2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_2$	—
2	3	$2\varepsilon_1-\varepsilon_2+\delta_1-2\delta_3$	—
≥ 4	2	$2\varepsilon_1-\varepsilon_m+\delta_1-2\delta_2$	$\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_m-\delta_1-\delta_2$
2	≥ 4	$2\varepsilon_1-\varepsilon_2+\delta_1-2\delta_n$	$\varepsilon_1+\varepsilon_2+\delta_1-\delta_{n-2}-\delta_{n-1}-\delta_n$
≥ 3	≥ 3	$2\varepsilon_1-\varepsilon_m+\delta_1-2\delta_n$ $\varepsilon_1+\varepsilon_2-\varepsilon_m+\delta_1-\delta_{n-1}-\delta_n$ $\varepsilon_1-\delta_n$ (if $m=n$)	—
2	2	$2\varepsilon_1-\varepsilon_2+\delta_1-2\delta_2$ $\varepsilon_1-\delta_2$	—

Table 14. Irreducible $\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)$submodules of $\mathfrak{g}_0\otimes\Lambda^2\mathfrak{g}^*_{-1}$.

$\mathfrak{gl}(\mathbf{m})\oplus\mathfrak{gl}(\mathbf{n})$-module	Highest weight	Highest weight vector
$V\otimes V^*/\mathbb{C}\otimes\Lambda^2U^*\otimes S^2V$	$3\varepsilon_1-\varepsilon_m-2\delta_n$	$(e_1\otimes\tilde{e}_m)\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)$
	$2\varepsilon_1-2\delta_n$	$v^1_\lambda=\sum\limits_{i=1}^m(e_i\otimes\tilde{e}_i)\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)-$ $m\sum\limits_{i=1}^m(e_1\otimes\tilde{e}_i)\otimes(\tilde{f}_n\otimes e_i)\wedge(\tilde{f}_n\otimes e_1)$
	$\varepsilon_1+\varepsilon_2-2\delta_n$	$v^1_\lambda=\sum\limits_{i=1}^m((e_1\otimes\tilde{e}_i)\otimes(\tilde{f}_n\otimes e_i)\wedge(\tilde{f}_n\otimes e_2)-$ $(e_2\otimes\tilde{e}_i)\otimes(\tilde{f}_n\otimes e_i)\wedge(\tilde{f}_n\otimes e_1))$
	$2\varepsilon_1+\varepsilon_2-\varepsilon_m-2\delta_n$ $(m\geq 3)$	$(e_1\otimes\tilde{e}_m)\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_n\otimes e_2)-$ $(e_2\otimes\tilde{e}_m)\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)$
$V\otimes V^*/\mathbb{C}\otimes S^2U^*\otimes\Lambda^2V$	$2\varepsilon_1+\varepsilon_2-\varepsilon_m-\delta_{n-1}-\delta_n$	$(e_1\otimes\tilde{e}_m)\otimes(\tilde{f}_{n-1}\otimes e_1)\wedge(\tilde{f}_n\otimes e_2)-$ $(e_1\otimes\tilde{e}_m)\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_{n-1}\otimes e_2)$
	$2\varepsilon_1-\delta_{n-1}-\delta_n$ $(m\geq 3)$	$v^1_\lambda=\sum\limits_{i=1}^m(e_1\otimes\tilde{e}_i)\otimes(\tilde{f}_{n-1}\otimes e_i)\wedge(\tilde{f}_n\otimes e_1)-$ $(e_1\otimes\tilde{e}_i)\otimes(\tilde{f}_n\otimes e_i)\wedge(\tilde{f}_{n-1}\otimes e_1)$
	$\varepsilon_1+\varepsilon_2-\delta_{n-1}-\delta_n$ $(m\geq 3)$	$v^1_\lambda=\sum\limits_{i=1}^m((e_1\otimes\tilde{e}_i)\otimes(\tilde{f}_{n-1}\otimes e_i)\wedge(\tilde{f}_n\otimes e_2)-$ $(e_1\otimes\tilde{e}_i)\otimes(\tilde{f}_n\otimes e_i)\wedge(\tilde{f}_{n-1}\otimes e_2)-$ $(e_2\otimes\tilde{e}_i)\otimes(\tilde{f}_{n-1}\otimes e_i)\wedge(\tilde{f}_n\otimes e_1)+$ $(e_2\otimes\tilde{e}_i)\otimes(\tilde{f}_n\otimes e_i)\wedge(\tilde{f}_{n-1}\otimes e_1))$
	$\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_m-\delta_{n-1}-\delta_n$ $(m\geq 4)$	$\sum\limits_{j=0}^2((e_{s^j(1)}\otimes\tilde{e}_m)\otimes(\tilde{f}_{n-1}\otimes e_{s^j(2)})\wedge(\tilde{f}_n\otimes e_{s^j(3)})-$ $(e_{s^j(1)}\otimes\tilde{e}_m)\otimes(\tilde{f}_n\otimes e_{s^j(2)})\wedge(\tilde{f}_{n-1}\otimes e_{s^j(3)}))$
$U\otimes U^*/\mathbb{C}\otimes\Lambda^2U^*\otimes S^2V$	$2\varepsilon_1+\delta_1-3\delta_n$	$(f_1\otimes\tilde{f}_n)\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)$
	$2\varepsilon_1-2\delta_n$	$v^2_\lambda=\sum\limits_{i=1}^n(f_i\otimes\tilde{f}_i)\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)-$ $n\sum\limits_{i=1}^n(f_i\otimes\tilde{f}_n)\otimes(\tilde{f}_i\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)$
	$2\varepsilon_1-\delta_{n-1}-\delta_n$	$v^2_\lambda=\sum\limits_{i=1}^n((f_i\otimes\tilde{f}_{n-1})\otimes(\tilde{f}_i\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)-$ $(f_i\otimes\tilde{f}_n)\otimes(\tilde{f}_i\otimes e_1)\wedge(\tilde{f}_{n-1}\otimes e_1))$
	$2\varepsilon_1+\delta_1-\delta_{n-1}-2\delta_n$ $(n\geq 3)$	$(f_1\otimes\tilde{f}_n)\otimes(\tilde{f}_{n-1}\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)-$ $(f_1\otimes\tilde{f}_{n-1})\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)$

Table 14 (cont). Irreducible $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$submodules of $\mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}^*_{-1}$.

$\mathfrak{gl}(\mathbf{m}) \oplus \mathfrak{gl}(\mathbf{n})$-module	**Highest weight**	**Highest weight vector**
$U \otimes U^*/\mathbb{C} \otimes S^2U^* \otimes \Lambda^2 V$	$\varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-1} - 2\delta_n$	$(f_1 \otimes \tilde{f}_n) \otimes (\tilde{f}_{n-1} \otimes e_1) \wedge (\tilde{f}_n \otimes e_2) -$ $(f_1 \otimes \tilde{f}_n) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_{n-1} \otimes e_2)$
	$\varepsilon_1 + \varepsilon_2 - 2\delta_n \ (n \geq 3)$	$v^2_\lambda = \sum\limits_{i=1}^{n} ((f_i \otimes \tilde{f}_n) \otimes (\tilde{f}_i \otimes e_1) \wedge (\tilde{f}_n \otimes e_2) -$ $(f_i \otimes \tilde{f}_n) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_i \otimes e_2))$
	$\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n \ (n \geq 3)$	$v^2_\lambda = \sum\limits_{i=1}^{n} ((f_i \otimes \tilde{f}_n) \otimes (\tilde{f}_i \otimes e_1) \wedge (\tilde{f}_{n-1} \otimes e_2) -$ $(f_i \otimes \tilde{f}_n) \otimes (\tilde{f}_i \otimes e_2) \wedge (\tilde{f}_{n-1} \otimes e_1) -$ $(f_i \otimes \tilde{f}_{n-1}) \otimes (\tilde{f}_i \otimes e_1) \wedge (\tilde{f}_n \otimes e_2) +$ $(f_i \otimes \tilde{f}_{n-1}) \otimes (\tilde{f}_i \otimes e_2) \wedge (\tilde{f}_n \otimes e_1))$
	$\varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-2} - \delta_{n-1} - \delta_n$ $(n \geq 4)$	$\sum\limits_{j=0}^{2} ((f_1 \otimes \tilde{f}_{t^j(n-2)}) \otimes (\tilde{f}_{t^j(n-1)} \otimes e_1) \wedge (\tilde{f}_{t^j(n)} \otimes e_2) -$ $(f_1 \otimes \tilde{f}_{t^j(n-2)}) \otimes (\tilde{f}_{t^j(n-1)} \otimes e_2) \wedge (\tilde{f}_{t^j(n)} \otimes e_1))$
$\mathbb{C} \otimes \Lambda^2 U^* \otimes S^2 V$	$2\varepsilon_1 - 2\delta_n$	$v^3_\lambda = (n \sum\limits_{i=1}^{m} e_i \otimes \tilde{e}_i +$ $+ m \sum\limits_{i=1}^{n} f_i \otimes \tilde{f}_i) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_1)$
$\mathbb{C} \otimes S^2 U^* \otimes \Lambda^2 V$	$\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$	$v^3_\lambda = (n \sum\limits_{i=1}^{m} e_i \otimes \tilde{e}_i +$ $+ m \sum\limits_{i=1}^{n} f_i \otimes \tilde{f}_i) \otimes ((\tilde{f}_{n-1} \otimes e_1) \wedge (\tilde{f}_n \otimes e_2) -$ $(\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_{n-1} \otimes e_2))$

Table 15. Irreducible $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-submodules of $\mathfrak{g}_1 \otimes \Lambda^2 \mathfrak{g}^*_{-1}$.

$\mathfrak{gl}(\mathbf{m}) \oplus \mathfrak{gl}(\mathbf{n})$-module	**Highest weight**	**Highest weight vector**
$(\Lambda^2 U^* \otimes U^*) \otimes (S^2 V \otimes V)$	$3\varepsilon_1 - 3\delta_n$	$(\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_1)$
	$2\varepsilon_1 + \varepsilon_2 - 3\delta_n$	$(\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_n \otimes e_2) \wedge (\tilde{f}_n \otimes e_1) -$ $(\tilde{f}_n \otimes e_2) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_1)$
	$3\varepsilon_1 - \delta_{n-1} - 2\delta_n$	$(\tilde{f}_{n-1} \otimes e_1) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_1) -$ $(\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_{n-1} \otimes e_1) \wedge (\tilde{f}_n \otimes e_1)$
	$2\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - 2\delta_n$	$v^1_\lambda = (\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_{n-1} \otimes e_1) \wedge (\tilde{f}_n \otimes e_2) -$ $(\tilde{f}_n \otimes e_2) \otimes (\tilde{f}_{n-1} \otimes e_1) \wedge (\tilde{f}_n \otimes e_1) -$ $(\tilde{f}_{n-1} \otimes e_1) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_2) +$ $(\tilde{f}_{n-1} \otimes e_2) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_1)$
$(S^2 U^* \otimes U^*) \otimes (\Lambda^2 V \otimes V)$	$2\varepsilon_2 + \varepsilon_2 - \delta_{n-1} - 2\delta_n$	$v^2_\lambda = (\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_{n-1} \otimes e_1) \wedge (\tilde{f}_n \otimes e_2) -$ $(\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_{n-1} \otimes e_2) \wedge (\tilde{f}_n \otimes e_1)$
	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \delta_{n-1} - 2\delta_n$ $(m \geq 3)$	$\sum\limits_{j=0}^{2} ((\tilde{f}_n \otimes e_{s^j(1)}) \otimes (\tilde{f}_{n-1} \otimes e_{s^j(2)}) \wedge (\tilde{f}_n \otimes e_{s^j(3)}) -$ $(\tilde{f}_n \otimes e_{s^j(1)}) \otimes (\tilde{f}_n \otimes e_{s^j(2)}) \wedge (\tilde{f}_{n-1} \otimes e_{s^j(3)})$
	$2\varepsilon_1 + \varepsilon_2 - \delta_{n-2} - \delta_{n-1} - \delta_n$ $(n \geq 3)$	$\sum\limits_{j=0}^{2} ((\tilde{f}_{t^j(n-2)} \otimes e_1) \otimes (\tilde{f}_{t^j(n-1)} \otimes e_2) \wedge (\tilde{f}_{t^j(n)} \otimes e_1) -$ $(\tilde{f}_{t^j(n-2)} \otimes e_1) \otimes (\tilde{f}_{t^j(n-1)} \otimes e_1) \wedge (\tilde{f}_{t^j(n)} \otimes e_2))$
	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \delta_{n-2} - \delta_{n-1} - \delta_n$ $(m, \ n \geq 3)$	$\sum\limits_{i=0}^{2} \sum\limits_{j=0}^{2} ((\tilde{f}_{t^j(n-2)} \otimes e_{s^i(1)}) \otimes (\tilde{f}_{t^j(n-1)} \otimes e_{s^i(2)}) \wedge$ $(\tilde{f}_{t^j(n)} \otimes e_{s^i(3)}) - (\tilde{f}_{t^j(n-2)} \otimes e_{s^i(1)}) \otimes$ $(\tilde{f}_{t^j(n)} \otimes e_{s^i(2)}) \wedge (\tilde{f}_{t^j(n-1)} \otimes e_{s^i(3)}))$

Table 16. Spencer cohomology of $\mathfrak{sl}(\mathrm{m}|\mathrm{n})$ endowed with a $\mathbb{Z}$-grading, where $\mathfrak{g}_0 = \mathfrak{c}(\mathfrak{sl}(m|q) \oplus \mathfrak{sl}(n-q))$.

$\mathbf{m}$	$\mathbf{q}$	$\mathbf{n-q}$	$\mathbf{H}^{1,2}_{\mathfrak{g}_0}$	$\mathbf{H}^{2,2}_{\mathfrak{g}_0}$
≥ 2	≥ 1	≥ 3	$m \neq q \pm 1$ $2\varepsilon_1 - \varepsilon_{m+q} + \delta_1 - 2\delta_{n-q}$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_{m+q} + \delta_1 - \delta_{n-q-1} - \delta_{n-q}$ $\varepsilon_1 - \delta_{n-q}$ (if $m = n$)	——
≥ 3	≥ 1	2	$m \neq q - 1$ $2\varepsilon_1 - \varepsilon_{m+q} + \delta_1 - 2\delta_2$ $\varepsilon_1 - \delta_2$ (if $m = n$)	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_{m+q} - \delta_1 - \delta_2$
2	≥ 2	2	$q \neq 3$ $2\varepsilon_1 - \varepsilon_{q+2} + \delta_1 - 2\delta_2$	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_{q+2} - \delta_1 - \delta_2$
2	1	2	$2\varepsilon_1 - \varepsilon_3 + \delta_1 - 2\delta_2$	$\varepsilon_1 + \varepsilon_3 - \delta_1 - \delta_2$
1	≥ 1	≥ 3	$q \neq 2$ $2\varepsilon_1 - \varepsilon_{q+1} + \delta_1 - 2\delta_{n-q}$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_{q+1} + \delta_1 - \delta_{n-q-1} - \delta_{n-q}$	——
1	≥ 1	2	$q \neq 2$ $2\varepsilon_1 - \varepsilon_{q+1} + \delta_1 - 2\delta_2$	$\varepsilon_1 + 2\varepsilon_2 - \varepsilon_{q+1} - \delta_1 - \delta_2$ ($q \neq 1$) $2\varepsilon_2 - \delta_1 - \delta_2$ ($q = 1$)
0	2	2	——	$3\varepsilon_1 - \varepsilon_2 - \delta_1 - \delta_2$ $\varepsilon_1 + \varepsilon_2 + \delta_1 - 3\delta_2$
0	2	≥ 3	$2\varepsilon_1 - \varepsilon_2 + \delta_1 - \delta_{n-3} - \delta_{n-2}$	$\varepsilon_1 + \varepsilon_2 + \delta_1 - 3\delta_{n-2}$
0	≥ 3	2	$\varepsilon_1 + \varepsilon_2 - \varepsilon_q + \delta_1 - 2\delta_2$	$3\varepsilon_1 - \varepsilon_q - \delta_1 - \delta_2$
0	≥ 3	≥ 3	$2\varepsilon_1 - \varepsilon_q + \delta_1 - \delta_{n-q-1} - \delta_{n-q}$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_q + \delta_1 - 2\delta_{n-q}$	——

Table 17. Spencer cohomology of $\mathfrak{sl}(m|n)$ endowed with a $\mathbb{Z}$-grading, where $\mathfrak{g}_0 = \mathfrak{c}(\mathfrak{sl}(m-p|q) \oplus \mathfrak{sl}(p|n-q))$.

$\mathbf{m-p}$	$\mathbf{q}$	$\mathbf{p}$	$\mathbf{n-q}$	$\mathbf{H}^{1,2}_{\mathfrak{g}_0}$	$\mathbf{H}^{2,2}_{\mathfrak{g}_0}$
0	2	2	0	$2\varepsilon_1 - \varepsilon_2 + \delta_1 - 2\delta_2$ $\varepsilon_1 - \delta_2$	——
0	2	3	0	$2\varepsilon_1 - \varepsilon_2 + \delta_1 - 2\delta_3$	——
0	3	2	0	$2\varepsilon_1 - \varepsilon_3 + \delta_1 - 2\delta_2$	——
0	2	≥ 4	0	$2\varepsilon_1 - \varepsilon_2 + \delta_1 - 2\delta_p$	$\varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{p-2} - \delta_{p-1} - \delta_p$
0	≥ 4	2	0	$2\varepsilon_1 - \varepsilon_q + \delta_1 - 2\delta_2$	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_q - \delta_1 - \delta_2$
0	≥ 3	≥ 3	0	$2\varepsilon_1 - \varepsilon_q + \delta_1 - 2\delta_p$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_q + \delta_1 - \delta_{p-1} - \delta_p$ $\varepsilon_1 - \delta_p$ ($m = n$)	——
0	2	≥ 1	≥ 1	$n \neq p + q + 1$ $2\varepsilon_1 - \varepsilon_2 + \delta_1 - \delta_{p+n-3} - \delta_{p+n-2}$ $\varepsilon_1 - \delta_{p+n-2}$ ($m = n$)	$\varepsilon_1 + \varepsilon_2 + \delta_1 - 3\delta_{p+n-2}$
≥ 1	≥ 1	2	0	$m \neq p + q + 1$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_{m-p+q} + \delta_1 - 2\delta_2$ $\varepsilon_1 - \delta_2$ ($m = n$)	$3\varepsilon_1 - \varepsilon_{m-p+q} - \delta_1 - \delta_2$
0	≥ 3	≥ 1	≥ 1	$n \neq p + q \pm 1$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_q + \delta_1 - 2\delta_{p+n-q}$ $2\varepsilon_1 - \varepsilon_q + \delta_1 - \delta_{p+n-q-1} - \delta_{p+n-q}$ $\varepsilon_1 - \delta_{p+n-q}$ ($m = n$)	——
≥ 1	≥ 1	≥ 3	0	$m \neq p + q \pm 1$ $2\varepsilon_1 - \varepsilon_{m-p+q} + \delta_1 - \delta_{p-1} - \delta_p$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_{m-p+q} + \delta_1 - 2\delta_p$ $\varepsilon_1 - \delta_p$ ($m = n$)	——
≥ 1	≥ 1	≥ 1	≥ 1	$m, n \neq p + q \pm 1$ $2\varepsilon_1 - \varepsilon_{m-p+q} + \delta_1 - 2\delta_{p+n-q}$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_{m-p+q} + \delta_1 - \delta_{p+n-q-1} - \delta_{p+n-q}$ $\varepsilon_1 - \delta_{p+n-q}$ ($m = n \geq 3$)	——

Table 18. Spencer cohomology of $\mathfrak{osp}(m|2n)$, where $m = 2r+2, 2r+3$.

r	n	$H^{2,2}_{\mathfrak{g}_0}$	$S^2(\mathfrak{g}_{-1})$
0	1	—	0 (if $m=2$) $0,\ \delta_1$ (if $m=3$)
0	≥ 2	$2\delta_1+2\delta_2$	$\delta_1+\delta_2,\ 0$
1	1	$\varepsilon_1+\delta_1$	$2\varepsilon_1,\ 0$
1	≥ 2	$2\varepsilon_1+\delta_1+\delta_2$	$2\varepsilon_1,\ 0$
≥ 2	≥ 1	$2\varepsilon_1+2\varepsilon_2$	$2\varepsilon_1,\ 0$

Chapter 9

The nonholonomic Riemann and Weyl tensors for flag manifolds (P. Grozman, D. Leites)

This chapter is an edited excerpt from [GL] which contains proof of Premet's important theorems and our calculations. In Chapter 8 the $\mathbb{Z}$-graded Lie (super)algebras of depth 1 were studied; here we tackle depth $d > 1$.

9.0. Introduction

H. Hertz [Hertz] coined the term *nonholonomic* during his attempts to geometrically describe motions in such a way as to exorcize the concept of "force". A manifold (phase space) is said to be *nonholonomic* if endowed with a nonintegrable distribution (a subbundle of the tangent bundle). A simplest example of a nonholonomic dynamical system is given by a body rolling without gliding over another body. Among various images that spring to mind, the simplest is a ball on a rough plane ([Poi]) or a bike on asphalt. At the tangency point of the wheel with asphalt, the velocity of the wheel is zero. (This is a linear constraint. We will not consider here more general non-linear constraints.) A famous theorem of Frobenius gives criteria of local integrability of the distribution: its sections should form a Lie algebra.

For a historical review of nonholonomic systems, see [VG2] and a very interesting paper by Vershik [Ve] with first rigorous mathematical formulations of nonholonomic geometry and indications to applications to various, partly unexpected at that time, areas (like optimal control or macro-economics, where *nonlinear* constraints are also natural, cf. [AS], [Bl], [S]); recent book by Kozlov [Koz] is extremely instructive. In [Ve], Vershik summarizes about 100 years of studies of nonholonomic geometry (Hertz, Carathéodory, Vrănceanu, Wagner, Schouten, Faddeev, Griffiths, Godbillon; now MathSciNet returns thousands entries for "nonholonomic" and its synonyms (anholonomic, "sub-Riemannian", "autoparallel") and particular cases leading to nonholonomic constraints ("Finsler", "cat's problem"). There seems to be "more", actually, nonholonomic dynamical systems than holonomic ones.

A relatively new theory, “supergravity” (the theory embodying Einstein’s dream of a Unified Field Theory), also deals exclusively with nonholonomic structures, albeit on *super*manifolds.)

At the end of [Ve] Vershik summarized futile attempts of the researchers to define an analog of the Riemann tensor for the general nonholonomic manifold in a conjecture that “though known in some cases, it is probably impossible to define such a general analog”.

However, in 1989, during his stay at IAS, DL gave such a general definition and lectured on it at various schools and conferences (ICTP, Euler Math. Inst., JINR, etc.), see [Le5], [LP0]; later we applied it to supergravity [GL31].

Main results of this chapter are (1) elucidation of this general definition of the nonholonomic counterparts of the Riemann tensor and its conformal, Weyl, analog; (2) Premet’s theorems that facilitate computation of these tensors in some cases (for flag varieties G/P, where G is a simple Lie group and P is its parabolic subalgebra). For the results of computation of these tensors in some of these cases, see [GL].

Tanaka [T] tackled the same, actually, problem for totally different reasons (in [T] and even in more recent [Y] and [YY] even the word “nonholonomic” is never used). Tanaka’s results (especially their lucid exposition in Yamaguchi’s paper [Y]) are easier to understand than the first attempts (by Schouten, Wagner, see [DG]) because, after some experiments, he used the hieroglyphics of Lie algebra cohomology which are much more graphic than coordinate tensor notations. Tanaka’s tensor coincides with the one we suggest.

We illustrate the main definitions by computing the nonholonomic analogs of the Riemann and Weyl tensor in several particular cases — the simplest analogs of “classical domains”. In doing so we rely on Premet’s theorems and a `Mathematica`-based package `SuperLie`.

Computations of nonholonomic analogs of Riemann tensor are rather difficult technically and the rare examples of works with actually computed **results** are [C1]–[C5], [GIOS], [HH], [Y, YY, EKMR, Ta] and refs. therein. In non-super setting, they used Tanaka’s definition of nonholonomic Riemann tensor, identical to ours, but lack Premet’s theorems and `SuperLie` and so could not compute as much as anybody is able to now with their help. [1)]

9.0.1. General description of classical tensors and our examples. In mid-1970s, Gindikin formulated a problem of local characterization of compact Hermitian symmetric domains $X = S/P$, where S is a simple Lie group and P its parabolic subgroup. Goncharov solved this problem [Go] having considered the fields of certain quadratic cones and having computed the **structure**

[1] In 2000, S. Vacaru informed us of his and mathematician’s from Vrănceanu’s school definitions partly summarized in [Va6] and refs. therein. It is not easy to see through the forest of non-invariant cumbrous tensor expressions that a number of components is lacking in [Va6] as well as in [DG], as compared with Tanaka’s or our definitions.

functions (obstructions to flatness) of the corresponding G-structures, where G is the Levi (reductive) part of P.

9.0.2. Examples. [18] Ol: ssbegin ili sssbegin? Let the ground field be $\mathbb{C}$. 18

1) For $S = O(n+2)$ and $G = CO(n) = O(n)\times\mathbb{C}^\times$, the structure functions were known; they constitute the Weyl tensor — the conformally invariant part of the Riemann curvature tensor.

2) For $S = SL(n+m)$ and $G = S(GL(n)\times GL(m))$, the structure functions are obstructions to integrability of multidimensional analogs of Penrose’s α- and β-planes on the Grassmannian Gr_n^{n+m} (Penrose considered Gr_2^4).

Not any simple complex Lie group S and its subgroup P can form a classical domain: S is any but $G(2)$ [2)], $F(4)$ and $E(8)$ and $P = P_i$ is a *maximal* parabolic subgroup generated by all Chevalley generators of S, but **one** (ith), say, negative. The group P, or which is the same, the ith Chevalley generator of S (in what follows referred to as *selected*) can not be arbitrary, either. To describe the admissible P’s, let us label the nodes of the Dynkin graph of S with the coefficients of the maximal root expressed in terms of simple roots. The selected generator may only correspond to the vertex with label 1 on the Dynkin graph.

It is natural to consider the following problems:

> **For any simple Lie group S, fix an arbitrary $\mathbb{Z}$-grading of its Lie algebra $\mathfrak{s} = \mathrm{Lie}(S)$.**
> **For any subgroup $P \subset S$, generated by elements of $\mathfrak{s}$ corresponding to nonnegative roots, what are the analogs of the Goncharov conformal structure and the corresponding analogs of Riemann and projective structures, which of these structures are flat, and what are the obstructions to their flatness?**

9.0.3. Remark. [19] Ol: ssbegin ili sssbegin? The adjective “arbitrary” ($\mathbb{Z}$-grading of $\mathfrak{s}$) in the above formulation appeared thanks to J. Bernstein who reminded us that parabolic subgroups are a particular case of such gradings. All $\mathbb{Z}$-gradings are obtained by setting $\deg X_i^\pm = \pm k_i$, where $k_i \in \mathbb{Z}$, for the Chevalley generators $X_i^\pm$ and parabolic subgroups appear if $k_i \geq 0$ for all i. Recently Kostant [Ko2] considered an analog of the Bott-Borel-Weil (BBW) theorem — one of our main tools — for the non-parabolic case, but the answer is not yet as algebraic as we need, so having answered the above displayed questions in full generality we calculate the nonholonomic invariants for parabolic subgroups only. 19

Modern descriptions of structure functions is usually given in terms of the *Spencer cohomology*, cf. [St2] (we will recall all definitions needed in (0.1) and

[2] We denote the exceptional groups and their Lie algebras in the same way as the serial ones, like $SL(n)$; we thus avoid confusing $\mathfrak{g}(2)$ with the second component $\mathfrak{g}_2$ of a $\mathbb{Z}$-grading of a Lie algebra $\mathfrak{g}$.

(0.2) in due course). Goncharov expressed the structure functions as tensors taking values in the vector bundle over $X = G/P$, whose fibers at every point $x \in X$ are isomorphic to each other and to

$$H^2(\mathfrak{g}_{-1}; (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*), \text{ where } \mathfrak{g}_0 = \mathrm{Lie}(G), \quad \mathfrak{g}_{-1} = T_xX,$$
$$\text{and where } (\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathop{\oplus}\limits_{i\geq -1} \mathfrak{g}_i \text{ is the Cartan prolong of } (\mathfrak{g}_{-1}, \mathfrak{g}_0). \tag{0.2}$$

The conventional representation of the structure functions as bigraded Spencer cohomology $H^{k,2}$ can be recovered any time as the homogeneous degree k component of $H^2(\mathfrak{g}_{-1}; (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*)$ corresponding to the $\mathbb{Z}$-grading of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$.

At about the same time Goncharov got his result, physicists trying to write down various supergravity equations (for standard or "exotic" N-SUGRAs, see [WB], [MaG], [GIOS], [HH]) bumped into the same problem (0.1) with the supergroup $S = \mathrm{SL}(4|N)$ for $N \leq 8$ and P generated by all the (analogs of the) Chevalley generators of G but **two**. The corresponding coset superspace X is a flag supervariety and the difficulties with SUGRAs spoken about, e.g., in [WB] ("we do not know how to define the analog of the Riemann tensor for $N > 2$", in other words: we do not know what might stand in the left-hand sides of the SUGRA equations), were caused not by a *super* nature of X but by its *nonholonomic* nature.

Shchepochkina introduced nonholonomic generalizations $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ of Cartan prolongation $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ for needs of our classification of simple infinite dimensional Lie superalgebras of vector fields ([?]). She introduced them (together with several new types of prolongation, e.g., *partial* prolongation) in [Sh1], [?], [?]. These generalizations are precisely what is needed to define the nonholonomic analog of the Weyl and Riemann tensors in the general case.

Observe that our nonholonomic invariants, though natural analogs of the curvature and torsion tensors, do not coincide on nonholonomic manifolds with the classical ones and bearing the same name. Indeed, on any nonholonomic manifold, there is, by definition, a nonzero classical torsion (the Frobenius form that to a pair of sections of the distribution assigns their bracket) while, for example, every contact manifold is flat in *our* sense. To avoid confusion, we should always add adjective "nonholonomic" for the invariants introduced below. Since this is too long, we will briefly say *N-curvature* tensor and specify its degree (=the order of the structure function) if needed; to require vanishing of the torsion is analogous of imposing Wess-Zumino constraints [WB].

The main thing is to answer the questions (0.1). Having done this (having given appropriate definitions in the general case of manifolds with nonholonomic structure) we explicitly compute the analogs of (0.2) — the space of nonholonomic structure functions — possible values of the nonholonomic versions of the Weyl and Riemann tensors. We do so for the simplest nonholonomic flag manifolds of the form S/P with **one** selected Chevalley generator. In most of our cases $(\mathfrak{g}_-, \mathfrak{g}_0)_* = \mathfrak{s}$, the Lie algebra of S, and therefore we can apply the BBW theorem (reproduced below; for a nice review, see [Wo]). If

$(\mathfrak{g}_-, \mathfrak{g}_0)_*$ strictly contains $\mathfrak{s}$, we consider the values of cocycles in $\mathfrak{s}$ as well as in $(\mathfrak{g}_-, \mathfrak{g}_0)_*$.

We cite Premet's theorems that show how to compute the N-Weyl and N-Riemann tensors and use the theorems to get an explicit answer.

The implicit form of the answer in [Go] hides phenomena manifest if the answer is explicit, as in [LPS], where, thanks to an explicit form of the answer we suggested some analogs of Einstein equations (EE) for certain Grassmannians. For the cases we consider here, a phenomenon similar to that observed in [LPS] is manifest, e.g., for the nodes at the base of the forks in $\mathfrak{e}(6)$ and $\mathfrak{o}(8)$. We intend to consider the related analogs of EE elsewhere.

We illustrate usefulness of computer-aided study by using `SuperLie` to compute the structure functions for the $G(2)$-structure, so popular lately, cf. [AW, B, FG]. The package `SuperLie` already proved useful in many instances (see [GL]), and is indispensable for Lie superalgebras: for practically all of them, there exists nothing as neat as the BBW theorem ([PS1]). We also apply `SuperLie` to compute the structure functions for a super version of the $G(2)$-structure on the projective superspace $\mathbb{C}P^{1,7}$ with a nonholonomic distribution.

9.1. Structure functions of *G*-structures

Let M^n be a manifold over a field $\mathbb{K}$. Let FM be the frame bundle over M, i.e., the principal $GL(n)$-bundle. Let $G \subset GL(n)$ be a Lie group. A *G-structure* on M is a reduction of the principal $GL(n)$-bundle to the principal G-bundle. Another formulation is more understandable: a *G-structure* is a selection of transition functions from one coordinate patch to another so that they belong to G for every intersecting pair of patches.

Thus, in the definition of G-structure the following characters participate: M^n and two vector bundles over it: TM and FM and the two groups $G \subset GL(n)$ both acting in each fiber of each bundle.

The simplest G-structure is the *flat* G-structure defined as follows. For a model manifold with the flat G-structure we take $V = \mathbb{C}^n$ with a fixed frame. The key moment is identification of the tangent spaces T_vV at distinct points v. This is performed by means of parallel translations along v. This means that we consider V as a commutative Lie group and identify the tangent spaces to it at various points with its Lie algebra, $\mathfrak{v}$. Thanks to commutativity:

$$\begin{array}{l}\mathfrak{v} \text{ can be naturally identified with } V \text{ itself;}\\ \text{it does not matter whether we use left or right translations.}\end{array} \tag{1.1}$$

In this way, we get a fixed frame in every T_vV. The *flat G-structure* is the bundle over V whose fiber over $v \in V$ consists of all frames obtained from the fixed one under the G-action. In textbooks on differential geometry (e.g., in [St2]), the obstructions to identification of the kth infinitesimal neighborhood of a point $m \in M$ on a manifold M with G-structure with the kth infinitesimal neighborhood of a point of the manifold V with the above flat G-structure are called *structure functions* of order k.

To precisely describe the structure functions, set

$$\mathfrak{g}_{-1} = T_m M, \quad \mathfrak{g}_0 = \mathfrak{g} = \mathrm{Lie}(G).$$

Recall that, for any (finite dimensional) vector space V, we have

$$\mathrm{Hom}(V, \mathrm{Hom}(V, \dots, \mathrm{Hom}(V, V) \dots)) \simeq L^i(V, V, \dots, V; V),$$

where L^i is the space of i-linear maps and we have $(i+1)$-many V's on both sides. Now, we recursively define, for any $i > 0$:

$$\mathfrak{g}_i = \{X \in \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{i-1}) \mid X(v_1)(v_2, v_3, ..., v_{i+1}) = X(v_2)(v_1, v_3, ..., v_{i+1}) \text{ where } v_1, \dots, v_{i+1} \in \mathfrak{g}_{-1}\}.$$

Let the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ be faithful. Then, clearly,

$$(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \subset \mathfrak{vect}(n) = \mathfrak{der}\, \mathbb{C}[x_1, \dots, x_n]], \quad \text{where } n = \dim\ \mathfrak{g}_{-1}.$$

It is subject to an easy verification that the Lie algebra structure on $\mathfrak{vect}(n)$ induces same on $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$. (It is also easy to see that even if $\mathfrak{g}_{-1}$ is not a faithful $\mathfrak{g}_0$-module $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ is a Lie algebra, but can not be embedded into $\mathfrak{vect}(\mathfrak{g}^*_{-1})$.) The Lie algebra $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ will be called the *Cartan's prolong* (the result of Cartan's *prolongation*) of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$. The Cartan prolong is the Lie algebra of symmetries of the G-structure in the space $T_m M$.

Let E^i be the operator of the ith exterior power, V^* the dual of V. Set

$$C^{k,s}_{(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*} = \mathfrak{g}_{k-s} \otimes E^s(\mathfrak{g}^*_{-1}).$$

The differential $\partial_s : C^{k,s}_{(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*} \longrightarrow C^{k,s+1}_{(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*}$ is given by (as usual, the slot with the hatted variable is to be ignored):

$$(\partial_s f)(v_1, \dots, v_{s+1}) = \sum_i (-1)^i [f(v_1, \dots, \widehat{v_{s+1-i}}, \dots, v_{s+1}), v_{s+1-i}]$$

for any $v_1, \dots, v_{s+1} \in \mathfrak{g}_{-1}$. As expected, $\partial_s \partial_{s+1} = 0$. The homology of this bicomplex is called *Spencer cohomology* of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ and denoted by $H^{k,s}_{(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*}$.

9.1.1. Proposition ([St2]). *The order k structure functions of the G-structure — obstructions to identification of the kth infinitesimal neighborhood of the point in a manifold with a flat G-structure with that at a given point $m \in M$ — span, for every m, the space $H^{k,2}_{(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*}$. These obstructions are defined provided obstructions of lesser orders vanish.*

9.1.2. Example. All structure functions of any $GL(n)$-structure vanish identically, so all $GL(n)$-structures are locally equivalent, in particular, locally flat. Indeed: by a theorem of Serre ([St2]) $H^2(V; (V, \mathfrak{gl}(V))_*) = 0$.

Clearly, the order of the structure functions of a given G-structure may run 1 to $N+2$ (or 1 to ∞ if $N = \infty$), where $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \bigoplus\limits_{i=-1}^{N} \mathfrak{g}_i$.

9.1.3. Example. Let $\mathfrak{g}_0 = \mathfrak{co}(V) := \mathfrak{o}(V) \oplus \mathbb{C}z$ be the Lie algebra of conformal transformations, $\mathfrak{g}_{-1} = V$, $\dim V = n$. For $n = 2$, let $V = V_1 \oplus V_2$ with basis ∂_x and ∂_y and let $\mathfrak{o}(V) := \mathbb{C}(x\partial_x - y\partial_y)$. Then (*Liouville's theorem*, [St2])

$$(V, \mathfrak{co}(V))_* = \begin{cases} \mathfrak{vect}(V^*) & \text{for } n = 1, \\ \mathfrak{vect}(V_1^*) \oplus \mathfrak{vect}(V_2^*) & \text{for } n = 2, \\ V \oplus \mathfrak{co}(V) \oplus V^* \simeq \mathfrak{o}(n+2) & \text{for } n > 2, \end{cases}$$

$$(V, \mathfrak{o}(V))_* = \begin{cases} V & \text{for } n = 1, \\ V \oplus \mathfrak{o}(V) & \text{for } n \geq 2\ . \end{cases}$$

The values of the Riemann tensor on any n-dimensional Riemannian manifold belong to $H^{2,2}_{(V, \mathfrak{o}(V))_*}$ whereas $H^{1,2}_{(V, \mathfrak{o}(V))_*} = 0$.

The fact that $H^{1,2}_{(V, \mathfrak{o}(V))_*} = 0$ (no torsion) is usually referred to as (a part of) the *Levi-Civita theorem*. It implies that, in the Taylor series expansion of the metric at some point (here η is the canonical form; x is the vector of coordinates, so x^2 is the vector of pairs of coordinates, etc.),

$$g(x) = \eta + s_1 x + s_2 x^2 + s_3 x^3 + \dots$$

the term s_1 can be eliminated by a choice of coordinates. Since there are no structure functions of orders > 2, all the s_i with $i \geq 2$ only depend on the Riemann tensor.

9.1.4. Remark (cf. [Go]). Let H^s_k be the degree k component of $H^s(\mathfrak{g}_{-1}; (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*)$ with respect to the $\mathbb{Z}$-grading induced by the $\mathbb{Z}$-grading of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$. Clearly, $H^{k,s}_{(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*} = H^s_k$, so

$$\bigoplus_k H^{k,s}_{(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*} = H^s(\mathfrak{g}_{-1}; (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*).$$

This remark considerably simplifies calculations, in particular, if the Lie algebra $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ is simple and finite dimensional, we can apply the BBW theorem. In the nonholonomic case considered in what follows we apply the remark to give a compact definition [3)] of structure functions. We can recover the bigrading at any moment but to work with just one grading is much simpler.

9.2. Structure functions of nonholonomic structures

To embrace contact-like structures, we have to slightly generalize the notion of Cartan prolongation: with the tangent bundle over every nonholonomic

[3] Cf. with the problems encountered in the pioneer papers [T], where only $d = 2$ is considered. Wagner's tensors (for any d) look even more horrible, see [DG], [Va6].

manifold there is naturally associated a bundle of graded nilpotent Lie algebras, cf. [VG], [M2]. For example, for any odd dimensional manifolds with a contact structure, this is a bundle of Heisenberg Lie algebras.

9.2.1. Nonholonomic manifolds ([VG, VG2]). Nonholonomic manifolds. Tanaka-Shchepochkina prolongs. Let M^n be an n-dimensional manifold with a nonintegrable distribution $\mathcal{D}$. Let

$$\mathcal{D} = \mathcal{D}_{-1} \subset \mathcal{D}_{-2} \subset \mathcal{D}_{-3} \dots \subset \mathcal{D}_{-d}$$

be the sequence of strict inclusions, where the fiber of $\mathcal{D}_{-i}$ at a point $x \in M$ is

$$\mathcal{D}_{-i+1}(x) + [\mathcal{D}_{-1}, \mathcal{D}_{-i+1}](x)$$

(here $[\mathcal{D}_{-1}, \mathcal{D}_{-i-1}] = \operatorname{Span}([X,Y] \mid X \in \Gamma(\mathcal{D}_{-1}), Y \in \Gamma(\mathcal{D}_{-i-1})))$ and d is the least number such that

$$\mathcal{D}_{-d}(x) + [\mathcal{D}_{-1}, \mathcal{D}_{-d}](x) = \mathcal{D}_{-d}(x).$$

In case $\mathcal{D}_{-d} = TM$ the distribution is called *completely nonholonomic*. The number $d = d(M)$ is called the *nonholonomicity degree*. A manifold M with a distribution $\mathcal{D}$ on it will be referred to as *nonholonomic* one if $d(M) \neq 1$. Let

$$n_i(x) = \dim \mathcal{D}_{-i}(x); \qquad n_0(x) = 0; \qquad n_d(x) = n - n_{d-1}. \tag{9.1}$$

The distribution $\mathcal{D}$ is said to be *regular* if all the dimensions n_i are constants on M. We will only consider regular, completely nonholonomic distributions, and, moreover, satisfying certain transitivity condition (9.2) introduced below.

To the tangent bundle over a nonholonomic manifold $(M, \mathcal{D})$ we assign a bundle of $\mathbb{Z}$-graded nilpotent Lie algebras as follows. Fix a point $pt \in M$. The usual adic filtration by powers of the maximal ideal $\mathfrak{m} := \mathfrak{m}_{pt}$ consisting of functions that vanish at pt should be modified because distinct coordinates may have distinct "degrees". The distribution $\mathcal{D}$ induces the following filtration in $\mathfrak{m}$:

$$\begin{aligned}\mathfrak{m}_k = \{f \in \mathfrak{m} \mid\ & X_1^{a_1} \dots X_n^{a_n}(f) = 0 \text{ for any } X_1, \dots, X_{n_1} \in \Gamma(\mathcal{D}_{-1}),\\ & X_{n_1+1}, \dots, X_{n_2} \in \Gamma(\mathcal{D}_{-2}), \dots, X_{n_{d-1}+1}, \dots, X_n \in \Gamma(\mathcal{D}_{-d})\\ & \text{such that } \sum_{1 \le i \le d} i \sum_{n_{i-1} < j \le n_i} a_j \le k\},\end{aligned} \tag{2.2}$$

where $\Gamma(\mathcal{D}_{-j})$ is the space of germs at pt of sections of the bundle $\mathcal{D}_{-j}$. Now, to a filtration

$$\mathcal{D} = \mathcal{D}_{-1} \subset \mathcal{D}_{-2} \subset \mathcal{D}_{-3} \dots \subset \mathcal{D}_{-d} = TM,$$

we assign the associated graded bundle

$$\operatorname{gr}(TM) = \oplus \operatorname{gr} \mathcal{D}_{-i}, \quad \text{where } \operatorname{gr} \mathcal{D}_{-i} = \mathcal{D}_{-i}/\mathcal{D}_{-i+1}$$

and the bracket of sections of $\operatorname{gr}(TM)$ is, by definition, the one induced by bracketing vector fields, the sections of TM. We assume a "transitivity condition": The Lie algebras

$$\operatorname{gr}(TM)|_{pt} \tag{9.2}$$

induced at each point $pt \in M$ are isomorphic.

The grading of the coordinates determines a nonstandard grading of $\mathfrak{vect}(n)$ (recall (9.1)):

$$\begin{aligned}&\deg x_1 = \ldots = \deg x_{n_1} = 1,\\ &\deg x_{n_1+1} = \ldots = \deg x_{n_2} = 2,\\ &\dots\dots\dots\dots\dots\dots\dots\dots\\ &\deg x_{n-n_{d-1}+1} = \ldots = \deg x_n = d.\end{aligned} \tag{9.3}$$

Denote by $\mathfrak{v} = \bigoplus_{i \ge -d} \mathfrak{v}_i$ the algebra $\mathfrak{vect}(n)$ with the grading (2.4). One can show that the "complete prolong" of $\mathfrak{g}_-$ to be defined shortly, i.e., $(\mathfrak{g}_-)_* := (\mathfrak{g}_-, \tilde{\mathfrak{g}}_0)_* \subset \mathfrak{v}$, where $\tilde{\mathfrak{g}}_0 := \mathfrak{der}_0 \mathfrak{g}_-$, preserves $\mathcal{D}$.

For nonholonomic manifolds, an analog of the group G from the term "G-structure", or rather of its Lie algebra, $\mathfrak{g} = \operatorname{Lie}(G)$, is the pair $(\mathfrak{g}_-, \mathfrak{g}_0)$, where $\mathfrak{g}_0$ is a subalgebra of the $\mathbb{Z}$-grading preserving Lie algebra of derivations of $\mathfrak{g}_-$, i.e., $\mathfrak{g}_0 \subset \mathfrak{der}_0\, \mathfrak{g}_-$. If $\mathfrak{g}_0$ is not explicitly indicated, we assume that $\mathfrak{g}_0 = \mathfrak{der}_0\, \mathfrak{g}_-$, i.e., is the largest possible.

Given a pair $(\mathfrak{g}_-, \mathfrak{g}_0)$ as above, define its *Tanaka-Shchepochkina prolong* to be the maximal subalgebra $(\mathfrak{g}_-, \mathfrak{g}_0)_* = \bigoplus_{k \ge -d} \mathfrak{g}_k$ of $\mathfrak{v}$ with given non-positive part $(\mathfrak{g}_-, \mathfrak{g}_0)$. For an explicit construction of the components, see [?], [Y], [Shch] and below.

Natural bases in T_mM: the D_i's and the Q_i's ([Shch]). Vershik and Gershkovich showed [VG] that every nonholonomic structure $\mathcal{D}$ on M determines a structure of $\mathbb{Z}$-graded nilpotent Lie algebra in $\operatorname{gr}(TM)$. We will only consider manifolds with a transitive action of the diffeomorphism group of M, i.e., the manifolds for which these Lie algebras are isomorphic.

A natural basis in every tangent space T_mM to any manifold M is given by partial derivatives. If M is endowed with a nonholonomic structure, then there are **two** types of natural bases in $\operatorname{gr} T_mM$. In physics literature on supersymmetry and supergravity, the elements of these two bases that generate the Lie algebra $\operatorname{gr} T_mM$ are denoted the D_i's and the Q_i's, respectively.

Let us consider the simplest example. Let $\dim M = 2n+1$ and let the nonholonomic structure on M be given by the contact form $\alpha = dt - \sum(p_i dq_i - q_i dp_i)$. The vector fields that **belong** to the distribution $\mathcal{D}$ are the fields

$$X = f\partial_t + \sum (g_i\partial_{q_i} + h_i\partial_{p_i}) \quad \text{such that} \quad \alpha(X) = f - \sum (p_ig_i + q_ih_i) = 0. \tag{2.5}$$

In particular, we see that neither ∂_{q_i} nor ∂_{p_i} belongs to $\mathcal{D}$, but rather

$$D_{p_i} = \partial_{q_i} + p_i\partial_t \quad \text{and} \quad D_{q_i} = \partial_{p_i} - q_i\partial_t.$$

These D_{p_i} and D_{q_i} are examples of the D-type basis vectors. They, and their brackets, span the space of sections of $gr(TM)$ at any given point m. By abuse of speech, we say that the D-vectors span T_mM, and same applies to Q-vectors defined below.

Now, the Lie algebra that **preserves** $\mathcal{D}$ consists of vector fields X such that (here L_X is the Lie derivative along X)

$$L_X(\alpha) = 0. \tag{2.6}$$

The corresponding vector fields in our particular case of the contact distribution are contact vector fields K_f generated by $f \in \mathbb{C}[t,p,q]$:

$$K_f = (2-E)(f)\frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t}E, \tag{2.7}$$

where $E = \sum\limits_i y_i\frac{\partial}{\partial y_i}$ (here the y_i are all the coordinates except t) is the *Euler operator*, and H_f is the Hamiltonian field with Hamiltonian f that preserves $d\alpha$:

$$H_f = \sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right).$$

(As is easy to see,

$$L_{K_f}(\alpha) = 2\frac{\partial f}{\partial t}\alpha, \tag{2.8}$$

where L_X is the Lie derivative along X.) The basis of the tangent space is spanned by

$$K_{p_i} = \partial_{q_i} - p_i\partial_t \quad \text{and} \quad K_{q_i} = \partial_{p_i} + q_i\partial_t$$

and their brackets. These K_{p_i} and K_{q_i} are examples of the Q-type basis vectors.

How to interpret the D-type and the Q-type vectors? Let

$$\mathfrak{n} = \bigoplus_{-d\leq i\leq -1}\mathfrak{n}_i$$

be a nilpotent Lie algebra generated by $\mathfrak{n}_{-1}$. Let $B = \{b_1, \dots, b_n\}$ be a graded basis of $\mathfrak{n}$ (the basis is said to be *graded* if its first $n_1 := \dim\mathfrak{n}_{-1}$ elements span $\mathfrak{g}_{-1}$, the next $n_2 := \dim\mathfrak{n}_{-2}$ elements span $\mathfrak{n}_{-2}$, and so on). Let N be the connected and simply connected Lie group with the Lie algebra $\mathfrak{n}$. On N, consider the two systems of vector fields: the left-invariant fields D_i and the right-invariant fields Q_i such that (e is the unit of N)

$$D_i(e) = Q_i(e) = b_i \quad \text{for all } i = 1, \dots, n.$$

NB: Here we deviate from the conventions of physical papers where the symbols D_i and Q_i are only applied to the generators of $\mathfrak{n}$, i.e., to the first n_1 elements.

Let $\mathfrak{g}_-$ be a realization of $\mathfrak{n}$ by **left-invariant** vector fields, so the vectors $D_i(e)$ span $\mathfrak{g}_-$. Let θ^i be **right-invariant** 1-forms on N such that

$$\theta^i(Q_j) = \delta^i_j.$$

Now, any vector field X on N is of the form

$$X = \sum_{i=1}^n \theta^i(X)Q_i. \tag{2.9}$$

Since each D_i commutes with each Q_j (if $\mathfrak{n}$ is a Lie superalgebra, they **super**commute), it follows that

$$\theta^i([D_j, X]) = D_j(\theta^i(X)).$$

Now, let us determine a **right-invariant** distribution $\mathcal{D}$ on N such that $\mathcal{D}|_e = \mathfrak{n}_{-1}$. Clearly, $\mathcal{D}$ is singled out in TN by eqs. for $X \in \mathfrak{vect}(n)$

$$\theta^{n_1+1}(X) = 0, \quad \dots, \quad \theta^n(X) = 0.$$

Since each D_i commutes with each Q_j, the algebra $\mathfrak{g}_-$ preserves $\mathcal{D}$. The coordinates (2.4) on N described above determine two embeddings of $\mathfrak{n}$ into $\mathfrak{vect}(n)$: one is spanned by the D_i and the other one by the Q_i.

Denote by $\mathfrak{g} = \bigoplus\limits_{i\geq -d}\mathfrak{g}_i$ the algebra $\mathfrak{vect}(n)$ with the grading (9.3). Then $\mathfrak{g}_- = \bigoplus\limits_{i<0}\mathfrak{g}_i$ preserves $\mathcal{D}$. We will show later that the "complete prolongation" of $\mathfrak{g}_-$, i.e., $(\mathfrak{g}_-)_* := (\mathfrak{g}_-, \tilde{\mathfrak{g}}_0)_*$, where $\tilde{\mathfrak{g}}_0 := \mathfrak{der}_0\mathfrak{g}_-$, also preserves $\mathcal{D}$.

Thus we see that, with every nonholonomic manifold $(M, \mathcal{D})$, a natural G-structure is associated, its Lie algebra is $\mathrm{Lie}(G) = \mathfrak{der}_0\,\mathfrak{g}_-$. But the structure functions of this G-structure do not reflect the nonholonomic nature of M.

Indeed, recall an example from [St2]. Let $W_1 \subset W$ be a subspace of dimension k and $G \subset GL(W)$ the parabolic subgroup that preserves the subspace. Then to determine a G-structure on M, where $\dim M = \dim W$, is the same as to determine a differential k-system or a k-dimensional distribution. A fixed frame f in T_mM determines an isomorphism $f : W \longrightarrow T_mM$. Given a G-structure on M, we set $\mathcal{D}(m) = f(W_1)$. Since G preserves W_1, the subspace $\mathcal{D}(m)$ indeed depends only on m, not on f.

The other way round, given a distribution $\mathcal{D}$, consider the frames f such that $f^{-1}(\mathcal{D}(m)) = W_1$. They form a G-structure. The flat G-structures correspond to integrable distributions.

To take the nonholonomic nature of M into account, we need something new — an analog of the above Proposition 1.1 for the case where the natural basis of the tangent space consists not of partial derivatives but rather of covariant derivatives corresponding to the connection determined by the same Pfaff equations that determine the distribution, and therefore instead of $T_mM = \mathfrak{g}_{-1}$ we have $(gr(TM))_m = \mathfrak{g}_-$. To be able to formulate such Proposition, we have to define (1) the simplest nonholonomic structure — the "flat" one,

(2) the analog of $\mathfrak{g}_0$ when $\mathfrak{g}_{-1}$ is replaced by $\mathfrak{g}_-$ and only distribution is given,

(3) what is the analog of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$,

(4) what is the analog of $H^{k,2}_{(\mathfrak{g}_{-1},\mathfrak{g}_0)_*}$.

Here are the answers:

1) Let $\mathcal{D}$ be a nonholonomic distribution in M, let F be the flag which $\mathcal{D}$ determines at a point $m \in M$. Let $N := \mathbb{K}^n$ with a fixed flag F and a fixed frame f. Having identified T_nN with N by means of the translation by n considered as an element of the *nilpotent* Lie group N whose Lie algebra is $\mathfrak{g}_-$ (since the group N is not commutative now, we select, say, left translations) we fix a frame and a flag — the images of f and F — in each T_nN. A *flat nonholonomic structure* on N is the *pair* of bundles (the frame bundle, the distribution $\mathcal{D}$); the fibers of both bundles over n are obtained from the fixed frame and flag, respectively, by means of the G-action, where G is the (connected and simply connected) Lie group whose Lie algebra $\mathfrak{g}_0$ is defined at the next step.

2) If only a distribution $\mathcal{D}$ is given, we set $\mathfrak{g}_0 := \mathfrak{der}_0\mathfrak{g}_-$; it is often interesting to consider an additional structure on the distribution, say Riemannian, cf. [VG2], as in the case of Carnot-Carathéodory metric in which case $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{der}_0\mathfrak{g}_-$, e.g., $\mathfrak{der}_0\mathfrak{g}_- \cap \mathfrak{o}(\mathfrak{g}_{-1})$.

3) Given a pair $(\mathfrak{g}_-, \mathfrak{g}_0)$ as above, define its kth *Tanaka-Shchepochkina prolong* (given simultaneously, although [Sh1] was published later than [T]; [Sh1] also embraces Lie superalgebras and various *partial prolongs*, see [?]) for $k > 0$ to be:

$$\mathfrak{g}_k = (i(S^{\bullet}(\mathfrak{g}_-^*) \otimes \mathfrak{g}_0) \cap j(S^{\bullet}(\mathfrak{g}_-^*) \otimes \mathfrak{g}_-))_k, \tag{2.10}$$

where the subscript singles out the component of degree k, where $S^{\bullet} = \oplus S^i$ and S^i denotes the operator of the ith symmetric power, and where

$$\begin{aligned} i &: S^{k+1}(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g}_{-1} \longrightarrow S^k(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}, \\ j &: S^k(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g}_0 \longrightarrow S^k(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1} \end{aligned}$$

are natural embeddings.

Similarly to the case where $\mathfrak{g}_-$ is commutative, define $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ to be $\bigoplus\limits_{k \geq -d} \mathfrak{g}_k$ with $\mathfrak{g}_k$ for $k > 0$ given by (2.10); then, as is easy to verify, $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ is a Lie algebra.

4) Arguments similar to those of [St2] should show that $H^2(\mathfrak{g}_-; (\mathfrak{g}_-, \mathfrak{g}_0)_*)$ is the space of values of all nonholonomic structure functions — obstructions to the identification of the infinitesimal neighborhood of a point m of the manifold M with a nonholonomic structure (given by $\mathfrak{g}_-$ and $\mathfrak{g}_0$) with the infinitesimal neighborhood of a point of a flat nonholonomic manifold with the same $\mathfrak{g}_-$ and $\mathfrak{g}_0$. We intend to give a detailed proof of this statement elsewhere.

The space $H^2(\mathfrak{g}_-; (\mathfrak{g}_-, \mathfrak{g}_0)_*)$ naturally splits into homogeneous components whose degrees will be called the *orders* of the structure functions; the orders run $2 - d$ to $N + 2d$ (or to ∞ if $N = \infty$). As in the case of a commutative $\mathfrak{g}_- = \mathfrak{g}_{-1}$, the structure functions of order k can be interpreted as obstructions to flatness of the nonholonomic manifold with the $(\mathfrak{g}_-, \mathfrak{g}_0)$-structure provided the obstructions of lesser orders vanish. Observe that, for nonholonomic manifolds, the order of structure functions is no more in direct relation with the orders of the infinitesimal neighborhoods of the points we wish to identify: distinct partial derivatives bear different "degrees".

Different filtered algebras L with the same graded $\mathfrak{g}_-$ are governed precisely by the coboundaries responsible for filtered deformations of $\mathfrak{g}_-$, and all of them vanish in cohomology, so the above nonholonomic structure functions are well-defined.

9.3. The Riemann and Weyl tensors. Projective structures

The conformal case. For the classical domains $X = S/P$ that Goncharov considered, the structure functions are generalizations of the Weyl tensor — the conformally invariant part of the Riemann tensor (the case $S = O(n+2)$ and $G = CO(n)$). In most of these cases

$$(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{s} \ (:= \mathrm{Lie}(S)) \tag{3.1}$$

and the description of the structure functions is a particular case of the BBW theorem. In particular, if (3.1) holds, the space $H^2(\mathfrak{g}_{-1}; (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*)$, considered as a $\mathfrak{g}_0$-module, has the same number of irreducible components and the same dimension as $E^2(\mathfrak{g}_{-1})$; only weights differ.

The generalized Riemannian case. When we reduce $\mathfrak{g}_0$, by retaining its semi-simple part $\hat{\mathfrak{g}_0}$ and deleting the center, we can not directly apply the BBW theorem because $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}_0})_* = \mathfrak{g}_{-1} \oplus \hat{\mathfrak{g}_0}$ is not simple but we can reduce the problem to the conformal case, since, as is known,

$$H^2(\mathfrak{g}_{-1}; (\mathfrak{g}_{-1}, \hat{\mathfrak{g}_0})_*) = H^2(\mathfrak{g}_{-1}; \mathfrak{s}) \oplus S^2(\mathfrak{g}_{-1}^*). \tag{3.2}$$

For the nonholonomic case, a similar reduction is given by Premet's theorem (below). Its general case, though sufficiently neat, is not as simple as (3.2). However, although the following analog of (3.2) is not always true

$$H^2(\mathfrak{g}_-; (\mathfrak{g}_-, \hat{\mathfrak{g}_0})_*) = H^2(\mathfrak{g}_-; \mathfrak{s}) \oplus S^2(\mathfrak{g}_{-1}^*), \tag{3.3}$$

it is still true in many cases of interest: for the "contact grading".

The projective case. Theorems of Serre and Yamaguchi. When (3.1) fails, $\mathfrak{s}$ is a proper subalgebra of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$. It is of interest therefore

(a) to list all the cases where, having started from a simple Lie (super)algebra $\mathfrak{s} = \bigoplus\limits_{i \geq -d} \mathfrak{s}_i$, we have the following analog of (3.1)

$$(\mathfrak{s}_-, \mathfrak{s}_0)_* = \mathfrak{s} \tag{3.4}$$

and

(b) find out *what* is the "complete prolongation" of $\mathfrak{s}_-$, i.e., what is $(\mathfrak{s}_-)_* := (\mathfrak{s}_-, \tilde{\mathfrak{s}}_0)_*$, where $\tilde{\mathfrak{s}}_0 := \mathfrak{der}_0 \mathfrak{s}_-$.

For the simple finite dimensional Lie algebras $\mathfrak{s}$, Yamaguchi [Y] gives the answer. It is rather interesting and we reproduce it. The answer for simple Lie superalgebras is obtained by Shchepochkina (unpublished). Comment: one would expect that $\tilde{\mathfrak{s}}_0$ strictly contains $\mathfrak{s}_0$, and hence $(\mathfrak{s}_-)_*$ should strictly contain $\mathfrak{s}$; instead they are equal (in particular, $\tilde{\mathfrak{s}}_0 = \mathfrak{s}_0$).

Theorem ([Y]). *Equality $(\mathfrak{s}_-)_* = \mathfrak{s}$ holds almost always. The exceptions are*

1) *$\mathfrak{s}$ with the grading of depth $d = 1$ (in which case $(\mathfrak{s}_-)_* = \mathfrak{vect}(\mathfrak{s}_-^*)$);*

2) *$\mathfrak{s}$ with the grading of depth $d = 2$ and $\dim \mathfrak{s}_{-2} = 1$, i.e., with the "contact" grading, in which case $(\mathfrak{s}_-)_* = \mathfrak{k}(\mathfrak{s}_-^*)$ (these cases correspond to exclusion of the nodes on the Dynkin graph connected with the node for the maximal root on the extended graph);*

3) *$\mathfrak{s}$ is either $\mathfrak{sl}(n+1)$ or $\mathfrak{sp}(2n)$ with the grading determined by "selecting" the first and the ith of simple coroots, where $1 < i < n$ for $\mathfrak{sl}(n+1)$ and $i = n$ for $\mathfrak{sp}(2n)$. (Observe that, in this case, $d = 2$ with $\dim \mathfrak{s}_{-2} > 1$ for $\mathfrak{sl}(n+1)$ and $d = 3$ for $\mathfrak{sp}(2n)$.)*

Moreover, $(\mathfrak{s}_-, \mathfrak{s}_0)_* = \mathfrak{s}$ is true almost always. The cases where this fails (the ones where a projective action is possible) are $\mathfrak{sl}(n+1)$ or $\mathfrak{sp}(2n)$ with the grading determined by "selecting" only one (the first) simple coroot.

Case 1) of Yamaguchi's theorem: for the conformal (Weyl) case, see [Go]; for the Riemannian case, see [LPS] .

For the classical domains $X = S/P$, (3.1) fails only for $S = SL(n+1)$ and $X = \mathbb{C}P^n$; in this case $\mathfrak{g}_0 = \mathfrak{gl}(n)$ and $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{vect}(n)$, the Lie algebra of vector fields in n indeterminates. The space of "total" structure functions $H^2(\mathfrak{g}_{-1}; \mathfrak{vect}(n))$ differs from $H^2(\mathfrak{g}_{-1}; \mathfrak{s})$, the latter structure functions correspond to obstructions to the *projective structure*. For many facets of projective structures, see [OT] and [BR].

The Riemannian version of this projective case, corresponds to $\hat{\mathfrak{g}}_0 = \mathfrak{sl}(n)$ and $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_* = \mathfrak{svect}(n)$, the Lie algebra of divergence free vector fields.

The cases of "complete prolongation" $(\mathfrak{s}_{-1})_* = \mathfrak{vect}(\mathfrak{s}_{-1}^*)$ and their "Riemannian version" $(\mathfrak{s}_{-1})_* = \mathfrak{svect}(\mathfrak{s}_{-1}^*)$, as well as $(\mathfrak{s}_{-1})_* = \mathfrak{h}(\mathfrak{s}_{-1}^*)$, were considered by Serre long ago, see [St2], and the answer is as follows:

Theorem (Serre, see [St2]; for super version, see [LPS] and [GLS3]).

1) $H^2(\mathfrak{s}_{-1}; \mathfrak{vect}(n)) = 0$ *and* $H^2(\mathfrak{s}_{-1}; \mathfrak{svect}(n)) = 0$.

2) $H^2(\mathfrak{s}_{-1}; \mathfrak{h}(2n)) = E^3(\mathfrak{s}_{-1}^*)$.

Case 2) of Yamaguchi's theorem is taken care of by one of Premet's theorems and formula (3.5) below.

Case 3) of Yamaguchi's theorem: see [GL].

In what follows, for manifolds $X = S/P$ with nonholonomic structure, we say "N-Weyl" or "N-conformal", for tensors corresponding to cohomology of $\mathfrak{g}_-$ with coefficients in $(\mathfrak{g}_-, \mathfrak{g}_0)_*$, "N-Riemannian" for nonholonomic structure functions $(\mathfrak{g}_-, \hat{\mathfrak{g}}_0)_*$, where $\hat{\mathfrak{g}}_0$ is the semi-simple part of $\mathfrak{g}_0$, and "N-projective" for the coefficients in $\mathfrak{s} = \mathrm{Lie}(S)$ whenever $\mathfrak{s}$ is smaller than $(\mathfrak{g}_-, \mathfrak{g}_0)_*$, for example, for *partial Cartan prolongs*, see [?].

The simplest examples (exclusion of the first simple coroot of $\mathfrak{sp}(2n+2)$). Let $\mathfrak{g}_- = \mathfrak{hei}(2n)$, the Heisenberg Lie algebra. Then $\mathfrak{g}_0 = \mathfrak{csp}(2n)$ (i.e., $\mathfrak{sp}(2n) \oplus \mathbb{C}z$) and $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ is the Lie algebra $\mathfrak{k}(2n+1)$ of contact vector fields.

So far, there is no analog of Serre's theorem on involutivity for simple $\mathbb{Z}$-graded Lie algebras of depth > 1, cf. [LPS], and examples from [GLS3] show that if exists, the theorem is much more involved.

The fact that

$$H^2(\mathfrak{hei}(2n); \mathfrak{k}(2n+1)) = 0 \tag{3.5}$$

explains why **the Pfaff equation $\alpha(X) = 0$ for $X \in \mathfrak{vect}(2n+1)$ can be reduced to a canonical form**, cf. [Z]. This fact is an easy corollary of a statement on cohomology of coinduced modules [FF2]. For the N-Riemannian tensor in this case, we have: $\hat{\mathfrak{g}}_0 = \mathfrak{sp}(2n)$ and $(\mathfrak{g}_-, \hat{\mathfrak{g}}_0)_*$ is the Poisson Lie algebra $\mathfrak{po}(2n)$. The Poisson Lie algebra is spanned by fields K_f, where $\dfrac{\partial f}{\partial t} = 0$. Now, from (3.5) and the short exact sequence

$$0 \longrightarrow \mathfrak{po}(2n) \longrightarrow \mathfrak{k}(2n+1) \xrightarrow{\frac{\partial}{\partial t}: K_f \mapsto \frac{\partial f}{\partial t}} \mathbb{C}[t, p, q] \longrightarrow 0$$

we easily deduce (using the corresponding long exact sequence, see [FF2]) that

$$H^2(\mathfrak{hei}(2n); \mathfrak{po}(2n)) = 0.$$

In our terms, this fact (usually called *Darboux's theorem* and proved by analytic means [Z]) is an explanation why **the contact form α can be reduced to a canonical form not only at any point but locally**.

Other examples. For numerous examples of N-projective structures in various instances, see [C1]–[C5] and [YY], and (in super setting) [MaG]. Armed with `SuperLie`, one can now easily perform the computations of relevant Lie algebra cohomology. Premet's theorems tell what to compute in the N-Riemannian case and again with **SuperLie** this will be easy: we just give a few samples (**one** selected simple coroot for every $\mathfrak{s}$ and **two** selected coroots for the two series of one of Yamaguchi's cases).

9.4. Premet's Theorems (from Premet's letter to DL, 10/17/1990)

In 1990, D.L. asked Alexander Premet: how to reduce computations of the space of values for a nonholonomic Riemann tensors to that for the nonholonomic Weyl tensor, as in (3.2)? Namely, is (3.3) always true?

Premet wrote two letters with a general answer. One letter is reproduced practically without changes below (D.L. is responsible for any mistakes left/inserted); it shows how to reduce the problem to computing (the 1st) cohomology of $\mathfrak{g}_-$ with coefficients in a certain $\mathfrak{g}_-$-module which is not a $\mathfrak{g}$-module. Little was known about such cohomology except theorems of Kostant (on H^1) and of Leger and Luks (on H^2) both for the case where $\mathfrak{g}_-$ is the maximal nilpotent subalgebra. Premet's second letter (reproduced in [LLS]) contained a mighty generalization of these theorems for H^i for any i and any $\mathfrak{g}_-$.

However, in nonholonomic cases, to derive an *explicit* answer from the BWB theorem is difficult "by hands", the extra terms in the Riemannian case (see sec. 4.4 below) add extra job. So Premet's theorems were put aside for 13 years. Now that a package `SuperLie` ([Gr]), originally designed for the purposes of supergravity, is sufficiently developed, we are able to give an explicit answer: see the next section. The cases we consider here (of the maximal parabolic subalgebras) required several minutes to compute. (But much longer to document the results, and it will require a while to interpret them, say as in [LPS].) To our regret, Premet looks at his theorems as a mere technical exercise ("a simple job for Kostant") not interesting enough to co-author the paper.

9.4.1. Terminological conventions. Let $\mathfrak{g}$ be a simple (finite dimensional) Lie algebra. Let L^λ denote the irreducible (finite dimensional) $\mathfrak{g}$-module with the highest weight λ; let E_μ be the subspace the module E of weight μ.

Let R be the root system of $\mathfrak{g}$ and B the base (system of simple roots). Let $W = W(R)$ be the Weyl group of $\mathfrak{g}$ and $l(w)$ the length of the element $w \in W$; let W_i be the subset of elements of length i. Let $R_I \subset R$ and let B_I be the base of R_I. Set (this is a definition of $k(i)$ as well)

$$W(I)_i = \{w_{i,1}, \dots, w_{i,k(i)} \in W_i \mid w_{i,j}^{-1}(B \setminus B_I) > 0 \text{ for all } 1 \le j \le k(i)\}. \tag{4.1}$$

Let the Dynkin graph of B be, for example, as follows:

● — ● — ○ — ● — ● — ● — ○ — ○ — ○ — ● — ● — ○ — ● — ● — ● — ○ — ● — ● — ○ — ○

and let B_I consist of roots corresponding to the black nodes. Let us represent B_I as the union of connected subgraphs:

$$B_I = B_I^{(1)} \coprod \dots \coprod B_I^{(s)},$$

where s (in our example $s = 5$) is the number of connected components of the Dynkin graph D_I of B_I and where $B_I^{(i)}$ corresponds to the ith connected components of D_I (counted from left to right). Set

$$c = \text{card } B, \quad c_i = \text{card } \{\alpha \in B \setminus B_I \mid (\alpha, B_I^{(i)}) \neq 0\} - 1.$$

Clearly, if $B_I \neq B$, then $c_i \in \{0, 1, 2\}$. For example, for the graph of $\mathfrak{o}(20)$ depicted above, we have:

$$c = 20, \quad c_1 = 0, \quad c_2 = 1, \quad c_3 = 1, \quad c_4 = 1, \quad c_5 = 2.$$

The following statement is obvious.

Statement. 1) $c_i = 2$ *if and only if R is of type D_n, E_6, E_7, E_8, one of the endpoints of $D_I^{(i)}$ is a branching point for D, and the remaining endpoint of $D_I^{(i)}$ is not an endpoint of D.*

2) $c_i = 0$ *if and only if all but one of the end vertices of the graph of $B_I^{(i)}$ are the end vertices for the graph of R.*

9.4.2. The Bott-Borel-Weil theorem. Let $\operatorname{rk}\mathfrak{g} = r > 1$, $I \subset \{1, \dots, r\}$; let $\mathfrak{p} = \mathfrak{p}_I$ be a parabolic subalgebra generated by the Chevalley generators $X_i^\pm$ of $\mathfrak{g}$ except the X_i^+, where $i \in I$. As is known, $\mathfrak{p} = \mathfrak{g}_- \oplus \mathfrak{l}$, where $\mathfrak{l}$ is the Levi (semi-simple) subalgebra generated by all the $X_i^\pm$, where $i \notin I$. Clearly, $\mathfrak{l} = \mathfrak{l}^{(1)} \oplus \mathfrak{z}$, where $\mathfrak{l}^{(1)}$ is the derived algebra of $\mathfrak{l}$, and $\mathfrak{z} = \mathfrak{z}(\mathfrak{l})$ is the center of $\mathfrak{l}$.

So, in terms of §3, $\mathfrak{g}_0 = \mathfrak{l}$, $\hat{\mathfrak{g}}_0 = \mathfrak{l}^{(1)}$.

Theorem (The BBW Theorem, see [BGG])**.** *Let $E = L^\lambda$ be an irreducible (finite dimensional) $\mathfrak{g}$-module with* **highest** *weight λ. Then $H^i(\mathfrak{g}_-; E)$ is the direct sum of $\mathfrak{l}$-modules with the* **lowest** *weights $-w_{i,j}(\lambda + \rho) + \rho$, where $w_{i,j} \in W(I)_i$, see* (4.1)*; each such module enters with multiplicity* 1.

The BBW theorem describes (for $i = 2$) nonholonomic analogs of Weyl tensors. Theorem 4.4 describes nonholonomic analogs of the Riemann tensors.

9.4.3. Theorem. $H^2(\mathfrak{g}_-; \mathfrak{g}_- \oplus \mathfrak{l}^{(1)}) = H^2(\mathfrak{g}_-; \mathfrak{g}) \oplus H^1(\mathfrak{g}_-; (\mathfrak{g}_- \oplus \mathfrak{z})^*)$.

Corollary. *Let $B_1 = B \setminus B_I$; let R_1 be the root system generated by B_1 and*

$$\mathfrak{g}_-^{ab} = (\mathfrak{g}_-/\mathfrak{g}_-^{(1)})^* = H^1(\mathfrak{g}_-).$$

1) *The following sequence is exact:*

$$0 \longrightarrow \mathfrak{g}_-^{ab} \longrightarrow \mathfrak{z}^* \otimes \mathfrak{g}_-^{ab} \longrightarrow H^1(\mathfrak{g}_-; \mathfrak{g}/(\mathfrak{g}_- \oplus \mathfrak{l}^{(1)})) \longrightarrow H^1(\mathfrak{g}_-; \mathfrak{g}_-^*) \longrightarrow$$
$$H^2(\mathfrak{g}_-) \oplus \bigoplus_{w \in W(R_1)_{(2)}} L^{\rho - w(\rho)} \longrightarrow 0.$$

2) *If* $\dim\ \mathfrak{z} = 1$, *then the sequence*

$$0 \longrightarrow H^1(\mathfrak{g}_-; (\mathfrak{g}_- \oplus \mathfrak{z})^*) \longrightarrow H^1(\mathfrak{g}_-; \mathfrak{g}_-^*) \longrightarrow H^2(\mathfrak{g}_-) \longrightarrow 0$$

is exact. In particular, if $\mathfrak{g}_-$ is a Heisenberg algebra (the case of contact grading), then

$$\boxed{H^2(\mathfrak{g}_-;\mathfrak{g}_-\oplus\mathfrak{l}^{(1)})\simeq H^2(\mathfrak{g}_-;\mathfrak{g})\oplus S^2(\mathfrak{g}_-/\mathfrak{z}(\mathfrak{g}_-))^*=H^2(\mathfrak{g}_-;\mathfrak{g})\oplus S^2(\mathfrak{g}_{-1}^*).}$$

3) *if $\mathfrak{g}_-=\mathfrak{g}_{-1}$ (is abelian), then*

$$H^2(\mathfrak{g}_-;\mathfrak{g}_-\oplus\mathfrak{l}^{(1)})\simeq H^2(\mathfrak{g}_-;\mathfrak{g})\oplus S^2(\mathfrak{g}_-^*)=H^2(\mathfrak{g}_-;\mathfrak{g})\oplus S^2(\mathfrak{g}_{-1}^*).$$

9.4.4. The number of $\mathfrak{g}_0$-modules. The following Theorem helps to verify the result. Let IR be the number of irreducible components in the $\mathfrak{g}_0$-module $H^2(\mathfrak{g}_-;(\mathfrak{g}_-,\mathfrak{g}_0)_*)$.

Theorem. $IR=\frac{1}{2}c(c+1)+\sum c_i$.

Chapter 10

Lie superalgebras of supermatrices of complex size (P. Grozman, D. Leites)

Summary. We distinguish a class of simple filtered Lie algebras $(U_\lambda(\mathfrak{g}))_L$ of polynomial growth with increasing filtration and whose associated graded Lie algebras are not simple. We describe presentations of such algebras. The Lie algebras $(U_\lambda(\mathfrak{g}))_L$, where λ runs over the projective space of dimension equal to the rank of $\mathfrak{g}$, are quantizations of the Lie algebras of functions on the orbits of the coadjoint representation of $\mathfrak{g}$.

The Lie algebra $\mathfrak{gl}(\lambda)$ of matrices of complex size is the simplest example; it is $(U_\lambda(\mathfrak{sl}(2)))_L$. The dynamical systems associated with it in the space of pseudodifferential operators in the same way as the KdV hierarchy is associated with $\mathfrak{sl}(n)$ are those studied by Gelfand–Dickey and Khesin–Malikov. For $\mathfrak{g}\neq\mathfrak{sl}(2)$ we get generalizations of $\mathfrak{gl}(\lambda)$ and the corresponding dynamical systems, in particular, their superized versions. The algebras $(U_\lambda(\mathfrak{sl}(2)))_L$ possess a trace and an invariant symmetric bilinear form, hence, with these Lie algebras associated are analogs of the Yang-Baxter equation, KdV, etc.

Our presentation of $(U_\lambda(\mathfrak{s}))_L$ for a simple $\mathfrak{s}$ is related to presentation of $\mathfrak{s}$ in terms of a certain pair of generators. For $\mathfrak{s}=\mathfrak{sl}(n)$ there are just 9 such relations.

This chapter reproduces (main points of) [GL2].

10.0. Introduction

10.0.0. History. About 1966, V. Kac and B. Weisfeiler began the study of simple *filtered* Lie algebras of *polynomial growth*. Kac first considered the $\mathbb{Z}$-*graded* Lie algebras associated with the filtered ones and classified *simple graded* Lie algebras of *polynomial growth* under a technical assumption and conjectured the inessential nature of the assumption. It took more than 20 years to get rid of the assumption: see very complicated papers by O. Mathieu, cf. [K3] and references therein. For a similar list of simple $\mathbb{Z}$-graded Lie *super*algebras of polynomial growth see [KS], [**?**] (for summary, see Chapter 2).

The Lie algebras Kac distinguished (or rather the algebras of derivations of their nontrivial central extensions, the *Kac–Moody* algebras) proved very interesting in applications. These algebras aroused such interest that the study of filtered algebras was arrested for two decades. Little by little, however, the simplest representative of the new class of simple filtered Lie superalgebras (of polynomial growth), namely, the Lie algebra $\mathfrak{gl}(\lambda)$ of matrices of complex size, and its projectivization, i.e., the quotient modulo the constants, $\mathfrak{pgl}(\lambda)$, drew its share of attention [F], [KM], [KR].

While we were typing [GL2], Shoikhet [Sho] published a description of representations of $\mathfrak{gl}(\lambda)$; we are thankful to M. Vasiliev who informed us of possible applications of generalizations of $\mathfrak{gl}(\lambda)$ in physics, see [BWV], [KV].

The paper [GL2] began a systematic study of a new class of Lie algebras: simple filtered Lie algebras of polynomial growth (SFLAPG) for which the graded Lie algebras associated with the filtration considered are not simple; $\mathfrak{sl}(\lambda)$ is our first example. Actually, an example of a Lie algebra of class SFLAPG was known even before the notion of Lie algebras was introduced. Indeed, the only deformation (physicists call it *quantization*) Q of the Poisson Lie algebra $\mathfrak{po}(2n)$ sends $\mathfrak{po}(2n)$ into $\mathfrak{diff}(n)$, the Lie algebra of differential operators with polynomial coefficients; the restriction of Q to $\mathfrak{h}(2n) = \mathfrak{po}(2n)/\mathfrak{center}$, the Lie algebra of Hamiltonian vector fields, sends $\mathfrak{h}(2n)$ to the projectivization $\mathfrak{pdiff}(n) = \mathfrak{diff}(n)/\mathbb{C}\cdot 1$ of $\mathfrak{diff}(n)$. The Lie algebra $\mathfrak{pdiff}(n)$ escaped Kac's classification, though it is the deform of an algebra from his list, because its intrinsically natural filtration given by $\deg q_i = -\deg \partial_{q_i} = 1$ is not of polynomial growth (the homogeneous components are of infinite dimension) while the graded Lie algebra associated with the filtration of polynomial growth (given by $\deg q_i = \deg \partial_{q_i} = 1$) is not simple.

Observe that from the point of view of dynamical systems the Lie algebra $\mathfrak{diff}(n)$ is not very interesting: it does not possesses a non-degenerate bilinear symmetric form; we will consider its subalgebras that do.

In what follows we will usually denote the associative (super)algebras by Latin letters; the Lie (super)algebras associated with them by Gothic letters; e.g., $\mathfrak{gl}(n) = \mathrm{Mat}(n)_L$, $\mathfrak{diff}(n) = \mathrm{Diff}(n)_L$, where the functor $_L$ replaces the dot product by the bracket.

10.0.1. The construction. Problems related. Each of our Lie algebras (and Lie superalgebras) $(U_\lambda(\mathfrak{g}))_L$ is realized as a quotient of the Lie algebra of global sections of the sheaf of twisted D-modules on the flag variety, cf. [Ka], [Di]. The general construction consists of the preparatory step 0), the main steps 1) and 2) and two extra steps 3) and 4).

We distinguish two cases: A) $\dim \mathfrak{g} < \infty$ and $\mathfrak{g}$ possesses a Cartan matrix and B) $\mathfrak{g}$ is a simple vectorial Lie (super)algebra.

Let $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$, where $\mathfrak{g}_+ = \mathop{\oplus}\limits_{\alpha>0} \mathfrak{g}_\alpha$ and $\mathfrak{g}_- = \mathop{\oplus}\limits_{\alpha<0} \mathfrak{g}_\alpha$, be one of the simple $\mathbb{Z}$-graded Lie algebras of polynomial growth, either finite dimensional or of vector fields, represented as the sum of its maximal torus (usually identical with the Cartan subalgebra) $\mathfrak{h}$ and the root subspaces $\mathfrak{g}_\alpha$ corresponding to an order in the set R of roots.

Observe that each order of R is in one-to-one correspondence with a system of simple roots. For the finite dimensional Lie algebras $\mathfrak{g}$ all systems of simple roots are equivalent, the equivalence is established by the Weyl group. For Lie superalgebras and infinite dimensional Lie algebras of vector fields there are inequivalent systems of simple roots; nevertheless, there is an analog of the Weyl group and the passage from system to system is described in [PS].

For vectorial Lie algebras and Lie superalgebras, even the dimension of the superspaces $X = (\mathfrak{g}_-)^*$ associated with systems of simple roots can vary. It is not clear if only *essential* (see [PS]) systems of simple roots are essential in the construction of Verma modules (roughly speaking, each Verma module is isomorphic to the space of functions on X) in which we will realize $(U_\lambda(\mathfrak{g}))_L$, but hopefully not all.

Step 0): From $\mathfrak{g}$ to $\tilde{\mathfrak{g}}$. From representation theory it is clear that there exists a realization of the elements of $\mathfrak{g}$ by differential operators of degree ≤ 1 on the space $X = (\mathfrak{g}_-)^*$. The realization has rank $\mathfrak{g}$ parameters (coordinates $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathfrak{h}^*$ of the highest weight of the $\mathfrak{g}$-module M^λ). For the algorithms of construction and its execution in some cases see Chapter 7 (as well as [BMP], [Bur], [BGLS]).

Let $\tilde{\mathfrak{g}}$ be the image of $\mathfrak{g}$ with respect to this realization. Let

$$\tilde{S}^\bullet(\tilde{\mathfrak{g}}) \ \text{ be the associative algebra generated by } \tilde{\mathfrak{g}}. \tag{10.1}$$

Clearly, $\tilde{S}^\bullet(\tilde{\mathfrak{g}}) \subset \mathfrak{diff}(\mathfrak{g}_-)$. Set

$$U_\lambda(\mathfrak{g}) = \tilde{S}^\bullet(\tilde{\mathfrak{g}})/J(\lambda), \ \text{ where } J(\lambda) \text{ is the maximal ideal.}$$

Observe that $J(\lambda) = 0$ for λ generic.

Roughly speaking, $U_\lambda(\mathfrak{g})$ is "Mat"(L^λ), where L^λ is the quotient of M^λ modulo the maximal submodule $I(\lambda)$ (it can be determined and described with the help of the Shapovalov form, see [K3]) and $\tilde{S}^\bullet(\tilde{\mathfrak{g}})$ is the subalgebra generated by $\tilde{\mathfrak{g}}$ in the symmetric algebra of $\tilde{\mathfrak{g}}$ **modulo the relations between differential operators**. Clearly,

$$\tilde{S}^\bullet(\tilde{\mathfrak{g}}) \ \text{ is smaller than } S^\bullet(\mathfrak{g}) \tag{10.2}$$

due to the relations between the differential operators that span $\tilde{\mathfrak{g}}$.

To explicitly describe the generators of $J(\lambda)$ is a main technical problem. We solve it here for $\operatorname{rk}\mathfrak{g} = 1$. The case of algebras of $\operatorname{rk} > 1$ is an *open problem*.

Step 1) From $U_\lambda(\mathfrak{g})$ to $(U_\lambda(\mathfrak{g}))_L$. Recall that $(U_\lambda(\mathfrak{g}))_L$ is the Lie algebra whose space is the same as that of $U_\lambda(\mathfrak{g})$ and the bracket is the commutator.

Step 2) The Montgomery functor. S. Montgomery suggested [M1] a construction of simple Lie superalgebras:

$$Mo\text{: a central simple } \mathbb{Z}\text{-graded algebra} \mapsto \text{a simple Lie superalgebra.} \tag{Mo}$$

Observe that the associative algebras $U_\lambda(\mathfrak{g})$ constructed from simple Lie algebras $\mathfrak{g}$ are central simple. To consider *Montgomery superalgebras* $Mo(U_\lambda(\mathfrak{g}))$ and compare them with the Lie superalgebras $(U_\mathfrak{s}(\lambda))_L$ constructed from Lie superalgebras $\mathfrak{s}$ is an *open problem*. The Montgomery functor can produce new Lie superalgebras, e.g., if $\mathfrak{g}$ is equal to $\mathfrak{f}(4)$ or $\mathfrak{e}(i)$, though not always:

$$Mo(U_\lambda(\mathfrak{sl}(2))) \cong (U_\lambda(\mathfrak{osp}(1|2)))_L.$$

Step 3) Twisted versions. An outer automorphism a of $\mathfrak{G} = (U_\lambda(\mathfrak{g}))_L$ or $Mo(U_\lambda(\mathfrak{g}))$ might single out a new simple Lie subsuperalgebra $\mathfrak{a}_0(\mathfrak{G})$, the set of fixed points of $\mathfrak{G}$ under a.

For example, the intersection of $(U_{\mathfrak{sl}(2)}(\lambda))_L$ with the set of anti-adjoint differential operators is a new Lie algebra $\mathfrak{o}/\mathfrak{sp}(\lambda)$ while the intersection of $Mo(U_{\mathfrak{sl}(2)}(\lambda)) = (U_{\mathfrak{osp}(1|2)}(\lambda))_L$ with the set of superanti-adjoint operators is the Lie superalgebra $\mathfrak{osp}(\lambda+1|\lambda)$. For the description of the outer automorphisms of $\mathfrak{gl}(\lambda)$ see [LSe]. In general even the definition is unclear.

Step 4) Deformations. The deformations of Lie algebras and Lie superalgebras obtained via steps 1) – 3) may lead to new algebras of class SFLAPG, cf. [Gol]. A. Sergeev posed the following interesting problem:

what Lie algebras and Lie superalgebras can we get by applying the above constructions 1) – 3) to the quantum deformation $U_q(\mathfrak{g})$ of $U(\mathfrak{g})$?

Remark. The above procedure can be also applied to (twisted) loop algebras $\mathfrak{g} = \mathfrak{h}^{(k)}$ and the stringy algebras; the result will be realized by differential operators of infinitely many indeterminates; these operators remind vertex operators. The algebra $(U_\lambda(\mathfrak{h}^{(k)}))_L$ is not of polynomial growth.

10.0.2. Another description of $U_\lambda(\mathfrak{g})$. For the finite dimensional simple $\mathfrak{g}$ there is an alternative description of $U_\lambda(\mathfrak{g})$ as the quotient of $U(\mathfrak{g})$ modulo the ideal $C_{(\lambda)}$ generated by rank $\mathfrak{g}$ elements $C_i - k_i(\lambda)$, where the C_i is the i-th Casimir element and the $k_i(\lambda)$ is the (computed by Harish–Chandra and Berezin) value of C_i on M^λ. This description of $U_\lambda(\mathfrak{g})$ goes back, perhaps, to Kostant, cf. [Ka]. From this description it is clear that, after the shift by ρ, the half sum of positive roots, we get

$$(U_{\sigma(\lambda)}(\mathfrak{g})_L \cong (U_\lambda(\mathfrak{g}))_L \quad \text{for any } \sigma \in W(\mathfrak{g}).$$

A similar isomorphism holds for $Mo(U_\lambda(\mathfrak{g}))$. In particular, over $\mathbb{R}$, it suffices to consider the λ that belong to one Weyl chamber only.

For vectorial Lie algebras, the description of $U_\lambda(\mathfrak{g})$ as $U(\mathfrak{g})/C_{(\lambda)}$ is unknown. For example, let $\mathfrak{g} = \mathfrak{vect}(n)$. The highest weight Verma modules are (for the standard filtration of $\mathfrak{g}$) identical with Verma modules over $\mathfrak{sl}(n+1)$, but the center of $U(\mathfrak{vect}(n))$ consists of constants only. It is a research problem to describe the generators of $C_{(\lambda)}$ in such cases.

Though the center of $U(\mathfrak{g})$ is completely described by A. Sergeev for all simple finite dimensional Lie superalgebras [Ser3], the problem

describe the generators of the ideal $C_{(\lambda)}$

is open for Lie **super**algebras $\mathfrak{g}$ even if $\mathfrak{g}$ is of the form $\mathfrak{g}(A)$ (i.e., if $\mathfrak{g}$ has Cartan matrix A) different from $\mathfrak{osp}(1|2n)$: for them the center of $U(\mathfrak{g})$ is not noetherian and it is *a priori* unclear if $C_{(\lambda)}$ has infinitely or finitely many generators.

10.0.3. Our main result. The main result is the statement of the fact that the above constructions 1) – 4) yield a new class of simple Lie (super) algebras of polynomial growth (some of which have nice properties).

Observe that our Lie algebras $(U_\lambda(\mathfrak{g}))_L$ are quantizations of the Lie algebras considered in [DGS1] which are also of class SFLAPG and are contractions of our algebras. Indeed, Donin, Gurevich and Shnider consider the Lie algebras of functions on the orbits of the coajoint representation of $\mathfrak{g}$ with respect to the Poisson bracket. These DGS Lie algebras are naturally realized as the quotients of the polynomial algebra modulo an inhomogeneous ideal that singles out the orbit; we realize the result of quantization of DGS Lie algebras (i.e., their deforms) by differential operators. Observe that while the polynomial Poisson Lie algebra has only one class of nontrivial deformations and all the deformed algebras are isomorphic, cf. [**?**], the dimension of the space of parameters of deformations of the DGS Lie algebras is equal to the rank of $\mathfrak{g}$ and all of the deforms are pairwise non-isomorphic, generally.

In this Chapter we consider the simplest case of the superization of this construction: we replace $\mathfrak{g} = \mathfrak{sl}(2)$ by $\mathfrak{osp}(1|2)$. To perform this construction explicitly for alis an open problem. The next *open problem* is to describe the structure of the algebras $(U_\lambda(\mathfrak{g}))_L$ (real forms, automorphisms, root systems).

An application: The Khesin–Malikov construction [KM] can be applied almost literally to the Lie (super)algebras $(U_\lambda(\mathfrak{g}))_L$ such that $\mathfrak{g}$ admits a (super)principal embedding, see, e.g., [GL2].

10.0.4. The defining relations. The notion of defining relations is clear for a nilpotent Lie algebra. This is one of the reasons why the most conventional way to present a simple Lie algebra $\mathfrak{g}$ is to split it into the direct sum of a (commutative) Cartan subalgebra and 2 maximal nilpotent subalgebras $\mathfrak{g}_\pm$ (positive and negative). There are about $(2\cdot \mathrm{rk}\,\mathfrak{g})^2$ relations between the $2\cdot\mathrm{rk}\,\mathfrak{g}$ generators of $\mathfrak{g}_\pm$. The generators of $\mathfrak{g}_+$ together with the generators of $\mathfrak{g}_-$ generate $\mathfrak{g}$ as well. In $\mathfrak{g}$, there are about $(3\cdot\mathrm{rk}\,\mathfrak{g})^2$ relations between these generators; the relations additional to those in $\mathfrak{g}_+$ or $\mathfrak{g}_-$, i.e., between the positive and the negative generators, are easy to grasp. Though numerous, these relations (called for $\mathfrak{g}_\pm$ *Serre relations*) are neat and this is another reason for their popularity. These relations are good to deal with not only for humans but for computers as well, cf. sec. 7.3.

Nevertheless, it so happens that the Chevalley-type generators and, therefore, the Serre relations are not always available. Besides, as we will see, there are problems in which other generators and relations naturally appear, cf. [GL2, Sa].

Though not so transparent as for nilpotent algebras, the notion of generators and relations makes sense in the general case. For instance, with the principal embeddings of $\mathfrak{sl}(2)$ into $\mathfrak{g}$ one can associate only **two** elements that generate $\mathfrak{g}$; we call them *Jacobson's generators*, see [GL21]. We explicitly describe the associated with the principal embeddings of $\mathfrak{sl}(2)$ presentations of simple Lie algebras, finite dimensional and certain infinite dimensional; namely, the Lie algebra "of matrices of complex size" realized as a subalgebra of the Lie algebra $\mathfrak{diff}(1)$ of differential operators in 1 indeterminate or of $\mathfrak{gl}_+(\infty)$, see §2.

The relations obtained are rather simple, especially for non-exceptional algebras. In contradistinction with the conventional presentation there are just 9 relations between Jacobson's generators for $\mathfrak{sl}(\lambda)$ series (actually, 8 if $\lambda \in \mathbb{C} \setminus \mathbb{Z}$) and not many more for the other algebras.

It is convenient to present $\mathfrak{sl}(\lambda)$ as the Lie algebra generated by two differential operators:

$$X^+ = u^2 \frac{d}{du} - (\lambda - 1)u \quad \text{and } Z_{\mathfrak{sl}} = \frac{d^2}{du^2};$$

its Lie subalgebra $\mathfrak{o}/\mathfrak{sp}(\lambda)$ of anti-adjoint operators — a hybrid of Lie algebras of series $\mathfrak{o}$ and $\mathfrak{sp}$ (do not confuse with the Lie superalgebra of $\mathfrak{osp}$ type!) — is generated by

$$X^+ = u^2 \frac{d}{du} - (\lambda - 1)u \quad \text{and } Z_{\mathfrak{o}/\mathfrak{sp}} = \frac{d^3}{du^3};$$

to make relations simpler, we always add the third generator $X^- = -\dfrac{d}{du}$.

For integer λ, each of these algebras has an ideal of finite codimension and the quotient modulo the ideal is the conventional $\mathfrak{sl}(n)$ (for $\lambda = n$ and $\mathfrak{gl}(\lambda)$) and either $\mathfrak{o}(2n+1)$ (for $\lambda = 2n+1$) or $\mathfrak{sp}(2n)$ (for $\lambda = 2n$), respectively , for $\mathfrak{o}/\mathfrak{sp}(\lambda)$.

In this Chapter we also superize [GL21]: replace $\mathfrak{sl}(2)$ by its closest relative, $\mathfrak{osp}(1|2)$. We denote by $\mathfrak{sl}(\lambda|\lambda+1)$the Lie superalgebra generated by

$$\nabla^+ = x\partial_\theta + x\theta\partial_x - \lambda\theta, \quad Z = \partial_x\partial_\theta - \theta{\partial_x}^2, \quad U = \partial_\theta - \theta\partial_x,$$

where x is an even indeterminate and θ is an odd one. We define $\mathfrak{osp}(\lambda|\lambda+1)$ as the Lie subsuperalgebra of $\mathfrak{sl}(\lambda|\lambda+1)$ generated by ∇^+ and Z. The presentations of $\mathfrak{sl}(\lambda|\lambda+1)$ and $\mathfrak{osp}(\lambda|\lambda+1)$ are associated with the *super-principal* embeddings of $\mathfrak{osp}(1|2)$. For $\lambda \in \mathbb{C} \setminus \mathbb{Z}$, these algebras are simple. For integer $\lambda = n$, each of these algebras has an ideal of finite codimension and the quotient modulo the ideal is the conventional $\mathfrak{sl}(n|n+1)$ and $\mathfrak{osp}(2n+1|2n)$, respectively.

10.0.5. Some applications. (1) Integrable systems like continuous Toda lattice or a generalization of the Drinfeld–Sokolov construction are based on the super-principal embeddings in the same way as the Khesin–Malikov construction [KM] is based on the principal embedding, cf. [GL2].

(2) To q-quantize the Lie algebras of type $\mathfrak{sl}(\lambda)$ à la Drinfeld, using only Chevalley generators, is impossible; our generators indicate a way to do it.

10.0.6. Related topics. We would like to draw attention of the reader to several other classes of Lie algebras. One of the reasons is that, though some of these classes have empty intersections with the class of Lie algebras we consider here, they naturally spring to mind and are, perhaps, deformations of our algebras in some, yet unknown, sense.

• *Krichever–Novikov algebras*, see [SH] and refs. therein. The KN-algebras are neither graded, nor filtered (at least, wrt the degree considered usually). Observe that so are our algebras $(U_\lambda(\mathfrak{g}))_L$ with respect to the degree induced from $U(\mathfrak{g})$, so a search for a better grading is a tempting problem.

• *Odessky* or *Sklyanin algebras*, see [FO] and refs. therein.

• *Continuum algebras*, see [SV] and refs. therein. In particular cases these algebras coincide with Kac–Moody or loop algebras, i.e., have a continuum analog of the Cartan matrix. But to suspect that $\mathfrak{gl}(\lambda)$ has a Cartan matrix is wrong, see sec. 2.2. Nevertheless, in the simplest cases, if $\operatorname{rk} \mathfrak{g} = 1$, the algebras $(U_\lambda(\mathfrak{g}))_L$ and their "relatives" obtained in steps 1) – 3) (and, perhaps, 4)) of sec 0.1 do possess Saveliev-Vershik's nonlinear *Cartan operator* which replaces the Cartan matrix.

10.1. Recapitulation: finite dimensional simple Lie algebras

This section is a continuation of [LP], where the case of the simplest base (system of simple roots) is considered and where non-Serre relations for simple Lie algebras first appear, though in a different setting. This Chapter is also the direct superization of [GL21]; we recall its results. For presentations of Lie superalgebras with Cartan matrix via Chevalley generators, see [LSa1], [GL1].

What are "natural" generators and relations for a *simple finite dimensional Lie algebra*? The answer is important in questions when it is needed to identify an algebra $\mathfrak{g}$ given its generators and relations. (Examples of such problems are connected with Estabrook–Wahlquist prolongations, Drinfeld's quantum algebras, symmetries of differential equations, integrable systems, etc.).

10.1.0. Defining relations. If $\mathfrak{g}$ is nilpotent, the problem of its presentation has a natural and unambiguous solution: representatives of the homology $H_1(\mathfrak{g}) \cong \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ are the generators of $\mathfrak{g}$ and the elements from $H_2(\mathfrak{g})$ correspond to non-trivial relations. (We will return to this in the study of presentations of simple modular Lie (super)algebras.)

On the other hand, if $\mathfrak{g}$ is simple, then $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ and there is no "most natural" way to select generators of $\mathfrak{g}$. The choice of generators is not unique.

Still, among algebras with the property $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ the simple ones are distinguished by the fact that their structure is very well known. By trial and error people discovered that for finite dimensional simple Lie algebras, there are certain "first among equal" sets of generators:

1) *Chevalley generators* corresponding to positive and negative simple roots;

2) a pair of generators that generate any finite dimensional simple Lie algebra associated with the *principal* $\mathfrak{sl}(2)$*-subalgebra* (considered below).

The relations associated with Chevalley generators are well-known, see e.g., [OV], [K3]. These relations (or, rather, their part involving the root elements of one sign only) are called *Serre relations*.

The possibility to generate any simple finite dimensional Lie algebra by two elements was first claimed by N. Jacobson; [BO] is not the first proof, but a rather detailed one and over various fields. [1)] We do not know what generators Jacobson had in mind; [BO] take for them linear combinations of positive and negative root vectors with generic coefficients; nothing like a "natural" choice that we suggest to refer to as *Jacobson's generators* was ever proposed.

To generate a simple algebra with only two elements is tempting but nobody yet had explicitly described relations between such generators, perhaps, because to check whether the relations between these elements are nice-looking is impossible without a modern computer (cf. an implicit description in [F]). As far as we could test, the relations for any other pair of generators chosen in a way distinct from ours are too complicated. There seem to be, however, one exception cf. [GL2].

10.1.1. The principal embeddings. There exists only one (up to equivalence) embedding $r : \mathfrak{sl}(2) \longrightarrow \mathfrak{g}$ such that $\mathfrak{g}$, considered as $\mathfrak{sl}(2)$-module, splits into $\operatorname{rk}\mathfrak{g}$ irreducible modules, cf. [Dy] or [OV]. This embedding is called *principal* and, sometimes, *minimal* because for the other embeddings (there are plenty of them) the number of irreducible $\mathfrak{sl}(2)$-modules is $> \operatorname{rk}\mathfrak{g}$. Example: for $\mathfrak{g} = \mathfrak{sl}(n)$, $\mathfrak{sp}(2n)$ or $\mathfrak{o}(2n+1)$ the principal embedding is the one corresponding to the irreducible representation of $\mathfrak{sl}(2)$ of dimension n, $2n$, $2n+1$, respectively.

For completeness, let us recall how the irreducible $\mathfrak{sl}(2)$-modules with highest weight look like. (They are all of the form L^μ, where $L^\mu = M^\mu$ if $\mu \notin \mathbb{Z}_+$, and $L^n = M^n/M^{-n-2}$ if $n \in \mathbb{Z}_+$, and where M^μ is described below.) Select the following basis in $\mathfrak{sl}(2)$:

$$X^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The $\mathfrak{sl}(2)$-module M^μ is illustrated with a graph whose nodes correspond to the eigenvectors $l_{\mu-2i}$ of H with the weight indicated;

[1] There are also similar statements on the Chevalley group level; and lately appeared papers with similar statement for the simple finite dimensional modular Lie algebras of vectorial type. It seems that the question is obvious for $\mathbb{Z}$-graded Lie algebras $\mathfrak{g}$: take the highest weight (w.r.t. $\mathfrak{g}_0$) element of $\mathfrak{g}_{-1}$ and the lowest weight vector of the highest component of $\mathfrak{g}$; they should do the trick. It only remain to verify this. The snag is: even if true, are the relations between these generators sufficiently simple to be useful?

$$\dots \overset{\mu-2i-2}{\circ} - \overset{\mu-2i}{\circ} - \dots - \overset{\mu-2}{\circ} - \overset{\mu}{\circ}$$

the edges depict the action of $X^\pm$ (the action of X^+ is directed to the right, that of X^- to the left: $X^- l_{\mu-2i} = l_{\mu-2i-2}$ and

$$X^+ l_{\mu-2i} = X^+((X^-)^i l_\mu) = i(\mu - i + 1) l_{\mu-2i+2}; \quad X^+(l_\mu) = 0. \tag{1.1}$$

As follows from (1.1), the module M^n for $n \in \mathbb{Z}_+$ has an irreducible submodule isomorphic to M^{-n-2}; the quotient, obviously irreducible, as follows from the same (1.1), will be denoted by L^n.

There are principal $\mathfrak{sl}(2)$-subalgebras in every finite dimensional simple Lie algebra, though, generally, not in infinite dimensional ones, e.g., not in affine Kac-Moody algebras. The construction is as follows. Let $X_1^\pm, \dots, X^\pm_{\operatorname{rk}\mathfrak{g}}$ be Chevalley generators of $\mathfrak{g}$, i.e., the generators corresponding to simple roots. Let the images of $X^\pm \in \mathfrak{sl}(2)$ in $\mathfrak{g}$ be

$$X^- \mapsto \sum X_i^-; \quad X^+ \mapsto \sum a_i X_i^+$$

and select the a_i from the relations $[[X^+, X^-], X^\pm] = \pm 2X^\pm$ true in $\mathfrak{sl}(2)$. For $\mathfrak{g} = \mathfrak{g}(A)$ constructed from a Cartan matrix A, there is a solution for a_i if and only if A is invertible.

In Table 1.1 a simple finite dimensional Lie algebra $\mathfrak{g}$ is described as the $\mathfrak{sl}(2)$-module corresponding to the principal embedding (cf. [OV], Table 4). The table introduces the number $2k_2$ used in relations. We set $k_1 = 1$.

10.1.2. Table. Simple $\mathfrak{g}$ as the $\mathfrak{sl}(2)$-module.

$\mathfrak{g}$	the $\mathfrak{sl}(2)$-spectrum of $\mathfrak{g} = L^2 \oplus L^{2k_2} \oplus L^{2k_3} \dots$	$2k_2$
$\mathfrak{sl}(n)$	$L^2 \oplus L^4 \oplus L^6 \dots \oplus L^{2n-2}$	4
$\mathfrak{o}(2n+1)$, $\mathfrak{sp}(2n)$	$L^2 \oplus L^6 \oplus L^{10} \dots \oplus L^{4n-2}$	6
$\mathfrak{o}(2n)$	$L^2 \oplus L^6 \oplus L^{10} \dots \oplus L^{4n-6} \quad \oplus L^{2n-2}$	6
$\mathfrak{g}(2)$	$L^2 \oplus L^{10}$	10
$\mathfrak{f}(4)$	$L^2 \oplus L^{10} \oplus L^{14} \oplus L^{22}$	10
$\mathfrak{e}(6)$	$L^2 \oplus L^8 \oplus L^{10} \oplus L^{14} \oplus L^{16} \oplus L^{22}$	8
$\mathfrak{e}(7)$	$L^2 \oplus L^{10} \oplus L^{14} \oplus L^{18} \oplus L^{22} \oplus L^{26} \oplus L^{34}$	10
$\mathfrak{e}(8)$	$L^2 \oplus L^{14} \oplus L^{22} \oplus L^{26} \oplus L^{34} \oplus L^{38} \oplus L^{46} \oplus L^{58}$	14

One can show that $\mathfrak{g}$ can be generated by two elements: $x := X^+ \in L^2 = \mathfrak{sl}(2)$ and a lowest weight vector $z := l_{-r}$ from an appropriate module L^r other than L^2 from Table 1.1. For the role of this L^r we take either L^{2k_2} if $\mathfrak{g} \neq \mathfrak{o}(2n)$ or the last module L^{2n-2} in the above table if $\mathfrak{g} = \mathfrak{o}(2n)$. (Clearly, z is defined up to proportionality; we will assume that a basis of L^r is fixed and denote $z = t \cdot l_{-r}$ for some $t \in \mathbb{C}$ that can be fixed at will.)

The exceptional choice for $\mathfrak{o}(2n)$ is occasioned by the fact that by choosing $z \in L^r$ for $r \neq 2n-2$ instead, we generate $\mathfrak{o}(2n-1)$.

We call the above x and z, together with $y := X^- \in L^2$ taken for good measure, *Jacobson's generators*. The presence of y considerably simplifies the form of the relations, though slightly increases their number. (One might think that taking the symmetric to z element l_r will improve the relations even more but in reality just the opposite happens.)

10.1.3. Relations between Jacobson's generators. First, observe that if an ideal of a free Lie algebra is homogeneous (with respect to the degrees of the generators of the algebra), then the number and the degrees of the defining relations (i.e., the generators of the ideal) is uniquely defined provided the relations are homogeneous. This is obvious.

A simple Lie algebra $\mathfrak{g}$, however, is the quotient of a free Lie algebra $\mathfrak{F}$ modulo a inhomogeneous ideal, $\mathfrak{I}$, the ideal without homogeneous generators. Therefore, we can speak about the number and the degrees of relations only conditionally. Our condition is the possibility to express any element $x \in \mathfrak{I}$ via the generators $g_1, \dots$ of $\mathfrak{I}$ by a formula of the form

$$x = \sum [c_i, g_i], \text{ where } c_i \in \mathfrak{F} \text{ and } \deg c_i + \deg g_i \leq \deg x \text{ for all } i. \tag{1.3}$$

Under condition (1.3) the number of relations and their degrees are uniquely determined. Now we can explain why do we need an extra generator y: without y the weight relations would have been of very high degree.

