

ASYMPTOTIC SHAPE IN A CONTINUUM GROWTH MODEL

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Abstract

A continuum growth model is introduced. The state at time t , S_t , is a subset of \mathbb{R}^d and consists of a connected union of randomly sized Euclidean balls, which emerge from outbursts at their center points. An outburst occurs somewhere in S_t after an exponentially distributed time with expected value $|S_t|^{-1}$ and the location of the outburst is uniformly distributed over S_t . The main result is that if the distribution of the radii of the outburst balls has bounded support, then S_t grows linearly and S_t/t has a non-random shape as $t \rightarrow \infty$. Due to rotation invariance the asymptotic shape must be a Euclidean ball.

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1 Introduction

There is a variety of random growth models defined in a discrete space, see for instance Eden (1961), Williams and Bjerknes (1972), Bramson and Griffeath (1981) and Lee and Cowan (1994). A particular one is the Richardson model, introduced in Richardson (1973). It describes a Markov process whose state at time t , S_t , is a subset of \mathbb{Z}^d . Each site in \mathbb{Z}^d is in either of two states, denoted 0 and 1, and S_t consists of the sites which are in state 1 at time t . A site in state 0 is transferred to state 1 at a rate proportional to the number of nearest neighbors in state 1, and once in state 1 it never returns to state 0. Thus, if sites in state 1 are thought of as infected sites and sites in state 0 as uninfected this dynamics defines a pure growth model. The main result, first proved in Richardson (1973), states that if S_0 consists of a single site then S_t/t has a non-random shape as $t \rightarrow \infty$. Generalizations of Richardson's result can be found in Cox and Durrett (1981), Kesten (1986) and Boivin (1990). Apart from the fact that the asymptotic shape is convex and compact, not much is known about its qualitative features.

In this paper we introduce a growth model defined in a continuous space, that is, the state at time t , still denoted by S_t , is a subset of \mathbb{R}^d instead of \mathbb{Z}^d . The process should be thought of as describing the spread of some kind of infection in a continuous medium and just as in the Richardson model, the set S_t specifies the region infected at time t . The growth takes place by way of outbursts in the infected region. Given the development of the infection up to time t , the time until an outburst occurs somewhere in S_t is exponentially distributed with parameter $|S_t|$ and the location of the outburst is uniformly distributed over S_t (the notation $|\cdot|$ will, throughout the paper, be used to denote both Lebesgue measure and Euclidean norm, but the meaning will always be obvious from the context). When an outburst occurs at an infected point it causes a ball of stochastic radius around the outburst point to be infected and the total infected region is enlarged by the amount of this ball that was not previously infected. Consequently the infected region is a connected union of Euclidean balls. The radii of these balls are assumed to be iid with distribution F and mean γ .

The main result in this paper is a shape theorem for the continuum model. An essential advantage of the continuum model as compared to the Richardson model is that it possesses rotational invariance, which forces the asymptotic shape to be a Euclidean ball. To formulate the theorem, let $B(x, r)$ denote a ball with radius r around the point $x \in \mathbb{R}^d$ and let S_t denote the infected region at time t in the d -dimensional continuum model starting at time zero from a ball with radius γ around the origin.

Theorem 1.1 (Shape theorem) *Assume that F has bounded support and let $S_0 = B(0, \gamma)$. Then, for any dimension d there is a real number $\mu > 0$ such that for any ε with $0 < \varepsilon < \mu^{-1}$, almost surely,*

$$(1 - \varepsilon)B(0, \mu^{-1}) \subset \frac{S_t}{t} \subset (1 + \varepsilon)B(0, \mu^{-1})$$

for all sufficiently large t .

Another example of a continuum model with a Euclidean ball as asymptotic shape is described in Howard and Newman (1997).

The condition that the support of F is bounded can probably be weakened but some assumption on the tail is certainly necessary to ensure that $\mu > 0$, i.e. to ensure that the growth is not faster than linear.

A natural question is how the asymptotic shape is affected if the region infected at time zero is chosen to be something other than a ball with radius γ around the origin. The answer is given in the following theorem:

Theorem 1.2 *The conclusion of the shape theorem holds for any bounded initial set S_0 with strictly positive Lebesgue measure. Also, the number μ is independent of the initial set.*

The rest of the paper is organized as follows. We start by describing the model more thoroughly in Section 2. In Section 3 we define some important quantities and prove a technical lemma. Section 4 contains results concerning the growth in a fixed direction. To obtain these results we formulate and prove a sharpened version of Liggett's subadditive ergodic theorem (Theorem 4.2). Section 5 is devoted to the proof of the shape theorem and the proof of Theorem 1.2.

2 Description of the model

In this section we construct the model more formally by defining a Markov process whose state at time t , S_t , is a subset of \mathbb{R}^d . The process may be thought of as describing the spread of an infection (with no recoveries) or the growth of a germ colony in a continuous medium. Points in S_t will be referred to as infected.

To define the model let N be a Poisson process on \mathbb{R}^{d+1} . The extra dimension represents the time dimension and the points of N are hence denoted (X_k, T_k) , where $X_k \in \mathbb{R}^d$ and the last coordinate T_k gives the location on the time axis. To each point in the Poisson process we associate a random radius R_k . The variables $\{R_k\}$ are assumed to be independent and identically distributed with mean γ . At time zero a ball with radius γ around the origin, denoted by B_0 , is infected. The idea now is to follow the cylinder $B_0 \times \mathbb{R}$ upwards along the time axis until a point in the Poisson process is found. An outburst then takes place at this point generating a new infection ball B_1 , whose size is given by the radius associated with this particular Poisson point. The new infected region is given by $B_0 \cup B_1$. Scanning within the cylinder $(B_0 \cup B_1) \times \mathbb{R}$ further upwards along the time axis we eventually hit a new Poisson point, representing a new outburst and corresponding enlargement of the infected region. And so on.

To make this description more formal, let $N_{S \times \mathbb{R}}$, $S \subset \mathbb{R}^d$, denote the restriction of N to $S \times \mathbb{R}$. The growth of the infected area takes place at time points $\{T_n\}$

by aid of outbursts with radii $\{R_n\}$ at points $\{X_n\}$ obtained from the following recursion:

1. Let $X_0 = 0, T_0 = 0, R_0 = \gamma$ and define $B_n = \{y \in \mathbb{R}^d; |X_n - y| \leq R_n\}$, i.e. B_n is a ball in \mathbb{R}^d with radius R_n centered at X_n .
2. Given $\{X_i; i \leq n\}, \{T_i; i \leq n\}$ and $\{R_i; i \leq n\}$, the time T_{n+1} is defined as

$$T_{n+1} = \inf_k \{T_k; T_k > T_n \text{ and } (X_k, T_k) \in N_{\cup_{i=0}^n B_i \times \mathbb{R}}\}$$

and X_{n+1} is the (a.s. unique) point in \mathbb{R}^d such that $(X_{n+1}, T_{n+1}) \in N_{\cup_{i=0}^n B_i \times \mathbb{R}}$. The radius R_{n+1} is given by the radius associated with (X_{n+1}, T_{n+1}) .

