Novikov's Inversion Formula for the Attenuated Radon Transform—A New Approach

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ABSTRACT. We study the inversion of weighted Radon transforms in two dimensions, $R_\rho f(L) = \int_L f(\cdot) \rho(L, \cdot) ds$, where the weight function $\rho(L, x)$, L a line and $x \in L$, has a special form. It was an important breakthrough when R.G. Novikov recently gave an explicit formula for the inverse of R_ρ when ρ has the form (1.2); in this case R_ρ is called the attenuated Radon transform. Here we prove similar results for a somewhat larger class of ρ using completely different and quite elementary methods.

1. Introduction

In so-called Emission Computed Tomography one is led to the inversion of the generalized Radon transform in two dimensions:

$$R_\rho f(\theta, p) = \int_{x \cdot \theta = p} f(x) \rho(x, \theta) ds, \quad (\theta, p) \in T \times \mathbf{R}, \quad (1.1)$$

where f is a continuous function with compact support, ds is arc length measure on the line $x \cdot \theta = p$, T denotes the set of unit vectors in $\mathbf{R}^2$, and $\rho(x, \theta)$ is a weight function defined by

$$\rho(x, \theta) = \exp \left(\int_0^\infty \mu(x + t\theta^\perp) dt \right). \quad (1.2)$$

Here, $\mu(x)$ is a given function with compact support and the operator $\perp$ is defined by $(\cos \alpha, \sin \alpha)^\perp = (-\sin \alpha, \cos \alpha)$. Physically, $f(x)$ is a radiation intensity to be determined, $-\mu(x)$ is a known attenuation coefficient, and $R_\rho f(\theta, p)$ is the measured radiation flux coming out of the body along the (oriented) line $x \cdot \theta = p$. The transform R_ρ with ρ given by (1.2) is called the attenuated Radon transform. If $\mu = 0$, then R_ρ reduces to the classical Radon transform. If μ is constant on the support of f , an inversion formula for R_ρ has been known for a long time. For about 20 years it has been an open problem whether R_ρ is always injective if $\rho(x, \theta)$ has the form (1.2). On the other hand, it is known that there exist smooth and positive $\rho(x, \theta)$ —not of the kind (1.2)—such that the transform $C_0^\infty \ni f \mapsto R_\rho f$ is not injective, [2].

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However, during the last few years considerable progress has been made on this problem. Using the theory of so-called A -analytic functions developed by Alexander Bukhgeim, injectivity of R_ρ for ρ of the type (1.2) was proved by Arbuzov, Bukhgeim, and Kazantsev in 1998 [1]. Although no inversion formula is given in [1], the method of Arbuzov, Bukhgeim, and Kazantsev easily gives an inversion formula, as explained in David Finch's survey article [3]. In spring 2000, inspired by arguments from classical scattering theory, Roman G. Novikov gave an explicit inversion formula for R_ρ for arbitrary Hölder continuous μ [5]. Slightly later, Frank Natterer supplied a simpler proof of Novikov's formula [4]. In [5] Novikov also proved injectivity for the attenuated Radon transform with the angle variable θ restricted to an arbitrarily small open set. Here we will prove Novikov's formula for a somewhat larger class of weight functions ρ than those given by (1.2). Our class contains all ρ of the form

$$\rho(x, \theta) = \exp \left(\int_0^\infty \langle \theta, \Psi(x + t\theta^\perp) \rangle dt \right), \quad (1.2')$$

where $\Psi(x)$ is a vector field with compact support and $\langle \cdot, \cdot \rangle$ denotes scalar product, as well as products of the expressions (1.2) and (1.2'); see (3.8). We shall also allow ρ to be complex valued (Theorem 2.5). Our method easily gives injectivity for the same class of R_ρ when θ is restricted to an open set (Theorem 2.8).

For more information on the recent developments on the attenuated Radon transform see [3].

Let H be the Hilbert transform in one real variable,

$$Hq(t) = \frac{1}{\pi} \int_{\mathbf{R}} \frac{1}{t-s} q(s) ds,$$

where the integral is understood in the principal value sense. Let R_ρ^* be the weighted adjoint Radon transform

$$(R_\rho^* \phi)(x) = \int_T \phi(\theta, x \cdot \theta) \rho(x, \theta) d\theta.$$

Here $d\theta$ is arc length measure on the circle T . Denote by Θ the operator of multiplication by $\theta_1 + i\theta_2 \in T$. To motivate the considerations below observe that

$$\left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) (R^* \Theta \phi)(x) = (R^* \partial_p \phi)(x),$$

hence the inversion formula, $f = (4\pi)^{-1} R^* \partial_p H R f$, for the classical Radon transform R can be written

$$f(x) = \frac{1}{4\pi} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) R^* \Theta H R f(x),$$

where R^* of course denotes the adjoint R_ρ^* with weight identically equal to 1.

Here is an outline of this article. Let Ω be an open subset of $\mathbf{R}^2$. First we prove an inversion formula (Corollary 2.3) for *complex-valued*, Hölder continuous weight functions $\rho(x, \theta)$ satisfying the condition

$$\begin{aligned} &\text{for every fixed } x \in \Omega \text{ the function } \theta \mapsto \rho(x, \theta) \text{ is the boundary value} \\ &\text{of a non-vanishing analytic function in } |\zeta| < 1, \end{aligned} \quad (1.3)$$

and one more condition (2.6), which will be automatically satisfied by the weight functions that we shall consider. For real-valued f the formula will be

$$f(x) = \frac{1}{4\pi} \Re \left(\left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \left(R_{1/\rho}^* \ominus H R_\rho f \right) (x) \right),$$

or a small modification of that formula, cf. (2.7). Then we will use the trivial observation that a factor $\tau(x, \theta)$ of $\rho(x, \theta)$ which is constant on all oriented lines $x \cdot \theta = p$ can be moved out of the integral defining R_ρ . In other words, if $\rho(x, \theta) = \tau(x, \theta) \rho_0(x, \theta)$, where $\tau(x, \theta) = \tau_0(\theta, x \cdot \theta)$ for some τ_0 depending only on θ and p , then for any f

$$R_\rho f = \tau_0 R_{\rho_0} f.$$

Thus, if $\rho_0 > 0$ is given and there exists a complex-valued function $\tau \neq 0$, which is constant on all lines $x \cdot \theta = p$, such that $\rho = \rho_0 \tau$ satisfies the condition (1.3), then we immediately obtain an inversion formula for R_{ρ_0} from the inversion formula for R_ρ . However, it is far from obvious how to determine if a given function $\rho_0(x, \theta)$ satisfies this condition. We therefore analyze this condition thoroughly in Section 3 and give directly verifiable equivalent conditions. In particular, we show that functions $\rho_0(x, \theta)$ of the forms (1.2) and (1.2') satisfy this condition. See Theorem 3.4 and the paragraph following it.

