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Infinitely many solutions of a semilinear problem for the Heisenberg Laplacian on the Heisenberg group

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Abstract. We study the semilinear equation

$$-\Delta_{\mathbb{H}} u(\eta) + u(\eta) = f(\eta, u(\eta)),$$

where $\Delta_{\mathbb{H}}$ is the Heisenberg Laplacian and $\mathbb{H}^N$ is the Heisenberg group. The function $f \in C^2(\mathbb{H}^N \times \mathbb{R}, \mathbb{R})$ is supposed to satisfy some (subcritical) growth conditions and to be left invariant under the action of the subgroup of $\mathbb{H}^N$ consisting of points with integer coordinates.. We show the existence of infinitely many solutions in the space $S_1^2(\mathbb{H}^N)$, which is the Heisenberg analogue of the Sobolev space $W^{1,2}(\mathbb{R}^N)$.

Contents

1. Introduction and results

Let $\mathbb{H}^N$ be the space $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ equipped with the following group operation:

$$\eta \circ \eta' = (x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2(x' \cdot y - x \cdot y')).$$

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where $\cdot$ denotes the usual inner product in $\mathbb{R}^N$. This operation endows $\mathbb{H}^N$ with the structure of a Lie group. The vector fields $X_1, \dots X_N, Y_1, \dots X_N, T$ given by

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \\ Y_j &= \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \\ T &= \frac{\partial}{\partial t}, \end{aligned}$$

form a basis for the tangent space at $\eta = (x, y, t)$. The commutation relations

$$\begin{aligned} &[X_i, Y_j] = -4\delta_{ij} T \\ &[X_i, X_j] = [Y_i, Y_j] = [X_i, T] = [Y_j, T] = 0 \end{aligned}$$

imply that the Lie algebra of left invariant vector fields of $\mathbb{H}^N$ is generated by X_1, $\dots X_N, Y_1, \dots Y_N$, and that they satisfy the Hörmander condition of order 1 (see [11]).

The Heisenberg Laplacian is by definition

$$\Delta_{\mathbb{H}} = \sum_{j=1}^{N} \left(X_j^2 + Y_j^2 \right)$$

and we use the notation $\nabla_{\mathbb{H}} u$ for the $2N$-vector $(X_1 u, \dots, X_N u, Y_1 u, \dots, Y_N u)$. The space $S_1^2(\mathbb{H}^N)$ is defined as the completion of $C_0^\infty(\mathbb{H}^N)$ in the norm $\|u\|_{S_1^2}$ given by

$$\|u\|_{S_1^2}^2 = \int_{\mathbb{H}^N} |u|^2 + \sum_{j=1}^{N} \int_{\mathbb{H}^N} \left(|X_j u|^2 + |Y_j u|^2 \right) = \int_{\mathbb{H}^N} \left(|u|^2 + |\nabla_{\mathbb{H}} u|^2 \right).$$

The left and right Haar measure on $\mathbb{H}^N$ is the Lebesgue measure, and the integral above is with respect to this measure. Let $Q = 2N + 2$ be the homogeneous dimension of $\mathbb{H}^N$ and let $2^* = 2Q/(Q-2) = 2 + 2/N$. There is an embedding theorem due to Folland and Stein [7], saying that $S_1^2(\mathbb{H}^N)$ is embedded in $L^p(\mathbb{H}^N)$ for $2 \le p \le 2^*$.

Let $\mathbb{H}_{\mathbb{Z}}^N$ be the subgroup of $\mathbb{H}^N$ consisting of points in $\mathbb{H}^N$ with integer coordinates, and consider the equation

$$\begin{aligned} -\Delta_{\mathbb{H}} u + u &= f(\eta, u), \\ u &\in S_1^2(\mathbb{H}^N) \end{aligned} \tag{1.1}$$

where $f \in C^2(\mathbb{H}^N \times \mathbb{R}, \mathbb{R})$ satisfies the following growth conditions:

(f_1) there is a constant $2 < p < 2^*$ and constants $a_0, a_1 > 0$ such that for any $\eta \in \mathbb{H}^N$ and $u \in \mathbb{R}$,

$$|f_u(\eta, u)| \le a_0 + a_1 |u|^{p-2},$$

(f_2) there is a constant $\mu > 2$ such that for any $\eta \in \mathbb{H}^N$ and $u \in \mathbb{R} \setminus \{0\}$,

$$uf(\eta, u) \geq \mu F(\eta, u) \equiv \mu \int_0^u f(\eta, \sigma)\, d\sigma > 0,$$

(f_3) $f(\eta, 0) = f_u(\eta, 0) = 0$.

(f_4) f is left invariant with respect to $\mathbb{H}^N_{\mathbb{Z}}$ in the η variable, i.e. for any $u \in \mathbb{R}$, $z \in \mathbb{H}^N$ and $\eta \in \mathbb{H}^N_{\mathbb{Z}}$,

$$f(\eta \circ z, u) = f(z, u).$$

We also require that $\mathbb{H}^N_{\mathbb{Z}}$ is the largest subgroup of $\mathbb{H}^N$ such that (f_4) holds.

Note that conditions (f_1), (f_3) and (f_4) imply that for every $\epsilon > 0$, there exists a constant $A_\epsilon > 0$ such that for every $\eta \in \mathbb{H}^N$ and $u \in \mathbb{R}$,

$$|f(\eta, u)| \leq \epsilon |u| + A_\epsilon |u|^{p-1}.$$

A simple example of a function f satisfying these conditions is

$$f(\eta, u) = a(\eta)|u|^{p-2}u,$$

where $a \in C^2(\mathbb{H}^N, \mathbb{R})$ is left invariant with respect to $\mathbb{H}^N_{\mathbb{Z}}$ and $p \in (2, 2^*)$.

Semilinear equations on the Heisenberg group have been studied by a number of authors, including Biagini, Birindelli, Capuzzo–Dolcetta, Citti, Cutrì, Garofalo, Lanconelli, Pohozaev, Ugozzoni, and Véron. See [1–4, 9, 13, 14, 17] for some of these results.

The functional on $S_1^2(\mathbb{H}^N)$ corresponding to equation (1.1) is given by

$$\varphi(u) = \frac{1}{2}\|u\|^2_{S_1^2} - \int_{\mathbb{H}^N} F(\eta, u(\eta))d\eta, \tag{1.2}$$

and the critical points of φ are exactly the solutions of (1.1).

Let K be the set of critical points of φ, i.e.

$$K = \{u \in S_1^2(\mathbb{H}^N);\ \varphi'(u) = 0\}.$$

For $\eta \in \mathbb{H}^N$, we define the action τ_η by

$$\tau_\eta u(z) = u(\eta^{-1} \circ z).$$

Note that K is invariant under the action τ_η, where $\eta \in \mathbb{H}^N_{\mathbb{Z}}$.

Two critical points u_1 and u_2 of φ are considered to be equivalent if there exists $\eta \in \mathbb{H}^N_{\mathbb{Z}}$ such that $u_1 = \tau_\eta u_2$. Let $\mathcal{F}$ be a set of representatives of K under this equivalence relation. Two critical points which are not equivalent are said to be *geometrically distinct*.

Note that this definition makes sense by the assumption that $\mathbb{H}^N_{\mathbb{Z}}$ is the largest subgroup of $\mathbb{H}^N$ such that (f_4) holds. The subgroup $\mathbb{H}^N_{\mathbb{Z}}$ could be replaced by another discrete subgroup $\mathcal{G}$ of $\mathbb{H}^N$ as long as $\sup_{\eta \in \mathbb{H}^N} d(\eta, \mathcal{G})$ is finite (for the definition of the Heisenberg distance function d, see section 2). In our deformation lemma (Lemma 5), it will also be needed that the infimum of the distance between elements in the subgroup is positive. In particular we see that this rules out the case when f is independent of η. Note however that if the problem was invariant with respect to another discrete subgroup than $\mathbb{H}^N_{\mathbb{Z}}$ (subject to the mentioned conditions), we could by a change of variables transform it into our problem. With this in mind, we see that there is no loss of generality assuming that the subgroup is $\mathbb{H}^N_{\mathbb{Z}}$.

Our main result is the following.

Theorem 1. *Suppose that* $f \in C^2(\mathbb{H}^N \times \mathbb{R}, \mathbb{R})$ *satisfies* (f_1), (f_2), (f_3) *and* (f_4). *Then the functional* φ *as defined in* (1.2) *has infinitely many geometrically distinct critical points and the corresponding equation* (1.1) *has infinitely many geometrically distinct solutions.*

A similar result for the classical Laplacian was proved by Coti-Zelati and Rabinowitz in [6]. In this paper, we use similar methods to obtain a proof of Theorem 1. Section 3 contains a deformation theorem which is needed in the proof of Theorem 1. Finally, the proof of Theorem 1 is given in Section 5.

2. Preliminaries and notation

There is a distance in the Heisenberg group, defined by

$$d(\xi, 0) = \left(\left(\sum_{i=1}^{N} x_i^2 + y_i^2 \right)^2 + t^2 \right)^{\frac{1}{4}}$$

and

$$d(\eta_1, \eta_2) = d(\eta_2^{-1} \circ \eta_1, 0).$$

The Heisenberg ball is the set

$$B_{\mathbb{H}}(\eta, r) = \{\xi \in \mathbb{H}^N;\ d(\xi, \eta) < r\},$$

and it plays the role of the Euclidean ball of $\mathbb{R}^N$. For instance,

$$|B_{\mathbb{H}}(\eta, r)| = r^Q |B_{\mathbb{H}}(0, 1)|,$$

where $Q = 2N + 2$ is the homogeneous dimension of the Heisenberg group. In some places we use the notation $d(\eta, A)$ for the Heisenberg distance between a point η and a set A. Likewise diam A is the diameter of A in terms of the distance d above. We will also use the notation $N_\delta(A)$ for the set $\{\eta \in \mathbb{H}^N; d(\eta, A) < \delta\}$.