From the sequence $\{X_n\}$ a new sequence $\{S_{(n)}\}$ is constructed by defining $S_{(n)} = \cup_{i=0}^n B_i$. The infected region at time t is now given by

$$S_t = S_{(n)} \quad \text{for } t \in [T_n, T_{n+1}).$$

Let us introduce the notation $\Delta_n = T_n - T_{n-1}, n \geq 1$, for the successive times between the outbursts. By construction of the model and properties of the Poisson process

$$\Delta_{n+1} | \mathcal{F}_n \sim \text{Exp}(|S_{(n)}|),$$

where $\mathcal{F}_n = \sigma(X_0, \dots, X_n, T_0, \dots, T_n, R_0, \dots, R_n)$. Given S_t , the memoryless property of the exponential distribution implies that the time until an outburst occurs somewhere in S_t is exponentially distributed with parameter $|S_t|$. The location of the outburst, X_n – where n is such that $t \in [T_{n-1}, T_n)$ – is uniformly distributed over S_t and the new infected region is given by $S_t \cup B_n$. Furthermore, the model is Markovian.

To guarantee that the model is defined for all t one detail remains to be checked: We have to make sure that the sequence $\{T_n\}$ does not have a finite limit point T_∞ , since this would cause problems defining S_t for $t > T_\infty$. The following proposition is what we need:

Proposition 2.1 *Assume that the radii distribution has finite moment of order d . Then, almost surely, $T_n \rightarrow \infty$ as $n \rightarrow \infty$.*

Proof: Let $\{E_k\}$ be independent, $E_k \sim \text{Exp}(k)$. It is left to the reader to show that it suffices to prove that $\sum_{k=1}^{\infty} E_k = \infty$ with probability one. To establish this, introduce $\tilde{E}_k = E_k - \mathbb{E}[E_k] = E_k - 1/k$. Using the fact that $\sum_{k=1}^{\infty} \mathbb{E}[\tilde{E}_k^2] = \sum_{k=1}^{\infty} 1/k^2 < \infty$, Kolmogorov's three series theorem implies that $\sum_{k=1}^{\infty} \tilde{E}_k$ converges almost surely. Thus

$$\sum_{k=1}^{\infty} E_k = \sum_{k=1}^{\infty} \tilde{E}_k + \sum_{k=1}^{\infty} \frac{1}{k} = \infty,$$

since $\sum_{k=1}^{\infty} 1/k = \infty$. □

3 Preliminaries

In this section we introduce some preliminary notation and results needed to prove the shape theorem.

To begin with, let $T(x)$ denote the time when the point x is infected, i.e.

$$T(x) = \inf\{t; x \in S_t\}.$$

Our first result is a lemma bounding the time it takes for the infection to reach the point x .

Lemma 3.1 *For any $x \in \mathbb{R}^d$ there exist iid exponential variables $\{E_k\}$ with parameter $\lambda = \lambda(d)$ such that*

$$T(x) \leq \sum_{k=1}^{2\lceil |x|\gamma^{-1} \rceil} E_k.$$

Proof: (We give the proof for $d = 2$. The case $d \geq 3$ is analogous.)

Fix $x \in \mathbb{R}^2$ and assume, without loss of generality, that x is located on the x -axis, that is, assume $x = (x', 0)$.

To begin with, fix $c \in \mathbb{R}$ and write B_k for the ball of radius c around the point $(k \cdot \gamma/2, 0)$, see Figure 1. If c is sufficiently small, say $c \leq \gamma/10$, then B_1 is contained in $B(0, \gamma)$, that is, B_1 is infected at time zero. Let E_1 denote the time from time zero until an outburst with radius at least γ occurs in B_1 . Clearly B_2 is infected by the time such an outburst has occurred, that is, $B_2 \subset S_{E_1}$. Now, the outburst points whose radius exceeds γ constitutes a Poisson process with intensity $p = P(R_k \geq \gamma)$ and thus $E_1 \sim \text{Exp}(p\pi c^2)$, where πc^2 is the area of B_1 .

The idea of how to continue should be clear: Let $E_0 = 0$ and define E_k , $k \geq 1$, recursively as the time until an outburst with radius larger than γ occurs in B_k counting from time $E_0 + \dots + E_{k-1}$. Since $B_k \subset S_{E_1 + \dots + E_{k-1}}$ we have that $E_k \sim \text{Exp}(p\pi c^2)$. When an outburst has occurred in $B_{2\lceil |x|\gamma^{-1} \rceil}$ it is clear that x must be infected, i.e. $x \in S_{E_1 + \dots + E_{2\lceil |x|\gamma^{-1} \rceil}}$. Hence

$$T(x) \leq \sum_{k=1}^{2\lceil |x|\gamma^{-1} \rceil} E_k$$

and the lemma is proved. \square

Remark 3.1 It follows from the proof of the lemma that the bound for $T(x)$ is valid also for the time until a small cube around x is infected.

By construction of the variables $\{E_k\}$, at time $\sum_{k=1}^{2\lceil |x|\gamma^{-1} \rceil} E_k$ an outburst with radius at least γ has occurred somewhere in a ball with radius $c/2$, $c \leq \gamma/10$, centered at a point within distance $\gamma/2$ from x . Thus a cube

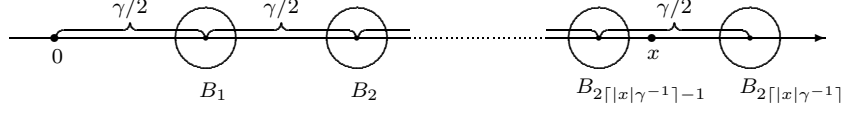


Figure 1: A chain of small balls located $\gamma/2$ units apart on the line segment joining the origin and x is constructed.

centered at x will be contained in the infected area at time $\sum_{k=1}^{2^{\lceil |x| \gamma^{-1} \rceil}} E_k$ if the side length of the cube is chosen small enough. This observation will be useful in proving Lemma 5.1.