2. The inversion formula

We first claim that the operator $A = R_{1/\rho}^* \ominus H R_\rho = R_{1/\rho}^* ((\theta_1 + i\theta_2) H R_\rho)$ can be written $Af(x) = \int K(x, y) f(y) dy$, where

$$K(x, y) = \frac{1}{\pi} \int_T \frac{\theta_1 + i\theta_2}{(x - y) \cdot \theta} \frac{\rho(y, \theta)}{\rho(x, \theta)} d\theta, \quad x \neq y. \quad (2.1)$$

The integrand is singular at two points where θ is perpendicular to $x - y$, and the integral should be interpreted in the principal value sense. If $\theta \mapsto \rho(x, \theta)$ is Hölder continuous for every x , the principal value integral exists. To prove (2.1) we replace H by convolution with $k_\varepsilon(t) = t/(\pi(t^2 + \varepsilon^2))$ for $\varepsilon > 0$, in which case the corresponding formula is a trivial application of Fubini's theorem, and then let $\varepsilon \rightarrow 0$. The kernel $K(x, y)$ is a singular integral operator, and the expression $\int K(x, y) f(y) dy$ should be interpreted in the sense of the theory of distributions.

When studying the operator A it will be convenient to identify with $\mathbb{C}$ that copy of $\mathbb{R}^2$ in which $f(x)$ is defined, so that $x \simeq x_1 + ix_2$, $\bar{x} \simeq x_1 - ix_2$, etc. However, we shall keep the notation $x \cdot \theta = x_1\theta_1 + x_2\theta_2$ for the real-valued scalar product in $\mathbb{C}$, viewed as a two-dimensional real vector space. We denote by Höl_δ the class of functions that are Hölder continuous with exponent $\delta > 0$.

Proposition 2.1. *Assume that $\rho(x, \theta) \neq 0$ is in Höl_δ in $\Omega \times T$ and satisfies (1.3). Let $K_+(x, y)$ and $K_-(x, y)$ be the Schwartz kernels of $R_{1/\rho}^* \ominus H R_\rho$ and $R_{1/\bar{\rho}}^* \ominus H R_{\bar{\rho}}$, respectively. For $x \neq y$ set $\eta = i(x - y)/|x - y| \simeq (x - y)^\perp/|x - y|$, a unit vector perpendicular to $x - y$, and set $m(x) = \text{mean } \rho(x, \cdot) = \rho(x, 0)$. Then for $x \neq y$*

$$K_+(x, y) = \frac{1}{\bar{x} - \bar{y}} Q(x, y, \eta), \quad K_-(x, y) = \frac{1}{x - y} \left(4 \frac{m(y)}{m(x)} - \overline{Q(x, y, \eta)} \right), \quad (2.2)$$

where

$$Q(x, y, \eta) = \frac{\rho(y, \eta)}{\rho(x, \eta)} + \frac{\rho(y, -\eta)}{\rho(x, -\eta)}.$$

The proof consists of an easy calculation of a complex integral using the residue theorem. We formulate it as a lemma.

Lemma 2.2. *Assume that $\varphi \in \text{Höl}_\delta(T)$ and that φ is the boundary value of an analytic function on the open unit disk, continuous on the closed disk. Let z be a non-zero complex number. Then*

$$\frac{1}{\pi} \int_T \frac{\theta_1 + i\theta_2}{z \cdot \theta} \varphi(\theta) d\theta = \frac{1}{z} (\varphi(iz/|z|) + \varphi(-iz/|z|)) ; \quad (2.3)$$

$$\frac{1}{\pi} \int_T \frac{\theta_1 - i\theta_2}{z \cdot \theta} \varphi(\theta) d\theta = \frac{4}{z} \varphi(0) - \frac{1}{z} (\varphi(iz/|z|) + \varphi(-iz/|z|)) . \quad (2.4)$$

(Recall that $d\theta$ denotes arc length measure on T .) The integrals should be understood in the principal value sense.

Proof. Writing $z = z_1 + iz_2$ and replacing $\theta_1 + i\theta_2$ by the complex variable ζ we have $d\theta = d\zeta/i\zeta$ and

$$z \cdot \theta = z_1\theta_1 + z_2\theta_2 = \Re z\bar{\zeta} = \frac{1}{2}(z\bar{\zeta} + \bar{z}\zeta) .$$

The integral in (2.3) can therefore be written

$$I_1 = \frac{2}{\pi} \int_T \frac{\zeta}{\bar{z}\zeta + z\bar{\zeta}} \varphi(\zeta) \frac{d\zeta}{i\zeta} = \frac{2}{i\pi} \int_T \frac{\zeta}{\zeta^2 \bar{z} + z} \varphi(\zeta) d\zeta .$$

The function $h(\zeta) = \zeta/(\zeta^2 \bar{z} + z)$ has simple poles at $\zeta = \pm iz/|z|$ with residues $1/2\bar{z}$. Let Γ_ε^0 be the unit circle oriented counterclockwise with two small segments where $|\zeta \pm iz/|z|| < \varepsilon$ omitted. Complete Γ_ε^0 to a simple closed curve Γ_ε by adjoining two circular arcs with radius ε in $|\zeta| < 1$, γ_ε^+ and γ_ε^- , respectively. By Cauchy's theorem, $\int_{\Gamma_\varepsilon} h(\zeta) \varphi(\zeta) d\zeta = 0$. Since φ is Hölder continuous on T , it is also Hölder continuous on the closed unit disk. Using this fact and the fact that the residue of $h(\zeta)$ at $\pm iz/|z|$ is $1/2\bar{z}$, it is easily seen that

$$\lim_{\varepsilon \rightarrow 0} \int_{\gamma_\varepsilon^+} h(\zeta) \varphi(\zeta) d\zeta = -\frac{1}{2} \cdot 2\pi i \cdot \frac{1}{2\bar{z}} \varphi(iz/|z|) = -\frac{\pi i}{2\bar{z}} \varphi(iz/|z|) .$$

Similarly,

$$\lim_{\varepsilon \rightarrow 0} \int_{\gamma_\varepsilon^-} h(\zeta) \varphi(\zeta) d\zeta = -\frac{\pi i}{2\bar{z}} \varphi(-iz/|z|) .$$

It follows that

$$\begin{aligned} I_1 &= \frac{2}{i\pi} \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon^0} h(\zeta) \varphi(\zeta) d\zeta = \frac{2}{i\pi} \lim_{\varepsilon \rightarrow 0} \left(\int_{\Gamma_\varepsilon} - \int_{\gamma_\varepsilon^+} - \int_{\gamma_\varepsilon^-} \right) \\ &= \frac{1}{z} (\varphi(iz/|z|) + \varphi(-iz/|z|)) . \end{aligned} \quad (2.5)$$

Similarly, the integral in (2.4) can be written

$$I_2 = \frac{2}{i\pi} \int \frac{\bar{\zeta}}{\zeta^2 \bar{z} + z} \varphi(\zeta) d\zeta = \frac{2}{i\pi} \int \frac{\varphi(\zeta) d\zeta}{\zeta(\zeta^2 \bar{z} + z)} .$$

