## PROBLEMS

### Chapter 2

**Problem 1.** *Prove that* $\lambda_0 = 0$ *is an eigenvalue for the standard Laplacian on any compact finite graph. What is the multiplicity of this eigenvalue?*

**Problem 2.** *Calculate the spectrum of the standard Laplacian of the compact star graph formed by three intervals of length* 1*.*

**Problem 3.** *Calculate the spectrum of the standard Laplacian on the* 8*-shape graph shown in Fig. 2 assuming that*

*(a) the lengths of the loops are equal* $\ell_1 = \ell_2 = \pi$
*(b) the lengths* $\ell_1$ *and* $\ell_2$ *are arbitrary.*

**Problem 4.** *Consider any compact metric graph and the standard Laplacian on it. What happens to the spectrum if one doubles the lengths of all edges?*

**Problem 5** (Kurasov-Stenberg)**.** *Prove that the scattering matrices for the Laplace operators on the graphs* $\Gamma$ *and* $\Gamma'$ *are equal. Calculate the scattering matrix for the graph* $\Gamma$*. Calculate the spectra of the Laplacians on* $\Gamma$ *and* $\Gamma'$*.*

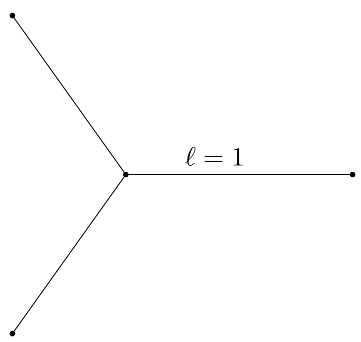


FIGURE 1. Compact star graph

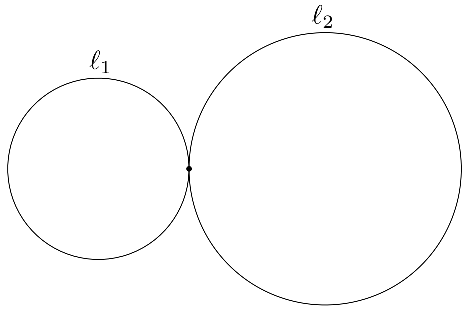


FIGURE 2. 8-shape graph

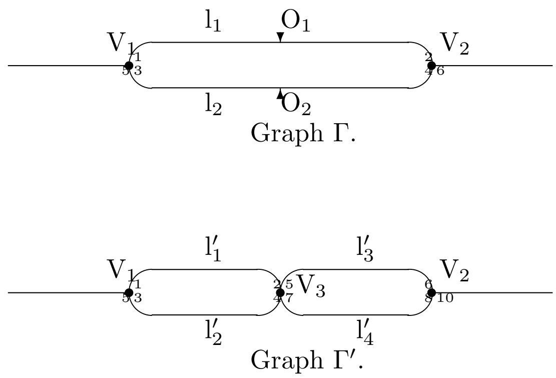


FIGURE 3. Two topologically different graphs having the same scattering matrix. (Arabic numbers indicate positions of the points $x_j$.)

**Problem 6*** (Gutkin-Smilansky). *The spectra of the Laplacians on the graphs presented in Fig. 4 are given by zeroes of the following two functions*

(1)
$$\begin{aligned} Z_I(k) &= \tan(2(a+b)k) \\ &\quad + \frac{2\tan ak + 2\tan bk + \tan(2a+b)k + \tan(a+2b)k}{1 - (2\tan ak + \tan bk)\left(\tan bk + \tan(2a+b)k + \tan(a+2b)k\right)}, \\ Z_{II}(k) &= \tan 2ak \\ &\quad + \frac{2\tan ak + 2\tan bk + \tan(a+2b)k + \tan(2a+3b)k}{1 - (\tan ak + \tan bk + \tan(a+2b)k)\left(\tan ak + \tan bk + \tan(2a+3b)k\right)}. \end{aligned}$$