For a domain $\Omega \subset \mathbb{H}^N$, the spaces $S_k^p(\Omega)$ and $\overset{\circ}{S}{}_k^p(\Omega)$ are the Heisenberg group analogues of the Sobolev spaces $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ for a domain $\Omega \subset \mathbb{R}^N$. We refer to Folland–Stein [7] for the exact definition of these spaces as well as the spaces $\Gamma^\beta(\Omega)$, which are the analogue of the Hölder spaces in $\mathbb{R}^N$.

The following theorem will be referred to as the Folland–Stein embedding theorem (see [7]):

Theorem 2. *Let* $\Omega \subset \mathbb{H}^N$ *be a bounded domain and let* $Q = 2N+2$, $1 < p, q < \infty$ *and* $k \geq 1$. *If* $q \leq Qp/(Q-kp)$, *then* $S_k^p(\Omega) \subset L^q(\Omega)$ *and there exists a constant* $C > 0$ *such that*

$$\|u\|_{L^q} \leq C\|u\|_{S_k^p}.$$

The embedding is compact when $q < Qp/(Q-kp)$.

For $k > Q/p$ *then* $S_k^p(\Omega) \subset \Gamma_\beta(\Omega)$, *where* $\beta = \alpha - Q/p$ *and there exists a constant* $C > 0$ *such that*

$$\|u\|_{\Gamma_\beta} \leq C\|u\|_{S_k^p}.$$

When $\Omega \subset \mathbb{H}^N$ *is an unbounded domain, then there is a continuous embedding of* $S_k^p(\Omega)$ *into* $L^p(\Omega)$ *when* $p \leq q \leq Qp/(Q-kp)$.

The weak maximum principle holds for $\Delta_{\mathbb{H}}$, and the proof is the same as for the classical Laplacian on $\mathbb{R}^N$ (see e.g. [10]).

3. A deformation lemma

Lemma 1. *Let* φ *be given by* (1.2), *where* $f \in C(\mathbb{H}^N \times \mathbb{R}, \mathbb{R})$ *satisfies condition* (f_2). *Let* $c \in \mathbb{R}$ *and let* u_j *be a (P–S)*$_c$*-sequence for* φ, *i.e.*

$$\varphi(u_j) \to c,$$
$$\varphi'(u_j) \to 0.$$

Then $c \geq 0$. *Moreover,* $\|u_j\|_{S_1^2}$ *is bounded and*

$$\limsup_{j\to\infty} \|u_j\|_{S_1^2}^2 \leq \frac{c}{\frac{1}{2} - \frac{1}{\mu}}.$$

The proof of this lemma is standard, and so we omit the proof.

Suppose that $F \subset S_1^2(\mathbb{H}^N)$ is a finite set. Let $[F, l] \subset S_1^2(\mathbb{H}^N)$ be the set defined by

$$[F, l] = \left\{ \sum_{n=1}^{m} \tau_{\alpha^{(n)}} w^{(n)};\ 1 \leq m \leq l,\ w^{(n)} \in F,\ \alpha^{(n)} \in \mathbb{H}_{\mathbb{Z}}^N \right\}.$$

Note that $[F, l]$ is left invariant with respect to $\mathbb{H}_{\mathbb{Z}}^N$.

Lemma 2. *Suppose that* $F \subset S_1^2(\mathbb{H}^N)$ *is a finite set, and let* $l \geq 1$. *Then the number*

$$\delta = \delta(l) \equiv \inf\{\|u - v\|_{L^p};\ u, v \in [F, l],\ u \neq v\}$$

is positive.

Proof. Let $\tilde{F} = F \cup (-F)$. The result follows if for $l \geq 1$,

$$\inf\{\|u\|_{L^p};\ u \in [\tilde{F}, l] \setminus \{0\}\} > 0.$$

We proceed by induction. Let $u_j \in [\tilde{F}, 1]\setminus\{0\}$ be a sequence such that $\|u_j\|_{L^p} \to 0$ as $j \to \infty$. Then we have $u_j = \tau_{\alpha_j} w_j$. Since $\tilde{F}$ is a finite set, we can after extracting a subsequence assume that w_j are independent of j. Moreover, since for any $\alpha \in \mathbb{H}^N_{\mathbb{Z}}$ and $u \in S^2_1(\mathbb{H}^N)$ we have $\|u\|_{L^p} = \|\tau_\alpha u\|_{L^p}$, we can assume that $\alpha_j = 0$. Thus $u_j = w$ for all j, but then $u_j = 0$, which is a contradiction.

Suppose now that the assertion is true for $1 \leq l \leq l_1 - 1$. Suppose that $u_j \in [\tilde{F}, l_1] \setminus \{0\}$ is such that $u_j \to 0$ in L^p. We can write

$$u_j = \sum_{n=1}^{m_j} \tau_{\alpha_j^{(n)}} w_j^{(n)},$$

where $m_j \leq l_1$, $\alpha_j^{(n)} \in \mathbb{H}^N_{\mathbb{Z}}$ and $w_j^{(n)} \in \tilde{F}$. Since $\tilde{F}$ is finite, we can extract a subsequence and assume that m_j and $w_j^{(n)}$ are independent of j. Since the L^p norm is unchanged under shifts τ_α where $\alpha \in \mathbb{H}^N_{\mathbb{Z}}$, we can also assume that $\alpha_j^{(1)} = 0$ for all j. Thus

$$u_j = \sum_{n=1}^{m} \tau_{\alpha_j^{(n)}} w^{(n)}.$$

After a renumbering, we are in one of the following two cases:

(i) $\alpha_j^{(n)}$ are bounded for $1 \leq n \leq m$. In this case, after taking a subsequence, $\alpha_j^{(n)} = \alpha^{(n)}$. Thus $\|u_j\|_{L^p}$ is constant, so it must be zero, but this is a contradiction.

(ii) there exists $1 \leq m_1 < m$ such that all $\alpha_j^{(n)}$ are bounded for $1 \leq n \leq m_1$ and unbounded for $m_1 + 1 \leq n \leq m$. Then we have

$$\|u_j\|^p_{L^p} = \left\| \sum_{n=1}^{m_1} \tau_{\alpha_j^{(n)}} w^{(n)} \right\|^p_{L^p} + \left\| \sum_{n=m_1+1}^{m} \tau_{\alpha_j^{(n)}} w^{(n)} \right\|^p_{L^p} + \epsilon_j,$$

where $\epsilon_j \to 0$ as $j \to \infty$. As in case (i), all $\alpha_j^{(n)}$ in the first sum can be assumed to be independent of j after taking a subsequence. Since $u_j \to 0$ in L^p, this first sum must be zero. But then

$$u_j = \sum_{n=m_1+1}^{m} \tau_{\alpha_j^{(n)}} w^{(n)},$$

and $u_j \in [\tilde{F}, l_1 - 1] \setminus \{0\}$, which is also impossible. □

The following lemma is a reformulation of Theorem 2.3, of [16] (see also [15]).

Lemma 3. *Let $u_j \in S_1^2(\mathbb{H}^N)$ be a bounded sequence. Then there exist $w^{(0)}$, $w^{(1)}$, $\ldots \in S_1^2(\mathbb{H}^N)$ and $\alpha_j^{(n)} \in \mathbb{H}_{\mathbb{Z}}^N$, $j, n \in \mathbb{N}$ such that on a subsequence,*

$$\tau_{\alpha_j^{(n)}} u_j \rightharpoonup w^{(n)}$$

$$u_j - \sum_{n=1}^{\infty} \tau_{(\alpha_j^{(n)})^{-1}} w^{(n)} \to 0 \textit{ in } L^p(\mathbb{H}^N), \ \textit{where } p \in (2, 2^*),$$

$$d(\alpha_j^{(m)}, \alpha_j^{(n)}) \to \infty \textit{ if } m \neq n,$$

$$\sum_{n=1}^{\infty} \|w^{(n)}\|_{S_1^2}^2 \leq \lim_{j\to\infty} \|u_j\|_{S_1^2}^2$$

$$\sum_{n=1}^{\infty} \|w^{(n)}\|_{L^p}^p = \lim_{j\to\infty} \|u_j\|_{L^p}^p, \ \textit{where } p \in (2, 2^*).$$

Let $\mathcal{F}$ be a set of representatives of K.

Lemma 4. *Suppose that $\mathcal{F}$ is finite. Let $c > 0$ and let $u_j \in S_1^2(\mathbb{H}^N)$ be a (P-S)$_c$ sequence for φ. Let*

$$m = \min_{u \in K \setminus \{0\}} \|u\|_{S_1^2}^2,$$

and let $p \in (2, 2^*)$. If

$$l \geq \frac{c}{m\left(\frac{1}{2} - \frac{1}{\mu}\right)},$$

then $d_{L^p}(u_j, [\mathcal{F}, l]) \to 0$.

Proof. By Lemma 1, the sequence u_j is bounded. Hence Lemma 3 is applicable. Let $w^{(n)}$ and $\alpha_j^{(n)}$, $j, n \in \mathbb{N}$ be as in Lemma 3. Since φ' is weakly continuous and equivariant under shifts τ_α, for $\alpha \in \mathbb{H}_{\mathbb{Z}}^N$,

$$\tau_{\alpha_j^{(n)}} \varphi'(u_j) = \varphi'(\tau_{\alpha_j^{(n)}} u_j) \to \varphi'(w^{(n)}).$$

Since $\tau_{\alpha_j^{(n)}} u_j$ is a (P–S)$_c$ sequence, it follows that $w^{(n)}$ is a critical point of φ.

