MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd Matematik
Clas Löfwall

Cyklisk homologi
VT 05

1 Introduction

Cyclic homology was introduced by Connes about 20 years ago in the context of non-commutative geometry. It may be seen as a refinement of the Hochschild homology of an associative algebra. Indeed, there is a long exact sequence which involves cyclic homology and Hochschild homology. In the case the algebra R is of the form $R = k \oplus R_1 \oplus R_2 \oplus \ldots$ where k is a field of characteristic zero, this long exact sequence gives in fact a decomposition of Hochschild homology into two pieces of cyclic homology. Hochschild homology was introduced by Hochschild in the 1940's. He considers especially the second cohomology group which characterizes the extensions of the algebra.

2 Hochschild homology

Literature: Loday, Cyclic homology, Ch. 1.0–1.3, 1.5; Cartan-Eilenberg, Homological algebra, Ch IX, XIV §2; Weibel, Ch. 9.1–9.3 (there are however lots of mistakes in Weibel).

2.1 What is an algebra?

Let k be a commutative ring with unit. Often k will be a field, but in general this is not necessary. An *associative unital k-algebra* R is a ring with unit together with a ring homomorphism $f : k \to R$ such that the image of f is contained in the center of R. Another way to express this is to say that R is a k-module and a ring with unit, such that $\lambda(ab) = (\lambda a)b = a(\lambda b)$ for all $\lambda \in k$ and $a, b \in R$. Usually we omit "associative unital" and just talk about k-algebras. Observe that this is in contrast to what we mean by a k-algebra when Lie-algebras are studied. In that case the term k-algebra is used for a module over k together with a bilinear product (not associative and without unit).

Any ring with unit is an algebra over $\mathbb{Z}$ (in a unique way). In general, a k-algebra may have many different k-algebra structures. E.g., a $\mathbb{Z}[x]$-algebra R is a ring with unit together with a special element in the center of R (corresponding to the image of x under the ring homomorphism $\mathbb{Z}[x] \to R$). A k-algebra may also be an l-algebra for different commutative rings k and l. The complex numbers, e.g., is a $\mathbb{Q}$-algebra but also an $\mathbb{R}$-algebra and we will see that the Hochschild homology is quite different in the two cases. The homology tells us something about the field extension $\mathbb{C}/\mathbb{Q}$ or $\mathbb{C}/\mathbb{R}$ and not about the complex numbers of its own. In one situation, which we will study intensively, this ambiguity is not present, namely for $\mathbb{N}$-*graded* (or, as we will say later, $\mathbb{N}$-*weighted*) *connected* k-algebras. Such an algebra is of the form

$$R = R_0 \oplus R_1 \oplus R_2 \oplus \ldots \quad \text{where} \quad R_0 = k \quad \text{and} \quad R_i R_j \subset R_{i+j}$$

In this case, k may be recovered as the elements of degree zero. A more general situation consists of *augmented* k-algebras. A k-algebra $f : k \to R$ is augmented if there is a ring homomorphism $g : R \to k$ such that $g \circ f = id$. Such an algebra is of the form

$$R = k \oplus I \quad \text{where} \quad I \quad \text{is an ideal in} \quad R$$

For an augmented k-algebra R we have that k is an R-module (any k-module is an R-module via $R \to k$), but in general this is not true. For the field extension $\mathbb{C}/\mathbb{Q}$, e.g., we have that $\mathbb{C}$ is a $\mathbb{Q}$-algebra but $\mathbb{Q}$ is not a $\mathbb{C}$-module.

If R is an augmented k-algebra and k is a field, then k is uniquely determined from R. Indeed, if R is an augmented k-algebra and R is an augmented l-algebra, where k and l are fields, then k and l are isomorphic (see exercise 1).

Any ring I without unit, which is also a k-module such that

$$\lambda(ab) = (\lambda a)b = a(\lambda b) \quad \text{for all} \quad \lambda \in k \quad \text{and} \quad a, b \in I$$

may be extended to an augmented k-algebra R, by defining

$$R = k \oplus I \quad \text{with multiplication} \quad (\lambda + a)(\mu + b) = \lambda\mu + \lambda b + \mu a + ab$$

A *homomorphism* from a k-algebra $f : k \to R$ to a k-algebra $g : k \to S$ is a ring homomorphism $\varphi : R \to S$ such that $\varphi \circ f = g$. The k-algebra $id : k \to k$ is an initial object in the category of k-algebras. It is also a final object in the subcategory of augmented k-algebras.