Using the identity

$$\frac{1}{\zeta(\zeta^2 \bar{z} + z)} = \frac{1}{z\zeta} - \frac{\bar{z}}{z} \cdot \frac{\zeta}{\zeta^2 \bar{z} + z}$$

and formula (2.5) we find that

$$I_2 = \frac{2}{i\pi} \frac{2\pi i}{z} \varphi(0) - \frac{\bar{z}}{z} \cdot I_1 = \frac{4}{z} \varphi(0) - \frac{1}{z} \left(\varphi\left(\frac{iz}{|z|}\right) + \varphi\left(-\frac{iz}{|z|}\right) \right),$$

which completes the proof of the lemma. $\square$

Proof of Proposition 2.1. We have seen that $K_+(x, y)$ is given by (2.1). With $x - y = z$ and $\rho(y, \zeta)/\rho(x, \zeta) = \varphi(\zeta)$ in Lemma 2.2 we immediately obtain the first formula in (2.2). To prove the second formula we observe that $\overline{K_-(x, y)}$ can be computed using the second part of Lemma 2.2:

$$\overline{K_-(x, y)} = \frac{1}{\pi} \int_T \frac{\theta_1 - i\theta_2}{(x - y) \cdot \theta} \frac{\rho(y, \theta)}{\rho(x, \theta)} d\theta = \frac{4}{x - y} \frac{\rho(y, 0)}{\rho(x, 0)} - \frac{1}{x - y} Q(x, y, \eta).$$

Taking complex conjugates proves the claim. $\square$

Corollary 2.3. Assume that $\rho(x, \theta)$ is in $\text{Höl}_\delta(\Omega \times T)$, satisfies (1.3) and that,

$$\arg \rho(x, \theta) \text{ is constant on all the oriented lines } x \cdot \theta = p \text{ for } x \in \Omega. \quad (2.6)$$

Set $m(x) = \overline{\rho(x, 0)} = \text{mean } \overline{\rho(x, \cdot)}$, and denote by Θ the operator of multiplication by $\theta_1 + i\theta_2$. Then for smooth complex-valued f supported in Ω

$$\left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \left(m(x) \left(R_{1/\rho}^* \Theta H R_\rho + R_{1/\bar{\rho}}^* \Theta H \bar{R}_\rho \right) f(x) \right) = 8\pi m(x) f(x). \quad (2.7)$$

Proof. Recalling that η is perpendicular to $x - y$ we observe that $\rho(y, \pm\eta)/\rho(x, \pm\eta)$ must be real if ρ satisfies (2.6). Hence by Proposition 2.1,

$$K_+(x, y) + K_-(x, y) = \frac{4}{\bar{x} - \bar{y}} \frac{m(y)}{m(x)}. \quad (2.8)$$

Multiplying by $m(x)$, applying the differential operator $\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2}$, and recalling that $1/(\pi \bar{x})$ is a fundamental solution of the differential operator $\frac{1}{2}(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2})$ completes the proof. $\square$

Theorem 2.4. Assume that $\rho_0(x, \theta) > 0$ is in $\text{Höl}_\delta(\Omega \times T)$ and that $\tau(x, \theta) \neq 0$ is a complex-valued function in $\text{Höl}_\delta(\Omega \times T)$, which is constant on all oriented lines $x \cdot \theta = p$, such that $\rho = \tau \rho_0$ satisfies (1.3). Then, if $m(x) = \text{mean } (\rho_0(x, \cdot) \overline{\tau(x, \cdot)}) = \overline{\rho(x, 0)}$ and Θ has the same meaning as in Corollary 2.3, the following formula holds in distribution sense

$$\left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \left(m(x) \left(R_{1/\rho_0}^* \Theta \tau^{-1} H \tau R_{\rho_0} + R_{1/\rho_0}^* \Theta \bar{\tau}^{-1} H \bar{\tau} R_{\rho_0} \right) f(x) \right) = 8\pi m(x) f(x) \quad (2.9)$$

for complex-valued functions $f \in L^1(\mathbf{R}^2)$ with compact support in Ω . If $\rho \in C^\infty(\Omega \times T)$, then the formula holds for all distributions with compact support in Ω . If, in addition, f is Hölder continuous in some neighborhood of a point $x_0 \in \Omega$, then the formula holds pointwise at x_0 .

Proof. Since $\rho_0 > 0$ and τ is constant on lines, $\rho = \tau \rho_0$ must satisfy (2.6), and $R_{1/\rho}^* \Theta H R_\rho = R_{1/\rho_0}^* \Theta \tau^{-1} H \tau R_{\rho_0}$, $R_{1/\bar{\rho}}^* \Theta H \bar{R}_\rho = R_{1/\rho_0}^* \Theta \bar{\tau}^{-1} H \bar{\tau} R_{\rho_0}$. For smooth f the assertion now follows from Corollary 2.3. For compactly supported $f \in L^1$ the expression $m(x) R_{1/\rho}^* \Theta H R_\rho f(x)$ is defined as a distribution by (note that $H^* = -H$)

$$\langle m R_{1/\rho}^* \Theta H R_\rho f, \varphi \rangle = -\langle f, R_\rho^* H \Theta R_{1/\rho}(m\varphi) \rangle$$

for all test functions $\varphi \in C_0^\infty$; here $\langle \cdot, \cdot \rangle$ denotes the action of a distribution on a test function. Since m and ρ are in Höl_δ and H maps Höl_δ into itself, we have $R_\rho^* H \Theta R_{1/\rho}(m\varphi) = \psi \in \text{Höl}_\delta$. Thus, $\langle f, \psi \rangle$ makes sense for $f \in L^1$, and the definition of the expression (2.9) can be extended by continuity from smooth f to compactly supported $f \in L^1$. The second statement is obvious. To prove the last statement recall that the Schwartz kernel of $R_{1/\rho}^* \Theta H R_\rho + R_{1/\bar{\rho}}^* \Theta H R_{\bar{\rho}}$ is given by (2.8), hence the expression between big brackets in (2.9) is equal to

$$\int \frac{4}{\bar{x} - \bar{y}} m(y) f(y) dy ,$$

which is continuously differentiable near x_0 if f is Hölder continuous near x_0 . $\square$

Remark. The regularity assumptions on τ in Theorem 2.4 are not essential for the existence of an inversion formula for R_{ρ_0} . In fact, we shall prove in Section 3 that if the assumptions of Theorem 2.4 are fulfilled with $\rho_0 \in \text{Höl}_\delta(\Omega \times T)$ and $\tau \in L^\infty(\Omega \times T)$, Ω is convex, and $\Omega_0 \subset\subset \Omega$ then there must exist $\tau \in \text{Höl}_\delta(\Omega_0 \times T)$ satisfying the same condition. This remark applies also to Theorem 2.5.

With the same arguments we can also prove an inversion formula for complex-valued weight functions $\rho_0(x, \theta)$, which contains Theorem 2.4 as a special case.