Note that by Lemma 1 and Lemma 3,

$$\frac{c}{\frac{1}{2} - \frac{1}{\mu}} \geq \limsup_{j\to\infty} \|u_j\|_{S_1^2}^2 \geq \sum_{n=1}^{\infty} \|w^{(n)}\|_{S_1^2}^2.$$

Since $w^{(n)} \in K$, and $\mathcal{F}$ is finite, it is clear that only finitely many $w^{(n)}$ are nonzero. Indeed, there are at most $c/(m(\frac{1}{2} - \frac{1}{\mu}))$ nonzero terms. The assertion follows from Lemma 3. □

Lemma 5. *Let $c > 0$ and assume that $\mathcal{F}$ is finite. Let $\rho > 0$ be given. Then if $\epsilon > 0$ is sufficiently small, then there exists $z(u,t) \in C(S_1^2(\mathbb{H}^N) \times [0,1], S_1^2(\mathbb{H}^N))$ such that*

(i) $z(u, 0) = u$ *for any* $u \in S_1^2(\mathbb{H}^N)$,
(ii) $z(u,t) = u$ *if* $u \notin \varphi^{-1}[c-\epsilon, c+\epsilon]$ *or if* $u \in N_{\rho/2}(K)$,
(iii) $\varphi(z(u,t)) \leq \varphi(u)$ *for any* $u \in S_1^2(\mathbb{H}^N)$ *and any* $t \in [0,1]$,
(iv) $\varphi(z(u,1)) \leq c - \epsilon/2$ *for any* $u \in \varphi^{-1}(-\infty, c+\epsilon/2] \setminus N_{2\rho}(K)$,
(v) $\|z(u,1) - u\|_{S_1^2} < \rho$ *for any* $u \in S_1^2(\mathbb{H}^N)$.

Proof. Since $\mathcal{F}$ is finite, the number $m = \min_{u \in K \setminus \{0\}} \|u\|^2$ is positive. Let l be an integer such that

$$ml \geq \frac{c+1}{\frac{1}{2} - \frac{1}{\mu}}.$$

By Lemma 2, there exists $\delta > 0$ such that

$$0 < \delta < \inf\{\|u - v\|_{L^p};\ u, v \in [\mathcal{F}, l], u \neq v\}.$$

By Lemma 4, there is a number $\nu > 0$ such that $\|\varphi'(u)\| > \nu$ whenever $u \in \varphi^{-1}[c+\epsilon, c+\epsilon]$ and $d(u, [\mathcal{F}, l]) > \rho/2$. Now, let $\epsilon \in (0,1)$ be such that

$$\epsilon < \frac{\nu\rho}{4}.$$

Recall that there exists a pseudogradient vector field for φ on $S_1^2(\mathbb{H}^N) \setminus K$, i.e. a locally Lipschitz mapping $W : S_1^2(\mathbb{H}^N) \setminus K \to S_1^2(\mathbb{H}^N)$ such that for each $u \in S_1^2(\mathbb{H}^N) \setminus K$ one has

a) $\|W(u)\| \leq 2\|\varphi'(u)\|$,
b) $\langle \varphi'(u), W(u) \rangle \geq \|\varphi'(u)\|^2$.

Let ψ_1 and $\psi_2 : S_1^2(\mathbb{H}^N) \to [0,1]$ be two locally Lipschitz continuous functions such that

$$\psi_1(u) = \begin{cases} 1 & \text{when } |\varphi(u) - c| < \epsilon/2, \\ 0 & \text{when } |\varphi(u) - c| > \epsilon. \end{cases}$$

and

$$\psi_2(u) = \begin{cases} 1 & \text{when } d(u, K) > \rho \\ 0 & \text{when } d(u, K) < \rho/2. \end{cases}$$

In particular, $\psi_2(u) = 1$ for $d(u, [\mathcal{F}, l]) > \rho$.

Let

$$V(u) = 4\epsilon\psi_1(u)\psi_2(u)\frac{W(u)}{\|W(u)\|^2}$$

and observe that $V : S_1^2(\mathbb{H}^N) \to S_1^2(\mathbb{H}^N)$ is a locally Lipschitz continuous vector field. Consider the initial value problem

$$\frac{dz(u,t)}{dt} = -V(z(u,t)),$$
$$z(u,0) = u. \tag{3.1}$$

By the theory of ordinary differential equations, (3.1) has a unique solution defined in a maximal right interval $[0, T)$. We claim that $T = \infty$. If $u \notin \varphi^{-1}[c-\epsilon, c+\epsilon] \setminus N_{\rho/2}(K)$, then this is certainly clear, because then $\psi_1(u)\psi_2(u) = 0$ and $z(u,t) = u$ for all $t \geq 0$. So we need only consider u such that $z(u,t) \in \varphi^{-1}[c-\epsilon, c+\epsilon] \setminus N_{\rho/2}(K)$ for all $t \in [0, T)$. We argue by contradiction, and suppose that $T < \infty$. Then there exists a sequence $t_j \to T$ such that $\|V(z(u,t_j))\| \to \infty$ as $t_j \to T$, since otherwise

$$\|z(u,t_n) - z(u,t_m)\|_{S_1^2} = \left\| \int_{t_m}^{t_n} V(z(u,t))\,dt \right\| \leq C|t_n - t_m|,$$

so that $z(u,t_n)$ is a Cauchy sequence in $S_1^2(\mathbb{H}^N)$. Then $z(u,t) \to z$ as $t \to T$, and the solution $z(u,t)$ can be extended beyond T. Hence $\|V(z(u,t_j))\| \to \infty$, and by the definition of V, $\|W(z(u,t_j))\| \to 0$ as $j \to \infty$. By property b) of the pseudogradient vector field,

$$\|\varphi'(z(u,t_j))\| \leq \|W(z(u,t_j))\|,$$

and so

$$\varphi(z(u,t_j)) \in [c-\epsilon, c+\epsilon],$$
$$\|\varphi'(z(u,t_j))\| \to 0.$$

Let $p \in (2, 2^*)$. Then, by Lemma 4, $d(z(u,t_j), [\mathcal{F}, l]) \to 0$ in $L^p(\mathbb{H}^N)$ on a subsequence, and by Lemma 2, $z(u,t)$ enters infinitely many L^p-balls $B_{L^p}(z_j, \delta/3)$, where $z_j \in [\mathcal{F}, l]$, as $j \to \infty$.

Let s_1 and s_2 be such that $z(u,t)$ leaves the ball $B_{L^p}(z_1, \delta/3)$ at $t = s_1$ and enters the ball $B_{L^p}(z_2, \delta/3)$ at $t = s_2$ and that $d_{L^p}(z(u,t), [\mathcal{F}, l]) \geq \delta/3$ for $t \in (s_1, s_2)$. By Lemma 4, there is a number $\tilde{\nu} > 0$ such that $\|\varphi'(u)\| > \tilde{\nu}$ whenever $u \in \varphi^{-1}[c-\epsilon, c+\epsilon]$ and $d_{L^p}(u, [\mathcal{F}, l]) \geq \delta/3$. Then

$$\begin{aligned}\frac{\delta}{3} &\leq \|z(u,s_2) - z(u,s_1)\|_{L^p} \leq S\|z(u,s_2) - z(u,s_1)\|_{S_1^2} \\ &\leq S\int_{s_1}^{s_2} \|V(z(u,t))\|_{S_1^2}\,dt \leq 4\epsilon S \int_{s_1}^{s_2} \frac{1}{\|W(z(u,t))\|_{S_1^2}}\,dt \\ &\leq 4\epsilon S \int_{s_1}^{s_2} \frac{1}{\|\varphi'(z(u,t))\|_{S_1^2}}\,dt \leq \frac{4\epsilon S}{\tilde{\nu}}|s_2 - s_1|. \end{aligned} \tag{3.2}$$

Now, since s_1 and s_2 can be chosen arbitrarily close to T, we get a contradiction. This shows that $T = \infty$.

It is clear that (i) and (ii) are true. To prove (iii), we use properties a) and b) of the pseudogradient vector field and compute

$$\begin{aligned}\frac{d\varphi(z(u,t))}{dt} &= \langle \varphi'(z(u,t)), \frac{dz}{dt}(u,t)\rangle \\ &= -\langle \varphi'(z(u,t)), V(z(u,t))\rangle \\ &= -4\epsilon \frac{\psi_1(z(u,t))\psi_2(z(u,t))}{\|W(z(u,t))\|^2}\langle \varphi'(z(u,t)), W(z(u,t))\rangle \\ &\leq -4\epsilon\psi_1(z(u,t))\psi_2(z(u,t))\frac{\|\varphi'(z(u,t))\|^2}{\|W(z(u,t))\|^2} \\ &\leq -\epsilon\psi_1(z(u,t))\psi_2(z(u,t)) \leq 0. \end{aligned} \tag{3.3}$$

This proves (iii).

To prove (v), a similar calculation as in (3.2) shows that

$$\|z(u,1) - u\| \leq \frac{4\epsilon}{\nu}.$$

By the choice of ϵ, (v) follows.

Note that by (iii), we have also proved (iv) for $u \in \varphi^{-1}(-\infty, c-\epsilon/2]\setminus N_{2\rho}(K)$. It remains to show (iv) for $u \in \varphi^{-1}(c-\epsilon/2, c+\epsilon/2]\setminus N_{2\rho}(K)$. By (iii) and (v), the only way for $z(u,t)$ to leave $\varphi^{-1}(c-\epsilon/2, c+\epsilon/2]\setminus N_\rho(K)$ is if $\varphi(z(u,t)) = c-\epsilon/2$ for some $t \in [0,1]$. But then, $\varphi(z(u,s)) \leq c - \epsilon/2$ for all $s \in [t,1]$, so that (iv) holds in this case. The only remaining case is when $z(u,t) \in \varphi^{-1}(c-\epsilon/2, c+\epsilon/2]\setminus N_\rho(K)$ for all $t \in [0,1]$. But then $\psi(z(u,t)) = 1$ for all $t \in [0,1]$, and so by (3.3),

$$\epsilon > \varphi(u) - \varphi(z(u,1)) = -\int_0^1 \frac{d\varphi(z(u,t))}{dt} \geq \epsilon,$$

and we get a contradiction. □

The following deformation lemma will also be needed in later sections (see [18]).