We now introduce some auxiliary quantities that will be of use later. Let $S_t^{(x,s)}$, $t \geq s$, denote the set of points that can be reached from x within time t if a new process is started at x at time s . That is, at time s all infection except a ball with radius γ around x is erased. The infection then evolves in time according to the same rules as for the original process, using the same $(d+1)$ -dimensional Poisson process. This gives rise to a new process, emanating from x , whose state at time t , $t \geq s$, is given by $S_t^{(x,s)}$. Now let

$$\tilde{T}(x) = \inf\{t; B(x, \gamma) \subset S_t\}$$

and

$$\tilde{T}(x, y) = \inf\left\{t; B(y, \gamma) \subset S_{\tilde{T}(x)+t}^{(x, \tilde{T}(x))}\right\}.$$

In words $\tilde{T}(x)$ is the time when the entire ball with radius γ around x is infected and $\tilde{T}(x, y)$ is the time it takes for the infection to invade the entire γ -ball around y if a new process is started at x at time $\tilde{T}(x)$. Note that the quantity $\tilde{T}(x, y)$ is independent of $\tilde{T}(x)$ and has the same distribution as $\tilde{T}(y-x)$. Furthermore it is clear that if a point is contained in the region infected at time $\tilde{T}(x) + t$ in the process started at x at time $\tilde{T}(x)$ then it is also contained in the region infected at time $\tilde{T}(x) + t$ in the original process, i.e.

$$S_{\tilde{T}(x)+t}^{(x, \tilde{T}(x))} \subset S_{\tilde{T}(x)+t}$$

and hence

$$\tilde{T}(y) \leq \tilde{T}(x) + \tilde{T}(x, y). \quad (1)$$

We close this section by anticipating that in what follows it will be convenient to have special notation for the quantity $\tilde{T}(mx, nx)$. Thus, let

$$\tilde{T}(mx, nx) = \tilde{T}_{m,n}(x).$$

Since $\tilde{T}(nx) = \tilde{T}_{0,n}(x)$ the subadditivity property (1) translates into

$$\tilde{T}_{0,n}(x) \leq \tilde{T}_{0,m}(x) + \tilde{T}_{m,n}(x). \quad (2)$$

4 Growth in a fixed direction

The proof of the shape theorem basically consists of two parts:

1. Show that S_t grows linearly in each fixed direction and that the asymptotic speed of the growth in each direction is an almost sure constant. By the rotational invariance of \mathbb{R}^d and the model, this constant must be the same for all directions.
2. Show that the linear growth of S_t holds for all directions simultaneously.

This section is devoted to the first part. The first task is to prove the following result:

Proposition 4.1 *For each $x \in \mathbb{R}^d$ we have*

- (a) $\mu(x) := \lim_{n \rightarrow \infty} \mathbb{E}[\tilde{T}(nx)]/n = \inf_{n \geq 1} \mathbb{E}[\tilde{T}(nx)]/n;$
- (b) $\lim_{n \rightarrow \infty} \tilde{T}(nx)/n = \mu(x)$ a.s.

Remark 4.1 We employ the convention that limits over n are taken over the positive integers while limits over t , which occur later on in the paper, are taken over all positive reals.

To prove Proposition 4.1 we will invoke the following theorem by Liggett:

Theorem 4.1 (Liggett's subadditive ergodic theorem) *Let $\{X_{m,n}\}$ be a collection of random variables indexed by integers satisfying $0 \leq m < n$. Suppose $\{X_{m,n}\}$ has the following properties:*

- (i) $X_{0,n} \leq X_{0,m} + X_{m,n}$.
- (ii) For each n , $\mathbb{E}|X_{0,n}| < \infty$ and $\mathbb{E}[X_{0,n}] \geq cn$ for some constant $c > -\infty$.
- (iii) The distribution of $\{X_{m,m+k}; k \geq 1\}$ does not depend on m .
- (iv) For each $k \geq 1$, $\{X_{nk,(n+1)k}; n \geq 0\}$ is a stationary sequence.

Then:

- (a) $\eta := \lim_{n \rightarrow \infty} \mathbb{E}[X_{0,n}]/n = \inf_{n \geq 1} \mathbb{E}[X_{0,n}]/n.$
- (b) $X := \lim_{n \rightarrow \infty} X_{0,n}/n$ exists a.s.
- (c) $\mathbb{E}[X] = \eta.$

Furthermore, if the stationary processes in (iv) are ergodic, then:

- (d) $X = \eta$ a.s.

A brief outline of the structure of the proof can be found below. For more detail we refer the reader to Liggett (1985).

We wish to apply this result to the variables $\{\tilde{T}_{m,n}(x)\}$. It turns out, however, that condition (iv) fails for the sequence $\{\tilde{T}_{nk,(n+1)k}(x); n \geq 0\}$. Luckily the assumption (iv) can be relaxed without weakening the conclusions:

Theorem 4.2 *Let $\{X_{m,n}\}$ be a collection of random variables satisfying (i)-(iii) of Theorem 4.1. Furthermore, suppose that*

(iv') $\limsup_{n \rightarrow \infty} X_{0,nk}/n \leq E[X_{0,k}]$ for each k .

Then (a)-(d) of Theorem 5.1 hold for $\{X_{m,n}\}$.

Remark 4.2 A trivial modification of the proof of Theorem 4.2 yields that if we confine ourselves with the conclusions (a)-(c), then (iv') can be replaced by the still weaker assumption

(iv'') $E[\limsup_{n \rightarrow \infty} X_{0,nk}/n] \leq E[X_{0,k}]$ for each k .

To prove Theorem 4.2 requires some knowledge of the structure of the proof of Theorem 4.1. This knowledge is provided in the following brief sketch:

Write

$$\bar{X} = \limsup_{n \rightarrow \infty} \frac{X_{0,n}}{n}$$

and

$$X = \liminf_{n \rightarrow \infty} \frac{X_{0,n}}{n}.$$

The proof of Theorem 4.1 is broken up into three steps:

(L1) $\eta = \lim_{n \rightarrow \infty} E[X_{0,n}]/n = \inf_{n \geq 1} E[X_{0,n}]/n$;

(L2) $E[\bar{X}] \leq \eta$, and if the stationary processes in (iv) are ergodic, then $\bar{X} \leq \eta$ almost surely;

(L3) $E[X] \geq \eta$.

From (L2) and (L3) it follows that $E[X] \geq E[\bar{X}]$. This implies that X and \bar{X} are equal, since trivially $X \leq \bar{X}$. Hence, once (L2) and (L3) are established it is clear that $X := \lim_{n \rightarrow \infty} X_{0,n}/n$ exists with probability one. It also follows from (L2) and (L3) that $E[X] = \eta$ and by (L1) that $\eta < \infty$. Furthermore, if $\bar{X} \leq \eta$ – which, according to (L2), for example is the case if the sequences $\{X_{nk,(n+1)k}; n \geq 0\}$ are ergodic – then we can deduce that $X = \eta$ almost surely.

For a proof of (L1)-(L3) we refer to Liggett (1985). The essential task for us is to identify the parts of the proof that makes use of the assumption (iv). Since this assumption is to be replaced by (iv'), these are the parts that have to be modified. The following table shows how the assumptions are used in the different steps:

Step	Assumptions used
(L1)	(i), (iii)
(L2)	(i)-(iv)
(L3)	(i)-(iii)

Since the proofs of (L1) and (L3) do not use (iv), these statements hold also if (iv) is dropped. In proving (L2) though, Liggett uses the assumption (iv) and thus a modification of Liggett's proof is necessary to establish that (L2) remains true when (iv) is replaced by (iv'). The modification is described in the following proof of Theorem 4.2.