2.2 Bimodules

In order to define Hochschild homology and cohomology we want to have "coefficients" like in other homology theories such as for topological spaces or groups. It turns out that the appropriate category to use for k-algebras are *bimodules*. A bimodule M over a k-algebra R is a k-module together with two commuting k-linear actions of R on M, or more precisely, there are two k-algebra homomorphisms $f, g : R \to \mathrm{Hom}_k(M, M)$ such that $f(r) \circ g(s) = g(s) \circ f(r)$ for all $r, s \in R$. A *homomorphism* from M to N where M, N are R-bimodules is a k-linear map from M to N which commutes with the actions of R.

Usually the two actions on a bimodule M are written as operations on the left and right side of M. Then the property that the actions commute looks like an associative law. Indeed, a standard example of a bimodule is R itself with left and right multiplication as the two actions. They commute because of the associative law in R.

If R is commutative and M is an R-module, we may consider M as a bimodule by defining the two actions to be the same. Indeed, if $f : R \to \mathrm{Hom}_k(M, M)$ is a k-algebra homomorphism, then

$$f(r) \circ f(s) = f(rs) = f(sr) = f(s) \circ f(r)$$

for all $r, s \in R$. There are examples of bimodules M, where the two actions are different (even if R is commutative). E.g., let A be any commutative ring and $a, b \in A$ two different elements in A, then A is a $\mathbb{Z}[x]$-bimodule via the two ring homomorphisms $\mathbb{Z}[x] \to \mathrm{Hom}_{\mathbb{Z}}(A, A)$ sending x to multiplication by a and b respectively.

An R-bimodule M with the property that the two actions of R on M coincide is called *symmetric*. If the operations on M are written as left and right actions, then M symmetric reads

$$rm = mr \quad \text{for all} \quad r \in R \text{ and } m \in M$$

One may also consider several k-algebras acting on a k-module. The only requirement is that the actions commute. If R and S are k-algebras acting in this way to the left on a k-module M, then M is a left $R \otimes S$-module and conversely (tensor products without index are always over k). A right R-module is the same as a left module over the *opposite* algebra R^{op}, which is equal to R as a k-module and with multiplication in the reversed order. Hence the category of left and right bimodules over a k-algebra R is the same as the category of left $R \otimes R^{op}$-modules. The algebra $R \otimes R^{op}$ is called the *enveloping algebra* of R and denoted R^e. Compare this with the enveloping algebra of a Lie algebra $\mathfrak{g}$, which converts $\mathfrak{g}$-modules to ordinary modules.

The R-bimodule R mentioned above is not free, not even projective in general, in the sense that the corresponding $R \otimes R^{op}$-module is not free (or projective). One may say that Hochschild homology measures the deviation from R being projective. We have that $R \otimes R$ is an R-bimodule by multiplication to the left and right side. This bimodule converted to a left $R \otimes R^{op}$-module is $R \otimes R^{op}$ itself (with left multiplication) and hence it is free. Moreover the multiplication map

$$R \otimes R \to R$$

is a surjective map of bimodules. Thus, this is a beginning of a free resolution of R as an R-bimodule. Generally a free R-bimodule is of the form $R \otimes V \otimes R$, where V is a free k-module.

2.3 Definition of homology and cohomology

We will in this section give a direct definition of Hochschild homology and cohomology which does not use homological algebra. In the next section resolutions will be introduced and we will see how the definitions can be obtained using homological algebra.

Let R be a k-algebra and M an R-bimodule (with left and right action). A complex $C_*(R, M)$ is defined as follows

$$\dots \xrightarrow{b} M \otimes R \otimes R \otimes R \xrightarrow{b} M \otimes R \otimes R \xrightarrow{b} M \otimes R \xrightarrow{b} M$$

where the boundary map b is defined to be an alternating sum of terms that put together two adjacent symbols (and the last term put the last element to the left of the first). A tensor in $M \otimes R^{\otimes n}$ will be written as $(m, a_1, a_2, \dots, a_n)$. With this notation we have

$$b(m, a_1, a_2, a_3) = (ma_1, a_2, a_3) - (m, a_1 a_2, a_3) + (m, a_1, a_2 a_3) - (a_3 m, a_1, a_2)$$