Theorem 2.5. *Let $\rho_0(x, \theta) \neq 0$ be complex-valued, $\rho_0 \in \text{Höl}_\delta(\Omega \times T)$, and assume that there exist functions $\tau_1(x, \theta)$ and $\tau_2(x, \theta)$, both in $\text{Höl}_\delta(\Omega \times T)$, non-vanishing, and constant on oriented lines $x \cdot \theta = p$, such that $\tau_1 \rho_0$ satisfies (1.3) and $\tau_2 \rho_0$ satisfies the analogous condition with “analytic” replaced by “anti-analytic.” Set $m(x) = \text{mean}(\tau_2(x, \cdot) \rho_0(x, \cdot))$. Then*

$$\left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \left(m(x) (R_{1/\rho_0}^* \Theta \tau_1^{-1} H \tau_1 R_{\rho_0} + R_{1/\rho_0}^* \Theta \tau_2^{-1} H \tau_2 R_{\rho_0} f)(x) \right) = 8\pi m(x) f(x)$$

holds in distribution sense for complex-valued functions $f \in L^1(\mathbf{R}^2)$ with compact support in Ω . If, in addition, f is Hölder continuous in some neighborhood of a point $x_0 \in \Omega$, then the formula holds pointwise at x_0 .

Proof. Set $\rho_1 = \tau_1 \rho_0$, $\rho_2 = \tau_2 \rho_0$. Let $K_+(x, y)$ be the kernel of $R_{1/\rho_1}^* \Theta H R_{\rho_1}$ and $K_-(x, y)$ the kernel of $R_{1/\rho_2}^* \Theta H R_{\rho_2}$. Applying Proposition 2.1 to ρ_1 we obtain $K_+(x, y) = (\bar{x} - \bar{y})^{-1} Q(x, y, \eta)$, where

$$Q(x, y, \eta) = \frac{\rho_0(y, \eta) \tau_1(y, \eta)}{\rho_0(x, \eta) \tau_1(x, \eta)} + \frac{\rho_0(y, -\eta) \tau_1(y, -\eta)}{\rho_0(x, -\eta) \tau_1(x, -\eta)} = \frac{\rho_0(y, \eta)}{\rho_0(x, \eta)} + \frac{\rho_0(y, -\eta)}{\rho_0(x, -\eta)} ,$$

since $\tau_1(x, \pm\eta) = \tau_1(y, \pm\eta)$. Similarly, with $\rho = \bar{\rho}_2$ in Proposition 2.1 we get

$$K_-(x, y) = \frac{1}{\bar{x} - \bar{y}} \left(4 \frac{\rho_2(y, 0)}{\rho_2(x, 0)} - Q(x, y, \eta) \right)$$

with the same Q . Hence, remembering that $\rho_2(x, 0) = m(x)$ we obtain

$$K_+(x, y) + K_-(x, y) = \frac{4}{\bar{x} - \bar{y}} \frac{m(y)}{m(x)} ,$$

and the proof is finished as before. $\square$

We will now show that an arbitrary attenuation weight ρ_0 of the form (1.2) satisfies the hypothesis of Theorem 2.4 with $\Omega = \mathbf{R}^2$, if μ is Hölder continuous and has compact support. We shall use the common notation

$$D\mu(x, \theta) = \int_0^\infty \mu(x + t\theta) dt .$$

Then the attenuation weight (1.2) can be written $\rho_0(x, \theta) = \exp(D\mu(x, \theta^\perp))$. Decompose $D\mu$ into even and odd part with respect to θ by writing $D\mu(x, \theta^\perp) = u(x, \theta) + w(x, \theta)$, where

$$u(x, \theta) = \frac{1}{2} \left(D\mu(x, \theta^\perp) + D\mu(x, -\theta^\perp) \right), \quad w(x, \theta) = \frac{1}{2} \left(D\mu(x, \theta^\perp) - D\mu(x, -\theta^\perp) \right) .$$

It is obvious that $2u(x, \theta)$ is equal to the Radon transform of μ evaluated at $(\theta, p) = (\theta, x \cdot \theta)$, hence as a function of (x, θ) it is constant on lines, and if μ is Hölder continuous and has compact support, then the same is true for $R\mu$, hence u must be Hölder continuous. For a given continuous function v on T let $\tilde{v}$ be that function on T with mean zero for which $\theta \mapsto v(\theta) + i\tilde{v}(\theta)$ is the boundary value (in distribution sense) of a holomorphic function on the unit disk. The function $\tilde{v}$ is called the conjugate function of v , and it can be computed as the convolution

$$\tilde{v}(\theta(\alpha)) = \frac{1}{2\pi} \int_0^{2\pi} v(\theta(\alpha - \beta)) \cot \frac{\beta}{2} d\beta , \quad (2.10)$$

where we have written $\theta(\alpha) = (\cos \alpha, \sin \alpha)$. It is known that $v \in \text{Höl}_\delta$ implies $\tilde{v} \in \text{Höl}_\delta$ (Privalov's theorem). We now claim that the conjugate function of the odd part w is equal to a constant multiple of the Hilbert transform of $p \mapsto R\mu(\theta, p)$, evaluated at $(\theta, p) = (\theta, x \cdot \theta)$. Indeed,

$$(HR\mu)(\theta, x \cdot \theta) = \frac{1}{\pi} \int_{\mathbf{R}} \frac{1}{x \cdot \theta - p} \left(\int_{y \cdot \theta = p} \mu(y) ds \right) dp = \frac{1}{\pi} \int_{\mathbf{R}^2} \frac{\mu(y)}{(x - y) \cdot \theta} dy .$$

Polar coordinates $y = x + r\omega$ transforms the last integral to

$$\frac{1}{\pi} \int_T \frac{-1}{\theta \cdot \omega} D\mu(x, \omega) d\omega . \quad (2.11)$$

Using the formula

$$\frac{1}{\cos \alpha} = \frac{1}{2} \left(\cot \frac{\alpha + \frac{\pi}{2}}{2} - \cot \frac{\alpha - \frac{\pi}{2}}{2} \right)$$

together with the expression (2.10) for the conjugate function we now see that the expression (2.11) can be written

$$-\widetilde{D\mu}(x, \theta^\perp) + \widetilde{D\mu}(x, -\theta^\perp) = -2\tilde{w}(x, \theta) ,$$

which proves that

$$2\tilde{w}(x, \theta) = -(HR\mu)(\theta, x \cdot \theta) . \quad (2.12)$$

Hence $\tilde{w}$ is constant on lines. (Note that the operator $\phi \mapsto \tilde{\phi}$ is translation invariant with respect to the group structure on the circle, so that the value of $\tilde{\phi}$ at the point $\theta^\perp$ is the same as the value of the conjugate function of $\theta \mapsto \phi(\theta^\perp)$ at the point θ .) It is clear that u and w are in Höl_δ if μ is, and by Privalov's theorem (with parameters) then also $\tilde{w} \in \text{Höl}_\delta$. Set $q(x, \theta) = D\mu(x, \theta^\perp)$.

Recalling that $q = u + w$ and observing that $w + i\tilde{w} = q - u + i\tilde{w}$ satisfies (1.3) we have proved that the assumptions of Theorem 2.4 are fulfilled with $\rho_0 = e^q$, $\tau = e^{-u+i\tilde{w}}$. Note also that w and $\tilde{w}$ are odd, so mean $\rho(x, \cdot) = e^0 = 1$.