Proof of Theorem 4.2: It suffices to show that $\bar{X} \leq \eta$ almost surely, that is, it suffices to show that

$$\limsup_{n \rightarrow \infty} \frac{X_{0,n}}{n} \leq \eta \quad \text{a.s.} \quad (3)$$

To achieve this, fix $\delta > 0$ and choose k large so that $E[X_{0,k}]/k \leq \eta + \delta$. We will show that for all j ,

$$\limsup_{n \rightarrow \infty} \frac{X_{0,nk+j}}{nk+j} \leq \frac{E[X_{0,k}]}{k} \quad \text{a.s.} \quad (4)$$

which yields

$$\limsup_{n \rightarrow \infty} \frac{X_{0,n}}{n} \leq \frac{E[X_{0,k}]}{k} \quad \text{a.s.}$$

Hence, once (4) has been established it follows from the choice of k that

$$\limsup_{n \rightarrow \infty} \frac{X_{0,n}}{n} \leq \eta + \delta$$

and, since $\delta > 0$ was arbitrary, this implies (3). To prove (4), fix j and use subadditivity to get

$$\frac{X_{0,nk+j}}{nk+j} \leq \frac{n}{nk+j} \cdot \frac{X_{0,nk}}{n} + \frac{n}{nk+j} \cdot \frac{X_{nk,nk+j}}{n}. \quad (5)$$

By (iii), the distribution of $X_{nk,nk+j}$ depends only on j and, by (ii), the first moment is finite. Thus,

$$\sum_{n=1}^{\infty} P(X_{nk,nk+j} > n\varepsilon) < \infty$$

for all $\varepsilon > 0$ and by the Borel-Cantelli lemma this implies that

$$\lim_{n \rightarrow \infty} \frac{X_{nk,nk+j}}{n} = 0 \quad \text{a.s.} \quad (6)$$

Using (iv'), (4) follows from (5) and (6). \square

We are now in a position to prove Proposition 4.1.

Proof of Proposition 4.1: Fix $x \in \mathbb{R}^d$. Since $\{\tilde{T}_{m,n}(x)\}$ satisfies the assumptions (i)-(iii) of Theorem 4.1 the proposition follows from Theorem 4.2 if we can show that (iv') holds for $\{\tilde{T}_{m,n}(x)\}$, that is, if we can show that

$$\limsup_{n \rightarrow \infty} \frac{\tilde{T}_{0,nk}(x)}{n} \leq \mathbb{E}[T_{0,k}] \quad \text{for all } k. \quad (7)$$

To do this it is necessary to introduce an auxiliary sequence $\{\tilde{T}'_{(i-1)k,ik}(x); i \geq 1\}$ defined recursively as follows:

Let $\tilde{T}'_{0,k}(x) = \tilde{T}_{0,k}(x)$. For $i \geq 2$, given $\{\tilde{T}'_{(l-1)k,lk}(x); l \leq i-1\}$, define

$$\tilde{T}'_{(i-1)k,ik}(x) = \inf \left\{ t; B(ikx, \gamma) \subset S_{\Phi_{i-1}^k + t}^{((i-1)kx, \Phi_{i-1}^k)} \right\},$$

where $\Phi_{i-1}^k = \sum_{l=1}^{i-1} \tilde{T}'_{(l-1)k,lk}(x)$.

Remember that $S_t^{(x,s)}$ is the area infected at time t , $t \geq s$, in a process started at time s emanating from a ball with radius γ around x . Thus $\tilde{T}'_{(i-1)k,ik}(x)$ is the time when the ball with radius γ around ikx is infected in a process started at $(i-1)kx$ at time Φ_{i-1}^k . Some thought reveals that the variables $\{\tilde{T}'_{(i-1)k,ik}(x)\}$ are independent and identically distributed with expected value $\mathbb{E}[\tilde{T}_{0,k}(x)]$. Hence, by the strong law of large numbers

$$\frac{1}{n} \sum_{i=1}^n \tilde{T}'_{(i-1)k,ik}(x) \rightarrow \mathbb{E}[\tilde{T}_{0,k}(x)] \quad \text{as } n \rightarrow \infty. \quad (8)$$

Furthermore it is readily seen that

$$\tilde{T}_{0,nk}(x) \leq \sum_{i=1}^n \tilde{T}'_{(i-1)k,ik}(x).$$

Dividing this inequality by n and using (8) we obtain (7). Theorem 4.2 now gives that all the conclusions (a)-(d) of Theorem 4.1 hold for $\{\tilde{T}_{m,n}(x)\}$. Part (a) of the proposition follows from 4.1(a) and part (b) follows from Theorem 4.1(b) and (d). \square

Our next task is to show that $\mu(x)$ is nonzero so that the growth is indeed linear.

Proposition 4.2 *If F has bounded support, then $0 < \mu(x) < \infty$ for each $x \in \mathbb{R}^d$, $x \neq 0$.*

Proof: Fix $x \in \mathbb{R}^d$. That $\mu(x) < \infty$ follows from Proposition 4.1(a) so it remains to show that $\mu(x) > 0$. Since F has bounded support there is a real number r_{max} such that with probability one the radius of an outburst ball does not

exceed r_{max} . Using this it follows from a minor modification of Lemma 3.1 in Penrose (2001) that there are constants $c_1, c_2 \in (0, \infty)$ such that

$$P(T(x) \leq c_1|x|) \leq 2 \cdot 3^{-c_2|x|}. \quad (9)$$

That $\mu(x)$ is nonzero follows from (9) and the fact that

$$\limsup_{n \rightarrow \infty} \frac{E[T(nx)]}{n} \leq \lim_{n \rightarrow \infty} \frac{E[\tilde{T}(nx)]}{n} = \mu(x) \quad (10)$$

by arguing as follows: Substituting x by nx in (9) yields

$$P\left(\frac{T(nx)}{n} \leq c_1|x|\right) \leq 2 \cdot 3^{-c_2n|x|}$$

which implies that

$$\frac{E[T(nx)]}{n} \geq c_1|x| \left(1 - 2 \cdot 3^{-c_2n|x|}\right).$$

Using (10) this gives that $\mu(x) \geq c_1|x|$. \square

The next step is to prove that the discrete limit in Proposition 4.1(b) can be replaced by a continuous one.