Lemma 6. *Let X be a Banach space, and let $\varphi \in C^1(X,\mathbb{R})$, $S \subset X$, $c \in \mathbb{R}$, ϵ, $\rho > 0$ be such that $\|\varphi'(u)\| \geq 8\epsilon/\rho$ whenever $u \in \varphi^{-1}([c-2\epsilon, c+2\epsilon]) \cap N_{2\rho}(S)$. Then there exists $z \in C(X\times[0,1], X)$ such that*

(i) *$z(u,t) = u$ if $t = 0$ or if $u \notin \varphi^{-1}([c-2\epsilon, c+2\epsilon]) \cap N_{2\rho}(S)$,*
(ii) *$z(\varphi^{-1}(-\infty, c+\epsilon] \cap S, 1) \subset \varphi^{-1}(-\infty, c-\epsilon]$,*
(iii) *$z(\cdot,t)$ is a homeomorphism of X for all $t \in [0,1]$,*
(iv) *$\|z(u,t) - u\| \leq \rho$ for all $u \in X$ and $t \in [0,1]$,*
(v) *$\varphi(z(u,\cdot))$ is non-increasing for all $u \in X$.*

4. An infinite sequence of minimax values

Lemma 7. *Suppose that* (f_1)–(f_4) *are satisfied. Then* 0 *is an isolated critical point of* φ *and*

$$\varphi(u) = \frac{1}{2}\|u\|^2_{S^2_1} + o(\|u\|^2_{S^2_1})$$

as $u \to 0$.

The proof is standard, and so it is omitted. Let $w \in S^2_1(\mathbb{H}^N) \setminus \{0\}$, then by (f_2), $\varphi(\beta w) \to -\infty$ as $\beta \to \infty$. By use of Lemma 7, this means that the class

$$\Gamma_1 = \{g \in C([0,1], S^2_1(\mathbb{H}^N));\ g(0) = 0,\ \varphi(g(1)) < 0\}$$

is nonempty. Hence, the problem has a mountain pass geometry, and we define our first minimax value to be the mountain pass value

$$c_1 = \inf_{g\in\Gamma_1} \max_{\theta\in[0,1]} \varphi(g(\theta)).$$

Note that $c_1 > 0$. Suppose that $\mathcal{F}$ is a finite set. Then the number

$$\alpha_1 = \sup\{\beta \in \mathbb{R};\ K \cap \varphi^{-1}[c_1 - \beta, c_1 + \beta] = K \cap \varphi^{-1}(c_1)\}$$

is positive. The following proposition shows that c_1 is a critical value of φ.

Proposition 1. *Let* φ *be given by* (1.2)*, where* $f \in C^2(\mathbb{H}^N \times \mathbb{R}, \mathbb{R})$ *satisfies* (f_1), (f_2), (f_3) *and* (f_4)*. Suppose that* $\mathcal{F}$ *is finite. Then there exists a finite set* $A \subset K \cap \varphi^{-1}(c_1)$ *such that for any* $\overline{\epsilon}_1 < \alpha_1/2$, $p \in \mathbb{N}$ *and* r_1 *sufficiently small, there exists* $\epsilon_1 \in (0, \overline{\epsilon}_1)$ *and* $g_1 \in \Gamma_1$ *such that*

(i) $\max_{\theta\in[0,1]} \varphi(g_1(\theta)) \le c_1 + \epsilon_1/p$,
(ii) *if* $\varphi(g_1(\theta)) > c_1 - \epsilon_1$, *then* $g_1(\theta) \in N_{r_1}(A)$.

Proof. Let $\tilde{g}_1 \in \Gamma_1$ be such that

$$\max_{\theta\in[0,1]} \varphi(\tilde{g}_1(\theta)) \le c_1 + \epsilon_1/p.$$

We invoke Lemma 5 with $\rho = r_1/3$ and $\epsilon = 2\epsilon_1$. Let $z \in C(S^2_1(\mathbb{H}^N) \times [0,1], S^2_1(\mathbb{H}^N))$ be as in Lemma 5, and put $g_1(\theta) = z(\tilde{g}_1(\theta), 1)$. By (iii) of Lemma 5,

$$\max_{\theta\in[0,1]} \varphi(g_1(\theta)) \le c_1 + \epsilon_1/p.$$

Moreover, by (iv) of Lemma 5 and the definition of α_1, if $\varphi(g_1(\theta)) \ge c_1 - \epsilon_1$, then $\tilde{g}_1(\theta) \in N_{2\rho}(K \cap \varphi^{-1}(c_1))$. Since by (v) of Lemma 5, $\|z(u,1) - u\| < r_1/3$, $\varphi(g_1(\theta)) \ge c_1 - \epsilon_1$ also implies $g_1(\theta) \in N_{r_1}(K \cap \varphi^{-1}(c_1))$.

If (ii) is not true, then there are sequences $v_m \in K \cap \varphi^{-1}(c_1)$ and $\theta_m \in [0,1]$ such that

$$\|g_1(\theta_m) - v_m\| \le r_1$$

and $\theta_m \to \theta \in [0, 1]$ on a subsequence. For these values of m, n, we have

$$\|v_m - v_n\| \leq \|v_m - g_1(\theta_m)\| + \|g_1(\theta_m) - g_1(\theta_n)\| + \|g_1(\theta_n) - v_n\|$$
$$\leq 2r_1 + \|g_1(\theta_m) - g_1(\theta_n)\| \to 2r_1$$

as $m, n \to \infty$. Thus, if we take

$$r_1 < \frac{1}{3S} \inf\{\|v - w\|_{L^p};\ v, w \in K,\ v \neq w\},$$

where S is the constant in the Folland–Stein embedding theorem, we get a contradiction against Lemma 2. □

We follow [6], and construct a sequence of minimax values as follows: For $k \geq 2$, let

$$\Gamma_k = \{G = g_1 + \cdots + g_k;\ g_i \text{ satisfies } (G_1)\text{–}(G_3),\ 1 \leq i \leq k\},$$

where

(G_1) $g_i \in C([0, 1]^k, S_1^2(\mathbb{H}^N))$, $1 \leq i \leq k$,
(G_2) $g_i(\theta_1, \ldots, \theta_{i-1}, 0, \theta_{i+1}, \ldots, \theta_k) = 0$ and $\varphi(g_i(\theta_1, \ldots, \theta_{i-1}, 1, \theta_{i+1}, \ldots, \theta_k)) < 0$ hold for $i = 1 \ldots k$ and for all $\theta \in [0, 1]^k$,
(G_3) For every $i = 1, \ldots, k$, there is an open set $\mathcal{O}_i \subset \mathbb{H}^N$, such that $\overline{\mathcal{O}_i} \cap \overline{\mathcal{O}_j} = \emptyset$ if $i \neq j$ and $\operatorname{supp} g_i(\theta) \subset \mathcal{O}_i$ for all $\theta \in [0, 1]^k$,

and let

$$c_k = \inf_{G \in \Gamma_k} \max_{\theta \in [0,1]^k} \varphi(G(\theta)).$$

To see that Γ_k is nonempty, note that it is a superset of

$$\{G(\theta) = g_1(\theta_1) + \cdots + g_k(\theta_k);\ g_i \in \Gamma_1, 1 \leq i \leq k,\ \text{and } (G_3) \text{ holds}\},$$

and that the latter is nonempty since there exists $g_1 \in \Gamma_1$ with compact support.

Lemma 8. *Let $k \geq 2$ and let g_i satisfy (G_1) and (G_2) for $i = 1 \ldots k$. Then there exists a $\bar{\theta} \in [0, 1]^k$ such that $\varphi(g_i(\bar{\theta})) \geq c_1$, $i = 1 \ldots k$.*

The proof is the same as that of Proposition 3.4. in [5], and so it is omitted.

Lemma 9. $c_k = kc_1$.

Proof. By (G_3), for every $\theta \in [0, 1]^k$,

$$\varphi(G(\theta)) = \sum_{i=1}^{k} \varphi(g_i(\theta)).$$

Thus by Lemma 8,

$$c_k \geq kc_1.$$

For the reverse inequality, note that the assertion follows if we can find $\hat{g} \in \Gamma_1$ and $R > 0$ such that $\operatorname{supp} \hat{g}(\tau) \subset B_{\mathbb{H}}(0, R)$ for all $\tau \in [0, 1]$, and

$$\max_{\tau \in [0,1]} \varphi(\hat{g}(\tau)) \leq c_1 + \epsilon/k$$

for all $\tau \in [0, 1]$. Then we may choose $\alpha_i \in \mathbb{H}^N_{\mathbb{Z}}$ such that

$$G(\theta) = \sum_{i=1}^{k} \tau_{\alpha^{(i)}} \hat{g}(\theta_i) \in \Gamma_k,$$

and

$$\max_{\theta \in [0,1]^k} \varphi(G(\theta)) \leq kc_1 + \epsilon.$$

To find such $\hat{g}$, choose $g \in \Gamma_1$ such that

$$\max_{\tau \in [0,1]} \varphi(g(\tau)) \leq c_1 + \epsilon/2k.$$

Let $R > 0$, an let $\chi_R \in C^\infty(\mathbb{R}^+, \mathbb{R})$ be such that $\chi_R(\rho) = 1$ if $\rho \leq R$, $|\chi_R'(\rho)| \leq 1$ and $\chi_R(\rho) = 0$ if $\rho > R + 2$. For R large, we let

$$\hat{g}(\tau)(\eta) = \chi_R(d(\eta, 0))g(\tau)(\eta).$$

To se that $\hat{g}$ satisfies the requirements, note that $\hat{g} \in C([0, 1], S^2_1(\mathbb{H}^N))$ and $\hat{g}(0) = 0$. Note that by the definition of Γ_1, $\varphi(g(1)) < 0$. Let

$$\gamma = \min(\epsilon/2k, -\varphi(g(1))).$$

Then if $\hat{g}$ satisfies

$$|\varphi(g(\tau)) - \varphi(\hat{g}(\tau))| < \gamma \tag{4.1}$$

for all $\tau \in [0, 1]$, then $\varphi(\hat{g}(1)) < 0$, and so $\hat{g} \in \Gamma_1$ and

$$\max_{\tau \in [0,1]} \varphi(\hat{g}(\tau)) \leq c_1 + \epsilon/k.$$

Finally, to verify (4.1), suppose that

$$\max_{\tau \in [0,1]} \|g(\tau)\|_{S^2_1(\mathbb{H}^N \setminus B_{\mathbb{H}}(0,R))} < \gamma_1,$$

where $\gamma_1 = \gamma_1(R)$ is to be specified later. Note that

$$\nabla_{\mathbb{H}}(\chi_R g) = \chi_R' \nabla_{\mathbb{H}} d(\eta, 0) g + \chi_R \nabla_{\mathbb{H}} g,$$