We divide the relations between the Jacobson generators into the types corresponding to the number of occurrences of z in them:

0. Relations in $L^2 = \mathfrak{sl}(2)$;
1. Relations coming from the $\mathfrak{sl}(2)$-action on L^{2k_2};
2. Relations coming from $L^{2k_1} \wedge L^{2k_2}$;
$\geq$ **3**. Relations coming from $L^{2k_2} \wedge L^{2k_2} \wedge L^{2k_2} \wedge \dots$ with ≥ 3 factors; among the latter relations we distinguish one — of type "∞" — the relation that shears the dimension. (For small rank $\mathfrak{g}$ the relation of type ∞ can be of the above types.)

Observe that, apart form relations of type ∞, the relations of type ≥ 3 are those of type 3 except for $\mathfrak{e}(7)$ which satisfies stray relations of types 4 and 5, cf. [GL21].

The relations of type 0 are the well-known relations in $\mathfrak{sl}(2)$

$$\mathbf{0.1}.\quad [[x,y],\, x] = 2x, \qquad \mathbf{0.2}.\quad [[x,y],\, y] = -2y. \tag{Rel 0}$$

The relations of type 1 mirror the fact that the space L^{2k_2} is the $(2k_2+1)$-dimensional $\mathfrak{sl}(2)$-module. To simplify notations we denote: $z_i = (\mathrm{ad}_x)^i z$. Then the type 1 relations are:

$$\begin{aligned} &\mathbf{1.1}.\quad [y,\, z] = 0,\\ &\mathbf{1.2}.\quad [[x,y],\, z] = -2k_2 z, \quad \text{with } 2k_2 \text{ from Table 10.1.2;}\\ &\mathbf{1.3}.\quad z_{2k_1+1} = 0. \end{aligned} \tag{Rel 1}$$

10.1.4. Theorem. *For the simple finite dimensional Lie algebras, all the relations between the Jacobson generators are the above relations* (Rel0), (Rel1) *and the relations from* [GL21]. (In §3, these relations from [GL21] are reproduced for the classical Lie algebras.)

10.2. The Lie algebra $\mathfrak{sl}(\lambda)$ as a quotient algebra of $\mathfrak{diff}(1)$ and a subalgebra of $\mathfrak{sl}_+(\infty)$

10.2.1. $\mathfrak{gl}(\lambda)$ is endowed with a trace. The Poincaré-Birkhoff-Witt theorem states that, as spaces, $U(\mathfrak{sl}(2)) \cong \mathbb{C}[X^-, H, X^+]$. We also know that to study representations of $\mathfrak{g}$ is the same as to study representations of $U(\mathfrak{g})$. Still, if we are interested in irreducible representations, we do not need the whole of $U(\mathfrak{g})$ and can do with a smaller algebra, easier to study.

This observation is used now and again; Feigin applied it in [F] writing, actually, (as deciphered in [PH], [GL21], [Sho]) that setting

$$X^- = -\frac{d}{du}, \quad H = 2u\frac{d}{du} - (\lambda - 1), \quad X^+ = u^2\frac{d}{du} - (\lambda-1)u \tag{2.1}$$

we obtain a morphism of $\mathfrak{sl}(2)$-modules and, moreover, of associative algebras: $U(\mathfrak{sl}(2)) \longrightarrow \mathbb{C}[u, \frac{d}{du}]$. The kernel of this morphism is the ideal generated by $\Delta - \lambda^2 + 1$, where $\Delta = 2(X^+X^- + X^-X^+) + H^2$. Observe, that this morphism is not an epimorphism, either. The image of this morphism is our Lie algebra of matrices of "complex size".

Remark. In their proof of certain statements from [F] that we will recall, [PH] used the well-known fact that the Casimir operator Δ acts on the irreducible $\mathfrak{sl}(2)$-module L^μ (see sec 1.1) as the scalar operator of multiplication by $\mu^2 + 2\mu$. The passage from [PH]'s λ to [F]'s μ is done with the help of a shift by the weight ρ, a half sum of positive roots, which for $\mathfrak{sl}(2)$, can be identified with 1, i.e., $(\lambda-1)^2 + 2(\lambda-1) = \lambda^2 - 1$ for $\lambda = \mu + 1$.

Consider the Lie algebra $(U(\mathfrak{sl}(2)))_L$ associated with the associative algebra $U(\mathfrak{sl}(2))$. Set

$$U_\lambda = U(\mathfrak{sl}(2))/(\Delta - \lambda^2 + 1). \tag{2.2}$$

The definition directly implies that $\mathfrak{gl}(-\lambda) \cong \mathfrak{gl}(\lambda)$, so speaking about real values of λ we can confine ourselves to the nonnegative values, cf. sec. 0.2. It is easy to see that, as $\mathfrak{sl}(2)$-module,

$$U_\lambda = L^0 \oplus L^2 \oplus L^4 \oplus \dots \oplus L^{2n} \oplus \dots \tag{2.3}$$

It is not difficult to show (see [PH] for details) that the Lie algebra $(U_n)_L$ for $n \in \mathbb{Z} \setminus \{0\}$ contains an ideal J_n and the quotient $(U_n)_L/J_n$ is the conventional $\mathfrak{gl}(n)$. In [PH] it is proved that, for $\lambda \neq \mathbb{Z} \setminus \{0\}$, the Lie algebra $(U_\lambda)_L$ is the direct sum of the two ideals — the center L^0 (spanned by constants) and its complement. Set

$$\mathfrak{pgl}(\lambda) = \mathfrak{gl}(\lambda)/L^0, \text{ where } \mathfrak{gl}(\lambda) = \begin{cases} (U_\lambda)_L & \text{for } \lambda \notin \mathbb{Z} \setminus \{0\} \\ (U_n/J_n)_L & \text{for } n \in \mathbb{Z} \setminus \{0\}. \end{cases} \tag{2.4}$$

Observe, that $\mathfrak{gl}(\lambda)$ is endowed with a trace. This follows directly from (2.3) and the fact that

$$\mathfrak{gl}(\lambda) \cong L^0 \oplus [\mathfrak{gl}(\lambda), \mathfrak{gl}(\lambda)].$$

Therefore, $\mathfrak{pgl}(\lambda)$ can be identified with $\mathfrak{sl}(\lambda)$, the subalgebra of the traceless matrices in $\mathfrak{gl}(\lambda)$. We can normalize the trace at will, for example, if we set $\mathrm{tr}(\mathrm{id}) = \lambda$, then the trace that our trace induces on the quotient of $(U_n(\mathfrak{sl}(2)))_L$ modulo $J(n)$ coincides with the usual trace on $\mathfrak{gl}(n)$ for $n \in \mathbb{N}$.

Another way to introduce the trace was suggested by J. Bernstein. We decipher its description in [KM] as follows. Look at the image $\rho_\lambda(H)$ of $H \in \mathfrak{sl}(2)$ in $\mathfrak{gl}(M^\lambda)$. Bernstein observed that though $\mathrm{tr}(\rho_\lambda(H))$ is an infinite sum, the sum of the first $D+1$ summands is a polynomial in D, call it $\mathrm{tr}_D(\rho_\lambda(H))$.

Exercise. Prove that $\mathrm{tr}_\lambda(\rho_\lambda(H)) = 0$.

Let $D(\lambda)$ be the value of the dimension of the irreducible finite dimensional $\mathfrak{g}$-module with highest weight λ, for an exact formula, see [Dy], [OV] (or volume 1). For *any* $x \in (U_\lambda(\mathfrak{g}))_L$ considered as an element of $\mathfrak{gl}(M^\lambda)$, set

$$\mathrm{tr}(x) := \mathrm{tr}(x; D(\lambda)) = \sum_{i=1}^{D(\lambda)} x_{ii};$$

as is easy to see, this formula determines the trace on $(U_\lambda(\mathfrak{g}))_L$ for arbitrary values of λ.

Observe that whereas for any irreducible finite dimensional module over the simple Lie algebra $\mathfrak{g}$, there is just one formula for $D(\lambda)$ (the H. Weyl dimension formulas) for Lie superalgebra there are *several* distinct formulas depending on how "typical" λ is.

10.2.2. There is no Cartan matrix for $\mathfrak{sl}(\lambda)$. What replaces it?. Are there *Chevalley generators* in $\mathfrak{sl}(\lambda)$? In other words are there elements $X_i^\pm$ of degree ± 2 and H_i of degree 0 (the *degree* is the weight with respect to the $\mathfrak{sl}(2) = L^2 \subset \mathfrak{sl}(\lambda)$) such that

$$[X_i^+, X_j^-] = \delta_{ij} H_i, \quad [H_i, H_j] = 0 \text{ and } [H_i, X_j^\pm] = \pm A_{ij} X_j^\pm? \tag{2.5}$$

The answer is **NO**: $\mathfrak{sl}(\lambda)$ is too small. To see what is the problem, consider the following elements of degree ± 2 from L^4 and L^6 of $\mathfrak{gl}(\lambda)$:

$$\begin{aligned} \deg = -2: &\ -4uD^2 - 2(\lambda-2)D \\ \deg = 2: &\ -4u^3D^2 + 6(\lambda-2)u^2D - 2(\lambda-1)(\lambda-2)u \\ \deg = -2: &\ 15u^2D^3 - 15(\lambda-3)uD^2 + 3(\lambda-2)(\lambda-3)D \\ \deg = 2: &\ 15u^4D^3 - 30(\lambda-3)u^3D^2 + \\ &\ 18(\lambda-2)(\lambda-3)u^2D - 3(\lambda-1)(\lambda-2)(\lambda-3)u \end{aligned}$$

To satisfy (2.5), we can complete $\mathfrak{gl}(\lambda)$ by considering infinite sums of its elements, but the completion erases the difference between different λ's:

Proposition. *For $\lambda \neq \rho$ the completion of $\mathfrak{sl}(\lambda)$ generated by Jacobson's generators* (see Tables) *is isomorphic to $\overline{\mathfrak{pdiff}(1)}$, the quotient of the Lie algebra of differential operators with formal coefficients modulo constants.*

Though there is no Cartan matrix, Saveliev and Vershik [SV] suggested an operator K which replaces Cartan matrix. For further details, see the paper by Shoihet and Vershik [ShV].

10.2.3. The outer automorphism of $(U_\lambda(\mathfrak{g}))_L$. The invariants of the mapping

$$X \mapsto -SX^tS \text{ for } X \in \mathfrak{gl}(n), \text{ where } S = \mathrm{antidiag}(1, -1, 1, -1\dots) \tag{2.6}$$

constitute $\mathfrak{o}(n)$ if $n \in 2\mathbb{N}+1$ and $\mathfrak{sp}(n)$ if $n \in 2\mathbb{N}$. By analogy, Feigin defined $\mathfrak{o}/\mathfrak{sp}(\lambda)$ (do not confuse with the Lie superalgebras of series $\mathfrak{osp}$) as the subalgebra of $\mathfrak{gl}(\lambda) = \bigoplus\limits_{k\geq 0} L^{2k}$ invariant with respect to the involution analogous to (2.6):

$$X \mapsto \begin{cases} -X & \text{if } X \in L^{4k} \\ X & \text{if } X \in L^{4k+2}. \end{cases} \tag{2.7}$$

It is clear that

$$\mathfrak{o}/\mathfrak{sp}(\lambda) = \begin{cases} \mathfrak{o}(\lambda) \ni I_\lambda & \text{if } \lambda \in 2\mathbb{N}+1, \\ \mathfrak{sp}(\lambda) \ni I_\lambda & \text{if } \lambda \in 2\mathbb{N}, \end{cases} \text{ where } I_\lambda \text{ is an ideal.}$$

In the realization of $\mathfrak{sl}(\lambda)$ by differential operators the transposition is the passage to the adjoint operator; hence, $\mathfrak{o}/\mathfrak{sp}(\lambda)$ is a subalgebra of $\mathfrak{sl}(\lambda)$ consisting of anti-self-adjoint operators with respect to the involution

$$a(u)\frac{d^k}{du^k} \mapsto (-1)^k \frac{d^k}{du^k} a(u)^*. \tag{2.8}$$

The superization of this formula is straightforward: via the Sign Rule.

10.2.4. The Lie algebra $\mathfrak{gl}(\lambda)$ as a subalgebra of $\mathfrak{gl}_+(\infty)$. Recall that $\mathfrak{gl}_+(\infty)$ often denotes the Lie algebra of infinite (in one direction; index + indicates that) matrices with nonzero elements inside a (depending on the matrix) strip along the main diagonal and containing it. The subalgebras $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$ are naturally defined, while the notation $\mathfrak{sl}(\infty)$ is, by abuse of language, sometimes used to denote $\mathfrak{pgl}(\infty)$.

When it comes to superization, one shall be very careful selecting an appropriate candidate for $\mathfrak{gl}(\infty|\infty)$ and its subalgebras: the "correct" answer depends on the situation and might turn out to be rather unexpected, cf. [E].

The realization (2.1) provides with an embedding $\mathfrak{sl}(\lambda) \subset \mathfrak{sl}_+(\infty) =$ "$\mathfrak{sl}(M^\lambda)$", so for $\lambda \neq \mathbb{N}$, the Verma module M^λ with highest weight μ is an irreducible $\mathfrak{sl}(\lambda)$-module.

Proposition. *The completion of* $\mathfrak{gl}(\lambda)$ *(generated by the elements of degree* ± 2 *with respect to* $H \in \mathfrak{sl}(2) \subset \mathfrak{gl}(\lambda)$*) is isomorphic for any noninteger* λ *to* $\mathfrak{gl}_+(\infty) =$ "$\mathfrak{gl}(M^\lambda)$".

10.2.5. The Lie algebras $\mathfrak{sl}(*)$ and $\mathfrak{o}/\mathfrak{sp}(*)$ for $* \in \mathbb{C}P^1 = \mathbb{C} \cup \{*\}$. The "dequantization" of the relations for $\mathfrak{sl}(\lambda)$ and $\mathfrak{o}/\mathfrak{sp}(\lambda)$ (see §3) is performed by passage to the limit as $\lambda \longrightarrow \infty$ under the change:

$$t \mapsto \begin{cases} \dfrac{t}{\lambda} & \text{for } \mathfrak{sl}(\lambda) \\ \dfrac{t}{\lambda^2} & \text{for } \mathfrak{o}/\mathfrak{sp}(\lambda). \end{cases} \tag{10.3}$$

So the parameter λ above can actually run over $\mathbb{C}P^1 = \mathbb{C} \cup \{*\}$, not just $\mathbb{C}$. In the realization with the help of deformation, cf. 2.7 below, this is obvious. Denote the limit algebras by $\mathfrak{sl}(*)$ and $\mathfrak{o}/\mathfrak{sp}(*)$ in order to distinguish them from $\mathfrak{sl}(\infty)$ and $\mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$ from sec. 2.4.

It seems impossible to embed $\mathfrak{sl}(*)$ and $\mathfrak{o}/\mathfrak{sp}(*)$ into the "quadrant" algebra $\mathfrak{sl}_+(\infty)$: indeed, $\mathfrak{sl}(*)$ and $\mathfrak{o}/\mathfrak{sp}(*)$ are subalgebras of the whole "plane" algebras $\mathfrak{sl}(\infty)$ and $\mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$.

10.2.6. Theorem. *For Lie algebras* $\mathfrak{sl}(\lambda)$ *and* $\mathfrak{o}/\mathfrak{sp}(\lambda)$*, where* $\lambda \in \mathbb{C}P^1$*, all the relations between the Jacobson generators are the relations of types* 0, 1 *with* $2k_2$ *found from Table* 1.1 *and the borrowed from* [GL21] *relations from* §3.

10.3. The Jacobson generators and relations between them

In what follows the E_{ij} are matrix units; $X_i^\pm$ stand for the conventional Chevalley generators of $\mathfrak{g}$. For $\mathfrak{sl}(\lambda)$ and $\mathfrak{o}/\mathfrak{sp}(\lambda)$, the generators $x = u^2\dfrac{d}{du} - (\lambda-1)u$ and $y = -\dfrac{d}{du}$ are common; $z_{\mathfrak{sl}} = t\dfrac{d^2}{du^2}$ while $z_{\mathfrak{o}/\mathfrak{sp}} = t\dfrac{d^3}{du^3}$. For $n \in \mathbb{C} \setminus \mathbb{Z}$, there is no shearing relation of type ∞; for $n = * \in \mathbb{C}P^1$ the relations are obtained by means of the substitution (10.3) with a subsequent passage to the limit as $\lambda \longrightarrow \infty$. The parameter t can be taken equal to 1; we kept it explicit to clarify how to "dequantize" the relations as $\lambda \longrightarrow \infty$.

$\underline{\mathfrak{sl}(*)}$.

2.1. $3[z_1, z_2] - 2[z, z_3] = 24y$,

3.1. $[z, [z, z_1]] = 0$,

3.2. $4[[z, z_1], z_3]]] + 3[z_2, [z, z_2]] = -576z$.

$\underline{\mathfrak{o}/\mathfrak{sp}(*)}$.

2.1. $2[z_1, z_2] - [z, z_3] = 72z$,

2.2. $9[z_2, z_3] - 5[z_1, z_4] = 216z_2 - 432y$,

3.1. $[z, [z, z_1]] = 0$,

3.2. $7[[z, z_1], z_3] + 6[z_2, [z, z_2]] = -720[z, z_1]$.

$\underline{\mathfrak{sl}(n) \text{ for } n \geq 3}$. Generators:

$$x = \sum_{1 \leq i \leq n-1} i(n-i)E_{i,i+1}, \quad y = \sum_{1 \leq i \leq n-1} E_{i+1,i}, \quad z = t \sum_{1 \leq i \leq n-2} E_{i+2,i}.$$

Relations:

2.1. $3[z_1, z_2] - 2[z, z_3] = 24t^2(n^2-4)y$,

3.1. $[z, [z, z_1]] = 0$,

3.2. $4[z_3, [z, z_1]] - 3[z_2, [z, z_2]] = 576t^2(n^2-9)z$.

$\infty = n - 1$. $(\mathrm{ad}_{z_1})^{n-2}z = 0$.

For $n = 3, 4$ the degree of the last relation is lower than the degree of some other relations, this yields simplifications.

$\underline{\mathfrak{o}(2n+1) \text{ for } n \geq 3}$. Generators:

$$\begin{aligned} x &= n(n+1)(E_{n+1,2n+1} - E_{n,n+1}) + \\ &\qquad \sum_{1 \leq i \leq n-1} i(2n+1-i)(E_{i,i+1} - E_{n+i+2,n+i+1}), \\ y &= (E_{2n+1,n+1} - E_{n+1,n}) + \sum_{1 \leq i \leq n-1} (E_{i+1,i} - E_{n+i+1,n+i+2}), \\ z &= t\big((E_{2n-1,n+1} - E_{n+1,n-2}) - (E_{2n+1,n-1} - E_{2n,n}) + \\ &\qquad \sum_{1 \leq i \leq n-3} (E_{i+3,i} - E_{n+i+1,n+i+4})\big). \end{aligned}$$

Relations:

2.1. $2[z_1, z_2] - [z, z_3] = 144t(2n^2+2n-9)z$,

2.2. $9[z_2, z_3] - 5[z_1, z_4] = 432t(2n^2+2n-9)z_2 +$
$1728t^2(n-1)(n+2)(2n-1)(2n+3)y$,

3.1. $[z, [z, z_1]] = 0$,

3.2. $7[z_3, [z, z_1]] - 6[z_2, [z, z_2]] = 2880t(n-3)(n+4)[z, z_1]$,

$\infty = n$. $(\mathrm{ad}_{z_1})^{n-1}z = 0$.

$\mathfrak{sp}(2n)$ for $n \geq 3$. Generators:

$$x = n^2 E_{n,2n} + \sum_{1 \leq i \leq n-1} i(2n-i)(E_{i,i+1} - E_{n+i+1,n+i}),$$
$$y = E_{2n,n} + \sum_{1 \leq i \leq n-1} (E_{i+1,i} - E_{n+i,n+i+1}),$$
$$z = t\Bigg((E_{2n,n-2} + E_{2n-2,n}) - E_{2n-1,n-1} + \sum_{1 \leq i \leq n-3} (E_{i+3,i} - E_{n+i,n+i+3})\Bigg).$$

Relations:

2.1. $2[z_1, z_2] - [z, z_3] = 72t(4n^2 - 19)z,$

2.2. $9[z_2, z_3] - 5[z_1, z_4] = 216t(4n^2 - 19)z_2 + 1728t^2(n^2 - 1)(4n^2 - 9)y,$

3.1. $[z, [z, z_1]] = 0,$

3.2. $7[z_3, [z, z_1]] - 6[z_2, [z, z_2]] = 720t(4n^2 - 49)[z, z_1],$

$\boldsymbol{\infty = n}$. $(\mathrm{ad}_{z_1})^{n-1} z = 0.$

For Jacobson generators and corresponding defining relations for the exceptional Lie algebras see [GL21].

10.4. The super-principal embeddings

We will need the orthosymplectic supermatrices in the alternating format; in this format we take the matrix $B_{m,2n}(\mathrm{alt}) = \mathrm{antidiag}(1, \dots, 1, -1, \dots, -1)$ with the only nonzero entries on the side diagonal, the last n being -1's. The Lie superalgebra of such supermatrices will be denoted by $\mathfrak{osp}(\mathrm{alt}_{m|2n})$, where, as is easy to see, either $m = 2n \pm 1$ or $m = 2n$.

There is a 1-parameter family of deformations $\mathfrak{osp}_\alpha(4|2)$ of the Lie superalgebra $\mathfrak{osp}(4|2)$; its only explicit description for $\alpha \neq 1$ we know (apart from [BGLS], of course) is in terms of Cartan matrix [GL1]. (It is also known that, for generic α, the irreducible $\mathfrak{osp}_\alpha(4|2)$-module of the least dimension is the adjoint one. For $\alpha = 1$, 2 and 3, there are other modules, see Ch. 8.)

Not every simple Lie superalgebra, even a finite dimensional one, hosts a super-principal $\mathfrak{osp}(1|2)$-subsuperalgebra. Let us describe those that do. (Aside: an interesting *open problem* is to describe *semiprincipal* embeddings into simple Lie superalgebras $\mathfrak{g}$, defined as the ones with the least possible number of irreducible components.)

We select the following basis in $\mathfrak{osp}(1|2) \subset \mathfrak{sl}(\bar{0}|\bar{1}|\bar{0})$:

$$X^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ X^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\nabla^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ \nabla^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The highest weight $\mathfrak{osp}(1|2)$-module $\mathcal{M}^\mu$ is illustrated with a graph whose nodes correspond to the eigenvectors l_i of H with the weight indicated; the horizontal edges depict the $X^\pm$-action (the X^+-action is directed to the right, that of X^- to the left; each horizontal string is an irreducible $\mathfrak{sl}(2)$-submodule; two such submodules are glued together into an $\mathfrak{osp}(1|2)$-module by the action of $\nabla^\pm$ (we set $\nabla^+(l_n) = 0$ and $\nabla^-(l_i) = l_{i-1}$; the corresponding edges are not depicted below); we additionally assume that $p(l_\mu) = \bar{0}$:

$$\dots \overset{\mu-2i}{\circ} \longleftrightarrow \overset{\mu-2i+2}{\circ} \longleftrightarrow \dots \quad \dots \quad \dots \longleftrightarrow \overset{\mu-2}{\circ} \longleftrightarrow \overset{\mu}{\circ}$$
$$\dots \overset{\mu-2i+1}{\circ} \longleftrightarrow \overset{\mu-2i+3}{\circ} \longleftrightarrow \dots \longleftrightarrow \overset{\mu-3}{\circ} \longleftrightarrow \overset{\mu-1}{\circ}$$

As follows from the relations of type 0 below in sec 4.2, the module $\mathcal{M}^n$ for $n \in \mathbb{Z}_+$ has an irreducible submodule isomorphic to $\Pi(\mathcal{M}^{-n-1})$; the quotient, obviously irreducible as follows from the same formulas, will be denoted by $\mathcal{L}^n$.

Serganova completely described super-principal embeddings of $\mathfrak{osp}(1|2)$ into a simple finite dimensional Lie superalgebra [LSS] (the main part of her result was independently obtained in [vJ]).

Each simple finite dimensional Lie superalgebra $\mathfrak{g}$ is is of the following form as the $\mathfrak{osp}(1|2)$-module corresponding to the super-principal embedding (the non-listed simple algebras $\mathfrak{g}$ do not contain any super-principal $\mathfrak{osp}(1|2)$):

10.4.1. Table. Simple $\mathfrak{g}$ as the $\mathfrak{osp}(1|2)$-module.

$\mathfrak{g}$	$\mathfrak{g} = \mathcal{L}^2 \oplus (\bigoplus\limits_{i>1} \mathcal{L}^{2k_i})$ for $i \geq 2$ $\oplus$	$(\bigoplus\limits_j \Pi(\mathcal{L}^{m_j}))$ for $j \geq 1$
$\mathfrak{sl}(n\|n+1)$	$\mathcal{L}^2 \oplus \mathcal{L}^4 \oplus \mathcal{L}^6 \dots \oplus \mathcal{L}^{2n-2}$	$\oplus \Pi(\mathcal{L}^1) \oplus \Pi(\mathcal{L}^3) \oplus \dots \oplus \Pi(\mathcal{L}^{2n-1})$
$\mathfrak{osp}(2n-1\|2n)$ $(n > 1)$	$\mathcal{L}^2 \oplus \mathcal{L}^6 \oplus \mathcal{L}^{10} \dots \oplus \mathcal{L}^{4n-6}$	$\oplus \Pi(\mathcal{L}^3) \oplus \Pi(\mathcal{L}^7) \oplus \dots \oplus \Pi(\mathcal{L}^{4n-1})$
$\mathfrak{osp}(2n+1\|2n)$	$\mathcal{L}^2 \oplus \mathcal{L}^6 \oplus \mathcal{L}^{10} \dots \oplus \mathcal{L}^{4n-2}$	$\oplus \Pi(\mathcal{L}^3) \oplus \Pi(\mathcal{L}^7) \oplus \dots \oplus \Pi(\mathcal{L}^{4n-1})$
$\mathfrak{osp}(2\|2) \cong \mathfrak{sl}(1\|2)$	$\mathcal{L}^2$	$\oplus \Pi(\mathcal{L}^1)$
$\mathfrak{osp}(4\|4)$	$\mathcal{L}^2 \oplus \mathcal{L}^6$	$\oplus \Pi(\mathcal{L}^3) \oplus \Pi(\mathcal{L}^3)$
$\mathfrak{osp}(2n\|2n)$	$\mathcal{L}^2 \oplus \mathcal{L}^6 \oplus \mathcal{L}^{10} \dots \oplus \mathcal{L}^{4n-2}$ $\quad \oplus \mathcal{L}^{2n-2}$	$\oplus \Pi(\mathcal{L}^3) \oplus \Pi(\mathcal{L}^7) \oplus \dots \oplus \Pi(\mathcal{L}^{4n-1})$
$\mathfrak{osp}(2n+2\|2n)$	$\mathcal{L}^2 \oplus \mathcal{L}^6 \oplus \mathcal{L}^{10} \dots \oplus \mathcal{L}^{4n+2}$ $\quad \oplus \mathcal{L}^{2n}$	$\oplus \Pi(\mathcal{L}^3) \oplus \Pi(\mathcal{L}^7) \oplus \dots \oplus \Pi(\mathcal{L}^{4n-1})$
$\mathfrak{osp}_\alpha(4\|2)$	$\mathcal{L}^2$ $\quad \oplus \mathcal{L}^2$	$\oplus \Pi(\mathcal{L}^3)$

The Lie superalgebra $\mathfrak{g}$ of type $\mathfrak{osp}$ that contains a super-principal subalgebra $\mathfrak{osp}(1|2)$ can be generated by two elements. For such elements we can take $X := \nabla^+ \in \mathcal{L}^2 = \mathfrak{osp}(1|2)$ and a lowest weight vector $Z := l_{-r}$ from the module $M = \mathcal{L}^r$ or $\Pi(\mathcal{L}^r)$, where for M we take $\Pi(\mathcal{L}^3)$ if $\mathfrak{g} \neq \mathfrak{osp}(2n|2m)$ or the last module with the even highest weight vector in the above table (i.e., $\mathcal{L}^{2n-2}$ if $\mathfrak{g} = \mathfrak{osp}(2n|2n)$ and $\mathcal{L}^{2n}$ if $\mathfrak{g} = \mathfrak{osp}(2n+2|2n)$).

To generate $\mathfrak{sl}(n|n+1)$ we have to add to the above X and Z a lowest weight vector U from $\Pi(\mathcal{L}^1)$. (Clearly, Z and U are defined up to factors that we can select at our convenience; we will assume that a basis of L^r is fixed and denote $Z = t \cdot l_{-r}$ and $U = s \cdot l_{-1}$ for $t, s \in \mathbb{C}$.)

We call the above X and Z, together with U, and fortified by $Y := X^- \in L^2$ the *Jacobson's generators*. The presence of Y considerably simplifies the form of the relations, though slightly increases the number of them.

10.4.2. Relations between Jacobson's generators. We repeat the arguments from sec. 1.2. Since we obtain the relations recurrently, it could happen that a relation of higher degree implies a relation of a lower degree. This did not happen when we studied $\mathfrak{sl}(\lambda)$, but does happen in what follows, namely, relation 1.2 implies relation 1.1.

We divide the relations between Jacobson's generators into the types corresponding the number of occurence of z in them:

0. Relations in $\mathfrak{sl}(1|2)$ or $\mathfrak{osp}(1|2)$;
1. Relations coming from the $\mathfrak{osp}(1|2)$-action on $\mathcal{L}^{2k_2}$;
2. Relations coming from $\mathcal{L}^{2k_1} \wedge \mathcal{L}^{2k_2}$;
3. Relations coming from $\mathcal{L}^{2k_2} \wedge \mathcal{L}^{2k_2}$;
∞. Relation that shear the dimension.

The relations of type 0 are the well-known relations in $\mathfrak{sl}(1|2)$, those of them that do not involve U (marked with an $*$) are the relations for $\mathfrak{osp}(1|2)$. The relations of type 1 that do not involve U express that the space $\mathcal{L}^{2k_2}$ is the $\mathfrak{osp}(1|2)$-module with highest weight $2k_2$. To simplify notations we denote: $Z_i = \mathrm{ad}_X^i Z$ and $Y_i = \mathrm{ad}_X^i Y$.

$$\mathbf{0.1}^*.\ [Y, Y_1] = 0,\qquad \mathbf{0.2}^*.\ [Y_2, Y] = 2Y,\qquad \mathbf{0.3}^*.\ [Y_2, X] = -X,$$
$$\mathbf{0.4}.\ [Y, U] = 0,\qquad \mathbf{0.5}.\ [U, U] = -2Y;\qquad \mathbf{0.6}.\ [U, Y_1] = 0,$$
$$\mathbf{0.7}.\ [[X, X], [X, U]] = 0,\qquad \mathbf{0.8}.\ [Y_2, U] = U.$$
$$\mathbf{1.1}.\ [Y,\ Z] = 0 \Longleftarrow \mathbf{1.2}.\ [[X, Y],\ Z] = 0,$$
$$\mathbf{1.3}.\ Z_{4k_1} = 0,\qquad \mathbf{1.4}.\ [Y_2, Z] = 3Z.$$

10.4.3. Theorem. *For the Lie superalgebras indicated, all the relations between Jacobson's generators are the above relations of types* 0, 1 *and the relations from* §6.

10.5. The Lie superalgebra $\mathfrak{gl}(\lambda|\lambda+1)$ as the quotient of $\mathfrak{diff}(1|1)$ and a subalgebra of $\mathfrak{sl}_+(\infty|\infty)$

There are several ways to superize $\mathfrak{sl}_+(\infty|\infty)$. For a description of "the best" one from a certain point of view see [E]. For our purposes any version of $\mathfrak{sl}_+(\infty|\infty)$ will do.

10.5.1. The Poincaré–Birkhoff–Witt theorem states that

$$U(\mathfrak{osp}(1|2)) \cong \mathbb{C}[X^-, \nabla^-, H, \nabla^+, X^+],$$

as superspaces. Set $U_\lambda = U(\mathfrak{osp}(1|2))/(\varDelta - \lambda^2 + \frac{9}{4})$. Denote: $\partial_x = \dfrac{\partial}{\partial x}$, $\partial_\theta = \dfrac{\partial}{\partial\theta}$ and set

$$X^- = -\partial_x,\quad H = 2x\partial_x + \theta\partial_\theta(\lambda-1),\quad X^+ = x^2\partial_x - (\lambda-1)x;$$
$$\nabla^- = \partial_\theta - \theta\partial_x,\quad \nabla^+ = x\partial_\theta + x\theta\partial_x - \lambda\theta.$$

These formulas establish a morphism of $\mathfrak{osp}(1|2)$-modules and, moreover, of associative superalgebras: $U_\lambda \longrightarrow \mathbb{C}[x, \theta, \partial_x, \partial_\theta]$.

In what follows we will need a well-known fact: the Casimir operator

$$\varDelta = 2(X^+X^- + X^-X^+) + \nabla^+\nabla^- - \nabla^-\nabla^+ + H^2$$

acts on the irreducible $\mathfrak{osp}(1|2)$-module $\mathcal{L}^\mu$ as the scalar operator of multiplication by $\mu^2 + 3\mu$. (The passage from μ to λ is done with the help of a shift by $\frac{3}{2}$.)

Consider the Lie superalgebra $(U(\mathfrak{osp}(1|2)))_L$ associated with the associative superalgebra U_λ. It is easy to see that, as an $\mathfrak{osp}(1|2)$-module,

$$(U_\lambda)_L = \mathcal{L}^0 \oplus \mathcal{L}^2 \oplus \dots \oplus \mathcal{L}^{2n} \oplus \dots \oplus \varPi(\mathcal{L}^1 \oplus \mathcal{L}^3 \oplus \dots) \tag{5.1}$$

In the same way as for Lie algebras we show that $(U_n)_L$ contains an ideal I_n for $n \in \mathbb{N}\setminus\{0\}$ and the quotient $(U_n)_L/I_n$ is the conventional $\mathfrak{sl}(n|n+1)$. It is clear that for $\lambda \neq \mathbb{Z}$ the Lie algebra $(U_\lambda)_L$ has only one ideal — the space $\mathcal{L}^0$ of constants and $(U_\lambda)_L = \mathcal{L}^0 \oplus (U_\lambda)'_L$; hence, there is a supertrace on $(U_\lambda)_L$. This justifies the following notations

$$\begin{aligned}&\mathfrak{sl}(\lambda|\lambda+1) = \mathfrak{gl}(\lambda|\lambda+1)/\mathcal{L}^0, \quad \text{where}\\ &\mathfrak{gl}(\lambda|\lambda+1) = \begin{cases}(U_\lambda)_L & \text{for } \lambda \neq \mathbb{N}\setminus\{0\}\\ (U_n)_L/I_n & \text{otherwise.}\end{cases}\end{aligned} \tag{5.2}$$

The definition directly implies that $\mathfrak{gl}(-\lambda|-\lambda+1) \cong \mathfrak{gl}(\lambda|\lambda+1)$, so speaking about real values of λ we can confine ourselves to the nonnegative values.

Define $\mathfrak{osp}(\lambda+1|\lambda)$ as the Lie subsuperalgebra of $\mathfrak{sl}(\lambda+1|\lambda)$ invariant with respect to the involution

$$X \to \begin{cases}-X & \text{if } X \in \mathcal{L}^{4k} \text{ or } X \in \varPi(\mathcal{L}^{4k\pm1})\\ X & \text{if } X \in \mathcal{L}^{4k\pm2} \text{ or } X \in \varPi(\mathcal{L}^{4k\pm3}),\end{cases} \tag{5.3}$$

which is the analogue of the map

$$X \to -X^{st} \quad \text{for} \quad X \in \mathfrak{gl}(m|n). \tag{5.4}$$

10.5.2. The Lie superalgebras $\mathfrak{sl}(*|*+1)$ and $\mathfrak{osp}(2*|*+1)$, for $* \in \mathbb{C}\boldsymbol{P}^1 = \mathbb{C}\cup\{*\}$. The "dequantization" of the relations for $\mathfrak{sl}(\lambda|\lambda+1)$ and $\mathfrak{osp}(\lambda+1|\lambda)$ is performed by passage to the limit as $\lambda \longrightarrow \infty$ under the change $t \mapsto \frac{t}{\lambda}$. We denote the limit algebras by $\mathfrak{sl}(*|*+1)$ and $\mathfrak{osp}(*+1|*)$ in order not to mix them with $\mathfrak{sl}(\infty|\infty+1)$ and $\mathfrak{osp}(\infty|\infty+1)$, respectively.

10.6. Tables. The Jacobson generators and relations between them

10.6.1. Table. The infinite dimensional case.

- $\underline{\mathfrak{osp}(\lambda|\lambda+1)}$. Generators:

$$X = x\partial_\theta + x\theta\partial_x - \lambda\theta, \;\; Y = \partial_x, \;\; Z = t(\partial_x\partial_\theta - \theta\partial_x{}^2).$$

Relations:

2.1. $3[Z, Z_3] + 2[Z_1, Z_2] = 6t(2\lambda+1)Z$,

2.2. $[Z_1, Z_3] = 2t^2(\lambda-1)(\lambda+2)Y + 2t(2\lambda+1)Z_1$,

3.1. $[Z_1, [Z, Z]] = 0$.

- $\underline{\mathfrak{osp}(*|*+1)}$. Relations: the same as in sec 4.2 plus the following relations:

2.1. $3[Z, Z_3] + 2[Z_1, Z_2] = 12tZ$,

2.2. $[Z_1, Z_3] = 2t^2Y + 4tZ_1$.

- $\underline{\mathfrak{sl}(\lambda+1|\lambda)}$ for $\lambda \in \mathbb{C}P^1$. Generators (for $\lambda \in \mathbb{C}$): the same as for $\mathfrak{osp}(\lambda|\lambda+1)$ and $U = \partial_\theta - \theta\partial_x$.

Relations: the same as for $\mathfrak{osp}(\lambda|\lambda+1)$ plus the following

1.5. $3[Z, [X, U]] - [U, Z_1] = 0$, **2.3**. $[Z, [U, Z]] = 0$,

1.6. $[[X, U], Z_1] = 0$, **2.4**. $[Z_1, [U, Z]] = 0$.

10.6.2. Table. Finite dimensional algebras. In this table E_{ij} are the matrix units; $X_i^\pm$ stand for the conventional Chevalley generators of $\mathfrak{g}$.

- $\underline{\mathfrak{sl}(n+1|n)}$ for $n \geq 3$. Generators:

$$X = \sum_{1\leq i\leq n}\big((n-i+1)E_{2i-1,2i} - iE_{2i,2i+1}\big), \quad Y = \sum_{1\leq i\leq 2n-1} E_{i+2,i},$$
$$U = \sum_{1\leq i\leq 2n}(-1)^{i+1}E_{i+1,i}, \quad Z = \sum_{1\leq i\leq 2n-2}(-1)^{i+1}E_{i+3,i}.$$

Relations: those for $\mathfrak{sl}(\lambda+1|\lambda)$ with $\lambda = n$ and an extra relation to shear the dimension:

$$(\mathrm{ad}_Z)^n([X, X]) = 0.$$

For $n = 1$ the relations degenerate in relations of type 0.

- $\underline{\mathfrak{osp}(2n+1|2n)}$. Generators:

$$X = \sum_{1\leq i\leq n}\big((2n-i+1)(E_{2i-1,2i} + E_{4n+2-2i,4n+3-2i}) - i(E_{2i,2i+1} - E_{4n+1-2i,4n+2-2i})\big),$$
$$Y = E_{2n+2,2n} + \sum_{1\leq i\leq 2n-1}(E_{i+2,i} - E_{4n+2-i,4n-i}),$$
$$Z = E_{2n+2,2n-1} - E_{2n+3,2n} + \sum_{1\leq i\leq 2n-2}\big((-1)^iE_{i+3,i} + E_{4n+2-i,4n-1-i}\big).$$

Relations: those for $\mathfrak{osp}(2\lambda+1|2\lambda)$ with $\lambda = n$ and an extra relation to shear the dimension (the form of the relation is identical to that for $\mathfrak{sl}(n+1|n)$).

- $\underline{\mathfrak{osp}_\alpha(4|2)}$. Generators: As $\mathfrak{osp}(1|2)$-module, the algebra $\mathfrak{osp}_\alpha(4|2)$ has 2 isomorphic submodules. The generators X and Y belong to one of them. It so happens that we can select Z from either of the remaining submodules and still generate the whole Lie superalgebra. The choice (a) is from $\Pi(\mathcal{L}^3)$; it is unique (up to a factor). The choices (b) and (c) are from $\mathcal{L}^2$; one of them gives simpler relations:

$$X = \frac{\alpha+1}{\alpha}X_1^+ + \frac{\alpha}{\alpha+1}X_2^+ + \frac{1}{\alpha(\alpha+1)}X_3^+,$$
$$Y = [X_1^-, X_2^-] + [X_1^-, X_3^-] + [X_2^-, X_3^-],$$
$$Z = \begin{cases} \text{a)} \;\; -[[X_1^-, X_2^-], X_3^-]; \\ \text{b)} \;\; -(1+2\alpha)[X_1^-, X_2^-] + \alpha^2(2+\alpha)[X_1^-, X_3^-] + (\alpha-1)(1+\alpha)^2[X_2^-, X_3^-]; \\ \text{c)} \;\; -[X_2^-, X_3^-] - (\alpha+1)[X_1^-, X_2^-]. \end{cases}$$

Relations of type 0 are common for cases a) – c):

0.1. $[Y, [Y, [X, X]]] = 4Y$; **0.2**. $[Y_1[X, X]]] = -2X$;

The other relations are as follows.

Relations a):

1.1. $[Y_1, Z_1] = 3Z$, **1.2**. $(\mathrm{ad}_{[X,X]})^3Z_1 = 0$;

2.1. $[Z, Z] = 0$; **2.2**. $[Z_1, [[X, X], Z]] = -4\dfrac{\alpha^2+\alpha+1}{\alpha(\alpha+1)}Z$,

3.1. $[\mathrm{ad}_{[X,X]}(Z_1), [Z_1, \mathrm{ad}_{[X,X]}(Z_1)]] =$
$$-\frac{16}{\alpha(\alpha+1)}Y + 8\frac{\alpha^2+\alpha+1}{\alpha(\alpha+1)}[Z_1, \mathrm{ad}_{[X,X]}(Z_1)] + 16\frac{(\alpha^2+\alpha+1)^2}{\alpha^2(\alpha+1)^2}Z_1.$$

Relations b):

1.1. $[Y_1, Z_1] = 2Z$; **1.2** $(\mathrm{ad}_{[X,X]})^2 Z_1 = 0$;

2.1*. $[Z_1, Z_1] = 2[Z, [Z, [X, X]]] - 18\alpha^2(1+\alpha)^2 Y + 4(1-\alpha)(2+\alpha)(1+2\alpha)Z$;

3.1. $(\mathrm{ad}_Z)^3 X = 0$,

3.2*. $[[Z, Z_1], (\mathrm{ad}_{[X,X]})^2 Z_1] = (-1+\alpha)(2+\alpha)(1+2\alpha)[Z, [Z, [X, X]]] + 12(1-\alpha)(2+\alpha)(1+2\alpha)\alpha^2(1+\alpha)^2 Y + 8(1-3\alpha^2-\alpha^3)(-1-3\alpha+\alpha^3)Z$.

Relations c): same as for b) except that the relations marked in b) by an $*$ should be replaced by the following ones

2.1. $[Z_1, Z_1] = 2[Z, [Z, [X, X]]] - 2\alpha^2 Y + 4(2+\alpha)Z$;

2.2. $(\mathrm{ad}_{[X,X]})^2 Z_1 = (-2-\alpha)[Z, [Z, [X, X]]] - 8(1+\alpha)Z + 4\alpha^2(2+\alpha)Y$.

10.7. Remarks and problems

10.7.1. On proof. For the exceptional Lie algebras and superalgebra $\mathfrak{osp}_\alpha(4|2)$, the proof is direct: the quotient of the free Lie algebra generated by x, y and z modulo our relations is the needed finite dimensional one. For rank $\mathfrak{g} \leq 12$ we similarly computed relations for $\mathfrak{g} = \mathfrak{sl}(n)$, $\mathfrak{o}(2n+1)$ and $\mathfrak{sp}(2n)$; as Post pointed out, together with the result of [PH] on deformation (cf. 2.7) this completes the proof for Lie algebras. The results of [PH] on deformations can be directly extended for the case of $\mathfrak{sl}(2)$ replaced by $\mathfrak{osp}(1|2)$; this proves Theorem 4.3.

Our Theorem 2.6 elucidates Proposition 2 of [F]; we just wrote relations explicitely. Feigin claimed [F] that for $\mathfrak{sl}(\lambda)$, the relations of type 3 follow from the decomposition of $L^{2k_2} \wedge L^{2k_2} \subset L^{2k_2} \wedge L^{2k_2} \wedge L^{2k_2}$. We verified that this is so not only in Feigin's case but for all the above-considered algebras except $\mathfrak{e}(6)$, $\mathfrak{e}(7)$ and $\mathfrak{e}(8)$: for them, one should consider the whole $L^{2k_2} \wedge L^{2k_2} \wedge L^{2k_2}$, cf. [GL21]. Theorem 4.3 is a direct superization of Theorem 2.6.

10.7.2. Problems. 1) How to present $\mathfrak{o}(2n)$ and $\mathfrak{osp}(2m|2n)$? One can select z as suggested in sec. 1.1. Clearly, the form of z (hence, relations of type 1) and the number of relations of type 3 depend on n in contradistinction with the algebras considered above. Besides, the relations are not as neat as for the above algebras. We should, perhaps, have taken the generators as for $\mathfrak{o}(2n-1)$ and add a generator from L^{2n-2}. We have no guiding idea; to try at random is frustrating, cf. the relations we got for $\mathfrak{osp}_\alpha(4|2)$.

2) We could have similarly realized the Lie algebra $\mathfrak{sl}(\lambda)$ as the quotient of $U(\mathfrak{vect}(1))_L$. However, $U(\mathfrak{vect}(1))$ has no center except the constants. What are the generators of the ideal modulo which we should factorize $U(\mathfrak{vect}(1))_L$ in order to get $\mathfrak{sl}(\lambda)$? (Observe that in case $U(\mathfrak{g})$, where $\mathfrak{g}$ is a simple finite dimensional Lie superalgebra such that $Z(U(\mathfrak{g}))$ is not noetherian, but the mysterious ideal is, nevertheless, finitely generated, cf. [GL2].)

3) Feigin realized $\mathfrak{sl}(*)$ on the space of functions on the open cell of $\mathbb{C}P^1$, a hyperboloid, see [F]. Examples of [DGS1] are similarly realized. Give any realization of $\mathfrak{o}/\mathfrak{sp}(*)$ and its superanalogs.

4) Other problems are listed in sec. 8.1–8.3 below.

10.7.3. Serre relations v. Jacobson ones. The following Table represents results of V. Kornyak's computations. N_{GB} is the number of relations in Groebner basis, N_{comm} is the number of non-zero commutators in the multiplication table, D_{GB} is a maximum degree of relations in GB, Space is measured in in bytes. The corresponding values for Chevalley generators/Serre relations are given in parenthesis.

alg	N_{GB}	N_{comm}	D_{GB}	Space	Time
$\mathfrak{sl}(3)$	23 (24)	21 (21)	9 (4)	1300 (1188)	$< 1sec$ ($< 1sec$)
$\mathfrak{sl}(4)$	69 (84)	70 (60)	17 (6)	3888 (3612)	$< 1sec$ ($< 1sec$)
$\mathfrak{sl}(5)$	193 (218)	220 (126)	25 (8)	13556 (8716)	$< 1sec$ ($< 1sec$)
$\mathfrak{sl}(6)$	444 (473)	476 (225)	33(10)	34692 (18088)	$2sec$ ($< 1sec$)
$\mathfrak{sl}(7)$	893 (908)	937 (363)	41(12)	80272 (33700)	$10sec$ ($1sec$)
$\mathfrak{sl}(8)$	1615(1594)	1632 (546)	49(14)	162128 (57908)	$34sec$ ($3sec$)
$\mathfrak{sl}(9)$	2705(2614)	2714 (780)	57(16)	314056 (93452)	$109sec$ ($6sec$)
$\mathfrak{sl}(10)$	4263 (4063)	4138 (1071)	65 (18)	534684 (143456)	$336sec$($10sec$)
$\mathfrak{sl}(11)$	6405 (6048)	6224 (1425)	73 (20)	921972 (211428)	$1058sec$ ($19sec$)

For the other Lie algebras, especially exceptional ones, the comparison is even more unfavorable. Nevertheless, for $\mathfrak{sl}(\lambda)$ with non-integer λ, there are only the Jacobson generators and we have to use them.

10.8. Lie (super)algebras of higher ranks. The exponents, and W-algebras

The following Tables 8.1 and 8.2 introduce the generators for the Lie algebras $U_\lambda(\mathfrak{g})$ and the analogues of the exponents that index the generalized W-algebras (for their definition in the simplest cases from different points of view see [FFr] and [KM]; we will follow the lines of [KM]).

Recall (10.1) that one of the definitions of $U_\lambda(\mathfrak{g})$ is as the associative algebra generated by $\tilde{\mathfrak{g}}$; we loosely denoted it by $\tilde{S}^\bullet(\tilde{\mathfrak{g}})$. For the generators of $(U_\lambda(\mathfrak{g}))_L$ we take the Chevalley generators of $\mathfrak{g}$ (since by sect. 7.3 they are more convenient) and the lowest weight vectors of the irreducible $\mathfrak{g}$-modules that constitute $\tilde{S}^2(\tilde{\mathfrak{g}})$.

10.8.1. The exponents. This section is just part of Table 1 from [OV] reproduced here for the convenience of the reader. Let $\mathfrak{g}$ be a simple (finite dimensional) Lie algebra, $W = W_\mathfrak{g}$ its Weyl group, $l = \mathrm{rk}\ \mathfrak{g}$, $\alpha_1, \ldots, \alpha_l$ the simple roots, α_0 the lowest root; the n_i the coefficients of linear relation among the α_i normed so that $n_0 = 1$; let $c = r_1 \cdot \dots \cdot r_l$, where the $r_i \in W$ are the reflections in simple roots, be the *Coxeter element*. The order h of c (the *Coxeter number*) is equal to $\sum\limits_{i>0} n_i$. The eigenvalues of c are $\varepsilon^{k_1}, \dots, \varepsilon^{k_l}$, where ε is a primitive h-th root of unity. The numbers k_i are called the *exponents*. Then (see [OV]):

The exponents k_i are given by Table 1.1, e.g., $k_1 = 1$. The number of roots of $\mathfrak{g}$ is equal to $l \sum\limits_{i>0} n_i = 2 \sum\limits_{i>0} k_i$. The order of W is equal to

$$zl! \prod n_i = \prod_{i>0} (k_i + 1),$$

where z is the number of 1's among the n_i's for $i > 0$ (the number z is also equal to the order of the centrum $Z(G)$ of the simply connected Lie group G whose Lie algebra is $\mathfrak{g}$). The algebra of W-invariant polynomials on the maximal diagonalizable (Cartan) subalgebra of $\mathfrak{g}$ is freely generated by homogeneous polynomials of degrees $k_i + 1$.

We will use the following notations:

For a given finite dimensional simple Lie algebra $\mathfrak{g}(A)$, let $R(\lambda)$ denote the irreducible representation with highest weight λ and $V(\lambda)$ the space of this representation;

$\rho = \frac{1}{2} \sum\limits_{\alpha>0} \alpha$ or ρ is a weight such that $\rho(\alpha_i) = A_{ii}$ for each simple root α_i.

The weights of the Lie algebras $\mathfrak{o}(2l)$ and $\mathfrak{o}(2l+1)$, $\mathfrak{sp}(2l)$ and $\mathfrak{f}(4)$ ($l = 4$) are expressed in terms of an orthonormal basis $\varepsilon_1, \dots, \varepsilon_l$ of the space $\mathfrak{h}^*$ over $\mathbb{Q}$. The weights of the Lie algebras $\mathfrak{sl}(l+1)$ as well as $\mathfrak{e}(7)$, $\mathfrak{e}(8)$ and $\mathfrak{g}(2)$ ($l = 7, 8$ and 2, respectively) are expressed in terms of vectors $\varepsilon_1, \dots, \varepsilon_{l+1}$ of the space $\mathfrak{h}^*$ over $\mathbb{Q}$ such that $\sum \varepsilon_i = 0$. For these vectors $(\varepsilon_i, \varepsilon_i) = \dfrac{l}{l+1}$ and $(\varepsilon_i, \varepsilon_j) = \dfrac{1}{l+1}$ for $i \neq j$. The indices in the expression of any weight are assumed to be different.

The analogues of the exponents for $(U_\lambda(\mathfrak{g}))_L$ are the highest weights of the representations that constitute $\tilde{S}^k(\tilde{\mathfrak{g}})$.

Problem. Interpret these exponents in terms of the analog of the Weyl group of $(U_\lambda(\mathfrak{g}))_L$ in the sense of [PS] and invariant polynomials on $(U_\lambda(\mathfrak{g}))_L$.

10.8.2. Table. The Lie algebras $(U_\lambda(\mathfrak{g}))_L$ as $\mathfrak{g}$-modules. Columns 2 and 3 of this Table are derived from Table 5 in [OV]. Columns 4 and 5 are results of a computer-aided study. To fill in the gaps is a research problem, cf. [GL2] for the Lie algebras different from $\mathfrak{sl}$ type.

$\mathfrak{g}$	ad	$\tilde{S}^2(\tilde{\mathfrak{g}})$	$\tilde{S}^3(\tilde{\mathfrak{g}})$	$\tilde{S}^k(\tilde{\mathfrak{g}})$
$\mathfrak{sl}(2)$	$R(2\pi)$	$R(4\pi)$	$R(6\pi)$	$R(2k\pi)$
$\mathfrak{sl}(3)$	$R(\pi_1+\pi_2)$	$R(2\pi_1+2\pi_2)$	$R(3\pi_1+3\pi_2)$	$R(k\pi_1+k\pi_2)$
		$R(\pi_1+\pi_2)$	$R(2\pi_1+2\pi_2)$	$R((k-1)\pi_1+(k-1)\pi_2)$

$\mathfrak{g}$	ad	$\tilde{S}^2(\tilde{\mathfrak{g}})$	$\tilde{S}^3(\tilde{\mathfrak{g}})$	$\tilde{S}^k(\tilde{\mathfrak{g}})$
$\mathfrak{sl}(4)$	$R(\pi_1+\pi_3)$	$R(2\pi_1+2\pi_3)$	$R(3\pi_1+3\pi_3)$	
		$R(\pi_1+\pi_3)$	$R(2\pi_1+2\pi_3)$	
		$R(2\pi_2)$	$R(2\pi_1+\pi_2)$	
			$R(\pi_2+2\pi_3)$	
			$R(\pi_1+\pi_3)$	
			$R(\pi_1+2\pi_2+\pi_3)$	
$\mathfrak{sl}(n+1)$	$R(\pi_1+\pi_n)$	$R(2\pi_1+2\pi_n)$	$R(3\pi_1+3\pi_n)$	
$n \geq 4$		$R(\pi_1+\pi_n)$	$R(2\pi_1+2\pi_n)$	
		$R(\pi_2+\pi_{n-1})$	$R(2\pi_1+\pi_{n-1})$	
			$R(\pi_2+\pi_{n-1})$	
			$R(\pi_2+2\pi_n)$	
			$R(\pi_1+\pi_n)$	
			$R(\pi_1+\pi_2+\pi_{n-1}+\pi_n)$	

The generators of $(U_\lambda(\mathfrak{g}))_L$ are the Chevalley generators $X_i^\pm$ of $\mathfrak{g}$, AND the lowest weight vectors z_i from $\tilde{S}^2$. Then the relations are (recall that $h_i = [X_i^+, X_i^-]$):

(type 0) the Serre relations in $\mathfrak{g}$

(type 1) The relations between $X_i^\pm$ and z_j, namely:

$$X_i^-(z_j) = 0; \quad h_i(z_j) = \text{weight}_i(z_j);$$
$$(\mathrm{ad}_{X_i^+})^{\text{the power determined by the weight of } z_j}(z_j) = 0.$$

Problem. Give an explicit form of the relations of higher types.

10.8.3. Tougher problems. Even if the explicit realization of the exceptional Lie algebras by differential operators on the base affine space were known at the moment, it is, nevertheless, a difficult computer problem to fill in the blank spaces in the above table and similar tables for Lie superalgebras. To make plausible conjectures we have to compute $\tilde{S}^k(\tilde{\mathfrak{g}})$ to, at least, $k = 4$.

Observe that, for simple Lie algebras $\mathfrak{g}$, we have a remarkable theorem by Kostant, which states that $U_\lambda(\mathfrak{g})$ contains every finite dimensional irreducible $\mathfrak{g}$-module V with multiplicity equal to the multiplicity of the zero weight in V; in view of this theorem, only the $\mathfrak{sl}(2)$-line is complete.

10.9. A connection with integrable dynamical systems

We recall the basic steps of the Khesin–Malikov construction. To superize them is a very interesting *research problem*.

10.9.1. The Hamilton reduction. Let (M^{2n}, ω) be a symplectic manifold with an action *act* of a Lie group G on M by symplectomorphisms (i.e., G preserves ω). The derivative of the G-action gives rise to a Lie algebra homomorphism $a : \mathfrak{g} = Lie(G) \longrightarrow \mathfrak{h}(2n)$. The action a is called a *Poisson* one, if a can be lifted to a Lie algebra homomorphism $\tilde{a} : \mathfrak{g} \longrightarrow \mathfrak{po}(2n)$.

For any Poisson G-action on M there arises a G-equivariant map $p : M \longrightarrow \mathfrak{g}^*$, called the *moment map*, given by the formula

$$\langle p(x), g\rangle = \tilde{a}(g)(x) \quad \text{for any} \quad x \in M, g \in \mathfrak{g}.$$

Fix $b \in \mathfrak{g}^*$; let $G_b \subset G$ be the stabilizer of b. Under certain regularity conditions (see [Ar]) $p^{-1}(b)/G_b$ is a manifold. This manifold is endowed with the symplectic form

$$\omega(\bar{v}, \bar{w}) = \omega(v, w) \quad \text{for arbitrary preimages} \quad v, w \text{ of } \bar{v}, \bar{w}, \text{ resp.}$$
$$\text{w.r.t. the natural projection} \quad T(p^{-1}(b)) \longrightarrow T(p^{-1}(b)/G_b).$$

The passage from M to $p^{-1}(b)/G_b$ is called *Hamilton reduction*. In the above picture M can be the *Poisson manifold*, i.e., ω is allowed to be degenerate on M; the submanifolds on which ω is non-degenerate are called *symplectic leaves*.

Example. Let $\mathfrak{g} = \mathfrak{sl}(n)$ and $M = \mathfrak{g}^*$, let G be the group N of upper-triangular matrices with 1's on the diagonal. The coadjoint N-action on $\mathfrak{g}^*$ is a Poisson one, the moment map is the natural projection $\mathfrak{g}^* \longrightarrow \mathfrak{n}^*$ and $\mathfrak{g}^*/N$ is a Poisson manifold.

10.9.2. The Drinfeld–Sokolov reduction. Let $\mathfrak{g} = \hat{\mathfrak{a}}^{(1)}$, where $\mathfrak{a}$ is a simple finite dimensional Lie algebra (the case $\mathfrak{a} = \mathfrak{sl}(n)$ is the one considered by Gelfand and Dickii), hat denotes the non-trivial central extension. The elements of $M = \mathfrak{g}^*$, can be identified with the $\mathfrak{a}$-valued differential operators:

$$(f(t)dt, az^*) \mapsto (tf(t) + at\frac{d}{dt}))\frac{dt}{t}.$$

Let N be the loop group with values in the group generated by positive roots of $\mathfrak{a}$. For the point b above take the element $y \in \mathfrak{a} \subset \hat{\mathfrak{g}}^*$ described in §3. If $\mathfrak{a} = \mathfrak{sl}(n)$, we can represent every element of $p^{-1}(b)/N$ in the form

$$t\frac{d}{dt} + y + \begin{pmatrix} b_1(t) & \dots & b_n(t) \\ 0 & \dots & 0 \\ 0 & \dots & 0 \end{pmatrix} \longleftrightarrow \frac{d^n}{d\varphi^n} + \tilde{b}_1(\varphi)\frac{d^{n-1}}{d\varphi^{n-1}} + \dots + \tilde{b}_n(\varphi),$$

To generalize the above to $\mathfrak{sl}(\lambda)$, Khesin and Zakharevich described the Poisson–Lie structure on symbols of pseudodifferential operators, see [KM] and refs therein. Let us recall the main formulas.

10.9.3. The Poisson bracket on symbols of ΨDO. Set $D = \frac{d}{dx}$; define

$$D^\lambda \circ f = fD^\lambda + \sum_{k\geq 1} \binom{\lambda}{k} f^{(k)} D^{\lambda-k}, \quad \text{where } \binom{\lambda}{k} = \frac{\lambda(\lambda-1)\dots(\lambda-k+1)}{k!}.$$

Set

$$G_\lambda = \left\{ D^\lambda(1 + \sum_{k\geq 1} u_k(x) D^{-k}) \right\}$$

and

$$TG_\lambda = \left\{ \sum_{k\geq 1} v_k(x) D^{-k} \right\} \circ D^\lambda, \quad T^*G_\lambda = D^{-\lambda} \circ DO.$$

For $X = D^{-\lambda} \circ \sum\limits_{k\geq 0} u_k(x) D^k \in T^*G_\lambda$ and $L = \left(\sum\limits_{k\geq 1} v_k(x) D^{-k}\right) \circ D^\lambda \in TG_\lambda$, define the pairing $\langle X, L\rangle$ to be

$$\langle X, L\rangle = \mathrm{tr}(L \circ X), \quad \text{where} \quad \mathrm{tr}(\sum w_k(x) D^k) = \mathrm{Res}\,|_{x=0} w_{-1}.$$

The Poisson bracket on the spae of psedodifferential symbols ΨDS_λ is defined on linear functionals by the formula

$$\{X, Y\}(L) = X(H_Y(L)), \quad \text{where} \quad H_Y(L) = (LY)_+L - L(YL)_+,$$

where L_+ is the differential part of the pseudodifferential differential operator L expandable in Laurent series in D.

Theorem (Khesin–Malikov)**.** *For $\mathfrak{a} = \mathfrak{sl}(\lambda)$ in the Drinfeld–Sokolov picture, the Poisson manifolds $p^{-1}(b)/N_b$ and ΨDS_λ are isomorphic. Each element of the Poisson leaf has a representative in the form*

$$t\frac{d}{dt} + y + \begin{pmatrix} b_1(t) & \dots & b_n(t) & \dots \\ 0 & \dots & 0 & \dots \\ 0 & \dots & 0 & \dots \end{pmatrix} \longleftrightarrow D^\lambda \left(1 + \sum_{k\geq 1} \tilde{b}_k(\varphi) D^{(-k)}\right).$$

The Drinfeld–Sokolov construction [DS], as well as its generalization to $\mathfrak{sl}(\lambda)$ and $\mathfrak{o}/\mathfrak{sp}(\lambda)$ ([KM]), hinges on a certain element that can be identified with the image of $X^+ \in \mathfrak{sl}(2)$ under the principal embedding. For the case of higher ranks, this is the image of the element $y \in \mathfrak{g}$, described in §3 for Lie algebras, in $U_\lambda(\mathfrak{g})$. In $\mathfrak{sl}(\lambda)$ and $\mathfrak{o}/\mathfrak{sp}(\lambda)$, this image is just $\frac{d}{dx}$ (or the matrix whose only nonzero entries are the 1's under the main diagonal in the realization of $\mathfrak{sl}(\lambda)$ and $\mathfrak{o}/\mathfrak{sp}(\lambda)$ by matrices).

10.9.4. Continuous Toda lattices. Khesin and Malikov [KM] considered straightforward generalizations of the Toda lattices — the dynamical systems on the orbits of the coadjoint representation of a simple finite dimensional Lie group G defined as follows. Let $\mathcal{X}$ be the image of $X^+ \in \mathfrak{sl}(2)$ in $\mathfrak{g} = Lie(G)$ under the principal embedding. Having identified $\mathfrak{g}$ with $\mathfrak{g}^*$ with the help of the invariant non-degenerate form, consider the orbit $\mathcal{O}_{\mathcal{X}}$. On $\mathcal{O}_{\mathcal{X}}$, the traces $H_i(A) = \mathrm{tr}(A + \mathcal{X})^i$ are the commuting Hamiltonians.

In our constructions we only have to consider in $(U_\lambda(\mathfrak{g}))_L$ either (for Lie algebras) the image of $\mathcal{X}$ or (for Lie superalgebras) the image of $\nabla^+ \in \mathfrak{osp}(1|2)$ under the super-principal embedding of $\mathfrak{osp}(1|2)$. For superalgebras, we also have to replace trace by the supertrace.

For the general description of dynamical systems on the orbits of the coadjoint representations of Lie supergroups, see [LST]. A possibility of odd mechanics is pointed out in [LST] and in the subsequent paper by R. Yu. Kirillova in the same Proceedings. To take such a possibility into account, we have to consider analogs of the principal embeddings for $\mathfrak{sq}(2)$. Even to define them is an *open problem*.

Chapter 11

Symmetries wider than supersymmetry and simple Volichenko algebras (D. Leites)

11.1. Introduction: Towards noncommutative geometry

This is a version of [LSa2]. Proof of the classification theorem is due to V. Serganova.

11.1.1. The gist of idea. To describe physical models, the least one needs is a triple $(X, F(X), L)$, consisting of the "phase space" X, the sheaf of functions on it, locally represented by the algebra $F(X)$ of "functions" on X — sections of this sheaf, and a Lie subalgebra L of the Lie algebra of of derivations of $F(X)$ considered as vector fields on X. Here X can be recovered from $F(X)$ as the collection $\mathrm{Spec}(F(X))$, called the *spectrum* and consisting of maximal ideals (or prime) of $F(X)$ (or some other type of ideals for noncommutative $F(X)$). Usually, X is endowed with a suitable topology.

After the discovery of quantum mechanics the attempts to replace $F(X)$ by the noncommutative ("quantum") algebra A became more and more popular. The first successful attempt was superization performed in [Le0], [BLS]. The road to it was prepared in the works of A. Weil, Leray, Grothendieck and Berezin. It turns out that having suitably generalized the notion of the tensor product and derivation (by inserting certain signs in the conventional formulas) we can reproduce on supermanifolds all the characters of differential geometry and actually obtain a much reacher and interesting plot than on manifolds. This picture proved to be a great success in theoretical physics since the language of supermanifolds and supergroups is a "natural" for a uniform description of Bose and Fermi particles. Today there is no doubt that this is the language of the Grand Unified Theories of all known fundamental forces. There are, however, no observable effects to prove it, see [**?**]. On a less ambitious scale, but with tangible results, Efetov [Ef] applied supersymmetry to problems of solid state physics.

Observe that physicists who, being unaware of [Le0], or even earlier, discovered supersymmetries (Stavraki, Golfand–Likhtman, Volkov–Akulov, Neveu–Schwartz, Ramond) were studying possibilities to enlarge the group of symmetries (or rather the Lie algebra of infinitesimal symmetries) of the known objects (in particular, objects described by Maxwell and Dirac equations). Finally, Wess and Zumino understood some of the consequences of supersymmetry and indicated several applications which resulted in mid 1970's in a euphoria "Einstein's dream — the Grand Unified theory — will soon be achieved! There are no divergences in SUSY GUTs!, etc.".

Though not all of these predictions are realized yet, we believe that basically they are true, and only "small technical problems" hamper the researchers world-wide for more than 30 years now. One of the mechanisms these and other wonders (for their list see, e.g., [D]) are due to the fact that SUSY is a hidden symmetry of known entities, a broader one than the used-to-be conventional ones.

We will show that the supergroups do not, nevertheless, constitute the largest symmetries of superspaces; one needs more noncommutativity than just a mere supercommutativity.

How noncommutative should $F(X)$ be? To define the space corresponding to an arbitrary algebra is very hard, see Manin's gloomy remarks in [Mn1], where he studies quadratic algebras as functions on "perhaps, nonexisting" noncommutative projective spaces.

Manin's idea that there hardly exists one uniform definition suitable for any noncommutative algebra (because there are several quite distinct types of them) was supported by A. Rosenberg's studies who managed to define several types of spectra in order to interpret ANY algebra as the algebra of functions on a suitable spectrum, see preprints of his two books [**?**], no. 25, and [**?**] nn. 26, 31 (the latter being expanded as [Ro]). In particular, there IS a space corresponding to a quadratic (or quadraticizable) algebra such as the so-called "quantum" deformation $U_q(\mathfrak{g})$ of the enveloping of a Lie algebra $\mathfrak{g}$, see [Dr].

Unlike numerous previous attempts, Rosenberg's theory is more natural; still, it is algebraic, without any real geometry (no differential equations, integration, etc.). For some noncommutative algebras, certain notions of differential geometry can be generalized: such is, now well-known, A. Connes's geometry, see [Mn2] and [Co]. Arbitrary algebras seem to be too noncommutative to allow to do any physics.

In contrast, the experience with the simplest non-commutative spaces, the superspaces, tells us that all constructions expressible in the language of differential geometry (these are used in physics) can be carried over to the super case. Still, supersymmetry has, as we will show, its shortcomings, which disappear in the theory we propose.

Specifically, we continue the study started under Berezin's influence in [Le0] (later suppressed under the same influence in [BLS], [DL2]), of algebras just slightly more general than supercommutative superalgebras, namely their arbitrary, not necessarily homogeneous, subalgebras. Thanks to Volichenko's theorem M ([**?**], no. 17) such subalgebras are precisely *metabelian* ones, i.e., those that satisfy the identity

$$[x,[y,z]]=0 \quad \text{(here } [\cdot,\cdot] \text{ is the usual commutator).} \tag{11.1}$$

Volichenko proved in his theorem M that every metabelian algebra can be embedded into a universal in a sense supercommutative superalgebra. As in noncommutative geometries, we think of metabelian algebras as "functions" on a what we will call *metaspace*.

Observe that the conventional superspaces considered as metaspaces have, together with the Lagrangeans currently considered in mathematical physics, **additional** symmetries as compared with supersymmetry.

11.1.2. The notion of Volichenko algebras. Thanks to Volichenko's Theorem M, we know a natural generalization of the supercommutativity. It remains to define the analogs of the tensor product and derivation (see [LSoS, IU]). I've conjectured that the analogs of Lie algebras in the new setting are *Volichenko algebras* defined here as inhomogeneous subalgebras of Lie superalgebras. For a proof of this conjecture, see [IU].

Our main mathematical result is the classification (under a technical hypothesis) of simple finite dimensional and vectorial Volichenko algebras.

Supersymmetry had been already justified for physicists when mathematicians' attention was drawn to it by the list of simple finite-dimensional Lie superalgebras: bar one exception it was discrete and looked miraculously like the list of simple Lie algebras. Similar is our list of simple Volichenko algebras.

Remarkably, Volichenko algebras are just deformations of Lie algebras though in an entirely new sense: in a category broader than that of Lie algebras or Lie superalgebras. This feature of Volichenko algebras could be significant for parastatistics because once we abandon Bose-Fermi statistics, there seem to be too many *ad hoc* ways to generalize. Our classification asserts that within the natural context of simple Volichenko algebras the set of possibilities consists only of discretely parameterized or 1-parameter (which are, anyway, describable!) algebras. This suggests a possibility of associating distinct types of particles to representations of these structures, so to define their representations is an important problem.

Our generalization of supersymmetry and its implications for parastatistics appear to be complementary to recent work on braid statistics in two dimensions [FRS] in the context of [DR], see also [JGW] and are expected to tie up at some stage.

Examples of what looks like non-simple Volichenko algebras recently appeared in another context in [Bea], [RS], [Sp] and [FIK].

11.1.3. The two intriguing examples. Each of the two super analogs of $\mathfrak{gl}(n)$ have a one-parameter Volichenko analog:

11.1.3.1. The general Volichenko algebra $\mathfrak{vgl}_\mu(p|q)$. Let the space $\mathfrak{h}$ of $\mathfrak{vgl}_\mu(p|q)$ be the space of $(p+q)\times(p+q)$-matrices divided into the direct sum of two subspaces $\mathfrak{h}_{\hat{0}}$ and $\mathfrak{h}_{\hat{1}}$:

$$\mathfrak{h}_{\hat{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \text{ where } \begin{matrix} A \in \mathfrak{gl}(p), \\ D \in \mathfrak{gl}(q) \end{matrix} \right\}; \ \mathfrak{h}_{\hat{1}} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \text{ for any } B \text{ and } C \right\}. \tag{11.2}$$

Here $\mathfrak{h}_{\hat{1}}$ is a natural $\mathfrak{h}_{\hat{0}}$-module with respect to the bracket of matrices; for any fixed $a, b \in \mathbb{C}$ such that $a : b = \mu$, we define the multiplication $\mathfrak{h}_{\hat{1}} \times \mathfrak{h}_{\hat{1}} \longrightarrow \mathfrak{h}_{\hat{0}}$ by the formula

$$[X, Y] = a[X, Y]_- + b[X, Y]_+ \text{ for } X, Y \in \mathfrak{h}_{\hat{1}}. \tag{11.3}$$

(The subscript $-$ or $+$ indicates the commutator and the anti-commutator, respectively; nothing "super".) As we sill see, $\mathfrak{h}$ is a simple Volichenko algebra for any a, b except for $ab = 0$ when it becomes isomorphic to either the Lie algebra $\mathfrak{gl}(p+q)$ (for $b = 0$) or the Lie superalgebra $\mathfrak{gl}(p|q)$ (for $a = 0$). Since we do not have an intrinsic definition of Volichenko algebras, to show that $\mathfrak{vgl}_\mu(p|q)$ is indeed a Volichenko algebra, we have to realize it as a subalgebra of a Lie superalgebra. This is done in the proof of heading 2 of Theorem 11.2.7.

11.1.3.2. The queer Volichenko algebra $\mathfrak{vq}_\mu(n)$. This is a subalgebra of $\mathfrak{vgl}_\mu(n|n)$ consisting of matrices

$$\mathfrak{h}_{\hat{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \text{ where } A \in \mathfrak{gl}(n) \right\}; \ \mathfrak{h}_{\hat{1}} = \left\{ \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \text{ for any } B \in \mathfrak{gl}(n) \right\}. \tag{11.4}$$

11.1.4. Selected open problems. (1) If we abandon the technical hypothesis on epimorphy, do we obtain any new simple Volichenko algebras? (Conjecture: we do not.)

(2) Describe Volichenko algebras intrinsically, via polynomial identities. This seems to be a difficult problem.

(3) Classify simple Volichenko algebras related to other known simple Lie superalgebras of interest (loop and stringy types, of Lie superalgebras of "supermatrices of complex size" type).

(4) Define the notion of representation for Volichenko algebra even for the epimorphic ones. To say "a *representation of a Volichenko algebra* is a through map: the composition of an embedding $\mathfrak{h} \subset \mathfrak{g}$ into a minimal ambient and a representation $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$" is too restrictive: the adjoint representation and homomorphisms of Volichenko algebras are ruled out.

(5) The changes of variables on a superdomain constitute a supergroup (of diffeomorphisms of the domain). Infinitesimally, they constitute a Lie superalgebra of vector fields. The functor of points and formula (11.5) *without* doubly underlined terms show that the functor of points assigns to every supercommutative superalgebra Λ the Lie algebra $(\mathfrak{vect}(n|m) \otimes \Lambda)_{\bar{0}}$.

What is the Lie algebra corresponding to all terms of (11.5) and functorially depending on Λ? To start with, answer the question in the absence of even variables. Conjecturally, we will obtain a functor representable by a Volichenko algebra.

(6) See problems at the end of sec. 11.2.5.

As is clear from the list of these problems, the notion of Volichenko algebras requires further study. At the beginning we shared a cautious attitude towards Volichenko algebras. The two facts that make us more optimistic are: the set of simple Volichenko algebras (under our proviso) is discrete; besides, the examples $\mathfrak{vgl}_\mu(p|q)$ and $\mathfrak{vq}_\mu(p|q)$ are so beautiful.

Finally, observe an obvious connection of Volichenko algebras with structures that become more and more fashionable lately, see [KW].

11.2. Metabelean algebra as the algebra of "functions". Volichenko algebras as analogs of Lie algebras

11.2.1. Two flaws of supersymmetry. 1) It was the desire to broaden the notion of a group that lead physicists to supersymmetry. However, in viewing supergroups as transformations of superspaces we consider only even, "statistics-preserving", maps: inhomogeneous "statistics-mixing" maps between superalgebras are explicitly excluded and this is why and how odd parameters of supergroups appear.

On the one hand, this exclusion is justified: since we consider graded objects why should we consider transformations that preserve these objects as abstract ones but destroy the grading? It seems to be inconsistent on our part.

On the other hand, if such parity violating transformations exist, albeit purely formally, they deserve to be studied, to ignore them is physically and mathematically an artificial restriction.

We would like to broaden the notion of supergroups and superalgebras to allow for the possibility of statistics-changing maps. Soon after Berezin published his description of automorphisms of the Grassmann algebra $\Lambda(n)$ [B1] it became clear that Berezin missed inhomogeneous automorphisms, but the complete description of automorphisms was unknown for a while. In 1977, L. Makar-Limanov gave us a correct description of Aut $\Lambda(n)$, Djokovic found another proof of this fact [Dj1] (and A. Vaintrob independently rediscovered it). A. A. Kirillov rediscovered both the fact and a proof while editing [B]. Now the most comprehensible exposition of the proof, due to V. Molotkov, is in [LSoS].

In [LSoS] it was shown that Λ-points of a generic automorphism of a supercommutative superalgebra of functions $\mathcal{F}(x, \theta)$ with n even generators $x_1, \dots, x_n$ and m odd ones $\theta_1, \dots, \theta_m$ is of the form (here $p(m)$ is parity of m, i.e., either 0 or 1)

$$\begin{aligned} x_i &\mapsto \Big((f_i + \sum_k f_i^{i_1 \dots i_{2k}} \theta_{i_1} \dots \theta_{i_{2k}}) + \underline{\sum_k f_i^{i_1 \dots i_{2k+1}} \theta_{i_1} \dots \theta_{i_{2k+1}}} \Big) \underline{\underline{(1 + F_i \theta_1 \dots \theta_m p(m))}} \\ \theta_j &\mapsto \left((\sum_k g_j^{i_1 \dots i_{2k+1}} \theta_{i_1} \dots \theta_{i_{2k+1}}) + \underline{g_j + \sum_k g_j^{i_1 \dots i_{2k}} \theta_{i_1} \dots \theta_{i_{2k}}} \right) \underline{\underline{(1 + g)}} \end{aligned} \tag{11.5}$$

where the f_i, F_i, $f_i^{i_1 \dots i_{2k}}$, and also $g_j^{i_1 \dots i_{2k+1}}$ are even elements (physicists call them superfields) of $\mathcal{F}(x, \theta) \otimes \Lambda$, whereas the $f_i^{i_1 \dots i_{2k+1}}$, g_j and $g_j^{i_1 \dots i_{2k}}$ and also

g and the F_i are odd superfields. (A mathematician would say that the odd superfields — underlined once — represent the odd parameters corresponding to the Λ-points (recall the definition of the functor of points) for the background supercommutative superalgebra Λ.) Notice that one g serves all the θ_j. The twice underlined factors constitute the extra symmetry as compared with supersymmetry.

Comment. We derive from (11.5) a consequence of fundamental importance:

supersymmetry is not the most broad symmetry of supercommutative superalgebras.

Its corollary: When the number of odd variables is even, as is usually the case in modern models of Minkowski superspace, there is only one extra functional parameter, g. And, on such supermanifolds, since the wave functions of Bose particles are even and those of Fermi particles are odd, we see that

the notion of a boson is coordinate-free, whereas that of a fermion depends on coordinates.

Apart from being not the widest possible symmetry, sypersymmetry has another quite unexpected flaw:

2) The category of supercommutative superalgebras is **not** closed with respect to complexification. It certainly is if $\mathbb{C}$ is understood naively, as a purely even space. Declaring $\sqrt{-1}$ to be odd we make $\mathbb{C}$ into a NONsupercommutative superalgebra.

This associative superalgebra over $\mathbb{R}$ is denoted by $Q(1;\mathbb{R})$. The complex structure given by an odd operator gives rise to a "queer" superanalogue of the matrix algebra, $Q(n;\mathbb{K})$ over any field $\mathbb{K}$. Its Lie version is $\mathfrak{q}(n;\mathbb{K}) := (Q(n;\mathbb{K}))_L$, the projectivization of its queertraceless Lie subsuperalgebra (first discovered by Gell-Mann, Mitchel and Radicatti, cf. [CNS]) is one of main examples of simple Lie superalgebras.

For *super*algebras over $\mathbb{C}$, Schur's Lemma and Burnside's theorem have two possibilities, not one as for algebras, and $Q(1)$ corresponds to one of them. Besides, an infinite dimensional representation of $Q(1)$ is crucial in A. Connes' noncommutative differential geometry. In short, this $Q(1)$ — the non-supercommutative complex structure — is an important structure.

How to modify definition of supermanifold to remedy the flaws?. Conjecturally, the answer is to consider arbitrary, not necessarily homogeneous subalgebras and quotients of supercommutative superalgebras. These are, clearly, metabelian algebras. But how to describe arbitrary metabelian algebras? In 1975, D.L. discussed this with V. Kac and V. Kac conjectured (see [LSoS]) that considering metabelian algebras we do not digress far from supercommutative superalgebras, namely, every metabelian algebra is a subalgebra of a supercommutative superalgebra. Volichenko had read [LSoS] and observed that "the [Kac's] conjecture is a well-known fact of the theory of varieties of associative algebras. From the context of [Le0], however, it is clear that the actual problem is, first of all, how to describe the variety of not necessarily homogeneous subalgebras which a priori can be smaller than the variety $\mathcal{G}$ of supercommutative superalgebras". His answer constituted Volichenko's Theorem M. Namely, Volichenko proved that *every metabelian algebra admits an embedding into a universal supercommutative superalgebra* and developed an analogue of Taylor series expansion.

Therefore, the most broad notion of morphisms of supercommutative superalgebras should only preserve their metabelianness but not parity. (Since $\mathbb{C}$, however understood, is metabelian, we get a category of algebras closed with respect to all algebra morphisms and complexifications.)

11.2.2. Problem. In mathematics and physics, spaces are needed almost exclusively to integrate over them. In problems where integration is not involved, we need sheaves of sections of various bundles over the spaces rather than the spaces themselves. Gauge fields, Lagrangeans, etc. are all sections of coherent sheaves, corresponding to sections of vector bundles. Now, more than 30 years after the definition of the scheme of a metabelian algebra (metavariety or metaspace) had been delivered at A. Kirillov's seminar ([Le0]), there is still no accepted definition of nice ("morally coherent" as Manin says) sheaves over such a scheme even for superspaces (for a discussion see [Bu]). As to candidates for such sheaves see Rosenberg's books on noncommutative geometry [**?**], nos. 25, 26, 31 and [Ro]) and § 9 in [Bu]. This § 9 is, besides all, a possible step towards "compactification in odd directions".

Until Volichenko proved his theorems, it was unclear how to *work* with metabelian algebras: are there any analogues of differential equations, of integral, in other words, is there any "real life" on metaspaces? Thanks to Volichenko, we can consider pairs (a metabelian algebra, its ambient supercommutative superalgebra) and corresponding (when we consider these algebras as algebras of functions) projections

$$\text{“superspace} \longrightarrow \text{metaspace”.}$$

Characterize metabelian algebras which are *quotients* of supercommutative superalgebras: in this case the corresponding metaspace can be embedded into the superspace and we can consider the induced structures (Lagrangeans, various differential equations, etc.).

But even if we would have been totally unable to work with metaspaces which are not superspaces, it is manifestly useful to consider superspaces as metaspaces. In so doing, we retain all the paraphernalia of the differential geometry for sure, and in addition get more transformations of the same entities.

For example, it is desirable to use the formula (first applied by Arnowitt, Coleman and Nath)

$$\mathrm{Ber}\ \exp\ X = \exp \mathrm{str}\ X \tag{11.6}$$

which extends the domain of the berezinian (superdeterminant) to inhomogeneous matrices X. Then we can consider the additional inhomogeneous transformations, like the ones described in (11.5). All supersymmetric Lagrangeans admit metasymmetry wider than supersymmetry.

11.2.3. Motivation of the notions of Volichenko algebra as an analogue of Lie algebra. A description of Volichenko algebras. It seemed natural to get for Lie superalgebras (whose elements are derivations of the algebras (of functions)) a result similar to the above Volichenko's Theorem M (concerning the algebras of functions themselves), i.e., describe arbitrary subalgebras of Lie superalgebras. Shortly before his untimely death, I. Volichenko (1955–88) announced such a description (see "Theorem" V below).

In his memory then, a *Volichenko algebra* is an inhomogeneous subalgebra $\mathfrak{h}$ of a Lie superalgebra $\mathfrak{g}$. The adjective "Lie" in front of a (super)algebra indicates that the algebra is not associative, likewise the adjective "Volichenko" reminds that the algebra is neither associative nor should it satisfy Jacobi or super-Jacobi identities. Thus, a Volichenko algebra $\mathfrak{h}$ is an inhomogeneous subspace of a Lie superalgebra $\mathfrak{g}$ closed with respect to the superbracket in $\mathfrak{g}$. How to describe $\mathfrak{h}$ by identities, i.e., in inner terms, without appealing to any ambient? Volichenko's "Theorem" V formulates an answer:

Let A be an algebra with multiplication denoted by juxtaposition. Define the Jordan elements $a\circ b := ab+ba$ and Jacobi elements

$$J(a,b,c) := a(bc)+c(ab)+b(ca) \quad \text{for any } a,b,c\in A. \tag{11.7}$$

Theorem (I. Volichenko, 1987)**.** *Suppose that*

(a) *A is Lie admissible, i.e., A is a Lie algebra with respect to the new product defined by the bracket* (not superbracket) $[a,b]=ab-ba$;

(b) *the subalgebra $A^{(JJ)}$ generated by all Jordan and Jacobi elements belongs to the anticenter of A, in other words*

$$ax+xa=0 \textit{ for any } a\in A^{(JJ)},\ x\in A; \tag{11.8}$$

(c) *$a(xy)=(ax)y+x(ay)$ for any $a\in A^{(JJ)}$, $x,y\in A$.*

Then

(1) *Any (not necessarily homogeneous) subalgebra $\mathfrak{h}$ of a Lie superalgebra $\mathfrak{g}$ satisfies the above conditions* (a) — (c).

(2) *If A satisfies* (a) — (c)*, then there exists a Lie superalgebra $SLie\ (A)$ and A is a subsuperalgebra (closed with respect to the superbracket) of $SLie\ (A)$.*

Heading (1) is subject to a direct verification.

Clearly, the parts of conditions (b) and (c) which involve Jordan (resp. Jacobi) elements replace the superanti-commutativity (resp. Jacobi identity). Condition (a) ensures that A is closed in $SLie(A)$ with respect to the bracket in the ambient.

Discussion. If true, Volichenko's "Theorem" V would have disproved a pessimistic conjecture of V. Markov cited in [DL2]: *the minimal set of polynomial identities that single out inhomogeneous subalgebras of Lie superalgebras is infinite*. Volichenko's scrap papers were destroyed after his death and no hint of his ideas remains.

Since the intrinsic definition of Volichenko algebras was not needed in our quest for s i m p l e Volichenko algebras we did not worry about Volichenko's theorem V. Several researchers tried to refute it.

A. Baranov showed [Ba] that Volichenko's "Theorem" V is wrong as stated (and that is why it is in quotation marks): one should add at least one more relation of degree 5. First, following Volichenko, Baranov introduced, instead of $J(a,b,c)$, more convenient linear combinations of the Jacobi elements

$$j(a,b,c) = [a,b\circ c]+[b,c\circ a]+[c,a\circ b] \text{ for any } a,b,c\in A. \tag{11.9}$$

Then Baranov rewrote identities (a)–(c) in the following equivalent but more transparent form (i)–(v):

(i) $[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0$;
(ii) $a\circ b\circ c=0$;
(iii) $j(a,b,c)\circ d=0$;
(iv) $[a\circ b,c\circ d]=[a\circ b,c]\circ d+[a\circ b,d]\circ c$;
(v) $[j(a,b,c),c\circ d]=[j(a,b,c),c]\circ d+[j(a,b,c),d]\circ c$.

Baranov's new degree 5 identity independent of (i) – (v) is somewhat implicit; it involves 49 monomials and no lucid expression for it is found yet.

Therefore, to describe Volichenko algebras in inner terms remains, for more than 30 years now, an *open problem*. I wonder if there exists a generating function F of an infinite number of identities satisfied by the algebras of the PI variety Markov spoke about, and in terms of F it will be more convenient to study these identities.

I. Volichenko did not investigate under which conditions a finite dimensional Volichenko algebra A can be embedded into a finite dimensional Lie superalgebra $\mathfrak{g}$, to find this out is another *open problem*.

11.2.4. On simplicity of Volichenko algebras. As we will see, the notion of Volichenko algebra is a totally new type of deformation of the usual Lie algebra. It also generalizes the notion of a Lie superalgebra in a sence that the Lie superalgebras are $\mathbb{Z}/2$-graded algebras (i.e., they are of the form $\mathfrak{g}=\bigoplus\limits_{i=\bar 0,\bar 1}\mathfrak{g}_i$ such that $[\mathfrak{g}_i,\mathfrak{g}_j]\subset\mathfrak{g}_{i+j}$) whereas Volichenko algebras are only 2-step filtered ones (i.e., they are of the form $\mathfrak{h}=\bigoplus\limits_{i=\hat 0,\hat 1}\mathfrak{h}_i$ as spaces and $\mathfrak{h}_{\hat 0}$ is a subalgebra. There are, however, several series of examples when Volichenko algebras are $\mathbb{Z}/2$-graded (as $\mathfrak{vgl}_\mu(p|q)$).

Hereafter $\mathfrak{g}$ is a Lie superalgebra over $\mathbb{C}$. Let $\mathfrak{h}\subset\mathfrak{g}$ be a Volichenko algebra, i.e., a subspace which is not a subsuperspace and closed with respect to the bracket in $\mathfrak{g}$.

A Volichenko algebra is said to be *simple* if it has no two-sided ideals and its dimension is $\neq 1$.

Remark. In 1989, when we knew even less about Volichenko algebras than now, P. Deligne argued that for algebras such as Volichenko ones, modules over which have no natural two-sided structure, the above definition seem to be too restricted: one should define simplicity by requiring the absence of *one-sided* ideals. As it turns out, neither of the simple Volichenko algebras we list in what follow has one-sided ideals either, so we will stick to the above (and at first glance preliminary) definition: it is easier to work with.

Hypothesis. For any simple Volichenko algebra $\mathfrak{h}$ defined as a subalgebra $\mathfrak{h}$ of a Lie superalgebra $\mathfrak{H}$, there exists a simple (modulo center) Lie subsuperalgebra $\mathfrak{g} \subset \mathfrak{H}$ such that $\mathfrak{h}$ can be embedded into $\mathfrak{g}$.

Comment. I can not recover the proof of this statement, initially formulated as a Lemma. Therefore, to prove it is an *open problem.*

So, we can (and will) assume that the ambient $\mathfrak{g}$ of a simple Volichenko algebra is simple. In what follows we will see that for a simple Volichenko algebra $\mathfrak{h}$ its simple ambient Lie superalgebra $\mathfrak{g}$ is unique.

11.2.5. The "epimorphy" condition. Denote by $p_i\colon \mathfrak{g} \longrightarrow \mathfrak{g}_i$ the projections to homogeneous components. A Volichenko algebra $\mathfrak{h} \subset \mathfrak{g}$ will be called *epimorphic* if $p_0(\mathfrak{h}) = \mathfrak{g}_{\bar 0}$. Not every Volichenko subalgebra is epimorphic: for example, the two extremes, Volichenko algebras with the zero bracket and free Volichenko algebras, are not epimorphic, generally. All simple finite dimensional Volichenko algebras that we know are, however, epimorphic.

Hypothesis. Every simple Volichenko algebra is epimorphic.

A case study of various simple Lie superalgebras of low dimensions reveals that they do not contain non-epimorphic simple Volichenko algebra. Still, we can not prove this hypothesis but will adopt it for it looks very natural at the moment.

Lemma. *Let $\mathfrak{h} \subset \mathfrak{g}$ be an epimorphic Volichenko algebra and $f\colon \mathfrak{g}_{\bar 0} \longrightarrow \mathfrak{g}_{\bar 1}$ the linear map that determines $\mathfrak{h}$, i.e., such that $\mathfrak{h} = \mathfrak{h}_f$, where*

$$\mathfrak{h}_f := \{a + f(a) \mid a \in \mathfrak{g}_{\bar 0}\}. \tag{11.10}$$

Then

1) *f is a 1-cocycle from $C^1(\mathfrak{g}_{\bar 0}; \mathfrak{g}_{\bar 1})$;*

2) *f can be uniquely extended to a derivation of $\mathfrak{g}$ (also denoted by f) such that $f(f(\mathfrak{g}_{\bar 0})) = 0$.*

Example. Recall, that an odd element such that $[x, x] = 0$ is called a *homologic* one. Let $x \in \mathfrak{g}_{\bar 1}$ be such that

$$[x, x] \in C(\mathfrak{g}), \tag{11.11}$$

where $C(\mathfrak{g})$ is the center of $\mathfrak{g}$. Clearly, the map $f = \mathrm{ad}_x$ satisfies Lemma 11.2.5 if x satisfies (11.11), i.e., is homologic modulo center.

A homologic modulo center element x will be said to *ensure non-triviality*

$$\text{(of the algebra } \mathfrak{h}_{\mathrm{ad}_x} = \{a + [a, x] \mid a \in \mathfrak{g}_{\bar 0}\}) \tag{11.12}$$

if

$$[[\mathfrak{g}_{\bar 0}, x], [\mathfrak{g}_{\bar 0}, x]] \neq 0, \tag{11.13}$$

i.e., if there exist elements $a, b \in \mathfrak{g}_{\bar 0}$ such that

$$[[a, x], [b, x]] \neq 0. \tag{11.14}$$

The meaning of this notion is as follows. Let $a, b \in \mathfrak{h}$, let $a = a_0 + a_1$ and $b = b_0 + b_1$, where $a_1 = [a_0, x]$, $b_1 = [b_0, x]$ for some $x \in \mathfrak{g}_{\bar 1}$. Notice that, for any x satisfying (11.11) we have

$$[[a_1, b_1], x] = 0. \tag{11.15}$$

If (11.14) holds, we have

$$\begin{aligned}&[a, b] = [a_0, b_0] + [a_1, b_1] + [a_0, b_1] + [a_1, b_0] =\\ &([a_0, b_0] + [a_1, b_1]) + [[a_0, b_0], x].\end{aligned} \tag{11.16}$$

It follows from (11.15) and (11.16) that if x is homologic modulo center, then $\mathfrak{h}$ is closed under the bracket of $\mathfrak{g}$; if this x does not ensure non-triviality, i.e., if $[a_1, b_1] = 0$ for all $a, b \in \mathfrak{h}$, then $\mathfrak{h}_{\mathrm{ad}_x}$ is just isomorphic to $\mathfrak{g}_{\bar 0}$.

In other words, an epimorphic Volichenko algebra is a deformation of the Lie algebra $\mathfrak{g}_{\bar 0}$ in a totally new sense: not in the class of Lie algebras, nor in in the class of Lie superalgebras but in the class of Volichenko algebras whose intrinsic description is to be given. (To see that an epimorphic Volichenko algebra $\mathfrak{h}$ is a result of a deformation of sorts, multiply the element x that determines $\mathfrak{h}$ by an even parameter, t. If t were odd, we would have obtained a deformation of $\mathfrak{g}_{\bar 0}$ in the class of Lie superalgebras.)

Exercise. Show why it is impossible to consider any other (inconsistent with parity) $\mathbb{Z}/2$-grading (call it deg) of $\mathfrak{g}$ and deform in a similar way the Lie subsuperalgebra of elements of degree 0 with respect to deg. (Hint. Use eq. (11.16).)

Any epimorphic Volichenko algebra $\mathfrak{h}_{\mathrm{ad}_x}$ is naturally 2-step filtered: the Lie algebra

$$(\mathfrak{h}_{\mathrm{ad}_x})_{\hat 0} := \mathrm{Ann}\,(x) = \{a \in \mathfrak{g}_{\bar 0} \mid [x, a] = 0\} \tag{11.17}$$

is a subalgebra.

Problems. 1) We have a sandwich: between Hopf (super)algebras, $U((\mathfrak{h}_{\mathrm{ad}_x})_{\hat 0})$ and $U(\mathfrak{g})$, a non-Hopf algebra, $U(\mathfrak{h}_{\mathrm{ad}_x})$ (the subalgebra of $U(\mathfrak{g})$ generated by $\mathfrak{h}$), is squeezed. How to measure its "non-Hopfness"? It seems to be an interesting invariant.

2) It is primarily real algebras and their representations that arise in applications. So what are these notions for Volichenko algebras?

11.2.6. For a vector field $D = \sum f_r \partial_r$ from $\mathfrak{vect}(m|n) = \mathfrak{der}\mathbb{C}[x, \theta]$, consider the nonstandard (if $m \neq 0$) grading induced by the grading of $\mathbb{C}[x, \theta]$ for which

$$\deg\ x_i = 0 \text{ and } \deg\ \theta_j = 1 \text{ for all } i \text{ and } j. \tag{11.18}$$

Define the *inverse order* inv.ord(f_r) of $f_r \in \mathbb{C}[x, \theta]$ with respect to (11.18) as the least of the degrees of monomials in the power series expansion of f_r.

Define the *inverse order* of $D \in \mathfrak{vect}(m|n)$ as the least of inv.ord(f_r).

Lemma. 1) *Let $\mathfrak{h} \subset \mathfrak{g}$ be a simple vectorial Volichenko subalgebra, i.e., a subalgebra of a simple vectorial Lie superalgebra. Then in the representation $\mathfrak{h} = \{a + f(a) \mid a \in \mathfrak{g}_{\bar 0}\}$ we have $f = \mathrm{ad}_x$, where x is homologic and inv.ord$(x) = -1$.*

2) *Table 11.2.9 contains all, up to* (Aut G_0)*-action, homologic elements of the minimal inverse order in the vectorial Lie superalgebras. (In particular, for $\widetilde{\mathfrak{svect}}(2n)$ there are none.)*

11.2.7. Theorem (Main Theorem)**.** *A simple epimorphic finite dimensional Volichenko algebra $\mathfrak{h} \subset \mathfrak{g}$, where $\mathfrak{g} \neq \mathfrak{psq}(n)$, can be only one of the form $\mathfrak{h}_{\mathrm{ad}_x}$, where:*

1) *x is an element from Table 11.2.9 or an element from Table 11.2.8 satisfying the condition ensuring non-triviality};*

2) *or a simple Volichenko subalgebra of $\mathfrak{g} = \mathfrak{psq}(n)$, there is one additional possibility beside the ones listed in case 1): namely, $x =$* antidiag (X, X)*, where*

$$X = \text{diag } (a1_p, b1_{n-p}) \textit{ with } ap + b(n-p) = 0. \tag{11.19}$$

11.2.8. Table. Homologic elements x and the condition when x ensures non-triviality of $\mathfrak{h}_{\mathrm{ad}_x}$ for Lie superalgebras $\mathfrak{g}$.

$\mathfrak{g}$	x	when x ensures non-triviality
$\mathfrak{sl}(m\|n)$, $m \leq n$	$x_q^p := \begin{pmatrix} 0 & 1^{(m,n,p)} \\ 1_{(m,n,q)} & 0 \end{pmatrix}$	$p, q > 0$, $p + q \leq m$
$\mathfrak{psl}(n\|n)$	same as for $\mathfrak{sl}(n\|n)$ and also antidiag$(1_n, 1_n)$	as above
$\mathfrak{osp}(2m\|2n)$	the image of the above $x_p^p \in \mathfrak{sl}(m\|n) \subset \mathfrak{osp}(2m\|2n)$,	$p > 0$, $2p \leq \min(m, n)$
$\mathfrak{osp}(2m+1\|2n)$	the image of the above x_p^p under the embedding $\mathfrak{osp}(2m\|2n) \subset \mathfrak{osp}(2m+1\|2n)$	as above
$\mathfrak{spe}(n)$	$y_q^p = \text{antidiag}(1^{(n,n,p)}, J_{(n,2q)})$	$p, q > 0$, $p + 2q \leq n$
$\mathfrak{psq}(n)$	$x_k = \text{antidiag}(X, X)$, where $X = \text{diag}(J_2(0), \dots, J_2(0), 0, \dots, 0)$ with k-many copies of$J_2(0) := \text{antidiag } (1, 0)$	$k > 0$, $2k \leq n$
$\mathfrak{ag}(2)$, $\mathfrak{ab}(3)$, $\mathfrak{osp}(4\|2; \alpha)$	the root vector corresponding to any isotropic (odd) simple root	never

In the following Table we have listed not only homologic elements — that is to say Volichenko subalgebras — of finite dimensional simple Lie superalgebras of vector fields but also simple Volichenko subalgebras of all non-exceptional simple Lie superalgebras of vector fields, see the list in [Le3], [?].

11.2.9. Table. Homologic elements x of minimal inverse order in simple vectorial Lie superalgebras $\mathfrak{g}$.

$\mathfrak{g}$	x
$\mathfrak{vect}(m\|n)$, where $mn \neq 0$, $n > 1$ or $m = 0$, $n > 2$; $\mathfrak{svect}(m\|n)$ for $m, n \neq 1$; $\mathfrak{svect}'(1\|n)$, $\mathfrak{svect}(0\|2n)$, $\mathfrak{le}(n)$, $\mathfrak{sle}'(n)$ for $n > 1$	$\dfrac{\partial}{\partial\theta_1}$
$\mathfrak{k}(2m+1\|n)$, where $n > 1$	K_{θ_1}
$\mathfrak{h}(2m\|n)$, where $mn \neq 0$, $n > 1$, and $\mathfrak{h}'(n)$ for $n > 3$	$\dfrac{\partial}{\partial\theta_1} = H_{\theta_1}$ and $(i = \sqrt{-1})$ $\dfrac{\partial}{\partial\theta_1} + i\dfrac{\partial}{\partial\theta_2} = H_{\theta_1 + i\theta_2}$
$\mathfrak{m}(n)$ for $n > 1$; $\mathfrak{b}_\lambda(n)$ for $\lambda \neq 0$, $n > 1$	M_1 and (only for $\mathfrak{b}_\lambda(2k)$) $M_{1+\theta_1 \dots \theta_{2k}}$
$\mathfrak{b}(n), n > 1$	Le_{q_1}
$\mathfrak{svect}(0\|2n+1)$, $n > 1$	$\dfrac{\partial}{\partial\theta_1}$ and $(1 + t\theta_2 \dots \theta_{2n+1})\dfrac{\partial}{\partial\theta_1}$, where $t \in \mathbb{C}$

Now, the final touch:

11.2.10. Proposition. *Simple epimorphic Volichenko algebras from Tables 11.2.8 and 11.2.9 have no one-sided ideals.*

Exercise. Prove this.

11.3. An explicit description of some simple Volichenko algebras (U. Iyer)

Here we explicitly describe most of the above epimorphic simple Volichenko algebras $\mathfrak{h}$ together with

(a) its simple ambient Lie superalgebra $\mathfrak{g}$ and an element $x \in \mathfrak{g}_{\bar 1}$ that determines it,

(b) the filtration $\mathfrak{h}_{\hat 0} = \text{Ann } (x) \subset \mathfrak{h}$,

(c) the quotient $\mathfrak{h}_{\hat 1} = \mathfrak{h}/\mathfrak{h}_{\hat 0}$ and the multiplication in $\mathfrak{h}$.

The most interesting cases are the ones where $\mathfrak{h}$ is $\mathbb{Z}/2$-graded. We start with them; the most complicated descriptions are left (so far) to the reader.

The $\mathbb{Z}/2$-graded cases.

$\underline{\mathfrak{vgl}_\mu(p|q)}$, the *general linear Volichenko algebra*, is described intrinsically in Introduction and as a subalgebra in a Lie superalgebra in sec. 11.4.9

$\underline{\mathfrak{vq}_\mu(n)}$, the *queer Volichenko algebra*; its description follows from those of $\mathfrak{vgl}_\mu(p|q)$.

$\underline{\mathfrak{vpsl}(n|n)}$, the *projective special linear Volichenko algebra*. The ambient is $\mathfrak{g} = \mathfrak{psl}(n|n)$; for $x = \Pi_{2n}$, we have

$$\mathfrak{h}_{\hat 0} = \left\{\begin{pmatrix} A & 0\\ 0 & A\end{pmatrix}\right\};\ \ \mathfrak{h}_{\hat 1} = \left\{\begin{pmatrix} B & 2\overline{B}\\ -2\overline{B} & -B\end{pmatrix}\right\}. \tag{11.20}$$

As is easy to see, $\mathfrak{h}_{\hat 1}$ is the adjoint $\mathfrak{h}_{\hat 0}$-module and the bracket $\mathfrak{h}_{\hat 1}\times\mathfrak{h}_{\hat 1} \longrightarrow \mathfrak{h}_{\hat 0}$ is given by the formula

$$\left[\begin{pmatrix} A & 2\overline{A}\\ -2\overline{A} & -A\end{pmatrix}, \begin{pmatrix} B & 2\overline{B}\\ -2\overline{B} & -B\end{pmatrix}\right] = \begin{pmatrix} [A,B]-4[\overline{A},\overline{B}]_+ & 0\\ 0 & [A,B]-4[\overline{A},\overline{B}]_+\end{pmatrix}. \tag{11.21}$$

There are two Hamiltonian Volichenko algebras:

$\underline{\mathfrak{vh}(2n|m)}$, the *main Hamiltonian Volichenko algebra* for which $x = H_{\theta_1}$. Since, $\{f,\theta_1\}_{P.b.} = -\dfrac{\partial f}{\partial\theta_1}$, we see that

$$\mathfrak{vh}(2n|m) = \{f_0 + \{f_0,\theta_1\}_{P.b.} \mid f_0 \in \mathbb{C}[p,q,\theta]_{\bar 0}\}.$$

If we let $\theta = \{\theta_1, \bar\theta\}$, then we can nicely express the “even” and “odd” elements of the Volichenko algebra as follows:

$$(\mathfrak{vh}(2n|m))_{\hat 0} = \mathbb{C}[p,q,\bar\theta]_{\bar 0},$$

and

$$(\mathfrak{vh}(2n|m))_{\hat 1} = \{f_1\theta_1 + f_1 \mid f_1 \in \mathbb{C}[p,q,\bar\theta]_{\bar 1}\}.$$

Explicit calculation of $\{\cdot,\cdot\}_{P.b.}\colon \mathfrak{v}_{\hat 1}\otimes\mathfrak{v}_{\hat 0}\to\mathfrak{v}_{\hat 1}$:

$$\{f_1\theta_1 + f_1,\ g_0\}_{P.b.} = \{f_1, g_0\}_{P.b.}\theta_1 + \{f_1, g_0\}_{P.b.}$$

Explicit calculation of $\{\cdot,\cdot\}_{P.b.}\colon \mathfrak{v}_{\hat 1}\otimes\mathfrak{v}_{\hat 1}\to\mathfrak{v}_{\hat 0}$:

$$\{f_1\theta_1 + f_1,\ g_1\theta_1 + g_1\}_{P.b.} = -f_1g_1 + \{f_1,\ g_1\}_{P.b.}.$$

$\underline{\mathfrak{vm}(n)}$, the *pericontact Volichenko* algebra. For it, $x = M_1$ and

$$\{f,\ 1\}_{m.b.} = -2\frac{\partial f}{\partial\tau}.$$

Hence,

$$\mathfrak{vm}(n) = \{M_g \in \mathfrak{m}(n) \mid g = f - 2\frac{\partial f}{\partial\tau},\ \text{where } p(f) = \bar 1\}.$$

The filtration on $\mathfrak{vm}(n)$ is given by

$$\begin{aligned}(\mathfrak{vm}(n))_{\hat 0} &= \{M_g \mid g \in \mathbb{C}[q,\theta]_{\bar 1}\},\\ (\mathfrak{vm}(n))_{\hat 1} &= \{M_g \mid g = f - 2\frac{\partial f}{\partial\tau},\ \ \text{where } f \in \mathbb{C}[q,\theta]_{\bar 0}\tau\}.\end{aligned}$$

To calculate $\{\cdot,\cdot\}_{m.b.}$, we introduce several notations. Let

$$\begin{aligned} &f = f_1 + f_0\tau,\ g = g_1 + g_0\tau \in \mathfrak{v}_{\hat 1},\ \text{where } p(f_i) = p(g_i) = \bar 1,\\ &\frac{\partial f_i}{\partial\tau} = \frac{\partial g_i}{\partial\tau} = \bar 0.\end{aligned} \tag{11.22}$$

Explicit calculation of $\{\cdot,\cdot\}_{m.b.}\colon \mathfrak{v}_{\hat 1}\otimes\mathfrak{v}_{\hat 0}\to\mathfrak{v}_{\hat 1}$:

$$\begin{aligned}&\{f_1 + f_0\tau - 2f_0, g_1\}_{M.b} =\\ &-\{f_1,\ g_1\}_{B.b.} - f_0(2-E)(g_1) - \{f_0, g_1\}_{B.b.}\tau + 2\{f_0,\ g_1\}_{B.b.}\end{aligned}$$

Explicit calculation of $\{\cdot,\cdot\}_{m.b.}\colon \mathfrak{v}_{\hat 1}\otimes\mathfrak{v}_{\hat 1}\to\mathfrak{v}$:

$$\begin{aligned}&\{f_1 + f_0\tau - 2f_0, g_1 + g_0\tau - 2g_0\}_{m.b.} =\\ &\quad(-\{f_1,\ g_1\}_{B.b.} - 4\{f_0,\ g_0\}_{B.b.} +\\ &\quad 2(f_1g_0 - f_0g_1) - (E(f_1)g_0 + f_0E(g_1))) +\\ &\quad(-\{f_1,\ g_0\}_{B.b.} - \{f_0,\ g_1\}_{B.b.} - (E(f_0)g_0 - f_0E(g_0)))\,\tau +\\ &\quad 2\,(\{f_1,\ g_0\}_{B.b.} + \{f_0,\ g_1\}_{B.b.} + (E(f_0)g_0 - f_0E(g_0)))\,.\end{aligned}$$

The $\mathbb{Z}/2$-filtered cases.