*Show that the zeroes of the two functions $Z_I(k)$ and $Z_{II}(k)$ coincide.*

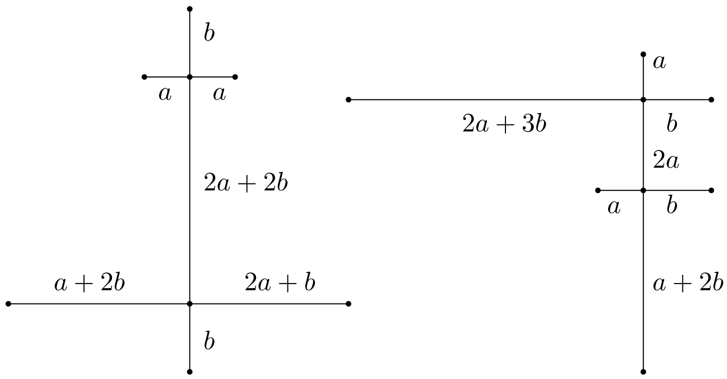


FIGURE 4. Gutkin-Smilansky isospectral graphs

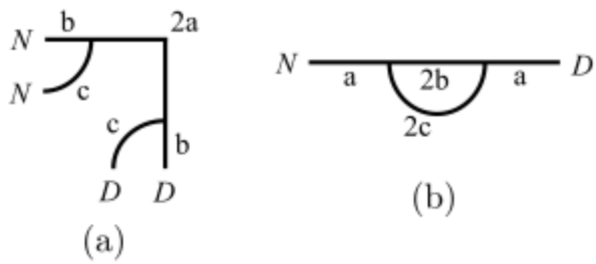


FIGURE 5. Parzanchevski-Band graphs

**Problem 7** (Parzanchevski-Band). *Consider the Laplace operator defined on the graphs depicted at Fig. 7. Dirichlet and Neumann conditions (indicated by letters D and N) are introduced at different boundary vertices. Standard matching conditions at all internal vertices. Prove that the operators are isospespectral assuming that the figure reflects the lengths of the edges correctly.*

**Problem 8.** *Let* $\Gamma_5$ *be a graph formed by* $4$ *edges* $[x_{2j-1}, x_{2j}], j = 1, 2, ..., 4$. *Let* $L$ *be the corresponding Laplace operator defined on the domain of functions satisfying* *the matching conditions:*

$$
\begin{aligned}
&\begin{pmatrix} 1 & 5 & -2 & -1 & 0 & 0 & 0 & -3 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 \\ 2 & 1 & 0 & -1 & 0 & -1 & 0 & -1 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 \\ 2 & 2 & 0 & 0 & 0 & -2 & 0 & -2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 & -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} u(x_1) \\ u(x_2) \\ u(x_3) \\ u(x_4) \\ u(x_5) \\ u(x_6) \\ u(x_7) \\ u(x_8) \end{pmatrix} \\
= &\begin{pmatrix} -1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} u'(x_1) \\ -u'(x_2) \\ u'(x_3) \\ -u'(x_4) \\ u'(x_5) \\ -u'(x_6) \\ u'(x_7) \\ -u'(x_8) \end{pmatrix}.
\end{aligned}
\tag{2}
$$

*The corresponding vertex scattering matrix is energy independent. Reconstruct the metric graph taking into account that the matching conditions respect connectivity of the graph.*

*Write the matching conditions using the other two standard parameterizations:*

- *via the vertex scattering matrix (canonical);*
- *via subspaces and Hermitian matrices (Kuchment).*

Hint *Use the fact that the matching conditions lead to an energy independent vertex scattering matrix and therefore can be written using projectors , hence it is enough to calculate the kernels of the matrices on the different sides of (3). The corresponding kernels should be orthogonal and span* $\mathbb{C}^8$.

**Problem 9.** *Let* $\Gamma_5$ *be a graph formed by* 4 *edges* $[x_{2j-1}, x_{2j}], j = 1, 2, ..., 4$. *Let* $L$ *be the corresponding Laplace operator defined on the domain of functions satisfying the matching conditions:*

$$
\begin{aligned}
&\begin{pmatrix} 1 & 5 & -2 & -1 & 0 & 0 & 0 & -3 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 \\ 2 & 1 & 0 & -1 & 0 & -1 & 0 & -1 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 \\ 2 & 2 & 0 & 0 & 0 & -2 & 0 & -2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 & -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} u(x_1) \\ u(x_2) \\ u(x_3) \\ u(x_4) \\ u(x_5) \\ u(x_6) \\ u(x_7) \\ u(x_8) \end{pmatrix} \\
= &\begin{pmatrix} -1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} u'(x_1) \\ -u'(x_2) \\ u'(x_3) \\ -u'(x_4) \\ u'(x_5) \\ -u'(x_6) \\ u'(x_7) \\ -u'(x_8) \end{pmatrix}.
\end{aligned}
\tag{3}
$$

*The corresponding vertex scattering matrix is energy independent. Reconstruct the metric graph taking into account that the matching conditions respect connectivity of the graph.*

*Write the matching conditions using the other two standard parameterizations:*

- *via the vertex scattering matrix (canonical);*
- *via subspaces and Hermitian matrices (Kuchment).*

Hint *Use the fact that the matching conditions lead to an energy independent vertex scattering matrix and therefore can be written using projectors , hence it is enough to calculate the kernels of the matrices on the different sides of (3). The corresponding kernels should be orthogonal and span* $\mathbb{C}^8$.

