Binomial posets with non-isomorphic intervals

Jörgen Backelin

Abstract

The confluent binomial posets with the atomic function $A(n) = \max(1, 2^{n-2})$ are classified. In particular, it is shown that in general there are non-isomorphic intervals of the same length.

1 Introduction

Recall that (following [5]) a binomial poset $(P, \leq)$ is a graded locally finite poset, where the number of maximal chains in an interval only depends on the length of that interval. In this article, we always consider directed infinite downwards bounded binomial posets. In other word, any poset P will have a minimal element $\widehat{0}$; for any $x, y \in P$, there is a $z \in P$ with $x \leq z$ and $y \leq z$; for any (by definition non-empty) interval $[x, y] \subset P$; each interval is a finite set; all inclusion maximal chains in $[x, y]$ have the same length $\ell([x, y])$; and the number of such chains for a given interval $[x, y]$ is only depending on P and $\ell([x, y])$. In fact, we mostly consider *(strongly) confluent* such posets, where in addition there is an infinite chain $x_1 < x_2 < x_3 < \ldots$ in P, such that $P = \bigcup_i [\widehat{0}, x_i]$. (The reader may verify that a downward bounded directed locally finite poset is strongly confluent if and only if it is countable as a set.)

Call $[x, y]$ an n-interval, if $\ell([x, y]) = n$; and let $B(P, n)$ (or $B(n)$, if there is no ambiguity) denote the number of maximal chains in an n-interval of P. The numbers $B(P, 0) = 1$, $B(P, 1) = 1$, $B(P, 2)$, $B(P, 3)$, $\ldots$, behave as a kind of generalised factorial functions, in the following senses: For any n-interval $[x, y]$, the number of elements therein that cover x is an integer $A(P, n)$ (or $A(n)$), only depending on P and n; $B(P, n) = \prod_{i=1}^{n} A(P, n)$; and for any integers j and n, such that $0 < j < n$, the number $B(P, n)/(B(P, j)\, B(P, n-j))$ counts the number of 2-chains $x < z < y$ in $[x, y]$, such that z has local rank j (i.e., such that $\ell([x, z]) = j$), and thus in particular is a positive integer, a 'generalised binomial coefficient'.

In the sequel, we ignore $B(0) = 1$ and $A(0) = 0$, but consider posets X with a given fixed sequence $B(1)$, $B(2)$, $\ldots$, and thus a fixed atomic sequence $A(1) = a_1 = 1$, $A(2) = a_2$, *et cetera*. The *atomic sequence* $\mathcal{A} := (a_i)_{i=1}^{\infty}$ must satisfy the *first binomial poset compatibility* condition

$$(1) \qquad 1 = a_1 \leq a_2 \leq a_3 \leq \ldots, \qquad \text{and} \qquad \frac{\prod_{l=j+1}^{j+i} a_l}{\prod_{l=1}^{i} a_l} \in \mathbb{N} = \{0, 1, 2, \ldots\}$$

for all $i, j \in \mathbb{P} = \{1, 2, 3, \ldots\}$.

Note that by the Ehrenborg-Readdy theorem [3], there are but two isomorphism classes of Eulerian (strongly confluent infinite binomial) posets, with the atomic sequences $(1, 2, 2, 2, 2 \ldots) = (1, 2^{\infty})$ and $(1, 2, 3, 4, 5, \ldots) = (i)_i$, respectively. In contrast, we may classify the strongly confluent infinite binomial posets with $\mathcal{A} = (1^2, 2^{\infty}) = (1, 1, 2, 2, 2, \ldots)$, by means of the infinite binary strings without consecutive 1's. In particular, there is an uncountable number of isomorphism classes of them.

Formally, given a poset X as above, we shall define a string $\phi(X) = (\iota_j)_{j=1}^{\infty} \in [2]^{\mathbb{P}}$, such that $\phi(X) = \phi(X') \iff X \simeq X'$, and that $S := \operatorname{Im} \phi = \{(\iota_j)_j \in [2]^{\mathbb{P}} : \max_{j \in \mathbb{P}}(\iota_j + \iota_{j+1}) \leq 3\}$.