Since $\nabla_{\mathbb{H}} d(\eta, 0)$ is homogeneous of degree 0 with respect to the Heisenberg dilations, it is bounded, and hence there is a constant $C > 0$ such that

$$|\nabla_{\mathbb{H}}(\chi_R g)| \leq C|g| + |\nabla_{\mathbb{H}} g|.$$

$$\begin{aligned}
|\varphi(\hat{g}(\tau)) - \varphi(g(\tau))| &\leq \left| \int_{d(\eta,0)>R} \left(\frac{1}{2} \left(|\nabla_{\mathbb{H}} g(\tau)|^2 + |g(\tau)|^2 \right) - F(\eta, g(\tau)) \right) d\eta \right| \\
&\quad + \left| \int_{R<d((\eta,0)<R+2} \left(\frac{1}{2} \left(|\nabla_{\mathbb{H}}(\chi_R g(\tau))|^2 + |\chi_R g(\tau)|^2 \right) \right.\right. \\
&\quad \left.\left. - F(\eta, \chi_R g(\tau)) \right) d\eta \right| \\
&\leq \frac{1}{2}\gamma_1^2 + \int_{d(\eta,0)>R} |F(\eta, g(\tau))| \, d\eta + \frac{C^2+1}{2}\gamma_1^2 \\
&\quad + \int_{R<d(\eta,0)<R+2} |F(\eta, \chi_R g(\tau))| \, d\eta. \qquad (4.2)
\end{aligned}$$

By condition (f_1),

$$|F(\eta, g(\tau)(\eta))| \leq A_1(|g(\tau)|^2 + |g(\tau)|^p),$$

and by the Folland–Stein embedding,

$$\|g(\tau)\|_{L^p(\mathbb{H}^N \setminus B_R(0))} \leq A_2 \|g(\tau)\|_{S_1^2(\mathbb{H}^N \setminus B_R(0))}$$

By applying a similar estimate to the last term of (4.2), we obtain

$$\max_{\tau \in [0,1]} |\varphi(g(\tau)) - \varphi(\hat{g}(\tau))| < \left(\frac{C^2}{2} + 1 \right) \gamma_1^2 + 2A_1A_2\gamma_1^p \equiv \psi(\gamma_1)$$

By choosing R sufficiently large, we can ensure that $\psi(\gamma_1) < \gamma$. This completes the proof. □

5. Infinitely many solutions

In this section, we prove Theorem 1.

Suppose that $\mathcal{F}$ is finite, and let k be so large that $\varphi'(u) \neq 0$ for every $y \in \varphi^{-1}[c_k - 1, \infty)$. Let $\beta > 0$ be a number to be specified later. During the argument we will need to increase β, but this will only happen finitely many times. For $i = 1 \ldots k$, we pick $\eta_i \in \mathbb{H}^N$ such that $d(\eta_i, \eta_l) \geq \beta + 2$ for $i \neq l$. Let

$$\mathcal{M} = \left\{ \sum_{j=1}^{k} \tau_{\eta_i} v_i; \; v_i \in A \right\},$$

where A is the finite set defined in Proposition 1.

Proposition 2. *Let (f_1), (f_2), (f_3) and (f_4) be satisfied, and suppose that $\mathcal{F}$ is finite. Then there exists $\delta > 0$ such that either*

(i) *there is a* $\nu > 0$ *such that* $\|\varphi'(w)\| \geq \nu$ *for all* $w \in N_\delta(\mathcal{M})$, *or*
(ii) *there is a* $w \in \overline{N_\delta(\mathcal{M})}$ *such that* $\varphi'(w) = 0$.

Proof. Suppose that (i) does not hold. Then there is a sequence $u_j \in N_\delta(\mathcal{M})$ such that $\varphi'(u_j) \to 0$. Let $v_1, \ldots, v_k \in A$ be such that

$$\left\| u_j - \sum_{i=1}^{k} \tau_{\eta_i} v_i \right\|_{S_1^2} < \delta.$$

By Lemma 3, there are $w_j^{(n)} \in S_1^2(\mathbb{H}^N)$ and $\alpha_j^{(n)} \in \mathbb{H}_{\mathbb{Z}}^N$ such that

$$\left\| u_j - \sum_{n=1}^{\infty} \tau_{\alpha_j^{(n)}} w^{(n)} \right\|_{L^p} \to 0.$$

and

$$d(\alpha_j^{(n)}, \alpha_j^{(m)}) \to \infty$$

for $n \neq m$. As in the proof of Lemma 4, we conclude that there are only finitely many terms (say l) in the sum above, and that $w^{(n)} \in \mathcal{F}$.

Thus,

$$\left\| \sum_{n=1}^{k} \tau_{\eta_n} v_n - \sum_{m=1}^{l} \tau_{\alpha_j^{(m)}} w^{(m)} \right\|_{L^p} < \delta + \epsilon_j,$$

where $\epsilon_j \to 0$ as $j \to \infty$.

By Lemma 2, choosing

$$\delta < \frac{1}{2} \inf\{\|u - v\|_{L^p};\ u, v \in [\mathcal{F}, \max(k, l)], u \neq v\},$$

we must have

$$\sum_{i=1}^{k} \tau_{\eta_i} v_i = \sum_{n=1}^{l} \tau_{\alpha_j^{(n)}} w^{(n)}$$

for j large, and then all terms for which $d(\alpha_j^{(n)}, 0) \to \infty$ must be 0. Thus, only one term remains, say $\tau_{\alpha_j^{(1)}} w_j^{(1)}$, and since $\alpha_j^{(1)}$ is bounded, it has a convergent subsequence. Since $\alpha_j^{(1)} \in \mathbb{H}_{\mathbb{Z}}^N$, $\alpha_j^{(1)}$ is eventually constant. Thus we have shown that

$$u_j = \tau_{\alpha_0}^{(1)} w^{(1)},$$

and since $\tau_{\alpha_0}^{(1)} w^{(1)} \in K \setminus \{0\}$, (ii) follows. □

Proposition 3. *There exists* $\delta = \delta_k > 0$ *such that if* $\beta > 0$ *is sufficiently large, then* $u \in N_\delta(\mathcal{M}) \cap K$ *implies* $\varphi(u) \in [c_k - \epsilon, c_k + \epsilon]$.

Proof. Let $u \in N_\delta(\mathcal{M}) \cap K$. Then there exist $v_i \in A$ such that

$$\left\| u - \sum_{i=1}^{k} \tau_{\eta_i} v_i \right\| \leq \delta.$$

Let

$$w = \sum_{i=1}^{k} \tau_{\eta_i} v_i .$$

Then by the mean value theorem, there is a $\lambda \in [0, 1]$ such that

$$\varphi(u) - \varphi(w) = \varphi'(\lambda u + (1 - \lambda)w)(u - w).$$

Requiring that $\delta < 1$, we have by Lemma 1,

$$\|\lambda u + (1 - \lambda)w\| < \sum_{i=1}^{k} \|v_i\| + 1 \leq k \left(\frac{c}{\frac{1}{2} - \frac{1}{\mu}} \right)^{1/2} + 1.$$

Let

$$B = \left\{ u \in S_1^2(\mathbb{H}^N);\ \|u\| \leq k \left(\frac{c}{\frac{1}{2} - \frac{1}{\mu}} \right)^{1/2} + 1 \right\}.$$

Since φ' is bounded on bounded sets,

$$\|\varphi'(\lambda u + (1 - \lambda)w)\| \leq \max_{v \in B} \|\varphi'(v)\| \equiv M.$$

Now we choose β so large that

$$\left| \varphi(w) - \sum_{i=1}^{k} \varphi(v_i) \right| = |\varphi(w) - c_k| < \frac{\epsilon}{2}.$$

We further require that $M\delta \leq \epsilon/2$. Then

$$|\varphi(u) - c_k| < \epsilon. \qquad \square$$

Proof of Theorem 1. Suppose that φ has only finitely many critical points. Let $\epsilon \in (0, 1)$ be given, and note that $\varphi'(u) \neq 0$ for every $u \in \varphi^{-1}[c_k - \epsilon, \infty)$. Proposition 3 and Proposition 2 together imply that there exist $\delta > 0$ and $\nu > 0$ such that $\|\varphi'(w)\| \geq \nu$ whenever $w \in N_\delta(\mathcal{M})$. Without loss of generality, we assume that $\epsilon < \delta\nu/32$.

By Proposition 1, there exists $g_1 \in \Gamma_1$ such that

$$\max_{t \in [0,1]} \varphi(g_1(t)) \leq c_1 + \frac{\epsilon}{3k}$$

and $g_1(t) \in N_{\frac{\delta}{16k}}(A)$ whenever $\varphi(g_1(t)) > c_1 - 2\epsilon$. By an approximation argument as in the proof of Lemma 9, there exists $g \in \Gamma_1$ and $R > 0$ such that

$$\|g_1(t) - g(t)\| \leq \frac{\delta}{16k},$$
$$|\varphi(g_1(t)) - \varphi(g(t))| < \frac{\epsilon}{6k}$$

and $\operatorname{supp} g(t) \subset B_{R/2}(0)$ for all $t \in [0, 1]$. We redefine β so that $\beta > R$.

Hence,

$$\max_{t \in [0,1]} \varphi(g(t)) \leq c_1 + \frac{\epsilon}{2k}.$$

Moreover, if $\varphi(g(t)) > c_1 - \frac{3\epsilon}{2}$, then

$$\varphi(g_1(t)) > c_1 - 2\epsilon.$$

Thus $g_1(t) \in N_{\frac{\delta}{16k}}(A)$, and so $g(t) \in N_{\frac{\delta}{8k}}(A)$.