$\underline{\mathfrak{vh}^i(2n|m)}$, the *other Hamiltonian Volichenko* algebra. For it, $x = H_{\theta_1 + i\theta_2}$. For $f \in \mathbb{C}[p,q,\theta]_0$, we have

$$\{f,\ x\}_{P.b.} = -\left(\frac{\partial f}{\partial\theta_1} + i\frac{\partial f}{\partial\theta_2}\right).$$

Let $D = \dfrac{\partial}{\partial\theta_1} + i\dfrac{\partial}{\partial\theta_2}$. Then

$$\begin{aligned}(\mathfrak{vh}^i(2n|m))_{\hat 0} &= \{f \in \mathbb{C}[p,q,\xi]_{\bar 0} \mid D(f) = 0\},\\ (\mathfrak{vh}^i(2n|m))_{\hat 1} &= \{f - D(f) \mid f \in \mathbb{C}[p,q,\xi]_{\bar 0},\ D(f) \neq 0\}.\end{aligned}$$

Explicit calculation of $\{\cdot,\cdot\}_{P.b.}\colon \mathfrak{v}_{\hat 1}\otimes\mathfrak{v}_{\hat 0}\to\mathfrak{v}$:

$$\{f - D(f),\ g\}_{P.b.} = \{f, g\}_{P.b.} - D(\{f,\ g\}_{P.b.}).$$

Explicit calculation of $\{\cdot,\cdot\}_{P.b.}\colon \mathfrak{v}_{\hat 1}\otimes\mathfrak{v}_{\hat 1}\to\mathfrak{v}$:

$$\{f - D(f),\ g - D(g)\}_{P.b.} = \{f,\ g\}_{P.b.} - D(\{f,\ g\}_{P.b.}) + \{D(f),\ D(g)\}_{P.b.}$$

$\underline{\mathfrak{vk}(2n+1|m)}$, the *contact Volichenko* algebra. For it, $x = K_{\theta_1}$ and for $f \in \mathbb{C}[t,p,q,\theta]_{\bar 0}$, we have

$$\{f,\ \theta_1\}_{K.b.} = -\frac{\partial f}{\partial t}\theta_1 + \frac{\partial f}{\partial\theta_1}.$$

Let $\theta = \{\theta_1, \bar\theta\}$. Then

$$(\mathfrak{vk}(2m+1|n))_{\hat 0} = \{K_f \mid f \in \mathbb{C}[p, q, \bar\theta]_0\},$$

Let $A = \mathbb{C}[p, q, \bar\theta]$. Then

$$(\mathfrak{vk}(2m+1|n))_{\hat 1} = \{K_f \mid f + \{f,\ \theta_1\}_{K.b.},\ f \in A_{\bar 0} \otimes \mathbb{C}[t]_{n\geq 1} \oplus A_{\bar 1} \otimes \mathbb{C}[t]\theta_1\}.$$

Explicit calculation of $\{\cdot,\cdot\}_{K.b.}\colon \mathfrak{v}_{\hat 1} \otimes \mathfrak{v}_{\hat 0} \to \mathfrak{v}$.

$$\begin{gathered}\{f - \frac{\partial f}{\partial t}\theta_1 + \frac{\partial f}{\partial \theta_1},\ g\}_{K.b.} = \{f,\ g\}_{K.b.} - \{\frac{\partial f}{\partial t}\theta_1,\ g\}_{K.b.} + \{\frac{\partial f}{\partial \theta_1},\ g\}_{K.b.} \\ = \left(-\frac{\partial f}{\partial t}(2-E)(g) - \{f,\ g\}_{P.b.}\right) - \frac{\partial}{\partial t}\left(-\frac{\partial f}{\partial t}(2-E)(g) - \{f,\ g\}_{P.b.}\right)\theta_1 + \\ \frac{\partial}{\partial \theta_1}\left(-\frac{\partial f}{\partial t}(2-E)(g) - \{f, g\}_{P.b.}\right).\end{gathered} \tag{11.23}$$

Explicit calculation of $\{\cdot,\cdot\}_{K.b.}\colon \mathfrak{v}_{\hat 1} \times \mathfrak{v}_{\hat 1} \to \mathfrak{v}$:

$$\{f - \frac{\partial f}{\partial t}\theta_1 + \frac{\partial f}{\partial \theta_1},\ g - \frac{\partial g}{\partial t}\theta_1 + \frac{\partial g}{\partial \theta_1}\}_{K.b.} = F - \frac{\partial F}{\partial t}\theta_1 + \frac{\partial F}{\partial \theta_1}, \tag{11.24}$$

where

$$\begin{gathered}F = \{f,\ g\}_{K.b.} - \frac{\partial f}{\partial t}\frac{\partial g}{\partial t} = \\ (2-E)\left(\frac{\partial f}{\partial \theta_1}\right)\frac{\partial^2 g}{\partial t \partial \theta_1} - (2-E)\left(\frac{\partial f}{\partial \theta_1}\right)\frac{\partial^2 g}{\partial\ t\partial\theta_1} - \{\frac{\partial f}{\partial \theta_1},\ \frac{\partial g}{\partial \theta_1}\}_{P.b.} \\ + \left(-E\left(\frac{\partial f}{\partial t}\right)\frac{\partial^2 g}{\partial t\partial\theta_1} - \frac{\partial^2 f}{\partial t\partial\theta_1}E\left(\frac{\partial g}{\partial t}\right)\right)\theta_1 \\ - \left(\frac{\partial^2 f}{\partial t^2}(2-E)\left(\frac{\partial g}{\partial \theta_1}\right) + (2-E)\left(\frac{\partial f}{\partial \theta_1}\right)\frac{\partial^2 g}{\partial t^2}\right)\theta_1 \\ + \left(\{\frac{\partial f}{\partial \theta_1},\ \frac{\partial g}{\partial t}\}_{P.b.} - \{\frac{\partial f}{\partial t},\ \frac{\partial g}{\partial \theta_1}\}_{P.b.}\right)\theta_1.\end{gathered} \tag{11.25}$$

$\underline{\mathfrak{vb}(n)}$, the *Buttin Volichenko algebra*. For it, $x = M_{q_1}$ and

$$\mathfrak{vb}(n) = \{f + \{f,\ q_1\}_{m.b.} \mid f \in \mathbb{C}[q, \theta]_{\bar 1}\}.$$

Set $\theta = \{\theta_1, \bar\theta\}$. Since $\{f, q_1\}_{m.b.} = \dfrac{\partial f}{\partial \theta_1}$, it follows that

$$(\mathfrak{vb}(n))_{\hat 0} = \{f \mid f \in \mathbb{C}[q, \bar\theta]_{\bar 1}\}$$

We then have

$$\begin{aligned}(\mathfrak{vb}(n))_{\hat 1} &= \{f + \frac{\partial f}{\partial \theta_1} \mid f \in \mathbb{C}[q, \bar\theta]_{\bar 0}\theta_1\} \\ &= \{f_0\theta_1 + f_0 \mid f_0 \in \mathbb{C}[q, \bar\theta]_{\bar 0}\}.\end{aligned}$$

Explicit calculation of $\{\cdot,\cdot\}_{m.b.}\colon \mathfrak{v}_{\hat 1} \otimes \mathfrak{v}_{\hat 0} \to \mathfrak{v}$:

$$\{f_0\theta_1 + f_0,\ g_1\}_{m.b.} = -\{f_0, g_1\}_{B.b.}\theta_1 + f_0\frac{\partial g_1}{\partial q_1} - \{f_0,\ g_1\}_{B.b.}.$$

Explicit calculation of $\{\cdot,\cdot\}_{m.b.}\colon \mathfrak{v}_{\hat 1} \otimes \mathfrak{v}_{\hat 1} \to \mathfrak{v}$:

$$\begin{gathered}\{f_0\theta_1 + f_0,\ g_0\theta_1 + g_0\}_{m.b.} \\ = \left(f_0\frac{\partial g_0}{\partial q_1} - \frac{\partial f_0}{\partial q_1}g_0\right)\theta_1 + \left(f_0\frac{\partial g_0}{\partial q_1} - \frac{\partial f_0}{\partial q_1}g_0\right) - \{f_0,\ g_0\}_{B.b.}.\end{gathered} \tag{11.26}$$

Exercise. Describe $\mathfrak{vle}$, the quotient of $\mathfrak{vb}$.

$\underline{\mathfrak{vvect}(m|n)}$, the *general vectorial Volichenko algebra*. For it, $x = \dfrac{\partial}{\partial \theta_1}$. Since $\mathfrak{vect}(m|n) = \mathfrak{der}(\mathbb{C}[x, \theta])$, so every element of $\mathfrak{vect}(m|n)$ can be expressed as $F = \sum_i f_i\dfrac{\partial}{\partial x_i} + \sum_j g_j\dfrac{\partial}{\partial \theta_j}$, where $f_i, g_j \in \mathbb{C}[x, \theta]$, it follows that

$$\{F,\ \frac{\partial}{\partial \theta_1}\} = -\left(\sum_i \frac{\partial f_i}{\partial \theta_1}\frac{\partial}{\partial x_i} + \sum_j \frac{\partial g_j}{\partial \theta_1}\frac{\partial}{\partial \theta_j}\right)$$

for $p(F) = 0$. This shows that

$$(\mathfrak{vvect}(m|n))_{\hat 0} = \{F \mid f_i \in \mathbb{C}[x, \bar\theta]_{\bar 0},\ g_j \in \mathbb{C}[x, \bar\theta]_{\bar 1}\}$$

where $\theta = \{\theta_1, \bar\theta\}$. To see further brackets in $\mathfrak{vvect}$, we use the following formulas:

$$\begin{aligned}\{f_a\frac{\partial}{\partial x_i},\ g_b\frac{\partial}{\partial x_j}\} &= f_a\left(\frac{\partial g_b}{\partial x_i}\right)\frac{\partial}{\partial x_j} - (-1)^{p(f_a)p(g_b)}g_b\left(\frac{\partial f_a}{\partial x_j}\right)\frac{\partial}{\partial x_i}, \\ \{f_a\frac{\partial}{\partial x_i}, g_b\frac{\partial}{\partial \theta_j}\} &= f_a\left(\frac{\partial g_b}{\partial x_i}\right)\frac{\partial}{\partial \theta_j} - (-1)^{p(f_a)(p(g_b)+1)}g_b\left(\frac{\partial f_a}{\partial \theta_j}\right)\frac{\partial}{\partial x_i}, \\ \{f_a\frac{\partial}{\partial \theta_i}, g_b\frac{\partial}{\partial \theta_j}\} &= f_a\left(\frac{\partial g_b}{\partial \theta_i}\right)\frac{\partial}{\partial \theta_j} - (-1)^{(p(f_a)+1)(p(g_b)+1)}g_b\left(\frac{\partial f_a}{\partial \theta_j}\right)\frac{\partial}{\partial \theta_i}.\end{aligned}$$

$\underline{\mathfrak{vsl}(m|n|^p_q)}$, the *special linear Volichenko algebra*. The ambient is $\mathfrak{g} = \mathfrak{sl}(m|n)$.

Convention. Hereafter the empty blocks are supposed to be filled in with zeros; for convenience of some calculations we bar the elements that occupy odd entries.

$$\mathfrak{h}_{\hat 0} = \left\{\begin{pmatrix} a & b & c & & & \\ & e & f & & & \\ & & k & & & \\ & & & a & & \\ & & & \delta & \varepsilon & \\ & & & \xi & \eta & k \end{pmatrix},\ \begin{gathered}\text{where}\ \ \mathrm{tr}\ e = \mathrm{tr}\ \varepsilon, \\ a \in \mathfrak{gl}(p),\ k \in \mathfrak{gl}(q) \\ e \in \mathfrak{gl}(m-p-q),\ \varepsilon \in \mathfrak{gl}(n-p-q)\end{gathered}\right\}; \tag{11.27}$$

$$\mathfrak{h}_{\hat 1} = \left\{\begin{pmatrix} \alpha & & & 2\overline{\alpha} & -\overline{\beta} & -\overline{\gamma} \\ d & & & \overline{d} & & \\ g & h & \kappa & \overline{g} & & \\ & & \overline{\gamma} & -\alpha & \beta & \gamma \\ & & & \overline{\psi} & & \psi \\ -\overline{g} & -\overline{h} & -2\overline{\kappa} & & & -\kappa \end{pmatrix},\ \text{where tr}\ \alpha + \text{tr}\ \kappa = 0\right\}. \tag{11.28}$$

Exercise. 1) Describe the *orthosymplectic Volichenko algebra* $\mathfrak{vosp}(m|n|p)$ corresponding to the elements x_p^p of Table 11.2.8.

2) Describe all queer Volichenko algebra $\mathfrak{vq}(n|k)$ of $\mathfrak{q}(n)$ corresponding to the element x_k of Table 11.2.8.

3) Describe all periplectic Volichenko algebra $\mathfrak{vspe}(n|p,q)$ of $\mathfrak{spe}(n)$ corresponding to the element y_q^p of Table 11.2.8.

11.4. Proofs

In what follows ∂_i is an abbreviation for $\dfrac{\partial}{\partial\xi_i}$.

11.4.1. Lemma. *For any epimorphic Volichenko algebra* $\mathfrak{h}\subset\mathfrak{g}$ *we have:*
1) $p_0(\mathfrak{h})$ *is a Lie subalgebra of* $\mathfrak{g}_{\bar 0}$*;*
2) $[p_1(\mathfrak{h}),p_1(\mathfrak{h})]\subseteq\mathfrak{h}$;
3) $\mathfrak{h}_{\bar 1}=\mathfrak{h}\cap\mathfrak{g}_{\bar 1}$ *is a* $p_0(\mathfrak{h})$*-module.*

Proof. 1) straightforward.

2) Let $a\in\mathfrak{h}$; $a=a_0+a_1$ its decomposition into homogeneous components. Then
$$[a,a]=[a_0+a_1,a_0+a_1]=[a_1,a_1] \tag{11.29}$$
which implies the statement thanks to the symmetry of the bracket on $\mathfrak{g}_{\bar 1}$ by appealing to polarization.

3) For $h\in\mathfrak{h}_{\bar 1}$ and $a\in\mathfrak{h}$ we have
$$[a,h]-[h,a]=[a_0,h]\in\mathfrak{h}_{\bar 1}.\quad\square \tag{11.30}$$

11.4.2. Proof of Lemma 11.2.5. 1) Let $a,b\in\mathfrak{h}$, $a=a_0+a_1$, $b=b_0+b_1$, $a_1=f(a_0)$, $b_1=f(b_0)$. Then
$$[a,b]=[a_0,b_0]+[a_1,b_1]+[a_0,a_1]=[b_0,a_1]. \tag{11.31}$$
Since $[a_1,a_1]\in\mathfrak{h}$, then by polarizing we get the cocycle condition
$$f([a_0,b_0])=[a_0,f(b_0)]-[b_0,f(a_0)]. \tag{11.32}$$

2) Follows from $f([p_1(\mathfrak{h}),p_1(\mathfrak{h})])=0$. $\square$

11.4.3. Lemma. *Let* $\mathfrak{g}$ *be a simple finite-dimensional Lie superalgebra. Then*
$$H^1(\mathfrak{g}_{\bar 0};\ \mathfrak{g}_{\bar 1})=\begin{cases}0 & \textit{if } \mathfrak{g}\neq\mathfrak{h}'(2n+1)\\ \operatorname{Span}(c) & \textit{if } \mathfrak{g}=\mathfrak{h}'(2n+1),\end{cases}$$
where
$$c(H_F)=H_{\frac{\partial^4(\xi_1\cdots\xi_{2n+1})}{\partial\xi_i\partial\xi_j\partial\xi_k\partial\xi_l}}\ \textit{if and only if } F=\xi_i\xi_j\xi_k\xi_l. \tag{11.33}$$

(Recall that in the realization of $\mathfrak{h}'$ by generating functions $\deg_{Lie}H_f=\deg f-2$.)

Proof. If $\mathfrak{g}_{\bar 0}$ is reductive, then $H^1(\mathfrak{g}_{\bar 0};\ \mathfrak{g}_{\bar 1})=0$ since $\mathfrak{g}_{\bar 1}$ has no $\mathfrak{g}_{\bar 0}$-invariants.

If $\mathfrak{g}$ is vectorial, the above argument implies that $c|\mathfrak{g}_0=0$ for any $c\in H^1(\mathfrak{g}_{\bar 0};\ \mathfrak{g}_{\bar 1})$ and the standard $\mathbb{Z}$-grading of $\mathfrak{g}$. Therefore, $c|\mathfrak{g}_2$ is a $\mathfrak{g}$-module morphism. If $c|\mathfrak{g}_2=0$, then $c=0$ since $\mathfrak{g}_0$ and $\mathfrak{g}_2$ generate $\mathfrak{g}_{\bar 0}$. The only possibility for $c|\mathfrak{g}_2\neq 0$ is realized for $\mathfrak{h}'(2n+1)$, when $\mathfrak{g}_2\cong\mathfrak{g}_{n-6}$, as $\mathfrak{g}_0$-modules. $\square$

11.4.4. Lemma. *Let* $\mathfrak{g}$ *be a simple vectorial Lie superalgebra,* $x\in\mathfrak{g}_{\bar 1}$*, inv.ord* $(x)=-1$ *and let* x *satisfy condition* 2) *of Lemma 11.2.5. Then* x *is homologic.*

Proof. $\mathfrak{g}=\mathfrak{vect}(\xi_1,\dots,\xi_n)$. Since *inv.ord* $x=-1$, then by $(Aut\ \mathfrak{g})$-action we can reduce x to the form
$$x=\partial_1+z,\quad\text{where } inv.ord\ z\geq 1. \tag{11.34}$$
Let
$$[x,x]=v=v_i+\dots, \tag{11.35}$$
where $\deg v_i=i$ with respect to the standard grading $\deg\xi_i=1$. Since $[x,v]=0$ by hypothesis and $[\partial_1,v_0]$ is a homogenous summand of the minimal degree in the expression (11.35), then $[\partial_1,v_0]=0$, hence,
$$v_0=\sum_{1\leq i\leq n}f_i(\xi_2,\dots,\xi_n)\partial_i. \tag{11.36}$$

Condition ensuring non-triviality implies that for any $g\in\mathfrak{g}_{\bar 0}$ (in the standard $\mathbb{Z}$-grading of $\mathfrak{g}$) we have
$$[x,[x,g],[x,g]]=[[v,g],[x,g]]=0, \tag{11.37}$$
and therefore
$$[[v_0,g],[\partial_1,g]]=0\ . \tag{11.38}$$
Let $g=\xi_1\partial_j-\xi_j\partial_1$. Then $[v_0,g]=\sum\limits_{1\leq i\leq n}\xi_1\dfrac{\partial f_i}{\partial\xi_j}\partial_i+f_1\partial_j-f_j\partial_1$ and $[x,g]=\partial_j$, so
$$\begin{gathered}[[v_0,g],[\partial_1,g]]=\frac{\partial f_1}{\partial\xi_j}\partial_j-\frac{\partial f_j}{\partial\xi_j}\partial_1=0,\ \text{i.e.,}\\ \frac{\partial f_1}{\partial\xi_j}=\frac{\partial f_j}{\partial\xi_j}=0\ \text{ for } j=2,\dots,n.\end{gathered} \tag{11.39}$$
Thus, $f_1=0$ and $\dfrac{\partial f_j}{\partial\xi_j}=0$ for $j=2,\dots,n$.

Let $g'=g+g_1$, where $g_1\in\mathfrak{vect}(\xi_2,\dots,\xi_n)\cap\mathfrak{g}_0$. Then
$$[[v_0,g+g_1],[\partial_1,g+g_1]]=[[v_0,g_1],\partial_j]=0 \tag{11.40}$$

for any $j = 2, \dots, n$ and $\mathfrak{g}_1 \in \mathfrak{vect}(\xi_2, \dots, \xi_n) \cap \mathfrak{g}_0$.

Since $v_0 \in \mathfrak{vect}(\xi_2, \dots, \xi_n) \cap \mathfrak{g}$, it follows from $[[v_0, g_1], \partial_j] = 0$ that

$$[v_0, g_1] = 0 \quad \text{for any} \quad g_1 \in \mathfrak{vect}(\xi_2, \dots, \xi_n) \cap \mathfrak{g}_0. \tag{11.41}$$

If $\mathfrak{g} = \mathfrak{vect}(\xi_1, \dots, \xi_n)$, then $v_0 = \lambda \sum\limits_{2 \le i \le n} \xi_i \partial_i$ and conditions $\dfrac{\partial f_i}{\partial \xi_j} = 0$ for $j = 2, \dots, n$ imply $v_0 = 0$.

$\underline{\mathfrak{g} = \mathfrak{svect}(\xi_1, \dots, \xi_n)}$ or $\underline{\widetilde{\mathfrak{svect}}(\xi_1, \dots, \xi_n)}$. Then $v_0 = 0$.

$\underline{\mathfrak{g} = \mathfrak{h}'(\xi_1, \dots, \xi_n)}$. Then v_0 is a vector field with generating function $\lambda \xi_2 \dots \xi_n$. Since $H_{\lambda \xi_2 \dots \xi_n} = [H_{\xi_1}, D]$ implies $D = H_{\lambda \xi_1 \dots \xi_n}$ but $H_{\xi_1 \dots \xi_n} \notin \mathfrak{h}'$, we deduce that $\lambda = 0$. Hence, $v_0 = 0$.

Thus, in all the cases $v_0 = 0$ implies that $v = 0$. □

11.4.5. Proof of Lemma 11.2.6. $\underline{\mathfrak{g} = \mathfrak{vect}(0|n)}$, x a homologic field. Since x must ensure non-triviality, x is a nondegenerate field. Now, apply Shander's rectifyability of the vector field theorem ([ShV]).

$\underline{\mathfrak{g} = \mathfrak{svect}(0|n)}$, x as above. We may assume that $x = \partial_1 + \sum f_i \partial_i$, where $\sum f_i \partial_i$ is homogenous with respect to the standard grading and $\deg\ f_i = p \ge 1$. Since $[x, x] = 0$, then $\dfrac{\partial f_i}{\partial \xi_1} = 0$ and since $\operatorname{div} x = 0$, then $\sum \dfrac{\partial f_i}{\partial \xi_i} = 0$.

Let $v = x - v_1$, where $v_1 \in \mathfrak{vect}(0|n-1)$ only depends on $\xi_2, \dots, \xi_n$ and $\operatorname{div} v_1 = f_1$. If $p < n - 1$, then such a vector field v always exists. The automorphism $\exp(\operatorname{ad}_{-v})$ reduces x to the form

$$x = \partial_1 + \sum g_i \partial_i, \quad \text{where} \quad \deg g_i > p. \tag{11.42}$$

By induction on p we bring any x to the form

$$x_a = \partial_1 + a \xi_2 \cdots \xi_n \partial_1. \tag{11.43}$$

If n is even, $a = 0$ since $p(x) = \bar{1}$.

If n is odd, the fields x_a are not conjugate to each other for different a. Indeed, suppose x_a is conjugate to x_b. Then a linear transformation which preservs ∂_1 and has determinant 1 should send x_a to x_b or the other way round. But, as is not difficult to show, this is impossible.

$\underline{\mathfrak{g} = \mathfrak{h}'(n)}$. As for $\mathfrak{svect}(0|n)$, let us represent x in the form $x = H_{\xi_1 + f + \dots}$, where $\deg f = p$ and the dots represent higher terms. Since $[x, x] = 0$, then $\dfrac{\partial f}{\partial \xi_1} = 0$. The automorphism $\exp(\operatorname{ad}_{-H_{\xi_1 f}})$ sends x into $x' = H_{\xi_1 + g}$, where $\deg g > p$. By induction on p we kill the summand g.

$\underline{\mathfrak{g} = \widetilde{\mathfrak{svect}}(2n)}$. Then x can be reduced to the form ($\Delta = \xi_1 \dots \xi_{2n}$):

$$x = (1 + \Delta)\partial_1 + v + g, \quad \text{where} \quad \deg v = p \ , \ \ \deg g > p. \tag{11.44}$$

Then the formula

$$[x, x] = \xi_2 \dots \xi_{2n} \partial_1 + 2[v, \partial_1] + [v, v] + \dots = 0 \tag{11.45}$$

implies that $[v, \partial_1] = 0$ and as in case $\mathfrak{svect}(0|n)$ applying the automorphism $\exp(\operatorname{ad}_{\xi_1 v + v_1})$, where v_1 is defined in the discussion of the $\mathfrak{svect}$ case, we get rid of v, provided $p \le 2n - 2$. Since p is odd, this is always the case; hence, x reduces to the form

$$x = (1 + \Delta)\partial_1. \tag{11.46}$$

But this x is not homologic, hence, the statement. □

11.4.6. Lemma. *Let $\mathfrak{g} = \mathfrak{h}'(2n+1)$, let φ be a cocycle $\mathfrak{g}_{\bar{0}} \Longrightarrow \mathfrak{g}_{\bar{1}}$ such that $\varphi(\mathfrak{g}_{\bar{0}}) \wedge \mathfrak{g}_{-1} \neq 0$ and*

$$\varphi([\varphi(g_1), \varphi(g_2)]) = 0 \quad \textit{for any} \ \ g_1, g_2 \in \mathfrak{g}_{\bar{0}}. \tag{11.47}$$

If $n \ge 3$, then $\varphi = dx$ for same $x \in \mathfrak{g}_{\bar{0}}$. If $n = 2$, then either $\varphi = dx$ or $\varphi = \lambda c$, where c is defined by formula (11.33).

Proof. By Lemma 11.2.5 φ is of the form

$$\varphi(g) = [g, x] + \lambda c(g) \tag{11.48}$$

and if $n \ge 3$, then $x = H_{\xi_1 + g}$, where $\deg(g) > 1$. Take

$$g_1 = H_{\xi_1 \dots \xi_n}, \quad g_2 = H_{\xi_1 \xi_2 \xi_5 \xi_6}.$$

Then

$$\varphi([\varphi(g_1), \varphi(g_2)]) \neq 0 \tag{11.49}$$

since $c[\varphi(g_1), \varphi(g_2)] = c(H_{\xi_3 \xi_4 \xi_5 \xi_6}) = H_{\xi_1 \xi_2 \xi_7 \dots \xi_{2n+1}}$.

For $n = 2$ by automorphisms of the form $\exp(\operatorname{ad}_g)$ for $g \in \mathfrak{g}_2$ we may reduce φ to one of the forms: either $\lambda c + dx$, with $\deg x = 1$ and $\lambda \neq 0$, or $\varphi = dx$.

Consider the first option. Let $x \neq 0$. Then there exists $g \in \mathfrak{g}_0$ such that

$$\varphi(x) = [x, g] = y \neq 0. \tag{11.50}$$

Select ξ_i so that $[y, \xi_i] \neq 0$ and set $z = H_{\frac{\partial \xi_1 \dots \xi_5}{\partial \xi_1}}$. Then

$$\varphi(z) = \xi_i \ \ , \quad \varphi([y, H_{\xi_i}]) = c([y, H_{\xi_i}] \neq 0\ . \tag{11.51}$$

Hence, $x = 0$. □

Remark. The cocycle $\varphi = \lambda c$ does not lead to a Volichenko algebra because

$$[c(x), c(y)] = 0 \quad \text{for any} \quad x, y \in \mathfrak{g}_{\bar{0}}\ . \tag{11.52}$$

11.4.7. Lemma. *Let $\mathfrak{g}$ be a simple Lie superalgebra with a reductive $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g} \neq \mathfrak{psq}$. If x ensures non-triviality, then x is homologic.*

Proof. Let $y = [x, x]$. Select a maximal torus $\mathfrak{t} \subset \mathfrak{g}$ so that $y \notin \mathfrak{t}$. (Since x ensures non-triviality, this is possible since y can not belong to the center of $\mathfrak{g}_{\bar{0}}$). Let R be the set of roots of $\mathfrak{g}$. Select $h \in \mathfrak{t}$ so that

$$\alpha(h) = \beta(h) \Longrightarrow \alpha = \beta \quad \text{for any} \quad \alpha, \beta \in R \tag{11.53}$$

and decompose x and y in sums of root vectors:

$$x = g_{\alpha_1} + \ldots + g_{\alpha_k} \quad , \quad y = g_{\beta_1} + \ldots + g_{\beta_l} \tag{11.54}$$

where

$$\alpha_1(h) < \ldots < \alpha_k(h) \quad , \quad \beta_1(h) < \ldots < \beta_l(h) \tag{11.55}$$

Clearly, h can always be chosen so that $\beta_1(h) \neq 0$. If $g_{\gamma_1}, g_{\gamma_2}$ are two even root vectors, then the condition

$$[x, [[x, g_{\gamma_1}], [x, g_{\gamma_2}]] = [[y, g_{\gamma_1}], [x, g_{\gamma_2}]] + [[y, g_{\gamma_2}], [x, g_{\gamma_1}]] = 0 \tag{11.56}$$

implies

$$[[g_{\beta_1}, g_{\gamma_1}], [g_{\alpha_1}, g_{\alpha_2}]] + [[g_{\beta_1}, g_{\gamma_2}], [g_{\alpha_1}, g_{\gamma_1}]] = 0 \; . \tag{11.57}$$

If α_1 is a root of type $\mathfrak{sl}(2)$, then $\beta_1 = 2\alpha_1$. Let $g_{\gamma_1} = g_{\gamma_2} = g_{-\beta_1}$. Then (11.57) fails:

$$2[[g_{\beta_1}, g_{-\beta_1}], [g_{\alpha_1}, g_{-\beta_1}]] = 2g_{-\alpha_1} \neq 0 \; . \tag{11.58}$$

If α_1 is a root of type $\mathfrak{sl}(1|1)$, then $\beta_1 \neq k\alpha_1$ and there exists $h_0 \in \mathfrak{t}$ such that $\alpha_1(h_0) = 1, \; \beta_1(h_0) = 0$. Let there exist $\gamma \in R_{\bar{0}}$ such that $[[g_{\beta_1}, g_\gamma], g_{\alpha_1}] \neq 0$.

Set $g_{\gamma_1} = h_0, g_{\gamma_2} = g_\gamma$. Then (11.57) takes the form

$$[[g_{\beta_1}, g_\alpha], [g_{\alpha_1}, h_0]] - [[g_{\beta_1}, h_0], [g_{\alpha_1}, g_\gamma]] = \tag{11.59}$$

$$= [[g_{\beta_1}, g_\gamma], g_{\alpha_1}] = 0, \tag{11.60}$$

which is impossible.

Let us show that there always exists $\gamma \in R_{\bar{0}}$ such that

$$[[g_{\beta_1}, g_\gamma], g_{\alpha_1}] \neq 0 \; . \tag{11.61}$$

Indeed, since $[x, y] = 0$, then $[g_{\beta_1}, g_{\alpha_1}] = 0$. Let $H_{\beta_1} = [g_{\beta_1}, g_{-\beta_1}]$. If $\alpha_1(h_{\beta_1}) \neq 0$, then take $\gamma = -\beta_1$. If $\alpha_1(h_{\beta_1}) = 0$, then select γ_0 so that $\alpha_1(h_{\gamma_0}) < 0$, $\beta_1(h_{\gamma_0}) \neq 0$. Now, set

$$\gamma = \begin{cases} \gamma_0 & \text{if } \beta_1(h_{\gamma_0}) < 0 \\ r_{\beta_1}(\gamma_0) & \text{otherwise.} \end{cases} \tag{11.62}$$

(Here r_{β_1} is the reflection with respect to β_1.) □

11.4.8. Lemma. *Let $\mathfrak{g} = \mathfrak{psq}(n)$ and let $x \in \mathfrak{g}_{\bar{1}}$ ensure non-triviality. Then either x is homologic or $x = \left(\begin{smallmatrix} 0 & X \\ X & 0 \end{smallmatrix}\right)$, where $X = diag(a1_p, b1_{n-p})$ with $ap + b(n-p) = 0$.*

Proof. The case of a homologic x had been partly dealt with already; see also sec. 3.10. Suppose that $[x, x] \neq 0$.

Reduce X to the Jordan form with Jordan blocks $j_1, \ldots, j_k$:

$$X = \operatorname{diag}(j_1, \ldots, j_k) \quad \text{with} \quad size(j_1) \geq \ldots \geq size(j_k) \; . \tag{11.63}$$

Suppose $size(j_1) = r > 2$. Then set $Z = (Z_{kr})$, where $Z_{i+1,i} = 1$ for $1 \leq i \leq r-1$ other entries being 0. Set $g = \operatorname{diag}\ (Z, Z)$; we should have

$$[x, [[x, g], [x, g]]] = \operatorname{antidiag}([[X, Z]^2, X], [[X, Z]^2, X]) = 0 \; . \tag{11.64}$$

Since $[X, Z] = (\delta_{1,n}) + (\delta_{r,r})$, then $[[X, Z]^2, X] \neq 0$ for $r > 2$. Therefore, $r = 1$ or 2.

Suppose $r = 2$. Take $g_1 = \operatorname{diag}(Z_1, Z_1)$, $g_2 = \operatorname{diag}(Z_2, Z_2)$, where $Z_1 = (\delta_{1,i})$ and $Z_2 = (\delta_{i,1})$ are $n \times n$ matrices with nonzero entries as indicated. We have

$$\begin{aligned} &[[x, g_1], [x, g_2]] = \operatorname{diag}(T, T), \quad \text{where} \\ &T = (x_{11} - x_{ii})^2((\delta_{11}) + (\delta_{ii})) + T_+ \end{aligned} \tag{11.65}$$

with an upper triangular matrix T_+. The condition 2 of Lemma 11.2.5, i.e.,

$$[x, [[x, g_1], [x, g_2]]] = 0, \tag{11.66}$$

implies $[T, X] = 0$. Let $Z = [T, X]$, then $Z_{12} = (x_{11} - x_{ii})^2 = 0$; hence, $x_{11} = \ldots = x_{nn} = 0$, and therefore $[x, x] = 0$.

Suppose $r = 1$, then $x = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Take

$$g = \begin{pmatrix} \delta_{ij} + \delta_{jk} & 0 \\ 0 & \delta_{ij} + \delta_{jk} \end{pmatrix} . \tag{11.67}$$

Then for $y = [x, x]$ we have

$$[x, [[x, g], [x, g]]] = 2[[y, g], [x, g]] = 0 \; . \tag{11.68}$$

Explicitely, we have

$$[(\lambda_i^2 - \lambda_j^2)\delta_{ij} + (\lambda_j^2 - \lambda_k^2)\delta_{jk} \;\; , \;\; (\lambda_i - \lambda_j)\delta_{ij} + (\lambda_j - \lambda_k)\delta_{jk}] = \tag{11.69}$$

$$= (\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_i - \lambda_k) \cdot \delta_{ik} = 0 \tag{11.70}$$

and, therefore, there are no more than two distinct eigenvalues; hence, there are exactly two of them and, since $X \in \mathfrak{psq}(n)_{\bar{1}}$, X is as in formulation of Lemma. □

11.4.9. The Volichenko algebra $\mathfrak{vgl}_\mu(p|q)$. Set $q = n-p$. It is convenient to pass from $\mathfrak{psq}(n)$ to $\mathfrak{q}(n)$ and take

$$x' = \text{antidiag}\ (X', X'),\ \text{where}\ X' = X + \lambda 1_n = \ \text{diag}\ (\mu 1_p, -\mu 1_q)\quad \text{with}\ \lambda = -\frac{a+b}{2},\ \mu = \frac{a-b}{2}. \tag{11.71}$$

Then $[x, g] = [x', g]$ for any $g \in \mathfrak{q}(n)_{\bar 0}$ and, therefore, condition 2 of Lemma 11.2.5 holds for x if and only if it holds for x'. But $[x', x']$ belongs to the center of $\mathfrak{q}(n)_{\bar 0}$, hence

$$\begin{gathered}2[x', [[x', g_1], [x', g_2]]] = \\ [[[x', x'], g_1], [x', g_2]] + [[x', g_1], [[x', x'], g_2]] = 0.\end{gathered} \tag{11.72}$$

11.4.10. Proof of Main Theorem.

Part A.

$\mathfrak{g} = \mathfrak{sl}(m|n)$, $x = \begin{pmatrix} 0_m & X \\ Y & 0_n \end{pmatrix}$. Since the automorphism $-\mathrm{st} : A \mapsto -A^{st}$ interchanges (and transposes) X and Y, we may assume that $p \le q$. Next, there exist invertible matrices A and B of sizes $m \times m$ and $n \times n$, respectively, such that

$$AXB = 1^{(m,n,p)}. \tag{11.73}$$

Hence, we may assume that $X = 1^{(m,n,p)}$. Since $[x, x] = 0$, it follows that $XY = YX = 0$; thus,

$$Y = \begin{pmatrix} 0_p & 0 \\ 0 & Z \end{pmatrix} \tag{11.74}$$

with the help of automorphisms of the form $\mathrm{Add}(A, B)$, where $A = \mathrm{diag}(1_p, A')$, $B = \mathrm{diag}(1_p, B')$, we reduce Y to the form $1_{(m,n,q)}$ and preserve X.

$\mathfrak{g} = \mathfrak{osp}(m|2n)$. Let V be the identity $\mathfrak{o}(m)$-module with inner product $(\cdot,\cdot)$, W the identity $\mathfrak{sp}(2n)$-module with inner product $<\cdot,\cdot>$. Any $x \in \mathfrak{g}_{\bar 1}$ is representable in the form $\sum\limits_{i\le s} v_i \otimes w_i$, where $v_1, \dots, v_s \in V$ are linearly independent, and $w_1, \dots, w_s \in W$ are also linearly independent. By definition

$$[x, x] = \sum_{i,j\le s} (v_i, v_j) w_i \circ w_j + v_i \wedge v_j < w_i, w_j > . \tag{11.75}$$

The linear independence of the v_i and w_i implies that

$$[x, x] = 0 \Longleftrightarrow (v_i, v_j) = 0,\ \ < w_i, w_j >= 0 \quad \text{for all} \quad i, j. \tag{11.76}$$

Let V_0 and W_0 be maximal isotropic subspaces in V and W relative to $(\cdot,\cdot)$ and $<\cdot,\cdot>$, respectively. Then $v_i \in V_0$, $w \in W_0$ for all i.

The group $GL(V_0) \times GL(W_0)$ is naturally embedded into Aut $\mathfrak{g}$. Let $e_1, \dots, e_{[\frac{m}{2}]}$ be a basis of V_0 and $f_1, \dots, f_n$ a basis of W_0. Obviously, x can be reduced to the form

$$x = e_1 \otimes f_1 + \dots + e_p \otimes f_p, \quad \text{where}\ \ p \le \min([\frac{m}{2}], n). \tag{11.77}$$

Thus $x \in \mathfrak{gl}([\frac{m}{2}]|n) \subset \mathfrak{osp}(m|2n)$ and we are in the situation of the above case.

$\mathfrak{g} = \mathfrak{psq}(n)$, $x = \left(\begin{smallmatrix} 0 & X \\ X & 0 \end{smallmatrix}\right)$. Since $[x, x] = 0$, then $X^2 = 0$. Now, reduce X to the Jordan form. Since $X^2 = 0$, the size of each block is ≤ 2.

$\mathfrak{g} = \mathfrak{spe}(n)$, $x = \left(\begin{smallmatrix} 0 & X \\ Y & 0 \end{smallmatrix}\right)$. By automorphisms of $\mathfrak{g}_{\bar 0}$ we reduce X to the form $\mathrm{diag}(1_p, 0)$. Hence, $Y = \mathrm{diag}(0_p, Z)$ with a skewsymmetric Z. The automorphisms of the form

$$\text{Add}\ (1_p, A, 1_p(A^t)^{-1}) \tag{11.78}$$

reduce Z to the form $\mathrm{diag}(0, J_{2q})$ and do not affect X.

$\mathfrak{g} = \mathfrak{osp}_\alpha(4|2)$. Since $\mathfrak{g}$ is a deformation of $\mathfrak{osp}(4|2)$, there is a linear map $\varphi\colon \mathfrak{g} \longrightarrow \mathfrak{osp}(4|2)$ which is an isomorphism of linear superspaces (but not of Lie superalgebras). However,

$$[x, x] = 0 \Longleftrightarrow [\varphi(x), \varphi(x)] = 0. \tag{11.79}$$

Indeed, look at the formula for multiplication $S^2(\mathfrak{g}_{\bar 1}) \longrightarrow \mathfrak{g}_{\bar 0}$ in $\mathfrak{g}$.

Recall that $\mathfrak{g}_{\bar 1} \cong V \otimes V \otimes V$, where V is the identity $\mathfrak{sl}(2)$-module. Then, in $\mathfrak{osp}(4|2)$,

$$[x, x] = 0 \Longleftrightarrow \text{the map } S^2(V \otimes V \otimes V) \longrightarrow \mathfrak{sl}(2)_i \subset \mathfrak{g}_{\bar 0} \text{ is zero.} \tag{11.80}$$

In $\mathfrak{g}$, these maps are multiplied by α (or $-1-\alpha$ or $\dfrac{1}{\alpha}$) but the result is still zero.

$\mathfrak{g} = \mathfrak{ag}(2)$, $x \in \mathfrak{g}_{\bar 1} = V \otimes W$, where V is the standard $\mathfrak{sl}(2)$-module, $W = \mathbb{O}^\circ$ is the $\mathfrak{g}(2)$-module corresponding to the first fundamental representation. Let $x = v_1 \otimes w_1 + v_2 \otimes w_2$, where v_1, v_2 is a basis of V. Then one of the following cases may occur:

1) $w_1 = w_2 = w$. Then $x = v \otimes w$ and $< w, w >= 0$, where $<\cdot,\cdot>$ is the inner product in W preserved by $\mathfrak{g}_2$. Since the highest weight vector of the identity $\mathfrak{o}(7)$-module is isotropic, and the G_2-orbit of the highest vector coincides with the $O(7)$-orbit, we may assume that v, w are highest weight vectors with respect to $\mathfrak{sl}(2)$ and $\mathfrak{g}(2)$, respectively. Hence, x is conjugate to a simple root of type $\mathfrak{sl}(1|1)$.

2) w_1 and w_2 are linearly independent. Then the condition $[x, x] = 0$ turns into (here ρ is multiplication in $\mathfrak{g}$)

$$\begin{gathered}< v_1, v_2 > \rho(w_1 \wedge w_2) + < w_1, w_1 > \rho(v_1^2) + \\ < w_1, w_2 > \rho(v_1 \cdot v_2) + < w_2, w_2 > \rho(v_2^2) = 0.\end{gathered} \tag{11.81}$$

Since, due to linear independence, each summand vanishes, we derive that

$$\rho(w_1 \wedge w_2) = 0, \quad < w_1, w_1 >=< w_1, w_2 >=< w_2, w_2 >= 0. \tag{11.82}$$

As in case 1) we may assume that w_1 is a highest weight vector with respect to $\mathfrak{g}(2)$.

Recall ([OV]), that, as $\mathfrak{g}(2)$ modules, $\Lambda^2 W = \mathrm{Ad}\otimes W_1$ for an irreducible W_1, hence, $\rho(w_1\wedge w_1)=0$ implies that $w_1\wedge w_2\in W_1$. Therefore, having applied several times elements from the maximal nilpotent (positive) subalgebra $\mathfrak{n}$ of $\mathfrak{g}$, we get the highest vector from W_1. On the other hand, we thus get an element of the form $w_1\wedge w$ for some $w\in W$ (since $\mathfrak{n}w_1=0$), which can only be highest in Ad. Thus, this case is impossible.

$\underline{\mathfrak{g}=\mathfrak{ab}(3)}$. Proof is completely analogous to that of $\mathfrak{ag}(2)$-case where $\mathfrak{g}(2)$ is replaced by $\mathfrak{o}(7)$ and W by $spin_7$.

Part B. In what follows $\mathfrak{g}\subset\mathfrak{vect}(m|n)$, $m,n>0$, where $\mathfrak{vect}(m|n)$ is considered in a nonstandard grading $\deg x_i=0$, $\deg\xi_j=1$ for all i,j.

Observe that $\mathfrak{vect}(m|n)_{\bar 0}=\mathfrak{vect}(m)\oplus I$, where I is the ideal spanned by vector fields of the form

$$\sum_{1\le i,j\le n} f_{ij}(x)\xi_i\partial_j+g,\quad\text{where}\quad \deg g>0\,. \tag{11.83}$$

If a homologic element $d\in\mathfrak{der}\,\mathfrak{g}\subset\mathfrak{vect}(m|n)$ (recall, that all derivations of any vectorial Lie superalgebra $\mathfrak{g}$ are contained in $\mathfrak{vect}$, see [K2]) determines a simple Volichenko algebra $\mathfrak{h}(d)\subset\mathfrak{g}$, i.e., $\mathfrak{h}(d)=\{\alpha+[a,d]\mid a\in\mathfrak{g}_{\bar 0}\}$, then there is an element $x\in\mathfrak{g}_{\bar 0}$ such that $[[x,d],\ [x,d]]\notin I$ (otherwise $I_d=\{a+[a,d]\mid a\in I\}$ is an ideal). This means that $inv.ord(d)=-1$. (Lemma 11.2.5 states this for $m=0$.)

Thus, we have to classify elements

$$d=d_{-1}+d_1+d_3+\dots\ \in(\mathfrak{der}\,\mathfrak{g})_{\bar 1}\ ,\quad d_{-1}\neq 0,\quad \deg d_i=i \tag{11.84}$$

such that $d^2=0$ up to (Aut $\mathfrak{g}$)-action. Recall that Aut $\mathfrak{g}$ is a subgroup in the group of coordinate transformations.

$\underline{\mathfrak{g}=\mathfrak{vect}(m|n)}$.

a) Let $d_{-1}=\sum f_i(x)\dfrac{\partial}{\partial\xi_i}$. The shift $x_j\mapsto x_j+c_j$, $c_j\in\mathbb{C}$, can assure that $f_i(0)\neq 0$ for at least one i. Let

$$d_{-1}=\sum_{1\le i\le n}\lambda_i\frac{\partial}{\partial\xi_i}+\sum_{1\le j\le n}g_j\frac{\partial}{\partial\xi_j},\quad\text{where }\lambda_i\in\mathbb{C}\ \text{ for all } i\quad\text{and } g_j(0)=0\quad\text{for all } j. \tag{11.85}$$

The transformation $\xi_1\mapsto\sum\lambda_i\xi_i$ reduces d_{-1} to the form

$$d_{-1}=\frac{\partial}{\partial\xi_1}+\sum_{1\le j\le n}\tilde g_j\frac{\partial}{\partial\xi_j},\quad\text{where }\ \tilde g_j(0)=0\quad\text{for all } j\,. \tag{11.86}$$

b) The transformation $\xi_1\mapsto\xi_1+\sum\tilde g_j\xi_j$ sends d_{-1} to $\dfrac{\partial}{\partial\xi_1}$. Thus, we may assume that $d_{-1}=\dfrac{\partial}{\partial\xi_1}$.

c) The condition $[d,d]=0$ implies $[\dfrac{\partial}{\partial\xi_1},d_1]=0$. Hence the automorphism $\exp(\mathrm{ad}_{-\xi_1 d_1})$ sends d to $\dfrac{\partial}{\partial\xi_1}+d_s^1+\dots$. Apply the automorphism $\exp(\mathrm{ad}_{-\xi_1 d_3^1})$ to this field, etc. Since our nonstandard grading is of finite length, we will eventually reduce d to the form $\dfrac{\partial}{\partial\xi_1}$.

The rest of the proof copies the above steps a)–c) for various subalgebras $\mathfrak{g}\subset\mathfrak{vect}(m|n)$; we only have to check that our transformations of indeterminates preserve the structure preserved by $\mathfrak{g}$.

$\underline{\mathfrak{g}=\mathfrak{svect}'(m|n)}$ (or $\mathfrak{svect}(1|n)$; since $\mathfrak{der}\,\mathfrak{svect}'(1|n)=\mathfrak{der}\,\mathfrak{svect}(1|n)$, proof is the same).

a) Completely copies the $\mathfrak{vect}$ case: the changes of variables multiply the volume form by a constant.

b) Same, since this transformation $\xi_1\longrightarrow\xi_1+\sum\tilde g_j\xi_j$ (and id on the other ξ_i) is volume-preserving.

c) If for some q we have div $\xi_1 d_q\neq 0$, we are in trouble. Notice that since div $d=0$, then div $d_q=0$. Since explicitly we have

$$d_q=\sum a_i\frac{\partial}{\partial x_i}+\sum b_j\frac{\partial}{\partial\xi_j}, \tag{11.87}$$

it follows that div $\xi_1 d_q=b_1$.

Set $c_1=\int b_1dx_1$ and instead of $\exp(\mathrm{ad}_{-\xi_1 d_q})$ take $\exp(\mathrm{ad}_{-\xi_1 d_q+c_1\frac{\partial}{\partial x_1}})$. Since $[\dfrac{\partial}{\partial\xi_i},c_1\dfrac{\partial}{\partial x_1}]=0$, the result of this replacement is the same while new change is volume-preserving, as required.

$\underline{\mathfrak{g}=\mathfrak{h}(m|n)}$.

a)–b) Consider the orthosymplectic form $\omega=\sum dp_jdq_j+\sum(\xi_i)^2$. Let $d_{-1}=H_{\sum_{i\le n}f_i(p,q)\xi_i}$. Since $[d,d]=0$, we see that $\sum f_i^2=const$.

11.4.10.1. Lemma. *Let $A(t,p,q)=(a_{ij})$ be an $n\times n$-matrix such that $A^{-1}A=\lambda\cdot 1_n$. Then there exists an ω-preserving coordinate transformation which multiplies the form*

$$\alpha=dt-\sum(p_jdq_j-q_jdp_j)-\sum\xi_id\xi_i \tag{11.88}$$

by a function and acts on the indeterminates as follows:

$$\xi_i\mapsto\sum_j a_{ij}\xi_j+a_i,\quad t\mapsto t+c,\quad p_j\mapsto p_j+b_j,\quad q_j\mapsto q_j+d_j, \tag{11.89}$$

where a_i, b_j, d_j and c are of $\deg>1$.

Proof. Consider the two possible cases.

i) $\sum f_i^2 \neq 0$. Select the matrix (a_{ij}) so that $a_{1j} = f_j$ and $(a_{ij})^{-1}(a_{ij}) = \lambda \cdot 1_n$. (From the standard courses of linear algebra we know that this is possible to perform.) Then we can transform d_{-1} to the form H_{ξ_1}.

ii) $\sum f_i^2 = 0$. With the help of shifts along p, q and linear changes of coordinates we can ascertain that $f_1(0) = 1$, $f_2(0) = \sqrt{-1}$, $f_j(0) = 0$ for $j > 2$. Then in a neighborhood of 0 we can solve the following equation in formal power series

$$\begin{gathered}\sum f_i a_i = 1, \quad \sum a_j^2 = 0,\\ a_1(0) = 1, \quad a_2(0) = -\sqrt{-1}\ , \quad a_j(0) = 0 \quad \text{for} \quad j > 2.\end{gathered} \tag{11.90}$$

Now, take the matrix a_{ij} such that

$$a_{1i} = f_i + a_i, \quad a_{2i} = (f_i - a_i)/\sqrt{-1} \tag{11.91}$$

and satisfying Lemma 11.4.10.1. Then we can reduce d_{-1} to the form $H_{\xi_1 + i\xi_2}$. □

c) In case i) just repeat the arguments for $\mathfrak{g} = \mathfrak{vect}$, in case ii) in these arguments replace ξ_1 by $\xi_1 - i\xi_2$.

$\underline{\mathfrak{g} = \mathfrak{k}(m|n)}$. This case reduces to the subcase ii) of the above case since $d_{-1} = K_{\sum f_i\xi_i}$ and $[d_{-1}, d_{-1}] = K_{\sum f_i^2} + L$, where $L \in \mathfrak{g}_0$.

$\underline{\mathfrak{g} = \mathfrak{le}(n)}$. In our grading, $d_{-1} = L_f$, where $f \in \mathbb{C}[[q]]/\mathbb{C} \cdot 1$.

11.4.10.2. Lemma. a) *Any invertible change of variables $q_i \mapsto f_i(q)$ extends to an automorphism of the algebra $\mathbb{C}[[q, \xi]]$ which preserves the form $\omega_1 = \sum dq_i d\xi_i$. This automorphism sends ξ_i to $\sum\limits_j a_{ij}(q)\xi_j$, where $(a_{ij}) = (b_{ij})^{-1}$ and $b_{ij} = \dfrac{\partial f_i}{\partial q_j}$.*

b) *If the change $q \mapsto f(q)$ preserves the volume form* $\operatorname{vol}(q)$, *then the extended change of variables preserves the form* $\operatorname{vol}(q, \xi)$.

c) *The extended transformation sends* $\mathrm{Le}_{f(q)}$ *into* $\mathrm{Le}_{f(f_1(q), \dots, f_n(q))}$

Proof. Steps a), b). It immediately follows from Lemma 11.4.10.1 that without loss of generality we may assume $d_{-1} = \mathrm{Le}_{q_1}$.

Step c). Let $d = \mathrm{Le}_{q_1} + d_r + d_{r+2} + \dots$. Then $[\mathrm{Le}_{q_1}, d_2] = 0$ since $[d, d] = 0$. If $d_r = \mathrm{Le}_f$, then $\dfrac{\partial f}{\partial \xi_1} = 0$.

A simple calculation shows that the automorphism $\exp(ad(-\mathrm{Le}_{\xi_1 f}))$ sends d into $\mathrm{Le}_{q_1} + d^1$, where $\deg d^1 > r$. The proof is completed by induction on r. □

$\underline{\mathfrak{g} = \mathfrak{sle}'(n)}$. In this case $d \in \mathfrak{sle}(n)$.

Steps a), b). It follows from Lemma 11.2.5 that $d = d_{-1} + d_1 + d_3 + \dots$, where $d_{-1} = \mathrm{Le}_{q_1}$.

It is important here that $n > 1$ since for $n = 1$ the automorphisms that send q to $f(q_1)$ do not preserve the volume form. Besides, if $n = 1$, then $\mathfrak{h}(d) = (\mathfrak{sle}(n))_{\bar 0}$, due to triviality of the bracket in this case, which is of no interest.

Step c). Here we may repeat the same arguments as in the case of $\mathfrak{le}(n)$ provided we ensure that $\operatorname{div}\ \mathrm{Le}_{\xi_1 f} = 0$. Let $\operatorname{div}\ \mathrm{Le}_{\xi_1 f} \neq 0$. Since $\operatorname{div}\ d = 0$, then $\operatorname{div} \mathrm{Le}_f = 0$ and, therefore,

$$\operatorname{div}\ \mathrm{Le}_{\xi_1 f} = 2 \sum \frac{\partial^2 \xi_1 f}{\partial \xi_i \partial q_i} = 2 \frac{\partial f}{\partial q_1}. \quad \square \tag{11.92}$$

11.4.10.3. Lemma. *Let $A = \mathbb{C}[[q_1, \dots, q_n, \xi_2, \dots, \xi_n]]$, $a \in A$ and $\sum\limits_{i \geq 2} \dfrac{\partial^2 a}{\partial q_i \partial \xi_i} = 0$. If $\deg a < n - 1$, then there exists $b \in A$ such that*

$$\sum_{i \geq 2} \frac{\partial^2 b}{\partial q_i \partial \xi_i} = a\ . \tag{11.93}$$

Proof. First, suppose that $\deg f < n - 1$. Set $a = \dfrac{\partial f}{\partial q_1}$ and apply Lemma 11.4.10.2. Then

$$[\mathrm{Le}_{\xi_1 f + b}, \mathrm{Le}_{q_1}] = [\mathrm{Le}_{\xi_1 f}, \mathrm{Le}_{q_1}]\ , \tag{11.94}$$

and

$$\operatorname{div}\ \mathrm{Le}_{\xi_1 f + \psi} = 0. \tag{11.95}$$

Therefore, we can replace the generator $\mathrm{Le}_{\xi_1 f}$ of the automorphism of $\mathfrak{le}'$ by $\mathrm{Le}_{\xi_1 f + b}$. Then $\exp(ad(\mathrm{Le}_{\xi_1 f + b}))$ preserves $\mathfrak{sle}'$.

If n is even, then $\deg f < n - 1$ since f is even. Hence, d can always be reduced to the form Le_{q_1}.

Let n be odd. Then d reduces to the form $\mathrm{Le}_{q_1 + \xi_2 \dots \xi_n f(q_1)}$. Indeed, f only depends on q_1 since

$$\operatorname{div}\ \mathrm{Le}_{q_1 + \xi_2 \cdots \xi_n f} = \sum_{i \geq 2} \left(\pm \xi_2 \cdots \hat{\xi}_i \cdots \xi_n \frac{\partial f}{\partial q_i} \right) = 0 \tag{11.96}$$

With the help of $\exp(\mathrm{ad}_{\lambda \mathrm{Le}_{\xi_1 \cdots \xi_n}})$ for an appropriate λ we can turn d into $\mathrm{Le}_{q_1 + \xi_2 \cdots \xi_n f(q_1)}$, where $f(0) = 0$.

By changes of indeterminates of the form $\xi_2 \mapsto \lambda \xi_2$, $q_2 \mapsto \lambda^{-1} q_2$ we can reduce d to the form

$$d_g = \mathrm{Le}_{q_1(1 + \xi_2 \dots \xi_n g(q_1))}, \tag{11.97}$$

where $g(q_1)) = q_1^m + \dots$.

Let us show that if $g_1 \neq g_2$, then d_{g_1} and d_{g_2} belong to different equivalence classes modulo Aut $\mathfrak{g}$-action. Indeed, if there is a change of indeterminates sending d_{g_1} to d_{g_2}, it is of the form

$$q_i \mapsto a_i, \quad \xi_i \mapsto \sum \xi_j b_{ij} \ \ \text{(see Lemma 11.4.10.2)}, \tag{11.98}$$

where

$$(b_{ij}) = \left(\frac{\partial a_j}{\partial q_i}\right)^{-1}. \tag{11.99}$$

This transformation maps the field d_{g_1} to

$$\mathrm{Le}_{a_1}(1 + \det((b_{ij})^n_{i,j=2}) \cdot \xi_2 \cdots \xi_n g_1(a_1)). \tag{11.100}$$

If d_{g_1} is mapped to d_{g_2}, then $a_1 = q_1$ and $\det((b_{ij})^n_{i,j=2}) = g_2/g_1$. Since the map (11.98) is volume-preserving, then

$$\det\left(\frac{\partial a_i}{\partial q_j}\right) \ / \ \det(b_{ij}) = 1 \tag{11.101}$$

which, taking (11.99) into account yields $(\det(b_{ij}))^2 = 1$. Since $\frac{\partial a_1}{\partial q_j} = \delta_{1j}$, then $b_{i1} = \delta_{1i}$ and $\det(b_{ij}) = \det(b_{ij})^n_{i,j=2} = \pm 1$, and therefore $g_2/g_1 = \pm 1$. Since $g_1(q_1) = q_1^{m_1} + \dots$ and $g_2(q_1) = q_1^{m_2} + \dots$, only the case $g_2/g_1 = 1$ is possible. This is a contradiction.

$\mathfrak{g} = \mathfrak{m}(n)$.

Steps a), b). Let $d = d_{-1} + \dots$, where $d_{-1} = M_{f(q)}$ and $f \in \mathbb{C}[[q_1, \dots, q_n]]$. With the help of the change of indeterminates of the form

$$q_i \mapsto q_i + q_i', \quad \tau \mapsto \tau + \sum q_i'\xi_i \tag{11.102}$$

we can guarantee that $f(0) \neq 0$.

11.4.10.4. Lemma. *For any $f \in \mathbb{C}[[q_1, \dots, q_n]]$ such that $f(0) \neq 0$, there exists an automorphism of the algebra $\mathbb{C}[[q_1, \dots, q_n, \xi_1, \dots, \xi_n, \tau]]$ such that $\tau \mapsto f(q)\tau$, volume is preserved, and the contact form α_0 is multiplied by a function. This change sends $M_{f(q)}$ to $M_1 + S$, where $\deg S > 0$.*

Proof is similar to that of Lemma 3.10.3. □

By Lemma 3.10.4 we can reduce d to the form

$$\alpha = M_1 + d_1 + \dots. \tag{11.103}$$

c) Let $d = M_1 + d_r + d_{r+2} + \dots$. Since $[d, d] = 0$, then $[M_1, d_r] = 0$. Therefore $d_r = M_f$ and $\frac{\partial f}{\partial \tau} = 0$. Consider the automorphism $\exp(\mathrm{ad}_{M_{\tau f/2}})$. It sends d to $M_1 + d_{r+2} + \dots$. We kill the rest of the d_k, $k \geq r + 2$, by induction.

$\mathfrak{g} = \mathfrak{sm}(n)$. Steps a), b). Similarly to $\mathfrak{m}(n)$ we reduce d to the form

$$d = M_1 + d_r + d_{r+2} + \dots. \tag{11.104}$$

c) By the same arguments as for $\mathfrak{g} = \mathfrak{m}(n)$ we can kill the term d_r if $r < n - 1$ with the help of the automorphism generated by $M_{\tau f/2 + g}$, where $g \in \mathbb{C}[[q_1, \dots, q_n, \xi_1, \dots, \xi_n]]$ is such that $\sum_i \frac{\partial^2 g}{\partial q_i \partial \xi_i} = -f$ (such g exists thanks to Lemma 11.4.10.3). Then $\operatorname{div} \ M_{\tau f/2 + g} = 0$.

There remains the case $r = n - 1$ which is only possible for n even. In this case we can reduce d to the form $M_{1 + \xi_1 \cdots \xi_n f(q_1, \dots, q_n)}$. Since $\operatorname{div} \ d = 0$, then $f = const$. The automorphism

$$\xi_1 \mapsto \lambda\xi_1, \quad q_1 \mapsto \lambda^{-1} q_1 \quad \text{for } \lambda \neq 0 \tag{11.105}$$

sends $M_{1 + \lambda\xi_1 \dots \xi_n}$ to $M_{1 + \xi_1 \dots \xi_n}$. Thus we arrive at the two cases of Theorem. □

11.4.11. Proof of Proposition 11.2.9. $\mathfrak{g} = \mathfrak{vect}(m|n)$. Let

$$\mathfrak{h} = \{a + [a, x] \mid a \in \mathfrak{g}_{\bar{0}}\},$$

let $\mathfrak{i} \subset \mathfrak{h}$ be a right ideal which is not a two-sided ideal. Set $\mathfrak{i}_0 = pr_{\bar{0}}\mathfrak{i}$. Then for any $z \in \mathfrak{i}_0$ and $a \in \mathfrak{g}_{\bar{0}}$, we have

$$[a + [x, a], z + [x, z]] = [a, z] + [[x, a], [x, z]] + [x, [a, z]] \in \mathfrak{i}_0, \tag{11.106}$$

hence, $[a, z] + [[x, a], [x, z]] \in \mathfrak{i}_0$. Take $a = E_{\bar{1}} = \sum \xi_i \frac{\partial}{\partial \xi_i}$. Then $[a, x] = \lambda(a)x$ for $\lambda(a) \in \mathbb{Z}$ implying $[a, z] \in \mathfrak{i}_0$.

Thus, $\mathfrak{i}_0$ is a $\mathbb{Z}$-graded subspace of $\mathfrak{g}_{\bar{0}}$ (with respect to the nonstandard grading $\deg x_i = 0$, $\deg \xi_j = 1$ for all i, j). Take a homogenous $g \in \mathfrak{i}_0$ such that $[a, g] \notin \mathfrak{i}_0$ for some homogenous (with respect to the $\mathbb{Z}$-grading) $a \in \mathfrak{h}$.

But since $\deg[a, g] \neq \deg[[x, a], [x, g]]$, we see that $[a, g] + [x, a], [x, g]] \notin \mathfrak{i}_0$. Now, there are no two-sided ideals in $\mathfrak{h}$ since if $\mathfrak{i}$ is such an ideal, then $\mathfrak{i}_0$ is an ideal in $\mathfrak{g}_{\bar{0}}$ and $[[\mathfrak{g}_0, x], [\mathfrak{i}_0, x]] \in \mathfrak{i}_0$ which, as we have seen, is impossible.

Similar arguments apply to other vectorial Volichenko algebras $\mathfrak{h}_{\mathrm{ad}\frac{\partial}{\partial \xi_1}}$. We only have to consider not just one $E_{\bar{1}}$ but the whole maximal torus $\mathfrak{t} \subset \mathfrak{g}_0$. Since $[h, x] = \lambda(h)x$ for any $h \in \mathfrak{t}$, then by (11.106) we have $[h, z] \in \mathfrak{i}$ for any $z \in \mathfrak{i}_0$, $h \in \mathfrak{t}$.

Let $\mathfrak{g}$ be not of hamiltonian series. Then $\mathfrak{g}_{\bar{0}} = \bigoplus_{\alpha \in \mathfrak{t}^*} \mathfrak{g}_\alpha$, where each root space belongs to a homogenous (with respect to the nonstandard grading) component $\mathfrak{g}_k$ for some $k \in \mathbb{Z}$. This implies that if $g = \sum_{a \in A} g_a \in \mathfrak{i}_0$, then $g_a \in \mathfrak{i}_0$ for all a.

Proof (a sketch of). Induction on the cardinality $|A|$ of the set of indices. Take $h \in \mathfrak{t}$ so that $a(h) \neq b(h)$ for all nonequal $a, b \in A$. Then

$$a(h)g - [h, g] = \sum_{b \in A \setminus \{a\}} a_b g_b \in \mathfrak{i}_0. \tag{11.107}$$

Having repeated the procedure for all $a \in A$ such that $a \neq a_0$, we get $g_{a_0} \in \mathfrak{i}_0$. In particular, $\mathfrak{i}_0 \subset \mathfrak{g}_{\bar{0}}$ is a homogenous (with respect to the nonstandard grading) subspace. The rest of the proof is similar to that for $\mathfrak{vect}$.

Problem. Consider the remaining case: $\mathfrak{h}(2m|n)$.

Part II

Modular Lie algebras and Lie superalgebras: Background and examples (A. Lebedev)

Chapter 12

Background: The modular case

12.1. Generalities

12.1.1. Definition of Lie superalgebras for $p = 2$. Define a Lie superalgebra for $p = 2$ as a superspace $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ such that

$\mathfrak{g}_{\bar 0}$ is a Lie algebra;

$\mathfrak{g}_{\bar 1}$ is a $\mathfrak{g}_{\bar 0}$-module (made into the two-sided one by symmetry; more exactly, by anti-symmetry, but if $p = 2$ it is the same);

on $\mathfrak{g}_{\bar 1}$ a *squaring* (roughly speaking, the halved bracket) is defined

$$\begin{array}{c} x \mapsto x^2 \quad \text{such that } (ax)^2 = a^2x^2 \text{ for any } x \in \mathfrak{g}_{\bar 1} \text{ and } a \in \mathbb{K}, \text{ and} \\ (x+y)^2 - x^2 - y^2 \text{ is a bilinear form on } \mathfrak{g}_{\bar 1} \text{ with values in } \mathfrak{g}_{\bar 0}. \end{array} \tag{12.1}$$

(We use a minus sign, so the definition also works for $p \neq 2$.) The origin of squaring is as follows: For any Lie superalgebra $\mathfrak{g}$, for $p \neq 2$ and for any odd element $x \in \mathfrak{g}_{\bar 1}$, there is an even element $x^2 := \frac{1}{2}[x, x] \in \mathfrak{g}_{\bar 0}$. It is desirable to keep this operation even if $p = 2$, but, since it can not be defined in the same way, we define it separately and define the bracket of odd elements to be (this equation is valid for $p \neq 2$ as well):

$$[x, y] := (x + y)^2 - x^2 - y^2. \tag{12.2}$$

We also assume, as usual, that

if $x, y \in \mathfrak{g}_{\bar 0}$, then $[x, y]$ is the bracket on the Lie algebra;

if $x \in \mathfrak{g}_{\bar 0}$ and $y \in \mathfrak{g}_{\bar 1}$, then $[x, y] := l_x(y) = -[y, x] = -r_x(y)$, where l and r are the left and right $\mathfrak{g}_{\bar 0}$-actions on $\mathfrak{g}_{\bar 1}$, respectively.

The *Jacobi identity* for one even and two odd elements has now the following form:

$$[x^2, y] = [x, [x, y]] \text{ for any } x \in \mathfrak{g}_{\bar 1}, y \in \mathfrak{g}_{\bar 0}. \tag{12.3}$$

The Jacobi identity for three odd elements now becomes

$$[x^2, y] = [x, [x, y]] \text{ for any } x, y \in \mathfrak{g}_{\bar 1}. \tag{12.4}$$

If $\mathbb{K} \neq \mathbb{Z}/2\mathbb{Z}$, we can replace the last condition by a simpler one:

$$[x, x^2] = 0 \text{ for any } x \in \mathfrak{g}_{\bar 1}. \tag{12.5}$$

Because of the squaring, several definitions involving brackets should be modified:

1) For any two subspaces A and B of a given Lie superalgebra $\mathfrak{g}$, set

$$\begin{aligned} [A, B] &:= \mathrm{Span}([a, b] \mid a \in A, b \in B); \\ \overline{[A, B]} &:= [A, B] + \mathrm{Span}(x^2 \mid x \in [A, B]_{\bar 1}). \end{aligned} \tag{12.6}$$

2) For any Lie superalgebra $\mathfrak{g}$, set ($\mathfrak{g}^{(0)} := \mathfrak{g}$)

$$\begin{aligned} \mathfrak{g}^{(1)} &:= [\mathfrak{g}, \mathfrak{g}] + \mathrm{Span}\{g^2 \mid g \in \mathfrak{g}_{\bar 1}\}, \\ \mathfrak{g}^{(i+1)} &:= [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] + \mathrm{Span}\{g^2 \mid g \in \mathfrak{g}^{(i)}_{\bar 1}\}. \end{aligned} \tag{12.7}$$

12.1.2. The *p*-structure. Let $\mathfrak{g}$ be a Lie algebra, $p > 0$. For every $x \in \mathfrak{g}$, the operator $(\mathrm{ad}_x)^p$ is a derivation of $\mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ is said to be *restricted* if there is given a map $[p] : \mathfrak{g} \to \mathfrak{g}$ (called *p-structure*) such that

$$(ax)^{[p]} = a^p x^{[p]} \quad \text{for all } a \in \mathbb{K},\ x \in \mathfrak{g}; \tag{12.8}$$

$$(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y) \quad \text{for all } x, y \in \mathfrak{g} \tag{12.9}$$

where $is_i(x, y)$ is the coefficient of λ^{i-1} in $(\mathrm{ad}_{\lambda x+y})^{p-1}(x)$, and also

$$[x^{[p]}, y] = (\mathrm{ad}_x)^p(y) \quad \text{for all } x, y \in \mathfrak{g}. \tag{12.10}$$

12.1.2.1. Remarks. 1) Note that the condition (12.10) means that, for every $x \in \mathfrak{g}$, the derivation $(\mathrm{ad}_x)^p$ is an inner one. If the algebra $\mathfrak{g}$ does not have a center, then the conditions (12.8) and (12.9) follow from (12.10).

2) The following condition is sufficient for a Lie algebra $\mathfrak{g}$ to possess a p-structure: For a basis $\{g_i\}$ of $\mathfrak{g}$, there are elements $g_i^{[p]}$ such that

$$[g_i^{[p]}, y] = (\mathrm{ad}_{g_i})^p(y) \quad \text{for all } y \in \mathfrak{g}$$

(i.e., for every i the derivation $(\mathrm{ad}_{g_i})^p$ is an inner one).

For a given Lie superalgebra $\mathfrak{g}$ of characteristic $p > 0$, let the Lie algebra $\mathfrak{g}_{\bar 0}$ be restricted and let

$$[x^{[p]}, y] = (\mathrm{ad}_x)^p(y) \quad \text{for all } x \in \mathfrak{g}_{\bar 0},\ y \in \mathfrak{g}, \tag{12.11}$$

This gives rise to the map

$$[2p] : \mathfrak{g}_{\bar 1} \to \mathfrak{g}_{\bar 0} : \quad x \mapsto (x^2)^{[p]} \tag{12.12}$$

satisfying the condition

$$[x^{[2p]}, y] = (\mathrm{ad}_x)^{2p}(y) \quad \text{for all } x \in \mathfrak{g}_{\bar 1},\ y \in \mathfrak{g}. \tag{12.13}$$

The pair of maps $[p]$ and $[2p]$, satisfying the condition (12.13), is called a *p-structure* (or, sometimes, a *$p|2p$-structure*) on $\mathfrak{g}$ and $\mathfrak{g}$ is said to be *restricted*.

12.1.2.2. 2|2-structure on Lie superalgebras. Let $p = 2$ and $\mathfrak{g}$ a restricted Lie superalgebra; let $\mathrm{Lie}(\mathfrak{g})$ be the Lie algebra one gets from $\mathfrak{g}$ by forgetting the superstructure and considering $[x, x] = 2x^2 = 0$ for x odd. Then $\mathrm{Lie}(\mathfrak{g})$ possesses a 2-structure given by

the "2" part of $2|4$-structure on $\mathfrak{g}_{\bar 0}$;

the squaring on $\mathfrak{g}_{\bar 1}$;

the rule $(x+y)^{[2]} = x^{[2]}+y^{[2]}+[x, y]$ on the former inhomogeneous elements of $\mathfrak{g}$.

So one can say that if $p = 2$, then the restricted Lie superalgebra (i.e., the one with a $2|4$-structure) also possesses a $2|2$-*structure* which is defined even on inhomogeneous elements (unlike $p|2p$-structures).

12.1.2.3. Remark. Note that the p-structure on $\mathfrak{g}_{\bar 0}$ does not have to generate a $p|2p$-structure on $\mathfrak{g}$. This happens if (12.11) is not satisfied. In other words, even if the actions of $(\mathrm{ad}_x)^p$ and $\mathrm{ad}_{x^{[p]}}$ coincide on $\mathfrak{g}_{\bar 0}$, they do not have to coincide on the whole of $\mathfrak{g}$. For example, let us consider $p = 2$ and $\mathfrak{g} = \mathfrak{oo}^{(1)}_{I\Pi}(1|2)$ (for the definition, see §13.7) with basis $\{X_-^2, X_-, H, X_+, X_+^2\}$, and relations

$$[H, X_\pm] = X_\pm; \quad [X_+, X_-] = H.$$

We can define a 2-structure on $\mathfrak{g}_{\bar 0} = \mathrm{Span}(X_-^2, H, X_+^2) \simeq \mathfrak{sl}(2)$ by setting

$$(X_-^2)^{[2]} = H; \quad H^{[2]} = H; \quad (X_+^2)^{[2]} = 0,$$

and extending it to the whole $\mathfrak{g}_{\bar 0}$ by properties (12.8) and (12.9). This 2-structure on $\mathfrak{g}_{\bar 0}$ can not be extended to a $2|4$-structure on $\mathfrak{g}$, since, for example,

$$[X_-^2, [X_-^2, X_-]] = 0 \neq [(X_-^2)^{[2]}, X_-] = X_-.$$

12.1.3. Analogs of functions and vector fields for $p > 0$. Let us consider the supercommutative superalgebra $\mathbb{C}[x]$ of polynomials in a indeterminates $x = (x_1, ..., x_a)$, for convenience ordered in a "standard format", i.e., so that the first m indeterminates are even and the rest n ones are odd $(m + n = a)$. Among the integer bases of $\mathbb{C}[x]$ (i.e., the bases, in which the structure constants are integers), There are two canonical ones, — the usual, monomial, one and the basis of *divided powers*, which is constructed in the following way.

For any multi-index $\underline{r} = (r_1, \dots, r_a)$, where $r_1, \dots, r_m$ are non-negative integers, and $r_{m+1}, \dots, r_n$ are 0 or 1, we set

$$u_i^{(r_i)} := \frac{x_i^{r_i}}{r_i!} \quad \text{and} \quad u^{(\underline{r})} := \prod_{i=1}^{a} u_i^{(r_i)}.$$

These $u^{(\underline{r})}$ form an integer basis of $\mathbb{C}[x]$. Clearly, their multiplication relations are

$$u^{(\underline{r})} \cdot u^{(\underline{s})} = \prod_{i=m+1}^{n} \min(1, 2-r_i-s_i) \cdot (-1)^{\sum\limits_{m<i<j\leq a} r_j s_i} \cdot \binom{\underline{r}+\underline{s}}{\underline{r}} u^{(\underline{r}+\underline{s})},$$
$$\text{where} \quad \binom{\underline{r}+\underline{s}}{\underline{r}} := \prod_{i=1}^{m} \binom{r_i+s_i}{r_i}. \tag{12.14}$$

In what follows, for clarity, we will sometimes write exponents of divided powers in parentheses, as above, especially if the usual exponents might be encountered as well.

Now, for an arbitrary field $\mathbb{K}$ of characteristic $p>0$, we may consider the supercommutative superalgebra $\mathbb{K}[u]$ spanned by elements $u^{(\underline{r})}$ with multiplication relations (12.14). For any m-tuple $\underline{N} = (N_1, ..., N_m)$, where N_i are either positive integers or infinity, denote

$$\mathcal{O}(m;\underline{N}) := \mathbb{K}[u;\underline{N}] := \mathrm{Span}_{\mathbb{K}}\left(u^{(\underline{r})} \mid r_i \begin{cases} < p^{N_i} & \text{for } i \leq m \\ = 0 \text{ or } 1 & \text{for } i > m \end{cases}\right) \tag{12.15}$$

(we assume that $p^\infty = \infty$). As is clear from (12.14), $\mathbb{K}[u;\underline{N}]$ is a subalgebra of $\mathbb{K}[u]$. The algebra $\mathbb{K}[u]$ and its subalgebras $\mathbb{K}[u;\underline{N}]$ are called the *algebras of divided powers;* they can be considered as analogs of the polynomial algebra.

12.1.4. Miscellanies. 1) The algebra of divided powers $\mathcal{O}(n;\underline{N})$ (the analog of the polynomial algebra for $p>0$), *and hence all CTS-prolongs* acquire one more — shearing — parameter: $\underline{N}$.

2) Only one of these numerous algebras of divided powers are indeed generated by the indeterminates declared: If $N_i = 1$ for all i. Otherwise, in addition to the u_i, we have to add $u_i^{p^{k_i}}$ for all $i \leq m$ and all k_i such that $1 < k_i < N_i$ to the list of generators. Since any derivation D of a given algebra is determined by the values of D on the generators, we see that $\mathfrak{der}(\mathcal{O}[m;\underline{N}])$ has more than m functional parameters (coefficients of the analogs of partial derivatives) if $N_i \neq 1$ for at least one i. Define *special partial derivatives* by setting

$$\partial_i(u_j^k) = \delta_{ij} u_j^{k-1} \quad \text{for all } k < p^{N_j}.$$

The simple vectorial Lie algebras over $\mathbb{C}$ have only one parameter: the number of indeterminates. If char $\mathbb{K} = p > 0$, the vectorial Lie algebras acquire one more parameter: $\underline{N}$. For Lie superalgebras, $\underline{N}$ only concerns the even indeterminates.

3) The Lie (super)algebra of all derivations $\mathfrak{der}(\mathcal{O}[m;\underline{N}])$ turns out to be not so interesting as its *Lie subsuperalgebra of special derivations*: Let

$$\mathfrak{vect}(m;\underline{N}|n) \text{ a.k.a } W(m;\underline{N}|n) \text{ a.k.a}$$
$$\mathfrak{sder}\ \mathbb{K}[u;\underline{N}] = \mathrm{Span}_{\mathbb{K}}\left(u^{\underline{r}}\partial_k \mid 1 \leq k \leq n,\ \ r_i \begin{cases} < p^{N_i} & \text{for } i \leq m \\ = 0 \text{ or } 1 & \text{for } i > m \end{cases}\right) \tag{12.16}$$

be the *general vectorial Lie algebra of special derivations*.

4) For the sake of generality, observe that in the super version of Block's description (1.1) of semi-simple modular Lie algebras, $\mathcal{F}(n_j)$ we should take the supercommutative superalgebra of divided powers in $n_j^{\bar{0}}$ even and $n_j^{\bar{1}}$ odd indeterminates, and set

$$\mathfrak{s} = \bigoplus_j (\mathfrak{s}_j \otimes \mathcal{F}(n_j)) \,. \tag{12.17}$$

12.1.5. CTS-prolongations in the modular case. A necessary condition for a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ of finite depth to be simple is $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$; so being interested in simple algebras, we note, that if $[\mathfrak{g}_{-1}, \mathfrak{g}_1] \neq \mathfrak{g}_0$ in $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$, we can replace $\mathfrak{g}_0$ by $[\mathfrak{g}_{-1}, \mathfrak{g}_1]$, and the resulting space is still a Lie algebra.

Obviously, for $p>0$, there is a series of Cartan prolongations labeled by the shearing parameter $\underline{N}$: Let the $(i,\underline{N})$-th prolong be

$$\mathfrak{g}_{i,\underline{N}} = \{X \in \mathfrak{vect}(m;\underline{N}) \mid \deg X = i,\ [X,\partial] \in \mathfrak{g}_{i-1,\underline{N}} \text{ for any } \partial \in \mathfrak{g}_{-1}\}.$$

12.1.5.1. Superizations of the Cartan prolongs and its Tanaka–Shchepochkina generalization. These superizations are straightforward bar one nuance:

If $p=2$, then we can not, in general, replace in $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ the Lie **super**algebra $\mathfrak{g}_0$ by $[\mathfrak{g}_1, \mathfrak{g}_{-1}]$, since $[\mathfrak{g}_1, \mathfrak{g}_{-1}]$ may be not closed under squaring. So, if we want to replace $\mathfrak{g}_0$ by the minimal possible space containing $[\mathfrak{g}_1, \mathfrak{g}_{-1}]$ and closed relative to the bracket and squaring, we should take

$$\overline{[\mathfrak{g}_1, \mathfrak{g}_{-1}]} := [\mathfrak{g}_1, \mathfrak{g}_{-1}] + \mathrm{Span}\{g^2 \mid g \in [\mathfrak{g}_1, \mathfrak{g}_{-1}]_{\bar{1}}\}. \tag{12.18}$$

12.1.6. Symmetric forms and exterior differential forms. Recall that, as is customary in supergeometry, we use the antisymmetric wedge product for the analogs of the exterior forms, and the symmetric $\circ$ product for the analogs of the metrics. Since the differentials of odd indeterminates commute, we can, in the super setting, consider the divided power versions of the exterior forms (as well as the divided power versions of the algebra in the dx_i relative the $\circ$ product). Usually we suppress the $\wedge$ or $\circ$ signs, since all is clear from the context, unless both multiplications are needed simultaneously.

Considering differential forms, we may also use divided powers $dx_i^{(\wedge k)}$ with multiplication relations (12.14), where the indeterminates are now the dx_i of parity $p(x_i)+\bar{1}$, and the Lie derivative along the vector field X given by the formula

$$L_X(dx_i^{(\wedge k)}) = (L_X dx_i) \wedge dx_i^{(\wedge k-1)}.$$

Note that if we consider divided power differential forms in characteristic 2, then for x_i odd we have $dx_i \wedge dx_i = 2(dx_i^{(\wedge 2)}) = 0$.

We can also use divided powers for chains and cochains of Lie superalgebras. This will affect the formula for the differential. For cochains of a given Lie superalgebra $\mathfrak{g}$, this only means that a divided power of an odd element must be differentiated as a whole:

$$d(\varphi^{(\wedge k)}) = d\varphi \wedge \varphi^{(\wedge(k-1))} \text{ for any } \varphi \in (\mathfrak{g}^*)_{\bar{1}}.$$

For chains, the modification is a little more involved: Let $g_1, \dots, g_n$ be a basis of $\mathfrak{g}$. Then for chains of $\mathfrak{g}$ with coefficients in a right module A, and $a \in A$, we have

$$\begin{aligned} d\left(a \otimes \bigwedge_{i=1}^{n} g_i^{(\wedge r_i)}\right) = &\sum_{p(g_k)=\bar{1},\ r_k \geq 2} a \otimes \bigwedge_{i<k} g_i^{(\wedge r_i)} \wedge g_k^2 \wedge g_k^{(\wedge(r_k-2))} \wedge \bigwedge_{i>k} g_i^{(\wedge r_i)} + \\ &\sum_{1\leq k<l\leq n,\ \ r_k, r_l \geq 1} (-1)^{\sum\limits_{k<m<l} r_m p(g_m)} a \otimes \bigwedge_{i<k} g_i^{(\wedge r_i)} \wedge [g_k, g_l] \wedge g_k^{(\wedge(r_k-1))} \wedge \\ &\bigwedge_{k<i<l} g_i^{(\wedge r_i)} \wedge g_l^{(\wedge(r_l-1))} \wedge \bigwedge_{i>l} g_i^{(\wedge r_i)} + \\ &\sum_{r_k \geq 1} (-1)^{p(g_k) \sum\limits_{m<k} r_m p(g_m)} (ag_k) \otimes \bigwedge_{i<k} g_i^{(\wedge r_i)} \wedge g_k^{(\wedge(r_k-1))} \wedge \bigwedge_{i>k} g_i^{(\wedge r_i)}. \end{aligned}$$

The divided powers of (co)chains naturally appear in the study of Lie **super**algebras for any p, even for $p = 0$, but at the moment their interpretation is unknown. We suggest to denote the corresponding spaces of (co)homology by

$$DPH_{(n,\underline{N})}(\mathfrak{g}; M) \text{ and } DPH^{(n,\underline{N})}(\mathfrak{g}; M).$$

Note that if $\mathfrak{g}$ is a Lie *super*algebra in characteristic $p = 2$, and we want to interpret its non-trivial infinitesimal deformations and its generating relations in terms of (co)homology, as we are used to, then we need elements of $DPH^{(2,\underline{N})}(\mathfrak{g}; \mathfrak{g})$ for deformations and those of $DPH_{(2,\underline{N})}(\mathfrak{g}) := DPH_{(2,\underline{N})}(\mathfrak{g}; \mathbb{K})$ for relations. We *can not do without* divided power (co)homology (with $\underline{N}$ such that $N_i \geq 2$ for all i): Otherwise we won't be able to take into account the deformations changing values of squares of odd elements or relations of the form $x^2 = 0$.

12.2. $p|2p$-structures on vectorial Lie superalgebras

12.2.1. Theorem. *The algebra* $\mathfrak{vect}(m; \underline{N}|n)$ *is restricted if and only if* $\underline{N} = (1, \dots, 1)$.

The corresponding $p|2p$-structure can be described as follows: Consider the standard monomial (perhaps, consisting of divided power monomials) basis of $\mathfrak{vect}(m; \underline{N}|n)$, consisting of elements of the form $x^{(r)}\partial_j$. Then

$$\begin{aligned} (x^{(r)}\partial_j)^{[p]} &= 0 && \text{if } x^{(r)} \neq x_j; \\ (x_j\partial_j)^{[p]} &= x_j\partial_j && \text{for all } j = 1, \dots, m+n. \end{aligned} \tag{12.19}$$

It is easy to see that if $\underline{N} \neq (1, \dots, 1)$, then the algebra $\mathfrak{vect}(m; \underline{N}|n)$ has no $p|2p$-structure: Let $N_i > 1$ for some i. Then the algebra contains the element $x_i^{(p)}\partial_i$, and we have

$$(\mathrm{ad}_{\partial_i})^p x_i^{(p)}\partial_i = \partial_i \neq 0.$$

Thus, $(\partial_i)^{[p]}$ has to be a non-zero element of degree $-p$ in the standard grading, while the minimal degree of the elements of $\mathfrak{vect}(m; \underline{N}|n)$ is -1.

12.2.2. Conjecture. *Simple finite dimensional modular vectorial Lie (super)algebras are restricted if and only if the shearing parameter is equal to* $(1, \dots, 1)$. *Their p- or p|2p-structures are restrictions of the corresponding p- or p|2p-structure on their ambient* $\mathfrak{vect}$.

12.2.2.1. Problem. Prove this Conjecture.

12.3. What $\mathfrak{g}(A)$ is

For simplicity, speaking about modular Lie (super)algebras we will assume them of finite dimension (except for the algebra $\tilde{\mathfrak{g}}(A, I)$ (see below) which is practically always infinite-dimensional).

12.3.1. Warning: Which $\mathfrak{sl}$ and $\mathfrak{psl}$ have no Cartan matrix and which of their relatives have them. For the most reasonable definition of Lie algebra with Cartan matrix over $\mathbb{C}$, see [K]. The same definition applies, practically literally, to Lie superalgebras and to modular Lie algebras and to modular Lie superalgebras. However, the usual sloppy practice is to attribute Cartan matrices to (usually simple) Lie (super)algebras none of which, strictly speaking, has a Cartan matrix!

Although it may look strange for those with non-super experience over $\mathbb{C}$, neither the simple modular Lie algebra $\mathfrak{psl}(pk)$, nor the simple modular Lie superalgebra $\mathfrak{psl}(a|pk + a)$, nor — in characteristic 0 — the simple Lie superalgebra $\mathfrak{psl}(a|a)$ possesses a Cartan matrix. Their central extensions — $\mathfrak{sl}(pk)$, the modular Lie superalgebra $\mathfrak{sl}(a|pk + a)$, and — in characteristic 0 — the Lie superalgebra $\mathfrak{sl}(a|a)$ — do not have Cartan matrix, either.

Their relatives possessing a Cartan matrix are, respectively, $\mathfrak{gl}(pk)$, $\mathfrak{gl}(a|pk + a)$, and $\mathfrak{gl}(a|a)$, and for the grading operator we take $E^{1,1}$.

Since often all the Lie (super)algebras involved (the simple one, its central extension, the derivation algebras thereof) are needed (and only representatives of one of the latter types of Lie (super)algebras are of the form $\mathfrak{g}(A)$), it is important to have (preferably short and easy to remember) notation for each of them. For example, in addition to $\mathfrak{psl}$, $\mathfrak{sl}$, $\mathfrak{pgl}$ and $\mathfrak{gl}$, we have:

<u>for $p = 3$</u>: $\mathfrak{e}(6)$ is of dimension 78, let us designate its CM version of dimension 79 by $\hat{\mathfrak{e}}(6)$, whereas the "simple core" is $\mathfrak{e}(6)/\mathfrak{c}$ of dimension 77;

$\mathfrak{g}(2)$ is not simple, its "simple core" is isomorphic to $\mathfrak{psl}(3)$;

<u>for $p = 2$</u>: $\mathfrak{e}(7)$ is of dimension 133, let us designate its CM version of dimension 134 by $\hat{\mathfrak{e}}(7)$, whereas the "simple core" is $\mathfrak{e}(7)/\mathfrak{c}$ of dimension 132;

$\mathfrak{g}(2)$ is not simple, its "simple core" is isomorphic to $\mathfrak{psl}(4)$;

the orthogonal Lie algebras and their super analogs are considered in detail later.

In what follows, the notation $D/d|B$ means that $\operatorname{sdim}\mathfrak{g}(A) = D|B$ whereas $\operatorname{sdim}\mathfrak{g}(A)^{(1)}/\mathfrak{c} = d|B$. The general formula is

$$d = D - 2(\operatorname{size}(A) - \operatorname{rk}(A)). \tag{12.20}$$

12.3.2. Generalities. Let us start with the construction of a Cartan matrix Lie (super)algebra (in what follows: CM Lie (super)algebra or even CMLA or CMLSA for short). Let $A = (A_{ij})$ be an $n \times n$-matrix. Let $\operatorname{rk} A = n - l$. It means that there exists an $l \times n$-matrix $T = (T_{ij})$ such that

$$\begin{aligned}&\text{a) the rows of } T \text{ are linearly independent;}\\ &\text{b) } TA = 0 \text{ (or, more precisely, “zero } l \times n\text{-matrix”).}\end{aligned} \tag{12.21}$$

Indeed, if $\operatorname{rk} A^T = \operatorname{rk} A = n - l$, then there exist l linearly independent vectors x_i such that $A^T x_i = 0$; set

$$T_{ij} = (x_i)_j.$$

Let the elements $e_i^\pm, h_i$ (where $i = 1, \dots, n$) generate a Lie superalgebra denoted $\tilde{\mathfrak{g}}(A, \mathrm{Par})$, where $\mathrm{Par} = (p_1, \dots p_n) \in (\mathbb{Z}/2)^n$ is a collection of parities ($p(e_i^\pm) = p_i$), free except for the relations

$$\begin{aligned}&[e_i^+, e_j^-] = \delta_{ij} h_i; \quad [h, e_j^\pm] = \pm\alpha_j(h) e_j^\pm \text{ for any } h \in \mathfrak{h} \text{ and any } i, j;\\ &[\mathfrak{h}, \mathfrak{h}] = 0.\end{aligned} \tag{12.22}$$

We often write I instead of Par, for brevity. The simple (and "relatives" of simple) Lie (super)algebras with Cartan matrix that we denote by $\mathfrak{g}(A, I)$ are quotients of $\tilde{\mathfrak{g}}(A, I)$ modulo the ideal we describe in general terms below (relations (20.14)) and precisely in [BGL1, BGL2, LCh].

Set

$$c_i = \sum_{j=1}^{n} T_{ij} h_j, \quad \text{where} \quad i = 1, \dots, l. \tag{12.23}$$

Then, from the properties of the matrix T, we deduce that

$$\begin{aligned}&\text{a) the elements } c_i \text{ are linearly independent;}\\ &\text{b) the elements } c_i \text{ are central, because}\\ &[c_i, e_j^\pm] = \pm\left(\sum_{k=1}^{n} T_{ik} A_{kj}\right) e_j^\pm = \pm (TA)_{ij} e_j^\pm.\end{aligned} \tag{12.24}$$

The existence of central elements means that the linear span of all the roots is only $(n-l)$-dimensional. (This can be explained even without central elements: The weights can be considered as column-vectors with i-th element being the corresponding eigenvalue of ad_{h_i}. The weight of e_i is the i-th column of A. Since $\operatorname{rk} A = n - l$, the linear span of all columns of A is $(n-l)$-dimensional (just by definition of the rank). Since any root is an (integral) linear combination of the weights of the e_i, the linear span of all roots is $(n-l)$-dimensional.) This means that some elements which we would like to see having different (even opposite if $p = 2$) weights have, actually, identical weights. To fix this, we do the following: Let B be an arbitrary $l \times n$-matrix such that

$$\text{the } (n+l) \times n\text{-matrix } \begin{pmatrix} A \\ B \end{pmatrix} \text{ has rank } n. \tag{12.25}$$

Let us add to the algebra the grading elements d_i, where $i = 1, \dots, l$, subject to the following relations:

$$[d_i, e_j^\pm] = \pm B_{ij} e_j; \quad [d_i, d_j] = 0; \quad [d_i, h_j] = 0 \tag{12.26}$$

(the last two relations mean that the d_i lie in the Cartan subalgebra, and even in the maximal torus).

Note that these d_i are *outer* derivations of $\mathfrak{g}$, i.e., they can not be obtained as linear combinations of brackets of elements of the algebra (i.e., they do not lie in $\mathfrak{g}^{(1)}$).

12.3.3. $\tilde{\mathfrak{g}}(A, I)$. First, recall, how to construct a Lie superalgebra from a Cartan matrix ([GL1]). Let $A = (A_{ij})$ be an arbitrary $n \times n$ matrix of rank l with entries in $\mathbb{K}$, such that

$$\text{if } A_{ij} = 0, \text{ then } A_{ji} = 0 \text{ for all } 1 \leq i, j \leq n. \tag{12.27}$$

Fix a (purely even) vector space $\mathfrak{h}$ of dimension $2n - l$ and its dual $\mathfrak{h}^*$, select a set [1] of n linearly independent vectors $h_i \in \mathfrak{h}$ and n linearly independent vectors $\alpha_j \in \mathfrak{h}^*$ so that $\alpha_i(h_j) = A_{ij}$.

Let $I = \{i_1, \dots, i_n\} \in (\mathbb{Z}/2\mathbb{Z})^n$; consider the free Lie superalgebra $\widetilde{\mathfrak{g}}(A, I)$ generated by $e_1^\pm, \dots, e_n^\pm$, where $p(e_j^\pm) = i_j$, and $\mathfrak{h}$, and defining relations(hereafter in similar occasions either all superscripts $\pm$ are $+$ or all are $-$)

$$[e_i^+, e_j^-] = \delta_{ij} h_i; \quad [h, e_j^\pm] = \pm\alpha_j(h) e_j^\pm \text{ for any } h \in \mathfrak{h} \text{ and any } i, j; \quad [\mathfrak{h}, \mathfrak{h}] = 0. \tag{12.28}$$

12.3.3.1. Remark. Observe that in (12.28) one may not replace an arbitrary h by the h_i (and then $\alpha_j(h_i)$ by A_{ji}), as in the case of non-degenerate Cartan matrix, since the h_i do not span $\mathfrak{h}$.

Clearly, the algebra $\widetilde{\mathfrak{g}}(A, I)$ possesses a $\mathbb{Z}^n$-grading such that

$$\begin{aligned}&\deg \mathfrak{h} = (0, \dots, 0);\\ &\deg e_i^\pm = (0, \dots, 0, \pm 1, 0, \dots, 0) \text{ where } \pm 1 \text{ stands in the } i\text{-th slot.}\end{aligned} \tag{12.29}$$

The following statements over $\mathbb{C}$ is well known for the Lie algebras [K]; for Lie superalgebras over $\mathbb{C}$, it is due to Serganova and van de Leur [Se1, vdL]; for Lie superalgebras over $p \neq 2$, see [CE2].

[1] For any two such systems of vectors $\{h_i, \alpha_j \mid i, j = 1, \dots, n\}$ and $\{h_i', \alpha_j' \mid i, j = 1, \dots, n\}$, there exists a linear map $L \in GL(\mathfrak{h})$ such that $Lh_i = h_i'$ and $L^*\alpha_j = \alpha_j'$ for all i, j. In this sense, such a system is unique.

12.3.3.2. Statement. a) *Let $\tilde{\mathfrak{g}}_+$ and $\tilde{\mathfrak{g}}_-$ be the superalgebras in $\tilde{\mathfrak{g}}(A,I)$ generated by $e_1^{\pm},\dots, e_n^{\pm}$; then $\tilde{\mathfrak{g}}(A,I)\cong\tilde{\mathfrak{g}}_+\oplus\mathfrak{h}\oplus\tilde{\mathfrak{g}}_-$, as vector superspaces.*

b) *Assume that $p\neq 2$ or if $i_j=\bar{1}$ for some $1\leq j\leq n$, then $A_{jk}\neq 0$ for some $k=1,\dots,n$. Then there exists a maximal ideal $\mathfrak{r}$ among the ideals of $\tilde{\mathfrak{g}}(A,I)$ whose intersection with $\mathfrak{h}$ is zero.*

c) *The ideal $\mathfrak{r}$ can be represented as a direct sum of spaces homogenous w.r.t. grading (12.29).*

Remark. The ideal $\mathfrak{r}$ is the direct sum of the ideals $\mathfrak{r}\bigcap\tilde{\mathfrak{g}}_+$ and $\mathfrak{r}\bigcap\tilde{\mathfrak{g}}_-$.

Set $\mathfrak{g}(A,I)=\tilde{\mathfrak{g}}(A,I)/\mathfrak{r}$. Both $\mathfrak{g}(A,I)$ and $\mathfrak{g}^{(1)}(A,I)$ may contain a center. As proved in [Se1, vdL], the centers $\mathfrak{c}$ of $\mathfrak{g}(A,I)$ and $\mathfrak{c}'$ of $\mathfrak{g}^{(1)}(A,I)$ consist of all $h\in\mathfrak{h}$ such that $\alpha_i(h)=0$ for all $i=1,\dots,n$; this is also true for $p>0$.

12.3.3.3. Remark. In the case of $p>0$, it may happen that $\mathfrak{h}$ is not a Cartan subalgebra of $\mathfrak{g}(A,I)$; see Remark 12.3.6.1.

Clearly, in relations (12.28),

$$\text{the rescaling } e_i^{\pm}\mapsto\sqrt{\lambda_i}e_i^{\pm}, \text{ sends } A \text{ to } A':=\operatorname{diag}(\lambda_1,\dots,\lambda_n)\cdot A. \tag{12.30}$$

12.3.3.4. Cartan matrices. Given a Lie superalgebra $\mathfrak{g}(A,I)$, the matrix A (more precisely, the pair (A,I)) is said to be a *Cartan matrix* of $\mathfrak{g}(A,I)$. Two pairs (A,I) and (A',I') are said to be *equivalent* if (A',I') is obtained from (A,I) by a composition of a permutation of indices and a rescaling

$$A'=\operatorname{diag}(\lambda_1,\dots,\lambda_n)\cdot A,\quad \text{ where } \lambda_1\dots\lambda_n\neq 0. \tag{12.31}$$

Clearly, equivalent pairs determine isomorphic Lie superalgebras. We will call a matrix A *symmetrizable* if it can be made symmetric by the operation (12.31) for some value of λ_i.

12.3.3.5. Roots. The elements of $\mathfrak{h}^*$ are called *weights*. For a given weight α, its *weight subspace* of a representation V of $\mathfrak{g}$ is defined as

$$V_\alpha:=\{x\in\mathfrak{g}\mid \text{exists } N\in\mathbb{Z}_+ \text{ such that } (\alpha(h)-\operatorname{ad}_h)^N x=0 \text{ for all } h\in\mathfrak{h}\}.$$

For Lie (super)algebras of the form $\mathfrak{g}(A)$, we may assume $N=1$. Further in this section $\mathfrak{g}$ denotes $\mathfrak{g}(A,I)$.

Any non-zero element $x\in V_\alpha$ is said to be *of weight* α. Over $\mathbb{C}$, it is customary to call the weights of the adjoint representation *roots*. In the modular case another definition seems to be more appropriate, we will give it shortly.

12.3.3.6. Statement ([K]). *The space of the Lie superalgebra $\mathfrak{g}$ can be represented as a direct sum of subspaces*

$$\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in\mathfrak{h}^*}\mathfrak{g}_\alpha.$$

This statement is well-known over $\mathbb{C}$; note that $\mathfrak{h}\neq\mathfrak{g}_0$ for $p=2$. We also have

$$\mathfrak{g}^{(1)}=\operatorname{Span}(h_1,\dots,h_n)\oplus\bigoplus_{\alpha\in\mathfrak{h}^*}\mathfrak{g}_\alpha.$$

By construction, the elements[2] $e_i^{\pm}$ with the same superscript (either $+$ or $-$) have linearly independent weights α_i, and any α such that $\mathfrak{g}_\alpha\neq 0$ lies in the $\mathbb{Z}$-span of $\{\alpha_1,\dots,\alpha_n\}$.

According to the part c) of Statement 12.3.3.2, the algebra $\mathfrak{g}$ has also a $\mathbb{Z}^n$-grading (12.29); in what follows we will consider it as $\mathbb{R}^n$-grading for simplicity of formulations. If $p=0$, this grading is equivalent to the weight grading of $\mathfrak{g}$. If $p>0$, this may be not true: in particular, if $p=2$, then the elements e_i^+ and e_i^- have the same weight. (That is why in what follows we consider roots as elements of $\mathbb{R}^n$, not as weights.)

Any non-zero element $\alpha\in\mathbb{R}^n$ is called *a root* if the corresponding space of grade α(which we denote $\mathfrak{g}_\alpha$ by abuse of notation) is non-zero. The set R of all roots is called *the root system* of $\mathfrak{g}$.

Clearly, the subspaces $\mathfrak{g}_\alpha$ are purely even or purely odd, and the corresponding roots are said to be *even* or *odd*. Moreover, we have the following

12.3.3.7. Statement. *Let $\mathfrak{g}=\mathfrak{g}(A)$. For any $\alpha\in R$, the space $\mathfrak{g}_\alpha$ is one-dimensional.*

Proof. This is true for Lie algebras for $p=0$, see [K3]. Since all finite-dimensional Lie (super)algebras of the form $\mathfrak{g}(A,I)$ for $p>3$ (except $\mathfrak{osp}(4|2;\alpha)$) can be represented in the form $\mathfrak{g}^0\otimes_{\mathbb{Z}}\mathbb{K}$, where $\mathfrak{g}^0$ is a Lie (super)algebra with CM over $\mathbb{Z}$ (see sect. 12.4.0.4), it is also true for them. It remains to check the cases of $p=2$ and exceptions for $p=5$, 3 and 2 (for the classification of Lie algebras, see [WK], for that of Lie superalgebras, see [BGL7]). This is performed case-by-case. □

For such definition of roots, there is no pairing (γ,h) for a root γ and element $h\in\mathfrak{h}$ anymore; we should replace it by a function

$$\operatorname{ev}: R\times\mathfrak{h}\to\mathbb{K},\quad \operatorname{ev}(\gamma,h)=\text{the eigenvalue of } \operatorname{ad}_h \text{ on } \mathfrak{g}_\gamma.$$

Clearly, this function ev is linear in the second argument in the usual sense, and it is $\mathbb{Z}$-linear in the first argument in the sense that

$$\operatorname{ev}(\sum_{i=1}^k c_i\gamma_i,h)=\sum_{i=1}^k c_i\operatorname{ev}(\gamma_i,h)$$

for any integers c_i (which are considered as elements of $\mathbb{Z}$ or $\mathbb{R}$ in the left-hand side and as elements of $\mathbb{K}$ in the right-hand side) and any roots γ_i such that

[2] We denote the images of $e_i^{\pm}$ and h_i in $\mathfrak{g}(A,I)$ and $\mathfrak{g}^{(1)}(A,I)$ also by $e_i^{\pm}$ and h_i by abuse of notation.

$\sum c_i\gamma_i$ is a root. And the other way round, if a function $f : R \to \mathbb{K}$ is such that

$$f(\sum_{i=1}^k c_i\gamma_i) = \sum_{i=1}^k c_i f(\gamma_i) \tag{12.32}$$

for any integers c_i and any roots α_i such that $\sum c_i\gamma_i$ is a root, then there is an element $h \in \mathfrak{h}$ such that $\mathrm{ev}(\gamma, h) = f(\gamma)$ for any root γ (it is enough to choose h so that the condition $\mathrm{ev}(\gamma, h) = f(\gamma)$ was satisfied for all roots γ corresponding to e_i^+ (such h exists because the elements $\alpha_j \in \mathfrak{h}^*$ are linearly independent); then, since any root can be represented as an integral linear combination of these roots and due to (12.32), this condition is satisfied for any root). Note that this may be not true for the Lie superalgebra $\mathfrak{g}^{(1)}(A, I)$.

12.3.4. $p|2p$-structure on $\mathfrak{g}$; restricted Lie superalgebras.

12.3.4.1. Proposition. *1) If $p > 2$ (or $p = 2$ but $A_{ii} \neq \bar{1}$ for all i) and $\mathfrak{g}(A)$ is finite-dimensional, then $\mathfrak{g}(A)$ has a $p|2p$-structure such that*

$$\begin{array}{l} (x_\alpha)^{[p]} = 0 \text{ for any even } \alpha \in R \text{ and } x_\alpha \in \mathfrak{g}_\alpha, \\ \mathfrak{h}^{[p]} \subset \mathfrak{h}. \end{array} \tag{12.33}$$

2) If all the entries of A are elements of $\mathbb{Z}/p\mathbb{Z}$, then we can set $h_i^{[p]} = h_i$ for all $i = 1, \dots, n$. In this case the algebra $\mathfrak{g}^{(1)}(A)$ also possesses a $p|2p$-structure. If A has entries not from $\mathbb{Z}/p\mathbb{Z}$, then $\mathfrak{g}^{(1)}$ may have no $p|2p$-structure even if $\mathfrak{g}(A)$ has one.

3) The quotient modulo center of $\mathfrak{g}(A)$ or $\mathfrak{g}^{(1)}(A)$ always inherits the p-structure of $\mathfrak{g}(A)$ or $\mathfrak{g}^{(1)}(A)$ (if any) whereas $\mathfrak{g}^{(1)}(A)$ does not necessarily inherit the p-structure of $\mathfrak{g}(A)$.