## Chapter 3

**Problem 10.** *Consider the star graph formed by three semi-infinite edges* $[x_j, \infty)$, $j = 1, 2, 3$. *Assume that the matching conditions are standard. Write conditions*

- *using matrices* $A$ *and* $B$ *,*
- *uisng the unitary matrix* $S$ *.*

*Are these matching conditions properly connecting and non-resonant?*

**Problem 11.** *Consider the lasso graph depicted in Fig. 1 with magnetic Schrdinger operator defined by the standard matching conditions at the vertex, i.e. the operator* $L^{\rm st}_{0,a}$. *Assume that the electric potential is zero* $q(x) = 0$ *everywhere on* $\Gamma$, *while magnetic potential is zero on the semi-infinite edge. Let us denote by* $\Phi$ *the flux of the magnetic field through the loop:* $\Phi = \int_{x_1}^{x_2} a(x)dx$. *Let* $U_a$ *be the unitary transformation removing the magnetic potential on the loop. Consider the Laplacian*

$$L_\Phi = U_a L_{0,a} U_a^{-1}.$$

*a) How do matching conditions for* $L_\Phi$ *depend on the magnetic flux* $\Phi$*?*
*b) Calculate the scattering matrix for the operator* $L_\Phi$.
*c) Determine the scattering matrix for the original operator* $L_{0,a}$.

## Chapter 4

**Problem 12.** *Prove that* $\det T(\lambda) \equiv 1$ *by using two functions* $f$ *and* $g$ *solving the differential equation (**??**) for* $\lambda$ *and* $\overline{\lambda}$, $\Im\lambda \neq 0$.

*Hint: use that* $T(\overline{\lambda}) = \overline{T(\lambda)}$ *by construction*

**Problem 13.** *Use developed formalism to obtain a characteristic equation for the standard magnetic Schrdinger equation on the* 8*-shape graph presented in Fig. 2. Show, that the spectrum depends on the fluxes of the magnetic field through the cycles* $\Phi_j = \int_{x_{2j-1}}^{x_{2j}} a(x)dx$, *but not on the particular form of the magnetic potential.*

*In the case of the standard Laplacian check that the spectrum you obtain coincides with the result of Problem* **??**.

**Problem 14.**

**Problem 15.** *Using representation*

$$M(\lambda) = \begin{pmatrix} -k\cot k\ell_1 & \dfrac{k}{\sin k\ell_1} \\ \dfrac{k}{\sin k\ell_1} & -k\cot k\ell_1 \end{pmatrix}. \tag{4}$$

*prove that $M(\lambda)$ is a Nevanlinna function, i.e. it satisfies conditions (1)-(3) in the definition of a Nevanlinna function.*

*Hint: Diagonalize $M(\lambda)$ and prove that both eigenvalues have nonnegative imaginary part.*

**Problem 16.** *What is the relation between the edge $M$-function just introduced and the edge scattering matrix $S_{\mathbf{e}}$?*

**Problem 17.** *Consider the ring graph. Calculate the spectrum of the standard Laplacian using all three methods from this chapter. Compare the results with the calculations carried our in Section* **??**. *Do you get all eigenvalues with correct multiplicities?*

**Problem 18.** *Consider the* 8*-shape graph given in Fig. 2. Calculate the spectrum of the standard Laplacian using all three methods from current chapter.*

**Problem 19.** *Let $\Gamma(\ell_1, \ell_2)$ be the graph formed by two edges of lengths $\ell_1$ and $\ell_2$ connected at their end points forming a loop of length $\ell_1 + \ell_2$. Consider the standard Laplacian $L^{\mathrm{st}}(\Gamma(\ell_1, \ell_2)$ and write characteristic equations on its spectrum using all three methods described.*

*a) Show that for $\ell_1$ and $\ell_2$ rationally independent all three equations determine all nonzero eigenvalues.*
*b) Prove that if $\ell_1$ and $\ell_2$ are rationally dependent, then there are exists non-zero eigenvalues that are not described by the last method.*
*c) Which methods determine eigenvalue $\lambda_0 = 0$ with correct multiplicity?*

**Problem 20.** *How does $M$-function depend on the magnetic potential? Derive an explicit formula connecting the $M$-functions corresponding to the same electric but different magnetic potentials. How to see from the third characteristic equation that the spectrum of a magnetic Schrdinger operator depends only on the fluxes of the magnetic field through the cycles.*

**Problem 21.** *Give another one explicit examples of a metric graph, such that the standard Laplacian has eigenvalues not determined by the third characteristic equation.*