Proposition 4.3 *For each $x \in \mathbb{R}^d$ we have $\lim_{t \rightarrow \infty} \tilde{T}(tx)/t = \mu(x)$, where the limit is taken along $t \in \mathbb{R}^+$.*

Proof: Fix $x \in \mathbb{R}^d$. We start by showing that

$$\lim_{n \rightarrow \infty} \frac{\tilde{T}(nqx)}{nq} = \mu(x) \quad \text{for all } q \in \mathbb{Q}, \quad (11)$$

that is, moving away from the origin in direction x using steps of arbitrary rational length yields the same limit. To this end, introduce the notation

$$\bar{T}_t = \frac{\tilde{T}(tx)}{t}$$

Now, since $q \in \mathbb{Q}$ we have $q = k/m$ for some integers k and m . The sequence $\{\bar{T}_{nk}\}$ is a subsequence of $\{\bar{T}_n\}$ and, hence, by Proposition 4.1(b),

$$\lim_{n \rightarrow \infty} \bar{T}_{nk} = \lim_{n \rightarrow \infty} \bar{T}_n = \mu(x) \quad \text{a.s.}$$

However, $\{\bar{T}_{nk}\}$ is also a subsequence of $\{\bar{T}_{nq}\}$ – obtained by considering only those points where n is a multiple of m – and since the sequence $\{\bar{T}_{nq}\}$ does indeed have a limit (this follows from Proposition 4.1(b) applied to the point qx) this implies that

$$\lim_{n \rightarrow \infty} \bar{T}_{nq} = \lim_{n \rightarrow \infty} \bar{T}_{nk} = \mu(x) \quad \text{a.s.}$$

To complete the proof we use (11) to show that

$$\limsup_{t \rightarrow \infty} \frac{\tilde{T}(tx)}{t} \leq \mu(x) \leq \liminf_{t \rightarrow \infty} \frac{\tilde{T}(tx)}{t}. \quad (12)$$

To establish (12) we will need a bound for the time from the moment when the γ -ball around an arbitrary point on the line segment between nqx and $(n+1)qx$ is infected until the γ -balls of all points on the line segment are infected. To obtain such a bound, let l_{nqx} denote the line segment between nqx and $(n+1)qx$ and write $\tilde{T}(l_{nqx})$ for the time when all points on l_{nqx} has their γ -balls infected, i.e.

$$\tilde{T}(l_{nqx}) = \inf\{t; B(z, \gamma) \subset S_t \text{ for all } z \in l_{nqx}\}.$$

Assume that at time t_0 there is a point $z_0 \in l_{nqx}$ such that $B(z_0, \gamma) \subset S_{t_0}$. For small $q \in \mathbb{Q}$ we will derive an upper bound for $\tilde{T}(l_{nqx}) - t_0$ expressed as the maximum of a number of iid exponential random variables. In two dimensions such a bound is easily obtained using a geometric construction displayed in Figure 2. Namely, let F be the outer ring of width $\gamma/10$ in $B(z_0, \gamma)$, that is, $F = B(z_0, \gamma) \setminus B(z_0, 9\gamma/10)$. Divide F into twelve disjoint pieces A_1, \dots, A_{12} of equal area as shown in Figure 2 and let E_k ($k = 1, \dots, 12$) be the time from time t_0 until an outburst with radius at least γ occurs in A_k . By construction the areas A_1, \dots, A_{12} are all infected at time t_0 , implying that $E_k \sim \text{Exp}(p|A_k|)$. Furthermore, since the A_k :s are disjoint, the variables $\{E_k\}$ are independent. Now, if q is small, then all points on l_{nqx} must have their γ -balls infected by the time an outburst whose radius exceeds γ has occurred in each of the areas A_k ($k = 1, \dots, 12$). Hence $B(z, \gamma) \subset S_{t_0 + \max\{E_1, \dots, E_{12}\}}$ for all $z \in l_{nqx}$ and we have proved that $\tilde{T}(l_{nqx}) - t_0 \leq \max\{E_1, \dots, E_{12}\}$ for all times t_0 such that there exists a point on l_{nqx} whose entire γ -ball is infected at time t_0 . The above reasoning easily generalizes to $d \geq 3$. We obtain

$$\tilde{T}(l_{nqx}) - t_0 \leq \max\{E_1, \dots, E_{k_0}\}, \quad (13)$$

where $k_0 = k_0(d)$ and $E_k \sim \text{Exp}(\lambda(d))$.

Now, to prove the first inequality in (12) let $q \in \mathbb{Q}$ be small enough to ensure (13) and let $\tilde{\varphi}_{nq}(x)$ be the time when the γ -balls of all points on l_{nqx} are infected counting from the time when the γ -ball around nqx is infected, that is, $\tilde{\varphi}_{nq}(x) = \tilde{T}(l_{nqx}) - \tilde{T}(nqx)$. By (13) there exist iid random variables E_1, \dots, E_{k_0} such that

$$\tilde{\varphi}_{nq}(x) \leq \max\{E_1, \dots, E_{k_0}\} := M_{k_0}. \quad (14)$$

Since $M_{k_0} \leq \sum_{k=1}^{k_0} E_k$ it follows that $E[M_{k_0}] \leq k_0 E[E_1] < \infty$ and, hence, for all $\varepsilon > 0$,

$$\sum_{n=0}^{\infty} P(M_{k_0} > n\varepsilon) < \infty.$$

Thus, by (14) and the Borel-Cantelli lemma,

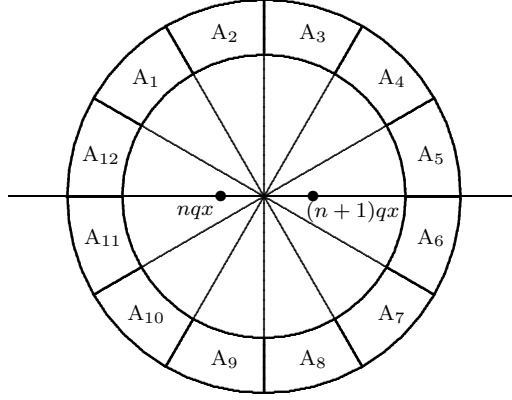


Figure 2: The front zone F of $B(z_0, \gamma)$ divided into pieces A_1, \dots, A_{12} .

$$\lim_{n \rightarrow \infty} \frac{\tilde{\varphi}_{nq}(x)}{n} = 0 \quad \text{a.s.} \quad (15)$$

Let n_t be such that $t \in [n_t q, (n_t + 1)q)$. Clearly $\tilde{T}(tx) \leq \tilde{T}(n_t qx) + \tilde{\varphi}_{n_t q}(x)$. Hence

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\tilde{T}(tx)}{t} &\leq \limsup_{t \rightarrow \infty} \frac{\tilde{T}(n_t qx) + \tilde{\varphi}_{n_t q}(x)}{n_t q} \\ &= \lim_{n \rightarrow \infty} \frac{\tilde{T}(nqx) + \tilde{\varphi}_{nq}(x)}{nq} \\ &= \mu(x), \end{aligned} \quad (16)$$

where the last equality follows from (11) and (15).