2 The main results

For any (strongly confluent infinite downwards bounded) binomial poset X, let $X_i := \{x \in X : \operatorname{rank}(x) = \ell([\widehat{0}, x]) = i\}$. For $0 \leq i \leq j$, let the *section* $X_{i,j}$ be $\bigcup_{l=i}^{j} X_l$. Name X *of bounded atomic number type*, if $a := \lim \mathcal{A} := \lim_{n\to\infty} a_i < \infty$. Call $\mathcal{A}$ *realised* by X. We have the following lemma:

Lemma. *For a strongly confluent infinite downwards bounded poset X with atomic sequence $\mathcal{A}$, et cetera, as above, we have*

(a) *The following conditions are equivalent:*
  i. $a < \infty$;
  ii. $|X_i| < \infty$ *for some* $i \in \mathbb{P}$;
  iii. All $|X_i| < \infty$;
  iv. $\sup_i |X_i| < \infty$.

(b) *If indeed X is of bounded atomic number type (i.e., the conditions in (a) are fulfilled), then*

$$|X_i| = \frac{a^i}{B(i)}; \quad \sup_i |X_i| = \lim_{i\to\infty} |X_i| = \prod_{i=1}^{\infty} (a a_i^{-1});$$

X is countable; and every element in X is covered by exactly a elements.

Proof. By the confluency condition, there is a chain $(x_i)_i$, such that for any finite sequence of different elements $y_1, \ldots, y_r \in X_1$, there is an $i_r \in \mathbb{P}$, such that $y_1, \ldots, y_r$ cover $\widehat{0}$ in $[\widehat{0}, x_{i_r}]$; and without loss of generality, we may assume that $x_{i_r} \in X_{j_r}$ for some $j_r \geq i_r$. Hence, if there is an infinite sequence $y_1, y_2, \ldots \in X_1$, then on the one hand $a = \lim_i a_{j_i} \geq \lim_i i = \infty$, while on the other $|X_i| \geq |X_1|$ and thus is infinite, for each $i \in \mathbb{P}$.

Conversely, assume that $X_1 = \{y_1, \ldots, y_r\}$. Then $a_i = |X_1|$ for $i \geq j_r$, whence $a = |X_1|$, indeed. Moreover, similar arguments hold for the elements covering any fixed $x \in X$, instead of X_1; whence then indeed x is covered by a elements. In particular, there are exactly a^i saturated chains of length i, starting at $\widehat{0}$. On the other hand, such a chain is a maximal chain in $[\widehat{0}, x]$ for some $x \in X_i$; whence there are $|X_i|\, B(i)$ different such chains. In particular indeed $|X_i| = a^i B(i)^{-1} < \infty$. Now, the remaining claims follow easily. □

Now, fix $a_1 = a_2 = 1$, $a_3 = \ldots = a = 2$. Then $|X_1| = 2$, and $|X_2| = |X_3| = \ldots = 4$. For any $i \in \mathbb{P}$, (the Hasse graph of) the section $X_{i+1,i+2}$ is a bipartite, 2-regular graph on 8 vertices, and thus is isomorphic to either C_8 or to $C_4 \cup C_4$ (distinguishable by the number of components). Put $\iota_i := 3 - \operatorname{comp}(X)_{i+1,i+2}$, and $\phi(X) := (\iota_i)_i$.

Obviously, $(X \simeq X' \implies \phi(X) = \phi(X'))$.

For the moment, fix $i \in \mathbb{P}$.

Recall that there are three possible $2 + 2$ partitions of a given 4-set. Now, any $y \in X_{i+2}$ covers exactly two elements in x_{i+1}; and if y covers x_1 and x_2, then there is a $y' \in X_{i+2}$ that covers $X \setminus \{x_1, x_2\}$, defining a $2 + 2$ 'cover' partition $\{\{x_1, x_2\},\ X \setminus \{x_1, x_2\}\}$ of X_{i+1}. Thus, $X_{i+1,i+2}$ induces $3 - \operatorname{comp}(X)_{i+1,i+2}$ different cover partitions of X_{i+1}; and symmetrically equally many 'co-cover' partitions of X_{i+2}. Likewise, $X_{1,2}$ induces one co-cover partition of X_2. More precisely, up to isomorphisms $X_{0,2} = \{\widehat{0}, z_1, z_2, w_1, w_2, w_3, w_4\}$ with the cover relations $\widehat{0} < z_\nu$, and $z_\nu < w_\mu$ for $\mu \equiv \nu \pmod 2$; inducing the partition $\{\{w_1, w_3\},\ \{w_2, w_4\}\}$. Given this, and given all *sets* X_j (and again letting i float), we get