12.3.4.2. Remark. 1) Note that for any integers $c_1, \dots, c_k$ and any roots $\gamma_1, \dots, \gamma_k$ such that $\gamma = \sum\limits_{i=1}^k c_i\gamma_i$ is a root, the eigenvalue of $(\mathrm{ad}_h)^p$ on $\mathfrak{g}_\gamma$ is equal to

$$\begin{array}{l} \mathrm{ev}\left(\sum\limits_{i=1}^k c_i\gamma_i, h\right)^p = \left(\sum\limits_{i=1}^k c_i \mathrm{ev}(\gamma_i, h)\right)^p = \\ \sum\limits_{i=1}^k (c_i \mathrm{ev}(\gamma_i, h))^p = \sum\limits_{i=1}^k c_i(ev(\gamma_i, h))^p, \end{array}$$

(since $c^p = c$ for any $c \in \mathbb{Z}/p\mathbb{Z}$). I.e., the function of eigenvalue of $(\mathrm{ad}_h)^p$ on $\mathfrak{g}_\gamma$ satisfies the condition (12.32). Thus, there really exists an element $h^{[p]} \in \mathfrak{h}$ such that $\mathrm{ad}_{h^{[p]}} = (\mathrm{ad}_h)^p$.

2) There are known examples of Lie superalgebras without Cartan matrix but with a 2-structure, e.g., $\mathfrak{psl}$, $\mathfrak{o}_I(n)$.

12.3.4.3. Example. 1) Observe that the center $\mathfrak{c}$ of $\mathfrak{wk}(3; a)$ is spanned by $ah_1 + h_3$. The 2-structure on $\mathfrak{wk}(3; a)$ is given by the conditions $(e_\alpha^\pm)^{[2]} = 0$ for all root vectors and the following ones:

a) For the matrix $B = (0, 0, 1)$ in (12.25) for the grading operator d, set:

$$\begin{array}{l} (\mathrm{ad}_{h_1})^{[2]} = (1 + at)h_1 + th_3 \equiv h_1 \pmod{\mathfrak{c}}, \\ (\mathrm{ad}_{h_2})^{[2]} = ath_1 + h_2 + th_3 + a(1+a)d \equiv h_2 + a(1+a)d \pmod{\mathfrak{c}}, \\ (\mathrm{ad}_{h_3})^{[2]} = (at + a^2)h_1 + th_3 \equiv a^2h_1 \pmod{\mathfrak{c}}, \\ (\mathrm{ad}_d)^{[2]} = ath_1 + th_3 + d \equiv d \pmod{\mathfrak{c}}, \end{array} \tag{12.34}$$

where t is a parameter.

b) Taking $B = (1, 0, 0)$ in (12.25) we get a more symmetric answer:

$$\begin{array}{l} (\mathrm{ad}_{h_1})^{[2]} = (1 + at)h_1 + th_3 \equiv h_1 \pmod{\mathfrak{c}}, \\ (\mathrm{ad}_{h_2})^{[2]} = ath_1 + ah_2 + th_3 + (1+a)d \equiv ah_2 + (1+a)d \pmod{\mathfrak{c}}, \\ (\mathrm{ad}_{h_3})^{[2]} = (at + a^2)h_1 + th_3 \equiv a^2h_1 \pmod{\mathfrak{c}}, \\ (\mathrm{ad}_d)^{[2]} = ath_1 + th_3 + d \equiv d \pmod{\mathfrak{c}}, \end{array} \tag{12.35}$$

(The expressions are somewhat different since we have chosen a different basis but on this simple Lie algebra the 2-structure is unique.)

2) The 2-structure on $\mathfrak{wk}(4; a)$ is given by the conditions $(e_\alpha^\pm)^{[2]} = 0$ for all root vectors and

$$\begin{array}{l} (\mathrm{ad}_{h_1})^{[2]} = ah_1 + (1+a)h_4, \\ (\mathrm{ad}_{h_2})^{[2]} = ah_2, \\ (\mathrm{ad}_{h_3})^{[2]} = h_3, \\ (\mathrm{ad}_{h_4})^{[2]} = h_4. \end{array} \tag{12.36}$$

12.3.4.4. $(2, 4)$- and $(2, -)$-structures on Lie algebras. If $p = 2$, we encounter a new phenomenon: a $2, 4$-*structure on Lie algebras.*[3] Namely, let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ be a $\mathbb{Z}/2\mathbb{Z}$-grading of a Lie algebra (not superalgebra) $\mathfrak{g}$. We say that $\mathfrak{g}$ has a $(2, -)$-*structure*, if there is a map $[2] : \mathfrak{g}_+ \to \mathfrak{g}_+$ such that (we consider the case of centerless $\mathfrak{g}$ for simplicity)

$$[x^{[2]}, y] = [x, [x, y]] \quad \text{for all } x \in \mathfrak{g}_+, y \in \mathfrak{g},$$

but there is no 2-structure on $\mathfrak{g}$. It sometimes happens that this $(2, -)$-structure can be extended to $(2, 4)$-*structure*, i.e., there is a map $[4] : \mathfrak{g}_- \to \mathfrak{g}_+$ such that

$$[x^{[4]}, y] = [x, [x, [x, [x, y]]]] \quad \text{for all } x \in \mathfrak{g}_-, y \in \mathfrak{g}.$$

Here is an example of such occurrence: if indecomposable symmetrizable matrix A of size $n > 1$ is such that

$$A_{nn} = \bar{1}; \quad A_{ii} = \bar{0} \text{ for } i < n,$$

and the Lie algebra $\mathfrak{g}(A)$ is finite-dimensional, then $\mathfrak{g}(A)$ has no 2-structure but has a $(2, 4)$-structure with the $\mathbb{Z}/2\mathbb{Z}$-grading given by:

$$\deg(\mathfrak{h}) = \bar{0}; \quad \deg(e_n^\pm) = \bar{0}; \quad \deg(e_i^\pm) = \bar{0} \text{ for } i < n.$$

In particular, the Lie algebra $\mathfrak{g} = \mathfrak{o}^{(1)}(2n + 1)$ with Cartan matrix

[3] Observe a slightly different notation: $(2, 4)$, not $2|4$.

$$\begin{pmatrix} \ddots & \ddots & \ddots & \vdots \\ \ddots & \bar 0 & 1 & 0 \\ \ddots & 1 & \bar 0 & 1 \\ \cdots & 0 & 1 & \bar 1 \end{pmatrix}$$

can be considered as the algebra of matrices of the form

$$\begin{pmatrix} A & X & B \\ Y^T & 0 & X^T \\ C & Y & A^T \end{pmatrix}, \text{ where } \begin{matrix} A \in \mathfrak{gl}(n); \quad B, C \in ZD(n); \\ X, Y \text{ are column } n\text{-vectors.} \end{matrix}$$

Then $\mathfrak{g}_+$ consists of matrices with $X = Y = 0$, and the map [2] is given by the squaring of matrices, $\mathfrak{g}_-$ consists of matrices with $A = B = C = 0$, and the map [4] is given by the fourth power of matrices.

12.3.4.5. (2, 4)|2-structure on Lie superalgebras. Similarly, the Lie superalgebra $\mathfrak{oo}^{(1)}_{I\Pi}(2k_{\bar 0} + 1|2k_{\bar 1})$ has a $(2,4)|2$-*structure* (i.e., the squaring on the odd part and a $(2,4)$-structure on the even part such that the conditions

$$\begin{aligned} (\mathrm{ad}_{x^{[2]}})y = (\mathrm{ad}_x)^2 y \quad &\text{for all } x \in (\mathfrak{oo}^{(1)}_{I\Pi}(2k_{\bar 0} + 1|2k_{\bar 1})_{\bar 0})_+ \text{ and} \\ (\mathrm{ad}_{x^{[4]}})y = (\mathrm{ad}_x)^4 y \quad &\text{for all } x \in (\mathfrak{oo}^{(1)}_{I\Pi}(2k_{\bar 0} + 1|2k_{\bar 1})_{\bar 0})_- \end{aligned} \tag{12.37}$$

are satisfied for any $y \in \mathfrak{oo}^{(1)}_{I\Pi}(2k_{\bar 0}+1|2k_{\bar 1})$, not only for $y \in \mathfrak{oo}^{(1)}_{I\Pi}(2k_{\bar 0}+1|2k_{\bar 1})_{\bar 0}$.

12.3.4.5a. Remark. Recently Dzhumadildaev investigated a phenomenon resembling p-structure: For the general and divergence-free Lie algebras of polynomial vector fields in n indeterminates over $\mathbb{C}$, he investigated for which $N = N(n)$ the anti-symmetrization of the map $D \longmapsto D^N$ (i.e., the expression $\sum\limits_{\sigma \in S_N} \mathrm{sign}(\sigma)\, X_{\sigma(1)} \dots X_{\sigma(N)}$) yields a vector field. For the answer in some cases, see [Dz]. Amazingly, in order to describe his new operations, he used certain hidden supersymmetry of the seemingly non-super problem.

Problem. Generalize Dzhumadildaev's result to other dimensions, to Lie superalgebras, to simple Lie (super)algebras of vector fields other than $\mathfrak{vect}$ or $\mathfrak{svect}$, and to modular Lie (super)algebras.

12.3.5. Systems of simple and positive roots. Let R be the root system of $\mathfrak{g} = \mathfrak{g}(A, I)$.

For any subset $B = \{\sigma_1, \dots, \sigma_m\} \subset R$, we set:

$$R^{\pm}_B = \{\alpha \in R \mid \alpha = \pm \sum n_i \sigma_i, \quad \text{where } n_i \in \mathbb{Z}_+\}.$$

The set B is called a *system of simple roots* of R (or $\mathfrak{g}$) if $\sigma_1, \dots, \sigma_m$ are linearly independent and $R = R^+_B \cup R^-_B$. Note that R contains basic coordinate vectors and, therefore, spans $\mathbb{R}^n$; thus, any system of simple roots contains exactly n elements.

A subset $R^+ \subset R$ is called a *system of positive roots* of R (or $\mathfrak{g}$) if there exists $x \in \mathbb{R}^n$ such that

$$\begin{aligned} &(\alpha, x) \in \mathbb{R}\backslash\{0\} \text{ for all } \alpha \in R, \\ &R^+ = \{\alpha \in R \mid (\alpha, x) > 0\}. \end{aligned}$$

(Here $(\cdot,\cdot)$ is the standard Euclidean inner product in $\mathbb{R}^n$).

By construction, any system B of simple roots is contained in exactly one system of positive roots, which is precisely R^+_B.

12.3.5.1. Statement. *Any system R^+ of positive roots of $\mathfrak{g}$ contains exactly one system of simple roots. This system consists of all the positive roots (i.e., elements of R^+) that can not be represented as a sum of two positive roots.*

Let $B = \{\sigma_1, \dots, \sigma_n\}$ be a system of simple roots. Choose non-zero elements $\tilde e^{\pm}_i \in \mathfrak{g}_{\pm\sigma_i}$; set $\tilde h_i = [\tilde e^+_i, \tilde e^-_i]$, $A_B = (A_{ij})$, where $A_{ij} = \sigma_i(\tilde h_j)$ and $I_B = \{p(\tilde e_1), \cdots, p(\tilde e_n)\}$. (The pair (A_B, I_B) constructed here is not uniquely defined by B, but all the pairs (A_B, I_B) are equivalent to each other, and for any such pair (A_B, I_B), we have $\mathfrak{g}(A_B, I_B) \simeq \mathfrak{g}(A, I)$.) Still, it may be impossible to construct the initial Cartan matrix (A, I) from a given system of simple roots B in such a way.

12.3.5.2. Equivalent systems of simple roots. Two systems of simple roots B_1 and B_2 are said to be *equivalent* if the pairs (A_{B_1}, I_{B_1}) and (A_{B_2}, I_{B_2}) are equivalent.

12.3.5.3. Convention. To be able to distinguish the case of the even simple root from the odd one where $A_{ii} = 0$, we write $A_{ii} = \bar 0$ and $A_{ii} = \bar 1$ instead of 0 and 1, respectively, if $p(\sigma_i) = \bar 0$.

12.3.6. Statement. *Let B be a system of simple roots of $\mathfrak{g}$, $\tilde e^{\pm}_i$ for $i = 1, \dots, n$ the corresponding set of generators and $A_B = (A_{ij})$ the Cartan matrix. Fix an i, and let $\mathfrak{f}_i$ be the Lie subsuperalgebra generated by the $e^{\pm}_i$. Then:*

a) *If $p(\sigma_i) = \bar 0$ and $A_{ii} \neq \bar 0$, then $2\sigma_i \notin R$ and $\mathfrak{f}_i$ is isomorphic to $\mathfrak{sl}(2)$.*

b) *If $p(\sigma_i) = \bar 0$ and $A_{ii} = \bar 0$, then $2\sigma_i \notin R$ and $\mathfrak{f}_i$ is isomorphic to $\mathfrak{hei}(2; p) \Subset \mathbb{K}d$, where $\mathfrak{hei}(2; p)$ is the modular Heisenberg Lie algebra on two generators (one creation and one annihilation) (we will simply write $\mathfrak{hei}(2n|m)$ instead of $\mathfrak{hei}(2n|m; p)$ if the characteristic p is known). Its natural (irreducible non-trivial) representation is realized in the Fock space of functions $\mathcal{O}(1; \underline{1})$.*

c) *If $p(\sigma_i) = \bar 1$ and $A_{ii} = 0$, then $2\sigma_i \notin R$ and $\mathfrak{f}_i$ is isomorphic to $\mathfrak{gl}(1|1)$.*

d) *If $p(\sigma_i) = \bar 1$ and $A_{ii} \neq 0$, then $3\sigma_i \notin R$ and $\mathfrak{f}_i$ is isomorphic to $\mathfrak{osp}(1|2)$ (or to $\mathfrak{oo}^{(1)}_{I\Pi}(1|2)$ if $p = 2$).*

Proof is subject to a direct verification.

12.3.6.1. Remark. In the case (d), the algebra $\mathfrak{f}_i$ (and, thus, $\mathfrak{g}$) contains elements $(e^{\pm}_i)^2$. If $p = 2$, then these elements have zero weights, and $\mathfrak{h}$ acts on them nilpotently (in the cases I know of — just by 0). Thus, in this case, $\mathfrak{h}$ is not a Cartan subalgebra in the usual sense (i.e., a maximal nilpotent subalgebra that coincides with its normalizer).

12.3.7. Normalization conventions. Analogs of the Dynkin diagrams. As it is said above, to a given system of simple roots correspond different (equivalent) pairs (A_B, I_B). It would be nice to find a convenient way to fix some distinguished pair (A_B, I_B) in the equivalence class. It is hardly possible to choose some "best" (first among equals) order of indices; usually we try to find such order, though, in order to minimize the value

$$\max_{i,j\in\{1,\dots,n\} \text{ such that } (A_B)_{ij}\neq 0} |i-j| \tag{12.38}$$

(i.e., gather the non-zero entries of A as close to the main diagonal as possible). Though not always: for the standard numerations of roots for Lie algebras $\mathfrak{e}(n)$ (for example, $\begin{smallmatrix}1\ 2\ 3\ 4\ 5\ 6\ 7\\ 8\end{smallmatrix}$ for $\mathfrak{e}(8)$) this value is equal to 3; one could make it equal to 2 by choosing the numeration $\begin{smallmatrix}1\ 2\ 3\ 4\ 5\ 7\ 8\\ 6\end{smallmatrix}$ for $\mathfrak{e}(8)$ and similar numerations $\begin{smallmatrix}1\ 2\ 3\ 5\ 6\\ 4\end{smallmatrix}$ for $\mathfrak{e}(6)$ and $\begin{smallmatrix}1\ 2\ 3\ 4\ 6\ 7\\ 5\end{smallmatrix}$ for $\mathfrak{e}(7)$.

The rescaling affects only the matrix A_B, not the set of indices I_B. In the case of finite dimensional (and for certain types of infinite dimensional) Lie algebras over $\mathbb{C}$, the Cartan matrices are rescaled so that $A_{ii} = 2$ for all i. In the case of Lie *super*algebras over fields of arbitrary characteristic, there is no universally accepted way of rescaling Cartan matrices. Still it is convenient to rescale A_B so that

$$(A_B)_{ii} = \begin{cases} 2 & \text{if } p\neq 2 \text{ and } p(\sigma_i)=\bar 0 \text{ and } A_{ii}\neq 0,\\ \bar 1 & \text{if } p=2 \text{ and } p(\sigma_i)=\bar 0 \text{ and } A_{ii}\neq \bar 0,\\ 1 & \text{if } p(\sigma_i)=\bar 1 \text{ and } A_{ii}\neq 0 \text{ or } \bar 0.\end{cases} \tag{12.39}$$

We call a Cartan matrix satisfying conditions (12.39) *normalized*. If the Lie superalgebra $\mathfrak{g}(A_B, I_B)$ corresponding to a normalized Cartan matrix is finite-dimensional, then the rows of the matrix with non-zero diagonal entries contain only integer entries. It means, in particular, that families of finite-dimensional CM Lie superalgebras generated by Cartan matrices with a parameter can not have this parameter in rows with non-zero diagonal entries.

The row with a 0 on the main diagonal can be multiplied by any nonzero factor; we usually multiply it so that A_B be symmetric, if possible. (A symmetrized but not normalized Cartan matrix is also useful: it gives the values of the inner product of simple roots (i.e., bilinear form on $\mathfrak{h}^*$ dual to the restriction of an invariant non-degenerate form on $\mathfrak{g}$ onto $\mathfrak{h}$), see [LSoS], and is needed to pass from one system of simple roots to another.)

For $p > 0$, for each simple **finite dimensional** Lie (super)algebra ([WK, BGL5]) of the form $\mathfrak{g}(A, I)$, the Cartan matrix A is symmetrizable.

We often denote the set of generators corresponding to a normalized matrix by $X_1^{\pm}, \dots, X_n^{\pm}$ instead of $e_1^{\pm}, \dots, e_n^{\pm}$; and call them, together with the elements $h_i := [X_i^+, X_i^-]$ for all i, the *Chevalley generators*.

12.3.8. Dynkin diagrams. A usual way to represent simple Lie algebras over $\mathbb{C}$ with integer Cartan matrices is via graphs called, in the finite dimensional case, *Dynkin diagrams*. The Cartan matrices of certain interesting infinite dimensional simple Lie *super*algebras $\mathfrak{g}$ (even over $\mathbb{C}$) can be non-symmetrizable or (for any p in the super case and for $p > 0$ in the non-super case) have entries belonging to the ground field $\mathbb{K}$. Still, it is always possible to assign an analog of the Dynkin diagram to both Lie superalgebras of polynomial growth, and to finite dimensional modular Lie algebras (if these Lie (super)algebras possess Cartan matrices). Perhaps, the edges and nodes of the graph should be rigged with an extra information. Although these analogs of the Dynkin graphs are not uniquely recovered from the Cartan matrix (and the other way round), they are helpful in graphic presentation of the Cartan matrices.

Namely, the *Dynkin–Kac diagram* of the matrix (A, I) is a set of n nodes connected by multiple edges, perhaps endowed with an arrow, according to the usual rules ([K]) or their modification, most naturally formulated by Serganova: compare [Se1, FLS] with [FSS]. In what follows, we recall these rules.

12.3.8.1. Nodes. The nodes of the diagram correspond to simple roots; the form of a node depends on the corresponding root as follows:

$$\begin{cases} \text{a node } \otimes & \text{if } p(\alpha_i)=\bar 1 \text{ and } A_{ii}=0,\\ \text{a node } \bullet & \text{if } p(\alpha_i)=\bar 1 \text{ and } A_{ii}=1;\\ \text{a node } * & \text{if } p(\alpha_i)=\bar 0 \text{ and } A_{ii}=\bar 1\\ \text{a node } \circ & \text{if } p(\alpha_i)=\bar 0 \text{ and } A_{ii}=2;\\ \text{a node } \odot & \text{if } p(\alpha_i)=\bar 0 \text{ and } A_{ii}=\bar 0.\end{cases} \tag{12.40}$$

12.3.8.2. Remarks. 1) The Lie algebra with Cartan matrix $(\bar 0)$ and the Lie superalgebra with Cartan matrix (0) are solvable of dim 4 and sdim $2|2$, respectively. Their derived algebras — *Heisenberg algebras* — are denoted $\mathfrak{hei}(2) \simeq \mathfrak{hei}(2|0)$ and $\mathfrak{hei}(0|2) \simeq \mathfrak{sl}(1|1)$, respectively; their (super)dimensions are 3 and $1|2$, respectively).

2) *A posteriori* (from the classification of simple Lie superalgebras with Cartan matrix and of polynomial growth for $p = 0$) we find out that the roots $\odot$ can only occur if $\mathfrak{g}(A, I)$ grows faster than polynomially. Thanks to classification again, if $\dim \mathfrak{g} < \infty$, the roots $\odot$ can not occur if $p > 3$; whereas for $p = 3$, the Brown Lie algebras are examples of $\mathfrak{g}(A)$ with a simple root of type $\odot$, see [BGL5].

12.3.9. Edges. If i-th and j-th nodes are not connected, then $A_{ij} = A_{ji} = 0$.

If i-th and j-th nodes are connected by a single edge rigged by a number a, then $A_{ij} = A_{ji} = a$. If there is no number a, then $A_{ij} = A_{ji} = -1$.

If i-th and j-th nodes are connected by $k > 0$ edges with an arrow pointing from i-th to j-th node, then $A_{ij} = -k$, $A_{ji} = -1$.

This all should be modified by the following condition: if both i-th and j-th nodes correspond to odd roots, and the i-th node is located to the left from the j-th node, then A_{ji} must be multiplied by -1.

More complicated cases are possible, but we don't use them in this thesis.

12.3.10. Reflections and Chevalley generators. Let R^+ be a system of positive roots of Lie superalgebra $\mathfrak{g}$, and let $B = \{\sigma_1, \dots, \sigma_n\}$ be the corresponding system of simple roots with some corresponding pair $(A = A_B, I = I_B)$. Then for any $k \in \{1, \dots, n\}$, the set $(R^+ \backslash (\{a\sigma_k \mid a > 0\} \cap R)) \coprod (\{-a\sigma_k \mid a > 0\} \cap R)$, equal to

$$\begin{cases} (R^+\backslash\{\sigma_k\}) \coprod \{-\sigma_k\} & \text{if } 2\sigma_k \notin R; \\ (R^+\backslash\{\sigma_k, 2\sigma_k\}) \coprod \{-\sigma_k, -2\sigma_k\} & \text{if } 2\sigma_k \in R, \end{cases}$$

is a system of positive roots. This operation is called *the reflection in* σ_k; it changes the system of simple roots by the formulas

$$r_{\sigma_k}(\sigma_j) = \begin{cases} -\sigma_j & \text{if } k = j, \\ \sigma_j + B_{kj}\sigma_k & \text{if } k \neq j, \end{cases} \tag{12.41}$$

where

$$B_{kj} = \begin{cases} -\dfrac{2A_{kj}}{A_{kk}} & \text{if } i_k = \bar{0}, A_{kk} \neq 0, \text{ and } -\dfrac{2A_{kj}}{A_{kk}} \in \mathbb{Z}/p\mathbb{Z}, \\ p-1 & \text{if } i_k = \bar{0}, A_{kk} \neq 0 \text{ and } -\dfrac{2A_{kj}}{A_{kk}} \notin \mathbb{Z}/p\mathbb{Z}, \\ -\dfrac{A_{kj}}{A_{kk}} & \text{if } i_k = \bar{1}, A_{kk} \neq 0, \text{ and } -\dfrac{A_{kj}}{A_{kk}} \in \mathbb{Z}/p\mathbb{Z}, \\ p-1 & \text{if } i_k = \bar{1}, A_{kk} \neq 0, \text{ and } -\dfrac{A_{kj}}{A_{kk}} \notin \mathbb{Z}/p\mathbb{Z}, \\ 1 & \text{if } i_k = \bar{1}, A_{kk} = 0, A_{kj} \neq 0, \\ 0 & \text{if } i_k = \bar{1}, A_{kk} = A_{kj} = 0, \\ p-1 & \text{if } i_k = \bar{0}, A_{kk} = \bar{0}, A_{kj} \neq 0, \\ 0 & \text{if } i_k = \bar{0}, A_{kk} = \bar{0}, A_{kj} = 0. \end{cases} \tag{12.42}$$

12.3.10.1. Remarks. 1) Here we consider $\mathbb{Z}/p\mathbb{Z}$ as the subfield of $\mathbb{K}$ generated by 1.

2) In the second, fourth and penultimate cases, the b_{ij} can, in principle, be equal to $(kp-1)$ for any $k \in \mathbb{N}$, and in the last case any element of $\mathbb{K}$ may occur. We may only hope at this stage that, at least for $\dim \mathfrak{g} < \infty$ such occurrences do not happen.

The values $-\dfrac{2A_{kj}}{A_{kk}}$ and $-\dfrac{A_{kj}}{A_{kk}}$ are elements of $\mathbb{K}$, while the roots are elements of a vector space over $\mathbb{R}$. Thus, these expressions here should be understood as "the minimal non-negative integer congruent to $-\dfrac{2A_{kj}}{A_{kk}}$ or $-\dfrac{A_{kj}}{A_{kk}}$, correspondingly". (If $\dim \mathfrak{g} < \infty$, these expressions are always congruent to integers.)

There is known just one exception: if $p = 2$, $A_{kk}, A_{jk} \neq 0$, then $-\dfrac{2A_{kj}}{A_{kk}}$ should be understood as 2, not 0.

The name "reflection" is used because in the case of (semi)simple finite-dimensional Lie algebras over $\mathbb{C}$ this action extended on the whole R by linearity is a map from R to R, and it does not depend on R^+, only on σ_k. This map is usually denoted by r_{σ_k} or just r_k. The map r_{σ_i} extended to the $\mathbb{R}$-span of R is reflection in the hyperplane orthogonal to σ_i relative the bilinear form dual to the Killing form.

The reflections in the even (odd) roots are referred to as *even* (*odd*) *reflections*. A simple root is called *isotropic*, if the corresponding row of the Cartan matrix has zero on the diagonal, and *non-isotropic* otherwise. The reflections that correspond to isotropic or non-isotropic roots will be referred to accordingly. Reflection in a non-isotropic even root σ_k such that $\dfrac{2A_{ki}}{A_{kk}} \in \mathbb{Z}/p\mathbb{Z}$ for all i maps a system of simple roots to an equivalent one (in the sense of definition 12.3.5.2); moreover, the corresponding Cartan matrix stays the same. Reflection in a non-isotropic even root σ_k such that $\dfrac{2A_{ki}}{A_{kk}} \notin \mathbb{Z}/p\mathbb{Z}$ for some i maps a system of simple roots to a non-equivalent one. In particular, let us consider the case of Brown algebra $\mathfrak{br}(2, a)$ (where $p \neq 0, -1$) in characteristic 3 with Cartan matrix

$$\begin{pmatrix} 2 & -1 \\ a & 2 \end{pmatrix}. \tag{12.43}$$

The reflection of a system of simple roots corresponding to this matrix in the second simple root maps this system to a system with Cartan matrix equivalent to

$$\begin{pmatrix} 2 & -1 \\ -(a+1) & 2 \end{pmatrix}$$

(and not equivalent to (12.43) unless $a = 1$).

For Lie superalgebras over $\mathbb{C}$, one can extend the action of reflections by linearity to the root lattice but this extension preserves the root system only for $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(2m+1|2n)$, cf. [Se1].

In the general case (of Lie superalgebras and $p > 0$), the action of isotropic or odd reflections can not be extended to a linear map $R \longrightarrow R$. These reflections just connect pair of "neighboring"' systems of simple roots and there is no reason to expect that we can multiply two distinct such reflections.

The reflection in the root σ_k sends one set of Chevalley generators into the new one as follows (up to scalars)

$$\tilde{X}_k^{\pm} = X_k^{\mp}; \quad \tilde{X}_j^{\pm} = (\mathrm{ad}_{X_k^{\pm}})^{B_{kj}} X_j^{\pm} \quad \text{for } j \neq k. \tag{12.44}$$

12.3.10.2. An analog of Serganova's lemma.

Lemma. *For any two systems of simple roots B_1 and B_2 of any finite dimensional Lie superalgebra with indecomposable Cartan matrix, there is always a chain of reflections connecting B_1 with B_2.*

Proof. The analogous statement for the systems of positive roots is clear, and the one-to-one correspondence between the systems of simple and positive roots gives us this statement. It allows one to find all Cartan matrices of a given algebra. □

12.4. The Kostrikin–Shafarevich conjecture and its generalizations

12.4.0.3. The KSh-conjecture. *For $p > 5$, each of simple finite dimensional restricted Lie algebras is obtained by one of the following methods:*

1) For each simple Lie algebra of the form $\mathfrak{g}(A)$ over $\mathbb{C}$ with Cartan matrix A normalized so that $A_{ii} = 2$ for each i, select a (unique up to signs) Chevalley $\mathbb{Z}$-form $\mathfrak{g}_{\mathbb{Z}}$; take $\mathfrak{g}_{\mathbb{K}} := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{K}$.

2) For each simple infinite dimensional vectorial Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ (preserving, perhaps, a tensor (volume, symplectic or contact form)), take for $\mathfrak{g}_{\mathbb{K}}$ the analog consisting of special derivations of $\mathcal{O}(n; \underline{(1, \dots, 1)})$ (preserving the modular analog of the same tensor).

These Lie algebras $\mathfrak{g}_{\mathbb{K}}$ are simple (in some cases, up to the center and up to taking the first or second derived algebra).

12.4.0.4. The generalized KSh-conjecture. *Together with deformations*[4] *of these examples we get in this way all simple finite dimensional Lie algebras over algebraically closed fields if $p > 5$. If $p = 5$, we should add Melikyan's examples (described in what follows) to the above list.*

Having built upon ca 30 years of work of several teams of researchers, and having added new ideas and lots of effort, Block, Wilson, Premet and Strade proved the generalized KSh-conjecture for $p > 3$, see [S]. For $p \leq 5$, the above KSh-procedure does not produce all simple finite dimensional Lie algebras; some other examples appear and old ones disappear. In [GL3], Grozman and Leites returned to É. Cartan's description of $\mathbb{Z}$-graded Lie algebras as CTS prolongs, i.e., as subalgebras of vectorial Lie algebras preserving certain non-integrable distributions; we thus interpreted the "mysterious" at that moment exceptional examples of simple Lie algebras for $p = 3$ (the Brown, Frank, Ermolaev and Skryabin algebras), further elucidated Kuznetsov's interpretation [Ku1] of Melikyan's algebras (as prolongs of the nonpositive part of the Lie algebra $\mathfrak{g}(2)$ in one of its $\mathbb{Z}$-gradings) and discovered three new series of simple Lie algebras. In [BjL], the same approach yielded $\mathfrak{bj}$, a simple super version of $\mathfrak{g}(2)$, and $\mathfrak{Bj}(1; \underline{N}|7)$, a simple super version of the Melikyan algebra. Both $\mathfrak{bj}$ and $\mathfrak{Bj}(1; \underline{N}|7)$ are indigenous to the case $p = 3$, the case where $\mathfrak{g}(2)$ is not simple.

[4] Although in [LL] there are given reasons why the conventional definition of the enveloping algebra should be modified, and therefore that of (co)homology, it seems that for restricted Lie superalgebras of the form $\mathfrak{g}(A)$ (and their "relatives") for $p \neq 2$ the infinitesimal deformations can be described in old terms of $H^2(\mathfrak{g}; \mathfrak{g})$, see [BGL4] and [Vi].

In [Le1], analogs of the KSh-conjecture for $p > 2$ (embracing Lie superalgebras) and for $p = 2$ are given and in [BGL5, BGL6] there are given arguments in its favor. Let us recall these analogs.

12.4.1. Classification of simple modular Lie superalgebras: Conjectures and results. In addition to the two types of algebras (of the form $\mathfrak{g}(A)$ and "vectorial" ones), in the super case there is one more type of "symmetric" Lie algebras, the *queer* type.

12.4.1.1. New examples of simple Lie superalgebras of the form $\mathfrak{g}(A)$ and related to them. 1) Elduque superalgebras Elduque suggested a new approach to the Freudenthal Magic Square — Elduque's Supermagic Square — a way to interpret the exceptional simple Lie algebras over $\mathbb{C}$; the modular version of his method leads to a discovery of 10 new simple (exceptional) Lie superalgebras for $p = 3$. For a description of the Elduque superalgebras, see [CE, El1, CE2, El2]; for their description in terms of Cartan matrices and analogs of Chevalley generators and Serre (and non-Serre) relations and notations we use in what follows, see [BGL1, BGL2].

2) Bouarroudj-Grozman-Leites (BGL) superalgebras Recent classification [BGL5] of finite dimensional Lie superalgebras with indecomposable Cartan matrices A, and related simple subquotients of such superalgebras brought about 12 new simple Lie superalgebras in characteristics 5, 3 and 2. In particular, for $p = 2$, the new examples are $\mathfrak{e}(6,1)$, $\mathfrak{e}(6,6)$, $\mathfrak{e}(7,1)$, $\mathfrak{e}(7,6)$, $\mathfrak{e}(7,7)$, $\mathfrak{e}(8,1)$, $\mathfrak{e}(8,8)$; $\mathfrak{bgl}(4;\alpha)$ and $\mathfrak{osp}(4|2;\alpha)$; their detailed description (in terms of Cartan matrices will be given later on, and, for each $\mathfrak{g}$, we will identify $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ as $\mathfrak{g}_{\bar{0}}$-module).

12.4.2. The super analog of the KSh-conjecture. We use standard notations of [FH, S]; for a precise definition (algorithm) of generalized Cartan–Tanaka–Shchepochkina (CTS) complete and partial prolongations, see [Shch]. Let $n\mathfrak{g}$ denote the incarnation of the Lie (super)algebra $\mathfrak{g}$ with the nth Cartan matrix taken from the lists of [GL3, BGL1, BGL2]. A $\mathbb{Z}$-grading $\deg e_i^{\pm} = \pm r_i$ for the Chevalley generators $e_i^{\pm}$ of $\mathfrak{g}(A)$ is said to be *simplest* if all but one coordinates of $r = (r_1, \dots, r_{\mathrm{rk}\,\mathfrak{g}})$ are equal to 0 and only one — *selected* — is equal to 1. For the definition of restricted Lie algebras and superalgebras, see Subsection 12.1.2.

In [Le1], the following analog of the KSh-conjecture, embracing Lie superalgebras, is formulated. It is based on an idea entirely different from that of KSh-conjecture. In it, same as in [GL3, BjL, BGL6], the CTS-prolongs play the main role.

12.4.2.1. Conjecture. *For every simple finite dimensional Lie (super)-algebra of the form $\mathfrak{g}(A)$ or $\mathfrak{psq}$, take its non-positive part with respect to a certain "simplest" $\mathbb{Z}$-grading, consider its complete and* **partial** *CTS-prolongs and take their simple subquotients obtained by passage to derived algebras and factorization modulo center. In this way, and by means of the KSh-procedure, all restricted simple Lie (super)algebras should be obtained.*

Together with deformations of these Lie (super)algebras we get all simple finite dimensional modular Lie (super)algebras for $p > 2$.

For $p = 2$, there are known several more ways to obtain new algebras from the ones obtained by the ways (12.4.2.1): To obtain

$$\begin{array}{ll}\text{Lie algebras, take} & \begin{cases}\text{(a)} & \mathbb{Z}\text{-forms of complex Lie superalgebras or} \\ & \text{Volichenko algebras (forgetting squaring),} \\ & \text{taking structure constants modulo 2;} \\ \text{(b)} & \text{Shen's "variations" [Shen1],} \\ \text{(c)} & \text{Jurman's construction [Ju],}\end{cases} \\ \text{Lie superalgebras, take} & \text{(d) queerification (17.1.1).}\end{array} \tag{12.45}$$

In addition to these, there is one more method, most difficult to formulate precisely:

(e) construct Lie algebras and Lie superalgebras "by analogy".

(And again, some of the algebras obtained by the above methods might be not simple; one has to select a simple subquotient.)

Whenever we can, we consider the results of all these methods: These results might be distinct. Sometimes, it is not clear how to apply the method (e): For example, for Lie (super)algebras preserving a non-degenerate bilinear form it is more or less clear what to do (although not immediately, cf. [Le1]), but what is an analog of the exceptional Lie algebra $\mathfrak{g}(2)$ if $p = 2$?

12.4.2.2. Conjecture. *Together with deformations of the examples obtained by means of (a)–(e), we obtain all simple finite dimensional Lie superalgebras over algebraically closed field for $p = 2$.*

Chapter 13

Non-degenerate bilinear forms in characteristic 2, related contact forms, simple Lie algebras and Lie superalgebras

13.1. Introduction

13.1.1. Motivations. Recall that to any bilinear form B on a given space V one can assign its *Gram matrix* by abuse of notations also denoted by $B = (B_{ij})$: in a fixed basis $x_1, \ldots, x_n$ of V we set

$$B_{ij} = B(x_i, x_j). \tag{13.1}$$

In what follows, we fix a basis of V and identify every bilinear form with its matrix.

Two bilinear forms B and C on V are said to be *equivalent* if there exists an invertible linear operator $A \in GL(V)$ such that

$$B(x, y) = C(Ax, Ay) \quad \text{for all } x, y \in V;$$

in this case,

$$B = ACA^T \tag{13.2}$$

for the matrices of B, C and A in the same basis.

A bilinear form B on V is said to be *symmetric* if $B(v, w) = B(w, v)$ for any $v, w \in V$; it is *anti-symmetric* if $B(v, v) = 0$ for any $v \in V$.

Given a bilinear form B, let

$$L(B) = \{F \in \text{End } V \mid B(Fx, y) + B(x, Fy) = 0\}.$$

be the Lie algebra that preserves B. If $p \neq 2$, some of the Lie algebras $L(B)$ are simple, for example, the orthogonal Lie algebras $\mathfrak{o}_B(n)$ that preserve non-degenerate symmetric forms and symplectic Lie algebras $\mathfrak{sp}_B(n)$ that preserve non-degenerate anti-symmetric forms.

If $p = 2$, either the derived algebras of $L(B)$ for non-degenerate forms B or their quotient modulo center are simple, so the canonical expressions of the forms B are needed as a step in classification of simple Lie algebras in characteristic 2 which is an open problem, and as a step in a version of this problem for Lie superalgebras, even more open.

The problem of describing preserved bilinear forms has two levels: we can consider *linear transformations* (Linear Algebra) and *arbitrary coordinate changes* (Differential Geometry). In the literature, both levels are completely investigated, except for the case where $p = 2$.

More precisely, for $p = 2$, there are obtained rather esoteric results such as classifications of quadratic forms over skew fields [ET], and of analogs of hermitian forms in infinite dimensional spaces [Gro], whereas (strangely enough for such a classically formulated problem) the non-degenerate bilinear forms over fields were never classified, except for symmetric forms. Moreover, the fact that the non-split and split forms of the Lie algebras that preserve the symmetric forms are not always isomorphic was never mentioned (although known on the Chevalley group level), cf. the latest papers with reviews of earlier results ([Sh, GG]).

Hamelink [H] considered simple Lie algebras over $\mathbb{K}$ but under too restrictive conditions (he considered only Lie algebras with a non-degenerate invariant form) and so missed many simple Lie algebras; besides, it seems he has missed several series of Lie algebras even satisfying his assumptions (such as $\mathfrak{h}'$).

The bilinear forms over fields of characteristic 2 also naturally appear in topological problems of the theory of real manifolds, for example, in singularity theory: As related to “symplectic analogs of Weyl groups” and related bilinear forms over $\mathbb{Z}/2$, cf. [I].

We also consider superspaces. The Lie superalgebras over $\mathbb{Z}/2$ were of huge interest in the 1930s–60s in relation with other applications in topology, see, e.g., [Ha, May]; lately, the interest comes back [Vo].

Let us review the known results and compare them with the new ones (§§3–8).

13.1.2. Known facts: The case $p \neq 2$. Having fixed a basis of the space on which bilinear or quadratic form is considered, we identify the form with its Gram matrix; this is understood throughout. Let me recall certain facts (both well known and not so well known).

Linear Algebra ([Pra], [Lang]). Any bilinear form B on a finite dimensional space V can be represented as the sum $B = S + K$ of a symmetric and a anti-symmetric form. Classics investigated $B = S + K$ by considering it as a member of the *pencil* $B(\lambda, \mu) = \lambda S + \mu K$, where $\lambda, \mu \in \mathbb{K}$, and studying invariants of $B(\lambda, \mu)$, cf. [Ga].

Besides, for $p \neq 2$, the space of bilinear forms is a direct sum of subspaces of symmetric and anti-symmetric forms.

If $p = 2$, we have an invariant subspace of symmetric forms and the quotient space of non-symmetric forms.

Over $\mathbb{C}$, there is only one class of symmetric forms and only one class of anti-symmetric forms ([Pra]). For a canonical form of the matrix of the form

B, one usually takes J_{2n} for the anti-symmetric forms and 1_n, Π_n or S_n for symmetric forms.

In order to have Cartan subalgebra of the orthogonal Lie algebra on the main diagonal (to have a split form of $\mathfrak{o}_B(n)$), one should take B of the shape Π_n or S_n, not 1_n. Over $\mathbb{R}$, the Lie algebra might have no split form; for purposes of representation theory, it is convenient therefore to take its form most close to the split one, see [FH].

Over $\mathbb{R}$, as well as over any ordered field, Sylvester's theorem states that the signature of the form is the only invariant ([Pra]).

Over an algebraically closed field $\mathbb{K}$ of characteristic $p \neq 2$, Ermolaev considered nondegenerate bilinear forms $B\colon V \times V \longrightarrow \mathbb{K}$, and gave the following description of the Lie algebras $L(B)$ (for details, see [Er]):

13.1.3. Statement ([Er])**.** *The Lie algebra $L(B)$ can not be represented as a direct sum of ideals (of type $L(B)$) if and only if all elementary divisors of the matrix B belong to the same point of the variety*

$$P = (\mathbb{K}^*/\{1, -1\}) \cup \{0\} \cup \{\infty\},$$

where $\mathbb{K}^*$ is the multiplicative group of $\mathbb{K}$. To each of the three different types of points in P (elementary divisors corresponding to 0, to ∞ or to a point of $\mathbb{K}^*/\{1, -1\}$), a series of Lie algebras $L(B)$ corresponds, and each of these algebras depends on a finite system of integer parameters.

Differential Geometry. Let the form $B = B(x)$ depend on a parameter x running over a (super)manifold M, and let this form be on the space T_x^*M, cotangent to this manifold. Locally, there are obstructions to reducing nondegenerate 2-form $B(x)$ on a (super)manifold to the canonical expression. These obstructions are the Riemann tensor if $B(x)$ is symmetric (metric) and dB if $B(x)$ is a anti-symmetric (differential) 2-form over $\mathbb{C}$ or $\mathbb{R}$; for these obstructions expressed in cohomological terms, see [LPS]. Elsewhere we intend to classify analogous obstructions to local reducibility of bilinear forms to canonical expressions found here.

13.1.4. Known facts: The case $p = 2$. 1) With any symmetric bilinear form B a quadratic form $Q(x) := B(x, x)$ is associated. The other way round, given a quadratic form Q, we define a symmetric bilinear form, called *the polar form* of Q, by setting

$$B_Q(x, y) = Q(x + y) - Q(x) - Q(y).$$

As we will see, the correspondence $Q \longleftrightarrow B_Q$ is one-to-one if and only if $p \neq 2$; moreover, if $p = 2$ it does not embrace non-symmetric forms.

Arf [Arf] has discovered *the Arf invariant* — an important invariant of nondegenerate quadratic forms in characteristic 2; for an exposition, see [Dye]. Two such forms are equivalent if and only if their Arf invariants are equal.

The Arf invariant, however, can not be used for classification of symmetric bilinear forms because one symmetric bilinear form can serve as the polar form for two non-equivalent (and having different Arf invariants) quadratic forms. Moreover, not every symmetric bilinear form can be represented as a polar form.

2) Albert [A] classified symmetric bilinear forms over a field of characteristic 2 and proved that

(1) two anti-symmetric symmetric forms (he calls a form B on V *alternate* if $B(x,x) = 0$ for every $x \in V$) of equal ranks are equivalent;

(2) every non-anti-symmetric form has a matrix which is equivalent to a diagonal matrix;

(3) if $\mathbb{K}$ is perfect (i.e., such that every element of $\mathbb{K}$ has a square root), then every two non-anti-symmetric forms of equal ranks are equivalent.

13.1.4.1. Remark. Let $p = 2$. Since $a^2 - b^2 = (a-b)^2$, it follows that no element can have two distinct square roots.

3) Albert also gave some results on the classification of quadratic forms over a field $\mathbb{K}$ of characteristic 2 (considered as elements of the quotient space of all bilinear forms by the space of anti-symmetric forms). In particular, he showed that if $\mathbb{K}$ is algebraically closed, then every quadratic form is equivalent to exactly one of the forms

$$x_1x_{r+1} + \dots + x_rx_{2r} \quad \text{or} \quad x_1x_{r+1} + \dots + x_rx_{2r} + x_{2r+1}^2, \tag{13.3}$$

where $2r$ is the rank of the form.

4) Albert also considered *semi-definite* bilinear forms, i.e., symmetric forms, which are equivalent to forms whose matrix is of the shape

$$\begin{pmatrix} 1_k & 0 \\ 0 & 0 \end{pmatrix}.$$

For $p = 2$, semi-definite forms constitute a linear space. In order not to have every non-anti-symmetric form semi-definite, the ground field should not be perfect. For this, $\mathbb{K}$ must be neither algebraically closed nor finite.

5) Skryabin [Sk] considered the case of the space V with a flag

$$\mathcal{F} : 0 = V_0 \subset V_1 \subset \dots \subset V_q = V,$$

and the equivalence of bilinear forms w.r.t. operators which, in addition to (13.2), *preserves* $\mathcal{F}$. He showed that under such equivalence the class of a (possibly, degenerate) anti-symmetric bilinear form is determined by parameters

$$n_{qr} = \dim(V_q \cap V_{r-1}^{\perp})/(V_q \cap V_r^{\perp} + V_{q-1} \cap V_{r-1}^{\perp})$$

for $q, r \geq 1$, where orthogonality is taken with respect to the form. This is true for any characteristic, but if $p = 2$, the anti-symmetric forms do not differ from zero-diagonal symmetric ones.

13.1.5. Lie (super)algebras in finite characteristic. There are different ways to generalize Lie (super)algebras defined over $\mathbb{C}$ or $\mathbb{R}$ to the case of finite characteristic. Here are the main ones:

1) By the meaning (structures they preserve). This way is used in this chapter for orthogonal algebras: the orthogonal algebra $\mathfrak{o}_B(n)$ is the algebra of linear transformations preserving the given non-degenerate symmetric bilinear form B.

2) By the multiplication table. Let $\mathcal{B} = \{g_i\}_{i \in I}$ be an *integer basis* of Lie algebra $\mathfrak{g}$ over a field of characteristic 0, i.e., all the structure constants of the algebra in this basis are integer. Let $\mathfrak{g}_{\mathcal{B},\mathbb{Z}}$ be the $\mathbb{Z}$-span of $\mathcal{B}$ in this basis. Then for any field $\mathbb{K}$, we see that

$$(\mathfrak{g}_{\mathcal{B},\mathbb{Z}})_{\mathbb{K}} = \mathfrak{g}_{\mathcal{B},\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{K}$$

is a Lie algebra over $\mathbb{K}$. The structure of this algebra essentially depends on the choice of the basis $\mathcal{B}$ (for example, if one multiplies all the elements of some integer basis by p, then the algebra constructed by the resulting basis over a field of characteristic p is commutative).

3) By generators and defining relations (all the coefficients in the relations must be integer). One should be careful in this case — for example, the Lie algebra in characteristic 2 constructed by the Serre relations for $\mathfrak{sl}(n)$ is infinite dimensional for $n \geq 4$, see Chapter on Presentations.

4) Lie algebras with Cartan matrix can be recovered from this matrix by more-or-less explicit rules.

In characteristic 2, anti-symmetric bilinear forms are symmetric, actually, so the definition of a symplectic Lie algebra "by the meaning" does not give anything new. Note that if we define the Lie algebra $\mathfrak{sp}(2n)$ consisting of linear transformations preserving the anti-symmetric bilinear form J_{2k} in characteristic 2 by the multiplication table, choosing the basis of matrix units, the resulting algebra is $\mathfrak{o}_{\Pi}(2n)$.

13.1.6. The structure of the Chapter. In §2 we reproduce Albert's results on classification of symmetric bilinear forms with respect to the classical equivalence (13.2).

In §3 we consider other approaches to the classification of non-symmetric bilinear forms, select the most interesting and adequate one (the "sociological" one) and describe the corresponding equivalence classes.

In §4 we classify bilinear forms on superspaces with respect to the classical and sociological equivalences.

In §5 we describe some relations between equivalences of bilinear forms and 1-forms.

In §6 we explicitly describe canonical forms of symmetric bilinear forms, related simple Lie algebras, and their Cartan subalgebras.

In §7 and §8, we give a super versions of §6 and §3, respectively.

13.1.7. Remarks. 1) In the study of simple Lie algebras over a field of characteristic $p > 0$, one usually takes an algebraically closed or sometimes finite ground field. Accordingly, these are the cases where bilinear or quadratic forms are to be considered first.

2) For quadratic forms in characteristic 2, we can also use Bourbaki's definition: q is quadratic if $q(ax) = a^2q(x)$ and $B(x, y) = q(x+y)-q(x)-q(y)$ is a bilinear form.

3) For some computations, connected with Lie algebras of linear transformations, preserving a given bilinear form (e.g., computations of Cartan prolongs), it is convenient to choose the form so that the corresponding Lie algebra has a Cartan subalgebra as close to the algebra of diagonal matrices as possible. It is shown in §6 that in the case of bilinear forms, equivalent to 1_n, over a space of even dimension, we need to take the form in one of the following shapes equivalent to 1_n:

$$\begin{pmatrix} 1_2 & 0 \\ 0 & S_{n-2} \end{pmatrix} \sim \begin{pmatrix} 1_2 & 0 \\ 0 & \Pi_{n-2} \end{pmatrix}.$$

The corresponding Cartan subalgebras consist of matrices of the following shape, respectively:

$$\left(\begin{array}{cc|c} 0 & a_0 & 0 \\ a_0 & 0 & 0 \\ \hline 0 & 0 & \mathrm{diag}_n(a_1, \dots, a_{k-1}, a_{k-1}, \dots, a_1) \end{array}\right)$$

or

$$\left(\begin{array}{cc|c} 0 & a_0 & 0 \\ a_0 & 0 & 0 \\ \hline 0 & 0 & \mathrm{diag}_n(a_1, \dots, a_{k-1}, a_1, \dots, a_{k-1}) \end{array}\right).$$

13.2. Symmetric bilinear forms (Linear Algebra)

13.2.1. Theorem. *Let $\mathbb{K}$ be a perfect field of characteristic 2. Let V be a n-dimensional space over $\mathbb{K}$.*

1) *For n odd, there is only one equivalence class of non-degenerate symmetric bilinear forms on V.*

2) *For n even, there are two equivalence classes of non-degenerate symmetric bilinear forms, one contains 1_n and the other one contains S_n.*

Later we show that, if n is even, a non-degenerate symmetric bilinear form is equivalent to S_n if and only if its matrix is zero-diagonal.

Observe that the fact that the bilinear forms are not equivalent does not imply that the Lie (super)algebras that preserve them are not isomorphic; therefore the next Lemma is non-trivial.

13.2.1.1. Lemma. 1) *The Lie algebras $\mathfrak{o}_I(2k)$ and $\mathfrak{o}_S(2k)$ are not isomorphic; the same applies to their derived algebras:*

2) $\mathfrak{o}_I^{(1)}(2k) \not\simeq \mathfrak{o}_S^{(1)}(2k)$;

3) $\mathfrak{o}_I^{(2)}(2k) \not\simeq \mathfrak{o}_S^{(2)}(2k)$.

13.2.2. Proof of Theorem 13.2.1. In what follows let E_{ij}, where $1 \le i, j \le n$, be a matrix unit, i.e., $(E_{i,j})_{kl} := \delta_{ik}\delta_{jl}$, and

$$T_{i,j} := I + E_{i,i} + E_{j,j} + E_{i,j} + E_{j,i}.$$

Note that the $T_{i,j}$ are invertible, and $T_{i,j} = T_{j,i} = (T_{i,j})^T$.

Note also, that any bilinear form B is equivalent to aB for any $a \in \mathbb{K}$ if $a \neq 0$. Indeed, since every element of $\mathbb{K}$ has a square root, $aB = (b1_n)B(b1_n)$, where $b \in \mathbb{K}$ is such that $b^2 = a$.

Now, let us first prove the following

13.2.3. Lemma. *Let B be a symmetric $n \times n$ matrix, and $\overline{B}$ be $n' \times n'$ matrix in the upper left corner of B, $n' < n$. Then, if $\overline{B}$ is invertible, B is equivalent to a matrix of the form*

$$\begin{pmatrix} \overline{B} & 0 \\ 0 & \widehat{B} \end{pmatrix}.$$

Proof. Let C be $(n-n') \times n'$ matrix in the lower left corner of B, and

$$M = \begin{pmatrix} 1_{n'} & 0 \\ C\overline{B}^{-1} & 1_{n-n'} \end{pmatrix}.$$

The matrix M is invertible because it is lower-triangular and has no zeros on the diagonal. Direct calculations show that the matrix MBM^T has the needed form. □

13.2.4. Lemma. *If B and C are $n \times n$ matrices, and*

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix},$$

where B_1 and C_1 are equivalent $n' \times n'$ matrices, and B_2 and C_2 are equivalent $(n-n') \times (n-n')$ matrices, then B and C are equivalent.

Proof. If $M_1B_1M_1^T = C_1$ and $M_2B_2M_2^T = C_2$, and $M = \mathrm{diag}(M_1, M_2)$, then $MBM^T = C$. □

Let us fix terminology. A bilinear form B is said to be *anti-symmetric* (Albert called them *alternate*) if the corresponding quadratic form $Q(x) = B(x, x)$ is identically equal to 0.

13.2.5. Lemma. *A bilinear form is anti-symmetric if and only if its matrix is zero-diagonal.*

Proof. Let $e_1, \dots, e_n$ be the basis in which matrix is taken. If B is anti-symmetric, the $B_{ii} = B(e_i, e_i) = 0$. On the other hand, if the matrix of B is zero-diagonal, and $e = \sum_i c_ie_i$, then

$$B(e, e) = \sum_{i,j} B_{ij}c_ic_j = 2\sum_{i<j} B_{ij}c_ic_j = 0. \quad \square$$

Since an anti-symmetric form can be equivalent only to an anti-symmetric form, we have

13.2.6. Corollary. *If matrices A and B are symmetric and equivalent, and A is zero-diagonal, then B is zero-diagonal.*

Now, let us prove the following part of Theorem 13.2.1:

If $n = 2k$, any non-degenerate symmetric zero-diagonal matrix $B \in GL(n)$ is equivalent to the matrix Z_{2k}.

We will prove a more general statement that will be needed later:

13.2.7. Lemma. *If B is a zero-diagonal $n\times n$ matrix (possibly, degenerate), then $r = \operatorname{rank} B$ is even, and B is equivalent to the matrix*

$$\tilde{Z}_{n,r} = \begin{pmatrix} Z_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. We will induct on n. In the cases $n = 1, 2$, the statement is evident.

If $B = 0$, the statement follows immediately. Otherwise, there exist i, j such that $B_{ij} \neq 0$, and B is equivalent to

$$C = (B_{i,j})^{-1} T_{2,j} T_{1,i} B T_{1,i} T_{2,j},$$

and $C_{12} = C_{21} = 1$, $C_{11} = C_{22} = 0$.

Then, by Lemma 13.2.3, C is equivalent to a matrix D of the form

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & D_1 \end{array}\right),$$

where D_1 is a $(n-2)\times(n-2)$ matrix. Since D is, by Corollary 13.2.6, symmetric and zero-diagonal, D_1 is also symmetric and zero-diagonal, and $\operatorname{rank} D_1 = r - 2$. Then, by the induction hypothesis, $r-2$ is even, and D_1 is equivalent to $\tilde{Z}_{n-2,r-2}$, and, by Lemma 13.2.4, D is equivalent to $\tilde{Z}_{n,r}$. Therefore, B is equivalent to $\tilde{Z}_{n,r}$. □

Now let us prove the following:

For n odd, any non-degenerate symmetric $n\times n$ matrix is equivalent to 1_n;

for n even, any non-degenerate symmetric $n\times n$ matrix which is not zero-diagonal is also equivalent to 1_n.

We will prove this by induction on n (simultaneously for n odd and even). For $n = 1$, the statement is evident.

Now, let n be even. If B is an invertible symmetric $n\times n$ matrix, $B_{ii} \neq 0$, then B is equivalent to $C = (B_{ii})^{-1} T_{1,i} B T_{1,i}$, and $C_{11} = 1$. Then, by Lemma 13.2.3, C is equivalent to matrix D of the form

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & D_1 \end{array}\right), \quad \text{where } D_1 \in \mathfrak{gl}(n-1). \tag{13.4}$$

Since B is symmetric and non-degenerate, D_1 is also symmetric and non-degenerate. Then, by induction hypothesis, D_1 is equivalent to 1_{n-1}, and, by Lemma 13.2.4, D is equivalent to 1_n, and B is also equivalent to 1_n.

If B is an invertible symmetric $n\times n$ matrix, and n is odd, then by Lemma 13.2.7, B has at least one non-zero element on the diagonal, and, similarly, we can show that B is equivalent to a matrix D of the form (13.4). Since B is symmetric and non-degenerate, D_1 is also symmetric and non-degenerate. Then, by induction hypothesis, D_1 is equivalent to either 1_{n-1} or Z_{n-1}, and, by Lemma 13.2.4, B is equivalent to either 1_n or

$$\widehat{Z}_n = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & Z_{n-1} \end{array}\right).$$

Let M be a $n\times n$ matrix such that

$$M_{ij} = \begin{cases} 1 & \text{if } i = 1 \text{ or } j = 1, \\ & \text{or if } j = i, \\ & \text{or if } j = i+1,\ i \text{ is odd}, \\ & \text{or if } j > i+1; \\ 0 & \text{if } j = i+1,\ i \text{ is even}, \\ & \text{or if } 1 < j < i. \end{cases}$$

Direct calculation shows, that $MM^T = \widehat{Z}_n$, so $\widehat{Z}_n$ is equivalent to 1_n, and B is equivalent to 1_n.

Now, to finish the proof of the theorem, we need to show that, for n even, 1_n and Z_n are not equivalent, which follows from Corollary 13.2.6. Theorem 13.2.1 is proved. □

13.2.8. Proof of Lemma 13.2.1.1. Direct computations show that (for details, see §6):

a) $\mathfrak{o}_I(n)$ is spanned by symmetric matrices;

b) $\mathfrak{o}_I^{(1)}(n)$ is spanned by zero-diagonal symmetric matrices;

c) $\mathfrak{o}_S(n)$ is spanned by matrices symmetric relative the side diagonal.

Indeed, let $C(\mathfrak{g})$ be the center of the Lie algebra $\mathfrak{g}$. We see that $1_n \in C(\mathfrak{o}_I(n))$, and $\dim C(\mathfrak{o}_I(n)) = 1$, because if $A \in \mathfrak{o}_I(n)$, and $A_{ii} \neq A_{jj}$, then $[A, E_{i,j} + E_{j,i}]_{ij} = A_{ii} + A_{jj} \neq 0$, and if $A_{ij} \neq 0$ for $i \neq j$, then $[A, E_{i,i}]_{ij} = A_{ij} \neq 0$. Since matrices from $\mathfrak{o}_I(n)^{(1)}$ are zero-diagonal ones, $\dim(C(\mathfrak{o}_I(n)) \cap \mathfrak{o}_I(n)^{(1)}) = 0$. At the same time,

$$1_n, \quad \sum_{i=1}^{k} E_{2i-1,2i}, \quad \sum_{i=1}^{k} E_{2i,2i-1} \in \mathfrak{o}_S(n); \qquad \text{and } 1_n \in C(\mathfrak{o}_S(n)),$$

but $[\sum_{i=1}^k E_{2i-1,2i},\ \ \sum_{i=1}^k E_{2i,2i-1}] = 1_n$, so $\dim(C(\mathfrak{o}_S(n)) \cap \mathfrak{o}_S(n)^{(1)}) \neq 0$, which shows that $\mathfrak{o}_I(n)$ and $\mathfrak{o}_S(n)$ are not isomorphic. Lemma is proved. □

13.3. Non-symmetric bilinear forms (Linear algebra)

13.3.1. Non-symmetric bilinear forms: Discussion. If $p = 2$, there is no canonical way to separate symmetric part of a given bilinear form from its non-symmetric part. In this subsection I list several more-or-less traditional equivalences before suggesting (in the next subsection) the one that looks the best.

1) The standard definition (13.2). This equivalence is too delicate: there are too many inequivalent forms: the classification problem looks wild.

2) The idea of classics (see, e.g., [Ga]) was to consider the following equivalence of non-degenerate bilinear forms regardless of their symmetry properties. Observe that any bilinear form B on V can be considered as an operator

$$\widetilde{B}\colon V \longrightarrow V^* \quad x \longmapsto B(x, \cdot). \tag{13.5}$$

If B is non-degenerate, then $\widetilde{B}$ is invertible. Two forms B and C are said to be *roughly equivalent*, if the operators $\widetilde{B}^{-1}\widetilde{B}^*$ and $\widetilde{C}^{-1}\widetilde{C}^*$ in V, where $*$ denotes the passage to the adjoint operator, are equivalent. This equivalence is, on the contrary, too rough: it does not distinguish between symmetric forms with non-zero entries on the diagonal and anti-symmetric forms, so *all* symmetric non-degenerate bilinear forms are roughly equivalent, for both odd- and even-dimensional V.

3) Leites suggested to call two bilinear forms B_1 and B_2 *Lie-equivalent* (we write $B_1 \simeq_L B_2$) if the Lie algebras that preserve them are isomorphic. This does reduce the number of non-equivalent forms but only slightly as compared with (13.2) and no general pattern is visible, see the following Examples for $n = 2, 3, 4$. So this equivalence is also, as (13.2), too delicate.

13.3.1.1. Examples. $\underline{n = 2,\ \mathbb{K} = \mathbb{Z}/2.}$ In this case, there exist only two non-symmetric non-degenerate matrices:

$$\begin{pmatrix}1&1\\0&1\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}1&0\\1&1\end{pmatrix},$$

and they are equivalent.

$\underline{n = 2,\ \mathbb{K}\text{ infinite.}}$ In this case, there exist *infinitely many* equivalence classes of non-symmetric non-degenerate forms. For example,

$$\mathrm{antidiag}_2(1, a) \sim \mathrm{antidiag}_2(1, b) \quad \text{only if either } a = b \text{ or } ab = 1.$$

But all these classes are Lie-equivalent: any non-symmetric non-degenerate 2×2 matrix is only preserved by scalar matrices.

$\underline{n = 3,\ \mathbb{K} = \mathbb{Z}/2.}$ In this case, there exist 3 equivalence classes with the following representatives:

$$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}, \quad \begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}, \quad \begin{pmatrix}0&0&1\\0&1&0\\1&1&0\end{pmatrix}.$$

Note that the last two matrices are equivalent as forms over an extension of $\mathbb{Z}/2$ with 4 elements. All these matrices are Lie-equivalent — the corresponding Lie algebras are 2-dimensional and, since they contain 1_3, commutative.

$\underline{n = 3,\ \mathbb{K}\text{ infinite.}}$ In this case, again, there exist infinitely many equivalence classes.

13.3.1.2. Conjecture. *All non-symmetric non-degenerate* 3×3 *matrices are Lie-equivalent and the corresponding Lie algebras are* 2*-dimensional and commutative.*

$\underline{n = 4,\ \mathbb{K} = \mathbb{Z}/2.}$ In this case, there exist 8 equivalence classes with the following representatives:

$$B_1 = \begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}; \quad B_2 = \begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix}; \quad B_3 = \begin{pmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{pmatrix}; \quad B_4 = \begin{pmatrix}1&1&0&0\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix};$$

$$B_5 = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&1&0&0\end{pmatrix}; \quad B_6 = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&1&0\\1&0&1&1\end{pmatrix}; \quad B_7 = \begin{pmatrix}0&0&0&1\\0&0&1&0\\1&0&0&0\\1&1&0&1\end{pmatrix}; \quad B_8 = \begin{pmatrix}0&0&0&1\\0&1&0&0\\0&1&1&0\\1&0&0&0\end{pmatrix};$$

The matrices in the pairs (B_1, B_4), (B_3, B_7), (B_5, B_6) are Lie-equivalent, so there are 5 Lie-equivalence classes.

$\underline{n = 4,\ \mathbb{K}\text{ infinite.}}$ Again, there exist infinitely many equivalence classes. There exist also at least 5 Lie-equivalence classes, described in the previous case.

13.3.1.3. Remark. The statements in these examples were obtained by computer calculations

13.3.2. A sociological approach to bilinear forms. Instead of considering non-symmetric forms individually, we can consider the quotient space $NB(n)$ of the space of all forms modulo the space of symmetric forms. We will denote the element of this quotient space with representative B, by $\{B\}$. We say that $\{B\}$ and $\{C\}$ are *equivalent* (and denote it $\{B\} \sim \{C\}$), if there exists an invertible matrix M such that

$$\{MBM^T\} = \{C\}, \text{ i.e., if } MBM^T - C \text{ is symmetric.}$$

13.3.2.1. Exercise. This definition does not depend on the choice of representatives B and C.

Any $\{B\}$ has both degenerate and non-degenerate representatives: the representative with non-zero elements only above the diagonal is degenerate, and if we add the unit matrix to it, we get a non-degenerate representative of $\{B\}$; both such representative are unique and characterize $\{B\}$.

Note that $\{B\}$ can be also characterized by the symmetric zero-diagonal matrix $B + B^T$. The rank of $B + B^T$ is said to be *the rank of* $\{B\}$. According to Lemma 13.2.7, it is always even. One can show that it is equal to doubled minimal rank of representatives of $\{B\}$.

13.3.3. Theorem. *The classes $\{B\}$ and $\{C\}$ are equivalent if and only of they have equal ranks.*

Proof. Let $\operatorname{rank}\{B\} = \operatorname{rank}\{C\}$, i.e., $\operatorname{rank}\,(B+B^T) = \operatorname{rank}\,(C+C^T)$. Since $B+B^T$ and $C+C^T$ are zero-diagonal, they are, according to Lemma 13.2.7, equivalent, i.e., there exists non-degenerate matrix M such that

$$M(B+B^T)M^T = C + C^T.$$

Then

$$MBM^T + C = (MBM^T + C)^T$$

is symmetric, and $\{B\} \sim \{C\}$. These arguments are reversible. $\square$

So, we see that $NB(n)$ has $[n/2]+1$ equivalence classes consisting of elements with ranks $0, 2, \dots, 2[n/2]$. As the representatives of these classes we can take $\{\tilde{S}^{n,m}\}$, where $m = 0, \dots, [n/2]$ and where $\tilde{S}^{n,m}$ is $n \times n$ matrix such that

$$\tilde{S}^{n,m} = \begin{array}{c c} & \begin{array}{cc} n-m & m \end{array} \\ \begin{array}{c} m \\ n-m \end{array} & \begin{pmatrix} 0 & S_m \\ 0 & 0 \end{pmatrix} \end{array}$$

The following definition of the linear transformations preserving an element of $NB(n)$ seems to be the most natural:

$$X \text{ preserves } \{B\} \text{ if } XB + BX^T \text{ is symmetric.} \tag{13.6}$$

Since

$$XB + BX^T + (XB+BX^T)^T = X(B+B^T) + (B+B^T)X^T,$$

X preserves $\{B\}$ if and only if X preserves $B+B^T$. Hence the transformations preserving $\{B\}$ do form a Lie algebra. One can check that they also form a Lie algebra if $p \neq 2$, and this algebra is the Lie algebra of transformations preserving the non-symmetric representative of $\{B\}$.

The Lie algebra $\mathfrak{o}_{\tilde{S}^{n,m}}$ consists of the matrices of the form

$$\left(\begin{array}{c|c|c} A & D & B \\ \hline 0 & E & 0 \\ \hline C & F & S_kAS_k \end{array}\right), \quad \begin{array}{l} \text{where } A \in \mathfrak{gl}(k);\ B, C \in \mathfrak{gl}(k) \text{ are} \\ \text{such that } B = S_kBS_k \text{ and } C = S_kCS_k; \\ D \text{ and } F \text{ are } (n-2k)\times k \text{ matrices; } E \in \mathfrak{gl}(n-2k). \end{array}$$

This Lie algebra is isomorphic to the semi-direct sum (the ideal on the right)

$$(\mathfrak{o}_S(2m) \oplus \mathfrak{gl}(n-2m)) \ni\!\!\!+ (R_{\mathfrak{o}} \otimes R^*_{\mathfrak{gl}}),$$

where $R_{\mathfrak{o}}$ and $R_{\mathfrak{gl}}$ are the spaces of the identity representations of $\mathfrak{o}_S(2m)$ and $\mathfrak{gl}(n-2m)$, respectively.

If $\operatorname{rank}\{B\} < n$, then the Lie algebra of linear transformations preserving $\{B\}$ is isomorphic to the Lie algebra of linear transformations preserving symmetric *degenerate* matrix $B + B^T$. So, it seems natural to call $\{B\}$ *non-degenerate* if and only if $\operatorname{rank}\{B\} = n$.

By Theorem 13.3.3, all non-degenerate elements of $NB(n)$ (they only exist if n is even) are equivalent. The Lie algebra of linear transformations preserving any of these forms is isomorphic to $\mathfrak{o}_S(2k)$.

13.4. Bilinear forms on superspaces (Linear algebra)

13.4.1. Canonical expressions of symmetric bilinear forms on superspaces and the Lie superalgebras that preserve them. For general background related to Linear Algebra in superspaces and proofs of the statements of this subsection, see [LSoS].

Speaking of superspaces we denote parity by Π, and superdimension by sdim. The operators and bilinear forms are represented by *supermatrices* which we will only consider here in the *standard format*. Recall only that to any bilinear form B on a given space V one can assign its *Gram matrix* also denoted $B = (B_{ij})$: in a fixed basis $x_1, \dots, x_n$ of V, we set (compare with (13.1); for $p = 2$, the sign disappears)

$$B_{ij} = (-1)^{\Pi(B)\Pi(x_i)} B(x_i, x_j). \tag{13.7}$$

In what follows, we fix a basis of V and identify a bilinear form with its matrix.

Two bilinear forms B and C on V are said to be *equivalent* if there exists an invertible *even* linear operator $A \in GL(V)$ such that $B(x,y) = C(Ax, Ay)$ for all $x, y \in V$; in this case, $B = ACA^T$ for the matrices of B, C and A in the same basis.

Generally, the symmetry of the bilinear forms involves signs which leads to the notion of *supertransposition* of the corresponding Gram supermatrices, but for $p = 2$ the supertransposition turns into transposition.

Recall also that, over superspaces, the parity change sends symmetric forms to anti-symmetric and the other way round, so if $p \neq 2$, it suffices to consider only symmetric forms.

In super setting, over $\mathbb{C}$, there is only one class of non-degenerate even symmetric (ortho-symplectic) forms and not more than one class of non-degenerate odd symmetric (periplectic) forms in each superdimension. Over $\mathbb{R}$, the invariants of even symmetric forms are pairs of invariants of the restriction of the form onto the even and odd subspaces and not more than one class of odd symmetric (periplectic) forms.

In order to have Cartan subalgebra of the ortho-symplectic Lie superalgebra on the main diagonal (to have a split form of $\mathfrak{osp}(n|2m)$), one should take for the canonical form of B the expression, for example,

$$\begin{pmatrix} \Pi_n & 0 \\ 0 & J_{2m} \end{pmatrix}.$$

A given symmetric even form on a superspace can be represented as a direct sum of two forms on the even subspace and the odd subspace. For each of these forms, Theorem 13.2.1 is applicable. Lemma 13.2.4 is also applicable in this case, so every even symmetric non-degenerate form on a superspace of dimension $(n_{\bar 0}|n_{\bar 1})$ over a perfect field of characteristic 2 is equivalent to a form of the shape (here: $i = \bar 0$ or $\bar 1$)

$$B = \begin{pmatrix} B_{\bar 0} & 0 \\ 0 & B_{\bar 1} \end{pmatrix}, \quad \text{where } B_i = \begin{cases} 1_{n_i} & \text{if } n_i \text{ is odd;} \\ 1_{n_i} \text{ or } Z_{n_i} & \text{if } n_i \text{ is even.} \end{cases}$$

The Lie superalgebra preserving B — by analogy with the ortho-symplectic Lie superalgebras $\mathfrak{osp}$ in characteristic 0 we call it *ortho-orthogonal* and denote $\mathfrak{oo}_B(n_{\bar 0}|n_{\bar 1})$ — is spanned by the supermatrices which in the standard format are of the form

$$\begin{pmatrix} A_{\bar 0} & B_{\bar 0}C^T B_{\bar 1}^{-1} \\ C & A_{\bar 1} \end{pmatrix}, \quad \begin{array}{l} \text{where } A_{\bar 0} \in \mathfrak{o}_{B_{\bar 0}},\ A_{\bar 1} \in \mathfrak{o}_{B_{\bar 1}}, \text{ and} \\ C \text{ is arbitrary } n_{\bar 1} \times n_{\bar 0} \text{ matrix.} \end{array}$$

For an odd symmetric form B on a superspace of dimension $(n_{\bar 0}|n_{\bar 1})$ over a field of characteristic 2 to be non-degenerate, we need $n_{\bar 0} = n_{\bar 1} = k$, so the matrix of B is of the shape

$$\begin{pmatrix} 0 & \overline{B} \\ \overline{B}^T & 0 \end{pmatrix},$$

where $\overline{B}$ is a square invertible matrix. Let us take

$$M = \begin{pmatrix} 1_k & 0 \\ 0 & \overline{B}^{-1} \end{pmatrix}$$

(here $k = n_{\bar 0} = n_{\bar 1}$), then B is equivalent to

$$MBM^T = \Pi_{k|k}.$$

This form is preserved by linear transformations with supermatrices in the standard format of the shape

$$\begin{pmatrix} A & C \\ D & A^T \end{pmatrix}, \quad \text{where } A \in \mathfrak{gl}(k),\ C \text{ and } D \text{ are symmetric } k \times k \text{ matrices.} \tag{13.8}$$

As over $\mathbb{C}$ or $\mathbb{R}$, the Lie superalgebra $\mathfrak{pe}(n)$ of supermatrices (20.4) (recall that $p = 2$) will be referred to as *periplectic*.

13.4.2. Non-symmetric forms on superspaces. If a non-symmetric form on a superspace is even, it can be again represented as a direct sum of two bilinear forms: one on the even subspace, and the other one on the odd subspace. These two forms can be independently transformed to canonical forms, see 13.3.

The situation with odd non-symmetric forms is more interesting. Such a form can be non-degenerate only on a space of superdimension $(k|k)$. In the standard format, the supermatrix of such a form has the shape

$$B = \begin{pmatrix} 0 & A \\ C & 0 \end{pmatrix},$$

where A and C are invertible matrices. Let M be an invertible matrix such that $L = MC(A^T)^{-1}M^{-1}$ is the Jordan normal form of $C(A^T)^{-1}$. Then B is equivalent to

$$\begin{pmatrix} (M^T)^{-1}A^{-1} & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} 0 & A \\ C & 0 \end{pmatrix} \begin{pmatrix} ((M^T)^{-1}A^{-1})^T & 0 \\ 0 & M^T \end{pmatrix} = \begin{pmatrix} 0 & 1_k \\ L & 0 \end{pmatrix}.$$

This expression (with L in the Jordan normal form) can be considered as a canonical form of a non-degenerate odd bilinear form.

13.4.3. Statement. *Two non-degenerate forms are equivalent if and only if they have equal canonical forms.*

13.5. Relation with 1-forms (Differential geometry)

13.5.1. Matrices and 1-forms. Recall that the 1-form α on a superdomain M is said to be *contact* if it singles out a non-integrable distribution in the tangent bundle TM and $d\alpha$ is non-degenerate on the fibers of this distribution; see [GL3] and [LPS].

Let B and B' be the matrices of bilinear forms on an n-dimensional space V over a field $\mathbb{K}$ of characteristic 2. Let $x_0, x_1, \dots, x_n$ be independent indeterminates; set

$$\deg x_0 = 2,\ \deg x_1 = \dots = \deg x_n = 1.$$

We say that B and B' are 1-*form-equivalent* if there exists a degree preserving transformation, i.e., a set of independent variables $x'_0, x'_1, \dots, x'_n$ such that

$$\deg x'_0 = 2, \quad \deg x'_1 = \dots = \deg x'_n = 1, \tag{13.9}$$

which are polynomials in $x_0, x_1, \dots, x_n$ in divided powers with shearing parameter

$$\underline{N} = (N_0, \dots, N_n) \text{ such that } N_i > 1 \text{ for every } i \text{ from 1 to } n, \tag{13.10}$$

and such that

$$dx_0 + \sum_{i,j=1}^{n} B_{ij} x_i dx_j = dx'_0 + \sum_{i,j=1}^{n} B'_{ij} x'_i dx'_j. \tag{13.11}$$

13.5.2. Lemma. *B and B' are* 1*-form-equivalent if and only if* $\{B\} \sim \{B'\}$.

Proof. By (13.9), we have

$$x'_0 = cx_0 + \sum_{i=1}^{n} A_{ii} x_i^{(2)} + \sum_{1\le i<j\le n} A_{ij} x_i x_j; \quad x'_i = \sum_{j=1}^{n} M_{ij} x_j, \tag{13.12}$$

where $c \neq 0$ and M is an invertible matrix. Thanks to (13.11), comparing coefficients of dx_0 in the left- and right-hand sides, we get $c = 1$. Let A be a symmetric $n \times n$ matrix with elements A_{ij} for $i \le j$ as in (13.12). Then

$$dx'_0 + \sum_{i,j=1}^{n} B'_{ij} x'_i dx'_j = dx_0 + \sum_{i,j=1}^{n} A_{ij} x_i dx_j + \sum_{i,j,k,l=1}^{n} M_{ki} B'_{kl} M_{lj} x_i dx_j,$$

i.e., $B = M^T B' M + A$, so $\{B\} \sim \{B'\}$. Since our arguments are invertible, the theorem is proved. □

13.5.3. The case of odd indeterminates. Let us modify the definition of 1-form-equivalence to adjust it to the super case where $x_1, \dots, x_n$ are all *odd*. In this case, we can only use divided powers with $\underline{N} = (N_0, 1, \dots, 1)$.

We say that B and B' are 1-*superform-equivalent* if there exists a set of indeterminates $x'_0, x'_1, \dots, x'_n$, which are polynomials in $x_0, x_1, \dots, x_n$, such that

$$\begin{array}{l} \Pi(x'_0) = \bar{0},\ \Pi(x'_1) = \dots = \Pi(x'_n) = \bar{1}, \\ \deg x'_0 = 2,\ \deg x'_1 = \dots = \deg x'_n = 1 \end{array} \tag{13.13}$$

and

$$dx_0 + \sum_{i,j=1}^{n} B_{ij} x_i dx_j = dx'_0 + \sum_{i,j=1}^{n} B'_{ij} x'_i dx'_j. \tag{13.14}$$

13.5.4. Lemma. *The matrices B and B' are 1-superform-equivalent if and only if exist an invertible matrix M and a symmetric zero-diagonal matrix A such that*

$$B = MB'M^T + A. \tag{13.15}$$

Proof. It is analogous to the proof of Lemma 13.5.2. □

Albert [A] considered the equivalence (13.15) as an equivalence of (matrices of) quadratic forms. In particular, he proved the following

13.5.5. Statement. *If $\mathbb{K}$ is algebraically closed, every matrix B is equivalent in the sense* (13.15) *to exactly one of the matrices*

$$Y(n,r) = \begin{pmatrix} 0_r & 1_r & 0 \\ 0_r & 0_r & 0 \\ 0 & 0 & 0_{n-2r} \end{pmatrix} \quad \textit{or} \quad \widetilde{Y}(n,r) = \begin{pmatrix} 0_r & 1_r & 0 & 0 \\ 0_r & 0_r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0_{n-2r-1} \end{pmatrix},$$

where $2r = rank(B+B^T)$. The corresponding quadratic form is non-degenerate if and only if either a) $n = 2r$, or b) $n = 2r + 1$ and the matrix is equivalent to $\widetilde{Y}(n,r)$.

If, in 1-form-equivalence, we consider divided powers with shearing parameter $\underline{N} = (N_0, 1, \dots, 1)$, it is the same as to consider 1-superform-equivalence.

13.5.5.1. Remark. Albert worked long before supersymmetry was defined. Serendipity of serious researchers is often amazing: how could one come to such a definition as equivalence (13.15)?!

13.5.6. Lemma. *Let $x_0, \dots, x_n$ be indeterminates,*

$$\Pi(x_0) = \bar{0}, \quad \Pi(x_1) = \dots = \Pi(x_n) = \Pi.$$

Then the 1-form on the $(n+1|0)$-dimensional (if $\Pi = \bar{0}$) or $(1|n)$-dimensional (if $\Pi = \bar{1}$) superspace

$$\alpha = dx_0 + \sum_{i,j=1}^{n} B_{ij} x_i dx_j \tag{13.16}$$

is contact if and only if one of the following conditions holds:

1) $\Pi = \bar{0}$, and $\{B\}$ is non-degenerate, i.e., $n = rank(B + B^T)$ (this rank is always even);

2) $\Pi = \bar{1}$, and the quadratic form corresponding to B is non-degenerate, i.e., either a) $n = rank(B + B^T)$ or b) B is not zero-diagonal and $n = rank(B + B^T) + 1$.

Proof. From Theorem 13.3.3, Statement 13.5.5 ([A]), and Lemmas 13.5.2 and 13.5.4 we know that, if $\Pi = \bar{0}$, every symmetric bilinear form is 1-form-equivalent to one of the forms $\widetilde{S}(n,r)$, $n \ge 2r$, and, if $\Pi = \bar{1}$, every symmetric bilinear form is 1-superform-equivalent to one of the forms $Y(n,r)$, where $n \ge 2r$, or $\widetilde{Y}(n,r)$, where $n \ge 2r+1$. Direct calculations show that if $\Pi = \bar{0}$, the 1-form (13.16) corresponding to $\widetilde{S}(n,r)$ is contact if and only if $n = 2r$; if $\Pi = \bar{1}$, then the 1-form (13.16) corresponding to $Y(n,r)$ is contact if and only if $n = 2r$ and the 1-form, corresponding to $\widetilde{Y}(n,r)$ is contact if and only if $n = 2r+1$. Since, by definition, two 1-forms that correspond to 1-(super)form-equivalent bilinear forms can be transformed into each other by a change of coordinates, we are done. □

From this, we get the following

13.5.6.1. Theorem. *The following are the canonical expressions of the odd contact forms if the indeterminates $x_1, \dots x_n$ are of the same parity* (for the general case, see Theorem 13.5.8)*:*

$$\alpha = dx_0 + \sum_{i=1}^{k} x_i dx_{k+i} \begin{cases} & \textit{for } n=2k \textit{ and } x_1,\dots,x_n \textit{ all even or all odd;} \\ +x_n dx_n & \textit{for } n=2k+1 \textit{ and } x_1,\dots x_n \textit{ odd.} \end{cases} \tag{13.17}$$

13.5.6.2. Remarks. 1) If $n > 1$ and $x_1, \dots x_n$ are odd, the 1-form $\alpha = dx_0 + \sum\limits_{i=1}^{n} x_i dx_i$ is not contact since (recall that $p = 2$)

$$\alpha = d\Big(x_0 + \sum_{i<j} x_i x_j\Big) + \Big(\sum_{i=1}^{n} x_i\Big) d\Big(\sum_{i=1}^{n} x_i\Big).$$

2) Let $p = 2$. Since there are two types of orthogonal Lie algebras if n is even, and orthogonal algebras coincide, in a sense, with symplectic ones, it seems natural to expect that there are also two types of the Lie algebras of hamiltonian vector fields (preserving I and S, respectively). This is indeed so, see [L2, ILL].

Are there two types of contact Lie algebras corresponding to these cases? The (somewhat unexpected) answer is NO:

The classes of 1-(super)form-equivalence of bilinear forms which correspond to contact forms have nothing to do with classes of classical equivalence of symmetric bilinear forms. The 1-forms, corresponding to symmetric bilinear forms are exact if $x_1, \dots x_n$ are even, and are of rank ≤ 2 if $x_1, \dots x_n$ are odd.

Recall the contact Lie superalgebra consists of the vector fields D that preserve the contact structure (non-integrable distribution given by a contact form α hereafter in the form (13.17)). Such fields satisfy

$$L_D(\alpha) = F_D\alpha \quad \text{for some } F_D \in \mathbb{C}[t, p, q, \theta].$$

For any $f \in \mathbb{C}[t, p, q, \theta]$, we set (the signs here are important only for $p \neq 2$):

$$K_f = (1 - E)(f)\frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t}E, \tag{13.18}$$

where $E = \sum\limits_i y_i \dfrac{\partial}{\partial y_i}$ (here the y_i are all the coordinates except t) is the *Euler operator*, and H_f is the hamiltonian field with Hamiltonian f that preserves $d\alpha$:

$$H_f = \sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right) - (-1)^{p(f)}\left(\sum_{j\leq m}\frac{\partial f}{\partial \theta_j}\frac{\partial}{\partial \theta_j}\right). \tag{13.19}$$

If one tries to build a contact algebra $\mathfrak{g}$ by means of a non-degenerate symmetric bilinear form B on the space V by setting (like it is done in characteristic 0) $\mathfrak{g}$ to be the generalized Cartan prolongation $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ (for the precise definition, see [Shch]), where the non-positive terms of $\mathfrak{g}$ are (here K_f is the contact vector field with the generating function f; for an exact formula, see, e.g., [GLS2]):

$$\mathfrak{g}_i = \begin{cases} 0 & \text{if } i \leq -3;\\ \mathbb{K}\cdot K_1 & \text{if } i = -2\\ V & \text{if } i = -1\\ \mathfrak{o}_B(V) \oplus \mathbb{K}K_t & \text{if } i = 0\end{cases}$$

and where the multiplication is given by the formulas

$$\begin{aligned}&[X, Y] = B(X, Y)K_1 \text{ for any } X, Y \in \mathfrak{g}_{-1};\\ &\mathfrak{o}_B(V) \text{ acts on } V \text{ via the standard action;}\\ &[\mathfrak{g}_0, \mathfrak{g}_{-2}] = 0;\\ &K_t \text{ acts as id on } \mathfrak{g}_{-1},\\ &[K_t, \mathfrak{o}_B(V)] = 0,\end{aligned}$$

then the form B must be zero-diagonal one (because $0 = [X, X] = B(X, X)K_1$ for $X \in \mathfrak{g}_{-1}$).

One can also try to construct a Lie superalgebra in a similar way by setting $\Pi(\mathfrak{g}_{-1}) = \bar{1}$ and

$$X^2 = B(X, X)K_1 \text{ for any } X \in \mathfrak{g}_{-1}. \tag{13.20}$$

Let us realize this Lie superalgebra by vector fields on a superspace of superdimension $(1|n)$ with basis $x_0, \dots, x_n$ such that

$$\Pi(x_0) = \bar{0}; \quad \Pi(x_i) = \bar{1} \text{ for } 1 \leq i \leq n.$$

If $e_1, \dots, e_n$ is a basis of V and we set $\Big(\text{here } \partial_i = \dfrac{\partial}{\partial x_i} \text{ for } i = 0, \dots, n\Big)$:

$$K_1 = \partial_0; \quad e_i = \partial_i + \sum_{j=1}^{n} A_{ij}x_j\partial_0 \text{ for } i = 1, \dots, n,$$

then, to satisfy relations (13.20), we need the following (here the Gram matrix B is taken in the basis $e_1, \dots, e_n$):

$$\begin{aligned}&A_{ii} = B_{ii} \text{ for } 1 \leq i \leq n;\\ &A_{ij} + A_{ji} = B_{ij} + B_{ji} \text{ for } 1 \leq i < j \leq n\end{aligned}$$

i.e., $A \in \{B\}$, where the equivalence class is taken with respect to zero-diagonal symmetric matrices.

These vector fields preserve the 1-form

$$\alpha = dx_0 + \sum_{i,j=1}^{n} A_{ij}x_i dx_j.$$

So, to get a contact Lie superalgebra in **this** way, one needs B to be non-symmetric with non-degenerate class $\{B\}$.

3) Lin [Lin1] considered an n-parameter family of simple Lie algebras for $p = 2$ preserving in dimension $2n + 1$ the distribution given by the contact form

$$\alpha = dt + \sum_{i=1}^{n} ((1-a_i)p_i dq_i + a_i q_i dp_i)\,, \text{ where } a_i \in \mathbb{K}.$$

Obviously, the linear change

$$t' = t + \sum a_i p_i q_i \quad \text{and identical on other indeterminates} \tag{13.21}$$

reduces α to the canonical form $dt + \sum\limits_{i=1}^{n} p_i dq_i$. So the parameters a_i can be eliminated. Although Lin mentioned the change (13.21) on p. 21 of [Lin1], its consequence was not formulated and, seven years after, Brown [Br] reproduced Lin's misleading n-parameter description of $\mathfrak{k}(2n+1)$.

13.5.7. The case of indeterminates of different parities.

13.5.7.1. The case of an odd 1-form. Let

$$\Pi(x_0) = \Pi(x_1) = \dots = \Pi(x_{n_{\bar 0}}) = \bar 0, \quad \Pi(x_{n_{\bar 0}+1}) = \dots = \Pi(x_n) = \bar 1.$$

This corresponds to the following equivalence (we call it 1-*superform-equivalence again*) of *even* bilinear forms on a superspace V of superdimension $(n_{\bar 0}|n_{\bar 1})$, where $n_{\bar 1} = n - n_{\bar 0}$: Two such forms B and B' are said to be 1-*superform-equivalent* if, for their supermatrices, we have (13.15), where $M \in GL(n_{\bar 0}|n_{\bar 1})$ and A is a symmetric even supermatrix such that the restriction of the bilinear form corresponding to it onto the odd subspace $V_{\bar 1}$ is anti-symmetric. This means that, in the standard format of supermatrices,

$$B = \begin{pmatrix} B_{\bar 0} & 0 \\ 0 & B_{\bar 1}\end{pmatrix} \quad \text{and} \quad B' = \begin{pmatrix} B'_{\bar 0} & 0 \\ 0 & B'_{\bar 1}\end{pmatrix}$$

are 1-superform-equivalent if and only if

(1) $B_{\bar 0}$ and $B'_{\bar 0}$ are 1-form-equivalent, and

(2) $B_{\bar 1}$ and $B'_{\bar 1}$ are 1-superform-equivalent. Then, from (13.17) we get the following

13.5.8. Theorem. *The following are the canonical expressions for an odd contact form on a superspace:*

$$dt + \sum_{i=1}^{k} p_i dq_i + \sum_{j=1}^{l} \xi_i d\eta_i \begin{cases} & \textit{for } n_{\bar 0} = 2k \textit{ and } n_{\bar 1} = 2l, \\ +\theta d\theta & \textit{for } n_{\bar 0} = 2k \textit{ and } n_{\bar 1} = 2l+1, \end{cases}$$

where $t = x_0$*;* $p_i = x_i$*,* $q_i = x_{k+i}$ *for* $1 \le i \le k$*;* $\xi_i = x_{n_{\bar 0}+i}$*,* $\eta_i = x_{n_{\bar 0}+l+i}$ *for* $1 \le i \le l$*;* $\theta = x_n$ *for* $n_{\bar 1} = 2l+1$*.*

(This also follows from the fact that the 1-form

$$dx_0 + \sum_{i,j=1}^{n_{\bar 0}} A_{ij} x_i dx_j + \sum_{i,j=1}^{n_{\bar 1}} B_{ij} x_{n_{\bar 0}+i} dx_{n_{\bar 0}+j}$$

is contact if and only if the forms

$$dx_0 + \sum_{i,j=1}^{n_{\bar 0}} A_{ij} x_i dx_j \quad \text{and} \quad dx_0 + \sum_{i,j=1}^{n_{\bar 1}} B_{ij} x_{n_{\bar 0}+i} dx_{n_{\bar 0}+j}$$

are contact on the superspaces of superdimension $(n_{\bar 0}+1|0)$ and $(1|n_{\bar 1})$, respectively.)

13.5.8.1. The case of an even 1-form. Let $\Pi(x_0) = \bar 1$. This corresponds to the following equivalence of *odd* bilinear forms on a superspace V of superdimension $(n_{\bar 0}|n_{\bar 1})$: two such forms B and B' are said to be 1-*superform-equivalent* if for their (super)matrices we have (13.15), where $M \in GL(n_{\bar 0}|n_{\bar 1})$ and A is a symmetric odd supermatrix. Then, since

$$\left(\begin{array}{c|c} 1_{n_{\bar 0}} & 0 \\ \hline 0 & M \end{array}\right) \left(B + \left(\begin{array}{c|c} 0 & C \\ \hline C^T & 0 \end{array}\right)\right) \left(\begin{array}{c|c} 1_{n_{\bar 0}} & 0 \\ \hline 0 & M^T \end{array}\right) = \left(\begin{array}{c|c} 0 & 0 \\ \hline X(D+C^T) & 0 \end{array}\right)$$

$$\text{for} \quad B = \left(\begin{array}{c|c} 0 & C \\ \hline D & 0 \end{array}\right),$$

any such B is equivalent to a form with a supermatrix of the shape (the indices above and to the left of the supermatrix are the sizes of the blocks)

$$\begin{array}{cc} & \begin{array}{ccc} r & n_{\bar 0}-r & n_{\bar 1} \end{array} \\ \begin{array}{c} n_{\bar 0} \\ r \\ n_{\bar 1}-r \end{array} & \left(\begin{array}{cc|c} 0 & 0 & 0 \\ \hline 1_r & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \end{array},$$

where $r = \operatorname{rank}(D + C^T)$. The corresponding form is contact if and only if $r = n_{\bar 0} = n_{\bar 1}$. Hence, we get the following somewhat unexpected result:

13.5.9. Theorem. *The following expressions for the canonical form of an even contact (pericontact) 1-form on a superspace of dimension $(k|k+1)$ are equivalent:*

$$d\tau + \sum_{i=1}^{k} \xi_i dq_i, \quad \textit{or} \;\; d\tau + \sum_{i=1}^{k} q_i d\xi_i, \quad \textit{or} \;\; d\tau + \sum_{i=1}^{l} \xi_i dq_i + \sum_{i=l+1}^{k} q_i d\xi_i,$$

where $\tau = x_0$*,* $\xi_i = x_{k+i}$*,* $q_i = x_i$ *for* $1 \le i \le k$*.*

13.6. Canonical expressions of symmetric bilinear forms. Related simple Lie algebras

If we want to have a canonical expression of a non-degenerate bilinear form B such that the intersection of the Cartan subalgebra of $\mathfrak{o}_B^{(1)}(n)$ or $\mathfrak{o}_B^{(2)}(n)$ with the space of diagonal matrices were of maximal possible dimension, we should take the following canonical forms of B. Each of the following subsections 13.6.1, 13.6.2, 13.6.3 contains two most convenient expressions of an equivalence class of bilinear forms.

13.6.1. $\boldsymbol{n = 2k + 1}$.

13.6.1.1. If $B = S_{2k+1}$, then $\mathfrak{o}_B(n)$ consists of the matrices, symmetric with respect to the side diagonal; it is convenient to express them in the block form

$$\begin{pmatrix} A & X & C \\ Y^T S_k & z & X^T S_k \\ D & Y & S_k A^T S_k \end{pmatrix},$$ where $A \in \mathfrak{gl}(k)$, C and D are symmetric with respect to the side diagonal, $X, Y \in \mathbb{K}^k$ are column-vectors, $z \in \mathbb{K}$.

The Lie algebra $\mathfrak{o}_B^{(1)}(n)$ consists of the elements of $\mathfrak{o}_B(n)$, which have only zeros on the side diagonal; the Cartan subalgebra of $\mathfrak{o}_B^{(1)}(n)$ of maximal dimension consists of the matrices

$$\mathrm{diag}_n(a_1, \dots, a_k, 0, a_k, \dots, a_1).$$

13.6.1.2. If $B = \Pi_{2k+1}$, then $\mathfrak{o}_B(n)$ consists of the matrices

$$\begin{pmatrix} A & X & C \\ Y^T & z & X^T \\ D & Y & A^T \end{pmatrix},$$ where $A \in \mathfrak{gl}(k)$, C and D are symmetric, $X, Y \in \mathbb{K}^k$ are column-vectors, $z \in \mathbb{K}$.

The Lie algebra $\mathfrak{o}_B^{(1)}(n)$ consists of the elements of $\mathfrak{o}_B$ such that C and D are zero-diagonal, $z = 0$; the Cartan subalgebra of $\mathfrak{o}_B^{(1)}(n)$ of maximal dimension consists of the matrices

$$\mathrm{diag}_n(a_1, \dots, a_k, 0, a_1, \dots, a_k).$$

13.6.2. $\boldsymbol{n = 2k}$ **and** $\boldsymbol{B}$ **equivalent to** $\boldsymbol{S_{2k}}$.

13.6.2.1. If $B = S_{2k}$, then $\mathfrak{o}_B(n)$ consists of the matrices, symmetric with respect to the side diagonal; it is convenient to express them in the block form

$$\begin{pmatrix} A & C \\ D & S_k A^T S_k \end{pmatrix},$$ where $A \in \mathfrak{gl}(k)$, C and D are symmetric with respect to the side diagonal.

The Cartan subalgebra of the related simple Lie algebra (it is described later) consists of the matrices

$$\mathrm{diag}_n(a_1, \dots, a_k, a_k, \dots, a_1) \text{ such that } a_1 + \dots + a_k = 0.$$

13.6.2.2. If $B = \Pi_{2k}$, then $\mathfrak{o}_B(n)$ is spanned by the matrices

$$\begin{pmatrix} A & C \\ D & A^T \end{pmatrix}, \quad \text{where } A \in \mathfrak{gl}(k),\ C \text{ and } D \text{ are symmetric.} \tag{13.22}$$

Observe that these matrices can be represented as $\Pi(2k)U$ or $V\Pi(2k)$, where U and V are symmetric.

The Cartan subalgebra of the related simple Lie algebra (it is described later) consists of the matrices

$$\mathrm{diag}_n(a_1, \dots, a_k, a_1, \dots, a_k) \text{ such that } a_1 + \dots + a_k = 0.$$

13.6.3. $\boldsymbol{n = 2k}$ **and** $\boldsymbol{B}$ **equivalent to** $\boldsymbol{1_n}$. We get the greatest dimension of the intersection of the Cartan subalgebra with the space of diagonal matrices if the matrix of B is of any of the following shapes:

13.6.3.1. If $B = \begin{pmatrix} 1_2 & 0 \\ 0 & S_{n-2} \end{pmatrix}$, then $\mathfrak{o}_B$ consists of the matrices

$$\begin{pmatrix} A & C \\ S_{n-2}B^T & D \end{pmatrix}$$ where $A \in \mathfrak{gl}(2)$ is symmetric, C is any $2 \times (n-2)$ matrix, $D \in \mathfrak{gl}(n-2)$ is symmetric with respect to the side diagonal.

The Lie algebra $\mathfrak{o}_B^{(1)}(n)$ consists of the elements of $\mathfrak{o}_B(n)$ such that A is zero-diagonal, D has only zeros on the side diagonal; the Cartan subalgebra of $\mathfrak{o}_B^{(1)}(n)$ of greatest dimension consists of the matrices

$$\left(\begin{array}{cc|c} 0 & a_0 & 0 \\ a_0 & 0 & 0 \\ \hline 0 & 0 & \mathrm{diag}_n(a_1, \dots, a_{k-1}, a_{k-1}, \dots, a_1) \end{array}\right).$$

13.6.3.2. If $B = \begin{pmatrix} 1_2 & 0 & 0 \\ 0 & 0 & 1_{k-1} \\ 0 & 1_{k-1} & 0 \end{pmatrix}$, then $\mathfrak{o}_B(n)$ is spanned by the matrices

$$\begin{pmatrix} X & Y & Z \\ Z^T & A & C \\ Y^T & D & A^T \end{pmatrix}$$ where $X \in \mathfrak{gl}(2)$ is symmetric, Y and Z are of size $2 \times (k-1)$, $A \in \mathfrak{gl}(k-1)$, $C, D \in \mathfrak{gl}(k-1)$ are symmetric.

The Lie algebra $\mathfrak{o}_B^{(1)}(n)$ consists of the elements of $\mathfrak{o}_B(n)$ such that X, C and D are zero-diagonal; the Cartan subalgebra of $\mathfrak{o}_B^{(1)}(n)$ greatest dimension consists of the matrices

$$\left(\begin{array}{cc|c} 0 & a_0 & 0 \\ a_0 & 0 & 0 \\ \hline 0 & 0 & \mathrm{diag}_n(a_1, \dots, a_{k-1}, a_1, \dots, a_{k-1}) \end{array}\right).$$

13.6.4. The derived Lie algebras of $\boldsymbol{\mathfrak{o}_I(n)}$. Direct calculation shows that

$$\mathfrak{o}_I^{(1)}(n) = \begin{cases} 0 & \text{if } n = 1 \\ \{\lambda S_2 \mid \lambda \in \mathbb{K}\} & \text{if } n = 2; \end{cases} \qquad \mathfrak{o}_I^{(2)}(n) = 0 \text{ if } n \le 2.$$

13.6.4.1. Lemma. *If $n > 2$, then*

i) $\mathfrak{o}_I^{(1)}(n) = ZD(n)$;

ii) $\mathfrak{o}_I^{(2)}(n) = \mathfrak{o}_I^{(1)}(n)$.

Proof. First, let us show that $\mathfrak{o}_I^{(1)}(n) \subset ZD(n)$. Indeed, if $A, A' \in \mathfrak{o}_I(n)$, then

$$[A, A']_{ii} = \sum_j A_{ij}A'_{ji} - A'_{ij}A_{ji} = 0$$

since A, A' are symmetric. So, matrices from $\mathfrak{o}_I^{(1)}(n)$ are zero-diagonal.

Let $F^{ij} = E_{ij} + E_{ji}$, where $1 \le i, j \le n$, $i \ne j$. These matrices are symmetric, so they are all in $ZD(n)$. Let us show that they also are in $\mathfrak{o}_I^{(1)}(n)$. Since, for $1 \le i < j \le n$, the matrices F^{ij} form a basis of $ZD(n)$, it follows that $ZD(n) \subset \mathfrak{o}_I^{(1)}(n)$, and it proves (i).

Direct calculation shows that if $1 \le k \le n$, $k \ne i, j$, then

$$[F^{ik}, F^{kj}] = F^{ij}, \tag{13.23}$$

so $F^{ij} \in \mathfrak{o}_I^{(1)}(n)$.

Moreover, once we have shown that $F^{ij} \in \mathfrak{o}_I^{(1)}(n)$, this computation also proves that $F^{ij} \in \mathfrak{o}_I^{(2)}(n)$. Since $\mathfrak{o}_I^{(2)}(n) \subset \mathfrak{o}_I^{(1)}(n) = ZD(n)$, it also proves (ii). □

13.6.4.2. Lemma. *If $n > 2$, then $\mathfrak{o}_I^{(1)}(n)$ is simple.*

Proof. Let $I \subset \mathfrak{o}_I^{(1)}(n)$ be an ideal, and $x \in I$ an element, such that its decomposition with respect to $\{F^{ij}\}$ contains F^{ab} with non-zero coefficient for some a, b. Let us note that

$$[F^{ij}, [F^{ij}, F^{kl}]] = \begin{cases} F^{kl} & \text{if } \operatorname{card}(\{i,j\} \cap \{k,l\}) = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{13.24}$$

Let us define an operator $P_{F^{ab}} \colon \mathfrak{o}_I^{(1)}(n) \to \mathfrak{o}_I^{(1)}(n)$ as follows:

$$P_{F^{ab}} = \begin{cases} (\operatorname{ad}\ F^{bc})^2 (\operatorname{ad}\ F^{ac})^2 \quad \text{for} \quad c \ne a, b, \quad 1 \le c \le 3 & \text{if } n = 3; \\ \prod\limits_{1 \le c \le n,\ c \ne a,b} (\operatorname{ad}\ F^{ac})^2 & \text{if } n > 3. \end{cases} \tag{13.25}$$

Then, from (13.24),

$$P_{F^{ab}} F^{cd} = \begin{cases} F^{cd} & \text{if } F^{cd} = F^{ab}; \\ 0 & \text{otherwise.} \end{cases} \tag{13.26}$$

So, $P_{F^{ab}} x$ is proportional (with non-zero coefficient) to F^{ab}, and $F^{ab} \in I$. Then, from (13.23), $F^{ib}, F^{ij} \in I$ for all i, j, $1 \le i, j \le n$, $i \ne j$, and $I = \mathfrak{o}_I^{(1)}(n)$. □

13.6.5. The derived Lie algebras of $\mathfrak{o}_\Pi(2n)$. Direct computations show that:

$\mathfrak{o}_\Pi^{(1)}(2) = \{\lambda \cdot 1_2 \mid \lambda \in \mathbb{K}\}$;
$\mathfrak{o}_\Pi^{(2)}(2) = 0$;
$\mathfrak{o}_\Pi^{(1)}(4) = \{$matrices of the shape (13.22) such that $B, C \in ZD(2)\}$;
$\mathfrak{o}_\Pi^{(2)}(4) = \{$matrices of $\mathfrak{o}_\Pi^{(1)}(4)$ such that $\operatorname{tr} A = 0\}$;
$\mathfrak{o}_\Pi^{(3)}(4) = \{\lambda \cdot 1_4 \mid \lambda \in \mathbb{K}\}$;
$\mathfrak{o}_\Pi^{(4)}(4) = 0$.

13.6.5.1. Lemma. *If $n \ge 3$, then*

i) $\mathfrak{o}_\Pi^{(1)}(2n) = \{$matrices of the shape (13.22) such that $B, C \in ZD(n)\}$;
ii) $\mathfrak{o}_\Pi^{(2)}(2n) = \{$matrices of the shape (13.22) such that$B, C \in ZD(n)$, and $\operatorname{tr}\ A = 0\}$;
iii) $\mathfrak{o}_\Pi^{(3)}(2n) = \mathfrak{o}_\Pi^{(2)}(2n)$.

Proof. Let M^1 and M^2 denote *conjectural* $\mathfrak{o}_\Pi^{(1)}(2n)$ and $\mathfrak{o}_\Pi^{(2)}(2n)$, respectively, as described in Lemma. First, let us prove that $\mathfrak{o}_\Pi^{(1)}(2n) \subset M^1$ and $\mathfrak{o}_\Pi^{(2)}(2n) \subset M^2$. Let

$$L = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}, \quad L' = \begin{pmatrix} A' & B' \\ C' & A'^T \end{pmatrix} \in \mathfrak{o}(2n), \quad \text{and} \quad L'' = [L, L'] = \begin{pmatrix} A'' & B'' \\ C'' & A''^T \end{pmatrix}.$$

Then, for any $i \in \overline{1, n}$, we have

$$B''_{ii} = \sum_{j=1}^{n} (A_{ij} B'_{ji} + B_{ij} A'_{ij} - A'_{ij} B_{ji} - B'_{ij} A_{ij}) = 0$$

since B, B' are symmetric. Analogically, $C''_{ii} = 0$, so $L'' \in M^1$. Hence, $\mathfrak{o}_\Pi^{(1)}(2n) \subset M^1$.

Now, if $L, L' \in \mathfrak{o}_\Pi^{(1)}(2n)$, then

$$\operatorname{tr} A'' = \sum_{i=1}^{n} A''_{ii} = \sum_{i,j=1}^{n} (A_{ij} A'_{ji} + B_{ij} C'_{ji} - A'_{ij} A_{ji} - B'_{ij} C_{ji}) = 0$$

since B, B', C, C' are symmetric and zero-diagonal. So, since

$$L'' \in \mathfrak{o}_\Pi^{(2)}(2n) \subset \mathfrak{o}_\Pi^{(1)}(2n) \subset M^1,$$

it follows that $L'' \in M^2$, and $\mathfrak{o}_\Pi^{(2)}(2n) \subset M^2$.

Let us introduce the following notations for matrices from $\mathfrak{o}_\Pi(2n)$:

F_1^{ij}, where $1 \le i, j \le n$, $i \ne j$, such that $A = C = 0$, $B = E_{ij} + E_{ji}$;
F_2^{ij}, where $1 \le i, j \le n$, $i \ne j$, such that $A = B = 0$, $C = E_{ij} + E_{ji}$;
G^{ij}, where $1 \le i, j \le n$, $i \ne j$, such that $B = C = 0$, $A = E_{ij}$;
H^{ij}, where $1 \le i, j \le n$, $i \ne j$, such that $B = C = 0$, $A = E_{ii} + E_{jj}$;
K_0 such that $B = C = 0$, $A = E_{11}$;
K_1 such that $A = C = 0$, $B = E_{11}$;
K_2 such that $A = B = 0$, $C = E_{11}$;

Observe that F_1^{ij}, F_2^{ij}, G^{ij}, and H^{ij} span M^2; whereas M^2 and K_0 span M^1.