To prove the last inequality in (12), let z_{nqx} be the first point on l_{nqx} whose γ -ball is infected and let $\tilde{\psi}_{nq}(x)$ be the time from when this occurs until the infection has invaded the γ -balls of all points on l_{nqx} , that is, $\tilde{\psi}_{nq}(x) = \tilde{T}(l_{nqx}) - \tilde{T}(z_{nqx})$. Using (13) and Borel-Cantelli it can be seen that

$$\lim_{n \rightarrow \infty} \frac{\tilde{\psi}_{nq}(x)}{n} = 0 \quad \text{a.s.} \quad (17)$$

Since $\tilde{T}(tx) + \tilde{\psi}_{n_t q}(x) \geq \tilde{T}(n_t qx)$ we have

$$\begin{aligned}
\liminf_{t \rightarrow \infty} \frac{\tilde{T}(tx)}{t} &\geq \liminf_{t \rightarrow \infty} \frac{\tilde{T}(n_t qx) - \tilde{\psi}_{n_t q}(x)}{(n_t + 1)} \\
&= \lim_{n \rightarrow \infty} \frac{\tilde{T}(nqx) - \tilde{\psi}_{nq}(x)}{(n + 1)q} \\
&= \mu(x),
\end{aligned} \tag{18}$$

where the last equality is a consequence of (11) and (17). Thereby (12) is established and the proposition follows. \square

It follows from Proposition 4.3 that $\mu(cx) = c\mu(x)$. This implies that $\mu(x) = |x|\mu(\hat{x})$, where $\hat{x} = x/|x|$, the unit vector in direction x . Due to rotational invariance of \mathbb{R}^d and the model it is clear that $\mu(\hat{x}) = \mu(\hat{y})$ for all $x, y \in \mathbb{R}^d$. Thus we can define a constant

$$\mu := \mu((1, 0, \dots, 0))$$

representing the asymptotic time it takes for the infection to travel a unit vector in an arbitrary direction. By Proposition 4.2, $\mu \in (0, \infty)$. We end up with the simple relation $\mu(x) = |x|\mu$ valid for all $x \in \mathbb{R}^d$.

To summarize the results obtained in the present section, we have deduced that there is a real number $\mu > 0$ such that, for each fixed $x \in \mathbb{R}^d$, almost surely

$$\lim_{t \rightarrow \infty} \frac{\tilde{T}(tx)}{t} = |x|\mu. \tag{19}$$

5 Proof of the shape theorem

The shape theorem asserts that $S_t \approx tB(0, \mu^{-1})$ for large t and in view of (19) this is indeed what to expect. However, it remains to show that the linear growth stipulated in (19) holds for all directions simultaneously. To this end we will need the following lemma, which asserts that with high probability the infected region in a process emanating from a point y will eventually contain a ball centered at y with radius proportional to time.

Lemma 5.1 *For small $\delta > 0$, there is a constant $c \in (0, \infty)$ and a time s_0 such that, for any $y \in \mathbb{R}^d$ and $s' \geq 0$, we have*

$$P(B(y, s\delta) \not\subset S_{s'+s}^{(y, s')}) \leq e^{-cs}$$

if $s > s_0$.

Proof of Lemma 5.1: Due to shift invariance of the model it suffices to prove the lemma for $y = 0$ and $s' = 0$, that is, it suffices to show that

$$P(B(0, s\delta) \not\subset S_s) \leq e^{-cs} \quad (20)$$

for large s . To achieve this, fix a real number $\alpha > 0$ and partition \mathbb{R}^d into cubes centered at the points $\alpha\mathbb{Z}^d$ and with vertices $(\alpha/2, \dots, \alpha/2) + \alpha\mathbb{Z}^d$. Furthermore, let $\hat{T}_\alpha(x)$, $x \in \alpha\mathbb{Z}^d$, denote the time when the entire cube centered at x is infected. For small α and δ we will show that there is a positive constant c' such that the estimate

$$P(\hat{T}_\alpha(x) > a|x|/\delta) \leq e^{-c'a|x|}, \quad (21)$$

holds simultaneously for all $a \geq 1/d$ and all $x \in \mathbb{Z}^d$. Given this estimate the lemma is readily established: Let $C(s\delta + \alpha)$ denote the cube with side length $2(s\delta + \alpha)$ centered at the origin and write $C_\alpha(s\delta + \alpha) = \alpha\mathbb{Z}^d \cap C(s\delta + \alpha)$. Note that $B(0, s\delta)$ is contained in the union of all α -cubes whose centers are in $C_\alpha(s\delta + \alpha)$. Thus, if $B(0, s\delta) \not\subset S_s$, then not all α -cubes with centers in $C_\alpha(s\delta + \alpha)$ are infected at time s , that is,

$$\begin{aligned} P(B(0, s\delta) \not\subset S_s) &\leq P\left(\bigcup_{x \in C_\alpha(s\delta + \alpha)} \{\hat{T}_\alpha(x) > s\}\right) \\ &\leq \sum_{x \in C_\alpha(s\delta + \alpha)} P(\hat{T}_\alpha(x) > s). \end{aligned}$$

Trivially

$$P(\hat{T}_\alpha(x) > s) = P\left(\hat{T}_\alpha(x) > \frac{s\delta}{|x|} \cdot \frac{|x|}{\delta}\right).$$

For $x \in C_\alpha(s\delta + \alpha)$ we have $|x| \leq \sqrt{d}(s\delta + \alpha)$ and, since $s\delta/\sqrt{d}(s\delta + \alpha) \rightarrow 1/\sqrt{d}$ as $s \rightarrow \infty$, it holds that $s\delta/|x| \geq 1/d$ for large s . Hence, if s is large and $x \in C_\alpha(s\delta + \alpha)$ it follows from (21) that

$$P\left(\hat{T}_\alpha(x) > \frac{s\delta}{|x|} \cdot \frac{|x|}{\delta}\right) \leq e^{-c's\delta}$$

and, consequently, for large s ,

$$P(B(0, s\delta) \not\subset S_s) \leq \sum_{x \in C_\alpha(s\delta + \alpha)} e^{-c's\delta}.$$

Since the number of points in $C_\alpha(s\delta + \alpha)$ grows only polynomially in s there is a time s_0 such that, for $s > s_0$, we have $P(B(0, s\delta) \not\subset S_s) \leq e^{-s^{c'\delta/2}}$ as desired.