Thus, we have proved the first part of the following proposition. The second statement follows in a similar way from (3.1) and (3.2).

Proposition 2.6. *Assume that $\rho_0(x, \theta) = \exp\left(\int_0^\infty \mu(x + t\theta^\perp) dt\right)$ with real-valued $\mu \in \text{Höl}_\delta(\mathbf{R}^2)$ of compact support. Set $\tau = e^{-u-iv}$, where*

$$u(x, \theta) = \frac{1}{2}R\mu(\theta, x \cdot \theta), \quad v(x, \theta) = \frac{1}{2}(HR\mu)(\theta, x \cdot \theta). \quad (2.13)$$

Then $\tau \in \text{Höl}_\delta(\mathbf{R}^2 \times T)$, $\rho = \rho_0\tau$ satisfies (1.3), and mean $\rho(x, \cdot) = 1$. More generally, if

$$\rho_0(x, \theta) = \exp\left(\int_0^\infty \left(\mu(x + t\theta^\perp) + \langle \theta, \Psi(x + t\theta^\perp) \rangle\right) dt\right),$$

where $\Psi \in \text{Höl}_\delta(\Omega)$ is a real-valued vector field with compact support, then $\rho = \rho_0\tau$ satisfies (1.3) with $\tau = e^{-u-iv}$ and

$$\begin{aligned} u(x, \theta) &= \frac{1}{2}R\mu(\theta, x \cdot \theta) + \frac{1}{2}R(\langle \theta, \Psi \rangle)(\theta, x \cdot \theta), \\ v(x, \theta) &= \frac{1}{2}(HR\mu)(\theta, x \cdot \theta) + \frac{1}{2}(HR(\langle \theta, \Psi \rangle))(\theta, x \cdot \theta). \end{aligned}$$

Combining the first part of Proposition 2.6 and Theorem 2.4 we obtain Novikov's result.

Corollary 2.7. *Assume that $\rho_0(x, \theta) = \exp\left(\int_0^\infty \mu(x + t\theta^\perp) dt\right)$ with $\mu \in \text{Höl}_\delta(\mathbf{R}^2)$ of compact support. Choose $\tau = e^{-u-iv}$ according to (2.13). Then for complex-valued f with compact support*

$$\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)\left(R_{1/\rho_0}^* \tau^{-1} \Theta H \tau R_{\rho_0} + R_{1/\rho_0}^* \bar{\tau}^{-1} \Theta H \bar{\tau} R_{\rho_0}\right)f(x) = 8\pi f(x). \quad (2.14)$$

Finally, we prove a limited-angle theorem for the transforms considered above.

Theorem 2.8. *Assume that ρ_0 satisfies the assumptions of Theorem 2.4 or Theorem 2.5. Assume, moreover, that f is continuous, has compact support in Ω , and that there is an open subset Γ of the circle T such that*

$$R_{\rho_0}f(\theta, p) = 0 \quad \text{for all } (\theta, p) \in \Gamma \times \mathbf{R}. \quad (2.15)$$

Then $f = 0$.

Proof. We shall consider the case when ρ_0 satisfies the assumptions of Theorem 2.4; the other case is similar. Define τ and $\rho = \tau\rho_0$ as in the proof of Theorem 2.4. Since τ is constant on lines it follows from the assumption (2.15) that the same is true with ρ instead of ρ_0 , that is, $R_\rho f(\theta, p) = 0$ for all $\theta \in \Gamma$ and for all $p \in \mathbf{R}$. Take an arbitrary polynomial $\psi(p)$ in one variable and form the function

$$G_\psi(\theta) = \int_{-\infty}^\infty \psi(p) R_\rho f(\theta, p) dp = \int_{\mathbf{R}^2} \psi(x \cdot \theta) f(x) \rho(x, \theta) dx, \quad \theta \in T. \quad (2.16)$$

By the assumptions $G_\psi(\theta) = 0$ on Γ . We shall show that $G_\psi(\theta)$ must vanish on all of T . Writing $\theta_1 = (\zeta + \zeta^{-1})/2$ and $\theta_2 = (\zeta - \zeta^{-1})/2i$ for $|\zeta| = 1$, we can continue the function $\theta \mapsto \psi(x \cdot \theta)$

to an analytic function in $\mathbb{C} \setminus \{0\}$. The integrand in the last expression of (2.16) must therefore for every fixed x be the boundary value of an analytic function in the punctured disk. Thus, $G_\psi(\theta)$ must be the boundary value of an analytic function in the same set. But such a function must vanish on all of the unit circle if it vanishes in an open set. This proves that $G_\psi(\theta) = 0$ for all $\theta \in T$.

Since f has compact support, the function $p \mapsto R_\rho f(\theta, p)$ must be supported in some interval $|p| \leq A$ for all θ , and the set of polynomials is dense in the space of all continuous functions on that interval. It follows that $R_\rho f(\theta, p) = 0$ for all θ and p . We saw earlier that R_ρ is injective, hence $f = 0$, and the theorem is proved. $\square$

Remark. By standard arguments we can easily replace the assumption that f is continuous with $f \in L^1$ (in which case $Rf(\omega, p)$ will only be defined almost everywhere), or even f being a distribution.

3. Analysis of the assumptions on $\rho_0(x, \theta)$

In this section we shall analyze the rather implicit condition on $\rho_0(x, \theta)$ occurring in Theorem 2.4 and the analogous condition in Theorem 2.5. We begin with the case of real-valued ρ_0 . The condition in question reads as follows:

there exists a function $\tau(x, \theta)$, $|\tau| > \varepsilon > 0$, constant on lines, such that
for every $x \in \Omega$ the function $T \ni \theta \mapsto \rho_0(x, \theta)\tau(x, \theta)$ is the boundary
value of some non-vanishing analytic function in $\{\zeta \in \mathbb{C}; |\zeta| < 1\}$. (P)

When no regularity conditions are explicitly mentioned, we shall assume that functions occurring are bounded and measurable. Boundary values of analytic functions should be interpreted in the distribution sense. In this section the domain Ω will be assumed convex.

Our main result (Theorem 3.4) in this section is a characterization of the functions $\rho_0(x, \theta)$ satisfying (P) in terms of a purely local condition, $\log \rho_0$ satisfying the differential equation (3.6). This condition shows that the assumptions of Theorem 2.4 are satisfied not only by weight functions $\rho_0(x, \theta)$ of the kind (1.2), but also by functions which differ in a non-trivial way from those; see Example 3.3 below.

We begin by reformulating the condition (P) in terms of $q = \log \rho_0$.

$q = \log \rho_0$ can be written $q(x, \theta) = u(x, \theta) + w(x, \theta)$,
where u is constant on lines and $\tilde{w}(x, \theta) = v(x, \theta) + h(x)$, (P1)
 v constant on lines and h independent of θ .