Direct computations give the following relations:

$$\begin{aligned} &\text{if } k \ne i, j, \text{ then } [H^{ik}, F_1^{ij}] = F_1^{ij}, \quad [H^{ik}, F_2^{ij}] = F_2^{ij}, \quad [H^{ik}, G^{ij}] = G^{ij}; \\ &[F_1^{ij}, F_2^{ij}] = H^{ij}; \\ &[K_1, K_2] = K_0. \end{aligned} \tag{13.27}$$

Since $F_1^{ij}, F_2^{ij}, G^{ij}, H^{ij}, K_1, K_2 \in \mathfrak{o}_\Pi(2n)$, it follows that F_1^{ij}, F_2^{ij}, G^{ij}, H^{ij}, $K_0 \in \mathfrak{o}_\Pi^{(1)}(2n)$. Hence, $M^1 \subset \mathfrak{o}_\Pi^{(1)}(2n)$, and $\mathfrak{o}_\Pi^{(1)}(2n) = M^1$. Relations (13.27) imply that $M^2 \subset [M^2, M^2]$, so $M^2 \subset [M^1, M^1] = \mathfrak{o}_\Pi^{(2)}(2n)$, and $\mathfrak{o}_\Pi^{(2)}(2n) = M^2$. Also, $M^2 \subset [M^2, M^2] = \mathfrak{o}_\Pi^{(3)}(2n)$, so $\mathfrak{o}_\Pi^{(3)}(2n) = M^2$. The lemma is proven. $\square$

13.6.5.2. Lemma. *If $n \geq 3$, then*

i) *if n is odd, then $\mathfrak{o}_\Pi^{(2)}(2n)$ is simple;*

ii) *if n is even, then the only non-trivial ideal of $\mathfrak{o}_\Pi^{(2)}(2n)$ is the center $Z = \{\lambda \cdot 1_{2n} \mid \lambda \in \mathbb{K}\}$ (thus, $\mathfrak{o}_\Pi^{(2)}(2n)/Z$ is simple).*

Proof. We use the notations of the previous Lemma. It follows from the relations

$$[F_1^{ij}, F_2^{ij}] = H^{ij};$$

$$[H^{ij}, X^{kl}] = \begin{cases} X^{kl} & \text{if } \mathrm{card}(\{i,j\} \cap \{k,l\}) = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } X^{kl} = F_1^{kl}, F_2^{kl}, G^{kl};$$

that if an ideal I of $\mathfrak{o}_\Pi^{(2)}(2n)$ contains any of the elements F_1^{ij}, F_2^{ij}, then $I = \mathfrak{o}_\Pi^{(2)}(2n)$.

Let $1 \leq i, j, k \leq n$, $i \neq j \neq k \neq i$. Direct computation shows that the operators

$$P_{F_1^{ij}} = \mathrm{ad}_{F_1^{jk}} \mathrm{ad}_{F_1^{ij}} \mathrm{ad}_{F_2^{ij}} \mathrm{ad}_{F_2^{jk}}; \quad P_{F_2^{ij}} = \mathrm{ad}_{F_2^{jk}} \mathrm{ad}_{F_2^{ij}} \mathrm{ad}_{F_1^{ij}} \mathrm{ad}_{F_1^{jk}}$$

on $\mathfrak{o}_\Pi^{(2)}(2n)$ act as follows: for X equal to one of the elements F_1^{lm}, F_2^{lm}, H^{lm}, G^{lm},

$$P_{F_1^{ij}} X = \begin{cases} X & \text{if } X = F_1^{ij} \\ 0 & \text{otherwise} \end{cases}; \qquad P_{F_2^{ij}} X = \begin{cases} X & \text{if } X = F_2^{ij} \\ 0 & \text{otherwise} \end{cases}.$$

It follows from these two facts that any element of a non-trivial ideal I of $\mathfrak{o}_\Pi^{(2)}(2n)$ must not contain F_1^{ij}, F_2^{ij} in its decomposition with respect to the basis of $F_1^{ij}, F_2^{ij}, H^{ij}, G^{ij}$ — i.e., it must have the shape

$$\begin{pmatrix} A & 0 \\ 0 & A^T \end{pmatrix}.$$

Then, for I to be an ideal, A must satisfy the following condition:

$$AB + BA^T = 0 \text{ for all } B \in ZD(n).$$

If A contains a non-zero non-diagonal entry A_{ij}, then

$$(A(E_{jk} + E_{kj}) + (E_{jk} + E_{kj})A^T)_{ik} = A_{ij} \neq 0$$

for $k \neq i, j$; if A contains two non-equal diagonal entries A_{ii} and A_{jj}, then

$$(A(E_{ij} + E_{ji}) + (E_{ij} + E_{ji})A^T)_{ij} = A_{ii} - A_{jj} \neq 0$$

So, A must be proportional to 1_n, and $1_{2n} \in \mathfrak{o}_\Pi^{(2)}(2n)$ if and only if n is even. $\square$

13.7. Canonical expressions of symmetric bilinear superforms. Related Lie superalgebras.

In this section we consider Lie superalgebras of linear transformations preserving bilinear forms on a superspace of superdimension $(n_{\bar{0}}|n_{\bar{1}})$ and their derived superalgebras. Since in the case where $n_{\bar{0}} = 0$ or $n_{\bar{1}} = 0$ these superalgebras are entirely even and do not differ from the corresponding Lie algebras, we do not consider this case.

As it was said in sec. 13.4.1, every even symmetric non-degenerate form on a superspace of superdimension $(n_{\bar{0}}|n_{\bar{1}})$ over a field of characteristic 2 is equivalent to a form of the shape (here: $i = \bar{0}$ or $\bar{1}$)

$$B = \begin{pmatrix} B_{\bar{0}} & 0 \\ 0 & B_{\bar{1}} \end{pmatrix}, \quad \text{where } B_i = \begin{cases} 1_{n_i} & \text{if } n_i \text{ is odd;} \\ 1_{n_i} \text{ or } \Pi_{n_i} & \text{if } n_i \text{ is even.} \end{cases} \tag{13.28}$$

We denote the *ortho-orthogonal* Lie superalgebras preserving this canonical bilinear forms by $\mathfrak{oo}_{II}(n_{\bar{0}}|n_{\bar{1}}), \mathfrak{oo}_{I\Pi}(n_{\bar{0}}|n_{\bar{1}}), \mathfrak{oo}_{\Pi\Pi}(n_{\bar{0}}|n_{\bar{1}})$, respectively (note that $\mathfrak{oo}_{\Pi I}(n_{\bar{0}}|n_{\bar{1}}) \simeq \mathfrak{oo}_{I\Pi}(n_{\bar{1}}|n_{\bar{0}})$). The Lie superalgebra $\mathfrak{oo}_B(n_{\bar{0}}|n_{\bar{1}})$ preserving B consists of the supermatrices which in the standard format are of the shape

$$\begin{pmatrix} A_{\bar{0}} & B_{\bar{0}} C^T B_{\bar{1}}^{-1} \\ C & A_{\bar{1}} \end{pmatrix}, \quad \begin{array}{c} \text{where } A_{\bar{0}} \in \mathfrak{o}_{B_{\bar{0}}}(n_{\bar{0}}),\ A_{\bar{1}} \in \mathfrak{o}_{B_{\bar{1}}}(n_{\bar{1}}), \text{ and} \\ C \text{ is arbitrary } n_{\bar{1}} \times n_{\bar{0}} \text{ matrix.} \end{array} \tag{13.29}$$

In what follows we use the fact that matrices and supermatrices of the same size behave identically with respect to multiplication and Lie (super)bracket — i.e., the entries of the product or Lie (super)bracket of two square supermatrices do not depend on their format (they must be of the same format for the product (bracket) be defined).

13.7.1. The derived Lie superalgebras of $\mathfrak{oo}_{II}(n_{\bar{0}}|n_{\bar{1}})$. Let B be of the shape (13.28) such that $B_i = 1_{n_i}$. We will denote Lie superalgebra preserving this form as $\mathfrak{oo}_{II}(n_{\bar{0}}|n_{\bar{1}})$; this superalgebra consists of symmetric supermatrices.

Direct calculation shows that

$$\mathfrak{oo}_{II}^{(i)}(1|1) = \begin{cases} \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{K} \right\} & \text{if } i = 1, \\ \{a \cdot 1_{1|1} \mid a \in \mathbb{K}\} & \text{if } i = 2, \\ 0 & \text{if } i \geq 3. \end{cases}$$

13.7.1.1. Lemma. *If $n = n_{\bar 0} + n_{\bar 1} \geq 3$, then*

i) $\mathfrak{oo}_{II}^{(1)}(n_{\bar 0}|n_{\bar 1})$ *consists of symmetric supermatrices of (super)trace* 0*;*

ii) $\mathfrak{oo}_{II}^{(2)}(n_{\bar 0}|n_{\bar 1}) = \mathfrak{oo}_{II}^{(1)}(n_{\bar 0}|n_{\bar 1})$.

Proof. It was shown in the proof of Lemma 13.6.4.1 that a (super)bracket of any two symmetric matrices is zero-diagonal, so to prove that supermatrices from $\mathfrak{oo}_{II}^{(1)}(n_{\bar 0}|n_{\bar 1})$ have trace 0, we only need to prove this for the squares of odd symmetric supermatrices. If L is an odd matrix of the shape (13.29), then

$$\operatorname{tr} L^2 = \sum_{i=1}^{n_{\bar 0}} \left(\sum_{j=1}^{n_{\bar 1}} C_{ij} \right)^2 + \sum_{j=1}^{n_{\bar 1}} \left(\sum_{i=1}^{n_{\bar 0}} C_{ij} \right)^2 = 2 \sum_{i=1}^{n_{\bar 0}} \sum_{j=1}^{n_{\bar 1}} C_{ij}^2 = 0.$$

Now let us introduce the following notations for matrices from $\mathfrak{oo}_{II}(n_{\bar 0}|n_{\bar 1})$:

$$\begin{aligned} F^{ij} &= E_{ij} + E_{ji} \quad \text{for } 1 \leq i, j \leq n,\ i \neq j;\\ H^{ij} &= E_{ii} + E_{jj} \quad \text{for } 1 \leq i \leq n_{\bar 0},\ 1 \leq j \leq n_{\bar 1}. \end{aligned}$$

These matrices span the space of symmetric matrices with trace 0. As it was shown in the proof of Lemma 13.6.4.1, if $n \geq 3$, then the matrices F^{ij} generate themselves. Now, if $1 \leq i \leq n_{\bar 0}$, $1 \leq j \leq n_{\bar 1}$ (so that F^{ij} is odd), then $(F^{ij})^2 = H^{ij}$. So all the F^{ij}, H^{ij} lie in $\mathfrak{oo}_{II}^{(k)}(n_{\bar 0}|n_{\bar 1})$ for any k. □

13.7.1.2. Lemma. *If $n = n_{\bar 0} + n_{\bar 1} \geq 3$, then*

i) *if n is odd, then* $\mathfrak{oo}_{II}^{(1)}(n_{\bar 0}|n_{\bar 1})$ *is simple;*

ii) *if n is even, then the only non-trivial ideal of* $\mathfrak{oo}_{II}^{(1)}(n_{\bar 0}|n_{\bar 1})$ *is the center* $C = \{\lambda \cdot 1_n \mid \lambda \in \mathbb{K}\}$ *(thus,* $\mathfrak{oo}_{II}^{(1)}(n_{\bar 0}|n_{\bar 1})/C$ *is simple).*

Proof. Let us define operators $P_{F^{ab}}$ as in (13.25). Then, due to (13.26) and the fact that

$$P_{F^{ab}} H^{kl} = 0,$$

we can show in the same way as in the proof of Lemma 13.6.4.2 that if an ideal of $\mathfrak{oo}_{II}^{(1)}(n_{\bar 0}|n_{\bar 1})$ contains a non-diagonal matrix, it contains all the F^{ij}. Since all the H^{ij} are squares of odd F^{ij}, such an ideal is trivial.

So, any non-trivial ideal of $\mathfrak{oo}_{II}^{(1)}(n_{\bar 0}|n_{\bar 1})$ is diagonal. For a diagonal matrix X,

$$[X, F^{ij}] = (X_{jj} - X_{ii}) F^{ij},$$

so all the elements of a non-trivial ideal must be proportional to 1_n, and $1_n \in \mathfrak{oo}_{II}^{(1)}(n_{\bar 0}|n_{\bar 1})$ if and only if n is even. □

13.7.2. The derived Lie superalgebras of $\mathfrak{oo}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$. Now let us consider the case where $n_{\bar 1}$ is even and B is of the shape (13.28) such that $B_{\bar 0} = 1_{n_{\bar 0}}$, $B_{\bar 1} = \Pi_{n_{\bar 1}}$. (The case where $n_{\bar 0}$ is even, $B_{\bar 0} = \Pi_{n_{\bar 0}}$, $B_{\bar 1} = 1_{n_{\bar 1}}$ is analogous to this one, so we will not consider it.) We will denote Lie superalgebra preserving this form by $\mathfrak{oo}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$; this superalgebra consists of supermatrices of the following shape:

$$\begin{pmatrix} A_{\bar 0} & C^T \Pi_{n_{\bar 1}} \\ C & \Pi_{n_{\bar 1}} A_{\bar 1} \end{pmatrix}, \quad \begin{array}{l} \text{where } A_{\bar 0}, A_{\bar 1} \text{ are symmetric,} \\ C \text{ is an arbitrary } n_{\bar 1} \times n_{\bar 0} \text{ matrix.} \end{array} \tag{13.30}$$

13.7.2.1. Lemma. i) $\mathfrak{oo}_{I\Pi}^{(1)}(n_{\bar 0}|n_{\bar 1})$ *consists of the matrices of the shape (13.30) such that $A_{\bar 0}$ is zero-diagonal;*

ii) $\mathfrak{oo}_{I\Pi}^{(2)}(n_{\bar 0}|n_{\bar 1}) = \mathfrak{oo}_{I\Pi}^{(1)}(n_{\bar 0}|n_{\bar 1})$.

Proof. Set $k_{\bar 1} = n_{\bar 1}/2$; let M be the *conjectural* space of $\mathfrak{oo}_{I\Pi}^{(1)}(n_{\bar 0}|n_{\bar 1})$ as it is described in the Lemma. First, let us prove that $\mathfrak{oo}_{I\Pi}^{(1)}(n_{\bar 0}|n_{\bar 1}) \subset M$. If

$$\begin{aligned} L &= \begin{pmatrix} A_{\bar 0} & C^T \Pi_{n_{\bar 1}} \\ C & A_{\bar 1} \end{pmatrix}, L' = \begin{pmatrix} A'_{\bar 0} & C'^T \Pi_{n_{\bar 1}} \\ C' & A'_{\bar 1} \end{pmatrix} \in \mathfrak{oo}_{I\Pi}(n_{\bar 0}|n_{\bar 1}), \text{ and}\\ L'' &= [L, L'] = \begin{pmatrix} A''_{\bar 0} & C''^T \Pi_{n_{\bar 1}} \\ C'' & A''_{\bar 1} \end{pmatrix}, \end{aligned}$$

then

$$\begin{aligned} &(A''_{\bar 0})_{ii} = ([A_{\bar 0}, A'_{\bar 0}] + C^T \Pi_{n_{\bar 1}} C' - C'^T \Pi_{n_{\bar 1}} C)_{ii} = \\ &\sum_{j=1}^{n_{\bar 0}} ((A_{\bar 0})_{ij} (A'_{\bar 0})_{ji} - (A'_{\bar 0})_{ij} (A_{\bar 0})_{ji}) + \\ &\sum_{j=1}^{k_{\bar 1}} (C_{ji} C'_{j+k_{\bar 1},i} + C_{j+k_{\bar 1},i} C'_{ji} - C'_{ji} C_{j+k_{\bar 1},i} + C'_{j+k_{\bar 1},i} C_{ji}) = 0 \end{aligned}$$

since $A_{\bar 0}, A'_{\bar 0}$ are symmetric. Now, if L is odd (i.e., $A_{\bar 0} = 0$, $A_{\bar 1} = 0$), then

$$L^2 = \begin{pmatrix} C^T \Pi_{n_{\bar 1}} C & 0 \\ 0 & C C^T \Pi_{n_{\bar 1}} \end{pmatrix},$$

and

$$(C^T \Pi_{n_{\bar 1}} C)_{ii} = \sum_{j=1}^{k_{\bar 1}} (C_{ji} C_{j+k_{\bar 1},i} + C_{j+k_{\bar 1},i} C_{ji}) = 0.$$

Let us introduce the following notations for matrices from $\mathfrak{oo}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$:

F^{ij}, where $1 \leq i, j \leq n_{\bar 0}$ and $i \neq j$, such that $C = 0$, $A_{\bar 1} = 0$, $A_{\bar 0} = E_{ij} + E_{ji}$;

G^{ij}, where $1 \leq i \leq n_{\bar 1}$, $1 \leq j \leq n_{\bar 0}$, such that $A_{\bar 0} = 0$, $A_{\bar 1} = 0$, $C = E_{ij}$;

H^{i}, where $1 \leq i \leq n_{\bar 1}$, such that

$$A_{\bar 0} = 0, \quad C = 0, \quad A_{\bar 1} = \begin{cases} E_{i,i+k_{\bar 1}} & \text{if } 1 \leq i \leq k_{\bar 1}; \\ E_{i,i-k_{\bar 1}} & \text{if } k_{\bar 1} + 1 \leq i \leq n_{\bar 1}; \end{cases}$$

$I_{\bar 1}$, such that $A_{\bar 0} = 0$, $C = 0$, $A_{\bar 1} = 1_{n_{\bar 1}}$.

Direct calculations give the following relations:

$$[G^{i1}, G^{k_{\bar 1}+1,j}] = F^{ij};$$
$$[I_{\bar 1}, G^{ij}] = G^{ij};$$
$$(G^{i1})^2 = H^i,$$

so we see that $F^{ij}, G^{ij}, H^i \in \mathfrak{oo}^{(1)}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$.

Let us also denote by K the subalgebra of $\mathfrak{oo}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$, consisting of all matrices of the form (13.30), such that $A_{\bar 0} = 0$, $C = 0$. Since $K \subset \mathfrak{oo}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$, it follows that $K^{(1)} \subset \mathfrak{oo}^{(1)}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$. We also have $K \simeq \mathfrak{o}_{\Pi}(n_{\bar 1})$, so, as it was shown in subsect. 13.6.5, $K^{(1)}$ consists of matrices from K, such that $A_{\bar 1}$ has the shape (13.22) (even if $n_{\bar 1} < 6$). Hence, $K^{(1)}$, F^{ij}, G^{ij}, H^i span M, so $M \subset \mathfrak{oo}^{(1)}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$, and $M = \mathfrak{oo}^{(1)}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$. Since F^{ij}, G^{ij}, H^{ij}, $I_{\bar 1} \in \mathfrak{oo}^{(1)}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$, and $K \subset \mathfrak{oo}^{(1)}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$ (because $K^{(1)}$ and H^i span K), we also see that $M \subset \mathfrak{oo}^{(2)}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$, and $M = \mathfrak{oo}^{(2)}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$. □

13.7.3. The derived Lie superalgebras of $\mathfrak{oo}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$. Now we consider the case where $n_{\bar 0}, n_{\bar 1}$ are even and B is of the shape (13.28) such that $B_i = \Pi_{n_i}$. We set $k_{\bar 0} = \frac{1}{2}n_{\bar 0}$, $k_{\bar 1} = \frac{1}{2}n_{\bar 1}$. We will denote the Lie superalgebra preserving this form B by $\mathfrak{oo}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$; it consists of supermatrices of the following shape:

$$\begin{pmatrix} \Pi_{n_{\bar 0}}A_{\bar 0} & \Pi_{n_{\bar 0}}C^T\Pi_{n_{\bar 1}} \\ C & \Pi_{n_{\bar 1}}A_{\bar 1}\end{pmatrix}, \quad \begin{array}{l}\text{where } A_{\bar 0}, A_{\bar 1} \text{ are symmetric,}\\ C \text{ is an arbitrary } n_{\bar 1}\times n_{\bar 0} \text{ matrix.}\end{array} \tag{13.31}$$

Direct computation shows that

$$\mathfrak{oo}^{(i)}_{\Pi\Pi}(2|2) = \begin{cases} \{\text{matrices of the shape (13.31) such that} & \\ \qquad\qquad A_{\bar 0}, A_{\bar 1} \in ZD(2)\}, & \text{if } i = 1,\\ \{\text{matrices of the shape (13.31) such that} & \\ \qquad\qquad \Pi_2 A_{\bar 0} = \Pi_2 A_{\bar 1} = \lambda\cdot 1_2\}, & \text{if } i = 2,\\ \{\lambda \cdot 1_{2|2} \mid \lambda\in\mathbb{K}\}, & \text{if } i = 3,\\ 0, & \text{if } i \geq 4.\end{cases}$$

13.7.3.1. Lemma. *If $n_{\bar 0} + n_{\bar 1} \geq 6$, then*

i) *$\mathfrak{oo}^{(1)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ consists of the matrices of the shape (13.31) such that $A_{\bar 0}, A_{\bar 1}$ are zero-diagonal;*

ii) *$\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ consists of matrices from $\mathfrak{oo}^{(1)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ such that*

$$\sum_{i=1}^{n_{\bar 0}/2}(\Pi_{n_{\bar 0}}A_{\bar 0})_{ii} + \sum_{i=1}^{n_{\bar 1}/2}(\Pi_{n_{\bar 1}}A_{\bar 1})_{ii} = 0,$$

i.e., the "half-supertrace" of the matrix vanishes;

iii) *$\mathfrak{oo}^{(3)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1}) = \mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$.*

Proof. Let $\tilde B = \mathrm{diag}_2(\Pi_{n_{\bar 0}}, \Pi_{n_{\bar 1}})$ be a (non-super) bilinear form on a space of dimension $n_{\bar 0} + n_{\bar 1}$. Denote:

$$\begin{aligned} M^k = \{L \in \mathfrak{gl}(n_{\bar 0}|n_{\bar 1}) \mid\ & \text{exists } L' \in \mathfrak{o}^{(k)}_{\tilde B}(n_{\bar 0}+n_{\bar 1}) \text{ such that}\\ & L_{ij} = L'_{ij} \quad \text{for all } i,j \in \overline{1, n_{\bar 0}+n_{\bar 1}}\};\\ N = \mathrm{Span}\{L^2 \mid L \in\ & \mathfrak{oo}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1}), L \text{ is odd }\}.\end{aligned}$$

As it was noticed before, matrices and supermatrices behave identically with respect to multiplication and Lie (super)bracket. So we get the following inclusion:

$$M^i \subset \mathfrak{oo}^{(i)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1}) \subset M^i + N.$$

Since $\tilde B$ is equivalent to $\Pi_{n_{\bar 0}+n_{\bar 1}}$, it follows from Lemma 13.6.5.1 that M^1, M^2, M^3 coincide with conjectural spaces $\mathfrak{oo}^{(1)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$, $\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$, $\mathfrak{oo}^{(3)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ as they are described in the lemma. So, to prove the lemma, it suffices to show that $N \subset M^2$. If

$$L = \begin{pmatrix} 0 & \Pi_{n_{\bar 0}}C^T\Pi_{n_{\bar 1}} \\ C & 0\end{pmatrix}$$

is an odd matrix from $\mathfrak{oo}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$, then

$$L^2 = \begin{pmatrix} \Pi_{n_{\bar 0}}C^T\Pi_{n_{\bar 1}}C & 0 \\ 0 & C\Pi_{n_{\bar 0}}C^T\Pi_{n_{\bar 1}}\end{pmatrix},$$

and

$$(C^T\Pi_{n_{\bar 1}}C)_{ii} = \sum_{j=1}^{k_{\bar 1}} C_{ji}C_{j+k_{\bar 1},i} + \sum_{j=k_{\bar 1}+1}^{n_{\bar 1}} C_{ji}C_{j-k_{\bar 1},i} = 0;$$

similarly, $\Pi_{n_{\bar 1}}C\Pi_{n_{\bar 0}}C^T\Pi_{n_{\bar 1}}$ is zero-diagonal, so $N \subset M^1$. Now,

$$\begin{aligned}&\sum_{i=1}^{k_{\bar 0}}(\Pi_{n_{\bar 0}}C^T\Pi_{n_{\bar 1}}C)_{ii} + \sum_{l=1}^{k_{\bar 1}}(C\Pi_{n_{\bar 0}}C^T\Pi_{n_{\bar 1}})_{ll} =\\ &\sum_{i=1}^{k_{\bar 0}}\left(\sum_{j=1}^{k_{\bar 1}} C_{ji}C_{j+k_{\bar 1},i} + \sum_{j=k_{\bar 1}+1}^{n_{\bar 1}} C_{ji}C_{j-k_{\bar 1},i}\right) +\\ &\sum_{l=1}^{k_{\bar 1}}\left(\sum_{m=1}^{k_{\bar 0}} C_{lm}C_{l,m+k_{\bar 0}} + \sum_{m=k_{\bar 0}+1}^{n_{\bar 0}} C_{lm}C_{l,m-k_{\bar 0}}\right) = 0,\end{aligned}$$

so $N \subset M^2$. □

13.7.3.2. Lemma. *If $n_{\bar 0} + n_{\bar 1} \geq 6$, then*

i) *if $k = (n_{\bar 0} + n_{\bar 1})/2$ is odd, then $\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ is simple;*

ii) *if k is even, then the only non-trivial ideal of $\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ is $Z = \{\lambda \cdot 1_{n_{\bar 0}|n_{\bar 1}} \mid \lambda \in \mathbb{K}\}$, its center, and hence $\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})/Z$ is simple.*

Proof. Let $\iota\colon \mathfrak{gl}(n_{\bar 0}|n_{\bar 1}) \to \mathfrak{gl}(n_{\bar 0}+n_{\bar 1})$ be a forgetful map that sends a supermatrix into the matrix with the same entries and superstructure forgotten. Since for $p=2$ matrices and supermatrices behave identically with respect to the Lie (super)bracket, $\iota(\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1}))$ is a Lie algebra, and $\iota(I)$ is an ideal in $\iota(\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1}))$ for any ideal $I \subset \mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$.

Set

$$M = \begin{pmatrix} 1_{k_{\bar 0}} & 0 & 0 & 0\\ 0 & 0 & 1_{k_{\bar 0}} & 0\\ 0 & 1_{k_{\bar 1}} & 0 & 0\\ 0 & 0 & 0 & 1_{k_{\bar 1}} \end{pmatrix}.$$

Then, according to Lemma 13.6.5.1 and 13.7.3.1, the map $X \mapsto MXM^{-1}$ gives us an isomorphism between $\iota(\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1}))$ and $\mathfrak{o}_{\Pi}(n_{\bar 0}+n_{\bar 1})$. So, according to Lemma 13.6.5.2, if k is odd, then $\iota(\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ is simple; if k is even, then the only non-trivial ideal of $\iota(\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1}))$ is

$$\{\lambda\cdot M^{-1}1_{n_{\bar 0}+n_{\bar 1}}M = \lambda\cdot 1_{n_{\bar 0}+n_{\bar 1}} \mid \lambda\in\mathbb{K}\} = \iota(Z).$$

Thus, since ι is invertible, $\mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ can have a non-trivial ideal only if k is even, and this ideal must be equal to Z; direct computation shows that Z is indeed an ideal. $\square$

13.7.4. The derived Lie superalgebras of $\mathfrak{pe}(k)$. As it was shown in subsect. 13.4.1, any non-degenerate odd symmetric bilinear form on a superspace of dimension $(k|k)$ (if dimensions of the even and odd parts of the space are not equal, there are no non-degenerate odd bilinear forms on it) is equivalent to the form with the matrix $\Pi_{k|k}$.

The Lie superalgebra $\mathfrak{pe}(k)$ preserving this form consists of the supermatrices of the shape

$$\begin{pmatrix} A & C\\ D & A^T\end{pmatrix},\quad \text{where } A\in\mathfrak{gl}(k),\ C \text{ and } D \text{ are symmetric } k\times k \text{ matrices.} \tag{13.32}$$

Direct computations show that:

$\mathfrak{pe}^{(1)}(1) = \{\lambda\cdot 1_{1|1} \mid \lambda\in\mathbb{K}\}$;
$\mathfrak{pe}^{(2)}(1) = 0$;
$\mathfrak{pe}^{(1)}(2) = \{$matrices of the shape (20.7) such that C and D are zero-diagonal$\}$;
$\mathfrak{pe}^{(2)}(2) = \{$matrices of $\mathfrak{pe}^{(1)}(2)$ such that $\operatorname{tr} A = 0\}$;
$\mathfrak{pe}^{(3)}(2) = \{\lambda\cdot 1_{2|2} \mid \lambda\in\mathbb{K}\}$;
$\mathfrak{pe}^{(4)}(2) = 0$.

13.7.4.1. Lemma. *If $k\geq 3$, then*

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i) $\mathfrak{pe}^{(1)}(k) = \{$matrices of the shape (20.7) such that C and D are zero-diagonal$\}$;
ii) $\mathfrak{pe}^{(2)}(k) = \{$matrices of $\mathfrak{pe}^{(1)}(k)$ such that $\operatorname{tr} A = 0\}$*;*
iii) $\mathfrak{pe}^{(3)}(k) = \mathfrak{pe}^{(2)}(k)$.

Proof. As it was noticed before, matrices and supermatrices behave identically with respect to multiplication and Lie (super)bracket. So, if we denote

$$M^i = \{L\in\mathfrak{gl}(k|k) \mid \text{exists } L'\in\mathfrak{o}_{\Pi}(2k) \text{ such that } L_{ij} = L'_{ij} \text{ for all } i,j\in\overline{1,2k}\};$$
$$N^i = \operatorname{Span}\{L^2 \mid L\in\mathfrak{pe}^{(i-1)}(k), L \text{ is odd }\},\quad \text{where } \mathfrak{pe}^{(0)}(k) = \mathfrak{pe}(k),$$

then we get the following inclusion:

$$M^i \subset \mathfrak{pe}^{(i)}(k) \subset M^i + N^i.$$

Now recall from the proof of Lemma 13.6.5.1 that M^1, M^2, M^3 coincide with conjectural spaces $\mathfrak{pe}^{(1)}(k)$, $\mathfrak{pe}^{(2)}(k)$, $\mathfrak{pe}^{(3)}(k)$ as they are described in the lemma. Notice also that $N^1\subset M^1$ (since N^1 is an even subspace of $\mathfrak{pe}(k)$), so $\mathfrak{pe}^{(1)}(k) = M^1$. If

$$L = \begin{pmatrix} 0 & C\\ D & 0\end{pmatrix}$$

is an odd matrix from $\mathfrak{pe}^{(1)}(k)$ (so C and D are symmetric and zero-diagonal), then

$$L^2 = \begin{pmatrix} CD & 0\\ 0 & DC\end{pmatrix},$$

and

$$\operatorname{tr} CD = \sum_{i,j=1}^{k} C_{ij}D_{ji} = 2\sum_{1\leq i<j\leq n} C_{ij}D_{ij} = 0,$$

so $N^2\subset M^2$, and $\mathfrak{pe}^{(2)}(k) = M^2$. Also, $N^3\subset N^2\subset M^2 = M^3$, and $\mathfrak{pe}^{(3)}(k) = M^3$. $\square$

13.8. Canonical expressions of non-symmetric bilinear superforms. Related Lie superalgebras.

As it was shown in sec. 13.4.2, any even non-symmetric bilinear form B on a superspace in the standard format has the shape

$$\begin{pmatrix} B_{\bar 0} & 0\\ 0 & B_{\bar 1}\end{pmatrix}.$$

This form is preserved by the matrices of the shape

$$\begin{pmatrix} A_{\bar 0} & C\\ D & A_{\bar 1}\end{pmatrix},\quad\text{where}\quad \begin{cases} A_{\bar 0} \text{ preserves the form } B_{\bar 0},\\ A_{\bar 1} \text{ preserves the form } B_{\bar 1},\\ CB_{\bar 1} + B_{\bar 0}D^T = 0,\\ DB_{\bar 0} + B_{\bar 1}C^T = 0.\end{cases}$$

Any odd non-degenerate non-symmetric bilinear form B on a superspace of dimension $(k|k)$ (non-degenerate odd form can exist only on a superspace with equal even and odd dimensions) is equivalent to a form of the shape

$$\begin{pmatrix} 0 & 1_k \\ J & 0 \end{pmatrix}, \text{ where } J \text{ is the Jordan normal form.}$$

Such a form is preserved by the matrices of the shape

$$\begin{pmatrix} A & C \\ D & A^T \end{pmatrix}, \text{ where } \begin{cases} AJ^T + J^T A = 0, \\ CJ + C^T = 0, \\ D + JD^T = 0. \end{cases}$$

13.8.1. Problem. The (first or second) derived Lie superalgebra of $\mathfrak{pe}_B(k|k)$ is simple (perhaps, modulo center) only if J consists of 1×1 blocks. What is an explicit structure in other cases? Compare with Ermolaev's description [Er].

Chapter 14

$\mathfrak{g}(A)$: Examples in characteristic 2

14.1. Ortho-orthogonal Lie superalgebras

In what follows we assume that $p = 2$ and $\mathbb{K}$ is perfect. We also assume that $n_{\bar{0}}, n_{\bar{1}} > 0$.

14.1.1. Non-degenerate bilinear forms and the Lie superalgebras that preserve them. For $p = 2$, there are up to four classes of equivalence of non-equivalent non-degenerate even supersymmetric bilinear forms on a given superspace. For discussion of various equivalences, see Section 13.3 of Chapter 13; here we just recall that we say that *two bilinear forms B and B' on a superspace V are equivalent* if there is an even non-degenerate linear map $M \colon V \to V$ such that

$$B'(x, y) = B(Mx, My) \quad \text{for any} \quad x, y \in V.$$

We fix some basis in V and identify a bilinear form with its *Gram matrix* in this basis; then two such matrices are equivalent if there is an even non-degenerate matrix M such that

$$B' = MBM^T.$$

The classes of equivalence of forms are as follows. Any such form B on a superspace V of superdimension $n_{\bar{0}}|n_{\bar{1}}$ can be decomposed as follows:

$$B = B_{\bar{0}} \oplus B_{\bar{1}},$$

where $B_{\bar{0}}$, $B_{\bar{1}}$ are symmetric non-degenerate forms on $V_{\bar{0}}$ and $V_{\bar{1}}$, respectively. For $i = \bar{0}, \bar{1}$, the form B_i is equivalent to 1_{n_i} if n_i is odd, and equivalent to 1_{n_i} or Π_{n_i} if n_i is even (we identify a bilinear form with its Gram matrix). So every non-degenerate even symmetric bilinear form is equivalent to one of the following forms (some of them are defined not for all dimensions):

$B_{II} = 1_{n_{\bar{0}}} \oplus 1_{n_{\bar{1}}}$; $\quad B_{I\Pi} = 1_{n_{\bar{0}}} \oplus \Pi_{n_{\bar{1}}}$ if $n_{\bar{1}}$ is even;
$B_{\Pi I} = \Pi_{n_{\bar{0}}} \oplus 1_{n_{\bar{1}}}$ if $n_{\bar{0}}$ is even; $B_{\Pi\Pi} = \Pi_{n_{\bar{0}}} \oplus \Pi_{n_{\bar{1}}}$ if $n_{\bar{0}}, n_{\bar{1}}$ are even.

For obvious reasons, the Lie superalgebras that preserve the respective forms are called *ortho-orthogonal*.

We denote them by $\mathfrak{oo}_{II}(n_{\bar 0}|n_{\bar 1})$, $\mathfrak{oo}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$, $\mathfrak{oo}_{\Pi I}(n_{\bar 0}|n_{\bar 1})$, $\mathfrak{oo}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$, respectively.

Now we describe these algebras.

14.1.1.1. $\mathfrak{oo}_{II}(n_{\bar 0}|n_{\bar 1})$. If $n \geq 3$, then the Lie superalgebra $\mathfrak{oo}^{(1)}_{II}(n_{\bar 0}|n_{\bar 1})$ is simple. This Lie superalgebra has a 2-structure; it has no Cartan matrix.

14.1.1.2. $\mathfrak{oo}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$ $(n_{\bar 1} = 2k_{\bar 1})$. The Lie algebra $\mathfrak{oo}^{(1)}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$ is simple.

The Lie algebra $\mathfrak{oo}^{(1)}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$ has a 2-structure if and only if $n_{\bar 0} = 1$; it has a Cartan matrix if and only if $n_{\bar 0}$ is odd; this matrix has the following form (up to format; the possible formats are described in the Table 20.10 below):

$$\begin{pmatrix} \ddots & \ddots & \ddots & \vdots \\ \ddots & * & 1 & 0 \\ \ddots & 1 & * & 1 \\ \cdots & 0 & 1 & 1 \end{pmatrix}$$

Since $\mathfrak{oo}_{\Pi I}(n_{\bar 0}|n_{\bar 1}) \simeq \mathfrak{oo}_{I\Pi}(n_{\bar 1}|n_{\bar 0})$, we do not have to consider the algebra $\mathfrak{oo}_{\Pi I}(n_{\bar 0}|n_{\bar 1})$ separately unless we consider CTS prolongations: Although the algebras are isomorphic, their prolongations from identity representations are non-isomorphic.

14.1.1.3. $\mathfrak{oo}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ $(n_{\bar 0} = 2k_{\bar 0}, n_{\bar 1} = 2k_{\bar 1})$. If $n = n_{\bar 0} + n_{\bar 1} \geq 6$, then

$$\begin{array}{l} \text{if } k_{\bar 0} + k_{\bar 1} \text{ is odd, then the Lie superalgebra } \mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1}) \text{ is simple;} \\ \text{if } k_{\bar 0} + k_{\bar 1} \text{ is even, then } \mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})/\operatorname{Span}(1_{n_{\bar 0}|n_{\bar 1}}) \text{ is simple.} \end{array} \tag{14.1}$$

Each of these simple Lie superalgebras has a 2|4-structure; they are also close to Lie superalgebras with Cartan matrix. To describe these CM superalgebras in most simple terms, we will choose a slightly different realization of $\mathfrak{oo}_{\Pi\Pi}(2k_{\bar 0}|2k_{\bar 1})$: Let us consider it as the algebra of linear transformations that preserve the bilinear form $\Pi(2k_{\bar 0} + 2k_{\bar 1})$ in format $k_{\bar 0}|k_{\bar 1}|k_{\bar 0}|k_{\bar 1}$. Then the algebra $\mathfrak{oo}^{(i)}_{\Pi\Pi}(2k_{\bar 0}|2k_{\bar 1})$ (we assume that $\mathfrak{g}^{(0)} = \mathfrak{g}$) is spanned by supermatrices of format $k_{\bar 0}|k_{\bar 1}|k_{\bar 0}|k_{\bar 1}$ and the form

$$\begin{pmatrix} A & C \\ D & A^T \end{pmatrix} \text{ where } \begin{array}{l} A \in \begin{cases} \mathfrak{gl}(k_{\bar 0}|k_{\bar 1}) & \text{if } i \leq 1, \\ \mathfrak{sl}(k_{\bar 0}|k_{\bar 1}) & \text{if } i \geq 2, \end{cases} \\ C, D \text{ are } \begin{cases} \text{symmetric matrices} & \text{if } i = 0; \\ \text{symmetric zero-diagonal matrices} & \text{if } i \geq 1. \end{cases} \end{array} \tag{14.2}$$

If $i \geq 1$, these derived algebras have a non-trivial central extension given by the following cocycle:

$$F\left(\begin{pmatrix} A & C \\ D & A^T \end{pmatrix}, \begin{pmatrix} A' & C' \\ D' & A'^T \end{pmatrix}\right) = \sum_{1 \leq i < j \leq k_{\bar 0} + k_{\bar 1}} (C_{ij} D'_{ij} + C'_{ij} D_{ij}) \tag{14.3}$$

(note that this expression resembles $\frac{1}{2}\operatorname{tr}(CD' + C'D)$). We will denote this central extension of $\mathfrak{oo}^{(i)}_{\Pi\Pi}(2k_{\bar 0}|2k_{\bar 1})$ by $\mathfrak{ooc}(i, 2k_{\bar 0}|2k_{\bar 1})$.

Let $I_0 := \operatorname{diag}(1_{k_{\bar 0}|k_{\bar 1}}, 0_{k_{\bar 0}|k_{\bar 1}})$. Then the corresponding CM Lie superalgebra is

$$\begin{array}{l} \mathfrak{ooc}(2, 2k_{\bar 0}|2k_{\bar 1}) \Subset \mathbb{K}I_0 \text{ if } k_{\bar 0} + k_{\bar 1} \text{ is odd;} \\ \mathfrak{ooc}(1, 2k_{\bar 0}|2k_{\bar 1}) \Subset \mathbb{K}I_0 \text{ if } k_{\bar 0} + k_{\bar 1} \text{ is even.} \end{array} \tag{14.4}$$

The corresponding Cartan matrix has the form (up to format; the possible formats are described in the Table 20.10 below):

$$\begin{pmatrix} \ddots & \ddots & \ddots & \vdots & \vdots \\ \ddots & 0 & 1 & 0 & 0 \\ \ddots & 1 & 0 & 1 & 1 \\ \cdots & 0 & 1 & 0 & 0 \\ \cdots & 0 & 1 & 0 & 0 \end{pmatrix} \tag{14.5}$$

14.2. Periplectic Lie superalgebras

If $n_{\bar 0} = n_{\bar 1}$, then one can also consider non-degenerate supersymmetric *odd* bilinear forms. All such forms are equivalent to $\Pi_{m|m}$, where $m = n_{\bar 0} = n_{\bar 1}$. We call the Lie superalgebra preserving a given non-degenerate symmetric odd bilinear form B *periplectic*, as A. Weil suggested, and denote it by $\mathfrak{pe}_B(m)$.

If $m \geq 3$, then

$$\begin{array}{l} \text{if } m \text{ is odd, then the Lie superalgebra } \mathfrak{pe}^{(2)}_B(m) \text{ is simple;} \\ \text{if } m \text{ is even, then } \mathfrak{pe}^{(2)}_B(m)/\operatorname{Span}(1_{m|m}) \text{ is simple.} \end{array} \tag{14.6}$$

If we choose the form B to be $\Pi_{m|m}$, then the algebras $\mathfrak{pe}^{(i)}_B(m)$ consist of matrices of the form (14.2); the only difference from $\mathfrak{oo}^{(i)}_{\Pi\Pi}$ is the format which in this case is $m|m$.

Each of these simple Lie superalgebras has a 2|4-structure. Note that if $p \neq 2$, then the Lie superalgebra $\mathfrak{pe}_B(m)$ and its derived algebras are not close to CM Lie (super)algebras (because, for example, their root system is not symmetric). If $p = 2$ and $m \geq 3$, then they **are** close to CM Lie superalgebras as we will see shortly.

The algebras $\mathfrak{pe}^{(i)}_B(m)$, where $i > 0$, have non-trivial central extensions with cocycles (14.3); we denote these central extensions by $\mathfrak{pec}(i, m)$. Let us introduce another matrix $I_0 = \operatorname{diag}(1_m, 0_m)$. Then the CM Lie superalgebras are

$$\begin{aligned} &\mathfrak{pec}(2, m) \Subset \mathbb{K}I_0 \text{ if } m \text{ is odd;}\\ &\mathfrak{pec}(1, m) \Subset \mathbb{K}I_0 \text{ if } m \text{ is even.}\end{aligned} \tag{14.7}$$

The corresponding Cartan matrix has the form (14.5); the only condition on its format is that the last two simple roots must have distinct parities. The corresponding Dynkin diagram is shown in the Table 20.10; all its nodes, except for the "horns", may be both $\otimes$ or $\odot$.

14.2.1. Dynkin diagrams. The following Dynkin diagrams correspond to CM Lie superalgebras close to ortho-orthogonal and periplectic Lie superalgebras. Each thin black dot may be $\otimes$ or $\odot$; the last five columns show conditions on the diagrams; what concerns the last four columns, it suffices to satisfy conditions in any one row. Horizontal lines in the last four columns separate the cases corresponding to different Dynkin diagrams. The notations are:

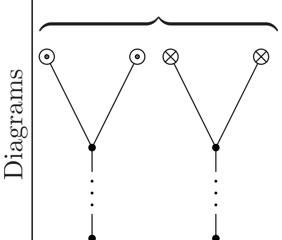
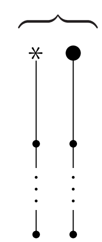
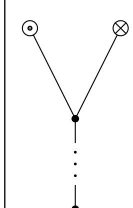
v is the total number of nodes in the diagram;

ng is the number of $\otimes$ among the thin black dots; png is the parity of this number;

ev and od are the number of thin black dots such that the number of $\otimes$ to the left from them is even and odd, respectively.

14.2.2. Superdimensions. The following are the superdimensions of the relatives of the ortho-orthogonal and periplectic Lie superalgebras that possess Cartan matrices. To get the superdimensions of the simple relatives, one should replace $+2$ and $+1$ by -2 and -1, respectively, in the two first lines and the four last ones.

$$\begin{array}{ll}
\dim \mathfrak{oc}(1; 2k) \Subset \mathbb{K}I_0 & = 2k^2 - k + 2 \ \text{ if } k \text{ is even;}\\
\dim \mathfrak{oc}(2; 2k) \Subset \mathbb{K}I_0 & = 2k^2 - k + 1 \ \text{ if } k \text{ is odd;}\\
\dim \mathfrak{o}^{(1)}(2k+1) & = 2k^2 + k\\
\operatorname{sdim} \mathfrak{oo}^{(1)}(2k_{\bar 0}+1|2k_{\bar 1}) & = 2k_{\bar 0}^2 + k_{\bar 0} + 2k_{\bar 1}^2 + k_{\bar 1} \mid 2k_{\bar 1}(2k_{\bar 0}+1)\\
\operatorname{sdim} \mathfrak{ooc}(1; 2k_{\bar 0}|2k_{\bar 1}) \Subset \mathbb{K}I_0 & = 2k_{\bar 0}^2 - k_{\bar 0} + 2k_{\bar 1}^2 - k_{\bar 1} + 2 \mid 4k_{\bar 0}k_{\bar 1} \ \text{ if } k_{\bar 0}+k_{\bar 1} \text{ is even;}\\
\operatorname{sdim} \mathfrak{ooc}(2; 2k_{\bar 0}|2k_{\bar 1}) \Subset \mathbb{K}I_0 & = 2k_{\bar 0}^2 - k_{\bar 0} + 2k_{\bar 1}^2 - k_{\bar 1} + 1 \mid 4k_{\bar 0}k_{\bar 1} \ \text{ if } k_{\bar 0}+k_{\bar 1} \text{ is odd;}\\
\operatorname{sdim} \mathfrak{pec}(1; m) \Subset \mathbb{K}I_0 & = m^2 + 2 \mid m^2 - m \ \text{ if } m \text{ is even;}\\
\operatorname{sdim} \mathfrak{pec}(2; m) \Subset \mathbb{K}I_0 & = m^2 + 1 \mid m^2 - m \ \text{ if } m \text{ is odd}
\end{array} \tag{14.8}$$

Diagrams	$\mathfrak{g}$	v	ev	od	png	$ng \le \min(*, *)$
	$\mathfrak{ooc}(2; 2k_{\bar 0}\|2k_{\bar 1}) \Subset \mathbb{K}I_0$ if $k_{\bar 0}+k_{\bar 1}$ is odd; $\mathfrak{ooc}(1; 2k_{\bar 0}\|2k_{\bar 1}) \Subset \mathbb{K}I_0$ if $k_{\bar 0}+k_{\bar 1}$ is even.	$k_{\bar 0}+k_{\bar 1}$	$k_{\bar 0}-2$	$k_{\bar 1}$	$\bar 0$	$2k_{\bar 0}-4, 2k_{\bar 1}$
			$k_{\bar 1}$	$k_{\bar 0}-2$	$\bar 1$	$2k_{\bar 0}-3, 2k_{\bar 1}-1$
			$k_{\bar 1}-2$	$k_{\bar 0}$	$\bar 0$	$2k_{\bar 0}, 2k_{\bar 1}-4$
			$k_{\bar 0}$	$k_{\bar 1}-2$	$\bar 1$	$2k_{\bar 0}-1, 2k_{\bar 1}-3$
			$k_{\bar 0}-1$	$k_{\bar 1}-1$		$2k_{\bar 0}-2, 2k_{\bar 1}-1$
			$k_{\bar 1}-1$	$k_{\bar 0}-1$		$2k_{\bar 0}-1, 2k_{\bar 1}-2$
	$\mathfrak{oo}^{(1)}_{I\Pi}(2k_{\bar 0}+1\|2k_{\bar 1})$	$k_{\bar 0}+k_{\bar 1}$	$k_{\bar 0}-1$	$k_{\bar 1}$	$\bar 0$	$2k_{\bar 0}-2, 2k_{\bar 1}$
			$k_{\bar 1}$	$k_{\bar 0}-1$	$\bar 1$	$2k_{\bar 0}-1, 2k_{\bar 1}-1$
			$k_{\bar 1}-1$	$k_{\bar 0}$	$\bar 0$	$2k_{\bar 0}, 2k_{\bar 1}-2$
			$k_{\bar 0}$	$k_{\bar 1}-1$	$\bar 1$	$2k_{\bar 0}-1, 2k_{\bar 1}-1$
	$\mathfrak{pec}(2; m) \Subset \mathbb{K}I_0$ if m is odd; $\mathfrak{pec}(1; m) \Subset \mathbb{K}I_0$ if m is even.	m				

14.3. The $\mathfrak{e}$-type superalgebras

Some of the results of this section are cited from [BGL5].

In characteristic 2 there exist super-analogs of Lie algebras $\mathfrak{e}(6)$, $\mathfrak{e}(7)$ and $\mathfrak{e}(8)$. These Lie superalgebras have Dynkin diagrams of $\mathfrak{e}(n)$ with some of the nodes changed to $\otimes$. We call them $\mathfrak{e}$-*type superalgebras* and denote them by their simplest Dynkin diagrams: $\mathfrak{e}(n,i)$ denotes the Lie superalgebra whose diagram is of the same shape as that of the Lie algebra $\mathfrak{e}(n)$ but with the only — i-th — node $\otimes$.

14.3.1. Certain notation. We enumerate the nodes of the Dynkin diagram of $\mathfrak{e}(n)$ as in [Bou, OV]: We first enumerate the nodes in the row corresponding to $\mathfrak{sl}(n)$ (from the end-point of the "longest" twig towards the branch point and further on along the second long twig), and the nth node is the end-point of the shortest "twig". Recall that for $\mathfrak{e}(6)$ and $\mathfrak{e}(7)$, the adjoint module is $R(\pi_6)$; for $\mathfrak{e}(8)$, the adjoint module is $R(\pi_1)$.

14.3.1.1. Remark. In what follows (and in Elduque's super constructions) the spinor representation appears. For $\mathfrak{o}^{(1)}(2k+1)$, the *spinor representation* should be *defined* to be the kth fundamental representation; for $\mathfrak{ooc}(2k)$, the *spinor representations* are the kth and the $(k-1)$st fundamental representations.

14.3.2. $\mathfrak{e}(6,1)\simeq\mathfrak{e}(6,5)$, sdim 46|32. We have $\mathfrak{g}_{\bar 0}\simeq\mathfrak{ooc}(2;10)\oplus\mathbb{K}Z$ and $\mathfrak{g}_{\bar 1}$ is a reducible module of the form $\text{spin}_{10}\oplus\text{spin}_{10}$ with two highest weight vectors

$$x_{36}=[[[x_4,x_5],[x_6,[x_2,x_3]]],[[x_3,[x_1,x_2]],[x_6,[x_3,x_4]]]]$$

and y_5. Denote the basis elements of the Cartan subalgebra by Z, h_1, h_2, h_3, h_4, h_6. The weights of x_{36} and y_5 are respectively, $(0,0,0,0,0,1)$ and $(0,0,0,0,1,0)$. The module generated by x_{36} gives all odd positive roots and the module generated by y_5 gives all odd negative roots.

14.3.3. $\mathfrak{e}(6,6)$, sdim 38|40. We have $\mathfrak{g}_{\bar 0}\simeq(\mathfrak{hei}(2)\oplus\mathfrak{psl}(6))\Subset\mathbb{K}Z$ (in fact, this Z is just h_6 from the Cartan subalgebra corresponding to the first Cartan matrix). The module $\mathfrak{g}_{\bar 1}$ is irreducible with highest weight vector

$$x_{35}=[[[x_3,x_6],[x_4,[x_2,x_3]]],[[x_4,x_5],[x_3,[x_1,x_2]]]]$$

of weight $(1,0,0,1,0,1)$ with respect to the Cartan subalgebra corresponding to the direct sum of Cartan subalgebra of $\mathfrak{hei}(2)$ spanned by $h_1+h_3+h_5$ and the Cartan subalgebra of $\mathfrak{psl}(6)$ spanned by h_1,h_2,h_3,h_4.

14.3.4. $\mathfrak{e}(7,1)$, sdim $=80|54$. Since the Cartan matrix above is of rank 6, a grading operator d_1 should be added. Now if we take $d_1=(1,0,0,0,0,0,0)$, then $\mathfrak{g}_{\bar 0}\simeq\mathfrak{e}(6)\oplus\mathbb{K}z\oplus\mathbb{K}I_0$. The Cartan subalgebra is generated by $h_1+h_3+h_7,h_2,h_3,h_4,h_5,h_6,h_7$ and d_1. So $\mathfrak{g}_{\bar 1}$ has the two highest weight vectors:

$$x_{63}=[[[[x_2,x_3],[x_4,x_7]],[[x_3,x_4],[x_5,x_6]]],[[[x_4,x_7],[x_5,x_6]],[[x_4,x_5],[x_3,[x_1,x_2]]]]]$$

and y_1. Their respective weights (if we take $d_1=(1,0,0,0,0,0,0)$) are $(0,0,0,0,0,1,0,1)$ and $(0,1,0,0,0,0,0,0)$. The module generated by x_{63} gives all odd positive roots and the module generated by y_1 gives all odd negative roots.

14.3.5. $\mathfrak{e}(7,6)$, sdim $=69|64$. We have

$$\mathfrak{g}_{\bar 0}\simeq(\mathfrak{hei}(2)\oplus\mathfrak{oc}(1;12)/\mathfrak{center})\Subset(\mathbb{K}z\oplus\mathbb{K}I_0).$$

The module $\mathfrak{g}_{\bar 1}$ is irreducible with the highest weight vector

$$x_{62}=[[[x_7,[x_5,[x_3,x_4]]],[[x_1,x_2],[x_3,x_4]]],[[[x_2,x_3],[x_4,x_5]],[[x_4,x_7],[x_5,x_6]]]].$$

The Cartan subalgebra is spanned by $h_1+h_3+h_5,h_1,h_2,h_3,h_4,h_7$ and also h_6 and d_1. The weight of x_{62} is $(1,0,0,0,0,0,1,0)$.

14.3.6. $\mathfrak{e}(7,7)$, sdim $=64|70$. This Cartan matrix is also of rank 6, so a grading operator should be added and is added. Then $\mathfrak{g}_{\bar 0}\simeq(\mathfrak{sl}(7)\oplus\mathbb{K})\Subset\mathbb{K}I_0$. The module $\mathfrak{g}_{\bar 1}$ has the two highest weight vectors:

$$x_{58}=[[[x_3,[x_1,x_2]],[x_6,[x_4,x_5]]],[[x_7,[x_3,x_4]],[[x_2,x_3],[x_4,x_5]]]]$$

and y_7. The Cartan subalgebra is spanned by h_1,h_2,h_3,h_4,h_5,h_6 and also $h_1+h_3+h_7$ and d_1. The weight of x_{58} is $(0,0,1,0,0,0,0,1)$ and the weight of y_7 is $(0,0,0,1,0,0,0,1)$. The module generated by x_{58} gives all odd positive roots and the module generated by y_7 gives all odd negative roots.

14.3.7. $\mathfrak{e}(8,1)$, sdim $=136|112$. We have $\mathfrak{g}_{\bar 0}\simeq(\mathfrak{hei}(2)\oplus\mathfrak{e}(7)/\mathfrak{center})\Subset\mathbb{K}z)$. The Cartan subalgebra is spanned by $h_2+h_4+h_8$ and h_2,h_3,h_4,h_5,h_6,h_7 and also h_1. The $\mathfrak{g}_{\bar 0}$-module $\mathfrak{g}_{\bar 1}$ is irreducible with one highest weight vector:

$$\begin{aligned}x_{119}=[[[[x_4,[x_2,x_3]],[[x_5,x_8],[x_6,x_7]]],[[x_8,[x_4,x_5]],[[x_3,x_4],[x_5,x_6]]]]\ ,\\ [[[x_7,[x_5,x_6]],[[x_1,x_2],[x_3,x_4]]],[[x_8,[x_5,x_6]],[[x_2,x_3],[x_4,x_5]]]]]\end{aligned}$$

of weight $(1,1,0,0,0,0,0,1)$ and one lowest weight vector y_{119} whose expression is as above the x's changed by the y's, of the same weight as that of x_{119}.

14.3.8. $\mathfrak{e}(8,8)$, sdim $=120|128$. We have $\mathfrak{g}_{\bar 0}\simeq\mathfrak{o}^1_{\Pi}(16)$ and $\mathfrak{g}_{\bar 1}\simeq R(\pi_7)$. In the $\mathbb{Z}$-grading with the 1st CM with $\deg e_8^{\pm}=\pm1$ and $\deg e_i^{\pm}=0$ for $i\neq 8$, we have $\mathfrak{g}_0=\mathfrak{gl}(8)=\mathfrak{gl}(V)$ and (as $\mathfrak{g}_0$-modules) $\mathfrak{g}_{\pm i}=\wedge^{(4-i)}(V)$ for $i=1,2,3$.

14.4. Systems of simple roots of the $\mathfrak{e}$-type Lie superalgebras

[Ol: zdes' index{$\mathfrak{e}(a,b)$}] 21

14.4.1. $\mathfrak{e}(6,1) \simeq \mathfrak{e}(6,5)$, sdim 46|32. All inequivalent Cartan matrices are as follows (symmetric ones are not excluded):

1) 000010	2) 010001	3) 100110	4) 000011
5) 010100	6) 101001	7) 000101	8) 011000
9) 101100	10) 000110	11) 011001	12) 110000
13) 000111	14) 011010	15) 110001	16) 001011
17) 011110	18) 110010	19) 001100	20) 100000
21) 110110	22) 001101	23) 100001	24) 111001
25) 001111	26) 100010	27) 111100	

(14.9)

14.4.2. $\mathfrak{e}(6,6)$, sdim 38|40. All inequivalent Cartan matrices are as follows:

1) 000001	2) 011011	3) 101110	4) 000100
5) 011100	6) 101111	7) 001000	8) 011101
9) 110011	10) 001001	11) 011111	12) 110100
13) 001010	14) 100011	15) 110101	16) 001110
17) 100100	18) 110111	19) 010000	20) 100101
21) 111000	22) 010010	23) 100111	24) 111010
25) 010011	26) 101000	27) 111011	28) 010101
29) 101010	30) 111101	31) 010110	32) 101011
33) 111110	34) 010111	35) 101101	36) 111111

(14.10)

14.4.3. $\mathfrak{e}(7,1)$, sdim = 80|54. All inequivalent Cartan matrices are as follows:

1) 1000000	2) 1000010	3) 1000110	4) 1001100
5) 1010001	6) 1011001	7) 1100000	8) 1100010
9) 1100110	10) 1101100	11) 1110001	12) 1111001
13) 0000011	14) 0000101	15) 0000111	16) 0001011
17) 0001101	18) 0001111	19) 0010100	20) 0011000
21) 0011010	22) 0011110	23) 0100001	24) 0101001
25) 0110000	26) 0110010	27) 0110110	28) 0111100

(14.11)

14.4.4. $\mathfrak{e}(7,6)$, sdim = 69|64. All inequivalent Cartan matrices are as follows:

1) 0000010	2) 0000100	3) 0000110	4) 0001000
5) 0001010	6) 0001100	7) 0001110	8) 0010001
9) 0010011	10) 0010101	11) 0010111	12) 0011001
13) 0011011	14) 0011101	15) 0011111	16) 0100000
17) 0100010	18) 0100100	19) 0100110	20) 0101000
21) 0101010	22) 0101100	23) 0101110	24) 0110001
25) 0110011	26) 0110101	27) 0110111	28) 0111001
29) 0111011	30) 0111101	31) 0111111	32) 1000001
33) 1000011	34) 1000101	35) 1000111	36) 1001001
37) 1001011	38) 1001101	39) 1001111	40) 1010000
41) 1010010	42) 1010100	43) 1010110	44) 1011000
45) 1011010	46) 1011100	47) 1011110	48) 1100001
49) 1100011	50) 1100101	51) 1100111	52) 1101001
53) 1101011	54) 1101101	55) 1101111	56) 1110000
57) 1110010	58) 1110100	59) 1110110	60 1111000
61) 1111010	62) 1111100	63) 1111110	

(14.12)

14.4.5. $\mathfrak{e}(7,7)$, sdim = 64|70. All inequivalent Cartan matrices are as follows:

1) 0000001	2) 0001001	3) 0010000
4) 0010010	5) 0010110	6) 0011100
7) 0100011	8) 0100101	9) 0100111
10) 0101011	11) 0101101	12) 0101111
13) 0110100	14) 0111000	15) 0111010
16) 0111110	17) 1000100	18) 1001000
19) 1001010	20) 1001110	21) 1010011
22) 1010101	23) 1010111	24) 1011011
25) 1011101	26) 1011111	27) 1100100
28) 1101000	29) 1101010	30) 1101110
31) 1110011	32) 1110101	33) 1110111
34) 1111011	35) 1111101	36) 1111111

(14.13)

14.4.6. $\mathfrak{e}(8,8)$, sdim = 120|128. All inequivalent Cartan matrices are as follows:

1)	00000001	2)	00000010	3)	00000110	4)	00001001
5)	00001100	6)	00010000	7)	00010001	8)	00010010
9)	00010110	10)	00011001	11)	00011100	12)	00100000
13)	00100010	14)	00100011	15)	00100101	16)	00100110
17)	00100111	18)	00101011	19)	00101100	20)	00101101
21)	00101111	22)	00110001	23)	00110100	24)	00111000
25)	00111001	26)	00111010	27)	00111110	28)	01000011
29)	01000100	30)	01000101	31)	01000111	32)	01001000
33)	01001010	34)	01001011	35)	01001101	36)	01001110
37)	01001111	38)	01010011	39)	01010100	40)	01010101
41)	01010111	42)	01011000	43)	01011010	44)	01011011
45)	01011101	46)	01011110	47)	01011111	48)	01100001
49)	01100100	50)	01101000	51)	01101001	52)	01101010
53)	01101110	54)	01110000	55)	01110010	56)	01110011
57)	01110101	58)	01110110	59)	01110111	60)	01111011
61)	01111100	62)	01111101	63)	01111111	64)	10000001
65)	10000100	66)	10001000	67)	10001001	68)	10001010
69)	10001110	70)	10010000	71)	10010010	72)	10010011
73)	10010101	74)	10010110	75)	10010111	76)	10011011
77)	10011100	78)	10011101	79)	10011111	80)	10100011
81)	10100100	82)	10100101	83)	10100111	84)	10101000
85)	10101010	86)	10101011	87)	10101101	88)	10101110
89)	10101111	90)	10110011	91)	10110100	92)	10110101
93)	10110111	94)	10111000	95)	10111010	96)	10111011
97)	10111101	98)	10111110	99)	10111111	100)	11000001
101)	11000100	102)	11001000	103)	11001001	104)	11001010
105)	11001110	106)	11010000	107)	11010010	108)	11010011
109)	11010101	110)	11010110	111)	11010111	112)	11011011
113)	11011100	114)	11011101	115)	11011111	116)	11100011
117)	11100100	118)	11100101	119)	11100111	120)	11101000
121)	11101010	122)	11101011	123)	11101101	124)	11101110
125)	11101111	126)	11110011	127)	11110100	128)	11110101
129)	11110111	130)	11111000	131)	11111010	132)	11111011
133)	11111101	134)	11111110	135)	11111111		

(14.14)

14.4.7. $\mathfrak{e}(8,1)$, sdim = 136|112. All inequivalent Cartan matrices are as follows:

1)	10000000	2)	10000010	3)	10000011	4)	10000101
5)	10000110	6)	10000111	7)	10001011	8)	10001100
9)	10001101	10)	10001111	11)	10010001	12)	10010100
13)	10011000	14)	10011001	15)	10011010	16)	10011110
17)	10100000	18)	10100001	19)	10100010	20)	10100110
21)	10101001	22)	10101100	23)	10110000	24)	10110001
25)	10110010	26)	10110110	27	10111001	28)	10111100
29)	11000000	30)	11000010	31)	11000011	32)	11000101
33)	11000110	34)	11000111	35)	11001011	36)	11001100
37)	11001101	38)	11001111	39)	11010001	40)	11010100
41)	11011000	42)	11011001	43)	11011010	44)	11011110
45)	11100000	46)	11100001	47)	11100010	48)	11100110
49)	11101001	50)	11101100	51)	11110000	52)	11110001
53)	11110010	54)	11110110	55)	11111001	56)	11111100
57)	00000011	58)	00000100	59)	00000101	60)	00000111
61)	00001000	62)	00001010	63)	00001011	64)	00001101
65)	00001110	66)	00001111	67)	00010011	68)	00010100
69)	00010101	70)	00010111	71)	00011000	72)	00011010
73)	00011011	74)	00011101	75)	00011110	76)	00011111
77)	00100001	78)	00100100	79)	00101000	80)	00101001
81)	00101010	82)	00101110	83)	00110000	84)	00110010
85)	00110011	86)	00110101	87)	00110110	88)	00110111
89)	00111011	90)	00111100	91)	00111101	92)	00111111
93)	01000000	94)	01000001	95)	01000010	96)	01000110
97)	01001001	98)	01001100	99)	01010000	100)	01010001
101)	01010010	102)	01010110	103)	01011001	104)	01011100
105)	01100000	106)	01100010	107)	01100011	108)	01100101
109)	01100110	110)	01100111	111)	01101011	112)	01101100
113)	01101101	114)	01101111	115)	01110001	116)	01110100
117)	01111000	118)	01111001	119)	01111010	120)	01111110

(14.15)

Chapter 15

Presentations of finite dimensional symmetric classical modular Lie algebras and superalgebras

15.1. Introduction

All spaces considered are finite dimensional over an algebraically closed field $\mathbb{K}$ of characteristic p.

15.1.1. Motivations. Recently we observe a rise of interest in presentations (by means of generators and defining relations) of simple (and close to simple) Lie (super)algebras occasioned by various applications of this technical result, see [GL1, LSe, Sa] and references therein.

This chapter can be considered as a first step towards solution of the problem of description of "Chevalley **super**groups"; for their first examples and a way to apply them to a description of the moduli superspaces of "super Riemannian surfaces", see [FG].

Representations of quantum groups — the deforms $U_q(\mathfrak{g})$ of the enveloping algebras — at q equal to a root of unity resemble, even over $\mathbb{C}$, representations of Lie algebras in positive characteristic and this is one more application that brought the modular Lie (super)algebras and their presentations to the limelight.

15.1.2. Disclaimer. Although presentation — description in terms of generators and relations — is one of the accepted ways to represent a given algebra, it seems that an *explicit* form of the presentation is worth the trouble to obtain only if this presentation is often in need, or (which is usually the same) is sufficiently neat. The Chevalley generators of simple finite dimensional Lie algebras over $\mathbb{C}$ satisfy simple and neat relations ("Serre relations") and are often needed for various calculations and theoretical discussions. Relations between their analogs in super case, although not so neat, are still tolerable, at least, for certain Cartan matrices (both Serre and "non-Serre relations").

Leites was the first (in 1975) to conjecture that simple infinite dimensional vectorial Lie algebras are finitely presented; he suggest analogs of the Chevalley generators for them and (roughly) described their defining relations; for details obtained later, see [LP]. These relations are too complicated to be used by humans and were of academic interest until lately Grozman's package SuperLie made the task of finding the explicit expression of the defining relations for many types of Lie algebras and superalgebras a routine exercise for anybody capable to use Mathematica.

Therefore I do not bother with vectorial type Lie (super)algebras, especially, in view of the shearing parameter $\underline{N}$, which makes the task undescribable in the general case.

Speaking about the general case, SuperLie, although great, can not prove by induction, and can only help to get presentations for a given algebra. Therefore, in this paper, I confine myself to the modular analogs of the simple finite dimensional Lie algebras over $\mathbb{C}$ only. It is still quite a job, hopefully worth the trouble.

15.1.3. Main results. Whereas Grozman, and even his package SuperLie [Gr], knew how to construct $\mathfrak{g}(A_p)$ for a decade (although not for $p = 3$ and 2), it is shown here for the first time how to construct a modular Lie superalgebra if it has no Cartan matrix, like $\mathfrak{o}_I(n)$.

Here, for $p > 0$, I describe the defining relations for the "symmetric" (for $p \neq 2$, these algebras can be described as having, together with any root α, a root $-\alpha$ of the same multiplicity) simple Lie (super)algebras obtained from the simple finite dimensional complex Lie algebras by means of the Kostrikin-Shafarevich procedure. I also consider certain "relatives" of simple Lie (super)algebras (e.g., their nontrivial central extensions); such relatives are often more interesting in applications than the simple algebras they originate from.

Having described in [L1] the orthogonal Lie algebras $\mathfrak{o}(n)$ (there are two non-isomorphic types for n even) and ortho-orthogonal Lie superalgebras for $p = 2$, I describe here presentations of them and of their simple subquotients.

15.1.4. Defining relations for Lie superalgebras $\mathfrak{g} = \mathfrak{g}(A)$. The simple Lie superalgebras of the form $\mathfrak{g} = \mathfrak{g}(A)$ and their relatives have several quite distinct sets of relations (cf. [Sa] and refs therein) but usually they are given by their *Chevalley generators* $X_i^{\pm}$ of degree ± 1 to which the elements $H_i = [X_i^+, X_i^-]$ are added for convenience. (Note that if A is degenerate, then the elements $X_i^{\pm}$ generate not the whole $\mathfrak{g}(A, I)$ but only $\mathfrak{g}^{(1)}(A)$. To generate $\mathfrak{g}(A, I)$, we need to add $\operatorname{size}(A) - \operatorname{rk}(A)$ generators from $\mathfrak{h}$. These generators do not give new type of relations, so in what follows we consider generating relations for $\mathfrak{g}^{(1)}(A)$. (This remark is rather important: It allows us to consider $\mathfrak{sl}(n)$ for any n, and not only the Lie algebras with Cartan matrices, like $\mathfrak{sl}(2n+1)$ or $\mathfrak{gl}(2n)$, and also $\mathfrak{e}(7)$.) These generators satisfy the following relations

$$[X_i^+, X_j^-] = \delta_{ij} H_i, \quad [H_i, H_j] = 0, \quad [H_i, X_j^{\pm}] = \pm A_{ij} X_j^{\pm}, \tag{15.1}$$

and additional relations $R_i = 0$ whose left sides are implicitly described, for a general Cartan matrix with entries in $\mathbb{K}$, as ([K3])

$$\text{"the } R_i \text{ that generate the ideal } \mathfrak{r} \text{ of } \tilde{g}(A) \text{ (see §12.3 of Chapter 12)"} \tag{15.2}$$

Serre has proved that for $p = 0$ and normalized Cartan matrices of simple finite dimensional Lie algebras this additional relations have the form

$$(\mathrm{ad}_{X_i^{\pm}})^{1-A_{ij}} X_j^{\pm} = 0. \tag{15.3}$$

If we consider normalized Cartan matrices of simple finite dimensional Lie superalgebras for $p = 0$, then we should add relations of the form

$$[X_j^{\pm}, X_j^{\pm}] = 0 \quad \text{for } 1 \leq j \leq n \text{ such that } i_j = \bar{1}, A_{jj} = 0. \tag{15.4}$$

Grozman and Leites ([GL1]) has shown that for $p = 0$ and finite-dimensional Lie superalgebras with Cartan matrix the powers $1 - A_{ij}$ in relations (15.3) must be replaced with $1 + b_{ij}$, where b_{ij} are as in (12.42); no additional relations appear. The same is true for the same types of Lie algebras and superalgebras if $p > 3$.

If $p = 3$, then one should also add the relations

$$[X_j^{\pm}, [X_i^{\pm}, X_i^{\pm}]] = 0 \quad \text{for } 1 \leq i \leq n \text{ such that } i_j = \bar{1}, A_{jj} \neq 0$$

(if $p = 3$, then this does not follow from the Jacobi identity).

Here we consider the case of $p = 2$. Clearly, in this case the relations (15.4) have the form

$$(X_j^{\pm})^2 = 0 \quad \text{for } 1 \leq j \leq n \text{ such that } i_j = \bar{1}, A_{jj} = 0$$

(of course, this form works for $p \neq 2$ as well). Still it turns out that in this case the above relations are not enough, and there appear relations of a new type (non-Serre relations). Here we describe them: for Lie algebras in Section 15.2 and for Lie superalgebras in Section 15.3. Note that we have proven them only for (super)algebras of $\mathfrak{sl}$ type and their relatives. Relations for the rest of the (super)algebras are results of computations with SuperLie. So for series of algebras (like $\mathfrak{o}, \mathfrak{oo}, \mathfrak{osp}$) they are conjectural.

15.2. Results: Lie algebras

Here we consider the classical Lie algebras and superalgebras as preserving the volume element or a non-degenerate bilinear form. We interpret the exceptional Lie (super)algebras as preserving a non-integrable distribution, cf. [Shch]. For each Lie (super)algebra there are several such interpretations, we consider the simplest ones: the most easy to describe. We hope to consider their other interpretations (as preserving a non-integrable distribution) elsewhere.

15.2.1. $\mathfrak{g} = \mathfrak{sl}(n+1)$. This is the algebra of traceless $(n+1) \times (n+1)$-matrices. Then $x_i = E^{i,i+1}, y_i = E^{i+1,i}, h_i = E^{i,i} - E^{i+1,i+1}$; the subalgebra $\mathfrak{n}$ consists of upper-triangular matrices.

15.2.2. Theorem. *In characteristic* > 2, *the Serre relations* (15.3) *define* $\mathfrak{n}$; *in characteristic* 2, *the following additional relations are required:*

$$[[x_{i-1}, x_i], [x_i, x_{i+1}]] = 0 \quad \text{for } 1 < i < n. \tag{15.5}$$

15.2.2.1. Remark. In characteristic $p > 0$, the Lie algebra $\mathfrak{sl}(pk)$ is not simple, since it contains the center $\mathfrak{c} = \{\lambda \cdot 1_{pk} \mid \lambda \in \mathbb{K}\}$. The corresponding simple Lie algebra $\mathfrak{sl}(pk)/\mathfrak{c}$ is denoted by $\mathfrak{psl}(pk)$. Since the reduction from $\mathfrak{sl}(pk)$ to $\mathfrak{psl}(pk)$ does not affect the structure of $\mathfrak{n}$, the non-Serre relations stay the same.

15.2.3. $\mathfrak{g} = \mathfrak{e}(n)$**.** Direct computer calculations prove the following theorem. Let the nodes of the Dynkin diagram of $\mathfrak{e}(8)$ be numbered as usual:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ & & & & 8 & & \end{matrix}$$

15.2.4. Theorem. *In characteristic* 2, *in the case of* $\mathfrak{g} = \mathfrak{e}(8)$, *the following list of relations must be added to the Serre relations:*

$$\begin{aligned}
&[[x_1, x_2], [x_2, x_3]] = 0;\\
&[[x_2, x_3], [x_3, x_4]] = 0;\\
&[[x_3, x_4], [x_4, x_5]] = 0;\\
&[[x_4, x_5], [x_5, x_6]] = 0;\\
&[[x_5, x_6], [x_6, x_7]] = 0;\\
&[[x_4, x_5], [x_5, x_8]] = 0;\\
&[[x_5, x_6], [x_5, x_8]] = 0;\\
&[[x_4, [x_5, x_6]], [x_4, [x_5, x_8]]] = 0;\\
&[[x_4, [x_5, x_6]], [x_8, [x_5, x_6]]] = 0;\\
&[[x_4, [x_5, x_8]], [x_8, [x_5, x_6]]] = 0;\\
&[[x_3, [x_4, [x_5, x_6]]], [x_3, [x_4, [x_5, x_8]]]] = 0;\\
&[[x_4, [x_5, [x_6, x_7]]], [x_8, [x_5, [x_6, x_7]]]] = 0;\\
&[[x_2, [x_3, [x_4, [x_5, x_6]]]], [x_2, [x_3, [x_4, [x_5, x_8]]]]] = 0;\\
&[[x_1, [x_2, [x_3, [x_4, [x_5, x_6]]]]], [x_1, [x_2, [x_3, [x_4, [x_5, x_8]]]]]] = 0.
\end{aligned} \tag{15.6}$$

To obtain the corresponding lists of relations for $\mathfrak{e}(6)$ *or* $\mathfrak{e}(7)$, *one should delete the relations containing the "extra"* x_i *and renumber the rest of the* x_i, *i.e:*

1) delete the relations containing x_1 *for* $\mathfrak{e}(7)$, x_1 *and* x_2 *for* $\mathfrak{e}(6)$;

2) decrease all indices of the x_i *by* 1 *for* $\mathfrak{e}(7)$, *by* 2 *for* $\mathfrak{e}(6)$.

15.2.4.1. Remark. The non-Serre generating relations for Lie superalgebras $\mathfrak{e}(n, i)$ (see Section 20.9.2) are the same as for the corresponding Lie algebras $\mathfrak{e}(n)$.

15.2.4.2. Remark. Here is a shorter way to describe these relations. Let *a chain of nodes* for a Dynkin diagram with n nodes be a sequence $i_1, \dots, i_k$, where $k \geq 2$ and

1) $i_j \in \overline{1, n}$ for all $j = 1, \dots, k$;

2) $i_j \neq i_{j'}$ for $j \neq j'$;

3) nodes with numbers i_j and i_{j+1} are connected for all $j = 1, \dots, k-1$.

The above non-Serre relations (both for $\mathfrak{sl}(n+1)$ and $\mathfrak{e}(n)$ can be represented in the form

$$\begin{aligned}
&[\,[x_{i_1}, [\dots, [x_{i_{k-1}}, x_{i_k}] \dots]],\ [x_{i_1}, [\dots, [x_{i_{k-1}}, x_{i'_k}] \dots]]\,] = 0,\\
&\text{where } i_1, \dots, i_{k-1}, i_k \text{ and } i_1, \dots, i_{k-1}, i'_k \text{ are two chains of nodes}\\
&\text{that differ only in the last element.}
\end{aligned} \tag{15.7}$$

All the relations that can be represented in the form (15.7) are necessary.

15.2.5. $\mathfrak{g} = \mathfrak{o}_B(2n)$**.** The orthogonal algebra is, by definition, the Lie algebra of linear transformations preserving a given non-degenerate symmetric bilinear form B. The bilinear form is usually taken with the Gram matrix 1_{2n} or Π_{2n}. In characteristic > 2, these two forms are equivalent over any perfect field. The corresponding Lie algebra has the same defining relations as in characteristic 0, so in this subsection we only consider $p = 2$.

It turns out ([L1]) that these two forms are not equivalent over any ground field $\mathbb{K}$ of characteristic 2. If $\mathbb{K}$ is perfect, then any non-degenerate symmetric bilinear form is equivalent to one of these two forms: It is equivalent to Π_n, if it is zero-diagonal; otherwise, it is equivalent to 1_n.

The orthogonal Lie algebras corresponding to these two forms (we denote them $\mathfrak{o}_I(n)$ and $\mathfrak{o}_\Pi(n)$, respectively) are not isomorphic and have different properties. In particular, **only** $\mathfrak{o}_\Pi(2n)$ **for** $n \geq 3$ **is close to an algebra with a Cartan matrix** (same as in characteristic 0). The corresponding algebra $\mathfrak{g}^{(1)}(A)$ is $\mathfrak{oc}(2; 2n)$ (i.e., the central extension of $\mathfrak{o}_\Pi^{(2)}(2n)$, given by the formula 14.3).

15.2.5.1. $\mathfrak{oc}(2; 2n)$**.** The algebra $\mathfrak{o}_\Pi^{(2)}(2n)$ (whose central extension is $\mathfrak{oc}(2; 2n)$) consists of matrices of the following form:

$$\begin{pmatrix} A & B \\ C & A^T \end{pmatrix}, \quad \begin{array}{l} \text{where } A \in \mathfrak{sl}(n); \\ B, C \in ZD(n). \end{array}$$

The Chevalley generators of $\mathfrak{oc}(2; 2n)$ are:

$$\begin{aligned}
&x_i = E^{i,i+1} + E^{n+i+1,n+i} \quad \text{for } 1 \leq i \leq n-1;\\
&x_n = E^{n-1,2n} + E^{n,2n-1};\\
&y_i = x_i^T \quad \text{for } 1 \leq i \leq n;\\
&h_i = E^{i,i} + E^{i+1,i+1} + E^{n+i,n+i} + E^{n+i+1,n+i+1} \quad \text{for } 1 \leq i \leq n-1;\\
&h_n = h_{n-1} + z,
\end{aligned}$$

where z is central element.

15.2.6. Theorem. *In characteristic* 2, *for* $\mathfrak{oc}(2;2n)$, *where* $n \geq 4$, *the defining relations for* $\mathfrak{n}$ *are Serre relations plus the following ones:*

$$\begin{aligned}
&[[x_{i-1}, x_i], [x_i, x_{i+1}]] = 0 \quad \textit{for } 2 \leq i \leq n-2;\\
&[[x_{n-3}, x_{n-2}], [x_{n-2}, x_n]] = 0;\\
&[[x_{n-2}, x_{n-1}], [x_{n-2}, x_n]] = 0;\\
&[[x_{n-3}, [x_{n-2}, x_{n-1}]], [x_n, [x_{n-1}, x_{n-2}]] = 0;\\
&[[x_{n-3}, [x_{n-2}, x_n]], [x_n, [x_{n-1}, x_{n-2}]] = 0;
\end{aligned}$$

and, for $1 \leq i \leq n-3$,

$$[[x_{n-1}, [x_i, [x_{i+1}, \dots, [x_{n-3}, x_{n-2}] \dots]]], [x_n, [x_i, [x_{i+1}, \dots, [x_{n-3}, x_{n-2}] \dots]]]] = 0.$$

(We don't consider the case of $n = 3$ in the theorem because $\mathfrak{oc}(2;6)$ is isomorphic to $\mathfrak{sl}(4)$.)

15.2.6.1. $\mathfrak{g} = \mathfrak{o}_I^{(1)}(2n)$**.** As shown in Lemma 13.6.4.1, if $n \geq 2$, then $\mathfrak{o}_I(2n) \neq \mathfrak{o}_I^{(1)}(2n) = \mathfrak{o}_I^{(2)}(2n)$ (and if $n = 1$, then the algebra $\mathfrak{o}_I(2n)$ is nilpotent). So any set of generators of $\mathfrak{o}_I(2n)$ contains "extra" (as compared with generators of $\mathfrak{o}_I^{(1)}(2n)$) generators $a_1, \dots, a_{2n}$. The relations containing these generators say nothing new about the structure of the simple (and, thus, more interesting) algebra $\mathfrak{o}_I^{(1)}(2n)$. Because of this and because we want to make the set of generators we use as small as possible, we consider the algebra $\mathfrak{o}_I^{(1)}(2n)$. It consists of symmetric zero-diagonal $2n \times 2n$-matrices. We can choose the following generators (for the whole algebra since in this case there is no $\mathfrak{n}$):

$$X_i = E^{i,i+1} + E^{i+1,i} \quad \text{for } 1 \leq i \leq 2n-1.$$

15.2.7. Theorem. *The following are the defining relations for* $\mathfrak{o}_I^{(1)}(2n)$, $n \geq 2$*:*

$$\begin{aligned}
&[X_i, X_j] = 0 && \textit{for } 1 \leq i,j \leq 2n-1, |i-j| \geq 2;\\
&[X_i, [X_i, X_{i+1}]] = x_{i+1}; && \\
&[X_{i+1}, [X_i, X_{i+1}]] = x_i && \textit{for } 1 \leq i \leq 2n-2;\\
&[[X_{i-1}, X_i], [X_i, X_{i+1}]] = 0 && \textit{for } 2 \leq i \leq 2n-2.
\end{aligned}$$

Proof. (Sketch of.) The algebra $\mathfrak{o}_I^{(1)}(2n)$ is filtered:

$$0 = L_0 \subset \dots \subset L_{2n-1},$$

where L_k consists of all symmetric zero-diagonal matrices M such that $M_{ij} = 0$ for all i, j such that $|i-j| > k$. The associated graded algebra is isomorphic to the algebra of upper-triangular matrices, i.e., a maximal nilpotent subalgebra of $\mathfrak{sl}(2n)$. So we can use Theorem 15.2.2. □

15.2.7.1. Remark. The similar result is true for the algebra $\mathfrak{o}_I^{(1)}(2n+1)$, $n \geq 1$.

15.2.8. $\mathfrak{g} = \mathfrak{o}_B(2n+1)$**.** For this algebra, again, the case of characteristic > 2 does not differ from the case of characteristic 0, so we only consider the case of characteristic 2. Then, if the ground field is perfect, all the non-degenerate symmetric bilinear form over a linear space of dimension $2n+1$ are equivalent. We choose the form Π_{2n+1}.

15.2.8.1. $\mathfrak{g} = \mathfrak{o}_\Pi(2n+1)$**.** It is easy to see that $\mathfrak{o}_\Pi(2n+1) \not\simeq \mathfrak{o}_\Pi^{(1)}(2n+1)$ and $\mathfrak{o}_\Pi^{(1)}(2n+1) = \mathfrak{o}_\Pi^{(2)}(2n+1)$ for $n \geq 1$. So, as for $\mathfrak{o}_I(2n)$, we consider the first derived algebra $\mathfrak{o}_\Pi^{(1)}(2n+1)$. The algebra $\mathfrak{o}_\Pi^{(1)}(2n+1)$ consists of matrices of the following form:

$$\begin{pmatrix} A & X & B \\ Y^T & 0 & X^T \\ C & Y & A^T \end{pmatrix}, \quad \begin{array}{l}\text{where } A \in \mathfrak{gl}(n); B, C \in ZD(n);\\ X, Y \text{ are } n\text{-vectors.}\end{array}$$

This algebra has a Cartan matrix. This affects the Serre relations. The Chevalley generators are:

$$\begin{aligned}
&x_i = E^{i,i+1} + E^{n+i+2,n+i+1} \quad \text{for } 1 \leq i \leq n-1;\\
&x_n = E^{n,n+2} + E^{n+1,2n+1};\\
&y_i = x_i^T \quad \text{for } 1 \leq i \leq n;\\
&h_i = E^{i,i} + E^{i+1,i+1} + E^{n+i+1,n+i+1} + E^{n+i+2,n+i+2} \quad \text{for } 1 \leq i \leq n-1;\\
&h_n = E^{n,n} + E^{2n+1,2n+1}.
\end{aligned}$$

15.2.9. Theorem. *In characteristic* 2, *for* $\mathfrak{g} = \mathfrak{o}_\Pi^{(1)}(2n+1)$, *the defining relations for* $\mathfrak{n}$ *are the Serre relations plus the following ones:*

$$[[x_{i-1}, x_i], [x_i, x_{i+1}]] = 0 \quad \textit{for } 2 \leq i \leq n-2.$$

15.2.10. $\mathfrak{g}(2)$**.** Selecting the Chevalley basis in the Lie algebra $\mathfrak{g}(2)$ (for its explicit description, see [FH], pp. 346) we find out that, although the Cartan matrix of $\mathfrak{g}(2)$ reduced modulo 2 coincides with Cartan matrix of $\mathfrak{sl}(3)$, the Lie algebra $\mathfrak{g}(2)_{\mathbb{K}}$ is still simple. This algebra, though, is isomorphic to $\mathfrak{psl}(4)$.

15.2.11. $\mathfrak{f}(4)$**.** There is no $\mathbb{Z}$-form of $\mathfrak{f}(4)$ such that the algebra $\mathfrak{f}(4)_{\mathbb{K}}$ is still simple.

15.2.12. $\mathfrak{wk}^{(1)}(3;a)$**, dim = 17.** According to [WK, BGL5] the Lie algebra $\mathfrak{wk}(3;a)$ has the following Cartan matrices

$$1) \begin{pmatrix} \bar{0} & 1 & 0 \\ a & \bar{0} & 1 \\ 0 & 1 & \bar{0} \end{pmatrix} \quad 2) \begin{pmatrix} \bar{0} & 1 & a \\ 1 & \bar{0} & a+1 \\ a & a+1 & \bar{0} \end{pmatrix}$$

The corresponding defining relations for $\mathfrak{n}$ are, in addition to the Serre relations:

For the first matrix:	For the second matrix:
$[[x_1, x_2], [x_3, [x_1, x_2]]] = 0$	$[x_2, [x_1, x_3]] = a[x_3, [x_1, x_2]]$
$[[x_2, x_3], [x_3, [x_1, x_2]]] = 0$	

$\mathfrak{wk}(4;a)$, dim $=34$ According to [WK] this Lie algebra has the following Cartan matrices:

$$1)\ \begin{pmatrix}\bar 0&1&0&0\\ a&\bar 0&1&0\\ 0&1&\bar 0&1\\ 0&0&1&\bar 0\end{pmatrix}\quad 2)\ \begin{pmatrix}\bar 0&1&a+1&0\\ 1&\bar 0&a&0\\ a+1&a&\bar 0&a\\ 0&0&a&\bar 0\end{pmatrix}\quad 3)\ \begin{pmatrix}\bar 0&a&0&0\\ a&\bar 0&a+1&0\\ 0&a+1&\bar 0&1\\ 0&0&1&\bar 0\end{pmatrix}$$

The non-Serre relations:

For the first matrix
$[[x_2,x_3],[x_3,x_4]]=0$
$[[x_1,x_2],[x_3,[x_1,x_2]]]=0$
$[[x_2,x_3],[x_3,[x_1,x_2]]]=0$
$[[x_4,[x_2,x_3]],[[x_1,x_2],[x_3,x_4]]]=0$
$[[[x_1,x_2],[x_2,x_3]],[[x_1,x_2],[x_4,[x_2,x_3]]]]=0$

For the second matrix
$[x_2,[x_1,x_3]]=(1+a)[x_3,[x_1,x_2]]$
$[[x_2,x_3],[x_3,x_4]]=0$
$[[x_1,x_3],[x_4,[x_1,x_3]]]=0$
$[[x_3,x_4],[x_4,[x_1,x_3]]]=0$

For the third matrix
$[[x_1,x_2],[x_3,[x_1,x_2]]]=0$
$[[x_2,x_3],[x_3,[x_1,x_2]]]=0$
$[[x_2,x_3],[x_4,[x_2,x_3]]]=0$
$[[x_3,x_4],[x_4,[x_2,x_3]]]=0$
$[[x_3,[x_1,x_2]],[x_4,[x_2,x_3]]]=(1+a)[[x_3,x_4],[[x_1,x_2],[x_2,x_3]]]$

15.3. Results: Lie superalgebras

15.3.1. $\mathfrak{g}=\mathfrak{sl}(n|m)$, $p=2$, $n\neq m$. Here the relations are the same as in the non-super case (apart from Serre ones for the odd generators):

$$[[x_i,x_{i+1}],[x_{i+1},x_{i+2}]]=0\quad\text{for all } i=1,\dots,m+n-3.$$

15.3.2. $\mathfrak{g}=\mathfrak{psl}(n|m)$, $p=2$, $m+n$ even. If we set x_i to be the image of $E^{i,i+1}$, the non-Serre relations are the same:

$$[[x_i,x_{i+1}],[x_{i+1},x_{i+2}]]=0\quad\text{for all } i=1,\dots,2n-3.$$

15.3.3. $\mathfrak{g}=\mathfrak{oo}(n|m)$, $p=2$. Here we consider CM Lie superalgebras close to some of the ortho-orthogonal algebras. There are two kinds of such CMs.

1) Cartan matrix

$$\begin{pmatrix}\ddots&\ddots&\ddots&\vdots\\ \ddots&0&1&0\\ \ddots&1&0&1\\ \dots&0&1&1\end{pmatrix}$$

generates $\mathfrak{oo}^{(1)}_{I\Pi}(2k_{\bar 0}|2k_{\bar 1}+1)$, $k_{\bar 0}+k_{\bar 1}=n$ (parities of the rows of the matrix may be different; the connection between these parities and $k_{\bar 0},k_{\bar 1}$ is described in Table 20.10). The corresponding non-Serre relations are as in Theorem 15.2.9.

2) Cartan matrix

$$\begin{pmatrix}\ddots&\ddots&\ddots&\vdots&\vdots\\ \ddots&\bar 0&1&0&0\\ \ddots&1&\bar 0&1&1\\ \cdots&0&1&\bar 0&0\\ \cdots&0&1&0&0\end{pmatrix}$$

generates an algebra close to $\mathfrak{oo}_{\Pi\Pi}(2k_{\bar 0}|2k_{\bar 1})$, $k_{\bar 0}+k_{\bar 1}=n$ (parities of the rows of the matrix may be different; the connection between these parities and $k_{\bar 0},k_{\bar 1}$ is described in Table 20.10; the exact description of the CM Lie superalgebra is in subsection 14.1.1.3). The corresponding non-Serre relations are as in Theorem 15.2.6.

15.3.4. $\mathfrak{g}=\mathfrak{ag}(2)$, $p=2$. The Cartan matrices for $p=0$ are

$$1)\ \begin{pmatrix}0&1&0\\ -1&2&-3\\ 0&-1&2\end{pmatrix}\quad 2)\ \begin{pmatrix}0&1&0\\ -1&0&3\\ 0&-1&2\end{pmatrix}\quad 3)\ \begin{pmatrix}0&-3&1\\ -3&0&2\\ -1&-2&2\end{pmatrix}\quad 4)\ \begin{pmatrix}2&-1&0\\ -3&0&2\\ 0&-1&1\end{pmatrix}$$

If $p=2$, these Cartan matrices do not produce anything "resembling" $\mathfrak{ag}(2)$ since they contain $-3\equiv -1 \pmod 2$. (In particular, the Lie superalgebra that corresponds to the matrices 1) and 2) is isomorphic to $\mathfrak{sl}(1|3)$.)

I do not know an integer basis of $\mathfrak{ag}(2)$ in which the corresponding Lie superalgebra in characteristics $p=2$ or 3 is simple. Elduque suggested its $p=3$ analog, see [CE].

15.3.5. $\mathfrak{g}=\mathfrak{ab}(3)$, $p=2$. The Cartan matrices for $p=0$ are

$$1)\ \begin{pmatrix}2&-1&0&0\\ -3&0&1&0\\ 0&-1&2&-2\\ 0&0&-1&2\end{pmatrix}\quad 2)\ \begin{pmatrix}0&-3&1&0\\ -3&0&2&0\\ 1&2&0&-2\\ 0&0&-1&2\end{pmatrix}\quad 3)\ \begin{pmatrix}2&-1&0&0\\ -1&2&-1&0\\ 0&-2&0&3\\ 0&0&-1&2\end{pmatrix}$$

$$4)\begin{pmatrix}2&-1&0&0\\-2&0&2&-1\\0&2&0&-1\\0&-1&-1&2\end{pmatrix}\quad 5)\begin{pmatrix}0&1&0&0\\-1&0&2&0\\0&-1&2&-1\\0&0&-1&2\end{pmatrix}\quad 6)\begin{pmatrix}2&-1&0&0\\-1&2&-1&0\\0&-2&2&-1\\0&0&-1&0\end{pmatrix}$$

This algebra can not be constructed in $p=2$ as a CM Lie superalgebra. I do not know any integer basis of $\mathfrak{ab}(2)$ such that the corresponding Lie superalgebra in characteristics $p=2$ or 3 is simple. Elduque suggested its $p=3$ analog, see [CE].

15.3.6. $\mathfrak{g}=\mathfrak{osp}(4|2;\alpha)$, $p=2$. The non-equivalent Cartan matrices for $p=0$ are

$$1)\begin{pmatrix}2&-1&0\\-\alpha&0&-1\\0&-1&2\end{pmatrix}\quad 2)\begin{pmatrix}0&-1&-\alpha\\1&0&-1-\alpha\\\alpha&-1-\alpha&0\end{pmatrix}$$

Observe that, up to parities,the Cartan matrices are the same as those of $\mathfrak{wk}(3,a)$. No wonder the relations are identical (except for the squares of odd root vectors): The defining relations for $\mathfrak{n}$ are, in addition to the Serre relations:

For the first matrix:	For the second matrix:
$[[x_1,x_2],[x_3,[x_1,x_2]]]=0$	$[x_2,[x_1,x_3]]=a[x_3,[x_1,x_2]]$
$[[x_2,x_3],[x_3,[x_1,x_2]]]=0$	

Looking at the description of $\mathfrak{osp}(4|2;\alpha)$ due to Kaplansky, A. Kirillov observed that the permutation group S_3 acts on the complex line of values of the parameter α; for the description of the fundamental domains of this action, see [LSoS].

15.4. Proofs: Lie algebras

15.4.1. $\mathfrak{g}=\mathfrak{sl}(n+1)$, $p=2$. The elements E^{ij}, where $1\le i<j\le n+1$, form a basis of the algebra $\mathfrak{n}$. In particular, $x_i=E^{i,i+1}$. Clearly, we have

$$[E^{ij},E^{kl}]=\delta_{jk}E^{il}+\delta_{il}E^{kj}.$$

Let $\overline{\mathfrak{h}}$ be the algebra of diagonal matrices. The elements E^{ii}, where $1\le i\le n+1$, form a basis of $\overline{\mathfrak{h}}$. Let the ω_i be the dual basis elements.

We consider the weights of $\mathfrak{n}$ with respect to $\overline{\mathfrak{h}}$. The weight of E^{ij} is equal to $\omega_i+\omega_j$.

Recall several facts about homology.

15.4.1.1. Lemma. *Let $c=E^{i_1j_1}\wedge\ldots\wedge E^{i_mj_m}$ be a basic chain. Set $M_c=\{E^{i_1j_1},\ldots,E^{i_mj_m}\}$. If for any $E^{ij}\in M_c$ and any k such that $i<k<j$, at least one of the elements E^{ik} and E^{kj} lies in M_c, then c can not appear with non-zero coefficient in decomposition of a boundary with respect to basic chains.*

Proof. Clearly, it suffices to show that c can not appear with non-zero coefficient in the decomposition of the differential of a basic chain w.r.t. basic chains. It follows from the formula for the differential d that any basic chains that appears with non-zero coefficient in decomposition of the differential of a basic chain F w.r.t. basic chains, can be obtained from F by replacing E^{ik} and E^{kj} by E^{ij} for some i,j,k. If c satisfies the hypothesis of the Lemma, then c can not be obtained in such a way from any F. $\square$

The elements of $C_2(\mathfrak{n};\mathbb{C})$ have weights of two types: $\omega_i+\omega_j$ and $\omega_i+\omega_j+\omega_k+\omega_l$. We consider them.

I. A weight $\alpha=\omega_i+\omega_j$, where $1\le i<j\le n+1$. The following chains form a basis of $C_2(\mathfrak{n};\mathbb{C})_\alpha$:

$$\begin{array}{ll}E^{ik}\wedge E^{kj},\ i<k<j; & d(E^{ik}\wedge E^{kj})=E^{ij};\\ E^{ki}\wedge E^{kj},\ 1\le k<i; & d(E^{ki}\wedge E^{kj})=0;\\ E^{ik}\wedge E^{jk},\ j<k\le n+1; & d(E^{ik}\wedge E^{jk})=0.\end{array}$$

Thus, the following cycles form a basis of $C_2(\mathfrak{n};\mathbb{C})_\alpha$:

$$\begin{array}{ll}E^{ik}\wedge E^{kj}+E^{i,k+1}\wedge E^{k+1,j}, & i<k<j-1;\\ E^{ik}\wedge E^{jk}, & j<k\le n+1.\end{array}$$

We consider them:

1) $E^{ik}\wedge E^{kj}+E^{i,k+1}\wedge E^{k+1,j}=d(E^{ik}\wedge E^{k,k+1}\wedge E^{k+1,j})$, so this is a boundary.

2) $E^{ki}\wedge E^{kj}$, where $1\le k<i$; in this case, we consider three subcases:

a) $j-i>1$: In this case, $E^{ki}\wedge E^{kj}=d(E^{ki}\wedge E^{k,j-1}\wedge E^{j-1,j})$.

b) $i-k>1$: In this case, $E^{ki}\wedge E^{kj}=d(E^{k,i-1}\wedge E^{i-1,i}\wedge E^{kj})$.

c) $i-k=j-i=1$, i.e., $i=k+1$; $j=k+2$. In this case, according to Lemma 15.4.1.1, the basic chain $E^{ki}\wedge E^{kj}$ can not appear with non-zero coefficient in decomposition of a boundary with respect to basic chains; so this is a non-trivial cycle. It gives us the relation

$$[E^{k,k+1},E^{k,k+2}]=0,\qquad \text{i.e., } [x_k,[x_k,x_{k+1}]]=0.$$

Here $k\in\overline{1,n-1}$.

3) This case is completely analogous to the previous one; it gives us the relation

$$[x_k,[x_{k-1},x_k]]=0,$$

where $k\in\overline{2,n}$.

II. A weight $\alpha=\omega_i+\omega_j+\omega_i+\omega_j$, where $1\le i<j<k<l\le n+1$. Clearly, $C_2(\mathfrak{n};\mathbb{C})_\alpha$ has the following basis:

$$c_{\alpha,1}=E^{ij}\wedge E^{kl},\qquad c_{\alpha,2}=E^{ik}\wedge E^{jl},\qquad c_{\alpha,3}=E^{il}\wedge E^{jk}.$$

All this three chains are cycles, i.e., $Z_2(\mathfrak{n};\mathbb{C})_\alpha=C_2(\mathfrak{n};\mathbb{C})_\alpha$. Here we have three subcases:

1) $j-i>1$. Then

$$\begin{aligned}c_{\alpha,1} &= d(E^{i,i+1}\wedge E^{i+1,j}\wedge E^{kl});\\ c_{\alpha,2} &= d(E^{i,i+1}\wedge E^{i+1,k}\wedge E^{jl});\\ c_{\alpha,3} &= d(E^{i,i+1}\wedge E^{i+1,l}\wedge E^{jk}).\end{aligned}$$

2) $l-k>1$. Then, similarly to the previous case,

$$\begin{aligned}c_{\alpha,1} &= d(E^{ij}\wedge E^{k,l-1}\wedge E^{l-1,l});\\ c_{\alpha,2} &= d(E^{ik}\wedge E^{j,l-1}\wedge E^{l-1,l});\\ c_{\alpha,3} &= d(E^{jk}\wedge E^{i,l-1}\wedge E^{l-1,l}).\end{aligned}$$

3) $j-i=l-k=1$, i.e., $j=i+1; l=k+1$. Then, from Lemma 15.4.1.1, $c_{\alpha,1}$ is a non-trivial cycle. It gives the relation

$$[E^{i,i+1}, E^{k,k+1}]=0, \qquad \text{i.e., } [x_i,x_k]=0.$$

Here $i,k\in\overline{1,n}$, and $k-i\geq 2$.

For the other cycles, we need to consider the two subcases:

a) $k-j>1$. Then

$$c_{\alpha,2}=d(E^{i,k-1}\wedge E^{k-1,k}\wedge E^{jl});\quad c_{\alpha,3}=d(E^{il}\wedge E^{j,k-1}\wedge E^{k-1,k}).$$

b) $k-j=1$, i.e., $i=j-1$; $k=j+1$; $l=i+2$. It is easy to see (like in the proof of Lemma 15.4.1.1) that the only two chains such that $c_{\alpha,2}$ or $c_{\alpha,3}$ appear with non-zero coefficients in the decomposition of their differentials w.r.t. basic chains are

$$E^{j-1,j}\wedge E^{j,j+1}\wedge E^{j,j+2}\quad\text{and}\quad E^{j-1,j+1}\wedge E^{j,j+1}\wedge E^{j+1,j+2}.$$

The differentials of both these chains are equal to $c_{\alpha,2}+c_{\alpha,3}$. So we can consider one of the chains $c_{\alpha,2}$ or $c_{\alpha,3}$ as a non-trivial cycle. The cycle $c_{\alpha,2}$ gives the relation

$$[E^{j-1,j+1}, E^{j,j+2}]=0, \qquad \text{i.e., } [[x_{j-1},x_j],[x_j,x_{j+1}]]=0,$$

and $c_{\alpha,2}$ gives an equivalent (taking other relations into account) relation

$$[E^{j-1,j+2}, E^{j,j+1}]=0 \qquad \text{i.e., } [[x_{j-1},[x_j,x_{j+1}]],x_j]=0.$$

Here $j\in\overline{2,n-1}$.

Chapter 16

Analogs of the Hamiltonian, Poisson, and contact Lie superalgebras in characteristic 2

16.1. Introduction

16.1.1. Main results. I describe the analogs of the Poisson bracket, Buttin bracket (a.k.a. anti-bracket) and contact brackets, cf. [LSh]. The quotients of the Poisson and Buttin Lie (super)algebras modulo center — analogs of Lie algebras of Hamiltonian vector fields — are also described.

Observe that the usual, valid for $p\neq 2$, interpretation of the Hamiltonian Lie superalgebra as the one preserving a non-degenerate closed differential 2-form is **not** applicable to all four analogs we introduce.

The particular case of Hamiltonian Lie algebras was partly investigated in [Lin2]; the particular case of Hamiltonian Lie superalgebras on purely odd space was (also partly) considered in [KL].

16.2. The Hamiltonian Lie superalgebras

Let $B=(B_{ij})$ be an even symmetric non-degenerate bilinear form on a superspace V of dimension $n_{\bar 0}|n_{\bar 1}$ with a basis $\{x_1,\dots,x_n\}$, where $n=n_{\bar 0}+n_{\bar 1}$, such that $P(x_1)=\dots=P(x_{n_{\bar 0}})=\bar 0$, $P(x_{n_{\bar 0}+1})=\dots=P(x_n)=\bar 1$. Then the Cartan prolong of the Lie superalgebra $\mathfrak{oo}_B$ is analogous to the Hamiltonian Lie superalgebra. However, it can be represented as a Lie superalgebra of vector fields preserving a given 2-form only if B is equivalent to $B_{\Pi\Pi}$; the corresponding 2-form is equal to

$$\omega_B=\sum_{1\leq i<j\leq n} B_{ij}dx_i\wedge dx_j. \tag{16.1}$$

If we consider divided power differential forms, then the above Cartan prolong of $\mathfrak{oo}_B$ can be represented as a Lie superalgebra of vector fields preserving a given 2-form if B is equivalent to $B_{\Pi\Pi}$ or $B_{\Pi I}$; the corresponding 2-form is

$$\omega_B=\sum_{1\leq i<j\leq n} B_{ij}dx_i\wedge dx_j+\sum_{i=n_{\bar 0}+1}^{n} B_{ii}dx_i^{(\wedge 2)} \tag{16.2}$$

(if B is equivalent to $B_{\Pi\Pi}$, this 2-form coincides with (16.1)).

As a linear space, the above Cartan prolong can be represented as

$$\mathrm{Reg}_B \oplus \mathrm{Irreg}^1_B \oplus \mathrm{Irreg}^2_B \tag{16.3}$$

where

$$\begin{gathered}\mathrm{Reg}_B = \left\{ \sum_{i,j=1}^{n} (B^{-1})_{ij} \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j} \mid f \in \mathbb{K}[x_1, \dots, x_n] \right\},\\ \mathrm{Irreg}^1_B = \mathrm{Span}\left(\sum_{j=1}^{n} (B^{-1})_{ij} x_i \frac{\partial}{\partial x_j} \mid n_{\bar 0} < i \le n \right)\\ \mathrm{Irreg}^2_B = \mathrm{Span}\left(\sum_{j=1}^{n} (B^{-1})_{ij} x_i^{(2^{\underline{N}_i}-1)} \frac{\partial}{\partial x_j} \mid i \in \overline{1, n_{\bar 0}} \text{ such that } \underline{N}_i < \infty \right).\end{gathered} \tag{16.4}$$

Note that $\mathrm{sdim}\ \mathrm{Irreg}^1_B = n_{\bar 1}|0$, and this space is spanned by elements "generated" by nonexisting "Hamiltonians" $x_i^{(2)}$, where $n_{\bar 0} < i \le n$; the space Irreg^2_B is spanned by elements "generated" by nonexisting "Hamiltonians" $x_i^{(2^{\underline{N}_i})}$, where $1 \le i \le n_{\bar 0}$.

This description implies, in particular, that the superdimensions of the prolongs do not depend on the type of the superalgebra (i.e., is it $\mathfrak{oo}_{II}$, $\mathfrak{oo}_{I\Pi}$, $\mathfrak{oo}_{\Pi I}$ or $\mathfrak{oo}_{\Pi\Pi}$) — they only depend on the superdimension $n_{\bar 0}|n_{\bar 1}$, the number of the prolong and the shearing parameter $\underline{N}$.

The general formula for the superdimensions seems to be complicated; for $\underline{N}_1 = \dots = \underline{N}_{n_{\bar 0}} = \infty$, the dimension of the k-th prolong is equal to a coefficient of the supercharacter of $\mathbb{K}[x_1, \dots, x_{n_{\bar 0}}, \xi_1, \dots, \xi_{n_{\bar 1}}]$, i.e.,

$$\sum_{i=0}^{n_{\bar 1}} \binom{n_{\bar 0}}{k+2-i}\binom{n_{\bar 1}}{i} = \text{the coefficient of } x^{k+2} \text{ in the Taylor series expansion of } \frac{(1+x)^{n_{\bar 1}}}{(1-x)^{n_{\bar 0}}} \text{ at } x = 0.$$

For $\underline{N}_1 = \dots = \underline{N}_{n_{\bar 0}} = 1$, the dimension of the k-th prolong is equal to $\binom{n}{k+2}$, and the dimension of the complete prolong is equal to $2^n + n - 1$.