For $\theta \in [0, 1]^k$, let

$$G(\theta) = \sum_{i=1}^{k} \tau_{\eta_i} g(\theta_i).$$

Then we have

$$\operatorname{supp} G(\theta) \subset \bigcup_{i=1}^{k} \tau_{\eta_i}(B_{\mathbb{H}}(0, R/2)).$$

It is clear that $G \in \Gamma_k$, and that

$$\varphi(G(\theta)) = \sum_{i=1}^{k} \varphi(g(\theta_i)) \leq kc_1 + \epsilon.$$

Moreover, if $\varphi(G(\theta)) > kc_1 - \epsilon$, we have for $1 \leq i \leq k$,

$$\varphi(g(\theta_i)) + (k-1)\left(c_1 + \frac{\epsilon}{2k}\right) > kc_1 - \epsilon,$$

and so

$$\varphi(g(\theta_i)) > c_1 - \frac{3\epsilon}{2}.$$

Hence $g(\theta_i) \in N_{\frac{\delta}{8k}}(A)$, and so

$$G(\theta) = \sum_{i=1}^{k} \tau_{\eta_i} g(\theta_i) \in N_{\frac{\delta}{8}}(\mathcal{M}).$$

Next, we apply Lemma 6 (with $S = N_{\delta/2}(\mathcal{M})$) to obtain a homeomorphism $w \in C(S_1^2(\mathbb{H}^N) \times [0, 1], S_1^2(\mathbb{H}^N))$ satisfying

(i) $w(u,t) = u$ if $t = 0$ or if $u \notin \varphi^{-1}([c_k - 2\epsilon, c_k + 2\epsilon]) \cap N_\delta(\mathcal{M})$,
(ii) $\varphi(w(u,1)) \le c - \epsilon$ if $\varphi(u) \le c + \epsilon$ and $u \in N_{\delta/2}(\mathcal{M})$,
(iii) $\|w(u,t) - u\| \le \delta/4$ for all $u \in S_1^2(\mathbb{H}^N)$ and $t \in [0,1]$.

Then the function $\overline{G}(\theta) = w(G(\theta), 1)$ satisfies

$$\max_{\theta \in [0,1]^k} \varphi(\overline{G}(\theta)) \le c_k - \epsilon.$$

If $\overline{G} \in \Gamma_k$ we would have a contradiction at this point. Unfortunately, this might not be the case, since condition (G_3) might not be satisfied. However,

$$\|\overline{G}(\theta)\|_{S_1^2(\mathbb{H}^N \setminus \cup_{i=1}^k S_i)} \le \frac{\delta}{4},$$

where $S_i = \tau_{\eta_i}(B_{R/2}(0))$.

By mollification, we can modify $\overline{G}(\theta)$ to $G^*(\theta)$, where $G^*(\theta) \in C^\infty(\mathbb{H}^N, \mathbb{R})$ satisfying

$$\|G^*(\theta) - G(\theta)\|_{S_1^2(\mathbb{H}^N)} \le \frac{\delta}{2}$$

and

$$\max_{\theta \in [0,1]^k} \varphi(G^*(\theta)) \le c_k - \frac{\epsilon}{2}.$$

By multiplying $G^*(\theta)$ with a smooth cutoff function $\chi_{\hat{R}}$ like in the proof of Lemma 9, we obtain a modified function $\hat{G}(\theta)$ with support in a ball $B_{\mathbb{H}}(0, \hat{R}+2)$ such that

$$\max_{\theta \in [0,1]^k} \varphi(\hat{G}(\theta)) \le c_k - \frac{\epsilon}{4}.$$

By choosing $\hat{R}$ large, we can also assure that

$$\|\hat{G}(\theta) - G(\theta)\|_{S_1^2(\mathbb{H}^N)} \le \delta.$$

The function $\hat{G}$ still does not belong to Γ_k. Note however, that (G_1) is satisfied, and that if we choose δ small enough, also (G_2) holds.

To construct a function $H \in \Gamma_k$, such that

$$\max_{\theta \in [0,1]^k} \varphi(H(\theta)) \le c_k - \frac{\epsilon}{8},$$

we will need to modify $\hat{G}$ once more. We start by solving the following minimization problem:

Lemma 10. *Let $k \geq 2$, and for $i = 1, \ldots, k$, let $W_i = B_{\mathbb{H}}(\eta_i, R)$, where $R > 0$. Let $\hat{R} > 0$ be so large that $\overline{W_i} \subset B_{\mathbb{H}}(0, \hat{R} + 2)$. Let*

$$W = B_{\mathbb{H}}(0, \hat{R} + 2) \setminus \cup_{i=1}^{k} \overline{W_i}.$$

Let

$$\hat{E}(\theta) = \{v \in S_1^2(W);\ v - \hat{G}(\theta) \in \overset{\circ}{S}{}_1^2(W),\ \textit{and}\ \|v\|_{S_1^2(W)} < 8\delta\},$$

and

$$\Psi(v) = \int_W \left(\frac{1}{2}(|\nabla_{\mathbb{H}} v(\eta)|^2 + (v(\eta))^2) - F(\eta, v(\eta)) \right) d\eta.$$

Then there is a unique minimum $v(\theta)$ of Ψ on $\hat{E}(\theta)$. Moreover, $v(\theta) \in \Gamma^{2+\gamma}(W)$ for all $\gamma \in (0, 1)$, and v depends continuously on $\theta \in [0, 1]^k$ in $S_1^2(W)$.

Proof. Note that since $\|\hat{G}(\theta)\|_{S_1^2(W)} < \delta$, the class $\hat{E}(\theta)$ is nonempty. By (f_1)–(f_4), there is a constant $c > 0$ such that for all $\eta \in \mathbb{H}^N$ and $u \in \mathbb{R}$,

$$F(\eta, u) \leq \frac{1}{8}|u|^2 + c|u|^{2^*}.$$

By the Folland–Stein embedding theorem, there is a constant $S > 0$ such that for all $u \in S_1^2(W)$

$$\|u\|_{L^{2^*}(W)} \leq S\|u\|_{S_1^2(W)}.$$

Hence,

$$\int_W F(\eta, u(\eta))\, d\eta \leq \frac{1}{8}\|u\|_{S_1^2(W)}^2 + cS^{2^*}\|u\|_{S_1^2(W)}^{2^*}.$$

Define $\hat{r}$ by

$$cS^{2^*}(8\hat{r})^{2^*-2} = \frac{1}{8}.$$

We restrict δ by requiring that $\delta < \hat{r}$. Then for $u \in \hat{E}(\theta)$,

$$\int_W F(\eta, u(\eta))\, d\eta \leq \frac{1}{4}\|u\|_{S_1^2(W)}^2$$

and

$$\Psi(u) \geq \frac{1}{4}\|u\|_{S_1^2(W)}^2.$$

Since $\hat{G}(\theta) \in \hat{E}(\theta)$,

$$\inf_{\hat{E}(\theta)} \Psi \leq \Psi(\hat{G}(\theta)) \leq \frac{1}{2}\|\hat{G}(\theta)\|_{S_1^2(W)}^2 \leq \frac{1}{2}\delta^2.$$

Thus, if $u \in \hat{E}(\theta)$ and $\|u\|_{S_1^2(W)} \geq 4\delta$, then we must have

$$\Psi(u) \geq 4\delta^2 > \Psi(\hat{G}(\theta)).$$

Consequently, the minimizer $v(\theta)$, if it exists, must occur at an interior point of $\hat{E}(\theta)$.

Let $v_j \in \hat{E}(\theta)$ be a minimizing sequence for Ψ. Then $\|v_j\|_{S_1^2(W)} \leq 4\delta$. This implies that v_j has a subsequence on which v_j converges weakly in $S_1^2(W)$ and strongly in $L^s(W)$ for $1 \leq s < 2^*$. Since Ψ is weakly lower semicontinuous and $\hat{E}(\theta)$ is weakly closed, $v(\theta)$ minimizes Ψ on $\hat{E}(\theta)$.

By applying Theorem 4.2 of [9] to the function $v(\theta) - \hat{G}(\theta) \in \overset{\circ}{S}{}_1^2(W)$, noting that $\hat{G}(\theta) \in C^\infty$, we see that $v(\theta)$ is a classical solution of

$$\begin{aligned} -\Delta_{\mathbb{H}} v(\eta) + v(\eta) &= f(\eta, v(\eta)), \quad \eta \in W \\ v &= \hat{G}(\theta), \quad \eta \in \partial W, \end{aligned} \tag{5.1}$$

and $v(\theta) \in \Gamma^{2+\beta}(W)$ for some $\beta \in (0, 1)$.

To see that the minimizer is unique, suppose that w and v are two minimizers of Ψ. Then by (5.1),

$$\begin{aligned} \|v - w\|^2_{S_1^2(W)} &= \int_W (f(\eta, v(\eta)) - f(\eta, w(\eta))(v(\eta) - w(\eta))\, d\eta \\ &= \int_W (v(\eta)-w(\eta))^2 \left(\int_0^1 f_u(\eta, w(\eta) + t(v(\eta) - w(\eta)))\, dt \right) d\eta. \end{aligned}$$

By (f_1)–(f_4), there is a constant c' such that for all $\eta \in \mathbb{H}^N$ and $u \in \mathbb{R}$,

$$|f_u(\eta, u)| \leq \frac{1}{8} + c'|u|^{2^*-2}.$$

Then

$$\begin{aligned} \|v - w\|^2_{S_1^2(W)} &\leq \frac{1}{8}\|v - w\|^2_{S_1^2(W)} + c' \int_W (v(\eta) - w(\eta))^2 (|v(\eta)| + |w(\eta)|)^{2^*-2}\, d\eta \\ &\leq \frac{1}{8}\|v - w\|^2_{S_1^2(W)} + c'S^2\|v - w\|^2_{S_1^2(W)} (\|v\|_{L^{2^*}(W)} + \|w\|_{L^{2^*}(W)})^{2^*-2} \\ &\leq \frac{1}{8}\|v - w\|^2_{S_1^2(W)} + c'S^2\|v - w\|^2_{S_1^2(W)} (S\|v\|_{S_1^2(W)} + S\|w\|^2_{S_1^2(W)})^{2^*-2} \\ &\leq \frac{1}{8}\|v - w\|^2_{S_1^2(W)} + c'S^{2^*}(4\delta)^{2^*-2}\|v - w\|^2_{S_1^2(W)}. \end{aligned}$$

Thus, we may further require that δ satisfies

$$c'S^{2^*}(4\delta)^{2^*-2} < \frac{7}{8}.$$

Then $v = w$, and uniqueness is proved.