Lemma 3.1. *The function $\rho_0(x, \theta) > 0$ satisfies (P) if and only if $q = \log \rho_0$ satisfies (P1).*

Proof. Assume that q satisfies (P1) and set $v = \tilde{w} - h$. Then $\widetilde{q - u} = \tilde{w} = v + h$, which shows that $q - u + i(v + h)$ is the boundary value of an analytic function for every x , and hence the same is true of $q - u + iv$. Thus, (P) is fulfilled with $\tau = e^{-u+iv}$. Conversely, assume (P) holds. Since $\Omega \times \{\zeta \in \mathbb{C}; |\zeta| < 1\}$ is simply connected, we can choose a continuous and single-valued branch of the logarithm, $\log \rho(x, \zeta)$, of the analytic extension of $\rho(x, \theta) = \rho_0(x, \theta)\tau(x, \theta)$. Write $\tau = e^{-u+iv}$. Then $u = -\Re \log \tau$ and $v = \Im \log \tau$ are constant on lines. Since $\theta \mapsto q(x, \theta) - u(x, \theta) + iv(x, \theta)$ is the boundary value of an analytic function, we must have $\widetilde{q - u} = v + h$ with $h(x) = -\text{mean } v(x, \cdot)$. This means that (P1) holds with $w = q - u$. $\square$

Geometrically, a weight function ρ is defined on the manifold of pairs (L, x) consisting of an oriented line L in the plane and a point x on L . This manifold can be fibered in two ways: either with $\mathbf{R}^2$ as base and the circle T as fiber, or with the space of lines as base and the set of points on a fixed line L as fiber. The functions constant on the first kind of fibers are the functions $q(x, \theta) = q(x)$ that are independent of θ , and the functions constant on the second kind of fibers are the functions of the form $u(x, \theta) = \phi(\theta, x \cdot \theta)$ for some $\phi(\theta, p)$ defined on the space of (oriented) lines, that is, the functions that are constant on lines. From a geometric point of view those two classes of functions are therefore quite analogous. Consider the operators of differentiation along the fibers, $\partial_\theta = \partial/\partial_\alpha$ and $\langle \theta^\perp, \partial_x \rangle$, respectively, where $\theta = (\cos \alpha, \sin \alpha)$. It is clear that the nullspace of ∂_θ is the set of functions $h(x)$ independent of θ , and that the nullspace of $\langle \theta^\perp, \partial_x \rangle$ is the set of functions constant on lines. Those operators do not commute. In fact, since $\partial_\theta(\theta^\perp) = -\theta$ we have

$$\partial_\theta \langle \theta^\perp, \partial_x \rangle - \langle \theta^\perp, \partial_x \rangle \partial_\theta = \langle \theta^\perp, \partial_x \rangle \partial_\theta - \langle \theta, \partial_x \rangle - \langle \theta^\perp, \partial_x \rangle \partial_\theta = -\langle \theta, \partial_x \rangle.$$

This identity shows that a function annihilated by both operators $\langle \theta^\perp, \partial_x \rangle$ and ∂_θ must also be annihilated by $\langle \theta, \partial_x \rangle$, hence must be constant. Therefore the ambiguity in a representation $q(x, \theta) = \phi(\theta, x \cdot \theta) + h(x)$ in a connected domain consists precisely of the constant functions.

Example 3.2. Let $\Omega = \mathbf{R}^2$ and set $q_\mu(x, \theta) = D\mu(x, \theta^\perp)$ for some $\mu \in \text{Höl}_\delta(\mathbf{R}^2)$ with compact support. As above we split $q_\mu(x, \theta)$ into θ -even and θ -odd parts,

$$q_{\mu, \text{even}}(x, \theta) = \frac{1}{2}(q_\mu(x, \theta) + q_\mu(x, -\theta)), \quad q_{\mu, \text{odd}}(x, \theta) = \frac{1}{2}(q_\mu(x, \theta) - q_\mu(x, -\theta)).$$

Then $q_{\mu, \text{even}} = \frac{1}{2}R\mu(\theta, x \cdot \theta)$ is constant on lines, and by (2.12) we know that $\widetilde{q_{\mu, \text{odd}}}(x, \theta) = -\frac{1}{2}(HR\mu)(\theta, x \cdot \theta)$, so $\widetilde{q_{\mu, \text{odd}}}$ is constant on lines, and hence q_μ satisfies (P1) with $h = 0$.

Example 3.3. Let $\Psi = (\psi_1, \psi_2) \in \text{Höl}_\delta(\mathbf{R}^2)$ be a vector field with compact support and set $p_\Psi(x, \theta) = \langle \theta, D\Psi(x, \theta^\perp) \rangle = \int_0^\infty \langle \theta, \Psi(x + t\theta^\perp) \rangle dt$. The odd part, $p_{\Psi, \text{odd}}$, of p_Ψ is constant on lines, because

$$p_{\Psi, \text{odd}}(x, \theta) = \frac{1}{2} \int_{-\infty}^{\infty} \langle \theta, \Psi(x + t\theta^\perp) \rangle dt = \frac{1}{2} R(\langle \theta, \Psi \rangle)(\theta, x \cdot \theta). \quad (3.1)$$

We shall show that $\widetilde{p_{\Psi, \text{even}}}$ is the sum of a function constant on lines and a function depending only on x . In fact, setting $F(x, \theta) = \frac{1}{2}(D\Psi(x, \theta^\perp) - D\Psi(x, -\theta^\perp))$ we can write

$$\widetilde{p_{\Psi, \text{even}}} = (\langle \theta, F \rangle)^\sim = \langle \theta, \widetilde{F} \rangle - CF, \quad (3.2)$$

where C is the commutator defined by $CW = \langle \theta, \widetilde{W} \rangle - (\langle \theta, W \rangle)^\sim$ for any vector valued function $W(\theta)$. The function $\langle \theta, \widetilde{F} \rangle$ is constant on lines, because $F = (q_{\psi_1, \text{odd}}, q_{\psi_2, \text{odd}})$ and $\widetilde{q_{\psi_j, \text{odd}}}$ is constant on lines by (2.12). Explicitly,

$$\begin{aligned} \langle \theta, \widetilde{F}(x, \theta) \rangle &= \theta_1 \widetilde{F}_1(x, \theta) + \theta_2 \widetilde{F}_2(x, \theta) \\ &= -\frac{1}{2}\theta_1(HR\psi_1)(\theta, x \cdot \theta) - \frac{1}{2}\theta_2(HR\psi_2)(\theta, x \cdot \theta) = -\frac{1}{2}(HR(\langle \theta, \Psi \rangle))(\theta, x \cdot \theta). \end{aligned}$$

The function CF must be equal to some function $h(x)$ depending only on x , because $\theta \mapsto CW(\theta)$ can at most have the Fourier coefficients of order $-1, 0$, and 1 different from zero (see the first part of the proof of Lemma 3.5 below), and since CF is also an even function of θ , in fact only the Fourier coefficient of order 0 can differ from zero, which means that CF is independent of θ . Thus, $p_\Psi = p_{\Psi, \text{odd}} + p_{\Psi, \text{even}}$, where $p_{\Psi, \text{odd}}$ is constant on lines and

$$\widetilde{p_{\Psi, \text{even}}}(x, \theta) = -\frac{1}{2}(HR(\langle \theta, \Psi \rangle))(\theta, x \cdot \theta) - h(x), \quad (3.3)$$

which shows that p_Ψ satisfies (P1).