16.3. The Poisson Lie superalgebras

Let

$$\begin{aligned} &p_i = x_i, \quad q_i = x_{k_{\bar 0}+i} && \text{for } n_{\bar 0} = 2k_{\bar 0} \text{ and } 1 \le i \le k_{\bar 0};\\ &\theta_i = x_{n_{\bar 0}+i} && \text{for } 1 \le i \le n_{\bar 1};\\ &\xi_i = x_{n_{\bar 0}+i}, \quad \eta_i = x_{n_{\bar 0}+k_{\bar 1}+i} && \text{for } n_{\bar 1} = 2k_{\bar 1} \text{ and } 1 \le i \le k_{\bar 1}. \end{aligned}$$

As it was said above, the space Reg_B consists of vector fields of the form

$$H_{B,f} = \sum_{i,j=1}^{n} (B^{-1})_{ij} \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}, \quad \text{where } f \in \mathbb{K}[x_1, \dots, x_n].$$

Here are the exact forms of these fields for the above bilinear forms:

$$\begin{aligned} H_{II,f} &:= \sum_{i=1}^{n_{\bar 0}} \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i} + \sum_{i=1}^{n_{\bar 1}} \frac{\partial f}{\partial \theta_i}\frac{\partial}{\partial \theta_i}\\ H_{I\Pi,f} &:= \sum_{i=1}^{n_{\bar 0}} \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i} + \sum_{i=1}^{k_{\bar 1}} \left(\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i}\frac{\partial}{\partial \xi_i} \right)\\ H_{\Pi I,f} &:= \sum_{i=1}^{k_{\bar 0}} \left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} + \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i} \right) + \sum_{i=1}^{n_{\bar 1}} \frac{\partial f}{\partial \theta_i}\frac{\partial}{\partial \theta_i}\\ H_{\Pi\Pi,f} &:= \sum_{i=1}^{k_{\bar 0}} \left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} + \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i} \right) + \sum_{i=1}^{k_{\bar 1}} \left(\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i}\frac{\partial}{\partial \xi_i} \right). \end{aligned} \tag{16.5}$$

The space Reg_B is closed under the Lie bracket (but may be not closed under squaring). The corresponding *Poisson bracket* of the nonexisting "generating functions" is of the form

$$[f, g]_B = \sum_{i,j=1}^{n} (B^{-1})_{ij} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}. \tag{16.6}$$

In particular, the Poisson brackets corresponding to the above bilinear forms B are of the form

$$\begin{aligned} \{f, g\}_{II} &:= \sum_{i=1}^{n_{\bar 0}} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i} + \sum_{i=1}^{n_{\bar 1}} \frac{\partial f}{\partial \theta_i}\frac{\partial g}{\partial \theta_i}\\ \{f, g\}_{I\Pi} &:= \sum_{i=1}^{n_{\bar 0}} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i} + \sum_{i=1}^{k_{\bar 1}} \left(\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i}\frac{\partial g}{\partial \xi_i} \right)\\ \{f, g\}_{\Pi I} &:= \sum_{i=1}^{k_{\bar 0}} \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} + \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right) + \sum_{i=1}^{n_{\bar 1}} \frac{\partial f}{\partial \theta_i}\frac{\partial g}{\partial \theta_i}\\ \{f, g\}_{\Pi\Pi} &:= \sum_{i=1}^{k_{\bar 0}} \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} + \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right) + \sum_{i=1}^{k_{\bar 1}} \left(\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i}\frac{\partial g}{\partial \xi_i} \right). \end{aligned} \tag{16.7}$$

In the cases ΠI and $\Pi\Pi$, if $\underline{N}$ is such that $N_i \ge 2$ for all i, then the space Reg_B is closed under squaring (i.e., it is a Lie superalgebra), and so $(H_f)^2 = H_{f^{[2]}}$, where the respective expressions of $f^{[2]}$ are

$$f^{[2]} := \begin{cases} \sum\limits_{i=1}^{k_{\bar 0}} \frac{\partial f}{\partial p_i}\frac{\partial f}{\partial q_i} + \sum\limits_{i=1}^{n_{\bar 1}} \left(\frac{\partial f}{\partial \theta_i}\right)^{(2)} & \text{for } \Pi I\\ \sum\limits_{i=1}^{k_{\bar 0}} \frac{\partial f}{\partial p_i}\frac{\partial f}{\partial q_i} + \sum\limits_{i=1}^{k_{\bar 1}} \frac{\partial f}{\partial \xi_i}\frac{\partial f}{\partial \eta_i} & \text{for } \Pi\Pi. \end{cases}$$

16.3.1. Remark. In these formulas we use divided square for arbitrary polynomials, not only for the indeterminates x_i, where $i \leq n_{\bar 0}$. We mean that $X^{(2)} = 0$ for any monomial X not proportional to x_i, $i \leq n_{\bar 0}$, and that the following relation holds:

$$(a + \lambda b)^{(2)} = a^{(2)} + \lambda^2 b^{(2)} + \lambda ab \quad \text{for any } a, b \in \mathbb{K}[x_1, \dots, x_n], \lambda \in \mathbb{K}.$$

16.3.1.1. Superdimensions. If $n_{\bar 0}, n_{\bar 1} > 0$, then the Cartan prolong of $\mathfrak{o}^{(1)}_{II}(n_{\bar 0}|n_{\bar 1})$ can be represented as

$$\mathrm{Reg}'_{II} \oplus \mathrm{Irreg}'^{1}_{II},$$

where

$$\mathrm{Reg}'_{II} = \left\{ H_{II,f} \mid f \in \mathbb{K}[x_1, \dots, x_n],\ \sum_{j=1}^{n_{\bar 0}} \frac{\partial^2 f}{\partial x_i^2} = 0 \right\},$$
$$\mathrm{Irreg}'^{1}_{II} = \mathrm{Span}\left(x_{i-1}\frac{\partial}{\partial x_{i-1}} + x_i \frac{\partial}{\partial x_i} \mid n_{\bar 0} < i \leq n \right).$$

If $n_{\bar 0} > 1$ and $\underline{N}_1 = \dots = \underline{N}_{n_{\bar 0}} = \infty$, then the dimension of the $(k, \underline{N})$-th prolong of $\mathfrak{o}^{(1)}_{II}(n_{\bar 0}|n_{\bar 1})$ is equal to the dimension of the k-th prolong of $\mathfrak{o}_{II}(n_{\bar 0} - 1|n_{\bar 1} + 1)$.

Let $n_{\bar 0} = 1$ and $\underline{N}_1 = \infty$. Then the dimension of the $(k, \underline{N})$-th prolong of $\mathfrak{o}^{(1)}_{II}(n_{\bar 0}|n_{\bar 1})$ is equal to $\binom{n_{\bar 1}+1}{k+2}$.

If $n_{\bar 0}, n_{\bar 1} > 0$, and $n_{\bar 1} = 2k_{\bar 1}$, then the Cartan prolong of $\mathfrak{o}^{(1)}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$ can be represented as

$$\mathrm{Reg}'_{I\Pi} \oplus \mathrm{Irreg}^{1}_{I\Pi},$$

where

$$\mathrm{Reg}'_{I\Pi} = \{ H_{I\Pi,f} \mid f \in \mathbb{K}[x_1, \dots, x_n],\ \deg f \leq 1 \quad \text{w.r.t. any } x_i,\ 1 \leq i \leq n \}.$$

So, independently of $\underline{N}$, the dimension of the k-th prolong of $\mathfrak{o}^{(1)}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$ is equal to $\binom{n}{k+2}$.

If $n_{\bar 0}, n_{\bar 1} > 0$, and $n_{\bar 0} = 2k_{\bar 0}$, then the Cartan prolong of $\mathfrak{o}^{(1)}_{\Pi I}(n_{\bar 0}|n_{\bar 1})$ is equal to

$$\mathrm{Reg}_{\Pi I} \oplus \mathrm{Irreg}^{2}_{\Pi I}.$$

Thus, the superdimension of the k-th prolong of $\mathfrak{o}^{(1)}_{\Pi I}(n_{\bar 0}|n_{\bar 1})$ is equal to the superdimension of the k-th prolong of $\mathfrak{o}_{\Pi I}(n_{\bar 0}|n_{\bar 1})$.

If $n_{\bar 0}, n_{\bar 1} > 0$, and $n_{\bar 0} = 2k_{\bar 0}$, $n_{\bar 1} = 2k_{\bar 1}$, then the Cartan prolong of $\mathfrak{o}^{(1)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ is equal to

$$\{ H_{\Pi\Pi,f} \mid f \in \mathbb{K}[x_1, \dots, x_n],\ f \text{ has degree} \leq 1 \text{ w.r.t. any } x_i,\ 1 \leq i \leq n \}.$$

So, independently of $\underline{N}$, the dimension of the k-th prolong of $\mathfrak{o}^{(1)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ is equal to $\binom{n}{k+2}$.

The Cartan prolong of $\mathfrak{o}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ consists of elements of the Cartan prolong of $\mathfrak{o}^{(1)}_{\Pi\Pi}$, generated by functions f such that

$$\sum_{i=0}^{k_{\bar 0}} \frac{\partial^2 f}{\partial p_i \partial q_i} + \sum_{i=0}^{k_{\bar 1}} \frac{\partial^2 f}{\partial \xi_i \partial \eta_i} = 0.$$

Observe that for the purposes of representation theory, it is desirable to have the Cartan subalgebra lying on the main diagonal. So it may be preferable to replace 1_{n_i} in the above bilinear forms by an equivalent form

$$\begin{array}{ll} \mathrm{diag}(1_2, \Pi_{2k-2}) & \text{if } n_i = 2k, \\ \mathrm{diag}(1, \Pi_{2k}) \text{ or } \Pi_{2k+1} & \text{if } n_i = 2k+1. \end{array} \tag{16.8}$$

16.4. The antibracket and the Buttin Lie superalgebras

The Cartan prolong of Lie superalgebra $\mathfrak{pe}_B$ also allows the description (16.3)-(16.4), so the dimensions of the prolongs are the same as of the prolongs of ortho-orthogonal superalgebras. The space Reg_B is closed under the Lie (super)bracket and under squaring. In particular, if $B = \Pi_{m|m}$, then Reg_B consists of vector fields of the form

$$H_{\mathfrak{pe},f} := \sum_{i=1}^{m} \left(\frac{\partial f}{\partial x_i}\frac{\partial}{\partial \theta_i} + \frac{\partial f}{\partial \theta_i}\frac{\partial}{\partial x_i} \right).$$

The corresponding *antibracket* and squaring of the generating functions are, respectively:

$$\{f, g\}_{\mathfrak{pe}} := \sum_{i=1}^{m} \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \theta_i} + \frac{\partial f}{\partial \theta_i}\frac{\partial g}{\partial x_i} \right); \quad f^{[2]} := \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial \theta_i}.$$

The Cartan prolong of $\mathfrak{pe}^{(1)}_{\Pi}(m)$ is equal to

$$\{ H_{\mathfrak{pe},f} \mid f \in \mathbb{K}[x_1, \dots, x_{2m}],\ f \text{ has degree} \leq 1 \text{ w.r.t. any } x_i,\ 1 \leq i \leq 2m \}.$$

So, independently of $\underline{N}$, the dimension of the k-th prolong of $\mathfrak{pe}^{(1)}_{\Pi}(m)$ is equal to $\binom{2m}{k+2}$.

The Cartan prolong of $\mathfrak{pe}^{(2)}_{\Pi}(m)$ consists of elements of the Cartan prolong of $\mathfrak{pe}^{(1)}_{\Pi}(m)$, generated by functions f such that

$$\sum_{i=0}^{m} \frac{\partial^2 f}{\partial x_i \partial \theta_i} = 0.$$

16.5. The contact brackets. Contact Lie superalgebras as CTS-prolongs

All the minuses in what follows are used in order to make expressions look like their analogs in characteristic $p \neq 2$ (if this analogs exist).

16.5.1. The odd (contact) form.

16.5.1.1. Notation. The superdimension of the superspace on which the contact structure is considered is equal to either $2k_{\bar 0}+1|2k_{\bar 1}$ or $2k_{\bar 0}+1|2k_{\bar 1}+1$. Set $k = k_{\bar 0} + k_{\bar 1}$.

The indeterminates are denoted by t, p_i, q_i, θ, where $i = 1, \dots, k$ and θ is present only if the superdimension is equal to $2k_{\bar 0}+1|2k_{\bar 1}+1$. The parities of the indeterminates are:

$$p(t) = \bar 0; \quad p(\theta) = \bar 1; \quad p(p_i) = p(q_i) = \begin{cases} \bar 0 & \text{if } i \leq k_{\bar 0}; \\ \bar 1 & \text{if } i > k_{\bar 0}. \end{cases}$$

The contact form is of the form

$$\alpha = dt + \sum_i p_i dq_i \ (+\theta d\theta).$$

16.5.2. Basis. The basis elements of the zeroth part $\mathfrak{g}_0$ of the contact Lie superalgebra $\mathfrak{g}$ in its standard $\mathbb{Z}$-grading are as follows:

	Element	Conditions
1	$t\partial_t + \sum p_i\partial_{p_i}$	$\mathrm{sdim} = 2k_{\bar 0}+1\vert 2k_{\bar 1}$
2	$p_i\partial_{q_j} - p_j\partial_{q_i}$	$1 \leq i, j \leq k$
3	$p_ip_j\partial_t - p_i\partial_{q_j} - p_j\partial_{q_i}$	$1 \leq i \neq j \leq k$
4	$q_iq_j\partial_t - q_i\partial_{p_j} - q_j\partial_{p_i}$	$1 \leq i \neq j \leq k$
5	$p_i^{(2)}\partial_t - p_i\partial_{q_i}$	$1 \leq i \leq k_{\bar 0}$
6	$q_i^{(2)}\partial_t - q_i\partial_{p_i}$	$1 \leq i \leq k_{\bar 0}$
7	$p_i\theta\partial_t - p_i\partial_\theta$	$\mathrm{sdim} = 2k_{\bar 0}+1\vert 2k_{\bar 1}+1, \quad 1 \leq i \leq k$
8	$q_i\theta\partial_t - q_i\partial_\theta$	$\mathrm{sdim} = 2k_{\bar 0}+1\vert 2k_{\bar 1}+1, \quad 1 \leq i \leq k$
9	$\theta\partial_\theta$	$\mathrm{sdim} = 2k_{\bar 0}+1\vert 2k_{\bar 1}+1$

16.5.3. Realization of $\mathfrak{g}_0$ in terms of ortho-orthogonal Lie superalgebras.

If $\mathrm{sdim} = 2k_{\bar 0}+1|2k_{\bar 1}$, then this algebra is the subalgebra of $\mathfrak{oo}_{\Pi\Pi}(2k_{\bar 0}|2k_{\bar 1})$ spanned by the grading operator $I_0 = \mathrm{diag}(1_{k_{\bar 0}|k_{\bar 1}}, 0_{k_{\bar 0}|k_{\bar 1}})$ and the supermatrices of format $k_{\bar 0}|k_{\bar 1}|k_{\bar 0}|k_{\bar 1}$ and having the form

$$\begin{pmatrix} A & C \\ D & A^T \end{pmatrix} \quad \begin{array}{l} \text{where } A \in \mathfrak{gl}(k_{\bar 0}|k_{\bar 1}), \\ C, D \text{ are symmetric}, \\ C_{ii} = D_{ii} = 0 \text{ for all } k_{\bar 0} < i \leq k. \end{array}$$

If $\mathrm{sdim} = 2k_{\bar 0}+1|2k_{\bar 1}+1$, then $\mathfrak{g}_0$ is NOT a subalgebra of $\mathfrak{oo}_{\Pi\Pi}(2k_{\bar 0}|2k_{\bar 1}+1)$. It is a subalgebra of the algebra of supermatrices preserving the form

$$\mathrm{antidiag}(1_{k_{\bar 0}|k_{\bar 1}}, 0, 1_{k_{\bar 0}|k_{\bar 1}}),$$

it is spanned by supermatrices of format $k_{\bar 0}|k_{\bar 1}+1|k_{\bar 0}|k_{\bar 1}$ and having the form

$$\begin{pmatrix} A & X & C \\ 0 & z & 0 \\ D & Y & A^T \end{pmatrix} \quad \begin{array}{l} \text{where } A \in \mathfrak{gl}(k_{\bar 0}|k_{\bar 1}), \\ C, D \text{ are symmetric}, \\ C_{ii} = D_{ii} = 0 \text{ for all } k_{\bar 0} < i \leq k, \\ X, Y \text{ are arbitrary } k_{\bar 0}|k_{\bar 1}\text{-vectors}, \\ z \in \mathbb{K}. \end{array}$$

16.5.4. The even (pericontact) form.

16.5.4.1. Notation. In this case the superdimension of the superspace is equal to $2k+1|2k$, the coordinates are t, p_i, ξ_i, where

$$p(t) = p(p_i) = \bar 0; \quad p(\xi_i) = \bar 1.$$

Let the pericontact form be of the form

$$\alpha = dt + \sum p_i d\xi_i.$$

16.5.5. Basis. The basis elements of the zeroth part $\mathfrak{g}_0$ of the pericontact Lie superalgebra $\mathfrak{g}$ in its standard $\mathbb{Z}$-grading are as follows:

	Element	Conditions
1	$t\partial_t + \sum p_i\partial_{p_i}$	—
2	$p_i\partial_{\xi_j} - p_j\partial_{\xi_i}$	$1 \leq i, j \leq k$
3	$p_ip_j\partial_t - p_i\partial_{\xi_j} - p_j\partial_{\xi_i}$	$1 \leq i \neq j \leq k$
4	$\xi_i\xi_j\partial_t - \xi_i\partial_{p_j} - \xi_j\partial_{p_i}$	$1 \leq i \neq j \leq k$
5	$p_i^{(2)}\partial_t - p_i\partial_{\xi_i}$	$1 \leq i \leq k$

16.5.6. Realization of $\mathfrak{g}_0$ in terms of $\mathfrak{pe}(k)$. The Lie superalgebra $\mathfrak{g}_0$ is the subalgebra of $\mathfrak{pe}(k)$ consisting of supermatrices of format $k|k$ and of the form

$$\begin{pmatrix} A & C \\ D & A^T \end{pmatrix} \quad \begin{array}{l} \text{where } A \in \mathfrak{gl}(k), \\ C \text{ is symmetric}, \\ D \in ZD(k). \end{array}$$

Chapter 17

Queerification

17.1.1. Queerification for $p > 2$. Let A be an associative (even) algebra. Then we can construct Lie superalgebra $\mathfrak{q}(A)$ which we call *the queerification* of A. As a linear space, $\mathfrak{q}(A)$ is equal to $A \oplus \Pi(A)$ with relations

$$[x,y] = xy - yx; \quad [x, \Pi(y)] = \Pi(xy - yx); \quad (\Pi(x))^2 = x^2 \quad \text{for all } \ x, y \in A.$$

The name is taken after the "queer" algebra $\mathfrak{q}(n)$, which is equal to $\mathfrak{q}(\mathrm{Mat}(n))$.

If A is an associative superalgebra, then we can similarly construct $\mathfrak{q}(A)$, but it coincides with the queerification of the algebra we get from A by forgetting the superstructure (because we don't use commutations relations of A).

17.1.2. Queerification for $p = 2$. If $p = 2$, and $\mathfrak{g}$ is a restricted Lie algebra (for the definition of restricted Lie (super)algebras, p- and $p|2p$-structures, see sect.12.1.2 of Chapter 12), then we can construct the queerification of $\mathfrak{g}$ even if $\mathfrak{g}$ is not the "Liefication" of any Lie-admissible (e.g., associative) algebra (for example, for $\mathfrak{g} = \mathfrak{sl}(n)$). We set $\mathfrak{q}(\mathfrak{g})_{\bar 0} = \mathfrak{g}$, $\mathfrak{q}(\mathfrak{g})_{\bar 1} = \Pi(\mathfrak{g})$, and define the bracket as follows:

$$[x, \Pi(y)] = \Pi([x,y]); \quad (\Pi(x))^2 = x^{[2]} \quad \text{for all } \ x, y \in \mathfrak{g}. \tag{17.1}$$

Clearly, if $I \subset \mathfrak{q}(\mathfrak{g})$ is an ideal, then $I_{\bar 0}$ and $\Pi(I_{\bar 1})$ are ideals in $\mathfrak{g}$. So, if $\mathfrak{g}$ is simple, then $\mathfrak{q}(\mathfrak{g})$ is a simple Lie superalgebra. (Note that $\mathfrak{g}$ has to be simple as a Lie algebra, not as a *restricted* Lie algebra.)

This construction brings up many new simple Lie superalgebras that have no analogs for $p \neq 2$.

17.1.2.1. Example: $\mathfrak{sq}(n)$ if $p = 2$. If $p = 2$, then on $\mathfrak{sq}(n)$, there is a new trace, an *even* one:

$$\mathrm{qtr}\colon \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mapsto \mathrm{tr}\, A \tag{17.2}$$

Note that $\mathfrak{q}(n)$ does not have this trace! So we can construct Lie superalgebra $\mathfrak{s}_e\mathfrak{sq}(n)$ consisting of elements of $\mathfrak{sq}(n)$ with zero (even) trace. If $n = 2k + 1$, then $\mathfrak{s}_e\mathfrak{sq}(n) = \mathfrak{psq}(n)$; if $n = 2k$, then $\mathfrak{s}_e\mathfrak{sq}(n)$ contains an ideal consisting of supermatrices of the form

$$\begin{pmatrix} a1_n & b1_n \\ b1_n & a1_n \end{pmatrix};$$

factorizing $\mathfrak{s}_e\mathfrak{sq}(n)$ by this ideal, we get Lie superalgebra $\mathfrak{ps}_e\mathfrak{psq}(2k)$ which is simple for $k > 1$. (Note that $\mathfrak{s}_e\mathfrak{sq}(2k)$ also contains a smaller ideal — the center consisting of scalar matrices; factorizing by the center, we get Lie
22 superalgebra [22] **DL: oboznachit' po drugomu! sejcas dve raznyh algebry oboznacheny odinakovo**] $\mathfrak{ps}_e\mathfrak{sq}(2k)$.)

We know that $\mathfrak{q}(n) = \mathfrak{q}(\mathfrak{gl}(n))$; we also see that

$$\mathfrak{s}_e\mathfrak{sq}(n) = \mathfrak{q}(\mathfrak{sl}(n)); \quad \mathfrak{ps}_e\mathfrak{psq}(2k) = \mathfrak{q}(\mathfrak{psl}(2k)).$$

17.1.3. Queerification of the exceptional Lie algebras. For $p = 2$, the Lie algebras $\mathfrak{e}(n)$ for $n = 6, 7, 8$, as well as $\mathfrak{wk}(3; a)$ and $\mathfrak{wk}(4; a)$, possess 2-structure described in Proposition 12.3.4.1 of Chapter 12.

The following theorem lists several simple Lie superalgebras in characteristic 2 obtained through queerification:

The algebra $\mathfrak{q}(\mathfrak{g})$ has a natural $2|4$-structure given by the 2-structure on $\mathfrak{g}$. Note that if $\mathfrak{g}$ is a Lie superalgebra with a $2|4$-structure, then we can similarly construct $\mathfrak{q}(\mathfrak{g})$, but it coincides with the queerification of the Lie algebra we get from $\mathfrak{g}$ by forgetting the superstructure.

No known new simple Lie superalgebras are obtained by queerification if $p \neq 2$.

17.1.4. Queerification of orthogonal Lie algebras $\mathfrak{q}(\mathfrak{o}_B(n))$. The Lie algebras $\mathfrak{o}_B(n)$ considered as algebras of matrices (or linear operators) have a natural 2-structure: if $XB + BX^T = 0$, then

$$X^2B + B(X^2)^T = X(XB + BX^T) + (XB + BX^T)X^T = 0,$$

thus, if $X \in \mathfrak{o}_B(n)$, then $X^2 \in \mathfrak{o}_B(n)$. So we can consider queerifications $\mathfrak{q}(\mathfrak{o}_B(n))$ of these algebras. We will find simple subquotients of these queerifications for n large enough; more specifically, $n \geq 3$ for $\mathfrak{q}(\mathfrak{o}_I(n))$, and $n \geq 6$ for $\mathfrak{q}(\mathfrak{o}_\Pi(n))$. We will first consider

$$\mathfrak{q}(\mathfrak{o}_B(n))^{(\infty)} := \bigcap_{i=1}^{\infty} \mathfrak{q}(\mathfrak{o}_B(n))^{(i)}.$$

Clearly, $\mathfrak{q}(\mathfrak{o}_B(n))^{(\infty)}_{\bar 0}$ and $\Pi(\mathfrak{q}(\mathfrak{o}_B(n))^{(\infty)}_{\bar 1})$ are subalgebras of $\mathfrak{o}_B(n)$.

Then we will need to find non-trivial ideals of $\mathfrak{q}(\mathfrak{o}_B(n))^{(\infty)}$. Clearly, for any ideal $I \subset \mathfrak{q}(\mathfrak{o}_B(n))^{(\infty)}$, we have

$I_{\bar 0}$ is an ideal of $\mathfrak{q}(\mathfrak{o}_B(n))^{(\infty)}_{\bar 0}$;

$\Pi(I_{\bar 1})$ is an ideal of $\Pi(\mathfrak{q}(\mathfrak{o}_B(n))^{(\infty)}_{\bar 1})$.

17.1.4.1. $\mathfrak{q}(\mathfrak{o}_I(n))$. As I have shown before, $\mathfrak{o}_I(n)$ is the algebra of all symmetric matrices, and $\mathfrak{o}_I(n)^{(\infty)} = \mathfrak{o}_I(n)^{(1)} = ZD(n)$.

Computations similar to the computations of $\mathfrak{o}_I(n)$ and $\mathfrak{oo}^{(i)}_{II}(m|n)$ show that

$$\begin{aligned} \mathfrak{q}(\mathfrak{o}_I(n))^{(1)} &= \mathfrak{o}_I(n) \oplus \Pi(ZD(n)); \\ \mathfrak{q}(\mathfrak{o}_I(n))^{(i)} &= (\mathfrak{o}_I(n) \cap \mathfrak{sl}(n)) \oplus \Pi(ZD(n)) \end{aligned}$$

As I have shown before, $ZD(n)$ is a simple Lie algebra, so any non-trivial ideal of $\mathfrak{q}(\mathfrak{o}_I(n))^{(\infty)}$ has zero odd part. Also (similarly to $\mathfrak{oo}_{II}(m|n)$) $\mathfrak{o}_I(n) \cap \mathfrak{sl}(n)$ considered as a restricted algebra has a non-trivial ideal if and only if n is even, and this ideal is equal to $\mathrm{Span}(1_n)$. Thus, **the simple part of $\mathfrak{q}(\mathfrak{o}_I(n))$ is equal to**

$$\begin{cases} (\mathfrak{o}_I(n) \cap \mathfrak{sl}(n)) \oplus \Pi(ZD(n)) & \text{if } n \text{ is odd,} \\ ((\mathfrak{o}_I(n) \cap \mathfrak{sl}(n)) \oplus \Pi(ZD(n)))/\mathbb{K}1_n & \text{if } n \text{ is even.} \end{cases}$$

17.1.4.2. $\mathfrak{q}(\mathfrak{o}_\Pi(2k))$. As I have shown before, the Lie algebra $\mathfrak{o}_\Pi(2k)$ consists of matrices of the following form:

$$\begin{pmatrix} A & C \\ D & A^T \end{pmatrix} \quad \begin{array}{c} \text{where } A \in \mathfrak{gl}(k), \\ C \text{ and } D \text{ are symmetric } k \times k\text{-matrices.} \end{array}$$

So the elements of $\mathfrak{q}(\mathfrak{o}_\Pi(2k))$ are of the form

$$\begin{pmatrix} A & C \\ D & A^T \end{pmatrix} \oplus \Pi\left(\begin{pmatrix} A' & C' \\ D' & A'^T \end{pmatrix}\right) \quad \begin{array}{c} \text{where } A, A' \in \mathfrak{gl}(k), \\ C, C', D, D' \text{ are symmetric } k \times k\text{-matrices.} \end{array}$$

Computations show that elements of $\mathfrak{q}(\mathfrak{o}_\Pi(2k))^{(i)}$ have the following conditions on them for different i:

$$\begin{aligned} &i = 1: A, A' \in \mathfrak{gl}(k), \quad C, C', D, D' \in ZD(k); \\ &i = 2: A \in \mathfrak{gl}(k), \quad A' \in \mathfrak{sl}(k), \quad C, C', D, D' \in ZD(k); \\ &i \geq 3: A, A' \in \mathfrak{sl}(k), \quad C, C', D, D' \in ZD(k). \end{aligned}$$

Thus,

$$\mathfrak{q}(\mathfrak{o}_\Pi(2k))^{(\infty)}_{\bar 0} \simeq \Pi(\mathfrak{q}(\mathfrak{o}_\Pi(2k))^{(\infty)}_{\bar 1}) = \Pi(\mathfrak{o}_\Pi(2k)^{(2)}) = \Pi(\mathfrak{o}_\Pi(2k)^{(\infty)}).$$

As I have shown before, $\mathfrak{o}_\Pi(2k)^{(2)}$ has a non-trivial ideal if and only if k is even, and this ideal is equal to $\mathrm{Span}(1_{2k})$. Thus, **the simple part of $\mathfrak{q}(\mathfrak{o}_\Pi(2k))$ is equal to**

$$\begin{cases} \mathfrak{o}_\Pi(2k)^{(2)} \oplus \Pi(\mathfrak{o}_\Pi(2k)^{(2)}) & \text{if } k \text{ is odd,} \\ (\mathfrak{o}_\Pi(2k)^{(2)} \oplus \Pi(\mathfrak{o}_\Pi(2k)^{(2)}))/(\mathbb{K}1_{2k} \oplus \mathbb{K}\Pi(1_{2k})) & \text{if } k \text{ is even.} \end{cases}$$

17.1.5. Queerification of vectorial Lie superalgebras. [[23] **DL: AL! Insert five series here; we (BJ&DL) have to check if $\mathfrak{q}(\mathfrak{vect}(0|n))$ is a CTS-prolong (actually, just C-) of $\mathfrak{q}$; our initial attempt to investigate this says NO, but at the moment I doubt it.**] [23]

17.1.6. Theorem. *The following Lie superalgebras are simple:*
$\mathfrak{psq}(2n+1)=\mathfrak{q}(\mathfrak{sl}(2n+1))$, $n\geq 1$;
$\mathfrak{pspsq}(2n)=\mathfrak{q}(\mathfrak{psl}(2n))$, $n\geq 2$;
$\mathfrak{q}(\mathfrak{e}(6))$, $\mathfrak{q}(\mathfrak{e}(8))$; $\mathfrak{q}(\mathfrak{e}^{(1)}(7)/\mathfrak{center})$;
$\mathfrak{q}^{(1)}(\mathfrak{wk}(3;a)/\mathfrak{center})$; $\mathfrak{q}(\mathfrak{wk}(4;a))$;
$\mathfrak{q}(\mathfrak{o}_I(2n+1))^{(1)}$, $n\geq 1$;
$\mathfrak{q}(\mathfrak{o}_I(2n))^{(1)}/\mathfrak{center}$, $n\geq 2$;
$\mathfrak{q}(\mathfrak{o}_{\Pi}^{(2)}(4n+2))$, $n\geq 1$;
$\mathfrak{q}(\mathfrak{o}_{\Pi}^{(2)}(4n)/\mathfrak{center})$, $n\geq 2$.

Proof. For the description and the proof of simplicity of the last 4 types of Lie superalgebras see §17.1.4 of Chapter 13.

The simplicity of Lie superalgebras $\mathfrak{q}(\mathfrak{sl}(2n+1))$, $\mathfrak{q}(\mathfrak{psl}(2n))$, $\mathfrak{q}(\mathfrak{e}(6))$, $\mathfrak{q}(\mathfrak{e}(8))$ follows from the simplicity of Lie algebras $\mathfrak{sl}(2n+1)$, $\mathfrak{psl}(2n)$, $\mathfrak{e}(6)$, $\mathfrak{e}(8)$. Lie algebra $\mathfrak{e}(7)$ (considered as the Lie algebra constructed by the corresponding Cartan matrix, see §12.3 of Chapter 12) in characteristic 2 is not simple since its Cartan matrix is degenerate. The simple subquotient of the algebra is $\mathfrak{e}^{(1)}(7)/fcenter$; so, thanks to Remark 12.3.4.2 the queerification of this subquotient is a simple Lie superalgebra.

Since $\mathfrak{wk}(3;a)/fcenter$ inherits the 2-structure of $\mathfrak{wk}(3;a)$, we may queerify this quotient. Select the grading operator d so that $B=(1,0,0)$ (for the definition of the matrix B, see 12.25); then although the operator of outer derivation d can not be obtained by bracketing it can be obtained thanks to the presence of the 2-structure: $d^{[2]}=d$. Therefore $\mathfrak{q}(\mathfrak{wk}(3;a)/\mathfrak{center})$ contains an ideal of codimension ε, i.e., the "simple core" is (as space:

$$\mathfrak{wk}(3;a)/\mathfrak{center}\oplus\Pi(\mathfrak{wk}^{(1)}(3;a)/\mathfrak{center}).$$

$\square$

17.1.7. Dynkin diagrams for the queer series. As indicated in [LSa1], in certain problems it is advisable to assign an analog of the Dynkin diagram to $\mathfrak{q}(n)$. Since the queer series is really queer to such extent that even its simple roots over $\mathbb{C}$ are of multiplicity $1|1$, Leites and Serganova denoted the nodes of the diagram by "square circles", each corresponding to $\mathfrak{q}(2)$. Having done this, we see that the Dynkin diagram for $\mathfrak{q}(n)$, does not differ in shape from that for $\mathfrak{sl}(n)$.

Part III

Modular Lie algebras and Lie superalgebras: Background (B. Clarke)

Chapter 18

Decompositions of the tensor products of irreducible $\mathfrak{sl}(2)$-modules in characteristic 3 (B. Clarke)

18.1. Introduction

Texts devoted to representations of Lie algebras in characteristic $p > 0$ are often prefaced by the disclaimer that the spaces considered are of dimension less than p. To the best of the author's knowledge, this restriction is always imposed on tensor products of irreducible modules when studying the analog of Klebsch-Gordon decompositions. That is, if V and W are two irreducible modules over a simple Lie algebra $\mathfrak{g}$ and one wishes to decompose $V \otimes W$ into indecomposable submodules, $\dim V \otimes W$ is always restricted to be less than p. In this paper, I remove this restriction and present a complete investigation of the decomposition of $V \otimes W$ for the case $\mathfrak{g} = \mathfrak{sl}(2)$ and $p = 3$ for any irreducible $\mathfrak{sl}(2)$-modules V and W.[1)]

In $\mathfrak{sl}(2)$, we consider the natural basis

$$X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.\binom{1}{2}$$

Hence, the structure constants are derived from the relations

$$[X_+, X_-] = H, \quad [H, X_\pm] = \pm 2X_\pm.$$

Let k be an algebraically closed field of characteristic 3. Irreducible $\mathfrak{sl}(2)$-modules in characteristic $p > 2$ were completely described by Rudakov

[1] Anton Cox wrote a letter to the author informing about the paper by Steve Doty and Anne Henke "Decomposition of tensor products of modular irreducibles for SL_2". Quarterly J. Math. vol. 56, (2005), 189–207. This result has little relation to ours since over fields of positive characteristic p there is no one-to-one correspondence between either Lie algebras and the "corresponding" Chevalley groups or between their representations; the smaller p, the more ephemeral this relation is. For example, if $p = 2$, then the Chevalley group $SL_2(\mathbb{F}_q)$ is simple, whereas Lie algebra $\mathfrak{sl}(2; \mathbb{F}_q)$ is solvable.

In arXiv the reader can find description of irreducibles over several Chevalley groups. —*D.L.*

and Shafarevich in [RSh]. These modules are all of dimension $D \le p$, and in the cases $D < p$, there is no difference from the case of characteristic zero (cf. [FH]). For $p = 3$ and $D < 3$, these are only the modules denoted by $\underline{1}$,

$$0 \xleftarrow[X_+]{} V_0 \xrightarrow{X_-} 0, \tag{18.1}$$

and $\underline{2}$,

$$0 \xleftarrow[X_+]{} V_1 \underset{X_+}{\overset{X_-}{\rightleftarrows}} V_{-1} \xrightarrow{X_-} 0. \tag{18.2}$$

In these diagrams, V_ρ denotes the 1-dimensional *weight space* of eigenvectors of H with eigenvalue ρ. The arrows indicate the action of the operators in the sub- or superscript.

18.1.1. Remark. More generally, over a field of prime characteristic p, we always have the irreducible $\mathfrak{sl}(2)$ modules $\underline{N}$ for $N \in \{1, \dots, p\}$, given diagrammatically by

$$0 \xleftarrow[X_+]{} V_{N-1} \underset{X_+}{\overset{X_-}{\rightleftarrows}} V_{N-3} \underset{X_+}{\overset{X_-}{\rightleftarrows}} \cdots \underset{X_+}{\overset{X_-}{\rightleftarrows}} V_{-N+3} \underset{X_+}{\overset{X_-}{\rightleftarrows}} V_{-N+1} \xrightarrow{X_-} 0. \tag{18.3}$$

Let us return to our case of characteristic $p = 3$. For $D = 3$, i.e., 3-dimensional irreducible representations, we have more than in the case of characteristic 0, where there is only the module $\underline{3}$,

$$0 \xleftarrow[X_+]{} V_{-1} \underset{X_+}{\overset{X_-}{\rightleftarrows}} V_0 \underset{X_+}{\overset{X_-}{\rightleftarrows}} V_1 \xrightarrow{X_-} 0. \tag{18.4}$$

There is in fact an entire family of irreducible representations, parametrized by a 3-dimensional variety, of which $\underline{3}$ is a special case. Writing the images of the generators of $\mathfrak{sl}(2)$ as matrices acting on a 3-dimensional vector space, these representations are given as follows. First, we have the irreducible modules that we denote by $T(b, c, d)$:

$$X_- = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} d-1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d+1 \end{pmatrix} \quad X_+ = \begin{pmatrix} 0 & a_1 & 0 \\ 0 & 0 & a_2 \\ b & 0 & 0 \end{pmatrix}, \tag{18.5}$$

where

$$\begin{aligned} a_1 &= bc + d - 1, \\ a_2 &= a_1 + d = bc - d - 1. \end{aligned} \tag{18.6}$$

We also have the family of "opposite" irreducible modules, where the forms of X_+ and X_- are exchanged, which we denote by $\widetilde{T}(b, c, d)$:

$$X_- = \begin{pmatrix} 0 & 0 & b \\ a_1 & 0 & 0 \\ 0 & a_2 & 0 \end{pmatrix} \quad H = \begin{pmatrix} d-1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d+1 \end{pmatrix} \quad X_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c & 0 & 0 \end{pmatrix}. \tag{18.7}$$

In both of these cases, b, c, and d are arbitrary elements of the ground field k, *however* we don't allow the cases

$$T(0, 0, 1) \quad \text{or} \quad T(0, 0, -1), \tag{18.8}$$

since in these cases the representation is not irreducible. Once the other parameters are chosen, a_1 and a_2 are necessarily given in terms of b, c, and d by (18.6) if the matrices (18.5) and (18.7) are to be representations of $\mathfrak{sl}(2)$. (To see this, one can explicitly solve, for example, the equation $[X_+, X_-] = H$ for a_1 and a_2 using the above matrix representations, then check that the relations $[H, X_\pm] = \pm 2X_\pm$ are satisfied).

In addition to these two kinds of irreducible modules, we have each of their duals, which we will denote by $T^*(b, c, d)$ and $\widetilde{T}^*(b, c, d)$.

18.1.2. Remark. Note that $\underline{3} \simeq T(0, 0, 0)$ as $\mathfrak{sl}(2)$-modules.

All of these irreducible modules can be glued into the following indecomposables.

We let $M \Supset \widetilde{M}$ denote the *semidirect sum* of the subspaces M and $\widetilde{M}$. By this, we mean that $\widetilde{M}$ is a submodule.

A diagram of subspaces (e.g. $M \to \widetilde{M}$) indicates something similar, but gives more information. A subspace that is the source of no arrows (in our example, $\widetilde{M}$) is a submodule. A subspace that is the source of some arrows becomes a submodule upon taking the quotient modulo the targets of those arrows (in our example, $M/\widetilde{M}$). Note that a subspace that is the source of some arrows cannot be selected uniquely, since it is a quotient space. Instead, we should think of it as the span of a collection of vectors, the representatives of which form a basis for the quotient space.

The direction of an arrow in a diagram also carries information. An arrow pointing to the right indicates that we get an element of the target by acting on an element of the source with X_-. An arrow pointing to the left indicates the same for X_+. Note that the "direction" of the arrow refers only to the left/right direction. That is, an arrow that points up/down and right is still thought of as an "arrow pointing to the right", and similarly for up/down and left. We also say, taking the example from the previous paragraph, that "M is glued in to $\widetilde{M}$ via X_-".

For later, we let M_1 denote the submodule (cf. [?]):

$$M_1 : \quad \begin{tikzcd} & \underline{2} \arrow[dl] \arrow[dr] & \\ \underline{1} \arrow[dr] & & \underline{1} \arrow[dl] \\ & \underline{2} & \end{tikzcd} .$$

To make the statements of the last two paragraphs concrete, we dissect this particular case. The symbol $\underline{2}$ at the bottom indicates an irreducible submodule. The symbol $\underline{1}$ on the left is represented by the span of a single vector v

with $X_+v = 0$ and X_-v a vector of the irreducible submodule $\underline{2}$ at the bottom. Similarly, the symbol $\underline{1}$ on the left is represented by the span of a single vector w with $X_-w = 0$ and X_+w a vector of the irreducible submodule $\underline{2}$ at the bottom. Finally, the symbol $\underline{2}$ at the top stands for the span of two vectors, v' and w', with

$$\begin{aligned} X_-v' &= w', \quad X_-w' = \mu w, \\ X_+v' &= \lambda v, \quad X_+w' = v', \end{aligned} \tag{18.9}$$

for some $\lambda, \mu \in k$, i.e. X_+v' is contained in the left "$\underline{1}$" and X_-w' is contained in the right "$\underline{1}$".

The main result of the paper is the following theorem.

18.1.3. Theorem. *The decompositions (into indecomposable submodules) of tensor products of all irreducible $\mathfrak{sl}(2)$-modules ($\underline{1}$, $\underline{2}$, $T(b,c,d)$, $\widetilde{T}(b,c,d)$, and their duals) are completely described by*

1) $\underline{1} \otimes V \simeq V$ *for any $\mathfrak{sl}(2)$-module V,*
2) $\underline{2} \otimes \underline{2} = \underline{1} \oplus \underline{3}$,
3) $\underline{2} \otimes \widetilde{T}\Big(b, \dfrac{1}{b}, 0\Big) = \widetilde{T}\Big(b, \dfrac{1}{b}, 0\Big) \oplus T(\dfrac{1}{b}, 0, 0)$,

together with Tables 18.3–18.7, found at the end of the paper.

18.1.4. Remark. Statement 1) of the theorem is obvious.

18.1.5. Remark. In § 18.2, we will explain how we arrived at the list of modules examined in Tables 18.3–18.7, as well as why we are allowed to seemingly ignore certain modules.

The paper is organized as follows. In § 18.2 we will closely examine the families of modules and reduce the number of different cases we must consider separately by demonstrating certain correspondences. Then, we will study the structure of the modules we are concerned with and, in particular, semidirect sums. In § 18.3, we will briefly present the notation we use for the calculations. Finally, in § 18.4, we break down the various tensor products we can form case by case and compute their decompositions.

All computations for this paper were made with the assistance of `SuperLie` ([Gr], [?]).

18.2. Preliminaries

Before we get into the meat of the paper, let us carefully describe the $\mathfrak{sl}(2)$-modules in characteristic 3, first the irreducible ones, then certain indecomposables (to describe all indecomposables is an open problem).

We begin by proving a couple of lemmas which will help us to reduce the amount of work we have to do. In particular, we can show (via a change of basis) that some seemingly different modules are actually isomorphic. In the end, we will only have to consider the cases given in the following lemma (in addition, of course, to $\underline{2}$):

18.2.1. Lemma. *We can represent every 3-dimensional irreducible $\mathfrak{sl}(2)$-module by a member of one of the following two families:*

1) $T(b,c,d)$, *where $b, c, d \in k$ are arbitrary subject to* (18.8)*;*
2) $\widetilde{T}\Big(b, \dfrac{1}{b}, 0\Big)$.

The proof of Lemma 18.2.1 is immediately implied by Lemmas 18.2.2, 18.2.3, and 18.2.4 below.

18.2.2. Lemma. *For the dual modules of $T(b,c,d)$ and $\widetilde{T}(b,c,d)$, given any $b, c, d \in k$, we have:*

1) $T^*(b,c,d) \simeq T(b',c',d')$ *for some $b', c', d' \in k$,*
2) $\widetilde{T}^*(b,c,d) \simeq \widetilde{T}(b',c',d')$ *for some $b', c', d' \in k$.*

Proof. 1) Recall that, given a matrix representation X of the action of an element of a Lie algebra on a module, the action of X on the dual module is given by $-X^t$, i.e., the negative transpose of the original matrix. So here, the action of $\mathfrak{sl}(2)$ on the dual module $T^*(b,c,d)$ is as follows:

$$X_- = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -c & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} -d+1 & 0 & 0 \\ 0 & -d & 0 \\ 0 & 0 & -d-1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 0 & -b \\ -a_1 & 0 & 0 \\ 0 & -a_2 & 0 \end{pmatrix}. \tag{18.10}$$

We apply a similarity transformation given by the matrix

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{18.11}$$

We then rename the resulting matrices to $X'_\pm := SX_\pm S^{-1}$ and $H' := SHS^{-1}$:

$$X'_- = \begin{pmatrix} 0 & 0 & c' \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad H' = \begin{pmatrix} d'-1 & 0 & 0 \\ 0 & d' & 0 \\ 0 & 0 & d'+1 \end{pmatrix}, \quad X'_+ = \begin{pmatrix} 0 & a'_1 & 0 \\ 0 & 0 & a'_2 \\ b' & 0 & 0 \end{pmatrix}. \tag{18.12}$$

Here one easily checks that $a'_1 = a_2$, $a'_2 = a_1$, $b' = -b$, $c' = -c$, and $d' = -d$. We have transformed our original representation $T^*(b,c,d)$ into one of the form $T(b',c',d')$.

2) The action on the dual module $\widetilde{T}^*(b,c,d)$ is given by:

$$X_- = \begin{pmatrix} 0 & -a_1 & 0 \\ 0 & 0 & -a_2 \\ -b & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} -d+1 & 0 & 0 \\ 0 & -d & 0 \\ 0 & 0 & -d-1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 0 & -c \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}. \tag{18.13}$$

Since this is in complete analogy with the previous case, we skip the detailed calculation. (We do note that one can, in fact, even use the same similarity tranformation as in (18.11) above.) □

18.2.3. Lemma. *For an appropriate choice of $b', c', d' \in k$ we have*

$$\widetilde{T}(b,c,d) \simeq T(b',c',d'), \tag{18.14}$$

if and only if at most one of a_1, a_2, and b are equal 0.

Proof. First, we note that if two of a_1, a_2, and b are zero, then we will clearly get no isomorphism, since for modules of the form $T(b,c,d)$, the matrix of X_- can have at most one eigenvector with eigenvalue 0.

The proof of the converse statement is broken up into different cases.

1) $a_1, a_2 \neq 0$. We apply the similarity transformation with the matrix

$$S = \begin{pmatrix} \frac{1}{a_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{a_1 a_2} \end{pmatrix}. \tag{18.15}$$

This yields matrices H', $X'_\pm$ of the same form as in (18.12) above. In this case, using the same notation as above, we have $a'_1 = a_1$, $a'_2 = a_2$, $b' = \frac{c}{a_1 a_2}$, $c' = a_1 a_2 b$, and $d' = d$.

2) $a_1 = 0$, $a_2, b \neq 0$. We apply the similarity transformation with the matrix

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{a_2} \\ \frac{1}{ba_2} & 0 & 0 \end{pmatrix}. \tag{18.16}$$

We get matrices as in the form of (18.12), where this time $a'_1 = a_2$, $a'_2 = bc$, $b' = \frac{1}{ba_2}$, $c' = 0$, and $d' = d + 1$.

3) $a_2 = 0$, $a_1, b \neq 0$. We apply the similarity transformation with the matrix

$$S = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{b} & 0 & 0 \\ 0 & \frac{1}{ba_1} & 0 \end{pmatrix}. \tag{18.17}$$

We again get matrices in the form of (18.12), where $a'_1 = bc$, $a'_2 = a_1$, $b' = \frac{1}{ba_1}$, $c' = 0$, and $d' = d - 1$. ☐

The following lemma will tell us more about $\widetilde{T}(b,c,d)$ when two of a_1, a_2, and b are zero.

18.2.4. Lemma. *If two of a_1, a_2, and b are zero, then $\widetilde{T}(b,c,d) \simeq \widetilde{T}\left(b, \frac{1}{b}, 0\right)$.*

Proof. There are three possibilities: $a_1, a_2 = 0$, $a_1, b = 0$, and $a_2, b = 0$. Let us examine each one in turn.

1) $a_1, a_2 = 0$. In this case, we have

$$\begin{aligned} bc + d - 1 &= 0, \\ bc - d - 1 &= 0. \end{aligned} \tag{18.18}$$

Subtracting the second equation from the first implies $d = 0$, which we can plug back into the first equation to get $bc - 1 = 0$, or $c = \frac{1}{b}$. Hence, the module must be of the form $\widetilde{T}\left(b, \frac{1}{b}, 0\right)$.

2) $a_1, b = 0$. Here we have $0 = a_1 = bc + d - 1 = d - 1$. Hence, $d = 1$. Furthermore, by Lemma 18.2.5 below, we can transform $T(0,c,1)$ into a representation with $d = 0$ provided that $c \neq 0$. Since X_- will still have two eigenvectors with eigenvalue 0, the transformed representation will necessarily be as in case 1). (When $a_2 = 0$ and $b = 0$, d is necessarily -1; see case 3) below.) Therefore, the only truly new case is $\widetilde{T}(0,0,1)$. However, $\widetilde{T}(0,0,1)$ is not irreducible (see the Introduction) and, since we assumed our module to be irreducible, is disallowed. So we have now completely reduced to case 1).

3) $a_2, b = 0$. In this case, we get $0 = a_2 = bc - d - 1 = -d - 1$, or $d = -1$. As above, we can reduce this to case 1) if and only if $c \neq 0$. Therefore, the only new case is $\widetilde{T}(0,0,-1)$. However, from the Introduction we know that $\widetilde{T}(0,0,-1)$ is not irreducible, and as in case 2) we have now completely reduced to case 1). ☐

We also want to examine the cases where $d = 0$ or ± 1 in more detail, since they will turn out to be special once we start tensoring. It turns out that all three of these cases correspond to a module where $d = 0$ unless $c = 0$:

18.2.5. Lemma. *Let $d = \pm 1$ and $c \neq 0$. Then*

$$T(b,c,d) \simeq T(b',c',0) \tag{18.19}$$

for an appropriate choice of b' and c'.

The statement remains true if we replace T by $\widetilde{T}$ everywhere above.

Proof. We will prove the statement for $T(b,c,1)$ (i.e. for $d = 1$). The other cases are completely analogous.

In this case, our representation is given by the following matrices:

$$X_- = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & a_1 & 0 \\ 0 & 0 & a_2 \\ b & 0 & 0 \end{pmatrix}. \tag{18.20}$$

We apply the similarity transformation with matrix

$$S = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{c} & 0 & 0 \\ 0 & \frac{1}{c} & 0 \end{pmatrix}, \tag{18.21}$$

renaming the resulting matrices to $X'_\pm := SX_\pm S^{-1}$ and $H' := SHS^{-1}$ to get

$$X'_- = \begin{pmatrix} 0 & 0 & c' \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad H' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad X'_+ = \begin{pmatrix} 0 & a'_1 & 0 \\ 0 & 0 & a'_2 \\ b' & 0 & 0 \end{pmatrix}. \tag{18.22}$$

Explicitly, we have $a'_1 = bc$, $a'_2 = a_1$, $b' = \frac{a_2}{c}$, and $c' = c$. So the change of basis gives us a representation of the form $T(b', c', 0)$. $\square$

Now that we have proved Lemma 18.2.1, we move on to studying semidirect sums and the structure of our $\mathfrak{sl}(2)$-modules. Let S and T be two 3-dimensional irreducible $\mathfrak{sl}(2)$-modules, and let us consider the tensor product $V = S \otimes T$ as an $\mathfrak{sl}(2)$-module. This is a 9-dimensional space, but we may divide it into three distinguished 3-dimensional subspaces, the *weight spaces*, i.e., the eigenspaces of H. (It is simple to check that H has three distinct eigenvalues for any such S and T.)

18.2.6. Remark. For the remainder of this section, all vectors are assumed to be weight vectors, i.e., eigenvectors of H.

Thus, instead of considering the full 9-dimensional space V, we restrict attention to one of the 3-dimensional weight spaces. We denote them by

$$V_\rho = \{v \in V \mid Hv = \rho v\}. \tag{18.23}$$

The element $X_+X_- \in U(\mathfrak{sl}(2))$ acts on each weight space, since if $Hv = \rho v$ for some $\rho \in k$, then

$$\begin{aligned} H(X_+X_-v) &= X_+HX_-v + [H, X_+]X_-v \\ &= X_+X_-Hv + X_+[H, X_-]v + 2X_+X_-v \\ &= \rho X_+X_-v. \end{aligned} \tag{18.24}$$

Therefore, we may consider X_+X_- as a linear transformation of one such space and look for its eigenvalues and eigenvectors. We relate this to semidirect sums in the following lemma:

18.2.7. Lemma. *Let X_+X_- have two distinct eigenvalues, λ_1 and λ_2, on the subspace V_ρ for some eigenvalue ρ of H. Suppose that V contains a semidirect sum $M \supsetplus \widetilde{M}$. Further, let $\widetilde{M} \cap V_\rho = \operatorname{span}(v_1, v_2)$, where $X_+X_-v_i = \lambda_i v_i$. Consider the action of $\mathfrak{sl}(2)$ on the quotient space and a vector $m \in M/\widetilde{M}$ with $Hm = \rho m$. Then $X_+X_-m = \lambda_i m$ for some i.*

Proof. Suppose, on the contrary, that $X_+X_-m = \mu m$ (equality being in the quotient space) for some $\mu \neq \lambda_i$ for all i. We will show that there is a vector $v \in V$ with $Hv = \rho v$ and $X_+X_-v = \mu v$, a contradiction.

For the remainder of the proof, the action of $\mathfrak{sl}(2)$ will be on the full space V, not the quotient space.

We know that $X_+X_-m = \mu m + \widetilde{m}$ for some $\widetilde{m} \in \widetilde{M}$. By the assumptions of the lemma, we can write $\widetilde{m} = \widetilde{m}_1 + \widetilde{m}_2$, where $X_+X_-\widetilde{m}_i = \lambda_i \widetilde{m}_i$. We set

$$v = m + \frac{1}{\mu - \lambda_1}\widetilde{m}_1 + \frac{1}{\mu - \lambda_2}\widetilde{m}_2. \tag{18.25}$$

Since $\mu \neq \lambda_1$ or λ_2, v is well-defined, and it is easily seen that $X_+X_-v = \mu v$. $\square$

We can consider other eigenvalue equations on the weight spaces of a tensor product, in particular for X_+^3 and X_-^3. The proof of the following lemma is straightforward and is left to the reader.

18.2.8. Lemma. *The action of X_+^3 and X_-^3 on any weight vector v in the module $V = S \otimes T$, where S and T are of the form $\widetilde{T}\Big(b, \frac{1}{b}, 0\Big)$ or $T(b, c, d)$, is given by:*

1) $\widetilde{T}\Big(b, \frac{1}{b}, 0\Big) \otimes \widetilde{T}(\beta, \frac{1}{\beta}, 0)$*:* $X_+^3 v = \frac{b + \beta}{b\beta}$*,* $X_-^3 v = 0$*;*
2) $\widetilde{T}\Big(b, \frac{1}{b}, 0\Big) \otimes T(\beta, \gamma, \delta)$*:* $X_+^3 v = (\frac{1}{b} + \beta\alpha_1\alpha_2)v$*,* $X_-^3 v = \gamma v$*;*
3) $T(b, c, d) \otimes T(\beta, \gamma, \delta)$*:* $X_+^3 v = (ba_1a_1 + \beta\alpha_1\alpha_2)v$*,* $X_-^3 v = (c + \gamma)v$*.*

Furthermore, assuming v is some weight vector, V contains highest weight vectors if and only if $X_+^3 v = 0$, and lowest weight vectors if and only if $X_-^3 v = 0$.

Combining Lemmas 18.2.7 and 18.2.8, we get:

18.2.9. Lemma. *Given a module $T(b, c, d)$ with no lowest weight vectors (i.e. $c \neq 0$), or a module $\widetilde{T}\Big(b, \frac{1}{b}, 0\Big)$, we can determine b, as well as c and d (where applicable), from the actions of X_+X_- and X_-^3 on the weight spaces. If $c = 0$ in $T(b, c, d)$, we may still determine b from the action of X_+^3 if we know d.*

Proof. Let us fix a basis $\{v_1, v_2, v_3\}$ such that the $\mathfrak{sl}(2)$-action is given by the matrices of (18.5) or (18.7).

For $\widetilde{T}\Big(b, \frac{1}{b}, 0\Big)$ we must simply note that $X_+^3 v_1 = \frac{1}{b} v_1$.

For $T(b, c, d)$, where $c \neq 0$ we note that the equations

$$\begin{aligned} X_+X_-v_1 &= (bc - 1 + d)v_1, \\ X_+X_-v_3 &= bcv_3, \\ X_-^3 v_1 &= cv_1 \end{aligned} \tag{18.26}$$

allow us to determine b, c, and d, since $c \neq 0$.

In the case of $T(b, 0, d)$ where d is known, we note that

$$X_+^3 v_1 = ba_1a_2v_1 = b(1 - d^2)v_1, \tag{18.27}$$

allowing us to determine b. $\square$

The number of linearly independent eigenvectors of X_+X_- will be important to us when determining the structure of a decomposition, as the following lemma shows.

18.2.10. Lemma. *Let there be either no highest or no lowest weight vectors in V.*

1) *If X_+X_- has three linearly independent eigenvectors v_1, v_2, and v_3 in V_ρ for an eigenvalue ρ of H, then there are 3-dimensional irreducible submodules $M, M', M'' \subset V$ such that $V = M \oplus M' \oplus M''$.*

2) *If X_+X_- has only two linearly independent eigenvectors v_1 and v_2 in V_ρ, then there are 3-dimensional submodules $M, M', M'' \subset V$ such that $V = M \supsetplus (M' \oplus M'')$. In this case, M' and M'' are irreducible.*

18.2.11. Remark. In the cases we consider, X_+X_- will always have at least two distinct eigenvectors on any V_ρ.

Proof. We will prove this for the case where $X_+v \neq 0$ for all $v \in V$. The other case is analogous.

Clearly, in an irreducible submodule of $\mathfrak{sl}(2)$, for any weight vector v, we have $X_+X_-v = \lambda v$ for some $\lambda \in k$. Further, a one-dimensional subspace belongs to at most one irreducible submodule. So we can have at most as many irreducible submodules as we have eigenvectors of X_+X_- in some V_ρ.

Furthermore, if $X_+X_-v = \lambda v$ for some $\lambda \in k$ and v a weight vector, then v, X_+v, and X_+^2v form an irreducible submodule. From Lemma 18.2.8, it is clear that X_+^3v is a nonzero multiple of v. From the relations of $\mathfrak{sl}(2)$, it is easy to check that $X_-X_+^nv$ is a multiple of $X_+^{n-1}v$ for any nonnegative integer n. So these three vectors do form a submodule, and it is irreducible because X_+^n sends any subspace to any other for some n.

This completes the proof of heading 1), since in that case we can build three 3-dimensional irreducible submodules by the procedure in the last paragraph. For heading 2), we can only build two such irreducible submodules M' and M'' this way. To complete the proof, we consider $M = V/(M' \oplus M'')$ and the action of $\mathfrak{sl}(2)$ on this quotient. We take any nonzero weight vector $m \in M$ and note again that m, X_+m, and X_+^2m form a basis of this quotient module, and that it is irreducible. They are nonzero because X_+^3m is a nonzero multiple of m. They are linearly independent because they are of different weights, and they form an irreducible submodule because there is no invariant subspace (X_+^n sends any subspace to any other for some n). □

18.3. Notation

For the sake of convenience, we fix the notation for our modules for the rest of the paper. We denote vectors in our modules according to the following scheme:

$$\begin{aligned}
&q_i \in \underline{2} \quad \text{for } i = 1,2;\\
&t_i \in \widetilde{T}\Big(b, \frac{1}{b}, 0\Big),\ u_i \in \widetilde{T}(\beta, \frac{1}{\beta}, 0) \quad \text{for } i = 1,2,3;\\
&v_i \in T(b,c,d),\ w_i \in T(\beta,\gamma,\delta) \quad \text{for } i = 1,2,3.
\end{aligned} \tag{18.28}$$

The index in the subscript refers to the weight of the vector. For $\underline{2}$, the vector q_1 is of weight 1 and q_2 is of weight -1. For t_i, u_i, v_i, and w_i, we refer to the matrix representations (18.5) and (18.7) and set $t_i = e_i$, $u_i = e_i$, etc., where $\{e_i \mid i = 1,2,3\}$ is the usual basis of k^3.

In addition, since we will need them so often and the notation becomes awkward, we will omit the "$\otimes$" when writing tensor products of vectors. For example, $q_1 \otimes v_2 \in \underline{2} \otimes T(b,c,d)$ will be expressed simply as q_1v_2.

18.4. Case-by-case calculations

All calculations were done with the assistance of the `Mathematica`-based package `SuperLie` ([Gr], [**?**]).

Due to space considerations, and for the flow of arguments, calculations will not be repeated here in detail. Instead, we refer to the `Mathematica` notebooks, which have been made available online at

`http://www.mis.mpg.de/~clarke/tensor`.

We have focused on producing explicit decompositions where possible. Many decompositions could have also been deduced using more general arguments based on the lemmas of §18.2, as was done in 18.4.5.10 or 18.4.6.19.

18.4.1. The case $V = \underline{2} \otimes \underline{2}$. This is the simplest case of all; in fact, there is no difference from the decomposition in characteristic 0. We start from the two highest weight vectors, r_1r_1 and $r_2r_1 - r_1r_2$. Applying X_- gives an irreducible submodule of the form $\underline{3}$ from the first, and $\underline{1}$ from the second. Hence, we have a direct sum of two irreducible modules,

$$\underline{2} \otimes \underline{2} = \underline{3} \oplus \underline{1}. \tag{18.29}$$

This could have been deduced from more general considerations as well. Working over fields of characteristic p, if we have two modules of the form $\underline{N}$ and $\underline{M}$, where $N, M \in \{1, \dots, p\}$ (defined in (18.3)), they have highest weights $N-1$ and $M-1$, respectively. Taking their tensor product $\underline{N} \otimes \underline{M}$ will give us vectors with weights at most $N+M-2$. If $N+M-2 < p$, then we will see behavior that is no different from the case of characteristic 0. Things start getting interesting when $N+M-2 \geq p$, which we will have in all following cases.

18.4.2. The case $V = \underline{2} \otimes \widetilde{T}\Big(b, \frac{1}{b}, 0\Big)$. In this case, we find two highest weight vectors, q_2t_1, and q_2t_2. Applying X_- to these, we construct two 3-dimensional irreducible submodules that supply a complete decomposition. From q_2t_1, we get the module $T(\frac{1}{b}, 0, 0)$, and from q_2t_2, we get the module $\widetilde{T}\Big(b, \frac{1}{b}, 0\Big)$. So

$$\underline{2} \otimes \widetilde{T}\Big(b, \frac{1}{b}, 0\Big) = T\Big(\frac{1}{b}, 0, 0\Big) \oplus \widetilde{T}\Big(b, \frac{1}{b}, 0\Big). \tag{18.30}$$

18.4.3. The case $V = \underline{2} \otimes T(b, c, d)$. In this case, we have the lowest weight vectors q_2v_3 and $q_1v_3 - q_2v_2$ if and only if $c = 0$. We have highest weight vectors if and only if we are in one of the following situations:

1. q_1v_1 and $q_1v_2 + (1-d)q_2v_1 \iff b = 0$,
2. q_1v_3 and $q_1v_1 - bq_2v_3 \iff 1 - bc + d = 0$,
3. q_1v_2 and $q_1v_3 + (1 - bc + d)q_2v_2 \iff 1 - bc - d = 0$.

The "only if" part of these statements is provided by Lemma 18.2.8.

Furthermore, the eigenvalues of H are d and $d \pm 1$. Hence, whenever $d = 0$, 1, or 2, we will have different behavior—e.g., the appearance of submodules like $\underline{3}$ and $\underline{1} \to \underline{2}$.

Therefore, we divide our computations into subcases. For the precise breakdown of these subcases, we note that for $c = 0$, whether we have the highest weight vectors described in (2) and (3) above depends only on d—we have (2) if and only if $d = -1$ and (3) if and only if $d = 1$. Furthermore, if $c \neq 0$, then we need not consider $d = \pm 1$ separately from $d = 0$, since in both of these cases $T(b, c, d)$ is isomorphic to a module with $T(b', c', 0)$ for some b' and c' by Lemma 18.2.5.

18.4.3.1. The subcase $c = 0$; $d = 0$; $b = 0$. In this case, by acting on the two lowest and highest weight vectors listed above, we compute a submodule of the form $\underline{1} \to \underline{2} \leftarrow \underline{1}$. We quickly verify that this contains all highest and lowest weight vectors.

The full module has dimension six, so we do not yet have a complete decomposition. Since any weight vectors not contained in $\underline{1} \to \underline{2} \leftarrow \underline{1}$ have weights 1 and -1, we know that they will form a module of the form $\underline{2}$ after quotienting by $\underline{1} \to \underline{2} \leftarrow \underline{1}$ (to form two modules $\underline{1}$, they would have to have weight 0). A direct computation shows that all together, the module decomposes as M_1.

18.4.3.2. The subcase $c = 0$; $d = 0$; $b \neq 0$. Here, we only have lowest weight vectors to work with. Acting on them by X_+ gives rise to one irreducible submodule, $M := \widetilde{T}(\frac{1}{b}, b, 0)$, which contains both lowest weight vectors.

Since there are no more highest or lowest weight vectors, there are now two possibilities. Either the remainder of the full module forms a submodule of the form $T(b', c', d')$ with no highest or lowest weight vectors, or the remainder forms irreducible submodules only upon taking some quotients.

The first possibility can hold if and only if X_+X_- has a basis of eigenvectors for each weight space. A quick check shows that the minimal polynomial of X_+X_- is $(\lambda - 1)^2$ when acting on the space of weight 1 vectors. Hence, the module contains no more irreducible submodules.

A direct computation shows us now that the quotient module V/M is again of the form $\widetilde{T}(\frac{1}{b}, b, 0)$, so we get the complete decomposition

$$\widetilde{T}\Big(\frac{1}{b}, b, 0\Big) \Supset \widetilde{T}\Big(\frac{1}{b}, b, 0\Big). \tag{18.31}$$

18.4.3.3. The subcase e$c = 0$; $d = 1$; $b = 0$. We begin by acting on the lowest weight vectors by X_+, which gives us two irreducible submodules, $\underline{3}$ and $\underline{1}$. This exhausts the lowest weight vectors, yet V is not yet completely decomposed — we are missing a 2-dimensional subspace.

Since $1 - bc - d = 1 - d = 0$, there is one remaining highest weight vector. Acting on it by X_- gives us the final submodule $\underline{2}$, which is glued into $\underline{1}$ via X_-. Hence, we get

$$\underline{3} \oplus (\underline{2} \to \underline{1}). \tag{18.32}$$

18.4.3.4. The subcase $c = 0$; $d = 1$; $b \neq 0$. As in the last case, we immediately get two irreducible submodules, $\underline{3}$ and $\underline{1}$. Here, there are no remaining highest weight vectors, but we note that X_+X_- has a basis of eigenvectors for the space of weight 1 vectors. We use this to compute that the remaining 2-dimensional submodule is irreducible after quotienting by $\underline{1}$, so we get

$$\underline{3} \oplus (\underline{2} \Supset \underline{1}). \tag{18.33}$$

18.4.3.5. The subcase $c = 0$; $d = 2$; $b = 0$. This case proceeds exactly analogously to Section 18.4.3.3. However, since we have different highest weight vectors (here $1 - bc + d = 1 + d = 0$), we get the slightly different decomposition

$$\underline{3} \oplus (\underline{1} \to \underline{2}). \tag{18.34}$$

18.4.3.6. The subcase $c = 0$; $d = 1$; $b \neq 0$. As above, this is completely analogous to Section 18.4.3.4, but we get the decomposition

$$\underline{3} \oplus (\underline{1} \Supset \underline{2}). \tag{18.35}$$

18.4.3.7. The subcase $c = 0$; $d \neq 0, 1, 2$. In this case, we may have highest weight vectors if $b = 0$, but it turns out that in either case, acting on the lowest weight vectors by X_+ immediately gives us a complete decomposition,

$$T\Big(\frac{b(d-1)}{d}, 0, d-1\Big) \oplus T\Big(\frac{b(d+1)}{d}, 0, d+1\Big). \tag{18.36}$$

18.4.3.8. The subcase $c \neq 0$; $d = 0$; $b = 0$. By acting on the highest weight vectors with X_-, we get a module of the form $M := T(0, c, 1)$, which exhausts the highest weight vectors. Since there are no more highest or lowest weight vectors, we have the same situation as in Section 18.4.3.2. As there, we can check that X_+X_- does not have a basis of eigenvectors for the space of weight 1 vectors. We then directly compute that V/M is again of the form $T(0, c, 1)$. In all, we get

$$T(0, c, 1) \Supset T(0, c, 1) \tag{18.37}$$

18.4.3.9. The subcase $c \neq 0$; $d = 0$; $b = \frac{1}{c}$. In this case there are three highest weight vectors, since $1 - bc + d = 1 - bc - d = 0$. By acting on these with X_-, we get a complete decomposition,

$$T(0, c, 1) \oplus T(0, c, 1). \tag{18.38}$$

18.4.3.10. The subcase $c \neq 0$; $d = 0$; $b \neq 0$ or $\frac{1}{c}$. Here there are no highest or lowest weight vectors. However, the minimal polynomial for X_+X_- acting on the space of weight 0 vectors is $\lambda^2 + bc\lambda + bc(bc-1)$, which has the two distinct roots $bc \pm \sqrt{bc}$. Hence, X_+X_- has a basis of eigenvectors for this space. Explicitly solving for these eigenvectors and acting on them by $X_\pm$ gives the decomposition

$$T\Big(b + \sqrt{\frac{b}{c}}, c, 1\Big) \oplus T\Big(b - \sqrt{\frac{b}{c}}, c, 1\Big). \tag{18.39}$$

18.4.3.11. The subcase $c \neq 0$; $d \neq 0, 1$ or 2; $b = 0$. In this case, acting by X_- on the two highest weight vectors gives us a complete decomposition,

$$T(0, c, d+1) \oplus T(0, c, d-1). \tag{18.40}$$

18.4.3.12. The subcase $c \neq 0$; $d \neq 0, 1$ or 2; $b = \frac{1}{c}$. Here again, acting by X_- on the two highest weight vectors gives us a complete decomposition,

$$T(0, c, d) \oplus T(0, c, d+1). \tag{18.41}$$

18.4.3.13. The subcase $c \neq 0$; $d \neq 0, 1$ or 2; $b = \frac{1}{c}$. Here again, acting by X_- on the two highest weight vectors gives us a complete decomposition,

$$T(0, c, d-1) \oplus T(0, c, d). \tag{18.42}$$

18.4.3.14. The subcase $c \neq 0$; $d \neq 0, 1$ or 2; $b \neq 0$ or $\frac{1}{c}$. As in Section 18.4.3.10, there are no highest or lowest weight vectors. Investigating the action of X_+X_- on the space of weight d vectors shows that it has eigenvalues $bc - d \pm \sqrt{bc + d^2}$, and that X_+X_- has a basis of eigenvectors for this space if and only if the eigenvalues are distinct, i.e., if $bc + d^2 \neq 0$.

If $bc + d^2 \neq 0$, we solve for these eigenvectors and act on them by $X_\pm$ to get a complete decomposition,

$$T\Big(b + \frac{d + \sqrt{bc + d^2}}{c}, c, d+1\Big) \oplus T\Big(b + \frac{d - \sqrt{bc + d^2}}{c}, c, d+1\Big). \tag{18.43}$$

If $bc + d^2 = 0$, the two eigenvectors from above degenerate to one, and acting on it we get a submodule $M := T(b + \frac{d}{c}, c, d+1)$. We can then explicitly compute that $V/M = T(b + \frac{d}{c}, c, d+1)$ as well. So in this case, the complete decomposition is

$$T\Big(b + \frac{d + \sqrt{bc + d^2}}{c}, c, d+1\Big) \ni T\Big(b + \frac{d - \sqrt{bc + d^2}}{c}, c, d+1\Big). \tag{18.44}$$

18.4.4. The case $V = \widetilde{T}\Big(b, \frac{1}{b}, 0\Big) \otimes \widetilde{T}\Big(\beta, \frac{1}{\beta}, 0\Big)$. In this case, by Lemma 18.2.8, there are always lowest weight vectors, namely t_1u_1, t_1u_2, t_2u_1, $t_1u_3 + t_3u_1$, and t_2u_2. We have the highest weight vectors $bt_1u_1 + t_2u_3 - t_3u_2$, $bt_1u_2 - bt_2u_1 - t_3u_3$, and $t_1u_3 - t_2u_2 + t_3u_1$ if and only if $\frac{b+\beta}{b\beta} = 0$, or equivalently $\beta = -b$. Therefore, we need to split this case into two subcases.

18.4.4.1. The subcase $\beta = -b$. Here, we start by acting on the highest weight vectors by X_-. This immediately generates three irreducible modules, $\underline{3}$, $\underline{1}$, and $\underline{2}$. This exhausts all of the highest weight vectors.

Since the full module has two lowest weight vectors not contained in the above submodules, there are two possibilities. Either these lowest weight vectors make up a module of the form $\widetilde{T}(b', \frac{1}{b'}, 0)$, or they form irreducible modules only upon quotienting. The first possibility is ruled out since X_+^3 of any weight vector is zero, which is not the case for any $\widetilde{T}(b', \frac{1}{b'}, 0)$. Hence, we have the second possibility.

By acting on the remaining lowest weight vectors by X_+, we obtain a module $\underline{2}$ glued into the module $\underline{1}$ via X_+, and a module $\underline{1}$ glued into the module $\underline{2}$ via X_+. Hence the full decomposition is

$$\underline{3} \oplus (\underline{2} \leftarrow \underline{1}) \oplus (\underline{1} \leftarrow \underline{2}). \tag{18.45}$$

18.4.4.2. The subcase $\beta \neq -b$. In this case, acting on the lowest weight vectors by X_+ immediately gives us a complete decomposition,

$$T\Big(\frac{b+\beta}{b\beta}, 0, 0\Big) \oplus \widetilde{T}\Big(\frac{b\beta}{b+\beta}, \frac{b+\beta}{b\beta}, 0\Big) \oplus \widetilde{T}\Big(\frac{b\beta}{b+\beta}, \frac{b+\beta}{b\beta}, 0\Big). \tag{18.46}$$

18.4.5. The case $V = \widetilde{T}\Big(b, \frac{1}{b}, 0\Big) \otimes T(\beta, \gamma, \delta)$. Again referring to Lemma 18.2.8, we determine that V contains the lowest weight vectors t_2w_3, $bt_1w_2 - t_3w_3$, and t_1w_3 if and only if $\gamma = 0$. Furthermore, V contains the highest weight vectors

$$\begin{aligned} &t_1w_1 - \beta t_2w_3 - \beta(1 - \beta\gamma + \delta)t_3w_2, \\ &t_1w_2 + (1 - \gamma - \delta)t_2w_1 - \beta(1 - \beta\gamma - \delta)t_3w_3, \\ &t_1w_3 + (1 - \beta\gamma + \delta)t_2w_1 + (1 - \beta\gamma + \delta)(1 - \beta\gamma - \delta)t_3w_1 \end{aligned} \tag{18.47}$$

if and only if

$$1 + b\beta(1 - \beta\gamma + \delta)(1 - \beta\gamma - \delta) = 0. \tag{18.48}$$

Furthermore, similarly to the case of $\underline{2} \otimes T(b, c, d)$, the eigenvalues of H are δ and $\delta \pm 1$.

Based on this, we split this case into subcases. Note that when we have both that $\gamma = 0$ and $\delta = \pm 1$, it is impossible to have highest weight vectors. In addition, if $\gamma \neq 0$, then we need not consider the cases $\delta = \pm 1$ separately from $\delta = 0$, since by Lemma 18.2.5, $T(\beta, \gamma, \pm 1)$ is isomorphic to a module $T(\beta', \gamma', 0)$ for some β', γ'.

18.4.5.1. The subcase $\gamma=0$; $\delta=0$; $\beta=-\frac{1}{b}$. In this case, we begin by acting on the three available lowest weight vectors by X_+, followed by acting on the highest weight vectors by X_-. This immediately allows us to compute two indecomposable submodules of the forms $\underline{3}$ and $\underline{1} \to \underline{2} \leftarrow \underline{1}$.

The above submodules have dimension seven, while V has dimension nine. The weight vectors linearly independent from the above submodules have weights 1 and -1. Since there are no more highest or lowest weight vectors, these vectors form an irreducible module of the form $\underline{2}$ only after passage to the quotient. A quick check of the minimal equation of X_+X_- acting on the space of vectors of weight 1 shows that X_+X_- has only two eigenvectors on this space. So we start with an arbitrary vector that is linearly independent from the submodules of the last paragraph and find that the complete decomposition is of the form

$$\underline{3} \oplus (\underline{2} \ni (\underline{1} \to \underline{2} \leftarrow \underline{1})). \tag{18.49}$$

18.4.5.2. The subcase $\gamma=0$; $\delta=0$; $\beta\neq-\frac{1}{b}$. Here, we have three lowest weight vectors and no highest weight vectors. Acting on the lowest weight vectors yields two submodules, $\widetilde{T}(\frac{b}{1+b\beta}, \frac{1+b\beta}{b}, 0)$ and $T(\frac{1+b\beta}{b}, 0, 0)$, which contain al of the lowest weight vectors.

One quickly checks that X_+X_- does not have a basis of eigenvectors for the space of weight 1 vectors. Therefore, the subspace of remaining vectors will be irreducible only upon quotienting. A direct computation shows that the quotient is of the form $\widetilde{T}(\frac{b}{1+b\beta}, \frac{1+b\beta}{b}, 0)$, giving the complete decomposition

$$\widetilde{T}\Big(\frac{b}{1+b\beta}, \frac{1+b\beta}{b}, 0\Big) \ni \Big(\widetilde{T}\Big(\frac{b}{1+b\beta}, \frac{1+b\beta}{b}, 0\Big) \oplus T\Big(\frac{1+b\beta}{b}, 0, 0\Big)\Big). \tag{18.50}$$

18.4.5.3. The subcase $\gamma=0$; $\delta=1$. In this case there are three lowest weight vectors but, as noted above, no highest weight vectors. By acting on the lowest weight vectors with X_+, we immediately get two submodules, $\widetilde{T}\Big(b, \frac{1}{b}, 0\Big)$ and $T(\frac{1}{b}, 0, 0)$.

The above submodules contain all of the highest and lowest weight vectors, and we can check that X_+X_- does not have a basis of eigenvectors for any weight space (actually, checking one particular weight space suffices). Hence, starting with an arbitrary weight vector that is linearly independent from the two submodules above, we compute the final submodule, which is irreducible upon quotienting and gives the decomposition

$$\widetilde{T}\Big(b, \frac{1}{b}, 0\Big) \ni \Big(\widetilde{T}\Big(b, \frac{1}{b}, 0\Big) \oplus T\Big(\frac{1}{b}, 0, 0\Big)\Big). \tag{18.51}$$

18.4.5.4. The subcase $\gamma=0$; $\delta=2$. This case is completely analogous to that of Section 18.4.5.3. The end result is also identical.

18.4.5.5. The subcase

$$\boldsymbol{\gamma = 0; \quad \delta \neq 0, 1 \text{ or } 2; \quad b\beta(1-\delta^2) = -1.}$$

For this case, please refer to Section 18.4.5.9 below. The calculation there is also completely valid for the case $\gamma = 0$.

18.4.5.6. The subcase

$$\boldsymbol{\gamma = 0; \quad \delta \neq 0, 1 \text{ or } 2; \quad b\beta(1-\delta^2) \neq -1.}$$

Here, there are no highest weight vectors, but acting on the three lowest weight vectors by X_+ gives the entire decomposition,

$$T(1+b\beta(1-\delta^2), 0, \delta-1) \oplus T(1+b\beta(1-\delta^2), 0, \delta) \oplus T(1+b\beta(1-\delta^2), 0, \delta+1). \tag{18.52}$$

18.4.5.7. The subcase $\gamma\neq0$; $\delta=0$; $b\beta(1-\beta\gamma)^2=-1$. Here there are no lowest weight vectors, but three highest weight vectors. Acting on these by X_- yields two 3-dimensional submodules, $T(0,\gamma,0)$ and $T(0,\gamma,-1)$.

A quick check then shows that X_+X_- does not have a basis of eigenvectors for any weight space. However, by selecting a weight vector linearly independent from the two submodules above and applying X_- to it, we find that the quotient of V by the two submodules above is of the form $T(0,\gamma,-1)$. In all, we get

$$T(0,\gamma,-1) \ni (T(0,\gamma,0) \otimes T(0,\gamma,-1)) \tag{18.53}$$

18.4.5.8. The subcase $\gamma\neq0$; $\delta=0$; $b\beta(1-\beta\gamma)^2\neq-1$. This case is covered by the calculation of Section 18.4.5.10 below. Note that in the current case, the condition (18.58) cannot hold. The assumption $\delta = 0$ implies $(\delta(\delta+1)(\delta-1))^2 = 0$ and we have assumed that

$$1 + b\beta(1-\beta\gamma+\delta)(1-\beta\gamma-\delta) \neq 0. \tag{18.54}$$

18.4.5.9. The subcase

$$\boldsymbol{\gamma \neq 0; \quad \delta \neq 0, 1 \text{ or } 2; \quad b\beta(1-\beta\gamma+\delta)(1-\beta\gamma-\delta) = -1.}$$

In this case, there are no lowest weight vectors, but three highest weight ones. Acting on them by X_- yields a full decomposition,

$$T(0,\gamma,\delta-1) \oplus T(0,\gamma,\delta) \oplus T(0,\gamma,\delta+1). \tag{18.55}$$

18.4.5.10. The subcase

$$\boldsymbol{\gamma \neq 0; \quad \delta \neq 0, 1 \text{ or } 2; \quad b\beta(1-\beta\gamma+\delta)(1-\beta\gamma-\delta) \neq -1.}$$

Here, we have neither highest nor lowest weight vectors to exploit. Hence, we must rely on Lemmas 18.2.8 and 18.2.10 and examine the eigenspaces of X_+X_-.

We begin with the space $V_{\delta+1}$. A straightforward computation shows that the characteristic polynomial of X_+X_- acting on $V_{\delta+1}$ is

$$\lambda^3 + (1-\delta^2)\lambda^2 + \lambda - \frac{\gamma}{b}(1 + b\beta(1-\beta\gamma+\delta)(1-\beta\gamma-\delta)). \tag{18.56}$$

To apply Lemma 18.2.10, we must know how many linearly independent eigenvectors X_+X_- has in $V_{\delta+1}$; that is, we must know what the *minimal* polynomial of X_+X_- is.

Noting that, in a field of characteristic 3, $(\lambda-\mu)^3 = \lambda^3 - \mu^3$, it is easily seen that (18.56) cannot be written as a perfect cube. So it must have at least two solutions.

We then note that

$$(\lambda-\mu)^2(\lambda-\rho) = \lambda^3 + (\mu-\rho)\lambda^2 + (\mu^2-\mu\rho)\lambda - \mu^2\rho. \tag{18.57}$$

By equating coefficients of λ, we deduce that (18.56) and (18.57) can be equal if and only if $\mu = 1-\delta^2$ and $\rho = -\delta^2$, *as well as* the following relation between the parameters of $\widetilde{T}\left(b, \frac{1}{b}, 0\right)$ and $T(\beta,\gamma,\delta)$ is satisfied:

$$\frac{\gamma}{b}(1 + b\beta(1-\beta\gamma+\delta)(1-\beta\gamma-\delta) = \mu^2\rho = -\delta^2(1-\delta)^2(1+\delta)^2. \tag{18.58}$$

Of course, even if the characteristic polynomial has a root of multiplicity two, X_+X_- may still have a basis of eigenvectors if its minimal polynomial factors into distinct linear factors, that is if X_+X_- satisfies the equation

$$(X_+X_- - (1-\delta^2 I))(X_+X_- + \delta^2 I) = 0, \tag{18.59}$$

where I is the identity operator on $V_{\delta+1}$. However, a direct computation shows that this is never the case.

This argumentation shows that we have a direct sum decomposition,

$$T(b_1,c_1,d_1) \oplus T(b_2,c_2,d_2) \oplus T(b_3,c_3,d_3), \tag{18.60}$$

if (18.58) does not hold. If (18.58) holds, we have a decomposition involving a semidirect sum,

$$T(b_3,c_3,d_3) \Supset (T(b_1,c_1,d_1) \oplus T(b_2,c_2,d_2)). \tag{18.61}$$

Now we are interested in determining the possible values of the parameters b_i, c_i, d_i. Let ρ_1, ρ_2, and ρ_3 be the (not necessarily distinct) roots of the polynomial (18.56), and assume that at least $\rho_1 \neq \rho_2$. Now, let $v_{3,i}$ be the distinct eigenvectors of X_+X_- in $V_{\delta+1}$, so that i ranges from one to the number of distinct eigenvectors. Finally, set $v_{2,i} = X_+v_{3,i}$ and $v_{1,i} = X_+^2 v_{3,i}$. We take $\{v_{1,i}, v_{2,i}, v_{3,i}\}$ as our basis for $T(b_i,c_i,d_i)$ in the matrix representation (18.5).

With this notation, and recalling Lemma 18.2.9, we can determine the parameters. We already know that $X_-^3 v_{j,i} = \gamma v_{j,i}$ for all i and j, so $c_i = \gamma$ for all i. Now, on the one hand, $X_+X_-v_{3,i} = b_ic_iv_{3,i}$, and the other hand, we know that $X_+X_-v_{3,i} = \rho_i v_{3,i}$. Therefore, $b_i = \dfrac{\rho_i}{\gamma}$. Finally, since each $v_{3,i}$ belongs to $V_{\delta+1}$, we know from the algebra equations for $\mathfrak{sl}(2)$ that $v_{2,i}$ is of weight $(\delta+1)+2 = \delta$. So $d_i = \delta$. This determines the parameters completely for the case that X_+X_- has a basis of eigenvectors for $V_{\delta+1}$.

If X_+X_- has only two distinct eigenvectors for $V_{\delta+1}$, b_3 is yet to be determined. To do this, we note the following general fact. Let $A\colon W \to W$ be a linear endomorphism of a finite-dimensional vector space W, and let U be an A-invariant subspace of W. Furthermore, let the minimal polynomial of $A\colon W \to W$ be m, that of $A\colon U \to U$ be m_1, and that of the induced map $A\colon W/U \to W/U$ be m_2. Then $m = m_1 \cdot m_2$.

With this in mind, assume we have nondistinct eigenvalues $\rho_1 = \rho_3 = 1-\delta^2$ and $\rho_2 = -\delta^2$. Then the minimal polynomial of X_+X_- on $V_{\delta+1}$ is

$$m = (\lambda - (1-\delta^2))^2(\lambda+\delta^2). \tag{18.62}$$

Using the notation of the previous paragraph with $A = X_+X_-$, $W = V_{\delta+1}$, and $U = \operatorname{span}(v_{3,1}, v_{3,2})$, we then have

$$m_1 = (\lambda - (1-\delta^2))(\lambda+\delta^2), \tag{18.63}$$

implying $m_2 = (\lambda - (1-\delta^2))$. Hence, letting $v_{3,3}$ be a representative for a nonzero vector in the quotient space W/U, we get X_+X_- is $X_+X_-v_{3,3} = (1-\delta^2)v_{3,3}$. Therefore,

$$b_3 = \frac{1-\delta^2}{c}. \tag{18.64}$$

Thus, we have determined all parameters for the decomposition.

18.4.6. The case $T(b,c,d) \otimes T(\beta,\gamma,\delta)$. By Lemma 18.2.8, we have lowest weight vectors if and only if $c+\gamma = 0$, and highest weight vectors if and only if $ba_1a_2 + \beta\alpha_1\alpha_2 = 0$, where

$$\begin{aligned} a_1 &= bc+d-1, & \alpha_1 &= \beta\gamma+\delta-1, \\ a_2 &= bc-d-1, & \alpha_2 &= \beta\gamma-\delta-1. \end{aligned} \tag{18.65}$$

The lowest weight vectors are then

$$\begin{aligned} &cv_1w_1 + v_2w_3 - v_3w_2, \\ &cv_1w_2 - cv_2w_1 - v_3w_3, \\ &v_1w_3 - v_2w_2 + v_3w_1, \end{aligned} \tag{18.66}$$

and the highest weight vectors are

$$\begin{aligned}&a_1a_2v_1w_1-\beta a_2v_2w_3+\beta\alpha_2v_3w_2,\\&a_1a_2v_1w_2-a_2\alpha_1v_2w_1+\beta\alpha_1v_3w_3,\\&a_1a_2v_1w_3-a_2\alpha_2v_2w_2+\alpha_1\alpha_2v_3w_1.\end{aligned}\tag{18.67}$$

The highest weight vectors were computed using the following method (here, as an example, for the weight space $\mathrm{span}(v_1w_1, v_2w_3, v_3w_2)$). We have, for the action of X_+ on this weight space,

$$\begin{aligned}X_+(v_1w_1)&=\beta v_1w_3+bv_3w_1,\\X_+(v_2w_3)&=a_1v_1w_3+\alpha_2v_2w_2,\\X_+(v_3w_2)&=a_2v_2w_2+\alpha_1v_3w_1.\end{aligned}\tag{18.68}$$

We first try to cancel the factors of v_1w_3, noting that

$$X_+(a_1v_1w_1-\beta v_1w_3)=-\beta\alpha_2v_2w_2+ba_1v_3w_1.\tag{18.69}$$

From here, we cancel the factors of v_2w_2:

$$X_+(a_2(a_1v_1w_1-\beta v_2w_3)+\beta\alpha_2v_3w_2)=(ba_1a_2+\beta\alpha_1\alpha_2)v_3w_1\tag{18.70}$$

Since we have assumed that $ba_1a_2+\beta\alpha_1\alpha_2=0$, this tells us that $a_1a_2v_1w_1-\beta a_2v_2w_3+\beta\alpha_2v_3w_2$ is a highest weight vector.

The problem with this method of computing the highest weight vector is that the vector we end up with might be the zero vector. We can try to rectify this by changing the order of the factors that we cancel. Doing so gives us two additional "representations" (by an abuse of language) for the highest weight vectors:

$$\begin{aligned}&a_1\alpha_1v_1w_1-\beta\alpha_1v_2w_3-ba_1v_3w_2,\\&\beta\alpha_2v_1w_2+ba_2v_2w_1-b\beta v_3w_3,\\&\beta\alpha_1v_1w_3+ba_2v_2w_2+b\alpha_1v_3w_1\end{aligned}\tag{18.71}$$

and

$$\begin{aligned}&\alpha_1\alpha_2v_1w_1+ba_2v_2w_3-b\alpha_2v_3w_2,\\&a_1\alpha_2v_1w_2-\alpha_1\alpha_2v_2w_1-ba_1v_3w_3,\\&\beta a_1v_1w_3-\beta\alpha_2v_2w_2+ba_1v_3w_1.\end{aligned}\tag{18.72}$$

We then hope that one of these "representations" is nonzero. However, for certain choices of the parameters, all three "representations" of some highest weight vector are zero. A lengthy but straightforward analysis shows that the only such choices are given by Table 18.1.

We do not have to consider all of these cases separately, however. Note that in all of these cases, $c\neq 0$. Whenever $b=\dfrac{1}{c}$, it is assumed, and whenever $b=0$, we have $d=\pm1$, and so we must have $c\neq 0$ if $T(b,c,d)$ is to be irreducible (see the Introduction). Likewise, $\gamma\neq 0$ in all of these cases. Therefore, we

Table 18.1 Parameter values for which all "representations" of some highest weight vector are zero

$\boldsymbol{b}$	0	0	0	0	0	0	$\frac{1}{c}$	$\frac{1}{c}$	$\frac{1}{c}$
$\boldsymbol{d}$	1	−1	1	−1	1	−1	0	0	0
$\boldsymbol{\beta}$	0	0	$\frac{1}{\gamma}$	0	0	$\frac{1}{\gamma}$	0	0	$\frac{1}{\gamma}$
$\boldsymbol{\delta}$	1	−1	0	1	−1	0	1	−1	0

may use Lemma 18.2.5 (and the explicit calculations of its proof) to see that $T(0,c,\pm1)\simeq T(\frac{1}{c},c,0)$ and $T(0,\gamma,\pm1)\simeq T(\frac{1}{\gamma},\gamma,0)$. Hence, all nine cases are equivalent to the case where $b=\frac{1}{c}$, $d=0$, $\beta=\frac{1}{\gamma}$, and $\delta=0$. So this means we must consider that particular case separately from the other cases where highest weight vectors are present.

One may ask the related question of whether there is only one highest weight vector in each weight space. For the lowest weight vectors this is clear by inspection, but in the case of highest weight vectors, it is not so easily seen directly whether this is true. We may, however, consider the characteristic polynomials of X_+X_- on the weight spaces. These are

$$\lambda^3+\lambda^2+((d+\delta)^2-1)\lambda+(c+\gamma)(ba_1a_2\beta\alpha_1\alpha_2)\text{ on }V_{d+\delta+1},\tag{18.73}$$

$$\lambda^3+\lambda^2+(d+\delta)((d+\delta)-1)\lambda+(c+\gamma)(ba_1a_2\beta\alpha_1\alpha_2)\text{ on }V_{d+\delta-1},\tag{18.74}$$

$$\lambda^3+\lambda^2+(d+\delta)((d+\delta)+1)\lambda+(c+\gamma)(ba_1a_2\beta\alpha_1\alpha_2)\text{ on }V_{d+\delta}.\tag{18.75}$$

From this, we clearly see that X_+X_- can have at most two eigenvectors with eigenvalue 0, since the geometric multiplicity of an eigenvalue is at most its algebraic multiplicity in the characteristic equation. Furthermore, we note that if $c+\gamma=0$, one of these zero eigenvectors will come from a vector v with $X_-v=0$. Therefore, we focus on the case where $c+\gamma\neq 0$. Another lengthy but straightforward calculation shows us that the only cases where X_+X_- has two eigenvectors with eigenvalue 0 in some weight space are again exactly given by Table 18.1. Hence, we have no extra cases here to consider specially.

For the action of H, we note that the possible weights for vectors in V are $d+\delta$ and $d+\delta\pm1$. As in the case of $\widetilde{T}\left(b,\frac{1}{b},0\right)\otimes T(\beta,\gamma,\delta)$, we will be concerned with when these weights can be 0, 1, or -1, since in these cases we will see phenomena that are not possible otherwise. Hence, we consider $d+\delta=0$ or ±1 separately.

However, for $\gamma\neq -c$ (i.e., when there are no lowest weight vectors), we may use Lemma 18.2.5 to reduce the cases $d+\delta=\pm1$ to the case $d+\delta=0$. This is because we can assume that either c or γ is nonzero. Without loss of generality, we assume $c\neq 0$. We can then use the isomorphisms

$$T(b,c,0) \simeq T(b',c',1) \simeq T(b'',c'',-1) \tag{18.76}$$

(for appropriate b', b'', c' and c'') to modify the value of $d+\delta$.

If $\gamma = -c$, on the other hand, we could have $\gamma = 0 = c$, in which case Lemma 18.2.5 is inapplicable.

Finally, we note the following factorizations for $K = ba_1a_2 + \beta\alpha_1\alpha_2$ for special values of $d+\delta$ when $\gamma = -c$:

$$\begin{aligned} d+\delta = 0 &\Longrightarrow K = (b+\beta)(1+bc+b^2c^2-d^2-c\beta-bc^2\beta+c^2\beta),\\ d+\delta = 1 &\Longrightarrow K = (1+d)((1-d)b-d\beta),\\ d+\delta = -1 &\Longrightarrow K = (1-d)((1+d)b+d\beta). \end{aligned} \tag{18.77}$$

With all of this in mind, we are now ready to break down the necessary subcases. Except in the two subcases where we explicitly state this to be the case, we assume that we do *not* have all of the conditions $b = \frac{1}{c}$, $d = 0$, $\beta = \frac{1}{\gamma}$, and $\delta = 0$. (The same assumption goes for Tables 18.6 and 18.7.)

18.4.6.1. The subcase $\boldsymbol{\gamma = -c;\quad b = \frac{1}{c};\quad d = 0;\quad \beta = \frac{1}{\gamma};\quad \delta = 0.}$ This computation is implied by that in 18.4.6.16 below. We note that $T(0,0,0) \simeq \underline{3}$, $T(0,0,1) \simeq \underline{1} \to \underline{2}$, and $T(0,0,-1) \simeq \underline{2} \to \underline{1}$.

18.4.6.2. The subcase

$$\boldsymbol{\gamma = -c;\quad d+\delta = 0;\quad \beta = -b;\quad 1+bc+b^2c^2-d^2-c\beta-bc^2\beta+c^2\beta^2 = 0.}$$

In this case, we have three lowest weight vectors. Acting on them by X_+, we obtain three submodules, of the forms $\underline{1}$, $\underline{2}$, and $\underline{3}$. This exhausts both the highest and lowest weight vectors.

Examining the action of X_+X_- on V_1, we notice by calculating the characteristic and minimal polynomials that X_+X_- has two eigenvectors with eigenvalue 1. We have only exploited one of these; acting on the other by X_+ and X_- gives a module that is of the form $\underline{2}$ after quotienting with the $\underline{1}$ from the last paragraph. Together, these submodules have dimension eight, while V has dimension nine, so we deduce the complete decomposition

$$\underline{1} \ni ((\underline{2} \ni \underline{1}) \oplus \underline{2} \oplus \underline{3}). \tag{18.78}$$

18.4.6.3. The subcase

$$\boldsymbol{\gamma = -c;\quad d+\delta = 0;\quad \beta = -b;\quad 1+bc+b^2c^2-d^2-c\beta-bc^2\beta+c^2\beta^2 \neq 0.}$$

As in the last case, we begin with acting by X_+ on the lowest weight vectors, which gives two submodules of the forms $\underline{3}$ and $\underline{1} \leftarrow \underline{2}$. These submodules contain all but one highest weight vector, which has weight 1. Acting on it by X_- gives a submodule of the form $\underline{2}$. Since there are no more highest or lowest weight vectors, the complete decomposition is of the form

$$\underline{1} \ni (\underline{2} \ni (\underline{3} \oplus (\underline{1} \leftarrow \underline{2}))). \tag{18.79}$$

18.4.6.4. The subcase

$$\boldsymbol{\gamma = -c;\quad d+\delta = 0;\quad \beta \neq -b;\quad 1+bc+b^2c^2-d^2-c\beta-bc^2\beta+c^2\beta^2 = 0.}$$

This case proceeds completely analogously to above, only after exploiting the lowest weight vectors, we have two submodules of the forms $\underline{3}$ and $\underline{2} \leftarrow \underline{1}$. The one remaining highest weight vector is of weight 0. Acting on it by X_- yields a vector in $\underline{3} \oplus (\underline{2} \leftarrow \underline{1})$. With no remaining highest or lowest weight vectors, the decomposition is of the form

$$\underline{2} \ni (\underline{1} \ni (\underline{3} \oplus (\underline{2} \leftarrow \underline{1}))). \tag{18.80}$$

18.4.6.5. The subcase

$$\boldsymbol{\gamma = -c;\quad d+\delta = 0;\quad \beta \neq -b;\quad 1+bc+b^2c^2-d^2-c\beta-bc^2\beta+c^2\beta^2 \neq 0.}$$

In this case there are no highest weight vectors. Acting on the lowest weight vectors by X_+ gives two irreducible submodules, $T(K,0,0)$ and $\widetilde{T}(\frac{1}{K},K,0)$. This exhausts all of the lowest weight vectors. By Lemma 18.2.8 there can be no module of the form $T(b,c,d)$ without highest or lowest weight vectors. Therefore, the remainder of the module can be irreducible only upon quotienting. Selecting an arbitrary vector from $V/(T(K,0,0) \oplus \widetilde{T}(\frac{1}{K},K,0))$ and acting on it by X_+ shows that the quotient is of the form $\widetilde{T}(\frac{1}{K},K,0)$. In all, we get

$$\widetilde{T}\Big(\frac{1}{K},K,0\Big) \ni \Big(T(K,0,0) \oplus \widetilde{T}\Big(\frac{1}{K},K,0\Big)\Big) \tag{18.81}$$

18.4.6.6. The subcase

$$\boldsymbol{\gamma = -c;\quad d+\delta = 1;\quad d = -1;\quad (1-d)b = d\beta.}$$

This case is done completely analogously to Section 18.4.6.2 and gives the same result.

18.4.6.7. The subcase

$$\boldsymbol{\gamma = -c;\quad d+\delta = 1;\quad d = -1;\quad (1-d)b \neq d\beta.}$$

This case proceeds as in Section 18.4.6.3. However, here the equations are a bit simpler, and we can achieve the somewhat sharper decomposition

$$\underline{1} \ni (\underline{3} \oplus (\underline{2} \to \underline{1} \leftarrow \underline{2})). \tag{18.82}$$

18.4.6.8. The subcase 18.4.6.3

$$\boldsymbol{\gamma = -c;\quad d+\delta = 1;\quad d \neq -1;\quad (1-d)b = d\beta.}$$

This case proceeds as in Section 18.4.6.4. As in the previous case, we can achieve a somewhat sharper decomposition,

$$\underline{2} \ni (\underline{3} \oplus (\underline{1} \to \underline{2} \leftarrow \underline{1})). \tag{18.83}$$

18.4.6.9. The subcase

$$\gamma = -c; \quad d+\delta = 1; \quad d \neq -1; \quad (1-d)b \neq d\beta.$$

This case is completely analogous to that of Section 18.4.6.5, and gives an identical result.

18.4.6.10. The subcase

$$\gamma = -c; \quad d+\delta = 2; \quad d = 1; \quad (1+d)b = -d\beta.$$

This case is handled just like Section 18.4.6.6 and gives the same result.

18.4.6.11. The subcase

$$\gamma = -c; \quad d+\delta = 2; \quad d = 1; \quad (1+d)b \neq -d\beta.$$

Here, we proceed as in Section 18.4.6.7 and get the same result.

18.4.6.12. The subcase

$$\gamma = -c; \quad d+\delta = 2; \quad d \neq 1; \quad (1+d)b = -d\beta.$$

This case is analogous to Section 18.4.6.8 and again gives the same result.

18.4.6.13. The subcase

$$\gamma = -c; \quad d+\delta = 2; \quad d \neq 1; \quad (1+d)b \neq -d\beta.$$

Here, we compute the same result as in Section 18.4.6.9 in exactly the same manner.

18.4.6.14. The subcase

$$\gamma = -c; \quad d+\delta \neq 0, 1 \text{ or } 2; \quad ba_1a_2 = -\beta\alpha_1\alpha_2.$$

The decomposition for this case is implied by that in Section 18.4.6.15 simply by substituting $K = 0$. The computations are all still valid.

18.4.6.15. The subcase

$$\gamma = -c; \quad d+\delta \neq 0, 1 \text{ or } 2; \quad ba_1a_2 \neq -\beta\alpha_1\alpha_2.$$

Here, we have no highest weight vectors, but there are three lowest weight vectors. Acting on them by X_+ immediately gives us the complete decomposition,

$$T(K,0,d+\delta-1) \oplus T(K,0,d+\delta) \oplus T(K,0,d+\delta+1). \tag{18.84}$$

18.4.6.16. The subcase

$$\gamma \neq -c; \quad b = \frac{1}{c}; \quad d = 0; \quad \beta = \frac{1}{\gamma}; \quad \delta = 0.$$

In this case, there are no lowest weight vectors, but we have five highest weight vectors to exploit. Acting on them by X_- gives the complete decomposition

$$T\Big(0, \frac{b+\beta}{b\beta}, -1\Big) \oplus T\Big(0, \frac{b+\beta}{b\beta}, 0\Big) \oplus T\Big(0, \frac{b+\beta}{b\beta}, 1\Big). \tag{18.85}$$

18.4.6.17. The subcase

$$\gamma \neq -c; \quad ba_1a_2 = -\beta\alpha_1\alpha_2; \quad d+\delta = 0.$$

In this case, there are three highest weight vectors. Acting on them by X_-, we obtain two submodules, $T(0, c+\gamma, -1)$ and $T(0, c+\gamma, 0)$. This exhausts all highest weight vectors.

The two remaining possibilities are that we have some module of the form $T(b', c', d')$ without highest or lowest weight vectors, or that the remainder of the module forms an irreducible module only after quotienting. Lemma 18.2.8 rules out the first possibility, so we must have the second. Furthermore, again by Lemma 18.2.8, we know that the quotient module must have the form $T(b', c', d')$ after quotienting, since it can have no lowest weight vectors. Furthermore, it must have $c' = c + \gamma$.

Let us examine the action of X_+X_- on V_0, and argue analogously to Section 18.4.5.10. The minimal polynomial of X_+X_- on this space is $\lambda^2(\lambda+1)$. There is one eigenvector of eigenvalue 0 and one of eigenvalue -1. The quotient of V_0 by the span of these eigenvectors is a 1-dimensional space, and the minimal polynomial of X_+X_- on this space is λ. Therefore, choosing a basis vector v_3' for the quotient, we have $X_+X_-v_3' = 0$ for the action on the quotient. Setting $v_2' = X_+v_3'$ and $v_1' = X_+^2v_3'$ and taking $\{v_1', v_2', v_3'\}$ as a basis for the quotient space, we determine that $b' = 0$ and $d' = 0 + 2 = -1$.

18.4.6.18. The subcase

$$\gamma \neq -c; \quad ba_1a_2 = -\beta\alpha_1\alpha_2; \quad d+\delta \neq 0, 1, 2.$$

Here, we have three highest weight vectors. Acting on them by X_- yields, after a lengthy but straightforward calculation, three irreducible submodules which form a complete decomposition,

$$T(0, c+\gamma, d+\delta-1) \oplus T(0, c+\gamma, d+\delta) \oplus T(0, c+\gamma, d+\delta+1). \tag{18.86}$$

18.4.6.19. The subcase $\gamma \neq -c$; $ba_1a_2 \neq -\beta\alpha_1\alpha_2$. Here, there are no highest or lowest weight vectors. Therefore, we want to proceed upon the lines of 18.4.5.10, trying to apply Lemma 18.2.10 to determine the structure of the decomposition.

The characteristic polynomial of X_+X_- acting on $V_{d+\delta+1}$ is given by

$$\lambda^3 + \lambda^2 + (1 - (d+\delta)^2)\lambda - (c+\gamma)(ba_1a_2 + \beta\alpha_1\alpha_2). \tag{18.87}$$

We are now interested in using this to determine how many linearly independent eigenvectors X_+X_- has on $V_{d+\delta+1}$.

We note, as in 18.4.5.10, that (18.87) must have at least two distinct roots, since $(\lambda-\mu)^3 = \lambda^3 - \mu^3$ in characteristic 3.

It is, however, possible that (18.87) has only two distinct roots. Equating coefficients of (18.87) and (18.57) gives that (18.87) is of the form $(\lambda-\mu)^2(\lambda-\rho)$ if and only if $\mu = -(d+\delta)(1+d+\delta)$ and $\rho = -1-(d+\delta)(1+d+\delta)$, *as well as* the following condition on the parameters b, c, d, etc. is satisfied:

$$(d+\delta)^2(1-(d+\delta)^2)^2 = -(c+\gamma)(ba_1a_2 + \beta\alpha_1\alpha_2). \tag{18.88}$$

We will still have a direct sum decomposition even if (18.88) is satisfied, provided the minimal polynomial of X_+X_- on $V_{d+\delta+1}$ factors into distinct linear factors, i.e.,

$$(X_+X_- - \mu I)(X_+X_- - \rho I) = 0, \tag{18.89}$$

This matrix equation can be viewed as a system of nine equations. Taking sums and differences of these equations, and using a good deal of brute force, we can reduce these nine equations to the following three conditions:

$$\begin{aligned} &bd(c+\gamma) = (d+\delta)(1-(d+\delta)^2), \\ &\beta\delta = bd, \\ &\gamma(d-d^3) = c(\delta-\delta^3). \end{aligned} \tag{18.90}$$

Therefore, we have the decomposition

$$T(b_3, c_3, d_3) \Supset (T(b_1, c_1, d_1) \oplus T(b_2, c_2, d_2)). \tag{18.91}$$

if and only if (18.88) is satisfied but (18.90) is not. Otherwise, we have the direct sum decomposition

$$T(b_1, c_1, d_1) \oplus T(b_2, c_2, d_2) \oplus T(b_3, c_3, d_3) \tag{18.92}$$

To determine the b_i, c_i, and d_i, let us repeat our considerations from Section 18.4.5.10 in this case. Let μ_1, μ_2, and μ_3 be the (not necessarily distinct) roots of the polynomial (18.87), and assume that at least $\mu_1 \neq \mu_2$. As before, let $v_{3,i}$ be the distinct eigenvectors of X_+X_- in $V_{d+\delta+1}$, so that i ranges from one to the number of distinct eigenvectors. Then set $v_{2,i} = X_+v_{3,i}$ and $v_{1,i} = X_+^2 v_{3,i}$. We take $\{v_{1,i}, v_{2,i}, v_{3,i}\}$ as our basis for $T(b_i, c_i, d_i)$ in the matrix representation (18.5).