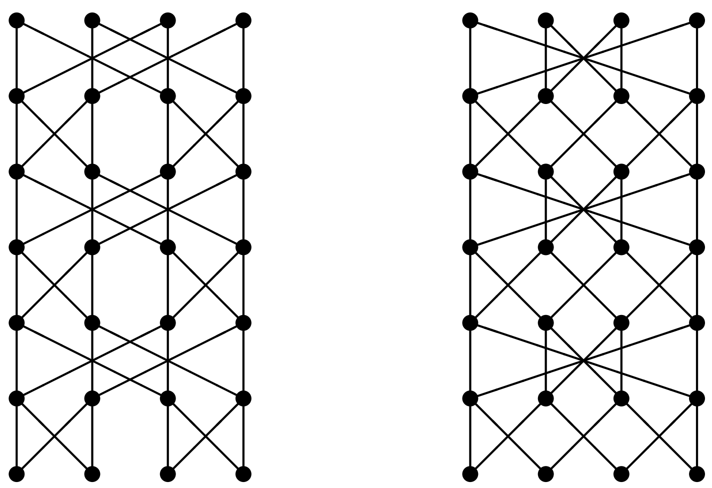
Lemma. *A definition of C_8 or $C_4 \cup C_4$ covering structures for all segments $X_{i+1,i+2}$ ($i \in \mathbb{P}$) defines a type $(1,1,2,2,2,\ldots)$ confluent binomial poset structure on X, iff there is no i and $2+2$ partition of X_{i+1} which is at the same time an induced cover and an induced co-cover partition.*

Proof. If the partition $\{\{b,c\},\ \{d,e\}\}$ of X_{i+2} is both a cover and a co-cover partition, then there are $f \in X_i$ and $g \in X_{i+2}$, such that the interval $[f,g] = \{f, b, c, g\}$, contradicting $A(2) = 1 \neq 2$. Thus the partition avoidance condition is necessary. For the sufficiency, inspect all interval maximal chain numbers, employing the fact that the partition avoidance condition in particular implies that
$((j - i \geq 2$ and $b \in X_i$ and $c \in X_j) \implies b < c)$. □

With X and $(\iota_i)_i$ as above, the lemma immediately yields that $\iota_i + \iota_{i+1} \leq |\{2+2$ partitions of $X_{i+2}\}| \leq 3$; but that this condition is sufficient for the existence of X. Finally, the implication $(\phi(X) = \phi(X') \implies X \simeq X')$ may be proved by constructing isomorphisms $X_{0,i} \simeq X'_{0,i}$ by induction on i. Thus the claim is proved.

Remark. Note, that the two kinds of sections, C_8 and $C_4 \cup C_4$ were employed in order to construct non-isomorphic strongly confluent finite binomial posets with the same 'factorial functions', or with non-isomorphic intervals of the same size, already in [1, Section 8.2]; cf. e.g. Figure 5 (p. 309) therein.

The following figure illustrates the posets with $\phi = (1^\infty)$ or $\phi = (1,2,1,2,1,2,\ldots)$, respectively. In a sense, these are the extremal possibilities. There are as many non-isomorphic posets (with atomic sequence $\mathcal{A} = (1^2, 2^\infty)$) as there are different ϕ, i.e., $\aleph_0$ many. Moreover, there is a ϕ, say $\overline{\phi}$, with the 'versal property' to contain each finite substring of $[2]^\infty$ without consecutive 2's. (You may e.g. enumerate all such substrings as $s_1, s_2, \ldots,$ and put $\overline{\phi} := (s_1, 1, s_2, 1, s_3, \ldots)$.) The corresponding poset contains copies of all possible intervals in posets with the given $\mathcal{A}$; and in particular contains a Fibonacci number of non-isomorphic intervals of any given positive length.



3 Further questions

This section contains some comments and questions that the 'experts' couldn't answer right on the spot, but seem to have thought of. Thus, I strongly suspect that the results I mention, or similar examples, are known (even if not published).