We will now show that any function q satisfying (P1) can be written $q = q_\mu + p_\Psi$ plus a function constant on lines.

The function q_μ in Example 3.2 satisfies the differential equation

$$\langle \theta^\perp, \partial_x \rangle q_\mu(x, \theta) = -\mu(x), \quad (3.4)$$

that is, $\langle \theta^\perp, \partial_x \rangle q_\mu(x, \theta)$ is independent of θ . For the function p_Ψ we have similarly

$$\langle \theta^\perp, \partial_x \rangle p_\Psi(x, \theta) = -\langle \theta, \Psi(x) \rangle. \quad (3.5)$$

Hence, if $q(x, \theta) = q_\mu(x, \theta) + p_\Psi(x, \theta) + u(x, \theta) + h(x)$, where u is constant on lines and h depends only on x , then $\langle \theta^\perp, \partial_x \rangle q$ has the form

$$\begin{aligned} \langle \theta^\perp, \partial_x \rangle q(x, \theta) &= -\mu(x) - \langle \theta, \Psi(x) \rangle + \langle \theta^\perp, \partial_x \rangle h(x) \\ &= -\mu(x) + \theta_1 d_1(x) + \theta_2 d_2(x), \end{aligned} \quad (3.6)$$

for some functions $\mu(x)$, $d_1(x)$, and $d_2(x)$.

Conversely, if q satisfies the differential equation (3.6) in $\Omega \times T$ and Ω is convex, then it is easy to see that q must be equal to a sum of functions of the types q_μ and p_Ψ plus a trivial term on any domain $\Omega_1 \times T$, where $\Omega_1 \subset \subset \Omega$. To prove this take $\chi \in C_0^\infty(\Omega)$, equal to 1 on Ω_1 , and choose

$$q_\mu(x, \theta) = D(\chi\mu)(x, \theta^\perp), \quad p_\Psi(x, \theta) = \langle \theta, D(\chi\Psi)(x, \theta^\perp) \rangle, \quad (3.7)$$

with $\Psi(x) = (-d_1(x), -d_2(x))$. Then $\langle \theta^\perp, \partial_x \rangle (q - q_\mu - p_\Psi) = 0$ on $\Omega_1 \times T$, so $q - q_\mu - p_\Psi$ must be constant on lines there.

A special case of p_Ψ is obtained if Ψ is a gradient field, say $\Psi(x) = \nabla\mu(x)$. Then

$$p_\Psi(x, \theta) = \int_0^\infty \langle \theta, \partial_x \rangle \mu(x + t\theta^\perp) dt = \langle \theta, \partial_x \rangle D\mu(x, \theta^\perp) = \langle \theta, \partial_x \rangle q_\mu(x, \theta).$$

However, this case is not so special, as the following argument shows. Let $\Psi \in C^1(\mathbf{R}^2)$ be an arbitrary compactly supported vector field and Ω a bounded set. Then we can find $\mu(x)$ and $v(x)$, compactly supported in $\mathbf{R}^2$, such that $\Psi(x) = \nabla\mu(x) + \nabla v(x)^\perp$ in Ω . Then

$$\begin{aligned} \langle \theta^\perp, \partial_x \rangle (p_\Psi(x, \theta) - p_{\nabla\mu}(x, \theta)) &= -\langle \theta, \Psi(x) - \nabla\mu(x) \rangle \\ &= -\langle \theta, \nabla v(x)^\perp \rangle = \langle \theta^\perp, \partial_x \rangle v(x) \end{aligned}$$

in $\Omega \times T$, so p_Ψ and $p_{\nabla\mu}$ can differ in Ω at most by the sum of a function constant on lines and a function depending only on x .

We have now shown several of the implications in the following theorem.

Theorem 3.4. *Assume Ω is convex. Then each of the following conditions is equivalent to (P1):*

on every $\Omega_1 \times T$, $\Omega_1 \subset \subset \Omega$, $q(x, \theta)$ can be written as a sum of

$$q_\mu(x, \theta) = \int_0^\infty \mu(x + t\theta^\perp) dt \quad \text{and} \quad p_\Psi(x, \theta) = \int_0^\infty \langle \theta, \Psi(x + t\theta^\perp) \rangle dt \quad (\text{A})$$

for some μ and Ψ plus a function $u(x, \theta)$ constant on lines ;

$$\begin{aligned} & q(x, \theta) \text{ satisfies (in the distribution sense) the differential equation (3.6)} \\ & \text{in } \Omega \times T \text{ for some distributions } \mu(x), d_1(x), d_2(x). \end{aligned} \quad (\text{B})$$

If $q \in \text{Höl}_\delta$, then the functions q_μ , p_Ψ , and u in condition (A) are also in Höl_δ .

Taking account of Lemma 3.1 we can now conclude that the assumption of Theorem 2.4 is satisfied by any $\rho_0(x, \theta)$ of the form

$$\rho_0(x, \theta) = e^{q_\mu(x, \theta)} e^{p_\Psi(x, \theta)} \quad (3.8)$$

for Hölder continuous $\mu(x)$ and $\Psi(x)$, and that any $\rho_0(x, \theta)$ satisfying the condition of Theorem 2.4 must be of this form, up to a factor which is constant on lines.

To prove that (P1) $\Rightarrow$ (B) we need to study $\langle \theta^\perp, \partial_x \rangle \tilde{v}(x, \theta)$ for an arbitrary function v which is constant on lines. This will be done in the next lemma. We have chosen to state this lemma for distributions, since regularity properties of functions is no issue at all here. The spaces of distributions in Ω and $\Omega \times T$ are denoted $\mathcal{D}'(\Omega)$ and $\mathcal{D}'(\Omega \times T)$, respectively.

Lemma 3.5. *The class of functions (distributions) $v(x, \theta)$ in $\Omega \times T$ for which $\langle \theta^\perp, \partial_x \rangle v(x, \theta)$, evaluated in distribution sense, is a first degree polynomial in θ_1 and θ_2 with coefficients in $\mathcal{D}'(\Omega)$,*

$$\langle \theta^\perp, \partial_x \rangle v(x, \theta) = a(x) + \theta_1 d_1(x) + \theta_2 d_2(x), \quad (3.9)$$

is closed under conjugation with respect to θ . In particular, if $v(x, \theta) = \phi(\theta, x \cdot \theta)$ in $\Omega \times T$ for some $\phi \in \mathcal{D}'(T \times \mathbf{R})$, then

$$\langle \theta^\perp, \partial_x \rangle \tilde{v}(x, \theta) = a(x) + \langle \theta, \partial_x \rangle b(x), \quad (3.10)$$

where

$$a(x) = -(2\pi)^{-1} R^* \phi'_p(x), \quad b(x) = -(2\pi)^{-1} R^* \phi(x). \quad (3.11)$$

Proof. We first prove the lemma assuming $v \in C^1(\Omega \times T)$. The conjugate of the right-hand side of (3.9) is $\theta_2 d_1(x) - \theta_1 d_2(x)$. We claim that, for every $u(x, \theta)$, the commutator

$$Cu = \langle \theta^\perp, \partial_x \rangle \tilde{u} - (\langle \theta^\perp, \partial_x \rangle u)^\sim$$

is a first degree polynomial in θ_1, θ_2 ; this implies the first statement of the lemma. To prove this claim it is enough to observe that conjugation commutes with differentiation (in distribution sense) and that the commutators $C_j : u \mapsto \theta_j \tilde{u} - \tilde{\theta_j u}$, $j = 1, 2$, annihilate each Fourier component $e^{in\alpha}$ with $|n| \geq 2$.