Note that the uniqueness of $v(\theta)$ also implies that v depends continuously on θ. The proof is complete. □

Lemma 11. *Let $v(\theta)$ be the minimizer obtained in Lemma 10. For $\rho > 0$, let*

$$\mathcal{D}_\rho = \{z \in W;\ d(z, \partial W) \geq \rho\}.$$

Then there exists a constant C, not depending on θ, R, $\hat{R}$ or β, such that

$$\|v(\theta)\|_{L^\infty(\mathcal{D}_\rho)} \leq C\|v(\theta)\|_{S_1^2(W)}.$$

Proof. Let $\mathcal{O} \subset\subset \hat{\mathcal{O}} \subset\subset \mathbb{H}^N$. By the L^p estimates (see [7]), there is a constant $K > 0$ such that

$$\|v\|_{S_2^q(\mathcal{O})} \leq K\left(\|f(\cdot, v)\|_{L^q(\hat{\mathcal{O}})} + \|v\|_{L^q(\hat{\mathcal{O}})}\right), \tag{5.2}$$

where K depends on k, q, N, diam $\hat{O}$ and $d(\mathcal{O}, \hat{\mathcal{O}})$. Let j be free for the moment, and let $i = 1 \dots j+1$. Let $B_i = B_{\mathbb{H}}(\eta, (i\rho)/(2j))$. We will use the estimate above with $\mathcal{O} = B_i$ and $\hat{\mathcal{O}} = B_m, m > i$.

Let $p_0 = 2^*/(p-1)$. By (f_1) and the Folland–Stein embedding,

$$\|f(\cdot, v)\|_{L^{p_0}(B_{j+1})} \leq \|v\|_{L^{p_0}(B_{j+1})} + a_3\|v\|^{p-1}_{L^{2^*}(B_{j+1})} \leq C\|v\|_{S_1^2(W)},$$

where $a_3 = a_1/(p-1)$ (we have used that $\|v\|_{S_1^2(W)} \leq 8\delta < 1$). Then by (5.2),

$$\|v\|_{S_2^{p_0}(B_i)} \leq C\|v\|_{S_1^2(W)}.$$

If $p_0 > N+1$, then Theorem 21.1 of [7] implies that

$$\|v\|_{L^\infty(B_1)} \leq C\|v\|_{S_1^2(W)},$$

and we are done.

If $p_0 = N+1$, then there exists $\overline{p} < p_0$ such that

$$\frac{1}{(p-1)(N+2)} = \frac{1}{\overline{p}} - \frac{1}{N+1}.$$

Thus by applying the Folland–Stein embedding theorem twice,

$$\|v\|_{L^{(p-1)(N+2)}(B_j)} \leq C\|v\|_{S_2^{\overline{p}}(B_j)} \leq C\|v\|_{S_1^2(W)}.$$

Again, by (f_1),

$$\|f(\cdot, v)\|_{L^{N+2}(B_j)} \leq C\|v\|_{S_1^2(W)}.$$

Then by (5.2),

$$\|v\|_{S_2^{p_0}(B_j)} \leq C\|v\|_{S_1^2(W)}.$$

Since $2 > (2N+2)/(N+2)$, Theorem 21.1 of [7] gives that

$$\|v\|_{L^\infty(B_1)} \leq C\|v\|_{S_1^2(W)}.$$

If $p_0 < N+1$, set

$$\frac{1}{t_0} = \frac{1}{p_0} - \frac{1}{N+1} = \frac{p-1}{2^*} - \frac{1}{N+1}.$$

Since $p < 2^*$, we have $t_0 > 2^*$. Choose j such that

$$j(p-1)\left(\frac{1}{2^*} - \frac{1}{t_0}\right) > \frac{1}{t_0}.$$

The same argument as above implies that

$$\|v\|_{L^{t_0}(B_j)} \le C\|v\|_{S_2^{p_0}(B_j)} \le C\|v\|_{S_1^2(W)}.$$

By (f_1) we then have

$$\|f(\cdot, v)\|_{L^{p_1}(B_j)} \le C\|v\|_{S_1^2(W)},$$

where $p_1 = t_0/(p-1) > p_0$. Then by (5.2),

$$\|v\|_{S_2^{p_1}(B_{j-1})} \le C\|v\|_{S_1^2(W)}$$

Now, if $p_1 > N+1$ we obtain as above

$$\|v\|_{L^\infty(B_1)} \le C\|v\|_{S_1^2(W)},$$

and we are finished. If $p_1 = N+1$, we argue as above to obtain the same conclusion.

If $p_1 < N+1$, we continue this process with

$$\frac{1}{t_i} = \frac{1}{p_i} - \frac{1}{N+1},$$
$$p_{i+1} = \frac{t_i}{p-1}.$$

We claim that in at most j steps we arrive at $p_j \ge N+1$, which implies

$$\|v\|_{L^\infty(B_1)} \le \|v\|_{S_1^2(W)}.$$

Indeed, if $p_j < N+1$, then

$$0 < t_j = \frac{1}{p_j} - \frac{1}{N+1} = \sum_{i=1}^{j}\left(\frac{1}{p_i} - \frac{1}{p_{i-1}}\right) + \frac{1}{p_0} - \frac{1}{N+1}.$$

But since

$$\begin{aligned}\frac{1}{p_{i+1}} - \frac{1}{p_i} &= (p-1)\left(\frac{1}{t_i} - \frac{1}{t_{i-1}}\right) = (p-1)\left(\frac{1}{p_i} - \frac{1}{p_{i-1}}\right)\\ &= (p-1)^i\left(\frac{1}{p_1} - \frac{1}{p_0}\right) = (p-1)^{i+1}\left(\frac{1}{t_0} - \frac{1}{2^*}\right),\end{aligned}$$

and since $p > 2$ and $t_0 > 2^*$ it follows that

$$0 < \sum_{i=1}^{j}(p-1)^i\left(\frac{1}{t_0} - \frac{1}{2^*}\right) + \frac{1}{t_0} < j(p-1)\left(\frac{1}{t_0} - \frac{1}{2^*}\right) + \frac{1}{t_0},$$

contrary to the choice of j. The proof is therefore complete. □

Lemma 12. *Let $v(\theta)$ be the minimizer obtained in Lemma 10. For $i = 1 \dots k$, let $\mathcal{A}_i$ be the annular region defined by*

$$\mathcal{A}_i = \{\eta \in \mathbb{H}^N;\ R + \frac{\beta}{2} - 1 < d(\eta, \eta_i) < R + \frac{\beta}{2} + 1\}.$$

Then there exists a constant $C > 0$ such that for $\eta \in \mathcal{A}_i$,

$$|v(\theta)(\eta)| \le Ce^{-\beta/8}. \tag{5.3}$$

Proof. By (f_1) and (f_4), there is a constant $r > 0$ such that $|u| < r$ implies

$$|f(\eta, u)| \le \frac{1}{2}|u|.$$

Restricting δ so that

$$\delta \le \frac{r}{8S},$$

where S is the constant in the Folland–Stein embedding theorem, Lemma 11 (with $\rho = 1$) implies that

$$\|v\|_{L^\infty(\mathcal{D}_1)} \le r.$$

Let $1 \le i \le k$, and let

$$\hat{S}_i = \{\eta \in \mathbb{H}^N;\ R + 1 \le d(\eta, \eta_i) \le R + \beta + 1\}.$$

Note that $\hat{S}_i \subset \mathcal{D}_1$, and observe that

$$\begin{aligned} -\Delta_{\mathbb{H}} v^2 + v^2 &= -2|\nabla_{\mathbb{H}} v|^2 - 2v\Delta_{\mathbb{H}} v + v^2 \\ &= -2|\nabla_{\mathbb{H}} v| + 2vf(\eta, v) - v^2. \end{aligned}$$

Since $|f(\eta, v)| \le \frac{1}{2}|v|$, we have

$$\begin{aligned} \Delta_{\mathbb{H}} v^2 - v^2 \ge 0, &\quad \text{in } \hat{S}_i \\ v^2 \le r^2, &\quad \text{on } \partial\hat{S}_i. \end{aligned}$$

Suppose that w satisfies

$$\begin{aligned} \Delta_{\mathbb{H}} w - w \le 0, &\quad \text{in } \hat{S}_i \\ w \ge v^2, &\quad \text{on } \partial\hat{S}_i. \end{aligned}$$

Then

$$\begin{aligned} \Delta_{\mathbb{H}}(v^2 - w) - (v^2 - w) \ge 0, &\quad \text{in } \hat{S}_i \\ v^2 - w \le 0, &\quad \text{on } \partial\hat{S}_i. \end{aligned}$$

The weak maximum principle then implies that

$$w \ge v^2 \quad \text{in } \hat{S}_i. \tag{5.4}$$

Next, we construct such a function w. For simplicity, we may assume that $\eta_i = 0$. Let $w_\pm$ be defined by

$$w_\pm(\eta) = e^{\pm\frac{1}{2}d(\eta,0)}.$$

Since $w_\pm$ is radially symmetric, we have by [8], p. 327 that

$$\Delta_{\mathbb{H}} w_\pm = \frac{\sum_1^N (x_j^2 + y_j^2)}{d(\eta,0)^2}\left(\frac{1}{4} \pm \frac{Q-1}{2d(\eta,0)}\right) w_\pm,$$

so that

$$\Delta_{\mathbb{H}} w_\pm - w_\pm \leq \left(-\frac{3}{4} + \frac{Q-1}{2d(\eta,0)}\right) w_\pm \leq 0$$

if $R \geq 2(Q-1)/3$, which can be assumed.

Now, set

$$\begin{aligned} w &= r^2\left(\exp\left\{-\frac{R+\beta+1}{2}\right\} w_+ + \exp\left\{\frac{R+1}{2}\right\} w_-\right) \\ &= r^2\left(\exp\left\{-\frac{R+\beta+1-d(\eta,0)}{2}\right\} + \exp\left\{\frac{R+1-d(\eta,0)}{2}\right\}\right). \quad (5.5) \end{aligned}$$

Hence $w = r^2\left(e^{-\beta/2}+1\right) \geq r^2$ on $\partial\hat{S}_i$. Thus, w has the required properties.