As is well-known, the first binomial compatibility condition (1) is not sufficient to guarantee realisability. I shall provide a brief proof, based on the following two lemmata:

LEMMA. *The infinite sequence $\mathcal{A}$ is realisable iff it is realised by some strongly confluent binomial poset.*

Proof. Suppose $\mathcal{A}$ is realised by P. For any $n \in \mathbb{N}$, let I_n be the set of isomorphism classes of intervals of length n in P. Each such interval has a fixed finite size, whence each I_n is finite. For each $n > 0$ and $Q \in I_n$, choose one element $f(Q) \in I_{n-1}$, such that any interval $[x, y]$ of type Q has a subinterval $[x, z]$ of type $f(Q)$. Make $I := \bigcup_n I_n$ to an infinite directed tree, by letting the children of $Q \in I_n$ be $f^{-1}(Q) \subseteq I_{n+1}$. By König's lemma, there is an infinite path in I. The direct limit of this path (in the natural sense) indeed is a strongly confluent binomial poset realising $\mathcal{A}$. □

LEMMA. *If the finite sequence $(a_1, a_2, \ldots, a_n)$ satisfies (1) for all $i, j \in \mathbb{P}$ such that $i + j \leq n$, then it may be extended to an infinite sequence $\mathcal{A}$ satisfying (1); and $\mathcal{A}$ may be chosen with* $\lim \mathcal{A} < \infty$.

PROOF. Put $a_{n+1} := a_{n+2} := \ldots := \operatorname{lcm}(a_1, \ldots, a_n)$. □

Now, note that $(1, 2, 3, 4, 4)$ fulfils the assumptions of the lemma, but cannot be the atomic numbers sequence for any binomial poset interval $X = [\widehat{0}, \widehat{1}]$ of rank 5. In fact, for any integers $n \geq 3$ and $a_n \geq n - 1$, the sequence $(1, 2, \ldots, n - 1, a_n)$ is realisable if and only if n divides a_n. (If $a_n = kn$, say, with $k \in \mathbb{P}$, then we may construct such an interval by stripping the boolean lattice on n atoms of its top and bottom elements, taking k disjoint copies of the resulting poset, and adding new top and bottom elements. Conversely, if we have a realising interval $X = [\widehat{0}, \widehat{1}]$ of the sequence, then we may define a binary relation R on the a_n-set X_1 by $xRx' \iff \exists\, y \in X_2 : x, x' \in [\widehat{0}, y]$. Now, by means of the fact that any proper subinterval of X is boolean,, we may prove that R is an equivalence relation, and that each equivalence class has the size n. Thus, if k is the number of equivalence classes, indeed we get $a_n = kn$.)

Thus and by the lemmata, there is an infinite sequence $\mathcal{A} = (1, 2, 3, 4, 4, \ldots)$ and fulfilling (1) but non-realisable as the atomic numbers sequence of any binomial poset whatsoever. In fact, the sequence $(1, 2, 3, 4^2, 6^\infty)$ satisfies (1).

Another example is provided by the fact that the atom number sequence $(a_1, a_2, a_3) = (1, m, m+1)$ is (uniquely) realised by the interval $\{\widehat{0}; x_1, \ldots, x_{m+1}; y_1, \ldots, y_{m+1}; \widehat{1}\}$, with $x_i < y_j$ if and only if $i \neq j$; but that for $m \geq 3$ this is not extendable to any binomial interval of larger length. Thus, $(1, m, m + 1, (m(m + 1))^\infty)$ satisfies (1), but is not realisable.

It would be interesting to find further general conditions that (finite or infinite) atomic number sequences must fulfil in order to be realisable; and optimally a necessary and sufficient set of numerical conditions.

It is fairly easy to see that (1) & $\lim \mathcal{A} \leq 3$ imply that $\mathcal{A}$ is realisable. In fact, any prime number $\mathcal{A}$ limit is covered by

LEMMA. *For any $m, n \in \mathbb{P}$, the sequence $(1^m, n^\infty)$ is realised by $X = \bigcup_i X_i$, defined by $X_i := \{i\} \times [n]^{\min(i,m)}$, with a cover relation $(i, s) < (i + 1, t)$ iff the maximal proper left substring of t is a right substring of s.* □

It is also fairly easy to extend this construction to any sequence $\mathcal{A} = (a_i)_{i=1}^{\infty} \in \{1\} \times \mathbb{P}^{\infty}$ such that $a_i | a_{i+1}$ for all $i \in \mathbb{P}$. In particular, this covers any sequence satisfying (1) and with a prime power limit. Thus and by the example *supra*, the smallest limit a for which (1) is not a sufficient condition is $a = 6$. I suspect that there are similar counterexamples for any limit a which has different prime factors.

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Matematiska Institutionen
Stockholms Universitet
SE-106 91 Stockholm
SWEDEN
joeb@math.su.se