The second statement is not needed for the proof of Theorem 3.4, but we think it has some independent interest, so we give a proof here. With a small calculation we find that if $u(\theta) = \sum a_n e^{in\alpha}$, then

$$\begin{aligned} C_1 u &= -\theta_2 a_0 + \frac{1}{2i}(a_1 - a_{-1}) = -\frac{\theta_2}{2\pi} \int u(\omega) d\omega - \frac{1}{2\pi} \int \omega_2 u(\omega) d\omega, \\ C_2 u &= \theta_1 a_0 + \frac{1}{2}(a_1 + a_{-1}) = \frac{\theta_1}{2\pi} \int u(\omega) d\omega + \frac{1}{2\pi} \int \omega_1 u(\omega) d\omega. \end{aligned}$$

Applying these formulas to $u = \partial_{x_1} v$ and $u = \partial_{x_2} v$, respectively, recalling that $v(x, \theta) = \phi(\theta, x \cdot \theta)$ we obtain

$$\begin{aligned} Cv &= -C_2 \partial_{x_1} v + C_1 \partial_{x_2} v = -\frac{1}{2\pi} \int \langle \theta, \partial_x \rangle v(x, \omega) d\omega - \frac{1}{2\pi} \int \langle \omega, \partial_x \rangle v(x, \omega) d\omega \\ &= -\frac{1}{2\pi} \langle \theta, \partial_x \rangle \int \phi(\omega, x \cdot \omega) d\omega - \frac{1}{2\pi} \int \langle \omega, \partial_x \rangle \phi(\omega, x \cdot \omega) d\omega \\ &= -\frac{1}{2\pi} \langle \theta, \partial_x \rangle R^* \phi(x) - \frac{1}{2\pi} R^* \phi'_p(x), \end{aligned}$$

which completes the proof for $v \in C^1$. To prove the general case take a sequence $v_n \in C^1(\Omega \times T)$ tending to v in $\mathcal{D}'(\Omega \times T)$ and note that $a_n(x)$, $d_{1,n}(x)$, and $d_{2,n}(x)$ must converge in $\mathcal{D}'(\Omega)$ if $a_n(x) + \theta_1 d_{1,n}(x) + \theta_2 d_{2,n}(x)$ converges. The proof of the second statement is similar. $\square$

We are now ready to complete the proof of Theorem 3.4.

Proof of Theorem 3.4. We have shown that (A) $\Rightarrow$ (P1) and that (A) $\Leftrightarrow$ (B). To prove the implication (P1) $\Rightarrow$ (B), assume q satisfies (P1), i.e., $q = u + w$ where u and $\tilde{w}(x, \theta) - h(x) = v(x, \theta)$ are constant on lines. Then $\tilde{w} = -w + g(x) = \tilde{v}$, where $g(x) = \text{mean } w(x, \cdot)$, so $w = -\tilde{v} + g$, and $q = u - \tilde{v} + g$. Since u is constant on lines we obtain

$$\langle \theta^\perp, \partial_x \rangle q(x, \theta) = 0 - \langle \theta^\perp, \partial_x \rangle \tilde{v}(x, \theta) + \langle \theta^\perp, \partial_x \rangle g(x),$$

and the assertion now follows immediately from Lemma 3.5.

To prove the last statement assume that $q \in \text{Höl}_\delta(\Omega \times T)$ satisfies (3.6). Let $q_{\text{odd}}(x, \theta)$ be the θ -odd part of $q(x, \theta)$. Then $\mu(x) = -\langle \theta^\perp, \partial_x \rangle q_{\text{odd}}(x, \theta)$ in distribution sense, where $\mu(x)$ may be a distribution of positive order. Observe that for smooth $\mu(x)$ the expression (3.7) can be rewritten

$$\begin{aligned} q_\mu(x, \theta) &= \int_0^\infty \chi(x + t\theta^\perp) \mu(x + t\theta^\perp) dt \\ &= - \int_0^\infty \chi(x + t\theta^\perp) \frac{d}{dt} q_{\text{odd}}(x + t\theta^\perp, \theta) dt \\ &= \chi(x) q_{\text{odd}}(x, \theta) + \int_0^\infty q_{\text{odd}}(x + t\theta^\perp, \theta) \frac{d}{dt} \chi(x + t\theta^\perp) dt. \end{aligned}$$

We can therefore take the last expression as definition of q_μ , which shows that $q_\mu \in \text{Höl}_\delta(\Omega \times T)$. Replacing q_{odd} by q_{even} we can treat the term p_Ψ similarly. $\square$

Finally, we make some brief comments on the case of complex-valued weights. Recall first that the conjugation operator $v \mapsto \tilde{v}$ is defined for complex-valued v by complex-linear extension from the case of real-valued v . Thus, the condition (P1) makes sense for complex-valued q .

The analog of Lemma 3.1 for complex-valued weight functions reads as follows.

Lemma 3.6. *The complex-valued function $\rho_0(x, \theta) = e^{q(x, \theta)}$, where $q \in \text{Höl}_\delta(\Omega \times T)$, satisfies the assumption of Theorem 2.5 if and only if q satisfies (P1).*

Proof. Write $q = s + it$. The assumption of Theorem 2.5 means that there exists $\tau_1 = \exp(u_1 + iv_1)$ and $\tau_2 = \exp(u_2 + iv_2)$, constant on lines, such that $s + u_1 + i(t + v_1)$ and $s + u_2 - i(t + v_2)$, as functions of θ , are boundary values of analytic functions in the unit disk for every fixed x . Taking linear combinations we get the equivalent condition that $2s + u_1 + u_2 + i(v_1 - v_2)$ and $2t + v_1 + v_2 + i(u_2 - u_1)$ have the same property. By Lemma 3.1 there exist such u_1, v_1, u_2, v_2 if and only if s and t satisfy (P1), or equivalently, $q = s + it$ satisfies (P1). $\square$

It is now clear that all of the analysis in this section, including Theorem 3.4, is valid also for complex-valued q . It follows that the assumptions of Theorem 2.5 are satisfied by any $\rho_0(x, \theta)$ of the form (3.8) for complex-valued $\mu(x)$ and $\Psi(x)$, and that any $\rho_0(x, \theta)$ satisfying the assumptions of Theorem 2.5 must have this form up to a trivial factor.

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