It remains to prove (21). Fix $x \in \alpha\mathbb{Z}^d$. By Lemma 3.1 and the remark following its proof, if α is small - say $\alpha \leq \gamma/10$ - we have

$$\hat{T}_\alpha(x) \leq \sum_{k=1}^{2\lceil |x|\gamma^{-1} \rceil} E_k,$$

where $\{E_k\}$ are iid exponential variables with parameter λ . Thus it suffices to find $c' > 0$ such that

$$P\left(\sum_{k=1}^{\lceil 2|x|\gamma^{-1} \rceil} E_k > \frac{a|x|}{\delta}\right) \leq e^{-c'a|x|}. \quad (22)$$

To this end, write $\lceil |x|\gamma^{-1} \rceil = m$ and introduce the notation $\Gamma_{2m} := \sum_{k=1}^{2m} E_k$. Using Markov's inequality and the fact that $\Gamma_{2m} \sim \text{Gamma}(2m, \lambda)$, we obtain

$$e^{\theta a|x|\delta^{-1}} P(\Gamma_{2m} > a|x|\delta^{-1}) \leq E[e^{\theta \Gamma_{2m}}] = (1 - \lambda\theta)^{-2m}$$

for $\theta \in (0, \lambda^{-1})$. Thus

$$P(\Gamma_{2m} > a|x|\delta^{-1}) \leq \exp\left\{-a|x|\left(\theta\delta^{-1} + \frac{2m}{a|x|}\log(1 - \theta\lambda)\right)\right\}. \quad (23)$$

We may assume that $|x| > \gamma - \alpha$, since for $x \in B(0, \gamma - \alpha)$ the α -cube centered at x is contained in $B(0, \gamma)$, implying that the left hand side in (21) equals zero and hence (21) is trivially true in this case. For $|x| > \gamma - \alpha$, the quotient $m/|x| = \lceil |x|\gamma^{-1} \rceil/|x|$ is bounded by $2\gamma^{-1}$. Substituting this in (23) and also using the fact that $a \geq 1/d$ yields

$$P(\Gamma_{2m} > a|x|\delta^{-1}) \leq e^{-a|x|f_\delta(\theta)},$$

where $f_\delta(\theta) = \theta\delta^{-1} + 4d\gamma^{-1}\log(1 - \lambda\theta)$. Now, $f_\delta(0) = 0$ and $f'_\delta(0) = \delta^{-1} - 4d\gamma^{-1}\lambda$. Thus, if δ is so small that $f'_\delta(0) > 0$, then we can pick θ small and get $f_\delta(\theta) > 0$. This proves (22). \square

Finally – equipped with the above lemma and the results from Section 4 – we are ready to prove the shape theorem.

Proof of Theorem 1.1: Fix $\varepsilon \in (0, \mu^{-1})$. We will prove the theorem in two steps:

- (i) There is almost surely a time T^1 such that $(1 - \varepsilon)tB(0, \mu^{-1}) \subset S_t$ for $t > T^1$.
- (ii) There is almost surely a time T^2 such that $S_t \subset (1 + \varepsilon)tB(0, \mu^{-1})$ for $t > T^2$.

As for (i) we will, in fact, prove the following:

- (i') There is almost surely a time \hat{T}^1 such that $(1 - \varepsilon/2)tB(0, \mu^{-1}) \subset S_t$ for $t > \hat{T}^1$, $t \in \mathbb{N}$.

From (i') it follows that $(1 - \varepsilon/2)\lfloor t \rfloor B(0, \mu^{-1}) \subset S_{\lfloor t \rfloor}$ for $t > \hat{T}^1 + 1$ and since $S_{\lfloor t \rfloor} \subset S_t$ for all t we obtain $(1 - \varepsilon/2)\lfloor t \rfloor B(0, \mu^{-1}) \subset S_t$. But, for large t , the

ball with radius $(1 - \varepsilon/2)[t]\mu^{-1}$ contains the ball with radius $(1 - \varepsilon)t\mu^{-1}$, and hence (i) follows from (i').

To prove (i'), note that since $(1 - \varepsilon/2)B(0, \mu^{-1})$ is compact, there are points $x_1, \dots, x_n \in (1 - \varepsilon/2)B(0, \mu^{-1})$ such that

$$(1 - \varepsilon/2)B(0, \mu^{-1}) \subset \bigcup_{i=1}^n B(x_i, \delta\varepsilon/4),$$

where $\delta > 0$ is chosen small enough to ensure that Lemma 5.1 holds. Clearly

$$(1 - \varepsilon/2)tB(0, \mu^{-1}) \subset \bigcup_{i=1}^n B(tx_i, t\delta\varepsilon/4). \quad (24)$$

Since $|x_i| \leq (1 - \varepsilon/2)\mu^{-1}$ it follows from (19) that $\lim_{t \rightarrow \infty} \tilde{T}(tx_i)/t \leq 1 - \varepsilon/2$ for each i almost surely. This implies that almost surely $\tilde{T}(tx_i) \leq t(1 - \varepsilon/4)$ for each i if t is large, that is,

$$B(tx_i, \gamma) \subset S_{t(1-\varepsilon/4)} \quad (25)$$

for large t . Furthermore, by Lemma 5.1,

$$\sum_{t \in \mathbb{N}} P\left(B(tx_i, t\delta\varepsilon/4) \not\subset S_t^{(tx_i, t(1-\varepsilon/4))}\right) \sim \sum_{t \in \mathbb{N}} e^{-ct\varepsilon/4} < \infty.$$

Thus, by the Borel-Cantelli lemma, for large integer times we have almost surely

$$B(tx_i, t\delta\varepsilon/4) \subset S_t^{(tx_i, t(1-\varepsilon/4))}. \quad (26)$$

Now, for each i , let $T(i)$ be such that both (25) and (26) hold for $t > T(i)$, $t \in \mathbb{N}$, and define $\hat{T}^1 = \max\{T(i)\}$. For $t > \hat{T}^1$ we have $B(tx_i, t\delta\varepsilon/4) \subset S_t$ for all i and $t \in \mathbb{N}$. Using (24) this implies that $(1 - \varepsilon/2)tB(0, \mu^{-1}) \subset S_t$ for all integer times larger than \hat{T}_1' , as desired.

Moving on to (ii), let A be an annulus of width $\mu^{-1}\varepsilon/2$ surrounding $(1 + \varepsilon/2)B(0, \mu^{-1})$, i.e.

$$A = (1 + \varepsilon)B(0, \mu^{-1}) \setminus (1 + \varepsilon/2)B(0, \mu^{-1}),$$

and let \hat{S}_t be the set of points whose entire γ -ball is infected at time t , i.e.

$$\hat{S}_t = \{x; B(x, \gamma) \subset S_t\}.$$

Points in \hat{S}_t will be referred to as strongly infected at time t . We begin by showing that almost surely $\hat{S}_t \cap tA = \emptyset$ for large t .