Since $X_-^3 v_{j,i} = (c+\gamma)v_{j,i}$ for all i and j, it follows that $c_i = c+\gamma$ for all i. We then have the two equations $X_+X_-v_{3,i} = b_ic_iv_{3,i}$ and $X_+X_-v_{3,i} = \mu_iv_{3,i}$. These imply that

$$b_i = \frac{\mu_i}{c+\gamma}. \tag{18.93}$$

To determine d_i, remember that each $v_{3,i}$ belongs to $V_{d+\delta+1}$, so $v_{2,i}$ is of weight $(d+\delta+1)+2 = d+\delta$. Hence, $d_i = d+\delta$.

As in Section 18.4.5.10, we must finally determine b_3 in the case that there are only two distinct eigenvectors, that is, when the minimal polynomial does not factor into linear factors with multiplicity one. In this case we have $\mu_1 = \mu_3 = -(d+\delta)(1+d+\delta)$ and $\mu_2 = -1-(d+\delta)(1+d+\delta)$. Here also, The minimal polynomial of X_+X_- on $V_{d+\delta+1}$ is

$$(\lambda + (d+\delta)(1+d+\delta))^2(\lambda + (1 + (d+\delta)(1+d+\delta))). \tag{18.94}$$

By the same arguments as in Section 18.4.5.10, we deduce that

$$b_3 = \frac{-(d+\delta)(1+d+\delta)}{c+\gamma}. \tag{18.95}$$

This completes our determination of the parameters for this decomposition, and thus our computations for Theorem 18.1.3.

Table 18.2 Symbols used in Tables 18.3–18.7

Symbol	Meaning
a_1	$bc+d-1$
a_2	$bc-d-1$
α_1	$\beta\gamma+\delta-1$
α_2	$\beta\gamma-\delta-1$
J	$1+\alpha_1\alpha_2 b\beta$
K	$a_1a_2b+\alpha_1\alpha_2\beta$
D	$(d+\delta)^2(1-(d+\delta)^2)^2$
Δ	$(d+\delta)(1+d+\delta)$
$\rho_1,\ \rho_2,\ \rho_3$	The roots of the polynomial $\lambda^3+(1-\delta^2)\lambda^2+\lambda-\frac{\gamma}{b}(1+b\beta(1-\beta\gamma+\delta)(1-\beta\gamma-\delta))$.
$\mu_1,\ \mu_2,\ \mu_3$	The roots of the polynomial $\lambda^3+\lambda^2+(1-(d+\delta)^2)\lambda-(c+\gamma)(ba_1a_2+\beta\alpha_1\alpha_2)$.
M_1	$\begin{tikzcd} & \underline{2} \arrow[dl] \arrow[dr] & \\ \underline{1} \arrow[dr] & & \underline{1} \arrow[dl] \\ & \underline{2} & \end{tikzcd}$

Table 18.3 $\underline{2}\otimes T(b,c,d)$

Relations				Decomposition
$c=0$	$d=0$	$b=0$		M_1
		$b\neq 0$		$\widetilde{T}(\frac{1}{b},b,0)\ni\widetilde{T}(\frac{1}{b},b,0)$
	$d=1$	$b=0$		$\underline{3}\oplus(\underline{2}\to\underline{1})$
		$b\neq 0$		$\underline{3}\oplus(\underline{2}\ni\underline{1})$
	$d=2$	$b=0$		$\underline{3}\oplus(\underline{1}\to\underline{2})$
		$b\neq 0$		$\underline{3}\oplus(\underline{1}\ni\underline{2})$
	$d\neq 0,1,2$			$T(\frac{b(d-1)}{d},0,d-1)\oplus T(\frac{b(d+1)}{d},0,d+1)$
$c\neq 0$	$d=0$	$b=0$		$T(0,c,1)\ni T(0,c,1)$
		$b=\frac{1}{c}$		$T(0,c,1)\oplus T(0,c,1)$
		$b\neq 0,\frac{1}{c}$		$T(b+\sqrt{\frac{b}{c}},c,1)\oplus T(b-\sqrt{\frac{b}{c}},c,1)$
	$d\neq 0,1,2$	$b=0$		$T(0,c,d+1)\oplus T(0,c,d-1)$
		$1-bc+d=0$		$T(0,c,d)\oplus T(0,c,d+1)$
		$1-bc-d=0$		$T(0,c,d-1)\oplus T(0,c,d)$
		$b\neq 0,$ $1-bc\pm d\neq 0$	$bc+d^2=0$	$T(b+\frac{d}{c},c,d+1)\ni T(b+\frac{d}{c},c,d+1)$
			$bc+d^2\neq 0$	$T(b+\frac{d+\sqrt{bc+d^2}}{c},c,d+1)\oplus T(b+\frac{d-\sqrt{bc+d^2}}{c},c,d+1)$

Table 18.4 $\widetilde{T}\Big(b,\frac{1}{b},0\Big)\otimes\widetilde{T}(\beta,\frac{1}{\beta},0)$

Relations	**Decomposition**
$b=-\beta$	$\underline{3}\oplus(\underline{2}\leftarrow\underline{1})\oplus(\underline{1}\leftarrow\underline{2})$
$b\neq-\beta$	$T(\frac{b+\beta}{b\beta},0,0)\oplus\widetilde{T}(\frac{b\beta}{b+\beta},\frac{b+\beta}{b\beta},0)\oplus\widetilde{T}(\frac{b\beta}{b+\beta},\frac{b+\beta}{b\beta},0)$

Table 18.5 $\widetilde{T}\Big(b,\frac{1}{b},0\Big)\otimes T(\beta,\gamma,\delta)$

Relations				**Decomposition**
$\gamma=0$	$\delta=0$	$\beta=-\frac{1}{b}$		$\underline{3}\oplus(\underline{2}\ni(\underline{1}\rightarrow\underline{2}\leftarrow\underline{1}))$
		$\beta\neq-\frac{1}{b}$		$\widetilde{T}(\frac{b}{1+b\beta},\frac{1+b\beta}{b},0)\ni(\widetilde{T}(\frac{b}{1+b\beta},\frac{1+b\beta}{b},0)\oplus T(\frac{1+b\beta}{b},0,0))$
	$\delta=1$			$\widetilde{T}\Big(b,\frac{1}{b},0\Big)\ni(\widetilde{T}\Big(b,\frac{1}{b},0\Big)\oplus T(\frac{1}{b},0,0))$
	$\delta=2$			$\widetilde{T}\Big(b,\frac{1}{b},0\Big)\ni(\widetilde{T}\Big(b,\frac{1}{b},0\Big)\oplus T(\frac{1}{b},0,0))$
	$\delta\neq0,1,2$	$b\beta(1-\delta^2)=-1$		$T(0,0,\delta-1)\oplus T(0,0,\delta)\oplus T(0,0,\delta+1)$
		$b\beta(1-\delta^2)\neq-1$		$T(J,0,\delta-1)\oplus T(J,0,\delta)\oplus T(J,0,\delta+1)$
$\gamma\neq0$	$\delta=0$	$b\beta(1-\beta\gamma)^2=-1$		$T(0,\gamma,-1)\ni(T(0,\gamma,0)\oplus T(0,\gamma,-1))$
		$b\beta(1-\beta\gamma)^2\neq-1$		$T(\rho_1,c,0)\oplus T(\rho_2,c,0)\oplus T(\rho_3,c,0)$
	$\delta\neq0,1,2$	$J=0$		$T(0,\gamma,\delta-1)\oplus T(0,\gamma,\delta)\oplus T(0,\gamma,\delta+1)$
		$J\neq0$	$-\frac{\gamma}{b}J\neq(\delta(\delta+1)(\delta-1))^2$	$T(\rho_1,c,\delta)\oplus T(\rho_2,c,\delta)\oplus T(\rho_3,c,\delta)$
			$-\frac{\gamma}{b}J=(\delta(\delta+1)(\delta-1))^2$	$T(\frac{1-\delta^2}{c},c,\delta)\ni(T(\frac{1-\delta^2}{c},c,\delta)\oplus T(-\frac{\delta^2}{c},c,\delta))$

Table 18.6 $T(b,c,d)\otimes T(\beta,\gamma,\delta)$ $(\gamma=-c)$

Relations			**Decomposition**
$d+\delta=0$	$b=\frac{1}{c},\ d=0,\ \beta=\frac{1}{\gamma},\ \delta=0$		$\underline{3}\oplus(\underline{1}\rightarrow\underline{2})\oplus(\underline{2}\rightarrow\underline{1})$
	$\beta=-b$	$1+bc+b^2c^2-d^2-c\beta-bc^2\beta+c^2\beta^2=0$	$\underline{1}\ni((\underline{2}\ni\underline{1})\oplus\underline{2}\oplus\underline{3})$
		$1+bc+b^2c^2-d^2-c\beta-bc^2\beta+c^2\beta^2\neq0$	$\underline{1}\ni(\underline{2}\ni(\underline{3}\oplus(\underline{1}\leftarrow\underline{2})))$
	$\beta\neq-b$	$1+bc+b^2c^2-d^2-c\beta-bc^2\beta+c^2\beta^2=0$	$\underline{2}\ni(\underline{1}\ni(\underline{3}\oplus(\underline{2}\leftarrow\underline{1})))$
		$1+bc+b^2c^2-d^2-c\beta-bc^2\beta+c^2\beta^2\neq0$	$\widetilde{T}(\frac{1}{K},K,0)\ni(T(K,0,0)\oplus\widetilde{T}(\frac{1}{K},K,0))$
$d+\delta=1$	$d=-1$	$\beta=b$	$\underline{1}\ni((\underline{2}\ni\underline{1})\oplus\underline{2}\oplus\underline{3})$
		$\beta\neq b$	$\underline{1}\ni(\underline{3}\oplus(\underline{2}\rightarrow\underline{1}\leftarrow\underline{2}))$
	$d\neq-1$	$(1-d)b=d\beta$	$\underline{2}\ni(\underline{3}\oplus(\underline{1}\rightarrow\underline{2}\leftarrow\underline{1}))$
		$(1-d)b\neq d\beta$	$\widetilde{T}(\frac{1}{K},K,0)\ni(T(K,0,0)\oplus\widetilde{T}(\frac{1}{K},K,0))$
$d+\delta=2$	$d=1$	$\beta=b$	$\underline{3}\oplus(\underline{2}\ni\underline{1})\oplus(\underline{1}\ni\underline{2})$
		$\beta\neq b$	$\underline{1}\ni(\underline{3}\oplus(\underline{2}\rightarrow\underline{1}\leftarrow\underline{2}))$
	$d\neq1$	$(1+d)b=-d\beta$	$\underline{2}\ni(\underline{1}\ni((\underline{1}\leftarrow\underline{2})\oplus\underline{3}))$
		$(1+d)b\neq-d\beta$	$\widetilde{T}(\frac{1}{K},K,0)\ni(T(K,0,0)\oplus\widetilde{T}(\frac{1}{K},K,0))$
$d+\delta\neq0,1,2$	$b(1-bc+d)(1-bc-d)=-\beta(1+c\beta+\delta)(1+c\beta-\delta)$		$T(0,0,d+\delta-1)\oplus T(0,0,d+\delta)\oplus T(0,0,d+\delta+1)$
	$b(1-bc+d)(1-bc-d)\neq-\beta(1+c\beta+\delta)(1+c\beta-\delta)$		$T(K,0,d+\delta-1)\oplus T(K,0,d+\delta)\oplus T(K,0,d+\delta+1)$
Note: Except where explicitly state this to be the case, we assume that we do *not* have all of the conditions $b=\frac{1}{c}$, $d=0$, $\beta=\frac{1}{\gamma}$, and $\delta=0$.			

Table 18.7 $T(b,c,d)\otimes T(\beta,\gamma,\delta)$ $(\gamma\neq c)$

Relations			**Decomposition**
$b=\dfrac{1}{c},\ d=0,\ \beta=\dfrac{1}{\gamma},\ \delta=0$			$T(0,\dfrac{b+\beta}{b\beta},-1)\oplus T(0,\dfrac{b+\beta}{b\beta},0)\oplus T(0,\dfrac{b+\beta}{b\beta},1)$
$K=0$	$d+\delta=0$		$T(0,c+\gamma,-1)\ni(T(0,c+\gamma,0)\oplus T(0,c+\gamma,-1))$
	$d+\delta\neq 0,1,2$		$T(0,c+\gamma,d+\delta-1)\oplus T(0,c+\gamma,d+\delta)\oplus T(0,c+\gamma,d+\delta+1)$
$K\neq 0$	$D=-K(c+\gamma)$	$bd(c+\gamma)=\sqrt{D}$, $\beta\delta=bd$, $\gamma(d-d^3)=c(\delta-\delta^3)$	$T(\dfrac{\mu_1}{c+\gamma},c+\gamma,d+\delta)\oplus T(\dfrac{\mu_2}{c+\gamma},c+\gamma,d+\delta)\oplus T(\dfrac{\mu_3}{c+\gamma},c+\gamma,d+\delta)$
		otherwise	$T(-\dfrac{\Delta}{c+\gamma},c+\gamma,d+\delta)\ni(T(-\dfrac{\Delta}{c+\gamma},c+\gamma,d+\delta)\oplus T(\dfrac{-1-\Delta}{c+\gamma},c+\gamma,d+\delta))$
	$D\neq -K(c+\gamma)$		$T(\dfrac{\mu_1}{c+\gamma},c+\gamma,d+\delta)\oplus T(\dfrac{\mu_2}{c+\gamma},c+\gamma,d+\delta)\oplus T(\dfrac{\mu_3}{c+\gamma},c+\gamma,d+\delta)$

Note: Except where explicitly state this to be the case, we assume that we do *not* have all of the conditions $b=\dfrac{1}{c}$, $d=0$, $\beta=\dfrac{1}{\gamma}$, and $\delta=0$.

Part IV

Quest for simple modular Lie algebras and Lie superalgebras (S. Bouarroudj, P. Grozman, D. Leites)

Chapter 19

Towards classification of simple finite dimensional modular Lie superalgebras (D. Leites)

Characteristic p is for the time when we retire.

Sasha Beilinson, when we all were young.

19.1. Introduction

The purpose of this transcript of the talk presented in March 2007 at the 3rd International Conference on 21st Century Mathematics 2007, School of Mathematical Sciences (SMS), Lahore, is to state problems, digestible to Ph.D. students (in particularly, the students at SMS) and worth (Ph.D. diplomas) to be studied (without waiting till retirement time), together even with ideas of their solution. In the process of formulating the problems, I'll overview the classical and latest results. To be able to squeeze the material into the prescribed 10 pages, all background is supplied by accessible references: We use standard notations of [FH, S]; for a precise definition of the Cartan prolongation and its generalizations (Cartan-Tanaka-Shchepochkina or CTS-prolongations), see [Shch]; see also [BGL3]–[BGL5], [L1, **?**]. Hereafter $\mathbb{K}$ is an algebraically closed (unless finite) field, $\operatorname{Char}\mathbb{K} = p$.

The works of S. Lie, Killing and Cartan, now classical, completed classification over $\mathbb{C}$ of **simple Lie algebras**

$$\begin{array}{c}\textbf{of finite dimension} \text{ and } \textbf{certain infinite dimensional}\\ \textbf{(of polynomial vector fields, or “vectorial” Lie algebras)}.\end{array} \tag{19.1}$$

In addition to the above two types, there are several more interesting types of simple Lie algebras but they do not contribute to the solution of our problem: **classification of simple finite dimensional modular Lie (super)algebras**, except one: the queer type described below (and, perhaps, examples, for $p = 2$, of the types described in [Ju, Shen1] and their generalizations, if any). Observe that all finite dimensional simple Lie algebras are of the form $\mathfrak{g}(A)$; for their definition embracing the modular case and the classification, see [BGL5].

Lie algebras and Lie superalgebras over fields in characteristic $p > 0$, a.k.a. *modular* Lie (super)algebras, were distinguished in topology in the 1930s. The **simple** Lie algebras drew attention (over finite fields $\mathbb{K}$) as a byproduct of classification of simple finite groups, cf. [St]. Lie **super**algebras, even simple ones, did not draw much attention of mathematicians until their (outstanding) usefulness was observed by physicists in the 1970s. Researchers discovered more and more of new examples of simple modular Lie algebras for decades until Kostrikin and Shafarevich ([KSh]) formulated a **conjecture** embracing all previously found examples for $p > 7$. The **generalized KSh–conjecture** states (for a detailed formulation, convenient to work with, see [?]):

Select a $\mathbb{Z}$-form $\mathfrak{g}_{\mathbb{Z}}$ of every $\mathfrak{g}$ of type[1] (19.1), *take $\mathfrak{g}_{\mathbb{K}} := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{K}$ and its simple subquotient $\mathfrak{si}(\mathfrak{g}_{\mathbb{K}})$ (for the Lie algebras of vector fields, there are several, depending on $\underline{N}$). Together with deforms (the results of deformations*[2]*) of these examples we get in this way all simple finite dimensional Lie algebras over algebraically closed fields if $p > 5$. If $p = 5$, Melikyan's examples*[3] *should be added to the examples obtained by the above method.*

After 30 years of work of several teams of researchers, Block, Wilson, Premet and Strade proved the generalized KSh conjecture for $p > 3$, see [S].

Even before the KSh conjecture was proved, its analog was offered in [KL] for $p = 2$. Although the KL conjecture was, as is clear now, a bit overoptimistic (in plain terms: wrong, as stated), it suggested a way to get such an abundance of examples (to verify which of them are really simple is one of the tasks still open) that Strade [S] cited [KL] as an indication that the case $p = 2$ is too far out of reach by modern means[4]. Still, [KL] made two interesting observations: It pointed at a striking similarity (especially for $p = 2$) between modular Lie algebras and Lie **super**algebras (even over $\mathbb{C}$), and it introduced totally new characters — *Volichenko algebras* (inhomogeneous with respect to parity subalgebras of Lie superalgebras); for the classification of simple Volichenko algebras (finite dimensional and infinite dimensional vectorial) over $\mathbb{C}$, see [LSa2] (where one of the most interesting examples is missed, the version of the proof with repair will be put in `arXiv` soon).

[1] Notice that the modular analog of the polynomial algebra—the algebra of divided powers—*and all prolongs* (vectorial Lie algebras) acquire for $p > 0$ one more (shearing) parameter $\underline{N}$.

[2] It is not clear, actually, if the conventional description of infinitesimal deformations in terms of $H^2(\mathfrak{g};\mathfrak{g})$ can always be applied if $p > 0$. This concerns both Lie algebras and Lie superalgebras (for the arguments, see [LL]); to give the correct (better say, universal) notion is an open problem, but we let it pass for the moment, besides, for $p \neq 2$ and $\mathfrak{g}$ with Cartan matrix, the conventional interpretation is applicable, see [BGL4].

[3] For their description as prolongs, and newly discovered super versions, see [GL3, BGL3].

[4] Contrarywise, **the "punch line" of this talk is: Cartan did not have the modern root technique, but got the complete list of simple Lie algebras; let's use his "old-fashioned" methods: they work!** Conjecture 2 expresses our hope in precise terms. How to *prove* the completeness of the list of examples we will have unearthed is another story.

Recently Strade had published a monograph [S] summarizing the description of newly classified simple finite dimensional Lie algebras over the algebraically closed fields $\mathbb{K}$ of characteristic $p > 3$, and also gave an overview of the "mysterious" examples (due to Brown, Frank, Ermolaev and Skryabin) of simple finite dimensional Lie algebras for $p = 3$ with no counterparts for $p > 3$. Several researchers started afresh to work on the cases where $p = 2$ and 3, and new examples of simple Lie algebras with no counterparts for $p \neq 2, 3$ started to appear ([Ju, GL3, L1], observe that the examples of [GG, Lin1] are erroneous as observed in MathRevies and [?], respectively). The "mysterious" examples of simple Lie algebras for $p = 3$ were interpreted as vectorial Lie algebras preserving certain distributions ([GL3]).

While writing [GL3] we realized, with considerable dismay, that there are reasons to put to doubt the universal applicability of the conventional definitions of the enveloping algebra $U(\mathfrak{g})$ (and its restricted version) of a given Lie algebra $\mathfrak{g}$, and hence doubt in applicability of the conventional definitions of Lie algebra representations and (co)homology to the modular case, cf. [LL]. But even accepting conventional definitions, there are plenty of problems to be solved before one will be able to start writing the proof of classification of simple modular Lie algebras, to wit: describe irreducible representations (as for vectorial Lie superalgebras, see [GLS2]), decompose the tensor product of irreducible representations into indecomposables, cf. [Cla], and many more; for a review, see [GL].

Classification of simple Lie **super**algebras for $p > 0$ and the study of their representations are of independent interest. A conjectural list of simple finite dimensional Lie superalgebras over an algebraically closed fields $\mathbb{K}$ for $p > 5$, known for some time, was recently cited in [BjL]:

Conjecture (Super KSh, $p > 5$). *Apply the steps of the KSh conjecture to the simple complex Lie* **super***algebras $\mathfrak{g}$ of types* (19.1). *The examples thus obtained exhaust all simple finite dimensional Lie superalgebras over algebraically closed fields if $p > 5$.*

The examples obtained by this procedure will be referred to as *KSh-type Lie superalgebras*. The first step towards obtaining the list of KSh-type Lie superalgebras is classification of simple Lie superalgebras of types (19.1) over $\mathbb{C}$. This is done by I.Shchepochkina and me; for summaries with somewhat different emphases, and proof, see [K2, Ka1, ?].

For a classification of finite dimensional simple modular Lie algebras with Cartan matrix, see [WK, KWK]. For a classification of finite dimensional simple modular Lie superalgebras with Cartan matrix, see [BGL5]. Not all finite dimensional Lie superalgebras over $\mathbb{C}$ are of the form $\mathfrak{g}(A)$; in addition to them, there are also queer types described below, and even simple vectorial.

I am sure that the same ideas of Block and Wilson that proved classification of simple *restricted* Lie algebras for $p > 5$ will work, if $p > 5$, *mutatis mutandis*, for Lie superalgebras and ideas of Premet and Strade will embrace the non-restricted superalgebras as well; although the definition of *restrictedness* and even of the Lie superalgebra itself acquire more features, especially for $p = 2$.

Here I will describe the cases $p \leq 5$ where the situation is different and suggest *another*, different from KSh, way to get simple examples.

19.2. How to construct simple Lie algebras and superalgebras

19.2.1. How to construct simple Lie algebras if $p = 0$. Let us recall how Cartan used to construct simple $\mathbb{Z}$-graded Lie algebras over $\mathbb{C}$ of polynomial growth [C] and finite depth. Now that they are classified (for examples of infinite depth, see [K3]), we know that, *all of them can be endowed with a* $\mathbb{Z}$*-grading* $\mathfrak{g} = \mathop{\oplus}\limits_{-d\leq i} \mathfrak{g}_i$ *of depth* $d = 1$ *or* 2 *so that* $\mathfrak{g}_0$ *is a simple Lie algebra* $\mathfrak{s}$ *or its trivial central extension* $\mathfrak{cs} = \mathfrak{s} \oplus \mathfrak{c}$*, where* $\mathfrak{c}$ *is a 1-dimensional center. Moreover, simplicity of* $\mathfrak{g}$ *requires* $\mathfrak{g}_{-1}$ *to be an irreducible* $\mathfrak{g}_0$*-module that generates* $\mathfrak{g}_- := \mathop{\oplus}\limits_{i<0} \mathfrak{g}_i$ *and* $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$.

Yamaguchi's theorem [Y], reproduced in [GL3, BjL], states that *for almost all simple finite dimensional Lie algebras* $\mathfrak{g}$ *over* $\mathbb{C}$ *and their* $\mathbb{Z}$*-gradings* $\mathfrak{g} = \mathop{\oplus}\limits_{-d\leq i} \mathfrak{g}_i$*, the generalized Cartan prolong of* $\mathfrak{g}_{\leq} = \mathop{\oplus}\limits_{-d\leq i\leq 0} \mathfrak{g}_i$ *is isomorphic to* $\mathfrak{g}$*, the rare exceptions being two of the four series of simple vectorial algebras; the other two series being* **partial** *prolongs (perhaps, after factorization modulo center).*

For illustration, we construct simple Lie algebras of type (19.1) over $\mathbb{C}$ by induction:

<u>**Depth** $d = 1$</u>. Here we use either usual or partial Cartan prolongations.

1) we start with 1-dimensional $\mathfrak{c}$, so $\dim \mathfrak{g}_{-1} = 1$ due to irreducibility. The complete prolong is isomorphic to $\mathfrak{vect}(1)$, the partial one to $\mathfrak{sl}(2)$.

2) Take $\mathfrak{g}_0 = \mathfrak{csl}(2) = \mathfrak{gl}(2)$ and its irreducible module $\mathfrak{g}_{-1}$. The component $\mathfrak{g}_1$ of the Cartan prolong is nontrivial only if $\mathfrak{g}_{-1}$ is $R(\varphi_1)$ or $R(2\varphi_1)$, where φ_i is the ith fundamental weight of the simple Lie algebra $\mathfrak{g}$ and $R(w)$ is the irreducible representation with highest weight w.

2a) If $\mathfrak{g}_{-1}$ is $R(\varphi_1)$, the component $\mathfrak{g}_1$ consists of two irreducible submodules, say $\mathfrak{g}_1'$ or $\mathfrak{g}_1''$. We can take any one of them or both; together with $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ this generates $\mathfrak{sl}(3)$ or $\mathfrak{svect}(2) \Subset \mathfrak{d}$, where $\mathfrak{d}$ is spanned by an outer derivation, or $\mathfrak{vect}(2)$, respectively.

2b) If $\mathfrak{gl}(2) \simeq \mathfrak{co}(3) \simeq \mathfrak{csp}(2)$-module $\mathfrak{g}_{-1}$ is $R(2\varphi_1)$, then $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \simeq \mathfrak{o}(5) \simeq \mathfrak{sp}(4)$.

3) Induction: Take $\mathfrak{g}_0 = \mathfrak{csl}(n) = \mathfrak{gl}(n)$ and its irreducible module $\mathfrak{g}_{-1}$. The component $\mathfrak{g}_1$ of the Cartan prolong is nontrivial only if $\mathfrak{g}_{-1}$ is $R(\varphi_1)$ or $R(2\varphi_1)$ or $R(\varphi_2)$.

3a) If $\mathfrak{g}_{-1} = R(\varphi_1)$, then $\mathfrak{g}_1$ consists of two irreducible submodules, $\mathfrak{g}_1'$ or $\mathfrak{g}_1''$. Take any of them or both; together with $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ this generates $\mathfrak{sl}(n+1)$ or $\mathfrak{svect}(n) \Subset \mathfrak{d}$, where $\mathfrak{d}$ is spanned by an outer derivation, or $\mathfrak{vect}(n)$, respectively.

3b) If $\mathfrak{g}_{-1} = R(2\varphi_1)$, then $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \simeq \mathfrak{sp}(2n)$.

3c) If $\mathfrak{g}_{-1} = R(\varphi_2)$, then $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \simeq \mathfrak{o}(2n)$.

4) The induction with $\mathfrak{g}_0 = \mathfrak{co}(2n-1)$-module $R(\varphi_1)$ returns $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \simeq \mathfrak{o}(2n+1)$. Observe that $\mathfrak{sl}(4) \simeq \mathfrak{o}(6)$. The induction with $\mathfrak{g}_0 = \mathfrak{co}(2n)$-module $R(\varphi_1)$ returns $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \simeq \mathfrak{o}(2n+2)$. (We have obtained $\mathfrak{o}(2n)$ twice; analogously, there many ways to obtain other simple Lie algebras as prolongs.)

5) The $\mathfrak{g}_0 = \mathfrak{sp}(2n)$-module $\mathfrak{g}_{-1} = R(\varphi_1)$ yields the Lie algebra $\mathfrak{h}(2n)$ of Hamiltonian vector fields.

<u>$\mathfrak{e}(6)$, $\mathfrak{e}(7)$</u>. The $\mathfrak{g}_0 = \mathfrak{co}(10)$-module $\mathfrak{g}_{-1} = R(\varphi_1)$ yields $\mathfrak{e}(6)$; the $\mathfrak{g}_0 = \mathfrak{ce}(6)$-module $\mathfrak{g}_{-1} = R(\varphi_1)$ yields $\mathfrak{e}(7)$.

<u>**Depth** $d = 2$</u>. Here we need **generalized** prolongations, see [Shch]. Again there are just a few algebras $\mathfrak{g}_0$ and $\mathfrak{g}_0$-modules $\mathfrak{g}_{-1}$ for which $\mathfrak{g}_1 \neq 0$ and $\mathfrak{g} = \oplus \mathfrak{g}_i$ is simple:

<u>$\mathfrak{g}(2)$; $\mathfrak{f}(4)$; $\mathfrak{e}(8)$</u>. These Lie algebras correspond to the prolongations of their non-positive part (with $\mathfrak{g}_0$ being isomorphic to $\mathfrak{gl}(2)$; $\mathfrak{o}(6)$ or $\mathfrak{sp}(6)$; $\mathfrak{e}(7)$ or $\mathfrak{o}(14)$, respectively) in the following $\mathbb{Z}$-gradings. Let the nodes of the Dynkin graph of $\mathfrak{g}$ be rigged out with the coefficients of linear dependence of the maximal root with respect to simple ones. If any end node is rigged out with a 2, mark it (mark only one node even if several are rigged out with 2's) and set the degrees of the Chevalley generators to be:

$$\deg X_i^{\pm} = \begin{cases} \pm 1 & \text{if the } i\text{th node is marked} \\ 0 & \text{otherwise.} \end{cases} \tag{19.2}$$

<u>$\mathfrak{k}(2n+1)$</u>. The cases where $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ is simple and of infinite dimension correspond to the prolongations of the non-positive part of $\mathfrak{sp}(2n+2)$ in the $\mathbb{Z}$-grading (19.2) with the last node marked. Then $\mathfrak{g}_0 = \mathfrak{csp}(2n)$, $\mathfrak{g}_{-1} = R(\varphi_1)$ and $\mathfrak{g}_{-1}$ is the trivial $\mathfrak{g}_0$-module. In these cases, $(\mathfrak{g}_-, \mathfrak{g}_0)_* = \mathfrak{k}(2n+1)$.

19.2.2. Superization.

19.2.2.1. Queerification. This is the functor $Q : A \longrightarrow Q(A) := A[\varepsilon]$, where

$$p(\varepsilon) = \bar{1},\ \varepsilon^2 = -1 \text{ and } \varepsilon a = (-1)^{p(a)} a\varepsilon \text{ for any } a \in A.$$

We set $\mathfrak{q}(n) = Q(\mathfrak{gl}(n))$.

19.2.2.2. Definition of Lie superalgebras for $p = 2$. A Lie superalgebra for $p = 2$ as a superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that $\mathfrak{g}_{\bar{0}}$ is a Lie algebra, $\mathfrak{g}_{\bar{1}}$ is an $\mathfrak{g}_{\bar{0}}$-module (made into the two-sided one by symmetry; more exactly, by anti-symmetry, but if $p = 2$, it is the same) and on $\mathfrak{g}_{\bar{1}}$ a *squaring* (roughly speaking, the halved bracket) is defined

$$\begin{gathered} x \mapsto x^2 \quad \text{such that } (ax)^2 = a^2x^2 \text{ for any } x \in \mathfrak{g}_{\bar{1}} \text{ and } a \in \mathbb{K}, \text{ and} \\ (x+y)^2 - x^2 - y^2 \text{ is a bilinear form on } \mathfrak{g}_{\bar{1}} \text{ with values in } \mathfrak{g}_{\bar{0}}. \end{gathered} \tag{19.3}$$

For any $x, y \in \mathfrak{g}_{\bar{1}}$, we set

$$[x, y] := (x+y)^2 - x^2 - y^2. \tag{19.4}$$

We also assume, as usual, that

if $x, y \in \mathfrak{g}_{\bar 0}$, then $[x, y]$ is the bracket on the Lie algebra;
if $x \in \mathfrak{g}_{\bar 0}$ and $y \in \mathfrak{g}_{\bar 1}$, then $[x, y] := l_x(y) = -[y, x] = -r_x(y)$, where l and r are the left and right $\mathfrak{g}_{\bar 0}$-actions on $\mathfrak{g}_{\bar 1}$, respectively.

The *Jacobi identity* involving odd element has now the following form:

$$[x^2, y] = [x, [x, y]] \text{ for any } x \in \mathfrak{g}_{\bar 1}, y \in \mathfrak{g}. \tag{19.5}$$

Conjecture (Amended KL = Super KSh, $p > 0$)**.** *For $p > 0$, to get all* **$\mathbb{Z}$-graded** *simple finite dimensional examples of Lie algebras and Lie superalgebras:*

(a) apply the KSh procedure to every simple Lie algebra of type (1) over $\mathbb{C}$ (if $p = 2$, apply the KSh procedure also to every simple Lie superalgebra of of type (1) over $\mathbb{C}$ and their simple Volichenko subalgebras described in [LSa2]),

*(b) if $p = 2$, apply queerification (as in [*__?__*]) to the results of (a);*

(c) if $p = 2$, take Jurman's examples [Ju] (and generalizations of the same construction, if any: It looks like a specific $p = 2$ non-super version of the queerification);

(d) take the non-positive part of every simple (up to center) finite dimensional $\mathbb{Z}$-graded algebra obtained at steps (a)–(c) and (for $p = 5$, 3 and 2) the exceptional ones of the form $\mathfrak{g}(A)$ listed in [BGL5], consider its complete and partial[5] *prolongs and distinguish their simple subquotients.*

To get non-graded examples, we have to take as a possible $\mathfrak{g}_0$ deforms of the simple algebras obtained at steps (a)–(d) and Shen's "variations" [Shen1] (unless they can be interpreted as deforms of the algebras obtained at earlier steps).

For preliminary results, see [GL3, BjL], [BGL3]–[BGL5], [ILL, **?**]. (For $p = 3$ and Lie algebras, this is how Grozman and me got an interpretation of all the "mysterious" exceptional simple vectorial Lie algebras known before [GL3] was published; we also found two (if not three) series of new simple algebras.) Having obtained a supply of such examples, we can sit down to compute certain cohomology in order to describe their deformations (provided we will be able to understand what we are computing, cf. footnote 2); for the already performed, see [KKCh, KuCh, Che, BGL4].

19.3. Further details

19.3.1. How to construct finite dimensional simple Lie algebras if $p \geq 5$. Observe that although in the modular case there is a wider variety of pairs $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ yielding nontrivial prolongs than for $p = 0$ (for the role of $\mathfrak{g}_0$ we can now take vectorial Lie algebras or their central extensions), *a posteriori* we know that we can always confine ourselves to the same pairs $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ as for $p = 0$. Melikyan's example looked as a deviation from the pattern, but Kuznetsov's observation [Ku1] elaborated in [GL3] shows that for $p \geq 5$ all is the same. Not so if $p \leq 3$:

[5] This term is too imprecise at the moment: it embraces Frank and Ermolaev Lie algebras, various exceptional Lie superalgebras ([BGL8, BGL7]).

19.3.2. New simple finite dimensional Lie algebras for $p = 3$. In [S], Strade listed known to him at that time examples of simple finite dimensional Lie algebras for $p = 3$. The construction of such algebras is usually subdivided into the following types and deforms of these types:

(1) algebras with Cartan matrix CM (sometimes encodable by Dynkin graphs, cf. [S, BGL5]),

(2) algebras of vectorial type (meaning that they have more roots of one sign than of the other with respect to a partition into positive and negative roots).

Case (1) was solved in [WK, KWK].

Conjecture 2 suggests to consider certain $\mathbb{Z}$-graded prolongs $\mathfrak{g}$. For Lie algebras and $p = 3$, Kuznetsov described various restrictions on the 0-th component of $\mathfrak{g}$ and the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ (for partial summary, see [GK, Ku1, Ku2], [BKK] and a correction in [GL3]). **What are these restrictions for Lie algebras for $p = 2$? What are they for Lie superalgebras for any $p > 0$?**

19.3.3. Exceptional simple finite dimensional Lie superalgebras for $p \leq 5$. Elduque investigated which spinor modules over orthogonal algebras can serve as the odd part of a simple Lie superalgebra and discovered an exceptional simple Lie superalgebra for $p = 5$. Elduque also superized the Freudenthal Magic Square and expressed it in a new way, and his approach yielded nine new simple (exceptional as we know now thanks to the classification [BGL5]) finite dimensional Lie superalgebras for $p = 3$, cf. [CE, El1, CE2]. These Lie superalgebras possess Cartan matrices (CM's) and we described all CMs and presentations of these algebras in terms of Chevalley generators, see [BGL5] and references in it. In [BGL5] 12 more examples of exceptional simple Lie superalgebras are discovered; in [BGL3], we considered some of their "most promising" (in terms of prolongations) $\mathbb{Z}$-gradings and discovered several new series of simple vectorial Lie superalgebras.

19.3.4. New simple finite dimensional Lie algebras and Lie superalgebras for $p = 2$. Lebedev [L1, **?**] offered a new series of examples of simple orthogonal Lie algebras without CM. Together with Iyer, we constructed their prolongations, missed in [Lin2], see [ILL]; queerifications of these orthogonal algebras (and of several more serial and exceptional Lie algebras, provided they are restricted) are totally new types of examples of simple Lie superalgebras. CTS prolongs of **some**[6] of these superalgebras and examples found in [BGL5] are considered in [BGL7].

[6] We are unable to CTS the superalgebras of dimension > 40 on computers available to us, whereas we need to be able to consider at least 250.

19.3.5. Conclusion. Passing to Lie superalgebras we see that even their definition, as well as that of their prolongations, are not quite straightforward for $p = 2$, but, having defined them ([LL, **?**]), it remains to apply the above-described procedures to get at least a supply of examples. To prove the completeness of the stock of examples for any p is a much more difficult task that requires serious preliminary study of the representations of the examples known and to be obtained — more topics for Ph.D. theses.

Chapter 20

Classification of simple finite dimensional modular Lie superalgebras with Cartan matrix (S. Bouarroudj, P. Grozman, D. Leites)

20.1. Introduction

The ground field $\mathbb{K}$ is algebraically closed of characteristic $p > 0$. (Algebraic closedness of $\mathbb{K}$ is only needed in the quest for parametric families.)

20.1.1. Main results. First of all, we clarify several key notions (of Lie superalgebra in characteristic 2, of Lie superalgebra with Cartan matrix, of weights and roots, and of restricted Lie (super)algebra). These clarifications are obtained by/with A. Lebedev.

Then we give an algorithm that, under certain, conjecturally immaterial, hypotheses, produces the complete list of all finite dimensional Lie algebras and Lie superalgebras possessing symmetrizable indecomposable Cartan matrices A, i.e., of the form $\mathfrak{g}(A)$. Our proof follows the lines of the proof of Weisfeiler and Kac, sketched for Lie algebras in [WK].

Observe that if a given indecomposable Cartan matrix A is invertible, the Lie (super)algebra $\mathfrak{g}(A)$ is simple, otherwise $\mathfrak{g}^{(1)}(A)/\mathfrak{c}$ — the quotient of its derived algebra modulo center — is simple if $\operatorname{rk} A > 1$; except for $p = 0$ and Lie algebras, this subtlety is never mentioned causing confusion: The conventional sloppy practice is to refer to the simple Lie (super)algebra $\mathfrak{g}^{(1)}(A)/\mathfrak{c}$ as "possessing a Cartan matrix" although it does not possess one.

The results for $p > 5$, $p = 5$, 3 and 2 are summarized in §§20.6, 20.7, 20.8 and 20.9, respectively.

For the new Lie superalgebras $\mathfrak{g}$ we have discovered ($\mathfrak{e}(6,1)$, $\mathfrak{e}(6,6)$, $\mathfrak{e}(7,1)$, $\mathfrak{e}(7,6)$, $\mathfrak{e}(7,7)$, $\mathfrak{e}(8,1)$, $\mathfrak{e}(8,8)$); $\mathfrak{bgl}(4;\alpha)$ and $\mathfrak{bgl}(3;\alpha)$ for $p = 2$; $\mathfrak{el}(5;3)$ for $p = 3$; and $\mathfrak{brj}(2;5)$ for $p = 5$), and for $\mathfrak{brj}(2;3)$ discovered in [El1], we list all Cartan matrices; for all the new Lie superalgebras of the form $\mathfrak{g}(A)$ (and the "old" $\mathfrak{brj}(2;3)$), we describe their structure: We identify $\mathfrak{g}_{\bar 0}$, and $\mathfrak{g}_{\bar 1}$ as $\mathfrak{g}_{\bar 0}$-module. *A posteriori* we see that for each finite dimensional Lie superalgebra $\mathfrak{g}(A)$

with indecomposable Cartan matrix, the module $\mathfrak{g}_{\bar 1}$ is a completely reducible $\mathfrak{g}_{\bar 0}$-module. [1)]

Elduque interpreted most of the exceptional (when their exceptional nature was only conjectured; now it is proved) simple Lie superalgebras in characteristic 3 [CE2] in terms of super analogs of division algebras and collected them into a Supermagic Square (an analog of Freudenthal's Magic Square); the rest of the exceptional examples for $p = 3$ and $p = 5$, not entering the Elduque Supermagic Square (the ones described here for the first time) are, nevertheless, somehow affiliated to the Elduque Supermagic Square [El3].

Very interesting, we think, is the situation in characteristic 2. *A posteriori* we see that the list of Lie superalgebras in characteristic 2 of the form $\mathfrak{g}(A)$ with an indecomposable matrix A is as follows: Take the classification of finite dimensional Lie algebras in characteristic 2 of the form $\mathfrak{g}(A)$ with indecomposable and symmetrizable Cartan matrices ([WK, Br3, KWK]) and declare some of its Chevalley generators odd (the corresponding diagonal elements of A should be changed accordingly $\bar 0$ to 0 and $\bar 1$ to 1). Do this for any of its inequivalent Cartan matrices and any distribution of parities. Construct Lie superalgebras $\mathfrak{g}(A)$ from these generators by the rules (20.14) explicitly described in [**?**, BGL1, BGL2]. In this way we obtain all finite dimensional Lie superalgebras with symmetrizable indecomposable Cartan matrices (they are simple if the Cartan matrix is invertible, otherwise the simple subquotient $\mathfrak{g}^{(1)}(A)/\mathfrak{c}$ is simple). In this way a given orthogonal Lie algebra may turn into ortho-orthogonal or periplectic Lie superalgebra and the three exceptional Lie algebras of $\mathfrak{e}$ type turn into seven non-isomorphic Lie superalgebras of $\mathfrak{e}$ type, whereas the $\mathfrak{wk}$ type algebras turn into $\mathfrak{bgl}$ type superalgebras.

We also classify here *all* inequivalent Cartan matrices A for each given Lie (super)algebra $\mathfrak{g}(A)$. Although the number of inequivalent Cartan matrices grows with the size of A, it is easy to list all possibilities for serial Lie (super)algebras. Certain exceptional Lie superalgebras have dozens of inequivalent Cartan matrices; nevertheless, there are at least two reasons to list them:

1) To classify all $\mathbb{Z}$-gradings of a given $\mathfrak{g}(A)$ (in particular, inequivalent Cartan matrices) is a very natural problem. Besides, sometimes the knowledge of the best, for the occasion, $\mathbb{Z}$-grading is important, cf. [RU] (all simple roots non-isotropic), [LSS] (all simple roots odd); for computations "by hand" the cases where only one simple root is odd are useful. In particular, the defining relations between the natural (Chevalley) generators of $\mathfrak{g}(A)$ are of completely different form for inequivalent $\mathbb{Z}$-gradings and this is used in [RU].

2) Distinct $\mathbb{Z}$-gradings yield distinct Cartan-Tanaka-Shchepochkina (CTS) prolongs (vectorial Lie (super)algebras). So to classify them is vital, for example, in the quest for simple vectorial Lie (super)algebras.

[1] For simple subquotient $\mathfrak{g} = \mathfrak{g}^{(1)}(A)/\mathfrak{c}$ of $\mathfrak{g}(A)$, complete reducibility of the $\mathfrak{g}_{\bar 0}$-module $\mathfrak{g}_{\bar 1}$ is sometimes violated.

Finally, we list the Lie superalgebras of fixed points of automorphisms corresponding to the symmetries of Dynkin diagrams and describe their simple subquotients. In characteristic 0 this is the way all Lie algebras whose Dynkin diagrams has multiple bonds (roots of different lengths) are obtained. Since, for $p = 2$, there are no multiple bonds or roots of different length (at least, this notion is not invariant), it is clear that this is the way to obtain something new, although, perhaps, not simple. Lemma 2.2 in [FrG] implicitly describes the ideal in the Lie algebra of fixed points of an automorphism of a Lie algebra, but one still has to describe the Lie algebra of fixed points explicitly. This explicit answer is given in the last section. No new simple Lie (super)algebras are obtained.

20.1.2. Related results. 1) For explicit presentations in terms of (the analogs of) Chevalley generators of the Lie algebras and superalgebras listed here, see [**?**] for $p = 2$ and [BGL1, BGL2] for $p = 3$, and 5. In addition to Serre-type relations there are always more complicated relations.

2) For deformations of the Lie (super)algebras of the form $\mathfrak{g}(A)$ with indecomposable and symmetrizable Cartan matrix A (and their simple subquotients $\mathfrak{g}^{(1)}(A)/\mathfrak{c}$), see [BGL4]. Observe that *whereas if $p > 3$, then the Lie (super)algebras with Cartan matrices of the same types that exist over $\mathbb{C}$ are either rigid or have deforms which also possess Cartan matrices, this is not the case with the simple modular Lie (super)algebras if $p = 3$ or 2.*

3) For generalized CTS prolongs of the simple Lie (super)algebras of the form $\mathfrak{g}(A)$, and the simple subquotients of such prolongs, see [BGL3, BGL7].

4) With restricted Lie algebras one can associate algebraic groups; analogously, with restricted Lie superalgebras one can associate algebraic supergroups. For this and other results of Lebedev's Ph.D. thesis pertaining to the classification of simple modular Lie superalgebras, see [LCh].

20.2. On Lie superalgebra in characteristic 2

20.2.1. Examples: Lie superalgebras preserving non-degenerate forms. Lebedev investigated various types of equivalence of bilinear forms for $p = 2$; we just recall the verdict and say that *two bilinear forms B and B' on a superspace V are equivalent* if there is an even non-degenerate linear map $M : V \to V$ such that

$$B'(x, y) = B(Mx, My) \quad \text{for all} \quad x, y \in V. \tag{20.1}$$

We fix some basis in V and identify a bilinear form with its *Gram matrix* in this basis; let us also identify any linear operator on V with its matrix. Then two bilinear forms (rather supermatrices) are equivalent if there is an even invertible matrix M such that

$$B' = MBM^T, \quad \text{where } T \text{ is for transposition.} \tag{20.2}$$

We often use the following matrices

$$J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \quad \Pi_n = \begin{cases} \begin{pmatrix} 0 & 1_k \\ 1_k & 0 \end{pmatrix} & \text{if } n = 2k, \\ \begin{pmatrix} 0 & 0 & 1_k \\ 0 & 1 & 0 \\ 1_k & 0 & 0 \end{pmatrix} & \text{if } n = 2k+1, \end{cases} \tag{20.3}$$

Let $J_{n|n}$ and $\Pi_{n|n}$ be the same as J_{2n} and Π_{2n} but considered as supermatrices.

Lebedev proved that, with respect to the above natural equivalence of forms (20.2), every even symmetric non-degenerate form on a superspace of dimension $n_{\bar 0}|n_{\bar 1}$ over a perfect field of characteristic 2 is equivalent to a form of the shape (here: $i = \bar 0$ or $\bar 1$ and each n_i may equal to 0)

$$B = \begin{pmatrix} B_{\bar 0} & 0 \\ 0 & B_{\bar 1} \end{pmatrix}, \quad \text{where } B_i = \begin{cases} 1_{n_i} & \text{if } n_i \text{ is odd;} \\ \text{either } 1_{n_i} \text{ or } \Pi_{n_i} & \text{if } n_i \text{ is even.} \end{cases}$$

In other words, the bilinear forms with matrices 1_n and Π_n are equivalent if n is odd and non-equivalent if n is even. The Lie superalgebra preserving B — by analogy with the orthosymplectic Lie superalgebras $\mathfrak{osp}$ in characteristic 0 we call it *ortho-orthogonal* and denote $\mathfrak{oo}_B(n_{\bar 0}|n_{\bar 1})$ — is spanned by the supermatrices which in the standard format are of the form

$$\begin{pmatrix} A_{\bar 0} & B_{\bar 0}C^T B_{\bar 1}^{-1} \\ C & A_{\bar 1} \end{pmatrix}, \quad \begin{array}{c} \text{where } A_{\bar 0} \in \mathfrak{o}_{B_{\bar 0}}(n_{\bar 0}),\ A_{\bar 1} \in \mathfrak{o}_{B_{\bar 1}}(n_{\bar 1}), \text{ and} \\ C \text{ is arbitrary } n_{\bar 1} \times n_{\bar 0} \text{ matrix.} \end{array}$$

Since, as is easy to see,

$$\mathfrak{oo}_{\Pi I}(n_{\bar 0}|n_{\bar 1}) \simeq \mathfrak{oo}_{I\Pi}(n_{\bar 1}|n_{\bar 0}),$$

we do not have to consider the Lie superalgebra $\mathfrak{oo}_{\Pi I}(n_{\bar 0}|n_{\bar 1})$ separately unless we study Cartan prolongations where the difference between these two incarnations of one algebra is vital.

For an odd symmetric form B on a superspace of dimension $(n_{\bar 0}|n_{\bar 1})$ over a field of characteristic 2 to be non-degenerate, we need $n_{\bar 0} = n_{\bar 1}$, and every such form B is equivalent to $\Pi_{k|k}$, where $k = n_{\bar 0} = n_{\bar 1}$. This form is preserved by linear transformations with supermatrices in the standard format of the shape

$$\begin{pmatrix} A & C \\ D & A^T \end{pmatrix}, \quad \text{where } A \in \mathfrak{gl}(k),\ C \text{ and } D \text{ are symmetric } k \times k \text{ matrices1.} \tag{20.4}$$

As over $\mathbb{C}$ or $\mathbb{R}$, the Lie superalgebra $\mathfrak{pe}(k)$ of supermatrices (20.4) (recall that $p = 2$) will be referred to as *periplectic*, as A. Weil suggested, and denote it by $\mathfrak{pe}_B(k)$ or just $\mathfrak{pe}(k)$.

The fact that two bilinear forms are inequivalent does not, generally imply that the Lie (super)algebras that preserve them are not isomorphic. Lebedev proved that for the non-degenerate symmetric forms this is, however, so, and therefore we have the following types of non-isomorphic Lie (super) algebras:

without CM	with CM
$\mathfrak{o}_I(2n)$; $\mathfrak{oo}_{II}(2n+1\|2m+1)$	$\mathfrak{o}_\Pi(2n)$, $\mathfrak{o}_I(2n+1)$;
$\mathfrak{oo}_{II}(2n\|2m)$, $\mathfrak{oo}_{I\Pi}(2n\|2m)$, $\mathfrak{oo}_{II}(2n+1\|2m)$	$\mathfrak{oo}_{\Pi\Pi}(2n\|2m)$, $\mathfrak{oo}_{I\Pi}(2n+1\|2m)$.

(20.5)

20.3. A careful study of an example

Let $p = 2$ and consider the Lie superalgebra $\mathfrak{pe}(k)$ (the situation with $\mathfrak{o}_\Pi(2k)$ and $\mathfrak{oo}_{\Pi\Pi}(2k_{\bar 0}|2k_{\bar 1})$ is the same). For the Cartan matrix we take (see Table §12)

$$A = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots \\ \cdots & * & 1 & 1 \\ \cdots & 1 & * & 0 \\ \cdots & 1 & 0 & \bar 0 \end{pmatrix} \tag{20.6}$$

The Lie superalgebra $\mathfrak{pe}^{(i)}(k)$ consists of supermatrices of the form

$$\begin{pmatrix} B & C \\ D & B^T \end{pmatrix},$$

where

$$\begin{array}{l} \text{for } i = 0, \text{ we have } B \in \mathfrak{gl}(k),\ C, D \text{ are symmetric;} \\ \text{for } i = 1, \text{ we have } B \in \mathfrak{gl}(k),\ C, D \text{ are symmetric zero-diagonal;} \\ \text{for } i = 2, \text{ we have } B \in \mathfrak{sl}(k),\ C, D \text{ are symmetric zero-diagonal.} \end{array} \tag{20.7}$$

We expect (by analogy with the orthogonal Lie algebras in characteristic $\neq 2$) that

$$\begin{array}{ll} e_i^+ = E^{i,i+1} + E^{k+i+1,k+i}; \quad e_i^- = E^{i+1,i} + E^{k+i,k+i+1} & \text{for } i = 1, \dots, k-1; \\ e_k^+ = E^{k-1,2k} + E^{k,2k-1}; \quad e_k^- = E^{2k-1,k} + E^{2k,k-1}. & \end{array} \tag{20.8}$$

Let us first consider the (simpler) <u>case of k odd</u>. Then $\operatorname{rk} A = k-1$ since the sum of the last two rows is zero. Let us start with the simple algebra $\mathfrak{pe}^{(2)}(k)$. The Cartan subalgebra (i.e., the subalgebra of diagonal matrices) is $(k-1)$-dimensional because the elements $[e_1^+, e_1^-], \dots, [e_{k-1}^+, e_{k-1}^-]$ are linearly independent, whereas $[e_k^+, e_k^-] = [e_{k-1}^+, e_{k-1}^-]$. Thus, we should first find a non-trivial central extension satisfying the condition

$$[e_k^+, e_k^-] + [e_{k-1}^+, e_{k-1}^-] = z. \tag{20.9}$$

Elucidation: The values of $e_i^\pm$ in (20.8) are what we *expect* them to be from their $p = 0$ analogs. But from the definition of CMLSA we see that the algebra must have a center z equal to the above expression (20.12). Thus, the CMLSA is not $\mathfrak{pe}^{(2)}(k)$ but is spanned by the central extension of $\mathfrak{pe}^{(2)}(k)$ plus the grading operator defined from (12.24). The extension $\mathfrak{pec}(2,k)$ described in (2.76) satisfies this condition.

Now let us choose B to be $(0,\dots,0,1)$. Then we need to add to the algebra a grading operator d such that

$$\begin{array}{l}[d, e_i^\pm] = 0 \text{ for all } i = 1,\dots,k-1;\\ [d, e_k^\pm] = e_k^\pm;\\ d \text{ commutes with all diagonal matrices.}\end{array} \tag{20.10}$$

The matrix $I_0 = \mathrm{diag}(1_k, 0_k)$ satisfies all these conditions. Thus, the corresponding CMLSA is

$$\mathfrak{pec}(2,k) \Subset \mathbb{K}I_0. \tag{20.11}$$

Remark. Recall that sometimes ideals of CM Lie (super)algebras that do not contain the outer grading operator(s) are needed, cf. subsect. 12.3.1. These ideals, such as $\mathfrak{pec}(2,k)$ or $\mathfrak{sl}(n|n)$, do not have Cartan matrix.

Now let us consider the case of k even. Then the simple algebra is $\mathfrak{pe}^{(2)}(k)/(\mathbb{K}1_{2k})$. The Cartan matrix is of rank $k-2$:

$$\begin{array}{l}\text{(a) the sum of the last two rows is zero;}\\ \text{(b) the sum of all the rows with odd numbers is zero.}\end{array} \tag{20.12}$$

The condition (20.12a) gives us the same central extension and the same grading operator an in the previous case.

To satisfy condition (20.12b), we should find a non-trivial central extension such that

$$z = \sum_{i \text{ is odd}} [e_i^+, e_i^-].$$

(This formula follows from (12.23) and the 2nd equality in (20.12).) But we can see that, in $\mathfrak{pe}^{(2)}(k)$, we have

$$\sum_{i \text{ is odd}} [e_i^+, e_i^-] = \sum_{i \text{ is odd}} (E^{i,i} + E^{i+1,i+1} + E^{k+i,k+i} + E^{k+i+1,k+i+1}) = 1_{2k}.$$

It means that the corresponding central extension of $\mathfrak{pe}^{(2)}(k)/(\mathbb{K}1_{2k})$ is just $\mathfrak{pe}^{(2)}(k)$.

Now, concerning the grading operator: Let the second row of B be $(1,0,\dots,0)$ (the first row is, as in the previous case, $(0,\dots,0,1)$). Then we need a grading operator d_2 such that

$$\begin{array}{l}[d_2, e_1^\pm] = e_1^\pm;\\ [d_2, e_i^\pm] = 0 \text{ for all } i > 1;\\ d_2 \text{ commutes with all diagonal matrices.}\end{array} \tag{20.13}$$

The matrix $d_2 := E^{1,1} + E^{k+1,k+1}$ satisfies these conditions. But

$$\mathfrak{pe}^{(2)}(k) \Subset \mathbb{K}(E^{1,1} + E^{k+1,k+1}) \simeq \mathfrak{pe}^{(1)}(k).$$

So, the resulting CMLSA is

$$\mathfrak{pec}(1,k) \Subset \mathbb{K}I_0.$$

20.4. Presentations of $\mathfrak{g}(A)$

Particular cases of the following statement are well known: For Lie algebras over $\mathbb{C}$, see [K3]; for Lie *super*algebras over $\mathbb{C}$, it is due to Serganova and van de Leur, see [Se, vdL]; in the modular case, it is due to Lebedev [LCh].

20.4.1. Statement. a) *Let $\tilde{\mathfrak{g}}(A)^\pm$ be the superalgebras in $\tilde{\mathfrak{g}}(A)$ generated by $e_1^\pm, \dots, e_n^\pm$; then $\tilde{\mathfrak{g}}(A) \cong \tilde{\mathfrak{g}}(A)^+ \oplus \mathfrak{h} \oplus \tilde{\mathfrak{g}}(A)^-$, as vector superspaces.*

b) *Assume that $p \neq 2$ or if $i_j = \bar{1}$ for some $1 \leq j \leq n$, then $A_{jk} \neq 0$ for some $k = 1,\dots,n$. Then there exists a maximal ideal $\mathfrak{r}$ among the ideals of $\tilde{\mathfrak{g}}(A, I)$ whose intersection with $\mathfrak{h}$ is 0.*

c) $\mathfrak{r} = \oplus(\mathfrak{r} \cap \tilde{\mathfrak{g}}_\alpha)$, *where $\tilde{\mathfrak{g}}_\alpha$ is the homogeneous component with respect to the $\mathbb{Z}^n$-grading by roots.*

20.4.2. Disclaimer. Although presentation — description in terms of generators and relations — is one of the accepted ways to represent a given algebra, it seems that an *explicit* form of the presentation is worth the trouble to obtain only if this presentation is often in need, or (which is usually the same) is sufficiently neat. The Chevalley generators of simple finite dimensional Lie algebras over $\mathbb{C}$ satisfy simple and neat relations ("Serre relations") and are often needed for various calculations and theoretical discussions. Relations between their analogs in super case, although not so neat, are still tolerable, at least, for certain Cartan matrices (both Serre and "non-Serre relations").

The simple Lie superalgebras of the form $\mathfrak{g} = \mathfrak{g}(A)$ have several quite distinct sets of generators (cf. [Sa, GL2] and refs. therein) but usually they are given in terms of their *Chevalley generators*. These generators satisfy the relations (12.22) and additional relations $R_i = 0$ whose left sides are implicitly described, for the general Cartan matrix with entries in $\mathbb{K}$, as follows ([K3]):

"the R_i that generate the maximal ideal $\mathfrak{r}$ (defined in Statement 20.4.1). (20.14)

To describe complicated presentations (like non-Serre relations) for all systems of simple roots is possible when there are a few inequivalent such systems for each Lie (super)algebra (see [GL1]), but unrealistic (and hardly needed) if there are dozens or even hundreds of systems of simple roots per algebra (as in the cases we have discovered). Nowadays the package `SuperLie` made the task of finding the explicit expression of the defining relations for many types of Lie algebras and superalgebras a routine exercise for anybody capable to use `Mathematica`. Therefore we mainly consider (in [BGL1, BGL2]) the simplest Cartan matrices — the ones with the only odd simple root (or if there are only a few inequivalent systems of simple roots).

In the rest of the section we recall how the division of the algebra into the sum of three nilpotent subalgebras ($\tilde{\mathfrak{g}}(A) \cong \tilde{\mathfrak{g}}(A)^+ \oplus \mathfrak{h} \oplus \tilde{\mathfrak{g}}(A)^-$) makes it possible to use cohomology in order to describe the relations. For simplicity, in this description, $p = 0$ and cohomology are defined as derived functors using the usual definition of $U(\mathfrak{g})$, not divided powers in the differentials of odd elements needed to describe relations and deformations for $p > 0$, cf. [LL].

20.4.2.1. Serre relations, see [GL1]. Let A be an $n \times n$ matrix. We find the defining relations by induction on n with the help of the Hochschild–Serre spectral sequence (for its description for Lie superalgebras, which has certain subtleties, see [Po]). For the basis of induction consider the following cases:

$$\begin{array}{lll} \circ \text{ or } \bullet & \text{no relations, i.e., } \mathfrak{g}^{\pm} \text{ are free Lie superalgebras if } p \neq 3; & \\ \bullet & \mathrm{ad}^2_{X^{\pm}}(X^{\pm}) = 0 \qquad\qquad\qquad\qquad \text{if } p = 3; & (20.15) \\ \otimes & [X^{\pm}, X^{\pm}] = 0. & \end{array}$$

Set $\deg X_i^{\pm} = 0$ for $1 \leq i \leq n-1$ and $\deg X_n^{\pm} = \pm 1$. Let $\mathfrak{g}^{\pm} = \oplus \mathfrak{g}_i^{\pm}$ and $\mathfrak{g} = \oplus \mathfrak{g}_i$ be the corresponding $\mathbb{Z}$-gradings. Set $\mathfrak{g}_{\pm} = \mathfrak{g}^{\pm}/\mathfrak{g}_0^{\pm}$. From the Hochschild–Serre spectral sequence for the pair $\mathfrak{g}_0^{\pm} \subset \mathfrak{g}^{\pm}$ we get:

$$H_2(\mathfrak{g}_{\pm}) \subset H_2(\mathfrak{g}_0^{\pm}) \oplus H_1(\mathfrak{g}_0^{\pm}; H_1(\mathfrak{g}_{\pm})) \oplus H_0(\mathfrak{g}_0^{\pm}; H_2(\mathfrak{g}_{\pm})). \tag{20.16}$$

It is clear that

$$H_1(\mathfrak{g}_{\pm}) = \mathfrak{g}_1^{\pm}, \quad H_2(\mathfrak{g}_{\pm}) = \wedge^2(\mathfrak{g}_1^{\pm})/\mathfrak{g}_2^{\pm}. \tag{20.17}$$

So, the second summand in (20.16) provides us with relations of the form:

$$\begin{array}{ll} (\mathrm{ad}_{X_n^{\pm}})^{k_{ni}}(X_i^{\pm}) = 0 & \text{if the } n\text{-th root is not } \otimes \\ [X_n, X_n] = 0 & \text{if the } n\text{-th root is } \otimes. \end{array} \tag{20.18}$$

while the third summand in (20.16) is spanned by the $\mathfrak{g}_0^{\pm}$-lowest vectors in

$$\wedge^2(\mathfrak{g}_1^{\pm})/(\mathfrak{g}_2^{\pm} + \mathfrak{g}^{\pm} \wedge^2 (\mathfrak{g}_1^{\pm})). \tag{20.19}$$

Let the matrix $B = (B_{ij})$ be as in formula (12.42). The following proposition, whose proof is straightforward, illustrates the usefulness of our normalization of Cartan matrices as compared with other options:

20.4.2.2. Proposition. *The numbers k_{in} and k_{ni} in* (20.18) *are expressed in terms of* (B_{ij}) *as follows:*

$$\begin{array}{ll} (\mathrm{ad}_{X_i^{\pm}})^{1+B_{ij}}(X_j^{\pm}) = 0 & \textit{for } i \neq j \\ [X_i^{\pm}, X_i^{\pm}] = 0 & \textit{if } A_{ii} = 0. \end{array} \tag{20.20}$$

The relations (12.22) and (20.20) will be called *Serre relations* for Lie superalgebra $\mathfrak{g}(A)$. If $p = 3$, then the relation

$$[X_i^{\pm}, [X_i^{\pm}, X_i^{\pm}]] = 0 \quad \text{for } X_i^{\pm} \text{ odd and } A_{ii} = 1 \tag{20.21}$$

is not a consequence of the Jacobi identity; for simplicity, however, we will include it in the set of Serre relations.

20.4.2.3. Non-Serre relations. These are relations that correspond to the third summand in (20.16). Let us consider the simplest case: $\mathfrak{sl}(m|n)$ in the realization with the system of simple roots

$$\circ - \cdots - \circ - \otimes - \circ - \cdots - \circ \tag{20.22}$$

Then $H_2(\mathfrak{g}_{\pm})$ from the third summand in (20.16) is just $\wedge^2(\mathfrak{g}_{\pm})$. For simplicity, we confine ourselves to the positive roots. Let $X_1, \dots, X_{m-1}$ and $Y_1, \dots, Y_{n-1}$ be the root vectors corresponding to even roots separated by the root vector Z corresponding to the root $\otimes$.

If $n = 1$ or $m = 1$, then $\wedge^2(\mathfrak{g})$ is an irreducible $\mathfrak{g}_{\bar{0}}$-module and there are no non-Serre relations. If $n \neq 1$ and $m \neq 1$, then $\wedge^2(\mathfrak{g})$ splits into 2 irreducible $\mathfrak{g}_{\bar{0}}$-modules. The lowest component of one of them corresponds to the relation $[Z, Z] = 0$, the other one corresponds to the non-Serre-type relation

$$[[X_{m-1}, Z], [Y_1, Z]] = 0. \tag{20.23}$$

If, instead of $\mathfrak{sl}(m|n)$, we would have considered the Lie algebra $\mathfrak{sl}(m+n)$, the same argument would have led us to the two relations, both of Serre type:

$$\mathrm{ad}^2_Z(X_{m-1}) = 0, \qquad \mathrm{ad}^2_Z(Y_1) = 0.$$

For explicit description of the defining relations in terms of the Chevalley generators of the algebras we consider, see [BGL1, BGL2, **?**].

20.5. Main steps of our classification

In this section we are dealing with Lie (super)algebras of the form $\mathfrak{g}(A)$ or their simple subquotients $\mathfrak{g}^{(1)}(A)/\mathfrak{c}$.

20.5.1. Step 1: An overview of known results. Lie algebras. There are known the two methods:

1) Over $\mathbb{C}$, Cartan [C] did not use any roots, instead he used what is nowadays called in his honor Cartan prolongations and a generalization (which he never formulated explicitly) of this procedure which we call *CTS-ing* (*Cartan-Tanaka-Shchepochkina prolonging*).

2) Nowadays, to get the shortest classification of the simple finite dimensional Lie algebras, everybody (e.g. [Bou, OV]) uses root technique and the non-degenerate invariant symmetric bilinear form (the Killing form).

In the **modular** case, as well as in the **super** case, and in the mixture of these cases we consider here, the Killing form might be identically zero. However, if the Cartan matrix A is symmetrizable (and indecomposable), on the Lie (super)algebra $\mathfrak{g}(A)$ if $\mathfrak{g}(A)$ is simple (or on $\mathfrak{g}^{(1)}(A)/\mathfrak{c}$ if $\mathfrak{g}(A)$ is not simple), there is a non-degenerate replacement of the Killing form. (Astonishingly, this replacement might sometimes be not coming from any representation, see [Ser]. Earlier, in his preprint [Kapp], Kaplansky observed a similar phenomenon (in the modular case) and associated the non-degenerate bilinear form with a *projective* representation.)

In the modular case, and in the super case for $p = 0$, this approach — to use a non-degenerate even invariant symmetric form in order to classify the simple algebras— was pursued by Kaplansky [Kapp].

For $p > 0$, Weisfeiler and Kac [WK] gave a classification, but although the idea of their proof is OK, the paper has several gaps and vague notions (the Brown algebra $\mathfrak{br}(3)$ was missed [Br3, KWK]; the notion of the Lie algebra with Cartan matrix nicely formulated in [K3] was not properly developed at the time [WK] was written; the Dynkin diagrams mentioned there were not defined at all in the modular case; the algebras $\mathfrak{g}(A)$ and $\mathfrak{g}(A)^{(1)}/\mathfrak{c}$ are sometimes identified). Therefore, the case $p > 3$ being completely investigated by Block, Wilson, Premet and Strade [S], we double-checked the cases where $p < 5$. The answer of [WK]∪[KWK] is correct.

Lie **super**algebras.

Over $\mathbb{C}$, for *any* Lie algebra $\mathfrak{g}_{\bar 0}$, Kac [K2] listed all

$$\mathfrak{g}_{\bar 0}\text{-modules } \mathfrak{g}_{\bar 1} \text{ such that the Lie superalgebra } \mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1} \text{ is simple.} \tag{20.24}$$

Kaplansky [Kapp, Kap], Djoković and Hochschild [Dj], and also Scheunert, Nahm and Rittenberg [SNR] had their own approaches to the problem (20.24) and solved it without gaps for various particular cases, but they did not investigate which of the simple finite dimensional Lie superalgebras possess Cartan matrix.

Kac observed that (1) some of the simple Lie superalgebras (20.24) possess analogs of Cartan matrix, (2) one Lie superalgebra may have several inequivalent Cartan matrices and tried to list all of them. His first list of inequivalent Cartan matrices (in other words, distinct $\mathbb{Z}$-gradings) for finite dimensional Lie superalgebras $\mathfrak{g}(A)$ in [K2] was with gaps; Serganova [Se] and (by a different method and only for symmetrizable matrices) van de Leur [vdL] fixed the gaps and even classified Lie superalgebras of polynomial growth (for the proof in the non-symmetrizable case, announced 20 years ago, see [HS]). Kac also suggested analogs of Dynkin diagrams to graphically encode the Cartan matrices.

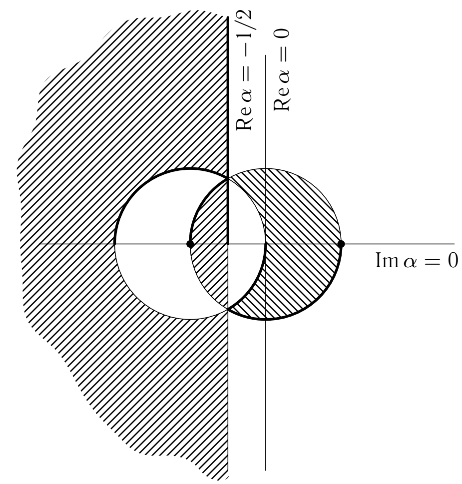
Kaplansky was the first (see his newsletters in [Kapp]) to discover the exceptional algebras $\mathfrak{ag}(2)$ and $\mathfrak{ab}(3)$ (he dubbed them Γ_2 and Γ_3, respectively) and a parametric family $\mathfrak{osp}(4|2;\alpha)$ (he dubbed it $\Gamma(A,B,C)$)); our notations reflect the fact that $\mathfrak{ag}(2)_{\bar 0} = \mathfrak{sl}(2) \oplus \mathfrak{g}(2)$ and $\mathfrak{ab}(3)_{\bar 0} = \mathfrak{sl}(2) \oplus \mathfrak{o}(7)$ ($\mathfrak{o}(7)$ is B_3 in Cartan's nomenclature). Kaplansky's description (irrelevant to us at the moment except for the fact that A, B and C are on equal footing) of what we now identify as $\mathfrak{osp}(4|2;\alpha)$, a parametric family of deforms of $\mathfrak{osp}(4|2)$, made an S_3-symmetry of the parameter manifest (to A. A. Kirillov, and he informed us, in 1976). Indeed, since $A + B + C = 0$, and $\alpha \in \mathbb{C} \cup \infty$ is the ratio of the two remaining parameters, we get an S_3-action on the plane $A + B + C = 0$ which in terms of α is generated by the transformations:

$$\alpha \longmapsto -1-\alpha, \qquad \alpha \longmapsto \frac{1}{\alpha}. \tag{20.25}$$

This symmetry should have immediately sprang to mind since $\mathfrak{osp}(4|2;\alpha)$ is strikingly similar to $\mathfrak{wk}(3;a)$ found 5 years earlier, cf. (20.28), and since $S_3 \simeq \mathrm{SL}(2;\mathbb{Z}/2)$.

The following figure depicts the fundamental domains of the S_3-action. The other transformations generated by (20.25) are

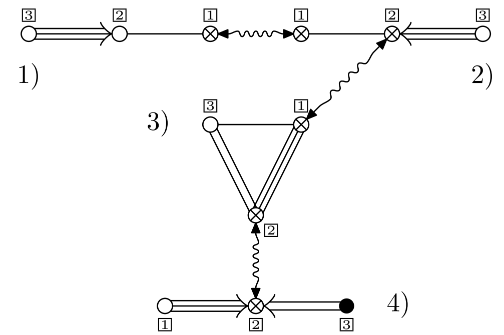
$$\alpha \longmapsto -\frac{1+\alpha}{\alpha}, \quad -\frac{1}{\alpha+1}, \quad -\frac{\alpha}{\alpha+1}.$$



20.5.1.1. Notation: On matrices with a "–" sign and other notations in the lists of inequivalent Cartan matrices. The rectangular matrix at the beginning of each list of inequivalent Cartan matrices for each Lie superalgebra shows the result of odd reflections (the number of the row is the number of the Cartan matrix in the list below, the number of the column is the number of the root (given by small boxed number) in which the reflection is made; the cells contain the results of reflections (the number of the Cartan matrix obtained) or a "–" if the reflection is not appropriate because $A_{ii} \neq 0$. Some of the Cartan matrices thus obtained are equivalent, as indicated.

The number of the matrix A such that $\mathfrak{g}(A)$ has only one odd simple root is $\boxed{\textit{boxed}}$, that with all simple roots odd is underlined. The nodes are numbered by small boxed numbers; the curly lines with arrows depict odd reflections.

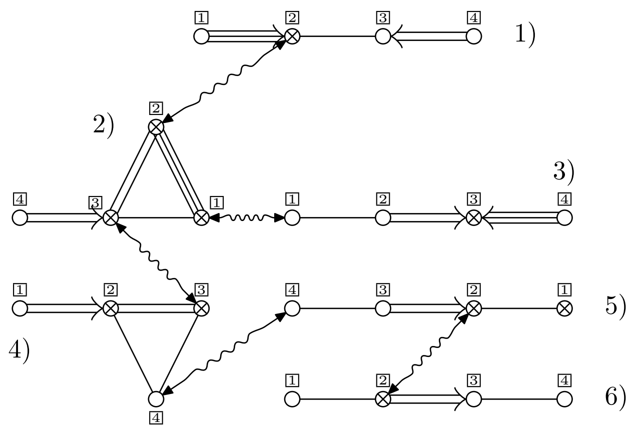
Recall that $\mathfrak{ag}(2)$ of sdim $= 17|14$ has the following Cartan matrices



$$\begin{pmatrix} 2 & - & - \\ 1 & 3 & - \\ - & 2 & 4 \\ - & - & 3 \end{pmatrix}$$

$$\boxed{1)}\ \begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}\quad 2)\ \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 3 \\ 0 & -1 & 2 \end{pmatrix}\quad 3)\ \begin{pmatrix} 0 & -3 & 1 \\ -3 & 0 & 2 \\ -1 & -2 & 2 \end{pmatrix}\quad 4)\ \begin{pmatrix} 2 & -1 & 0 \\ -3 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix} \tag{20.26}$$

Recall that $\mathfrak{ab}(3)$ of sdim $= 24|16$ has the following Cartan matrices



$$\begin{pmatrix} - & 2 & - & - \\ 3 & 1 & 4 & - \\ 2 & - & - & - \\ - & - & 2 & 5 \\ - & 6 & - & 4 \\ - & 5 & - & - \end{pmatrix}$$

$$\begin{array}{l} \boxed{1)}\ \begin{pmatrix} 2 & -1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}\quad 2)\ \begin{pmatrix} 0 & -3 & 1 & 0 \\ -3 & 0 & 2 & 0 \\ 1 & 2 & 0 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}\quad \boxed{3)}\ \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & -1 & 2 \end{pmatrix} \\ 4)\ \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 0 & 2 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}\quad 5)\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}\quad \boxed{6)}\ \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \end{array} \tag{20.27}$$

Modular Lie algebras and Lie superalgebras.

$p=2$, Lie algebras. Weisfeiler and Kac [WK] discovered two new parametric families that we denote $\mathfrak{wk}(3;a)$ and $\mathfrak{wk}(4;a)$ (*Weisfeiler and Kac algebras*).

$\mathfrak{wk}(3;a)$, where $a\neq 0,-1$, of dim 18 is a non-super version of $\mathfrak{osp}(4|2;a)$ (although no $\mathfrak{osp}$ exists for $p=2$); the dimension of its simple subquotient $\mathfrak{wk}(3;a)^{(1)}/\mathfrak{c}$ is equal to 16; the inequivalent Cartan matrices are:

$$1)\ \begin{pmatrix} \bar 0 & a & 0 \\ a & \bar 0 & 1 \\ 0 & 1 & \bar 0 \end{pmatrix},\ 2)\ \begin{pmatrix} \bar 0 & 1+a & a \\ 1+a & \bar 0 & 1 \\ a & 1 & \bar 0 \end{pmatrix}$$

$\mathfrak{wk}(4;a)$, where $a\neq 0,-1$, of dim $=34$; the inequivalent Cartan matrices are:

$$1)\ \begin{pmatrix} \bar 0 & a & 0 & 0 \\ a & \bar 0 & 1 & 0 \\ 0 & 1 & \bar 0 & 1 \\ 0 & 0 & 1 & \bar 0 \end{pmatrix},\ 2)\ \begin{pmatrix} \bar 0 & 1 & 1+a & 0 \\ 1 & \bar 0 & a & 0 \\ a+1 & a & \bar 0 & a \\ 0 & 0 & a & \bar 0 \end{pmatrix},\ 3)\ \begin{pmatrix} \bar 0 & a & 0 & 0 \\ a & \bar 0 & a+1 & 0 \\ 0 & a+1 & \bar 0 & 1 \\ 0 & 0 & 1 & \bar 0 \end{pmatrix}$$

Weisfeiler and Kac investigated also which of these algebras are isomorphic and the answer is as follows:

$$\begin{array}{l} \mathfrak{wk}(3;a)\simeq\mathfrak{wk}(3;a') \Longleftrightarrow a'=\dfrac{\alpha a+\beta}{\gamma a+\delta},\ \text{where}\ \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL(2;\mathbb{Z}/2) \\ \mathfrak{wk}(4;a)\simeq\mathfrak{wk}(4;a') \Longleftrightarrow a'=\dfrac{1}{a}. \end{array} \tag{20.28}$$

20.5.1.2. 2-structures on $\mathfrak{wk}$ algebras. 1) Observe that the center $\mathfrak{c}$ of $\mathfrak{wk}(3;a)$ is spanned by ah_1+h_3. The 2-structure on $\mathfrak{wk}(3;a)$ is given by the conditions $(e_\alpha^\pm)^{[2]}=0$ for all root vectors and the following ones:

a) For the matrix $B=(0,0,1)$ in (12.25) for the grading operator d, set:

$$\begin{array}{l} (\mathrm{ad}_{h_1})^{[2]}=(1+at)h_1+th_3\equiv h_1 \pmod{\mathfrak{c}}, \\ (\mathrm{ad}_{h_2})^{[2]}=ath_1+h_2+th_3+a(1+a)d\equiv h_2+a(1+a)d \pmod{\mathfrak{c}}, \\ (\mathrm{ad}_{h_3})^{[2]}=(at+a^2)h_1+th_3\equiv a^2h_1 \pmod{\mathfrak{c}}, \\ (\mathrm{ad}_{d})^{[2]}=ath_1+th_3+d\equiv d \pmod{\mathfrak{c}}, \end{array} \tag{20.29}$$

where t is a parameter.

b) Taking $B=(1,0,0)$ in (12.25) we get a more symmetric answer:

$$\begin{array}{l} (\mathrm{ad}_{h_1})^{[2]}=(1+at)h_1+th_3\equiv h_1 \pmod{\mathfrak{c}}, \\ (\mathrm{ad}_{h_2})^{[2]}=ath_1+ah_2+th_3+(1+a)d\equiv ah_2+(1+a)d \pmod{\mathfrak{c}}, \\ (\mathrm{ad}_{h_3})^{[2]}=(at+a^2)h_1+th_3\equiv a^2h_1 \pmod{\mathfrak{c}}, \\ (\mathrm{ad}_{d})^{[2]}=ath_1+th_3+d\equiv d \pmod{\mathfrak{c}}, \end{array} \tag{20.30}$$

(The expressions are somewhat different since we have chosen a different basis but on this simple Lie algebra the 2-structure is unique.)

2) The 2-structure on $\mathfrak{wk}(4;a)$ is given by the conditions $(e_\alpha^\pm)^{[2]}=0$ for all root vectors and

$$
\begin{aligned}
(\mathrm{ad}_{h_1})^{[2]} &= ah_1 + (1+a)h_4,\\
(\mathrm{ad}_{h_2})^{[2]} &= ah_2,\\
(\mathrm{ad}_{h_3})^{[2]} &= h_3,\\
(\mathrm{ad}_{h_4})^{[2]} &= h_4.
\end{aligned}
\tag{20.31}
$$

$p=3$, Lie algebras. *Brown algebras*:

$$
\mathfrak{br}(2,a) \text{ with CM } \begin{pmatrix} 2 & -1\\ a & 2\end{pmatrix} \text{ and } \mathfrak{br}(2) = \lim_{-\frac{2}{a}\longrightarrow 0} \mathfrak{br}(2,a) \text{ with CM } \begin{pmatrix} 2 & -1\\ -1 & 0\end{pmatrix}
\tag{20.32}
$$

The reflections change the value of the parameter, so

$$
\mathfrak{br}(2,a) \simeq \mathfrak{br}(2,a') \Longleftrightarrow a' = -(1+a).
\tag{20.33}
$$

$$
1\mathfrak{br}(3) \text{ with CM } \begin{pmatrix} 2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & \bar 0\end{pmatrix} \text{ and } 2\mathfrak{br}(3) \text{ with CM } \begin{pmatrix} 2 & -1 & 0\\ -2 & 2 & -1\\ 0 & -1 & \bar 0\end{pmatrix}
\tag{20.34}
$$

$p=3$, Lie superalgebras.

Brown superalgebra $\mathfrak{brj}(2;3)$ of sdim $=10|8$ recently discovered in [El1] (Theorem 3.2(i); its Cartan matrices are first listed in [BGL3]) has the following Cartan matrices

$$
1)\ \begin{pmatrix} 0 & -1\\ -2 & 1\end{pmatrix},\quad 2)\ \begin{pmatrix} 0 & -1\\ -1 & \bar 0\end{pmatrix},\quad 3)\ \begin{pmatrix} 1 & -1\\ -1 & \bar 0\end{pmatrix}.
$$

The Lie superalgebra $\mathfrak{brj}(2;3)$ is a super analog of the Brown algebra $\mathfrak{br}(2) = \mathfrak{brj}(2;3)_{\bar 0}$, its even part; $\mathfrak{brj}(2;3)_{\bar 1} = R(2\pi_1)$ is irreducible $\mathfrak{brj}(2;3)_{\bar 0}$-module.

Elduque [El1, El2, CE, CE2] considered a particular case of the problem (20.24). He arranged the Lie (super)algebras he obtained in the Elduque Supermagic Square — an analog of the Freudenthal Magic Square.

All these examples turned out to be of the form $\mathfrak{g}(A)$. These *Elduque and Cunha superalgebras* are, indeed, exceptional ones. For the complete list of their inequivalent Cartan matrices, reproduced here, see [BGL1], where their presentation are also given; we also reproduce the description of the even and odd parts of these Lie superalgebras (all but one discovered by Elduque and whose description in terms of symmetric composition algebras is due to Elduque and Cunha), see subsect. 20.8.1.

$p=5$, Lie superalgebras. *Brown superalgebra* $\mathfrak{brj}(2;5)$ of sdim $=10|12$, recently discovered in [BGL3], such that $\mathfrak{brj}(2;5)_{\bar 0} = \mathfrak{sp}(4)$ and $\mathfrak{brj}(2;5)_{\bar 1} = R(\pi_1+\pi_2)$ is an irreducible $\mathfrak{brj}(2;5)_{\bar 0}$-module. [2] The Lie superalgebra $\mathfrak{brj}(2;5)$ has the

[2] To the incredulous reader: The Cartan subalgebra of $\mathfrak{sp}(4)$ is generated by h_2 and $2h_1+h_2$. The highest weight vector is $x_{10} = [[x_2,\ [x_2,\ [x_1,\ x_2]]],\ [[x_1,\ x_2],\ [x_1,\ x_2]]]$ and its weight is not a multiple of a fundamental weight, but $(1,1)$. We encounter several more instances of non-fundamental weights in descriptions of exceptions for $p=2$.

following Cartan matrices:

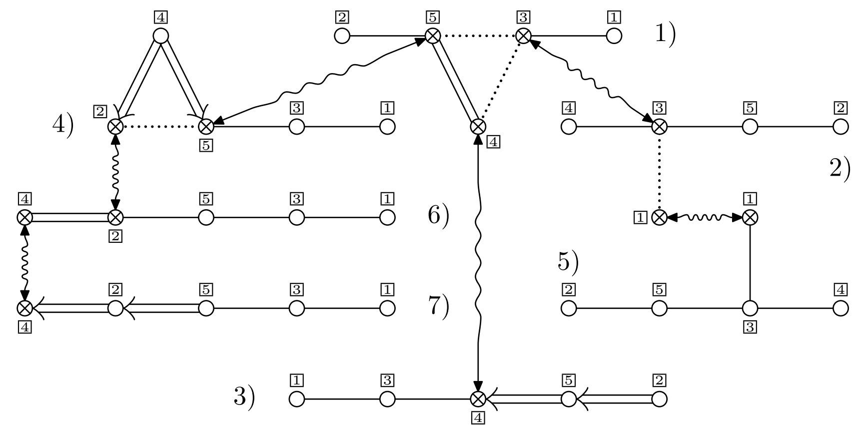
$$
\begin{pmatrix} 2 & -\\ 1 & -\end{pmatrix}\qquad 1)\ \begin{pmatrix} 0 & -1\\ -2 & 1\end{pmatrix},\quad 2)\ \begin{pmatrix} 0 & -1\\ -3 & 2\end{pmatrix}.
$$

Elduque superalgebra $\mathfrak{el}(5;5)$ of sdim $=55|32$, where $\mathfrak{el}(5;5)_{\bar 0} = \mathfrak{o}(11)$ and $\mathfrak{el}(5;5)_{\bar 1} = \mathrm{spin}_{11}$. Its inequivalent Cartan matrices, first described in [BGL2], are as follows:

Instead of joining nodes by four segments in the cases where

$$
A_{ij} = A_{ji} = 1 \equiv -4 \mod 5
$$

we use one dotted segment.



$$
1)\ \begin{pmatrix} 2 & 0 & -1 & 0 & 0\\ 0 & 2 & 0 & 0 & -1\\ -1 & 0 & 0 & -4 & -4\\ 0 & 0 & -4 & 0 & -2\\ 0 & -1 & -4 & -2 & 0\end{pmatrix}\quad
2)\ \begin{pmatrix} 0 & 0 & -4 & 0 & 0\\ 0 & 2 & 0 & 0 & -1\\ -4 & 0 & 0 & -1 & -1\\ 0 & 0 & -1 & 2 & 0\\ 0 & -1 & -1 & 0 & 2\end{pmatrix}\quad
\boxed{3)}\ \begin{pmatrix} 2 & 0 & -1 & 0 & 0\\ 0 & 2 & 0 & 0 & -1\\ -1 & 0 & 2 & -1 & 0\\ 0 & 0 & -1 & 0 & 2\\ 0 & -2 & 0 & -1 & 2\end{pmatrix}
$$

$$
4)\ \begin{pmatrix} 2 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 2 & -4\\ -1 & 0 & 2 & 0 & -1\\ 0 & -1 & 0 & 2 & -1\\ 0 & -4 & -1 & 2 & 0\end{pmatrix}\quad
\boxed{5)}\ \begin{pmatrix} 0 & 0 & -1 & 0 & 0\\ 0 & 2 & 0 & 0 & -1\\ -1 & 0 & 2 & -1 & -1\\ 0 & 0 & -1 & 2 & 0\\ 0 & -1 & -1 & 0 & 2\end{pmatrix}\quad
6)\ \begin{pmatrix} 2 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & -2 & -1\\ -1 & 0 & 2 & 0 & -1\\ 0 & -2 & 0 & 0 & 0\\ 0 & -1 & -1 & 0 & 2\end{pmatrix}
$$

$$
\boxed{7)}\ \begin{pmatrix} 2 & 0 & -1 & 0 & 0\\ 0 & 2 & 0 & -1 & -2\\ -1 & 0 & 2 & 0 & -1\\ 0 & 2 & 0 & 0 & 0\\ 0 & -1 & -1 & 0 & 2\end{pmatrix}\quad
8)\ \begin{pmatrix} - & - & 2 & 3 & 4\\ 5 & - & 1 & - & -\\ - & - & - & 1 & -\\ - & 6 & - & - & 1\\ 2 & - & - & - & -\\ - & 4 & - & 7 & -\\ - & - & - & 6 & -\end{pmatrix}
$$

20.5.2. Step 2: Studying 2×2 and 3×3 Cartan matrices. 1) We ask `Mathematica` to construct all possible matrices of a specific size. The matrices are not normalized and they must not be symmetrizable: we can not eliminate non-symmetrizable matrices at this stage. Fortunately, all 2×2 matrices are symmetrizable.

2) We ask `Mathematica` to eliminate the matrices with the following properties:

$$\begin{array}{l}\text{a) Matrices } A \text{ for whose submatrix } B \text{ we know that } \dim\mathfrak{g}(B)=\infty;\\ \text{b) decomposable matrices.}\end{array}\tag{20.35}$$

3) Matrices with a row in each that differ from each other by a nonzero factor are counted once, e.g.,

$$\begin{pmatrix}1&1\\3&2\end{pmatrix}\cong\begin{pmatrix}2&2\\3&2\end{pmatrix}\cong\begin{pmatrix}6&6\\6&4\end{pmatrix}.$$

4) Equivalent matrices are counted once, where equivalence means that one matrix can be obtained from the other one by simultaneous transposition of rows and columns with the same numbers and *the same parity*. For example,

$$\begin{pmatrix}0&\alpha&0&0\\ \alpha&\bar 0&0&1\\ 0&0&\bar 0&1\\ 0&1&1&\bar 0\end{pmatrix}\sim\begin{pmatrix}0&\alpha&1&0\\ \alpha&\bar 0&0&0\\ 1&0&\bar 0&1\\ 0&0&1&\bar 0\end{pmatrix}\sim\begin{pmatrix}0&0&0&1\\ 0&\bar 0&\alpha&1\\ 0&\alpha&\bar 0&0\\ 1&1&0&\bar 0\end{pmatrix}$$

At the substeps 1.1)–1.4) we thus get a store of Cartan matrices to be tested further.

5) Now, we ask `SuperLie` to construct the Lie superalgebras $\mathfrak{g}(A)$ up to certain dimension (say, 256). Having stored the Lie superalgebras $\mathfrak{g}(A)$ of dimension <256 we increase the range again if there are any algebras left (say, to 1024 or 2048). At this step, we **conjecture** that the dimension of any finite dimensional simple Lie (super)algebra of the form $\mathfrak{g}(A)$, where A is of size $n\times n$, does not grow too rapidly with n. Say, at least, not as fast as n^{10}.

If the dimension of $\mathfrak{g}(A)$ increases accordingly, then we **conjecture** that $\mathfrak{g}(A)$ is infinite dimensional and this Lie superalgebra is put away for a while (but not completely eliminated as decomposable matrices that correspond to non-simple algebras: The progress of science might require soon to investigate **how fast** the dimension grows with n: polynomially or faster).

6) For the stored Cartan matrices A, we have $\dim\mathfrak{g}(A)<\infty$. Once we get the full list all of such Cartan matrices of a given size, we have to check if $\mathfrak{g}(A)$ is simple, one by one.

7) The vectors of parities of the generators $\mathrm{Pty}=(p_1,\dots,p_n)$ are only considered of the form $(\bar 1,\dots,\bar 1,\bar 0,\dots,\bar 0)$.

20.5.2.1. The case of 2×2 Cartan matrices. On the diagonal we may have 2, $\bar 1$ or $\bar 0$, if the corresponding root is even; 0 or 1 if the root is odd. To be on the safe side, we redid the purely even case. We have the following options to consider:

$$\begin{array}{lll}\mathrm{Pty}=(\bar 0,\bar 0) & \mathfrak{a}_1 & \begin{pmatrix}2&2a\\2b&2\end{pmatrix}\simeq\begin{pmatrix}2&2a\\b&\bar 1\end{pmatrix}\simeq\begin{pmatrix}\bar 1&a\\2b&2\end{pmatrix}\simeq\begin{pmatrix}\bar 1&a\\b&\bar 1\end{pmatrix}\simeq\begin{pmatrix}b&ab\\ab&a\end{pmatrix}\\ & \mathfrak{a}_2 & \begin{pmatrix}\bar 1&2a\\-1&\bar 0\end{pmatrix}\qquad \mathfrak{a}_3\quad \begin{pmatrix}\bar 0&-1\\-1&\bar 0\end{pmatrix}\end{array}\tag{20.36}$$

$$\begin{array}{lll}\mathrm{Pty}=(\bar 1,\bar 0) & \mathfrak{a}_4 & \begin{pmatrix}0&-1\\2a&2\end{pmatrix}\simeq\begin{pmatrix}0&-1\\a&\bar 1\end{pmatrix}\\ & \mathfrak{a}_5 & \begin{pmatrix}\bar 0&-1\\-1&0\end{pmatrix}\qquad \mathfrak{a}_6\quad \begin{pmatrix}1&a\\-1&\bar 0\end{pmatrix}\\ & \mathfrak{a}_7 & \begin{pmatrix}1&a\\2b&2\end{pmatrix}\simeq\begin{pmatrix}1&a\\b&\bar 1\end{pmatrix}\end{array}\tag{20.37}$$

$$\begin{array}{lll}\mathrm{Pty}=(\bar 1,\bar 1) & \mathfrak{a}_8 & \begin{pmatrix}1&a\\b&1\end{pmatrix}\simeq\begin{pmatrix}b&ab\\ab&a\end{pmatrix}\\ & \mathfrak{a}_9 & \begin{pmatrix}0&-1\\-1&0\end{pmatrix}\qquad \mathfrak{a}_{10}\quad \begin{pmatrix}0&-1\\b&1\end{pmatrix}\end{array}\tag{20.38}$$

Obviously, some of these CMs had appeared in the study of (twisted) loops and the corresponding Kac-Moody Lie (super)algebras. One could expect that the reduction of the entries of A modulo p might yield a finite dimensional algebra, but this does not happen.

20.5.2.2. Conjecture. *If A is non-symmetrizable, then* $\dim\mathfrak{g}(A)=\infty$.

This is true *a posteriori* for $p=0$. We prove this by inspection for 3×3 matrices, but, regrettably, the general case does not follow by reduction and induction: For example, for $p=2$ and the non-symmetrizable matrix

$$\begin{pmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&1&a&1\end{pmatrix},$$

where $a\neq 0,1$, or analogous $n\times n$ matrix whose Dynkin diagram is a loop, any 3×3 submatrix is symmetrizable.

20.5.3. Step 3: Studying $n\times n$ Cartan matrices for $n>3$. Putting our faith into Conjecture 20.5.2.2, we will assume that A is symmetric. The idea is to use induction and the information found at each step.

Since we are sure that each Lie superalgebra of the form $\mathfrak{g}(A)$ possesses a "simplest" Dynkin diagram — the one with only one odd node, we are going to use one more hypothesis:

20.5.3.1. Hypothesis. Passing from $n\times n$ Cartan matrices to $(n+1)\times(n+1)$ Cartan matrices it suffices to consider just two types of $n\times n$ Cartan matrices: Purely even ones and the "simplest" ones — with only one odd node on their Dynkin diagrams. To the latter ones only even node should be added.

20.5.3.2. Further simplification of the algorithm. Enlarging Cartan matrices by adding new row and column, we let, for $n>4$, its only non-zero elements occupy at most two slots (apart from the diagonal). Justification: Lemmas from §3 in [WK].

Even this simplification still leaves lots of cases: To the 5 cases to be enlarged for Cartan matrices of size ≤ 8 that we encounter for $p=0$, we have to add ≤ 16 super cases, each producing tens of possibilities in each of the major cases $p=2$, 3 and 5. To save several pages per each n for each p, we have omitted the results of enlargements of each Cartan matrix and give only the final summary.

20.5.4. On a quest for parametric families. Even for 2×2 Cartan matrices we could have proceeded by "enlarging" but to be on the safe side we performed the selection independently. We considered only one or two parameters using the function called **ParamSolve** (of `SuperLie`). It shows all cases where the division by an expression possibly equal to zero occurred. Every time `SuperLie` shows such a possibility we check it by hand; these possibility are algebraic equations of the form $\beta=f(\alpha)$, where α and β are the parameters of the CM. We saw that whenever α and β are generic $\dim\mathfrak{g}(A)$ grows too fast as compared with the *height* of the element (i.e., the number of brackets in expressions like $[a,[b,[c,d],],]$) that `SuperLie` should not exceed constructing a Lie (super)algebra. We did not investigate if the growth is polynomial or exponential, but definitely $\dim\mathfrak{g}(A)=\infty$. For each pair of singular values of parameters $\beta=f(\alpha)$, we repeat the computations again. In most cases, the algebra is infinite-dimensional, the exceptions being $\beta=\alpha+1$ that nicely correspond to some of CMs we already know, like $\mathfrak{wk}$ algebras.

For three parameters, we have equations of the form $\gamma=f(\alpha,\beta)$. For generic α and β. the Lie superalgebra $\mathfrak{g}(A)$ is infinite-dimensional. For the singular cases given by `SuperLie`, the constraints are of the form $\beta=g(\alpha)$. Now we face two possibilities: If γ is a constant, then we just use the result of the previous step, when we dealt with two parameters. In the rare cases where γ is not a constant and depends on the parameter α, we have to recompute again and again the $\dim\mathfrak{g}(A)$ is infinite in these cases.

We find Cartan matrices of size 4×4 and larger by "enlarging". For $p=2$, we see that 3×3 CMs with parameters can be extended to 4×4 CMs. However, 4×4 CMs cannot be extended to 5×5 CMs whose Lie (super)algebras are of finite dimension. For $p>2$, even 3×3 CMs cannot be extended.

20.5.5. Super and modular cases: Summary of new features (as compared with simple Lie algebras over $\mathbb{C}$). The super case, $p=0$.

1) There are three types of nodes (●, $\otimes$ and ○),

2) there may occur a loop but only of length 3;

3) there is at most 1 parameter, but 1 parameter may occur;

4) to one algebra several inequivalent Cartan matrices can correspond.

The modular case. For Lie algebras, new features are same as in the $p=0$ super case; additionally there appear new types of nodes ($\odot$ and $*$).

20.6. The answer: The case where $p>5$

This case is the simplest one since it does not differ much from the $p=0$ case, where the answer is known.

Simple Lie algebras:

1) Lie algebras obtained from their $p=0$ analogs by reducing modulo p. We thus get

the CM versions of $\mathfrak{sl}$, namely: either simple $\mathfrak{sl}(n)$ or $\mathfrak{gl}(pn)$ whose "simple core" is $\mathfrak{psl}(pn)$;

the orthogonal algebras $\mathfrak{o}(2n+1)$ and $\mathfrak{o}(2n)$;

the symplectic algebras $\mathfrak{sp}(2n)$;

the exceptional algebras are $\mathfrak{g}(2)$, $\mathfrak{f}(4)$, $\mathfrak{e}(6)$, $\mathfrak{e}(7)$, $\mathfrak{e}(8)$.

Simple Lie superalgebras

Lie superalgebras obtained from their $p=0$ analogs by reducing modulo p. We thus get

1) the CM versions of $\mathfrak{sl}$, namely: either simple $\mathfrak{sl}(m|n)$ or $\mathfrak{gl}(a|pk+a)$ whose "simple core" is $\mathfrak{psl}(a|pk+a)$ and $\mathfrak{psl}(a|pk+a)^{(1)}$ if $a=kn$;

2) the ortho-symplectic algebras $\mathfrak{osp}(m|2n)$;

3) a parametric family $\mathfrak{osp}(4|2;a)$;

4) the exceptional algebras are $\mathfrak{ag}(2)$ and $\mathfrak{ab}(3)$.

20.7. The answer: The case where $p=5$

Simple Lie algebras:

1) same as in §20.6 for $p=5$.

Simple Lie superalgebras

1) same as in §20.6 for $p=5$ and several new exceptions:

2) The Brown superalgebras [BGL3]: $\mathfrak{brj}(2;5)$ such that $\mathfrak{brj}(2;5)_{\bar 0}=\mathfrak{sp}(4)$ and the $\mathfrak{brj}(2;5)_{\bar 0}$-module $\mathfrak{brj}(2;5)_{\bar 1}=R(\pi_1+\pi_2)$ is irreducible with the highest weight vector

$$x_{10}=[[x_2,\,[x_2,\,[x_1,\,x_2]]],\,[[x_1,\,x_2],\,[x_1,\,x_2]]]$$

(for the CM 2): with the two Cartan matrices

$$\begin{pmatrix}2&-\\1&-\end{pmatrix}\qquad 1)\ \begin{pmatrix}0&-1\\-2&1\end{pmatrix},\quad 2)\ \begin{pmatrix}0&-1\\-3&2\end{pmatrix}$$

3) The Elduque superalgebra $\mathfrak{el}(5;5)$. Having found out one Cartan matrix of $\mathfrak{el}(5;5)$, we have listed them all, see 20.5.1.2.

20.8. The answer: The case where $p = 3$

Simple Lie algebras:

1) same as in §20.6 for $p = 3$, except $\mathfrak{g}(2)$ which is not simple but contains a unique minimal ideal isomorphic to $\mathfrak{psl}(3)$, and the following additional exceptions:

2) the Brown algebras $\mathfrak{br}(2; a)$ and $\mathfrak{br}(2)$ as well as $\mathfrak{br}(3)$, see subsect. 20.5.1.

Simple Lie superalgebras

1) same as in §20.6 for $p = 3$ and $\mathfrak{e}(6)$ (with CM) which is not simple but has a "simple core" $\mathfrak{e}(6)/\mathfrak{c}$;

2) the Brown superalgebras, see subsect. 20.5.1;

3) the Elduque and Cunha superalgebras, see [CE2, BGL1]. They are respective "enlargements" of the following Lie algebras (but can be also obtained by enlarging certain Lie superalgebras):

$\mathfrak{g}(2,3)$ ($\mathfrak{gl}(3)$ yields $2\mathfrak{g}(1,6)$ and $1\mathfrak{g}(2,3)$) (with CM) has a simple core $\mathfrak{bj} := \mathfrak{g}(2,3))/\mathfrak{c}$;

$\mathfrak{g}(3,6)$ ($\mathfrak{sl}(4)$ yields $7\mathfrak{g}(3,6)$);

$\mathfrak{g}(3,3)$ ($\mathfrak{sp}(6)$ yields $1\mathfrak{g}(3,3)$ and $10\mathfrak{g}(3,3)$);

$\mathfrak{g}(4,3)$ ($\mathfrak{o}(7)$ yields $1\mathfrak{g}(4,3)$);

$\mathfrak{g}(8,3)$ ($\mathfrak{f}(4)$ yields $1\mathfrak{g}(8,3)$);

$\mathfrak{g}(2,6)$ ($\mathfrak{sl}(5)$ yields $3\mathfrak{g}(2,6)$) (with CM) has a simple core $\mathfrak{g}(2,6))/\mathfrak{c}$;

$\mathfrak{g}(4,6)$ ($\mathfrak{gl}(6)$ yields $3\mathfrak{g}(4,6)$ and $\mathfrak{o}(10)$ yields $7\mathfrak{g}(4,6)$);

$\mathfrak{g}(6,6)$ ($\mathfrak{o}(11)$ yields $21\mathfrak{g}(6,6)$);

$\mathfrak{g}(8,6)$ ($\mathfrak{sl}(7)$ yields $8\mathfrak{g}(8,6)$ and $\mathfrak{e}(6)$ yields $3\mathfrak{g}(8,6)$);

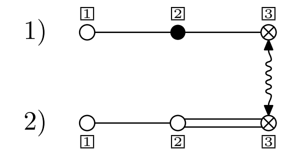
4) the Lie superalgebra $\mathfrak{el}(5;3)$ we have discovered is a $p = 3$ version of the Elduque superalgebra $\mathfrak{el}(5;5)$: Their Cartan matrices (whose elements are represented by non-positive integers) 7) for $\mathfrak{el}(5;5)$ and 1) for $\mathfrak{el}(5;3)$ are identical after a permutation of indices that is why we baptized $\mathfrak{el}(5;3)$ so. It can be obtained as an "enlargement" of any of the following Lie (super)algebras: $\mathfrak{sp}(8)$, $\mathfrak{sl}(1|4)$, $\mathfrak{sl}(2|3)$, $\mathfrak{osp}(4|4)$, $\mathfrak{osp}(6|2)$, $\mathfrak{g}(3,3)$.

20.8.1. Elduque and Cunha superalgebras: Systems of simple roots. For details of description of Elduque and Cunha superalgebras in terms of symmetric composition algebras, see [El1, CE, CE2]. Here we consider the simple Elduque and Cunha superalgebras with Cartan matrix for $p = 3$. In what follows, we list them using somewhat shorter notations as compared with the original ones: Hereafter $\mathfrak{g}(A, B)$ denotes the superalgebra occupying (A, B)th slot in the Elduque Supermagic Square; the first Cartan matrix is usually the one given in [CE], where only one Cartan matrix is given; the other matrices are obtained from the first one by means of reflections. Accordingly, $i\mathfrak{g}(A, B)$ is the shorthand for the realization of $\mathfrak{g}(A, B)$ by means of the ith Cartan matrix.

There are no instances of isotropic even reflections. On notation in the following tables, see subsect. 20.5.1.1.

20.8.1.1. $\mathfrak{g}(1,6)$ of sdim $= 21|14$. We have $\mathfrak{g}(1,6)_{\bar 0} = \mathfrak{sp}(6)$ and $\mathfrak{g}(1,6)_{\bar 1} = R(\pi_3)$.