By (5.5) and (5.4),

$$v^2(x) \leq 2r^2 e^{-\beta/4}\cosh\frac{1}{2},$$

and the proof is complete. □

Now we complete the proof of Theorem 1. For each $\theta \in [0,1]^k$, define

$$U(\theta)(\eta) = \begin{cases} \hat{G}(\theta)(\eta) & \text{if } \eta \notin W, \\ v(\theta)(\eta) & \text{if } \eta \in W, \end{cases}$$

where $v(\theta)$ is the minimizer given by lemma 10.

For $1 \leq i \leq k$, let

$$h_i(\theta)(z) = \begin{cases} U(\theta)(z) \text{ if } d(z,\eta_i) < R + \frac{\beta}{2}, \\ 2\left|d(z,\eta_i) - \left(R+\frac{\beta}{2}+\frac{1}{2}\right)\right| U(\theta)(z) \text{ if } R+\frac{\beta}{2} < d(z,\eta_i) < R+\frac{\beta}{2}+\frac{1}{2} \\ 0 \text{ otherwise.} \end{cases}$$

It is clear that h_i satisfies (G_1) and (G_3) if $\beta > 1$. Define

$$H(\theta) = \sum_{i=1}^{k} h_i(\theta).$$

We claim that H also satisfies (G_2), and so $H \in \Gamma_k$. By the definition of h_i, this follows if $\hat{G}$ and v satisfy (G_2). We have already checked this for $\hat{G}$. Since

$$\hat{G}(\theta_1, \ldots, \theta_{i-1}, 0, \theta_{i+1}, \ldots, \theta_k) = 0$$

for all $\theta \in [0, 1]^k$, it follows that

$$v(\theta_1, \ldots, \theta_{i-1}, 0, \theta_{i+1}, \ldots, \theta_k) = 0 \quad \text{on } \partial W.$$

Since 0 solves equation (5.1), and by the proof of Lemma 10, this solution is unique. Thus

$$v(\theta_1, \ldots, \theta_{i-1}, 0, \theta_{i+1}, \ldots, \theta_k) = 0 \quad \text{in } W.$$

Likewise, since $v(\theta)$ is the minimizer of Ψ on $E(\theta)$ and $\hat{G}(\theta) \in E(\theta)$, we have

$$\Psi(v(\theta)) \leq \Psi(\hat{G}(\theta)).$$

Now it follows that (G_2) is satisfied.

It remains to show that $\varphi(H(\theta)) < c_k$, in order to get a contradiction. Note that

$$\begin{aligned}\varphi(H(\theta)) &\leq \varphi(U(\theta)) + (\varphi(H(\theta)) - \varphi(U(\theta))) \\ &\leq c_k - \frac{\epsilon}{4} + |\varphi(H(\theta)) - \varphi(U(\theta))|.\end{aligned}$$

Thus it suffices if we show that

$$|\varphi(H(\theta)) - \varphi(U(\theta))| \leq \frac{\epsilon}{8}. \tag{5.6}$$

Note that $H(\theta)(\eta) = U(\theta)(\eta)$ when $\eta \in \underline{W} \equiv \cup_{i=1}^k \tau_{\eta_i}(B_{\mathbb{H}}(0, R + \beta/2))$ and that $\operatorname{supp} H \cup \operatorname{supp} U \subset W \cup \underline{W}$. Thus, (5.6) follows if we show that

$$\varphi_H \equiv \left| \int_{W \setminus \underline{W}} \left(\frac{1}{2}(|\nabla_{\mathbb{H}} H(\theta)(\eta)|^2 + |H(\theta)(\eta)|^2) - F(\eta, H(\theta)(\eta)) \right) \right| \leq \frac{\epsilon}{16}$$

$$\varphi_U \equiv \left| \int_{W \setminus \underline{W}} \left(\frac{1}{2}(|\nabla_{\mathbb{H}} U(\theta)(\eta)|^2 + |U(\theta)(\eta)|^2) - F(\eta, U(\theta)(\eta)) \right) \right| \leq \frac{\epsilon}{16}.$$

First, we attend to φ_H, and we start by choosing β so large so that

$$Ce^{-\beta/4} \leq 1,$$

where C is the constant in Lemma 12. This guarantees that $|v(\theta)(\eta)| \leq 1$ for $\eta \in \mathcal{A}_i$. Thus, as in the proof of Lemma 12,

$$F(\eta, h_i(\theta)(\eta)) \leq \frac{1}{2}|h_i(\theta)(\eta)|^2.$$

Hence,

$$\varphi_H \leq \left| \sum_{i=1}^{k} \int_{\mathcal{A}_i} \left(\frac{1}{2} (|\nabla_{\mathbb{H}} h_i(\theta)(\eta)|^2 + |h_i(\theta)(\eta)|^2) - F(\eta, h_i(\theta)(\eta)) \right) d\eta \right|$$
$$\leq \frac{1}{2} \sum_{i=1}^{k} \|h_i\|_{S_1^2(\mathcal{A}_i)}.$$

Let

$$\mathcal{B}_i = \{\eta \in \mathbb{H}^N;\ R + \beta/2 < d(\eta, \eta_i) < R + \beta/2 + 1/2\}.$$

For $\eta \in \cup_{i=1}^{k} \mathcal{B}_i$, by (5.2), we have the estimate

$$\|v(\theta)\|_{S_2^s(B_{\mathbb{H}}(\eta,1/4))} \leq K e^{-\beta/8}$$

for $s > 1$. We choose $s > Q$ and the Folland–Stein embedding theorem to conclude that

$$\|v(\theta)\|_{C^1(B_{\mathbb{H}}(\eta,1/4))} \leq K e^{-\beta/8}. \tag{5.7}$$

Since this estimate holds for all $\eta \in \mathcal{B}_i$, we obtain

$$\varphi_H \leq K e^{-\beta/4} \sum_{i=1}^{k} |\mathcal{B}_i| \leq K' \left(R + \frac{\beta}{2} + 1 \right)^{Q} e^{-\beta/4}.$$

Now, we may choose β so large that

$$K' \left(R + \frac{\beta}{2} + 1 \right)^{Q} e^{-\beta/4} \leq \frac{\epsilon}{16}.$$

It then follows that

$$\varphi_H \leq \frac{\epsilon}{16}.$$

Now we turn to the estimate of φ_U. Let $\mathcal{D} = W \setminus \underline{W}$. By (f_1), (f_3) and (f_4), there is a constant $c_1 > 0$ such that

$$F(\eta, u) \leq \frac{1}{4}|u|^2 + c_1 |u|^{2^*}.$$

Hence,

$$\int_{\mathcal{D}} F(\eta, v(\theta)(\eta))\, d\eta \leq \frac{1}{4} \int_{\mathcal{D}} |v(\theta)(\eta)|^2\, d\eta + c_1 \int_{\mathcal{D}} |v(\theta)(\eta)|^{2^*-2}\, d\eta$$
$$\leq \left(\frac{1}{4} + c_2 \|v(\theta)\|_{S_1^2(W)}^{2^*-2} \right) \|v(\theta)\|_{S_1^2(\mathcal{D})}^2.$$

Recalling that $\|v(\theta)\|_{S_1^2(W)} \leq 4\delta$, and requiring that δ satisfies

$$\frac{1}{4} + c_2 (4\delta)^{2^*-2} \leq \frac{1}{2},$$

this implies

$$\int_{\mathcal{D}} F(\eta, v(\theta)(\eta))\, d\eta \leq \frac{1}{2}\|v(\theta)\|^2_{S_1^2(\mathcal{D})}.$$

This leads to the estimate

$$\varphi_U \leq \frac{1}{2}\|v(\theta)\|^2_{S_1^2(\mathcal{D})}.$$

Observe that $\Delta_{\mathbb{H}} u = \operatorname{div}(A\nabla u)$, where A is the matrix

$$A = \begin{pmatrix} I & 0 & 2y^T \\ 0 & I & -2x^T \\ 2y & -2x & 4\sum(x_i^2 + y_i^2) \end{pmatrix}.$$

Thus the Gauss–Green formula holds for $\Delta_{\mathbb{H}}$. This implies that

$$\|v(\theta)\|^2_{S_1^2(\mathcal{D})} = \int_{\mathcal{D}} v(\theta)(\eta) f(\eta, v(\theta)(\eta))\, d\eta - \int_{\partial\mathcal{D}} v(\theta) A\nabla v(\theta) \cdot \overline{n}\, d\omega,$$

where $\overline{n}$ is the outward unit normal to ∂D at η. A similar estimate as above shows that

$$\int_{\mathcal{D}} v(\theta) f(\eta, v(\theta)(\eta)) d\eta \leq \frac{1}{2}\|v(\theta)\|^2_{S_1^2(\mathcal{D})}.$$

Since

$$\partial\mathcal{D} = \partial B_{\mathbb{H}}(0, \hat{R}+2) \cup \bigcup_{i=1}^{k} \partial B_{\mathbb{H}}(\eta_i, R+\beta/2)$$

and since $v(\theta) = 0$ on $\partial B_{\mathbb{H}}(0, \hat{R}+2)$, we may use the estimate (5.7) to show that

$$|v(\theta)| \leq K e^{-\beta/4},$$
$$|\nabla v(\theta)| \leq K e^{-\beta/4}$$

on $\partial D \setminus \partial B_{\mathbb{H}}(0, \hat{R}+2)$, so that

$$\left|\int_{\partial\mathcal{D}} v(\theta) A\nabla v(\theta) \cdot \overline{n}\, d\omega\right| \leq K' \left(R + \frac{\beta}{2}\right)^{Q+2} e^{-\beta/2}.$$

By combining the above estimates, we obtain

$$\|v(\theta)\|^2_{S_1^2(\mathcal{D})} \leq 2K' \left(R + \frac{\beta}{2}\right)^{Q+2} e^{-\beta/2}.$$

Choosing β large enough, this estimate guarantees that $\varphi_U \leq \epsilon/16$. The proof is complete. □

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