Let δ be small enough to ensure that Lemma 5.1 holds and pick $x_1, \dots, x_n \in A$ such that

$$A \subset \bigcup_{i=1}^n B(x_i, \delta\varepsilon/8). \quad (27)$$

By (19), $\lim_{t \rightarrow \infty} \tilde{T}(tx_i)/t \geq 1 + \varepsilon/2$ for each i . Hence, for large t almost surely $\tilde{T}(tx_i) \geq t(1 + \varepsilon/4)$ for each i , implying that for every $c \in (0, 1)$ we have

$$P\left(B(tx_i, \gamma) \subset S_{t(1+\varepsilon/4)} \text{ for some } i = 1, \dots, n\right) \leq c \quad (28)$$

if t is large. The idea of the proof is that, if tA contains strongly infected points for large t , then with high probability some point tx_i will be strongly infected within time $t\varepsilon/4$ and this conflicts with (28). To formalize this intuition, let

$$p = P\left(\hat{S}_t \cap tA \neq \emptyset \text{ for arbitrarily large } t\right)$$

and assume for contradiction that $p > 0$. For fixed t write $E_t = \{t' \geq t; \hat{S}_{t'} \cap t'A \neq \emptyset\}$ and define

$$T_t = \begin{cases} \inf E_t & \text{if } E_t \neq \emptyset, \\ \infty & \text{if } E_t = \emptyset. \end{cases}$$

Note that $P(T_t < \infty) \geq p > 0$ for each t . Consequently, we can condition on the event that $T_t < \infty$ and pick y_t uniformly on $S_{T_t} \cap T_t A$. By Lemma 5.1,

$$P\left(B(y_t, T_t \delta \varepsilon/4) \subset S_{T_t(1+\varepsilon/4)}^{(y_t, T_t)} \mid T_t < \infty\right) \geq 1 - e^{-ct\varepsilon/4},$$

that is, the ball with radius $T_t \delta \varepsilon/4$ around y_t is with high probability infected at time $T_t(1 + \varepsilon/4)$ in a process started from y_t at time T_t . If t is large this implies that the ball with radius $T_t \delta \varepsilon/8$ is strongly infected with high probability. Formally,

$$P\left(B(y_t, T_t \delta \varepsilon/8) \subset \hat{S}_{T_t(1+\varepsilon/4)}^{(y_t, T_t)} \mid T_t < \infty\right) \geq 1 - e^{-ct\varepsilon/4}.$$

Since y_t is strongly infected at time T_t we obtain

$$P\left(B(y_t, T_t \delta \varepsilon/8) \subset \hat{S}_{T_t(1+\varepsilon/4)} \mid T_t < \infty\right) \rightarrow 1 \quad \text{as } t \rightarrow \infty.$$

Now, by (27) $T_t A$ is covered by the balls $B(T_t x_i, T_t \delta \varepsilon/8)$. Hence, since $y_t \in T_t A$ we can find at least one point x_i such that $T_t x_i \in B(y_t, T_t \delta \varepsilon/8)$ and, consequently,

$$P\left(B(T_t x_i, \gamma) \subset S_{T_t(1+\varepsilon/4)} \text{ for some } i = 1, \dots, n \mid T_t < \infty\right) \rightarrow 1 \quad \text{as } t \rightarrow \infty.$$

Pick t large so that the above probability is greater than $1/2$ and so that (28) holds for $c = p/4$. Use the fact that $P(T_t < \infty) \geq p$ for each t to obtain

$$\begin{aligned} & P(\exists t' \geq t \text{ such that } B(t'x_i, \gamma) \subset S_{t'(1+\varepsilon/4)} \text{ for some } i = 1, \dots, n) \\ & \geq P\left(B(T_t x_i, \gamma) \subset S_{T_t(1+\varepsilon/4)} \text{ for some } i = 1, \dots, n \mid T_t < \infty\right) P(T_t < \infty) \\ & > p/2. \end{aligned}$$

This contradicts (28). Hence we must have $p = 0$, that is, almost surely $\hat{S}_t \cap tA = \emptyset$ for large t . It remains to show that (ii) follows from this. To this end let Γ

denote the set of outbursts that occur in tA for some t during the progress of the growth, that is, Γ is the set of outbursts (X_n, T_n) such that $X_n \in T_n A$. Furthermore, let Γ_γ be those outbursts in Γ whose radius is at least γ . Assume that (ii) fails so that with positive probability $S_t \cap [(1 + \varepsilon)tB(0, \mu^{-1})]^c \neq \emptyset$ for arbitrarily large t . This implies that $P(|\Gamma| = \infty) > 0$. Now, each time an outburst takes place we condition on the process up to that time, including the fact that an outburst takes place at that time at that location, but excluding the radius of the outburst. Since the radii of the outbursts are iid it follows from Levy's version of the Borel-Cantelli lemma (see Williams (1991), section 12.5) that we can not have $|\Gamma| = \infty$ and $|\Gamma_\gamma| < \infty$. Hence the fact that $P(|\Gamma| = \infty) > 0$ implies that $P(|\Gamma_\gamma| = \infty) > 0$ as well. But if $|\Gamma_\gamma| = \infty$ the region tA must contain strongly infected points for arbitrarily large t and hence we have derived a contradiction.

At this point (i) and (ii) are established and all that remains is to note that for $t > \max\{T^1, T^2\}$ we have

$$(1 - \varepsilon)tB(0, \mu^{-1}) \subset S_t \subset (1 + \varepsilon)tB(0, \mu^{-1}).$$

The shape theorem is proved. \square

We end by proving Theorem 1.2, that is, by proving that the asymptotic shape is independent of the state at time zero.

Proof of Theorem 1.2: For an arbitrary set $\Pi \subset \mathbb{R}^d$, introduce generalized versions of the quantities S_t and $S_t^{(x,s)}$ as follows: Let S_t^Π denote the region infected at time t in a process started from the set Π and let $S_t^{\Pi,s}$ denote the region infected at time t , $t \geq s$, in a process started from the set Π at time s . We will show that, for any $\varepsilon > 0$, we have

$$(1 - \varepsilon)tB(0, \mu^{-1}) \subset \frac{S_t^\Pi}{t} \subset (1 + \varepsilon)tB(0, \mu^{-1}) \quad (29)$$

if t is large. To this end, let τ be the first time when Π is contained in S_t , and let τ' be the first time counting from τ when $B(0, \gamma)$ is contained in $S_t^{\Pi,\tau}$, that is,

$$\tau = \inf\{t; \Pi \subset S_t\}$$

and

$$\tau' = \inf\{t; B(0, \gamma) \subset S_{\tau+t}^{\Pi,\tau}\}.$$

If Π is bounded and has strictly positive Lebesgue measure these times are both finite with probability one. Furthermore,

$$S_{\tau+\tau'+t}^{(B(0,\gamma),\tau+\tau')} \subset S_{\tau+\tau'+t}^{\Pi,\tau} \subset S_{\tau+\tau'+t}$$

which implies that

$$\frac{S_{\tau+\tau'+t}^{(B(0,\gamma),\tau+\tau')}}{t} \subset \frac{\tau' + t}{t} \cdot \frac{S_{\tau+\tau'+t}^{\Pi,\tau}}{\tau' + t} \subset \frac{\tau + \tau' + t}{t} \cdot \frac{S_{\tau+\tau'+t}}{\tau + \tau' + t}.$$

The quotients $(\tau'+t)/t$ and $(\tau+\tau'+t)/t$ tends to 1 as $t \rightarrow \infty$ and by Theorem 1.1 the asymptotic shape for the processes $S_{\tau+\tau'+t}^{(B(0,\gamma),\tau+\tau')}/t$ and $S_{\tau+\tau'+t}/(\tau+\tau'+t)$ is $B(0, \mu^{-1})$. This establishes (29). \square

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