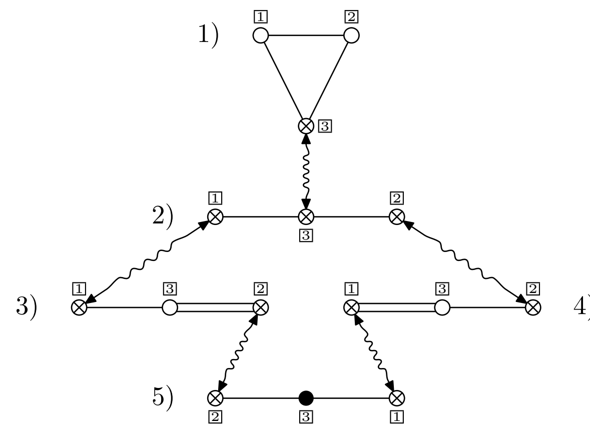
$$\begin{pmatrix} - & - & 2 \\ - & - & 1 \end{pmatrix}$$



$$1)\ \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad \boxed{2)}\ \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 0 \end{pmatrix}$$

20.8.1.2. $\mathfrak{g}(2,3)$ of sdim $= 12/10|14$. We have $\mathfrak{g}(2,3)_{\bar 0} = \mathfrak{gl}(3) \oplus \mathfrak{sl}(2)$ and $\mathfrak{g}(2,3)_{\bar 1} = \mathfrak{psl}(3) \otimes \mathrm{id}$.

$$\begin{pmatrix} - & - & 2 \\ 3 & 4 & 1 \\ 2 & 5 & - \\ 5 & 2 & - \\ 4 & 3 & - \end{pmatrix}$$

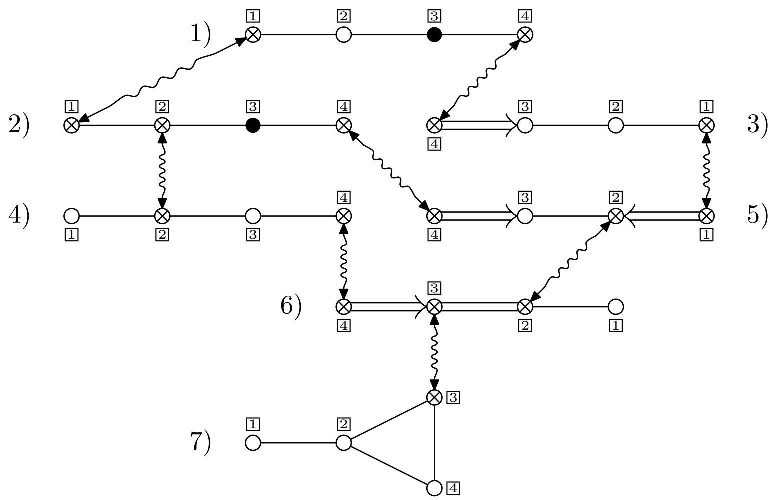


$$\boxed{1)}\ \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix} \quad \underline{2)}\ \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \quad 3)\ \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -2 \\ -1 & -2 & 2 \end{pmatrix}$$

$$4)\ \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & -1 \\ -2 & -1 & 2 \end{pmatrix} \quad \underline{5)}\ \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

20.8.1.3. $\mathfrak{g}(3,6)$ of sdim $= 36|40$. We have $\mathfrak{g}(3,6)_{\bar 0} = \mathfrak{sp}(8)$ and $\mathfrak{g}(3,6)_{\bar 1} = R(\pi_3)$.

$$\begin{pmatrix} 2 & - & - & 3 \\ 1 & 4 & - & 5 \\ 5 & - & - & 1 \\ - & 2 & - & 6 \\ 3 & 6 & - & 2 \\ - & 5 & 7 & 4 \\ - & - & 6 & - \end{pmatrix} \quad 1)\ \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \underline{2)}\ \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$



$$3)\begin{pmatrix}0&-1&0&0\\-1&2&-1&0\\0&-1&2&-2\\0&0&-1&0\end{pmatrix}\quad 4)\begin{pmatrix}2&-1&0&0\\-1&0&-2&0\\0&-2&2&-1\\0&0&-1&0\end{pmatrix}\quad 5)\begin{pmatrix}0&-1&0&0\\-2&0&-1&0\\0&-1&2&-2\\0&0&-1&0\end{pmatrix}$$

$$6)\begin{pmatrix}2&-1&0&0\\-1&0&-2&0\\0&-2&0&-2\\0&0&-1&0\end{pmatrix}\quad \boxed{7)}\begin{pmatrix}2&-1&0&0\\-1&2&-1&-1\\0&-1&0&-1\\0&-1&-1&2\end{pmatrix}$$

20.8.1.4. $\mathfrak{g}(3,3)$ of sdim = 23/21|16. We have $\mathfrak{g}(3,3)_{\bar 0}=(\mathfrak{o}(7)\oplus\mathbb{K}z)\oplus\mathbb{K}d$ and $\mathfrak{g}(3,3)_{\bar 1}=(\mathrm{spin}_7)_+\oplus(\mathrm{spin}_7)_-$; the action of d separates the summands — identical $\mathfrak{o}(7)$-modules spin_7, acting on one as the scalar multiplication by 1, on the other one by -1.

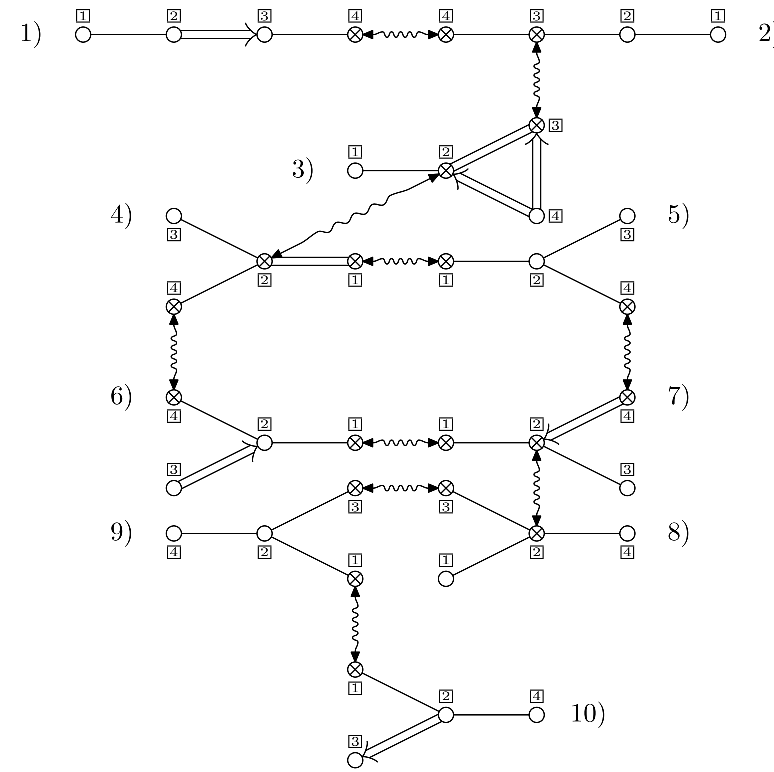
$$\begin{pmatrix}-&-&-&2\\-&-&3&1\\-&4&2&-\\5&3&-&6\\4&-&-&7\\7&-&-&4\\6&8&-&5\\-&7&9&-\\10&-&8&-\\9&-&-&-\end{pmatrix}\quad \boxed{1)}\begin{pmatrix}2&-1&0&0\\-1&2&-1&0\\0&-2&2&-1\\0&0&-1&0\end{pmatrix}\quad 2)\begin{pmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&0&-1\\0&0&-1&0\end{pmatrix}$$

$$3)\begin{pmatrix}2&-1&0&0\\-1&0&-2&-2\\0&-2&0&-2\\0&-1&-1&2\end{pmatrix}\quad 4)\begin{pmatrix}0&-1&0&0\\-2&0&-1&-1\\0&-1&2&0\\0&-1&0&0\end{pmatrix}$$

$$5)\begin{pmatrix}0&-1&0&0\\-1&2&-1&-1\\0&-1&2&0\\0&-1&0&0\end{pmatrix}\quad 6)\begin{pmatrix}0&-1&0&0\\-1&2&-2&-1\\0&-1&2&0\\0&-1&0&0\end{pmatrix}\quad 7)\begin{pmatrix}0&-1&0&0\\-1&0&-1&-2\\0&-1&2&0\\0&-1&0&0\end{pmatrix}$$

$$8)\begin{pmatrix}2&-1&-1&0\\-2&0&-2&-1\\-1&-1&0&0\\0&-1&0&2\end{pmatrix}\quad 9)\begin{pmatrix}0&0&-1&0\\0&2&-1&-1\\-1&-1&0&0\\0&-1&0&2\end{pmatrix}\quad \boxed{10)}\begin{pmatrix}0&0&-1&0\\0&2&-1&-1\\-1&-2&2&0\\0&-1&0&2\end{pmatrix}$$

20.8.1.5. $\mathfrak{g}(4,3)$ of sdim = 24|26. We have $\mathfrak{g}(4,3)_{\bar 0}=\mathfrak{o}(7)\oplus\mathfrak{sl}(2)$ and $\mathfrak{g}(4,3)_{\bar 1}=R(\pi_2)\otimes\mathrm{id}$.



$$\begin{pmatrix}-&-&-&2\\-&3&-&1\\4&2&5&-\\3&-&6&-\\6&-&3&7\\5&8&4&9\\9&-&-&5\\-&6&-&10\\7&10&-&6\\-&9&-&8\end{pmatrix}\quad \boxed{1)}\begin{pmatrix}2&-1&0&0\\-1&2&-2&-1\\0&-1&2&0\\0&-1&0&0\end{pmatrix}\quad 2)\begin{pmatrix}2&-1&0&0\\-1&0&-2&-2\\0&-1&2&0\\0&-1&0&0\end{pmatrix}$$

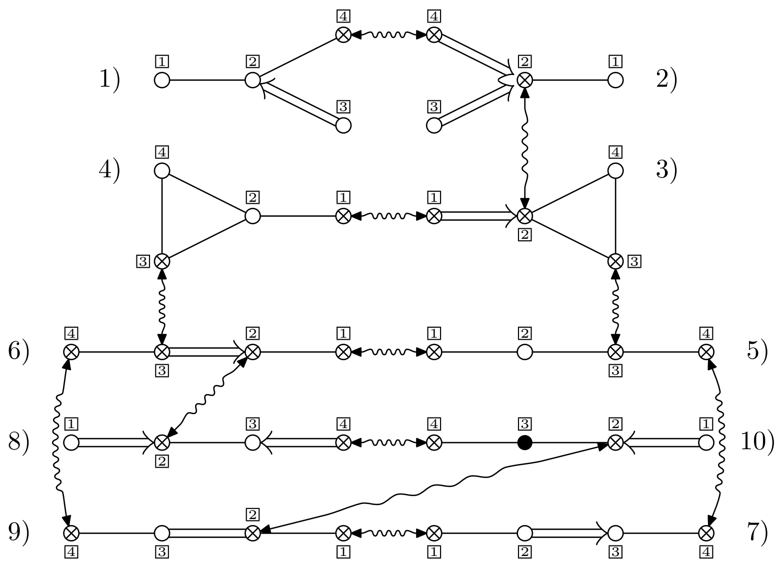
$$3)\begin{pmatrix}0&-1&0&0\\-2&0&-1&-1\\0&-1&0&-1\\0&-1&-1&2\end{pmatrix}\quad 4)\begin{pmatrix}0&-1&0&0\\-1&2&-1&-1\\0&-1&0&-1\\0&-1&-1&2\end{pmatrix}$$

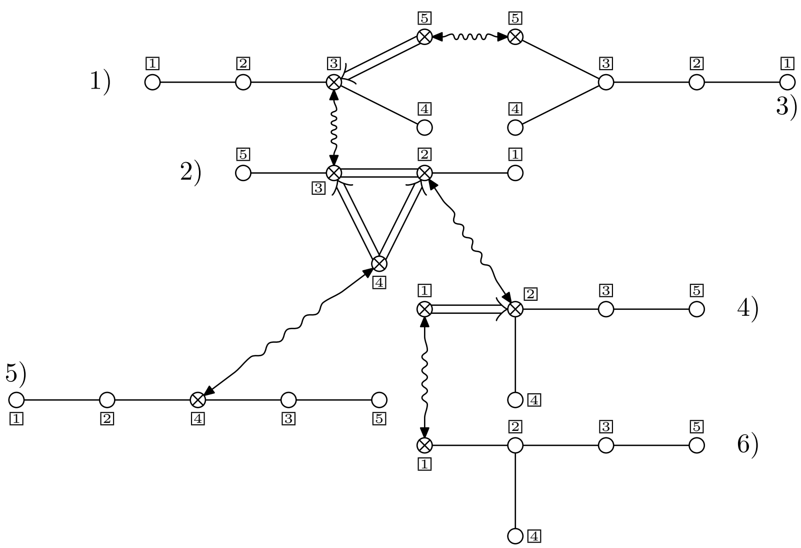
$$5)\begin{pmatrix}0&-1&0&0\\-1&2&-1&0\\0&-1&0&-1\\0&0&-1&0\end{pmatrix}\quad \underline{6)}\begin{pmatrix}0&-1&0&0\\-1&0&-2&0\\0&-1&0&-1\\0&0&-1&0\end{pmatrix}\quad 7)\begin{pmatrix}0&-1&0&0\\-1&2&-1&0\\0&-2&2&-1\\0&0&-1&0\end{pmatrix}$$

$$8)\begin{pmatrix}2&-1&0&0\\-2&0&-1&0\\0&-1&2&-2\\0&0&-1&0\end{pmatrix}\quad 9)\begin{pmatrix}0&-1&0&0\\-1&0&-2&0\\0&-2&2&-1\\0&0&-1&0\end{pmatrix}\quad 10)\begin{pmatrix}2&-1&0&0\\-2&0&-1&0\\0&-1&1&-1\\0&0&-1&0\end{pmatrix}$$

20.8.1.6. $\mathfrak{g}(2,6)$ of sdim = 36/34|20. We have $\mathfrak{g}(2,6)_{\bar 0}=\mathfrak{gl}(6)$ and $\mathfrak{g}(2,6)_{\bar 1}=R(\pi_3)$.

$$\begin{pmatrix}-&-&2&-&3\\-&4&1&5&-\\-&-&-&-&1\\6&2&-&-&-\\-&-&-&2&-\\4&-&-&-&-\end{pmatrix}\quad 1)\begin{pmatrix}2&-1&0&0&0\\-1&2&-1&0&0\\0&-1&0&-1&-2\\0&0&-1&2&0\\0&0&-1&0&0\end{pmatrix}\quad 2)\begin{pmatrix}2&-1&0&0&0\\-1&0&-2&-2&0\\0&-2&0&-2&-1\\0&-1&-1&0&0\\0&0&-1&0&2\end{pmatrix}$$





$$\boxed{3)}\begin{pmatrix}2&-1&0&0&0\\-1&2&-1&0&0\\0&-1&2&-1&-1\\0&0&-1&2&0\\0&0&-1&0&0\end{pmatrix}\sim\boxed{6)}\begin{pmatrix}0&-1&0&0&0\\-1&2&-1&-1&0\\0&-1&2&0&-1\\0&-1&0&2&0\\0&0&-1&0&2\end{pmatrix}$$

$$4)\ \begin{pmatrix}0&-1&0&0&0\\-2&0&-1&-1&0\\0&-1&2&0&-1\\0&-1&0&2&0\\0&0&-1&0&2\end{pmatrix}\qquad\boxed{5)}\begin{pmatrix}2&-1&0&0&0\\-1&2&0&-1&0\\0&0&2&-1&-1\\0&-1&-1&0&0\\0&0&-1&0&2\end{pmatrix}$$

20.8.1.7. $\mathfrak{g}(8,3)$ of sdim $= 55|50$. We have $\mathfrak{g}(8,3)_{\bar 0}=\mathfrak{f}(4)\oplus\mathfrak{sl}(2)$ and $\mathfrak{g}(8,3)_{\bar 1}=R(\pi_4)\otimes\mathrm{id}$.

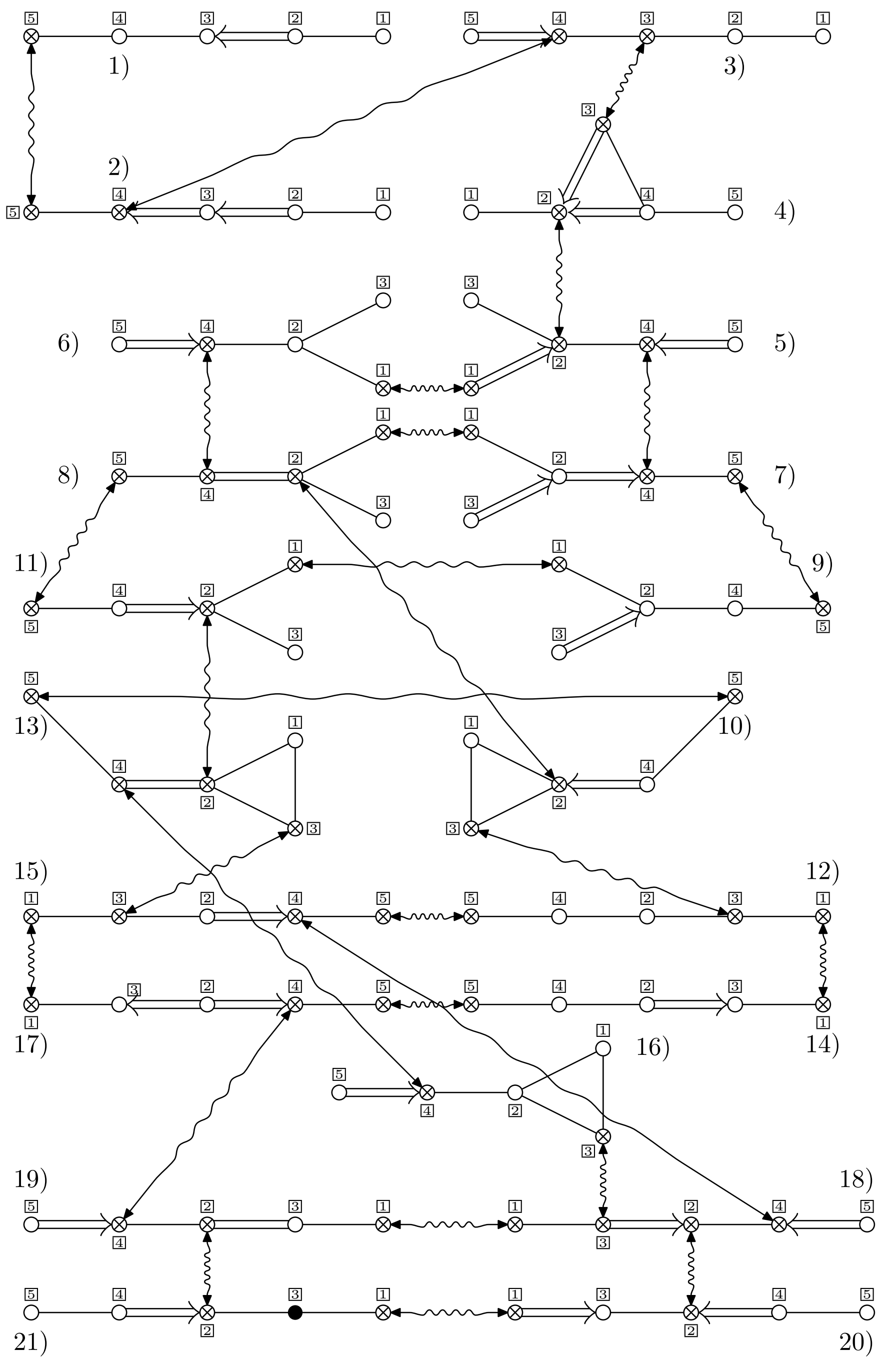
$$\begin{pmatrix}-&-&-&-&2\\-&-&-&3&1\\-&-&4&2&-\\-&5&3&-&-\\6&4&-&7&-\\5&-&-&8&-\\8&-&-&5&9\\7&10&-&6&11\\11&-&-&-&7\\-&8&12&-&13\\9&13&-&-&8\\14&-&10&-&15\\-&11&15&16&10\\12&-&-&-&17\\17&-&13&18&12\\-&-&18&13&-\\15&-&-&19&14\\19&20&16&15&-\\18&21&-&17&-\\21&18&-&-&-\\20&19&-&-&-\end{pmatrix}$$

$$\boxed{1)}\begin{pmatrix}2&-1&0&0&0\\-1&2&-1&0&0\\0&-2&2&-1&0\\0&0&-1&2&-1\\0&0&0&1&0\end{pmatrix}\quad 5)\ \begin{pmatrix}0&-1&0&0&0\\-2&0&-1&-1&0\\0&-1&2&0&0\\0&-1&0&0&-2\\0&0&0&-1&2\end{pmatrix}$$

$$2)\ \begin{pmatrix}2&-1&0&0&0\\-1&2&-1&0&0\\0&-2&2&-1&0\\0&0&-2&0&-1\\0&0&0&-1&0\end{pmatrix}\quad 6)\ \begin{pmatrix}0&-1&0&0&0\\-1&2&-1&-1&0\\0&-1&2&0&0\\0&-1&0&0&-2\\0&0&0&-1&2\end{pmatrix}$$

$$3)\ \begin{pmatrix}2&-1&0&0&0\\-1&2&-1&0&0\\0&-1&0&-1&0\\0&0&-1&0&-2\\0&0&0&-1&2\end{pmatrix}\quad 7)\ \begin{pmatrix}0&-1&0&0&0\\-1&2&-2&-1&0\\0&-1&2&0&0\\0&-2&0&0&-1\\0&0&0&-1&0\end{pmatrix}$$

$$4)\ \begin{pmatrix}2&-1&0&0&0\\-1&0&-2&-2&0\\0&-1&0&-1&0\\0&-1&-1&2&-1\\0&0&0&-1&2\end{pmatrix}\quad 8)\ \begin{pmatrix}0&-1&0&0&0\\-1&0&-1&-2&0\\0&-1&2&0&0\\0&-2&0&0&-1\\0&0&0&-1&0\end{pmatrix}$$



$$9)\ \begin{pmatrix} 0 & -1 & 0 & 0 & 0\\ -1 & 2 & -2 & -1 & 0\\ 0 & -1 & 2 & 0 & 0\\ 0 & -1 & 0 & 2 & -1\\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad 10)\ \begin{pmatrix} 2 & -1 & -1 & 0 & 0\\ 1 & 0 & 1 & 2 & 0\\ 1 & 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 2 & -1\\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad 11)\ \begin{pmatrix} 0 & -1 & 0 & 0 & 0\\ -1 & 0 & -1 & -2 & 0\\ 0 & -1 & 2 & 0 & 0\\ 0 & -1 & 0 & 2 & -1\\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$12)\ \begin{pmatrix} 0 & 0 & -1 & 0 & 0\\ 0 & 2 & -1 & -1 & 0\\ -1 & -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 2 & -1\\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad 13)\ \begin{pmatrix} 2 & -1 & -1 & 0 & 0\\ -1 & 0 & -1 & -2 & 0\\ -1 & -1 & 0 & 0 & 0\\ 0 & -2 & 0 & 0 & -1\\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad 14)\ \begin{pmatrix} 0 & 0 & -1 & 0 & 0\\ 0 & 2 & -1 & -1 & 0\\ -1 & -2 & 2 & 0 & 0\\ 0 & -1 & 0 & 2 & -1\\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$15)\ \begin{pmatrix} 0 & 0 & -1 & 0 & 0\\ 0 & 2 & -1 & -1 & 0\\ -1 & -1 & 0 & 0 & 0\\ 0 & -2 & 0 & 0 & -1\\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad 16)\ \begin{pmatrix} 2 & -1 & -1 & 0 & 0\\ -1 & 2 & -1 & -1 & 0\\ -1 & -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & -2\\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} \quad 17)\ \begin{pmatrix} 0 & 0 & -1 & 0 & 0\\ 0 & 2 & -1 & -1 & 0\\ -1 & -2 & 2 & 0 & 0\\ 0 & -2 & 0 & 0 & -1\\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$18)\ \begin{pmatrix} 0 & 0 & -1 & 0 & 0\\ 0 & 0 & -2 & -1 & 0\\ -1 & -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & -2\\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} \quad 19)\ \begin{pmatrix} 0 & 0 & -1 & 0 & 0\\ 0 & 0 & -2 & -1 & 0\\ -1 & -2 & 2 & 0 & 0\\ 0 & -1 & 0 & 0 & -2\\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$20)\ \begin{pmatrix} 0 & 0 & -1 & 0 & 0\\ 0 & 0 & -1 & -2 & 0\\ -2 & -1 & 2 & 0 & 0\\ 0 & -1 & 0 & 2 & -1\\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} \quad 21)\ \begin{pmatrix} 0 & 0 & -1 & 0 & 0\\ 0 & 0 & -1 & -2 & 0\\ -1 & -1 & 1 & 0 & 0\\ 0 & -1 & 0 & 2 & -1\\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

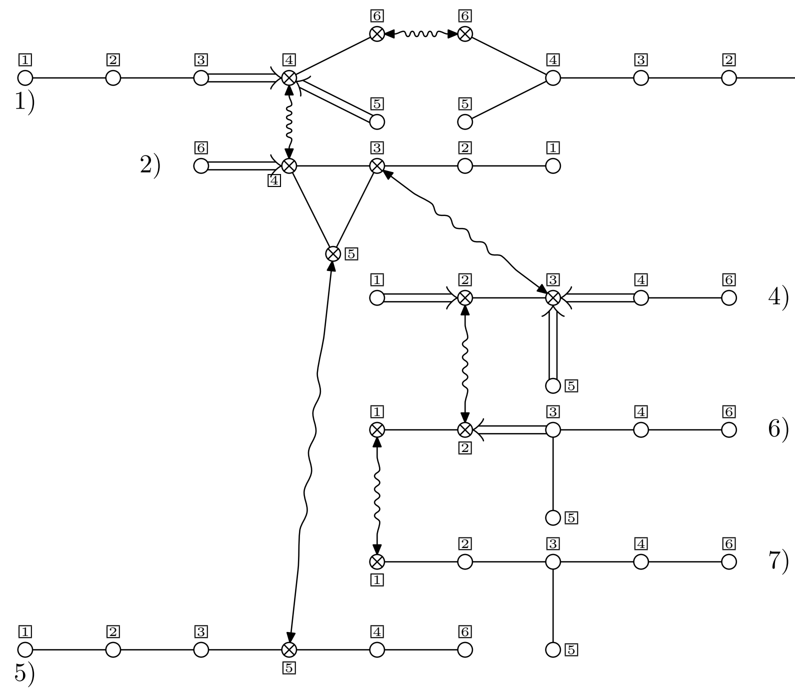
20.8.1.8. $\mathfrak{g}(4,6)$ of sdim $= 66|32$. We have $\mathfrak{g}(4,6)_{\bar 0} = \mathfrak{o}(12)$ and $\mathfrak{g}(4,6)_{\bar 1} = R(\pi_5)$.

$$\begin{pmatrix} - & - & - & 2 & - & 3\\ - & - & 4 & 1 & 5 & -\\ - & - & - & - & - & 1\\ - & 6 & 2 & - & - & -\\ - & - & - & - & 2 & -\\ 7 & 4 & - & - & - & -\\ 6 & - & - & - & - & - \end{pmatrix} \quad 1)\ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & 0 & 0\\ 0 & 0 & -2 & 0 & -2 & -1\\ 0 & 0 & 0 & -1 & 2 & 0\\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$2)\ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0\\ 0 & -2 & 0 & -1 & -1 & 0\\ 0 & 0 & -1 & 0 & -1 & -2\\ 0 & 0 & -1 & -1 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix} \quad \boxed{3)}\ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & 0 & 0\\ 0 & 0 & -1 & 2 & -1 & -1\\ 0 & 0 & 0 & -1 & 2 & 0\\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \quad 4)\ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0\\ -2 & 0 & -1 & 0 & 0 & 0\\ 0 & -1 & 0 & -2 & -2 & 0\\ 0 & 0 & -1 & 2 & 0 & -1\\ 0 & 0 & -1 & 0 & 2 & 0\\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$\boxed{5)}\ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0\\ 0 & -1 & 2 & 0 & -1 & 0\\ 0 & 0 & 0 & 2 & -1 & -1\\ 0 & 0 & -1 & -1 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix} \quad 6)\ \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0\\ -1 & 0 & -2 & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & -1 & 0\\ 0 & 0 & -1 & 2 & 0 & -1\\ 0 & 0 & -1 & 0 & 2 & 0\\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix} \quad \boxed{7)}\ \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & -1 & 0\\ 0 & 0 & -1 & 2 & 0 & -1\\ 0 & 0 & -1 & 0 & 2 & 0\\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

20.8.1.9. $\mathfrak{g}(6,6)$ of sdim $= 78|64$. We have $\mathfrak{g}(6,6)_{\bar 0} = \mathfrak{o}(13)$ and $\mathfrak{g}(6,6)_{\bar 1} = \mathrm{spin}_{13}$.



$$\begin{pmatrix} 2 & 3 & - & 4 & - & 5 \\ 1 & - & - & 6 & - & 7 \\ - & 1 & 8 & 9 & - & 10 \\ 6 & 9 & 11 & 1 & 12 & - \\ 7 & 10 & - & - & - & 1 \\ 4 & - & 13 & 2 & 14 & - \\ 5 & - & - & - & - & 2 \\ - & - & 3 & - & - & 15 \\ - & 4 & - & 3 & 16 & - \\ - & 5 & 15 & - & - & 3 \\ 13 & - & 4 & - & - & - \\ 14 & 16 & - & - & 4 & - \\ 11 & 17 & 6 & - & - & - \\ 12 & - & - & - & 6 & - \\ - & - & 10 & 18 & - & 8 \\ - & 12 & 19 & - & 9 & - \\ - & 13 & - & - & - & - \\ - & - & - & 15 & 20 & - \\ - & - & 16 & - & - & - \\ - & - & - & - & 18 & 21 \\ - & - & - & - & - & 20 \end{pmatrix}$$

$$1)\ \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 & -2 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$2)\ \begin{pmatrix} 0 & -2 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 & -2 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$3)\ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 & -2 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$4)\ \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix} \quad 5)\ \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \quad 6)\ \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

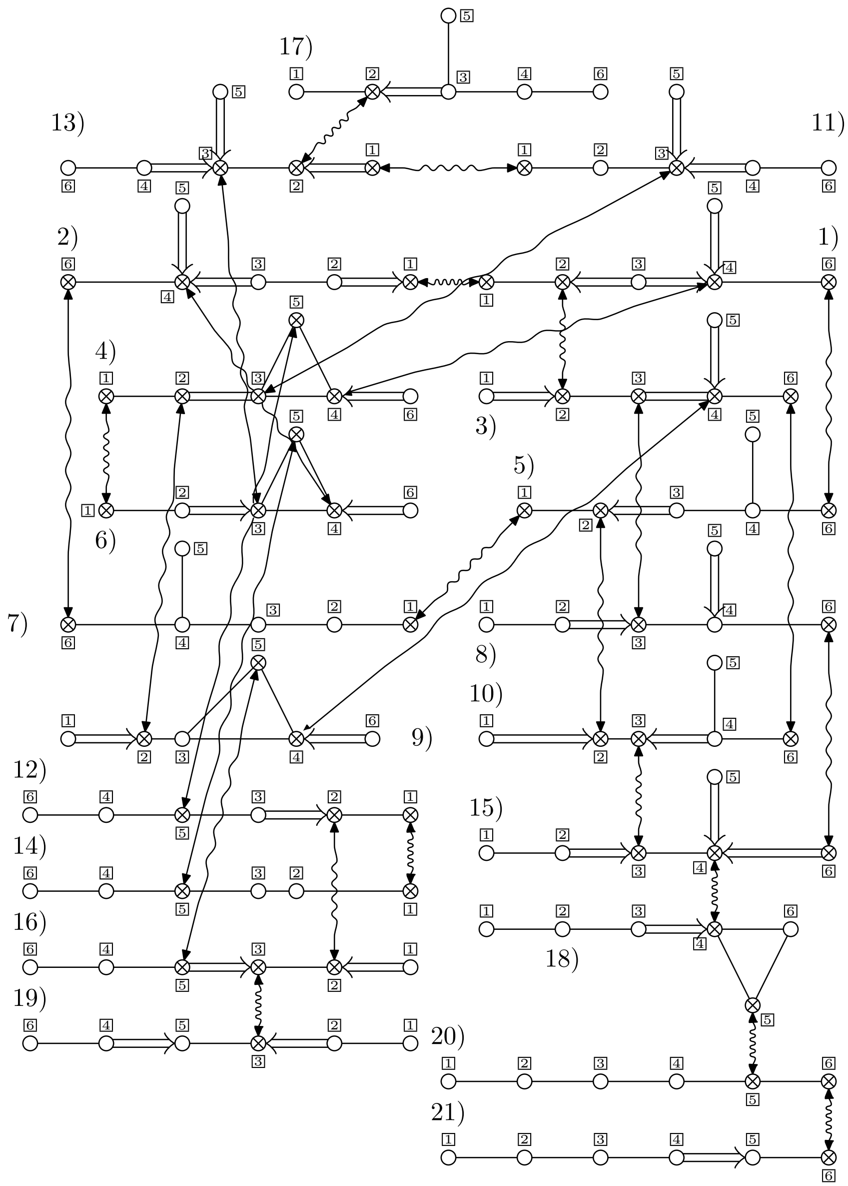
$$7)\ \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \quad 8)\ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -2 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \quad 9)\ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$10)\ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \quad 11)\ \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 & -2 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix} \quad 12)\ \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$13)\ \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 & -2 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix} \quad 14)\ \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix} \quad 15)\ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -2 & -2 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$16)\ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix} \quad \boxed{17)}\ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix} \quad 18)\ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$$\boxed{19)}\ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & -1 & -2 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix} \quad 20)\ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad \boxed{21)}\ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -2 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$



20.8.1.10. $\mathfrak{g}(8,6)$ of sdim $=$ 133|56. We have $\mathfrak{g}(8,6)_{\bar 0} = \mathfrak{e}(7)$ and $\mathfrak{g}(8,6)_{\bar 1} = R(\pi_1)$.

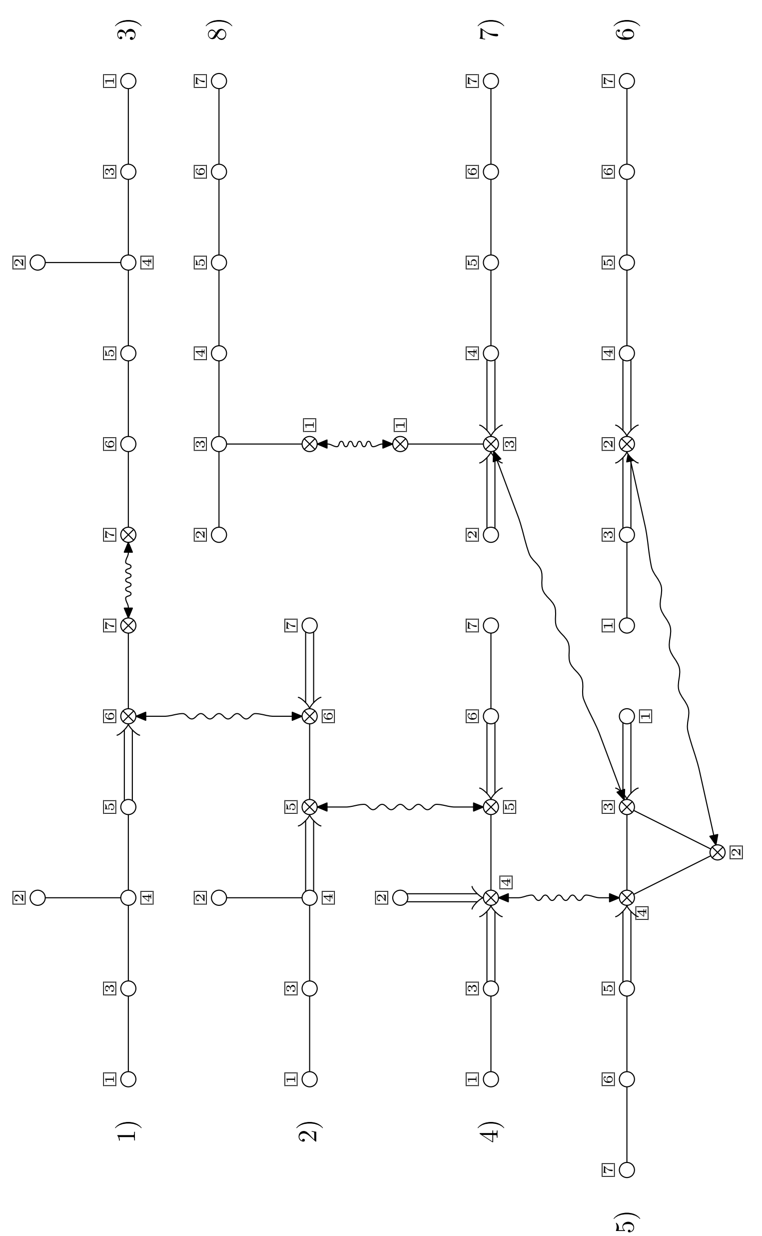
$$\begin{pmatrix} - & - & - & - & - & 2 & 3\\ - & - & - & - & 4 & 1 & -\\ - & - & - & - & - & - & 1\\ - & - & - & 5 & 2 & - & -\\ - & 6 & 7 & 4 & - & - & -\\ - & 5 & - & - & - & - & -\\ 8 & - & 5 & - & - & - & -\\ 7 & - & - & - & - & - & - \end{pmatrix}$$

$$1)\ \begin{pmatrix} 2&0&-1&0&0&0&0\\ 0&2&0&-1&0&0&0\\ -1&0&2&-1&0&0&0\\ 0&-1&-1&2&-1&0&0\\ 0&0&0&-1&2&-1&0\\ 0&0&0&0&-2&0&-1\\ 0&0&0&0&0&-1&0 \end{pmatrix} \qquad 2)\ \begin{pmatrix} 2&0&-1&0&0&0&0\\ 0&2&0&-1&0&0&0\\ -1&0&2&-1&0&0&0\\ 0&-1&-1&2&-1&0&0\\ 0&0&0&-2&0&-1&0\\ 0&0&0&0&-1&0&-2\\ 0&0&0&0&0&-1&2 \end{pmatrix}$$

$$\boxed{3)}\ \begin{pmatrix} 2&0&-1&0&0&0&0\\ 0&2&0&-1&0&0&0\\ -1&0&2&-1&0&0&0\\ 0&-1&-1&2&-1&0&0\\ 0&0&0&-1&2&-1&0\\ 0&0&0&0&-1&2&-1\\ 0&0&0&0&0&-1&0 \end{pmatrix} \qquad 4)\ \begin{pmatrix} 2&0&-1&0&0&0&0\\ 0&2&0&-1&0&0&0\\ -1&0&2&-1&0&0&0\\ 0&-2&-2&0&-1&0&0\\ 0&0&0&-1&0&-2&0\\ 0&0&0&0&-1&2&-1\\ 0&0&0&0&0&-1&2 \end{pmatrix}$$

$$5)\ \begin{pmatrix} 2&0&-1&0&0&0&0\\ 0&0&-1&-1&0&0&0\\ -2&-1&0&-1&0&0&0\\ 0&-1&-1&0&-2&0&0\\ 0&0&0&-1&2&-1&0\\ 0&0&0&0&-1&2&-1\\ 0&0&0&0&0&-1&2 \end{pmatrix} \qquad \boxed{6)}\ \begin{pmatrix} 2&0&-1&0&0&0&0\\ 0&0&-2&-2&0&0&0\\ -1&-1&2&0&0&0&0\\ 0&-1&0&2&-1&0&0\\ 0&0&0&-1&2&-1&0\\ 0&0&0&0&-1&2&-1\\ 0&0&0&0&0&-1&2 \end{pmatrix}$$

$$7)\ \begin{pmatrix} 0&0&-1&0&0&0&0\\ 0&2&-1&0&0&0&0\\ -1&-2&0&-2&0&0&0\\ 0&0&-1&2&-1&0&0\\ 0&0&0&-1&2&-1&0\\ 0&0&0&0&-1&2&-1\\ 0&0&0&0&0&-1&2 \end{pmatrix} \qquad \boxed{8)}\ \begin{pmatrix} 0&0&-1&0&0&0&0\\ 0&2&-1&0&0&0&0\\ -1&-1&2&-1&0&0&0\\ 0&0&-1&2&-1&0&0\\ 0&0&0&-1&2&-1&0\\ 0&0&0&0&-1&2&-1\\ 0&0&0&0&0&-1&2 \end{pmatrix}$$



20.8.2. The Elduque superalgebra $\mathfrak{el}(5;3)$: Systems of simple roots. Its superdimension is $39|32$; the even part is $\mathfrak{el}(5;3)_{\bar 0} = \mathfrak{sp}(8)\oplus\mathfrak{sl}(2)$ and its odd part is irreducible: $\mathfrak{el}(5;3)_{\bar 1} = R(\pi_4)\otimes \mathrm{id}$.

The following are all its Cartan matrices (on the rectangular matrix, see subsect. 20.5.1.1):

$$\begin{pmatrix} 2&3&-&-&-\\ 1&-&-&-&-\\ -&1&-&4&-\\ 5&-&6&3&-\\ 4&-&7&-&-\\ 7&-&4&-&8\\ 6&-&5&9&10\\ 10&-&-&-&6\\ -&11&-&7&12\\ 8&-&-&12&7\\ -&9&13&-&14\\ -&14&15&10&9\\ -&-&11&-&-\\ -&12&-&-&11\\ -&-&12&-&- \end{pmatrix}$$

$$1)\ \begin{pmatrix} 0&-1&0&0&0\\ -1&0&0&-1&0\\ 0&0&2&-1&-1\\ 0&-1&-1&2&0\\ 0&0&-1&0&2 \end{pmatrix} \quad \boxed{2)}\ \begin{pmatrix} 0&-2&0&0&0\\ -1&2&0&-2&0\\ 0&0&2&-1&-1\\ 0&-1&-1&2&0\\ 0&0&-1&0&2 \end{pmatrix}$$

$$3)\ \begin{pmatrix} 2&-1&0&-1&0\\ -2&0&0&-2&0\\ 0&0&2&-1&-1\\ -2&-2&-1&0&0\\ 0&0&-1&0&2 \end{pmatrix} \quad 4)\ \begin{pmatrix} 0&0&0&-1&0\\ 0&2&0&-1&0\\ 0&0&0&-2&-1\\ -1&-1&-2&0&0\\ 0&0&-1&0&2 \end{pmatrix}$$

$$5)\ \begin{pmatrix} 0&0&0&-2&0\\ 0&2&0&-1&0\\ 0&0&0&-2&-1\\ -1&-2&-1&2&0\\ 0&0&-1&0&2 \end{pmatrix} \quad 6)\ \begin{pmatrix} 0&0&0&-1&0\\ 0&2&0&-1&0\\ 0&0&0&-1&-2\\ -1&-1&-1&2&0\\ 0&0&-2&0&0 \end{pmatrix}$$

$$7)\ \begin{pmatrix} 0&0&0&-2&0\\ 0&2&0&-1&0\\ 0&0&0&-1&-2\\ -2&-1&-1&0&0\\ 0&0&-2&0&0 \end{pmatrix} \quad 8)\ \begin{pmatrix} 0&0&0&-1&0\\ 0&2&0&-1&0\\ 0&0&2&-1&-1\\ -1&-1&-1&2&0\\ 0&0&-1&0&0 \end{pmatrix} \quad 9)\ \begin{pmatrix} 2&0&0&-1&0\\ 0&0&-2&-2&0\\ 0&-1&2&-1&-1\\ -1&-2&-2&0&0\\ 0&0&-2&0&0 \end{pmatrix}$$

$$10)\ \begin{pmatrix} 0&0&0&-2&0\\ 0&2&0&-1&0\\ 0&0&2&-1&-1\\ -2&-1&-1&0&0\\ 0&0&-1&0&0 \end{pmatrix} \quad 11)\ \begin{pmatrix} 2&0&0&-1&0\\ 0&0&-1&-1&0\\ 0&-1&0&0&-2\\ -1&-1&0&2&0\\ 0&0&-2&0&0 \end{pmatrix} \quad 12)\ \begin{pmatrix} 2&0&0&-1&0\\ 0&0&-2&-2&0\\ 0&-2&0&-2&-1\\ -1&-2&-2&0&0\\ 0&0&-1&0&0 \end{pmatrix}$$

$$\boxed{13)}\ \begin{pmatrix} 2&0&0&-1&0\\ 0&2&-1&-2&0\\ 0&-2&0&0&-1\\ -1&-1&0&2&0\\ 0&0&-1&0&2 \end{pmatrix} \quad 14)\ \begin{pmatrix} 2&0&0&-1&0\\ 0&0&-1&-1&0\\ 0&-1&2&0&-1\\ -1&-1&0&2&0\\ 0&0&-1&0&0 \end{pmatrix} \quad \boxed{15)}\ \begin{pmatrix} 2&0&0&-1&0\\ 0&2&-1&0&0\\ 0&-1&0&-1&-2\\ -1&0&-1&2&0\\ 0&0&-1&0&2 \end{pmatrix}$$

20.9. The answer: The case where $p = 2$

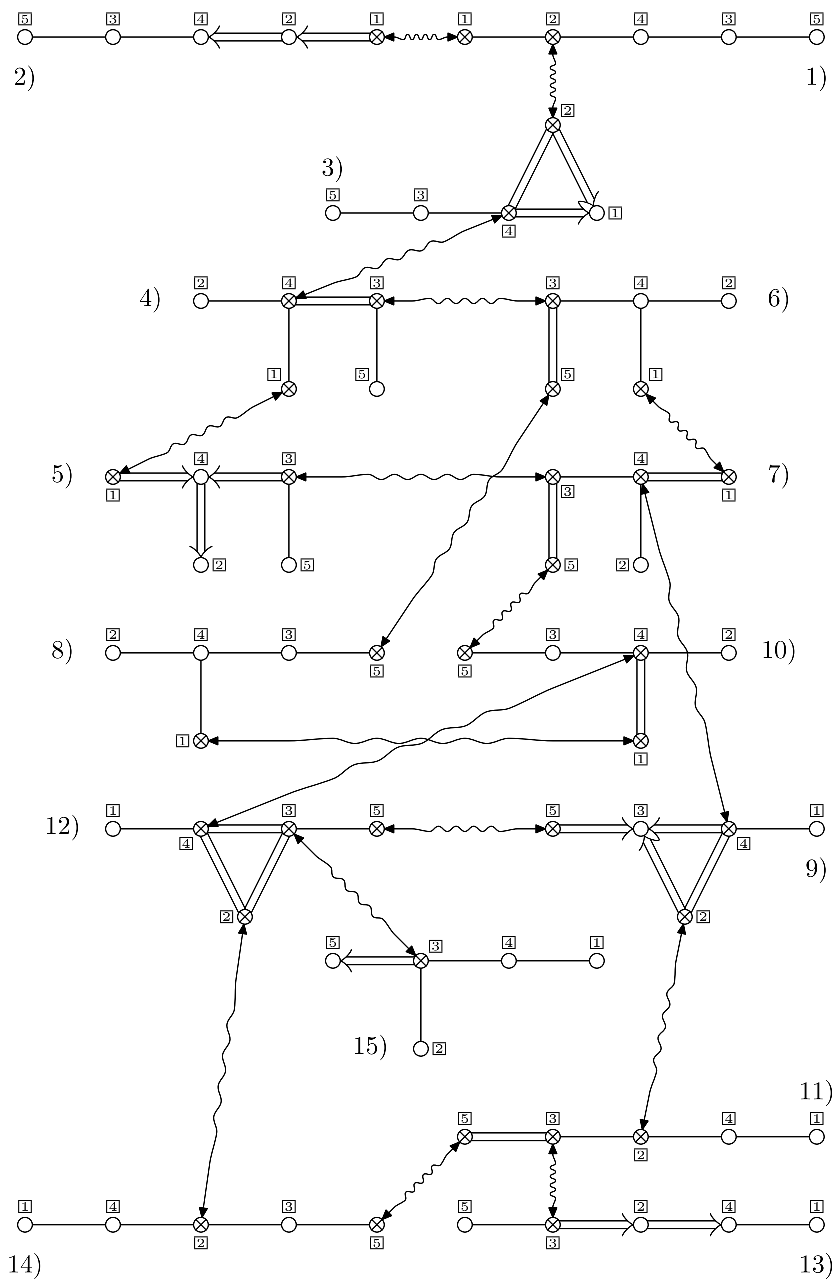
Simple Lie algebras:

1) The Lie algebras obtained from their Cartan matrices by reducing modulo 2 (for $\mathfrak{o}(2n+1)$ one has, first of all, to divide the last row by 2 in order to adequately normalize CM). We thus get:

the CM versions of $\mathfrak{sl}$, namely: $\mathfrak{sl}(2n+1)$, and $\mathfrak{gl}(2n)$ whose "simple core" is $\mathfrak{psl}(2n)$; in the "second" integer basis of $\mathfrak{g}(2)$ given in [FH], p. 346, all structure constants are integer and $\mathfrak{g}(2)$ becomes, after reduction modulo 2, a simple Lie algebra $\mathfrak{psl}(4)$ (without Cartan matrix, as we know);

the "simple cores" of the orthogonal algebras, namely, of $\mathfrak{o}^{(1)}(2n+1)$ and $\mathfrak{oc}(2n)$;

$\mathfrak{e}(6)$, $\mathfrak{e}(7)/\mathfrak{c}$, $\mathfrak{e}(8)$;



2) the Weisfeiler and Kac algebras $\mathfrak{wk}(3;a)^{(1)}/\mathfrak{c}$ and $\mathfrak{wk}(4;a)$.

<u>Simple Lie superalgebras</u>

In the list below the term "super version" of a Lie algebra $\mathfrak{g}(A)$ stands for a Lie superalgebra with the "same" root system as that of $\mathfrak{g}(A)$ but with some of the simple roots considered odd.

1) The Lie superalgebras obtained from their $p = 0$ analogs that have no -2 in off-diagonal slots of the Cartan matrix by reducing the structure constants modulo 2 (for $\mathfrak{osp}(2n+1|2m)$ one has, first of all, to divide the last row by 2 in order to normalize CM), we thus get

the CM versions of $\mathfrak{sl}$, namely: either simple $\mathfrak{sl}(a|a+2k+1)$ or $\mathfrak{gl}(a|2k+a)$ whose "simple core" is $\mathfrak{psl}(a|a+2k)$ and $\mathfrak{psl}(a|a+2k)^{(1)}$ if $a = 2n$;

2) the ortho-orthogonal algebras, namely: $\mathfrak{oo}^{(1)}$ and $\mathfrak{ooc}$;

3) $\mathfrak{bgl}(3;a)^{(1)}/\mathfrak{c}$ which is an analog of $\mathfrak{wk}(3;a)^{(1)}/\mathfrak{c}$ with "same" Cartan matrices but different root systems;

4) the CM versions of periplectic algebras, namely: $\mathfrak{pec}$; these are at the same time super versions of $\mathfrak{oc}$;

5) a super version of $\mathfrak{wk}(4;a)$, namely: $\mathfrak{bgl}(4;a)$;

6) the super versions of $\mathfrak{e}(6)$, namely: $\mathfrak{e}(6,1)$, $\mathfrak{e}(6,6)$;

7) the super versions of $\mathfrak{e}(7)$, namely: $\mathfrak{e}(7,1)$, $\mathfrak{e}(7,6)$, $\mathfrak{e}(7,7)$ whose "simple cores" are described in the next subsection;

8) the super versions of $\mathfrak{e}(8)$, namely: $\mathfrak{e}(8,1)$, $\mathfrak{e}(8,8)$.

20.9.1. On the structure of $\mathfrak{bgl}(3;\alpha)$, $\mathfrak{bgl}(4;\alpha)$, and $\mathfrak{e}(a,b)$. In this section we describe the even parts $\mathfrak{g}_{\bar 0}$ of the new Lie superalgebras $\mathfrak{g} = \mathfrak{g}(A)$ and their odd parts $\mathfrak{g}_{\bar 1}$ as $\mathfrak{g}_{\bar 0}$-modules. `SuperLie` enumerates the elements of the Chevalley basis the x_i (positive), starting with the generators, then their brackets, etc., and the y_i are negative root vectors opposite to the x_i. Since the irreducible representations of the Lie algebras may have neither highest nor lowest weight, observe that the $\mathfrak{g}_{\bar 0}$-modules $\mathfrak{g}_{\bar 1}$ always have both highest and lowest weights.

20.9.1.1. Notation $\mathfrak{A} \oplus_c \mathfrak{B}$ needed to describe $\mathfrak{bgl}(4;\alpha)$, $\mathfrak{e}(6,6)$, $\mathfrak{e}(7,6)$, and $\mathfrak{e}(8,1)$. This notation describes the case where $\mathfrak{A}$ and $\mathfrak{B}$ are nontrivial central extensions of the Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$, respectively, and $\mathfrak{A} \oplus_c \mathfrak{B}$ — a nontrivial central extension of $\mathfrak{a} \oplus \mathfrak{b}$ (or, perhaps, a more complicated $\mathfrak{a} \Subset \mathfrak{b}$) with 1-dimensional center spanned by c — is such that the restriction of the extension of $\mathfrak{a} \oplus \mathfrak{b}$ to $\mathfrak{a}$ gives $\mathfrak{A}$ and that to $\mathfrak{b}$ gives $\mathfrak{B}$. (In other words, the situation resembles the (nontrivial) central extension of the Lie algebra of derivations of the loop algebra, namely, $\mathfrak{g} \otimes \mathbb{C}[t^{-1}, t] \Subset \mathfrak{der}(\mathbb{C}[t^{-1}, t])$, where one central element serves both central extensions: Those of $\mathfrak{g} \otimes \mathbb{C}[t^{-1}, t]$ and of $\mathfrak{der}(\mathbb{C}[t^{-1}, t])$.)

In these four cases, the even part of $\mathfrak{g}(A)$ is of the form

$$\mathfrak{g}(B) \oplus_c \mathfrak{hei}(2) \simeq \mathfrak{g}(B) \oplus \operatorname{Span}(X^+, X^-),$$

where the matrix B is degenerate (so $\mathfrak{g}(B)$ has a grading element d and a central element c), and where X^+, X^- and c span the Heisenberg Lie algebra $\mathfrak{hei}(2)$. The brackets are:

$$\begin{array}{ll} [\mathfrak{g}^{(1)}(B), X^{\pm}] = 0; & \\ [d, X^{\pm}] = X^{\pm}; & ([d, X^{\pm}] = \alpha X^{\pm} \text{ for } \mathfrak{bgl}(4;\alpha)) \\ [X^+, X^-] = c. & \end{array} \tag{20.39}$$

The odd part of $\mathfrak{g}(A)$ (at least in two of the four cases) consists of two copies of the same $\mathfrak{g}(B)$-module N, the operators $\mathrm{ad}_{X^\pm}$ permute these copies, and $\mathrm{ad}^2_{X^\pm} = 0$, so each of the operators maps one of the copies to the other, and this other copy to zero.

20.9.1.2. $\mathfrak{bgl}(3;\alpha)$, where $\alpha \neq 0, 1$. This Lie superalgebra is of $\mathrm{sdim} = 10|8$, so sdim of the simple subquotient $\mathfrak{bgl}(3;\alpha)^{(1)}/\mathfrak{c}$ is equal to $8|8$. We consider the following Cartan matrix and the corresponding positive root vectors (odd | even)

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & \bar{0} & \alpha \\ 0 & \alpha & \bar{0} \end{pmatrix} \quad \begin{array}{l} x_1 \mid x_2, \quad x_3, \\ x_4 = [x_1,\ x_2] \mid x_5 = [x_2,\ x_3], \quad x_6 = [x_3, [x_1,\ x_2]], \mid \\ x_7 = [[x_1,\ x_2],\ [x_2, x_3]] \mid \end{array}$$

Then $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{gl}(3) \oplus \mathbb{K}Z$. The $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is reducible, with the two highest weight vectors, x_7 and y_1. The Cartan subalgebra of $\mathfrak{gl}(3) \oplus \mathbb{K}Z$ is spanned by $\alpha h_1 + h_3, h_2, h_3$ and Z. In this basis, the weight of x_7 is $(0, 1+\alpha, 0, 1)$. The weight of y_1 is $(0, 1, 0, 1)$, if for the grading operator we take $(1, 0, 0) \in \mathfrak{gl}(3)$.

The lowest weight vectors of these modules are x_1 and y_7 and their weights are $(0, 1, 0, 1)$ and $(0, 1+\alpha, 0, 1)$.

The module generated by x_7 is $\mathrm{Span}\,\{x_1, x_4, x_6, x_7\}$. The module generated by y_1 is $\mathrm{Span}\,\{y_1, y_4, y_6, y_7\}$.

All inequivalent Cartan matrices are

$$\begin{pmatrix} d_1 & \alpha & 1 \\ \alpha & d_2 & 0 \\ 1 & 0 & d_3 \end{pmatrix}, \quad \begin{pmatrix} d_1 & \alpha & 1+\alpha \\ \alpha & d_2 & 1 \\ 1+\alpha & 1 & d_3 \end{pmatrix},$$

where (d_1, d_2, d_3) is any distribution of 0's and $\bar{0}$'s, except $(\bar{0}, \bar{0}, \bar{0})$.

20.9.1.3. $\mathfrak{bgl}(4;\alpha)$, where $\alpha \neq 0, 1$, of $\mathrm{sdim} = 18|16$. We consider the following Cartan matrix and the corresponding positive root vectors (odd | even)

$$\begin{pmatrix} 0 & \alpha & 1 & 0 \\ \alpha & \bar{0} & 0 & 0 \\ 1 & 0 & \bar{0} & 1 \\ 0 & 0 & 1 & \bar{0} \end{pmatrix} \quad \begin{array}{l} x_1 \mid x_2, \quad x_3, \quad x_4, \\ x_5 = [x_1,\ x_2], \quad x_6 = [x_1, x_3] \mid x_7 = [x_3,\ x_4], \\ x_8 = [x_3,\ [x_1, x_2]], \quad x_9 = [x_4,\ [x_1,\ x_3]] \mid \\ x_{11} = [[x_1,\ x_2], [x_3,\ x_4]] \mid x_{10} = [[x_1, x_2],\ [x_1,\ x_3]] \\ \mid x_{12} = [[x_1,\ x_2],\ [x_4, [x_1,\ x_3]]], \\ \mid x_{13} = [[x_3,\ [x_1,\ x_2]], [x_4,\ [x_1,\ x_3]]], \\ x_{14} = [[x_4,\ [x_1, x_3]],\ [[x_1,\ x_2],\ [x_1, x_3]]] \mid \\ x_{15} = [[[x_1,\ x_2],\ [x_1,\ x_3]], [[x_1,\ x_2],\ [x_3, x_4]]] \mid \end{array}$$

In this case $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{gl}(4) \oplus_c \mathfrak{hei}(2)$ see (20.9.1.1) with commutation relations (20.39). The $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is irreducible: $\mathfrak{g}_{\bar{1}} \simeq N \otimes \mathrm{id}$, where id is the standard 2-dimensional $\mathfrak{hei}(2)$-module and N is an 8-dimensional $\mathfrak{gl}(4)$-module.

The highest weight vector x_{15} has weight $(\alpha, 0, 0, 0, \alpha)$ with respect to

$$c = h_2, \quad d = h_1, \quad H_1 = h_3, \quad H_2 = h_3, \quad H_3 = h_2 + h_3,$$

where the h_i's are the Chevalley generators of the Cartan subalgebra of $\mathfrak{bgl}(4;\alpha)$. The lowest weight vector is y_{15} of the same weight as x_{15}.

All inequivalent Cartan matrices of $\mathfrak{bgl}(4;\alpha)$ are

$$\begin{pmatrix} d_1 & \alpha & 0 & 0 \\ \alpha & d_2 & 1 & 0 \\ 0 & 1 & d_3 & 1 \\ 0 & 0 & 1 & d_4 \end{pmatrix}, \quad \begin{pmatrix} d_1 & 1 & 1+\alpha & 0 \\ 1 & d_2 & \alpha & 0 \\ \alpha+1 & \alpha & d_3 & \alpha \\ 0 & 0 & \alpha & d_4 \end{pmatrix}, \quad \begin{pmatrix} d_1 & \alpha & 0 & 0 \\ \alpha & d_2 & \alpha+1 & 0 \\ 0 & \alpha+1 & d_3 & 1 \\ 0 & 0 & 1 & d_4 \end{pmatrix},$$

where $\{d_1, d_2, d_3, d_4\}$ is any distribution of 0's and $\bar{0}$'s, except $\{\bar{0}, \bar{0}, \bar{0}, \bar{0}\}$.

20.9.1.4. Proposition (Cf. (20.28) and (20.25))**.** *1) We have*

$$\begin{array}{l} \mathfrak{bgl}(3;a) \simeq \mathfrak{bgl}(3;a') \Longleftrightarrow a' = \dfrac{\alpha a + \beta}{\gamma a + \delta}, \text{ where } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2;\mathbb{Z}/2) \\ \mathfrak{bgl}(4;a) \simeq \mathfrak{bgl}(4;a') \Longleftrightarrow a' = \dfrac{1}{a}. \end{array} \tag{20.40}$$

2) The $2|4$-structures on $\mathfrak{bgl}(3;a)$ and $\mathfrak{bgl}(4;a)$ are given by the same formulas (20.5.1.2) as for $\mathfrak{wk}(3;a)$ and $\mathfrak{wk}(4;a)$ with the amendment: $(e^{\pm}_{\alpha})^{[2]} = 0$ for all even root vectors and $(e^{\pm}_{\alpha})^{[4]} = ((e^{\pm}_{\alpha})^2)^{[2]}$ for all odd root vectors.

20.9.1.5. The $\mathfrak{e}$-type superalgebras. Notation: The $\mathfrak{e}$-type superalgebras will be denoted by their simplest Dynkin diagrams: $\mathfrak{e}(n,i)$ denotes the Lie superalgebra whose diagram is of the same shape as that of the Lie algebra $\mathfrak{e}(n)$ but with the only — ith — node $\otimes$. This, and other "simplest", Cartan matrices are boxed. We enumerate the nodes of the Dynkin diagram of $\mathfrak{e}(n)$ as in [Bou, OV]: We first enumerate the nodes in the row corresponding to $\mathfrak{sl}(n)$ (from the end-point of the "longest" twig towards the branch point and further on along the second long twig), and the nth node is the end-point of the shortest "twig".

20.9.1.5a. $\mathfrak{e}(6,1) \simeq \mathfrak{e}(6,5)$ of $\mathrm{sdim} = 46|32$. We have $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{oc}(2;10) \oplus \mathbb{K}Z$ and $\mathfrak{g}_{\bar{1}}$ is a reducible module of the form $R(\pi_4) \oplus R(\pi_5)$ with the two highest weight vectors

$$x_{36} = [[[x_4, x_5], [x_6, [x_2, x_3]]], [[x_3, [x_1, x_2]], [x_6, [x_3, x_4]]]]$$

and y_5. Let $Z, h_1, h_2, h_3, h_4, h_6$ be basis elements of the Cartan subalgebra. The weights of x_{36} and y_5 are respectively, $(0, 0, 0, 0, 0, 1)$ and $(0, 0, 0, 0, 1, 0)$. The module generated by x_{36} gives all odd positive roots and the module generated by y_5 gives all odd negative roots.

20.9.1.5b. $\mathfrak{e}(6,6)$ of sdim $=$ 38|40. In this case, $\mathfrak{g}(B) \simeq \mathfrak{gl}(6)$, see (20.9.1.1). The module $\mathfrak{g}_{\bar{1}}$ is irreducible with the highest weight vector

$$x_{35} = [[[x_3, x_6], [x_4, [x_2, x_3]]], [[x_4, x_5], [x_3, [x_1, x_2]]]]$$

In this case, the highest weight with respect to $h_1, \dots, h_6$ (which correspond to $E^{11} - E^{22}, \dots, E^{55} - E^{66}, E^{11} + E^{22} + E^{33}$ in $\mathfrak{gl}(6)$; well, actually, we can express h_6 as $E^{11}+E^{22}+E^{33}+ac$ for any $a \in \mathbb{K}$, where c is the central element of $\mathfrak{g}(B)$) is $(0,0,1,0,0,1)$. If we set $h_6 = E^{11} + E^{22} + E^{33}$, then $M = \bigwedge^3(\mathrm{id})$ as a $\mathfrak{gl}(6)$-module (note that it is not enough to write $M = R(\pi_3)$ since this only describes M as an $\mathfrak{sl}(6)$-module).

20.9.1.5c. $\mathfrak{e}(7,1)$ of sdim $=$ 80/78|54. Since the Cartan matrix of this Lie superalgebra is of rank 6, a grading operator d_1 should be (and is) added. Now if we take $d_1 = (1,0,0,0,0,0,0)$, then $\mathfrak{g}_{\bar{0}} \simeq (\mathfrak{e}(6) \oplus \mathbb{K}z) \oplus \mathbb{K}I_0$. The Cartan subalgebra is spanned by $h_1+h_3+h_7, h_2, h_3, h_4, h_5, h_6, h_7$ and d_1. We see that $\mathfrak{g}_{\bar{1}}$ has the two highest weight vectors:

$$\begin{aligned} x_{63} = [[[[x_2, x_3], [x_4, x_7]], [[x_3, x_4], [x_5, x_6]]], \\ [[[x_4, x_7], [x_5, x_6]], [[x_4, x_5], [x_3, [x_1, x_2]]]]] \end{aligned}$$

and y_1. Their respective weights (if we take $d_1 = (1,0,0,0,0,0,0)$) are $(0,0,0,0,0,1,0,1)$ and $(0,1,0,0,0,0,0,0)$. The module generated by x_{63} gives all odd positive roots and the module generated by y_1 gives all odd negative roots.

20.9.1.5d. $\mathfrak{e}(7,6)$ of sdim $=$ 70/68|64. We are in the same situation as before (sect. 20.9.1.1). We have

$$\mathfrak{g}(B) \simeq \mathfrak{oc}(1;12) \in \mathbb{K}I_0.$$

Note that in this case $\mathrm{size}(B) - \mathrm{rk}(B) = 2$, so the center of $\mathfrak{g}(B)$ is 2-dimensional, and $\dim \mathfrak{g}(B) - \dim \mathfrak{g}^{(1)}(B) = 2$. So we should be a bit more specific than in (20.39); namely, we have

$$\begin{aligned} &[\mathfrak{oc}(1;12), X^{\pm}] = 0;\\ &[I_0, X^{\pm}] = X^{\pm};\\ &[X^+, X^-] = h_1 + h_3 + h_5 \quad \text{(which corresponds to } 1_{12} \text{ in } \mathfrak{oc}(1;12)). \end{aligned}$$

The module $\mathfrak{g}_{\bar{1}}$ is irreducible with the highest weight vector

$$x_{62} = [[[x_7, [x_5, [x_3, x_4]]], [[x_1, x_2], [x_3, x_4]]], [[[x_2, x_3], [x_4, x_5]], [[x_4, x_7], [x_5, x_6]]]].$$

The Cartan subalgebra is spanned by $h_1 + h_3 + h_5, h_1, h_2, h_3, h_4, h_7$ and also h_6 and d_1. The weight of x_{62} is $(1,0,0,0,0,0,1,0)$. The highest weight vector of $\mathfrak{g}_{\bar{1}}$ is the highest weight vector of one of the copies of the $\mathfrak{g}(B)$-module N, see 20.9.1.1, so the highest weight of N is the same as the highest weight of $\mathfrak{g}_{\bar{1}}$. (Of course, this is true for the other two similar cases as well; in the case of $\mathfrak{e}(6,6)$, we used Lebedev's choice — another basis of $\mathfrak{h}$ — and expressed the weight with respect to it.)

20.9.1.5e. $\mathfrak{e}(7,7)$ of sdim $=$ 64/62|70. Since the Cartan matrix of this Lie superalgebra is of rank 6, a grading operator d_1 should be (and is) added. Then $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{gl}(8)$. The module $\mathfrak{g}_{\bar{1}}$ has the two highest weight vectors:

$$x_{58} = [[[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]], [[x_7, [x_3, x_4]], [[x_2, x_3], [x_4, x_5]]]]$$

and y_7. The Cartan subalgebra is spanned by $h_1, h_2, h_3, h_4, h_5, h_6$ and also $h_1+h_3+h_7$ and d_1. The weight of x_{58} with respect to these elements of the Cartan subalgebra is $(0,0,1,0,0,0,0,1)$ and the weight of y_7 is $(0,0,0,1,0,0,0,1)$. The module generated by x_{58} gives all odd positive roots and the module generated by y_7 gives all odd negative roots.

20.9.1.5f. $\mathfrak{e}(8,1)$ of sdim $=$ 136|112. We have (cf. sect. 20.9.1.1) $\mathfrak{g}(B) \simeq \hat{\mathfrak{e}}(7)$. (Recall that, in our notation, $\mathfrak{e}(7)$ has a center but not the grading operator, see section "Warning" 12.3.1.) The Cartan subalgebra is spanned by $h_2 + h_4 + h_8$ and $h_1, h_2, h_3, h_4, h_5, h_6, h_7$. The $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is irreducible with the highest weight vector:

$$\begin{aligned} x_{119} = [[[[x_4, [x_2, x_3]], [[x_5, x_8], [x_6, x_7]]], [[x_8, [x_4, x_5]], [[x_3, x_4], [x_5, x_6]]]], \\ [[[x_7, [x_5, x_6]], [[x_1, x_2], [x_3, x_4]]], [[x_8, [x_5, x_6]], [[x_2, x_3], [x_4, x_5]]]]] \end{aligned}$$

of weight $(1,1,0,0,0,0,0,1)$ and one lowest weight vector y_{119} whose expression is as above the x's changed by the y's, of the same weight as that of x_{119}. (Again, the highest weight of the $\mathfrak{g}(B)$-module N, see 20.9.1.1, is the same as the highest weight of $\mathfrak{g}_{\bar{1}}$.)

20.9.1.5g. $\mathfrak{e}(8,8)$ of sdim $=$ 120|128. In the $\mathbb{Z}$-grading with the 1st CM with $\deg e_8^{\pm} = \pm 1$ and $\deg e_i^{\pm} = 0$ for $i \neq 8$, we have $\mathfrak{g}_0 = \mathfrak{gl}(8) = \mathfrak{gl}(V)$. There are different isomorphisms between $\mathfrak{g}_0$ and $\mathfrak{gl}(8)$; using the one where $h_i = E^{i,i} + E^{i+1,i+1}$ for all $i = 1, \dots, 7$, and $h_8 = E^{6,6} + E^{7,7} + E^{8,8}$, we see that, as modules over $\mathfrak{gl}(V)$,

$$\begin{aligned} &\mathfrak{g}_1 = \textstyle\bigwedge^5 V^*; \ \mathfrak{g}_2 = \bigwedge^6 V; \quad \mathfrak{g}_3 = V;\\ &\mathfrak{g}_{-1} = \textstyle\bigwedge^5 V; \ \mathfrak{g}_{-2} = \bigwedge^6 V^*; \ \mathfrak{g}_{-3} = V^*. \end{aligned}$$

We could also set, e.g., $h_8 = E^{1,1} + E^{2,2} + E^{3,3} + E^{4,4} + E^{5,5}$. Then we would get

$$\begin{aligned} &\mathfrak{g}_1 = \textstyle\bigwedge^3 V; \quad \mathfrak{g}_2 = \bigwedge^6 V; \quad \mathfrak{g}_3 = \bigwedge^7 V^*;\\ &\mathfrak{g}_{-1} = \textstyle\bigwedge^3 V^*; \ \mathfrak{g}_{-2} = \bigwedge^6 V^*; \ \mathfrak{g}_{-3} = \bigwedge^7 V. \end{aligned}$$

The algebra $\mathfrak{g}_{\bar{0}}$ is isomorphic to $\mathfrak{o}_{\Pi}^{(2)}(16) \in \mathbb{K}d$, where $d = E^{6,6} + \dots + E^{13,13}$, and $\mathfrak{g}_{\bar{1}}$ is an irreducible $\mathfrak{g}_{\bar{0}}$-module with the highest weight the highest weight element x_{120} of weight $(1,0,\dots,0)$ (with respect to $h_1, \dots, h_8$); $\mathfrak{g}_{\bar{1}}$ also possesses a lowest weight vector.

20.9.2. Systems of simple roots of the $\mathfrak{e}$-type Lie superalgebras.

20.9.2.1. Remark. Observe that if $p = 2$ and the Cartan matrix has no parameters, the reflections do not change the shape of the diagram. Therefore, for the $\mathfrak{e}$-superalgebras, it suffices to list distributions of parities of the nodes in order to describe the diagrams. Since there are tens and even hundreds of diagrams in these cases, this possibility saves a lot of space, see the lists of all inequivalent Cartan matrices of the $\mathfrak{e}$-type Lie superalgebras.

20.9.2.2. $\mathfrak{e}(6,1) \simeq \mathfrak{e}(6,5)$ of sdim 46|32. All inequivalent Cartan matrices are as follows (none of the matrices corresponding to the symmetric pairs of Dynkin diagrams is excluded but are placed one under the other for clarity, followed by symmetric diagrams):

$\boxed{1)}$	000010	2)	010001	3)	100110	4)	000011	17)	000110	19)	000111
$\boxed{5)}$	100000	6)	000101	7)	110010	8)	100001	21)	110000	23)	110001
9)	111001	10)	101001	11)	011000	12)	101100	18)	011001	20)	011110
13)	001111	14)	001011	15)	001100	16)	011010	22)	001101	24)	111100
25)	010100	26)	100010	27)	110110						

20.9.2.3. $\mathfrak{e}(6,6)$ of sdim $=$ 38|40. All inequivalent Cartan matrices are as follows:

$\boxed{1)}$	000001	$\boxed{4)}$	000100	$\boxed{7)}$	001000	$\boxed{19)}$	010000	2)	011011	3)	101110	33)	111110
5)	011100	6)	101111	8)	011101	29)	101010	30)	111101	31)	010110	32)	101011
9)	110011	10)	001001	11)	011111	12)	110100	25)	010011	26)	101000	27)	111011
13)	001010	14)	100011	15)	110101	16)	001110	21)	111000	22)	010010	23)	100111
17)	100100	18)	110111	20)	100101	24)	111010	28)	010101	34)	010111	35)	101101
36)	111111												

20.9.2.4. $\mathfrak{e}(7,1)$ of sdim $=$ 80/78|54. All inequivalent Cartan matrices are as follows:

$\boxed{1)}$	1000000	2)	1000010	3)	1000110	4)	1001100	25)	0110000	26)	0110010	27)	0110110
5)	1010001	6)	1011001	7)	1100000	8)	1100010	21)	0011010	22)	0011110	23)	0100001
9)	1100110	10)	1101100	11)	1110001	12)	1111001	17)	0001101	18)	0001111	19)	0010100
13)	0000011	14)	0000101	15)	0000111	16)	0001011	28)	0111100	24)	0101001	20)	0011000

20.9.2.5. $\mathfrak{e}(7,6)$ of sdim $=$ 70/68|64. All inequivalent Cartan matrices are as follows:

$\boxed{1)}$	0000010	$\boxed{2)}$	0000100	3)	0000110	$\boxed{4)}$	0001000	62)	1111100	63)	1111110
5)	0001010	6)	0001100	7)	0001110	8)	0010001	60	1111000	61)	1111010
9)	0010011	10)	0010101	11)	0010111	12)	0011001	58)	1110100	59)	1110110
13)	0011011	14)	0011101	15)	0011111	$\boxed{16)}$	0100000	56)	1110000	57)	1110010
17)	0100010	18)	0100100	19)	0100110	20)	0101000	54)	1101101	55)	1101111
21)	0101010	22)	0101100	23)	0101110	24)	0110001	52)	1101001	53)	1101011
25)	0110011	26)	0110101	27)	0110111	28)	0111001	50)	1100101	51)	1100111
29)	0111011	30)	0111101	31)	0111111	32)	1000001	48)	1100001	49)	1100011
33)	1000011	34)	1000101	35)	1000111	36)	1001001	46)	1011100	47)	1011110
37)	1001011	38)	1001101	39)	1001111	40)	1010000	44)	1011000	45)	1011010
41)	1010010	42)	1010100	43)	1010110						

20.9.2.6. $\mathfrak{e}(7,7)$ of sdim $=$ 64/62|70. All inequivalent Cartan matrices are as follows:

$\boxed{1)}$	0000001	2)	0001001	$\boxed{3)}$	0010000	4)	0010010	34)	1111011	35)	1111101
5)	0010110	6)	0011100	7)	0100011	8)	0100101	32)	1110101	33)	1110111
9)	0100111	10)	0101011	11)	0101101	12)	0101111	30)	1101110	31)	1110011
13)	0110100	14)	0111000	15)	0111010	16)	0111110	28)	1101000	29)	1101010
17)	1000100	18)	1001000	19)	1001010	20)	1001110	26)	1011111	27)	1100100
21)	1010011	22)	1010101	23)	1010111	24)	1011011	25)	1011101		

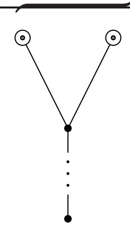
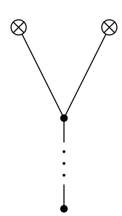
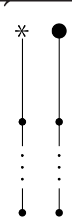
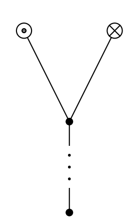
20.9.2.7. $\mathfrak{e}(8,1)$ of sdim $=$ 136|112. All inequivalent Cartan matrices are as follows:

$\boxed{1)}$	10000000	2)	10000010	3)	10000011	4)	10000101	120)	01111110
5)	10000110	6)	10000111	7)	10001011	8)	10001100	119)	01111010
9)	10001101	10)	10001111	11)	10010001	12)	10010100	118)	01111001
13)	10011000	14)	10011001	15)	10011010	16)	10011110	117)	01111000
17)	10100000	18)	10100001	19)	10100010	20)	10100110	116)	01110100
21)	10101001	22)	10101100	23)	10110000	24)	10110001	115)	01110001
25)	10110010	26)	10110110	27	10111001	28)	10111100	114)	01101111
29)	11000000	30)	11000010	31)	11000011	32)	11000101	113)	01101101
33)	11000110	34)	11000111	35)	11001011	36)	11001100	112)	01101100
37)	11001101	38)	11001111	39)	11010001	40)	11010100	111)	01101011
41)	11011000	42)	11011001	43)	11011010	44)	11011110	110)	01100111
45)	11100000	46)	11100001	47)	11100010	48)	11100110	109)	01100110
49)	11101001	50)	11101100	51)	11110000	52)	11110001	108)	01100101
53)	11110010	54)	11110110	55)	11111001	56)	11111100	107)	01100011
57)	00000011	$\boxed{58)}$	00000100	59)	00000101	60)	00000111	106)	01100010
$\boxed{61)}$	00001000	62)	00001010	63)	00001011	64)	00001101	105)	01100000
65)	00001110	66)	00001111	67)	00010011	68)	00010100	104)	01011100
69)	00010101	70)	00010111	71)	00011000	72)	00011010	103)	01011001
73)	00011011	74)	00011101	75)	00011110	76)	00011111	102)	01010110
77)	00100001	78)	00100100	79)	00101000	80)	00101001	101)	01010010
81)	00101010	82)	00101110	83)	00110000	84)	00110010	100)	01010001
85)	00110011	86)	00110101	87)	00110110	88)	00110111	99)	01010000
89)	00111011	90)	00111100	91)	00111101	92)	00111111	98)	01001100
$\boxed{93)}$	01000000	94)	01000001	95)	01000010	96)	01000110	97)	01001001

20.9.2.8. $\mathfrak{e}(8,8)$ of sdim $=$ 120|128. All inequivalent Cartan matrices are as follows:

$\boxed{1)}$)	00000001	$\boxed{2)}$	00000010	$\boxed{12)}$	00100000	$\boxed{6)}$	00010000	109)	11010101
5)	00001100	4)	00001001	7)	00010001	8)	00010010	110)	11010110
9)	00010110	10)	00011001	11)	00011100	3)	00000110	111)	11010111
13)	00100010	14)	00100011	15)	00100101	16)	00100110	112)	11011011
17)	00100111	18)	00101011	19)	00101100	20)	00101101	113)	11011100
21)	00101111	22)	00110001	23)	00110100	24)	00111000	114)	11011101
25)	00111001	26)	00111010	27)	00111110	28)	01000011	115)	11011111
29)	01000100	30)	01000101	31)	01000111	32)	01001000	116)	11100011
33)	01001010	34)	01001011	35)	01001101	36)	01001110	117)	11100100
37)	01001111	38)	01010011	39)	01010100	40)	01010101	118)	11100101
41)	01010111	42)	01011000	43)	01011010	44)	01011011	119)	11100111
45)	01011101	46)	01011110	47)	01011111	48)	01100001	120)	11101000
49)	01100100	50)	01101000	51)	01101001	52)	01101010	121)	11101010
53)	01101110	54)	01110000	55)	01110010	56)	01110011	122)	11101011
57)	01110101	58)	01110110	59)	01110111	60)	01111011	123)	11101101
61)	01111100	62)	01111101	63)	01111111	64)	10000001	124)	11101110
65)	10000100	66)	10001000	67)	10001001	68)	10001010	125)	11101111
69)	10001110	70)	10010000	71)	10010010	72)	10010011	126)	11110011
73)	10010101	74)	10010110	75)	10010111	76)	10011011	127)	11110100
77)	10011100	78)	10011101	79)	10011111	80)	10100011	128)	11110101
81)	10100100	82)	10100101	83)	10100111	84)	10101000	129)	11110111
85)	10101010	86)	10101011	87)	10101101	88)	10101110	130)	11111000
89)	10101111	90)	10110011	91)	10110100	92)	10110101	131)	11111010
93)	10110111	94)	10111000	95)	10111010	96)	10111011	132)	11111011
97)	10111101	98)	10111110	99)	10111111	100)	11000001	133)	11111101
101)	11000100	102)	11001000	103)	11001001	104)	11001010	134)	11111110
105)	11001110	106)	11010000	107)	11010010	108)	11010011	<u>135)</u>	<u>11111111</u>

20.10. Table. Dynkin diagrams for $p = 2$

Diagrams	$\mathfrak{g}$	v	ev	od	png	$ng \leq \min(*,*)$
1) 	$\mathfrak{ooc}(2; 2k_{\bar 0}\|2k_{\bar 1}) \in \mathbb{K}I_0$ if $k_{\bar 0}+k_{\bar 1}$ is odd; $\mathfrak{ooc}(1; 2k_{\bar 0}\|2k_{\bar 1}) \in \mathbb{K}I_0$ if $k_{\bar 0}+k_{\bar 1}$ is even.	$k_{\bar 0}+k_{\bar 1}$	$k_{\bar 0}-2$	$k_{\bar 1}$	$\bar 0$	$2k_{\bar 0}-4, 2k_{\bar 1}$
			$k_{\bar 1}$	$k_{\bar 0}-2$	$\bar 1$	$2k_{\bar 0}-3, 2k_{\bar 1}-1$
			$k_{\bar 1}-2$	$k_{\bar 0}$	$\bar 0$	$2k_{\bar 0}, 2k_{\bar 1}-4$
			$k_{\bar 0}$	$k_{\bar 1}-2$	$\bar 1$	$2k_{\bar 0}-1, 2k_{\bar 1}-3$
2) 			$k_{\bar 0}-1$	$k_{\bar 1}-1$		$2k_{\bar 0}-2, 2k_{\bar 1}-1$
			$k_{\bar 1}-1$	$k_{\bar 0}-1$		$2k_{\bar 0}-1, 2k_{\bar 1}-2$
3) 4) 	$\mathfrak{oo}_{I\Pi}(2k_{\bar 0}+1\|2k_{\bar 1})$	$k_{\bar 0}+k_{\bar 1}$	$k_{\bar 0}-1$	$k_{\bar 1}$	$\bar 0$	$2k_{\bar 0}-2, 2k_{\bar 1}$
			$k_{\bar 1}$	$k_{\bar 0}-1$	$\bar 1$	$2k_{\bar 0}-1, 2k_{\bar 1}-1$
			$k_{\bar 1}-1$	$k_{\bar 0}$	$\bar 0$	$2k_{\bar 0}, 2k_{\bar 1}-2$
			$k_{\bar 0}$	$k_{\bar 1}-1$	$\bar 1$	$2k_{\bar 0}-1, 2k_{\bar 1}-1$
5) 	$\mathfrak{pec}(2; m) \in \mathbb{K}I_0$ if m is odd; $\mathfrak{pec}(1; m) \in \mathbb{K}I_0$ if m is even.	m				

20.10.1. Notation. The Dynkin diagrams in Table correspond to CM Lie superalgebras close to ortho-orthogonal and periplectic Lie superalgebras. Each thin black dot may be $\otimes$ or $\odot$; the last five columns show conditions on the diagrams; what concerns the last four columns, it suffices to satisfy conditions in any one row. Horizontal lines in the last four columns separate the cases corresponding to different Dynkin diagrams. The notations are:
v is the total number of nodes in the diagram;
ng is the number of "grey" nodes $\otimes$'s among the thin black dots; png is the parity of this number;
ev and od are the number of thin black dots such that the number of $\otimes$'s to the left from them is even and odd, respectively.

20.11. Fixed points of symmetries of the Dynkin diagrams

20.11.1. Recapitulation. For $p = 0$, it is well known that the Lie algebras of series B and C and the exceptions F and G are obtained as the sets of fixed points of the outer automorphism of an appropriate Lie algebra of ADE series. All these automorphisms correspond to the symmetries of the respective Dynkin diagram. Not all simple finite dimensional Lie superalgebras can be obtained as the sets of fixed points of the symmetry of an appropriate Dynkin diagram, but many of them can, see [FSS].

Recall Serganova's result [Se] on outer automorphisms (i.e., modulo the connected component of the unity of the automorphism group) of simple finite dimensional Lie superalgebras for $p = 0$. The symmetry of the Dynkin diagram of $\mathfrak{sl}(n)$ corresponds to the transposition with respect to the side diagonal, conjugate in the group of automorphisms of $\mathfrak{sl}(n)$ to the "minus transposition" $X \mapsto -X^t$. In the super case, this automorphism becomes $X \longmapsto -X^{st}$, where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{st} = \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix}$$

This automorphism, seemingly of order 4, is actually of order 2 modulo the connected component of the unity of the automorphism group, and is of order 4 only for $\mathfrak{sl}(2n+1|2m+1)$.

The queer Lie superalgebra $\mathfrak{q}(n)$ is obtained as the set of fixed points of the automorphism

$$\Pi : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \begin{pmatrix} D & C \\ B & A \end{pmatrix}$$

of $\mathfrak{gl}(n|n)$ corresponding to the symmetry of the Dynkin diagram

$$\bigcirc_{11} - \cdots - \bigcirc_{1n} - \otimes - \bigcirc_{21} - \cdots - \bigcirc_{2n} \longmapsto \bigcirc_{21} - \cdots - \bigcirc_{2n} - \otimes - \bigcirc_{11} - \cdots - \bigcirc_{1n}$$

which interchanges the identical maximal parts $\bigcirc - \cdots - \bigcirc$ preserving the order of nodes; whereas $\mathfrak{pe}(n)$ is the set of fixed points of the composition automorphism $\Pi \circ (-st)$.

20.11.2. New result. The modular version of the above statements is given in the next Theorem in which, speaking about ortho-orthogonal and periplectic superalgebras, we distinguish the cases where the fork node is grey or white ($g\mathfrak{g}(A)$ and $w\mathfrak{g}(A)$, respectively); to squeeze the data in the table, we write $\widehat{\mathfrak{g}}$ instead of $\mathfrak{g} \Subset \mathbb{K}I_0$. We also need the following decomposable Cartan matrices ($p = 2$):

$$\mathcal{N} := \begin{pmatrix} \bar{0} & 1 & 0 & 0 \\ 1 & \bar{0} & 0 & 0 \\ 0 & 1 & \bar{0} & 1 \\ 0 & 0 & 1 & \bar{0} \end{pmatrix}, \quad \mathcal{M} := \begin{pmatrix} \bar{0} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & \bar{0} & 1 \\ 0 & 0 & 1 & \bar{0} \end{pmatrix}$$

20.11.2.0a. The Lie algebra $\mathfrak{g}(\mathcal{N})$. It is of dim 34 and not simple; it contains a simple ideal of dim $= 26$ which is $\mathfrak{o}(1;8)^{(1)}/\mathfrak{c}$ and the quotient is isomorphic to $\mathfrak{sl}(3)$.

20.11.2.0b. The Lie superalgebra $\mathfrak{g}(\mathcal{M})$. It is of sdim $18|16$ and not simple. Its even part is $\mathfrak{hei}(2) \oplus_c \mathfrak{g}(C)$, where $\mathfrak{hei}(2) = \mathrm{Span}\{X^\pm, c\}$ and c is the center of the Lie algebra $\mathfrak{g}(C)$, where

$$C := \begin{pmatrix} \bar{0} & 0 & 0 \\ 0 & \bar{0} & 1 \\ 0 & 1 & \bar{0} \end{pmatrix}$$

The brackets are as follows:

$$[X^\pm, \mathfrak{g}(C)^{(1)}] = 0; \quad [X^\pm, d] = X^\pm; \quad [X^+, X^-] = c,$$

where d is the grading operator of the Lie algebra $\mathfrak{g}(C)$.

Now the Cartan subalgebra of $\mathfrak{g}(\mathcal{M})$ is generated by $h_3, h_6, h_1+h_5, h_2+h_4$ and the highest weight vector of the module $\mathfrak{g}(\mathcal{M})_{\bar{1}}$ is $x_{32}+x_{33}$, where

$$\begin{aligned} x_{32} &= [[[x_1, x_2], [x_3, x_4]], [[x_3, x_6], [x_4, x_5]]], \\ x_{33} &= [[[x_1, x_2], [x_3, x_6]], [[x_2, x_3], [x_4, x_5]]] \end{aligned}$$

Its weight is $(0,0,1,0)$ (according to the ordering of the generators of the Cartan subalgebra as above).

The restriction of the module to $\mathfrak{hei}(2)$ consists of 8 copies of the 2-dimensional irreducible Fock module; the restriction to $\mathfrak{g}(C)$ consists of 2 copies of an irreducible 8-dimensional module.

The lowest weight vector is $y_{32}+y_{33}$ with weight $(0,0,1,0)$.

The Lie superalgebra $\mathfrak{g}(\mathcal{M})$ has a simple ideal, of sdim $= 10|16$ which is $\mathfrak{oo}(1;4|4)^{(1)}/\mathfrak{c}$ (to be described separately below) and the quotient is isomorphic to $\mathfrak{sl}(3)$.

20.11.2.0c. Theorem. *If the Dynkin diagram of $i\mathfrak{g}(A)$ is symmetric, it gives rise to an outer automorphism σ whose fixed points constitute the Lie superalgebra $(i\mathfrak{g}(A))^\sigma$ which occupies the slot under $i\mathfrak{g}(A)$ in the following tables* (20.41), (20.42), (20.43).

1) *The order 2 automorphisms of the $\mathfrak{sl}$ series corresponding to the symmetries of Dynkin diagrams give the following fixed points:*

$\mathfrak{sl}(2n+1)$	$\mathfrak{gl}(2n)$	$\begin{cases} \mathfrak{sl}(2k+1\vert 2m+1) & \text{for } k+m \text{ odd;} \\ \mathfrak{gl}(2k+1\vert 2m+1) & \text{for } k+m \text{ even.} \end{cases}$	$\mathfrak{sl}(2k+1\vert 2m)$
$\mathfrak{o}(2n+1)$	$\mathfrak{o}(2n)$	$\mathfrak{oo}(2k+1\vert 2m+1)$	$\mathfrak{oo}(2k+1\vert 2m)$
$\mathfrak{gl}(n\vert n)$, Π	$\mathfrak{gl}(n\vert n)$, $\Pi\circ(t)$	$\mathfrak{gl}(2k\vert 2m)$	
$\mathfrak{q}(n)$	$\mathfrak{pe}(n)$	$\mathfrak{oo}(2k\vert 2m)$	

(20.41)

2) *The order 2 automorphisms of the orthogonal and ortho-orthogonal series give the following fixed points:*

$\widehat{\mathfrak{ooc}(2; 2k_{\bar{0}}\vert 2k_{\bar{1}})}$ *for* $k_{\bar{0}}+k_{\bar{1}}$ *odd;*	$\widehat{\mathfrak{oc}(2; 2k)}$ *for* k *odd;*
$\widehat{\mathfrak{ooc}(1; 2k_{\bar{0}}\vert 2k_{\bar{1}})}$ *for* $k_{\bar{0}}+k_{\bar{1}}$ *even.*	$\widehat{\mathfrak{oc}(1; 2k)}$ *for* k *even.*

(20.42)

3) *The following are the fixed points of order 2 automorphisms of the exceptional Lie (super)algebras and periplectic superalgebras, and of order 3 automorphisms of the orthogonal algebra and ortho-orthogonal superalgebras.*

$1\mathfrak{g}(2,3)$	$2\mathfrak{g}(2,3)$	$5\mathfrak{g}(2,3)$	$5\mathfrak{g}(2,6)$	$2\mathfrak{g}(2,6)$	$\widehat{\mathfrak{oc}(1;8)}$
$\mathfrak{psl}(2\vert 2)$	$\mathfrak{sl}(1\vert 2)$	$\mathfrak{osp}(3\vert 2)$	$\mathfrak{g}(1,6)$	$\mathfrak{g}(1,6)$	$\mathfrak{gl}(4)$
$g\widehat{\mathfrak{ooc}(1;4\vert 4)}$	$w\widehat{\mathfrak{ooc}(1;4\vert 4)}$	$g\widehat{\mathfrak{pec}(1;4)}$	$w\widehat{\mathfrak{pec}(1;4)}$	$g\widehat{\mathfrak{ooc}(2;6\vert 2)}$	$w\widehat{\mathfrak{ooc}(2;6\vert 2)}$
$\mathfrak{gl}(2\vert 2)$	$\mathfrak{gl}(2\vert 2)$	$\mathfrak{gl}(1\vert 3)$	$\mathfrak{gl}(1\vert 3)$	$\mathfrak{gl}(2\vert 2)$	$\mathfrak{gl}(2\vert 2)$

(20.43)

Besides, $\mathfrak{e}(6)^\sigma = \mathfrak{g}(\mathcal{N})$, *whereas*

$$\begin{aligned} &25\mathfrak{e}(6,1)^\sigma \simeq 26\mathfrak{e}(6,1)^\sigma \simeq 27\mathfrak{e}(6,1)^\sigma \simeq \\ &1\mathfrak{e}(6,6)^\sigma \simeq 7\mathfrak{e}(6,6)^\sigma \simeq 5\mathfrak{e}(6,6)^\sigma \simeq 33\mathfrak{e}(6,6)^\sigma \simeq 8\mathfrak{e}(6,6)^\sigma \simeq 29\mathfrak{e}(6,6)^\sigma \simeq \\ &32\mathfrak{e}(6,6)^\sigma \simeq 10\mathfrak{e}(6,6)^\sigma \simeq 14\mathfrak{e}(6,6)^\sigma \simeq 18\mathfrak{e}(6,6)^\sigma \simeq 28\mathfrak{e}(6,6)^\sigma \simeq 36\mathfrak{e}(6,6)^\sigma \simeq \mathfrak{g}(\mathcal{M}). \end{aligned}$$

20.11.3. The structure of the deforms at the exceptional values of parameter. The result (it is not new for $p = 0$ for $\mathfrak{osp}(4|2;a)$ and is given for comparison) obtained by tending the parameter to the limit (denoted $\lim\limits_{a \longrightarrow a_0} \mathfrak{g}(A(a))$) differs, sometimes, from the one obtained by constructing $\mathfrak{g}(A(a_0))$ at the exceptional value of parameter (even their dimensions differ):

Statement.

$$
\begin{array}{ll}
\mathfrak{wk}(3;0) \simeq \mathfrak{gl}(2) \oplus \mathfrak{sl}(3) & \mathfrak{wk}(3;1) \simeq \mathfrak{gl}(4) \\
\lim\limits_{a\longrightarrow 0} \mathfrak{wk}(3;a) \simeq & \lim\limits_{a\longrightarrow 1} \mathfrak{wk}(3;a) \simeq \\
\mathfrak{bgl}(3;0) \simeq \mathfrak{gl}(1|1) \oplus \mathfrak{sl}(3) & \mathfrak{bgl}(3;1) \simeq \mathfrak{gl}(2|2) \\
\lim\limits_{a\longrightarrow 0} \mathfrak{bgl}(3;a) \simeq & \lim\limits_{a\longrightarrow 1} \mathfrak{bgl}(3;a) \simeq \\
\mathfrak{osp}(4|2;0) \simeq \mathfrak{sl}(2) \mathbin{\ni\!\!\!\!+} \mathfrak{psl}(2|2) & \mathfrak{osp}(4|2;-1) \simeq \mathfrak{psl}(2|2) \mathbin{\ni\!\!\!\!+} \mathfrak{sl}(2) \\
\mathfrak{wk}(4;0) \simeq \mathfrak{gl}(2) \oplus \mathfrak{gl}(4) & \mathfrak{wk}(4;1) \simeq \mathfrak{sl}(5) \\
\lim\limits_{a\longrightarrow 0} \mathfrak{wk}(4;a) \simeq & \lim\limits_{a\longrightarrow 1} \mathfrak{wk}(4;a) \simeq \\
\mathfrak{bgl}(4;0) \simeq \mathfrak{gl}(1|1) \oplus \mathfrak{gl}(4) & \mathfrak{bgl}(4;1) \simeq \mathfrak{sl}(1|4) \\
\lim\limits_{a\longrightarrow 0} \mathfrak{bgl}(4;a) \simeq & \lim\limits_{a\longrightarrow 1} \mathfrak{bgl}(3;a) \simeq
\end{array}
\tag{20.44}
$$

20.11.4. A realization of $\mathfrak{g} = \mathfrak{oo}(4|4)^{(1)}/\mathfrak{c}$. This simple Lie superalgebra $\mathfrak{g}$ admits an unexpected realization in which $\mathfrak{g}_{\bar 0} \simeq \mathfrak{hei}(8) \mathbin{\in\!\!\!\!+} \mathbb{K}E$, where $\mathfrak{hei}(8) = \mathrm{Span}(p, q, c)$ with $p = (p_1, \dots, p_4)$, $q = (q_1, \dots, q_4)$ and c being the center of $\mathfrak{hei}$ and $\mathfrak{g}_{\bar 1}$ being a copy of the Fock space considered purely odd, i.e., as $\Pi(\mathbb{K}[p]/(p_1^2, \dots, p_4^2))$, and $E := \sum (p_i \partial_{p_i} + q_i \partial_{q_i})$.

Indeed, consider the following isomorphism

$$
\begin{aligned}
&\varphi : \Pi(\mathbb{K}[p]/(p_1^2, \dots, p_4^2)) \longrightarrow \mathrm{Span}(\varphi_0, \dots, \varphi_{1234}) \\
&\varphi_0 := \Pi(1),\ \varphi_i := \Pi(p_i),\ \varphi_{ij} := \Pi(p_i p_j), \dots,\ \varphi_{1234} := \Pi(p_1 p_2 p_3 p_4).
\end{aligned}
\tag{20.45}
$$

Now the multiplication is given by the following two tables, where $D := c + E$ to save space:

	c	E	p_1	p_2	p_3	p_4	q_1	q_2	q_3	q_4
φ_0	φ_0	φ_0	φ_1	φ_2	φ_3	φ_4	0	0	0	0
φ_1	φ_1	0	0	φ_{12}	φ_{13}	φ_{14}	φ_0	0	0	0
φ_2	φ_2	0	φ_{12}	0	φ_{23}	φ_{24}	0	φ_0	0	0
φ_3	φ_3	0	φ_{13}	φ_{23}	0	φ_{34}	0	0	φ_0	0
φ_4	φ_4	0	φ_{14}	φ_{24}	φ_{34}	0	0	0	0	φ_0
φ_{12}	φ_{12}	φ_{12}	0	0	φ_{123}	φ_{124}	φ_2	φ_1	0	0
φ_{13}	φ_{13}	φ_{13}	0	φ_{123}	0	φ_{134}	φ_3	0	φ_1	0
φ_{14}	φ_{14}	φ_{14}	0	φ_{124}	φ_{134}	0	φ_4	0	0	φ_1
φ_{23}	φ_{23}	φ_{23}	φ_{123}	0	0	φ_{234}	0	φ_3	φ_2	0
φ_{24}	φ_{24}	φ_{24}	φ_{124}	0	φ_{234}	0	0	φ_4	0	φ_2
φ_{34}	φ_{34}	φ_{34}	φ_{134}	φ_{234}	0	0	0	0	φ_4	φ_3
φ_{123}	φ_{123}	0	0	0	0	φ_{1234}	φ_{23}	φ_{13}	φ_{12}	0
φ_{124}	φ_{124}	0	0	0	φ_{1234}	0	φ_{24}	φ_{14}	0	φ_{12}
φ_{134}	φ_{134}	0	0	φ_{1234}	0	0	φ_{34}	0	φ_{14}	φ_{13}
φ_{234}	φ_{234}	0	φ_{1234}	0	0	0	0	φ_{34}	φ_{24}	φ_{23}
φ_{1234}	φ_{1234}	φ_{1234}	0	0	0	0	φ_{234}	φ_{134}	φ_{124}	φ_{123}

	φ_{1234}	φ_{234}	φ_{134}	φ_{124}	φ_{123}	φ_{34}	φ_{24}	φ_{23}	φ_{14}	φ_{13}	φ_{12}	φ_4	φ_3	φ_2	φ_1	φ_0
φ_{1234}	0	0	0	0	0	0	0	0	0	0	0	p_4	p_3	p_2	p_1	D
φ_{234}	0	0	0	0	0	0	0	0	p_4	p_3	p_2	0	0	0	E	q_1
φ_{134}	0	0	0	0	0	0	p_4	p_3	0	0	p_1	0	0	E	0	q_2
φ_{124}	0	0	0	0	0	p_4	0	p_2	0	p_1	0	0	E	0	0	q_3
φ_{123}	0	0	0	0	0	p_3	p_2	0	p_1	0	0	E	0	0	0	q_4
φ_{34}	0	0	0	p_4	p_3	0	0	0	0	0	D	0	0	q_1	q_2	0
φ_{24}	0	0	p_4	0	p_2	0	0	0	0	D	0	0	q_1	0	q_3	0
φ_{23}	0	0	p_3	p_2	0	0	0	0	D	0	0	q_1	0	0	q_4	0
φ_{14}	0	p_4	0	0	p_1	0	0	D	0	0	0	0	q_2	q_3	0	0
φ_{13}	0	p_3	0	p_1	0	0	D	0	0	0	0	q_2	0	q_4	0	0
φ_{12}	0	p_2	p_1	0	0	D	0	0	0	0	0	q_3	q_4	0	0	0
φ_4	p_4	0	0	0	E	0	0	q_1	0	q_2	q_3	0	0	0	0	0
φ_3	p_3	0	0	E	0	0	q_1	0	q_2	0	q_4	0	0	0	0	0
φ_2	p_2	0	E	0	0	q_1	0	0	q_3	q_4	0	0	0	0	0	0
φ_1	p_1	E	0	0	0	q_2	q_3	q_4	0	0	0	0	0	0	0	0
φ_0	D	q_1	q_2	q_3	q_4	0	0	0	0	0	0	0	0	0	0	0

20.11.4.1. Remark. If $p = 0$, every irreducible module over a solvable Lie algebra is 1-dimensional. A theorem, based on this fact, states that *any Lie superalgebra $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}_{\bar 0}$ is solvable*. The example above shows that if $p > 0$, life is much more interesting.

We are unable to answer: Are there (hopefully, simple) Lie superalgebras $\mathfrak{g}$ with $\mathfrak{g}_{\bar 0} \simeq \mathfrak{hei}(2n) \mathbin{\in\!\!\!\!+} \mathbb{K}E$ and $\mathfrak{g}_{\bar 1} \simeq \Pi(\text{Fock module over } \mathfrak{hei}(2n))$ for $n \neq 4$?

Chapter 21

Selected problems (D. Leites)

21.1. Representations

A serious problem: What is universal enveloping algebra in the modular case?. I do not understand why the conventional theory only accepts the "usual" $U(\mathfrak{g})$ and restricted $u(\mathfrak{g})$ notions of the universal enveloping algebras. Let us explain how definition of $U(\mathfrak{g})$ affects both the definition of (co)homology and the notion of (co)induced representations. The scientific definition of the Lie (super)algebra cohomology and homology are

$$H^i(\mathfrak{g}; M) := \mathrm{Ext}^i_{U(\mathfrak{g})}(\mathbb{K}; M), \qquad H_i(\mathfrak{g}; M) := \mathrm{Tor}_i^{U(\mathfrak{g})}(\mathbb{K}; M). \tag{21.1}$$

So it is clear, actually, how to *approach* the problem, at least for the modular Lie algebras obtained by means of the Kostrikin-Shafarevich approach (and its super analog), i.e., the ones that have analogs over $\mathbb{C}$ with a basis in which all structure constants are integer: Speaking about non-super cases, take any book (e.g., [St]) in which a convenient $\mathbb{Z}$-form $U_{\mathbb{Z}}(\mathfrak{g})$ of $U(\mathfrak{g})$ is described for any simple complex $\mathfrak{g}$, and introduce $\underline{N}$ (similar to the $\underline{N}$ in the definition of the algebra of divided powers $\mathcal{O}(n; \underline{N})$) by setting something like

$$\begin{array}{ll} U(\mathfrak{g}; \underline{N}) := & \text{subalgebra of } U_{\mathbb{Z}}(\mathfrak{g}) \text{ constructed} \\ & \text{"similarly to the algebra of divided powers" } \mathcal{O}(n; \underline{N}); \\ H^i_{\underline{N}}(\mathfrak{g}; M) := \mathrm{Ext}^i_{U(\mathfrak{g}; \underline{N})}(\mathbb{K}; M). & \end{array} \tag{21.2}$$

How to perform this "similar construction" of "something like" is the whole point.

Absolutely correct — in terms of the conventional definition (21.1) — Dzhumadildaev's computations (elucidated in [Vi]) imply that

$$\mathfrak{vect}(n; \underline{N}) := \mathfrak{der}(\mathcal{O}(n; \underline{N})),$$

where only "special derivatives" are considered, is not rigid. I find this result "ideologically wrong" and believe that the cause is buried in the definitions used. Recall the arguments in favor of rigidity of $\mathfrak{vect}$, see [LL]:

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. For any $\mathfrak{h}$-module V, we define a series of coinduced $\mathfrak{g}$-modules:

$$\mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}}(V;\underline{N}) := \mathrm{Hom}_{U(\mathfrak{h};\underline{N})}(U(\mathfrak{g};\underline{N}),V); \tag{21.3}$$

Then, in terms of the conjectural definition (21.2), we should have the following analog of the well-known isomorphism:

$$H^i_{\underline{N}}(\mathfrak{g};\mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}}(V;\underline{N})) \simeq H^i(\mathfrak{h};V). \tag{21.4}$$

This isomorphism implies that, for $\mathfrak{g} = \mathfrak{vect}(n;\underline{N})$, we should have $H^2_{\underline{N}}(\mathfrak{g};\mathfrak{g}) \simeq H^2(\mathfrak{gl}(V);V) = 0$, where $\dim V = n$, at least, if n is not divisible by p.

The situation is opposite in a sense to that with the Kac-Moody groups that "did not exist" until a correct definition of cohomology was used; or with Dirac's δ-function which is not a function in the conventional sense.

Dzhumadildaev [Dz2] (also Farnsteiner and Strade [FS]) showed that for $p>0$ the conventional analog of the statement (21.4), a.k.a. Shapiro's lemma, should be formulated differently because $H^i(\mathfrak{g};\mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}}(V))$ strictly contains $H^i(\mathfrak{h};V)$. I hope that one can get rid of these extra cocycles in an appropriate theory.

The Lie (super)algebra (co)homology can also be defined "naively", as a generalization (and dualization) of the de Rham complex. In this approach the enveloping algebra does not appear explicitly and the divided powers we tried to introduce above seem to disappear.

All of the following problems are formulated as if the above problem does not exist, i.e., we use the conventional definition of representations.

21.1.1. Problem. 1) Describe the irreducible representations of infinite dimensional solvable Lie superalgebras over $\mathbb{C}$, thus having superized the results of [D] and generalized Chapter 6.

2) Describe the irreducible representations of finite dimensional solvable Lie superalgebras over fields of positive characteristic.

21.1.2. Problem. Following [RSh], describe the irreducible representations of the simplest of simple Lie superalgebras for $p>0$: that of $\mathfrak{osp}(1|2)$ for $p>2$ and of $\mathfrak{oo}(1|2)$ for $p=2$ (cf. also [Do] who did not bother to refer to [RSh] whose technique he repeats literally).

Since [RSh] was published when the journal *Matematicheskie Zametki* was not cover-to-cover translated, and the translation (of [Do]) is not easily available either, let me give a gist of the relevant ideas and results.

21.1.3. Problem. The realization of the spinor representations by means of quantization of the Poisson (super)algebra can be defined, but only for the restricted version of the Poisson superalgebra. Even this can only be performed for $p>2$. To describe all deformations of the Poisson superalgebras $\mathfrak{po}_B(n_{\bar 0}|n_{\bar 1};\underline{N})$ — in particular, to quantize it (here: realize by differential operators in the space of "functions" of sorts — for arbitrary values of $\underline{N}$.

The same problem for $p=2$ is hardly more difficult but the answer is distinct from that for $p>2$: Indeed, we already know that $\mathfrak{o}(3)$ has no 2-dimensional representations.

21.1.4. Problem. Consider the same problem as that solved by B. Clarke but for $p>3$. For example, for $p=5$. If the answer will reveal a pattern, one could make a conjecture concerning the structure of the answer for any $p>3$. The number of cases to consider grows with p.

I expect the answer is somewhat different for $p=2$. It could be easier to get than for $p=5$ since one has fewer cases to consider.

21.1.5. Problem. A) Over $\mathbb{C}$:

1) Describe (unary and binary) differential operators invariant with respect to non-standard regradings of the (simple) vectorial Lie superalgebras.

2) Describe at least, the primitive forms in these cases, and for the exceptional simple vectorial Lie superalgebras.

B) Same problem for $p>0$. Something is already done by S. Krylyuk.[24] 24

21.2. Lie (super)algebras. Their structure

21.2.1. Problem. One of the roughest invariants of a given (super)algebra is its (super)dimension. Hence, this problem: Give a precise formula for the superdimension of the k-th prolong of $\mathfrak{o}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$. Lebedev showed that this dimension depends only on $n=n_{\bar 0}+n_{\bar 1}$ and k — not on $n_{\bar 0}$ and $n_{\bar 1}$ but was unable to derive a precise formula in the general case.

The following table shows the known dimensions for small n and k:

$n \setminus k$	1	2	3	4	5
4	0	0	0	0	0
6	14	1	0	0	0
8	48	43	8	1	0

21.3. Quest for simple Volichenko algebras

21.3.1. Problem. A) Over $\mathbb{C}$:

1) Describe simple Volichenko subalgebras of the exceptional simple vectorial Lie superalgebras.

2) Describe simple Volichenko subalgebras of the simple stringy Lie superalgebras.

3) Describe simple Volichenko subalgebras of the loop Lie superalgebras.

B) Same problem for $p>0$.

21.4. Miscellanies

21.4.1. Problem. Generalize Dzhumadildaev's result ([Dz]) to other dimensions, or to Lie superalgebras, or to simple Lie (super)algebras of vector fields other than $\mathfrak{vect}$ or $\mathfrak{svect}$, or to the modular Lie (super)algebras.

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