The fact that this defines a complex, i.e., $b \circ b = 0$, follows from the following identities (for b defined on $M \otimes R^{\otimes n}$)

$$b = \sum_{i=0}^{n} (-1)^i d_i \quad \text{and} \quad d_i d_j = d_{j-1} d_i \quad \text{for all} \quad 0 \le i < j \le n$$

Letting $M \otimes R^{\otimes n}$ have homological degree n, we make the following definition of the Hochschild homology of R with coefficients in M.

$$\mathrm{H}_n(R, M) = \mathrm{H}_n(C_*(R, M))$$

Observe that the homology groups are just k-modules in general. In particular $\mathrm{H}_0(R, M) = M/[R, M]$, where $[R, M]$ denotes the k-module generated by $\{rm - mr;\ r \in R,\ m \in M\}$, is not an R-module. In particular $[R, R]$ is not an ideal in R, so $\mathrm{H}_0(R, R) = R/[R, R]$ may not be interpreted as a commutative ring. If $R = k\langle x, y\rangle$, the free associative k-algebra on x, y, then $\mathrm{H}_0(R, R)$ is not equal to $k[x, y]$ (even if this is stated in Loday, Theorem 3.1.6!). In fact, the equivalence relation on R imposed by $[R, R]$ is in this case the same as allowing cyclic permutations of monomials in x, y. E.g., we have $xyxy \neq xxyy$ in $R/[R, R]$. This is the starting point of cyclic homology, which we will come back to later.

However, if R is commutative the homology groups are R-modules, since the boundary map is R-linear in this case (R is acting to the left). If R is commutative $\mathrm{H}_0(R, M)$ may be seen as the symmetrized R-bimodule of M (i.e., a symmetric bimodule together with a map from M to it, which is universal for maps from M to symmetric bimodules). See also exercise 2.

The homology groups are functors in the following sense. Let $\alpha : R \to R'$ be a map of k-algebras, M an R-bimodule, M' an R'-bimodule and $\varphi : M \to M'$ a map of R-bimodules, where M' is considered as an R-bimodule via α. Then there is a natural map of complexes $C_*(R, M) \to C_*(R', M')$ which induces a map $\mathrm{H}_n(R, M) \to \mathrm{H}_n(R', M')$. It is evident that this makes H_n to a functor from the category of pairs (R, M), where R is a k-algebra, M is an R-bimodule and maps are defined as above, to the category of k-modules.

The case when $M = R$ is of particular interest and a special notation is used

$$\mathrm{HH}_n(R) = \mathrm{H}_n(R, R)$$

which is called the Hochschild homology of the algebra R. For each n, we get a functor, HH_n, from the category of k-algebras to the category of k-modules.

In order to define cohomology, a cocomplex, $C^*(R, M)$, is introduced as follows. Let $C^0(R, M) = M$ and for $n \geq 1$,

$$C^n(R, M) = \mathrm{Hom}_k(R^{\otimes n}, M)$$

i.e., $C^n(R, M)$ consists of k-multilinear maps from R to M in n arguments. The coboundary map β is defined analogous to b above. We give the formula for $\beta : C^2(R, M) \to C^3(R, M)$:

$$\beta(f)(a_1, a_2, a_3) = a_1 f(a_2, a_3) - f(a_1 a_2, a_3) + f(a_1, a_2 a_3) - f(a_1, a_2) a_3$$

The proof of $\beta \circ \beta = 0$ is the same as above for b. We get cohomology groups

$$\mathrm{H}^n(R, M) = \mathrm{H}^n(C^*(R, M))$$

which in general are k-modules, but when R is commutative they are also R-modules. We have

$$\mathrm{H}^0(R, M) = \{m \in M;\ rm = mr \text{ for all } r \in R\}$$

which is just a k-module in the general case, but a symmetric sub-bimodule of M if R is commutative. The fact that $\mathrm{H}^0(R, M)$ is a symmetric R-bimodule is true also for the higher cohomology (and homology) groups, see exercise 3.

The cohomology groups are functors in the following way. Let $\alpha : R \to R'$ be a map of k-algebras, M an R-bimodule, M' an R'-bimodule and $\varphi : M' \to M$ (observe the order!) a map of R-bimodules, where M' is considered as an R-bimodule via α. Then there is a natural map of complexes $C^*(R', M') \to C^*(R, M)$ which induces a map $\mathrm{H}^n(R', M') \to \mathrm{H}^n(R, M)$. It is evident that this makes H^n to a contravariant functor from the category of pairs (R, M), where R is a k-algebra, M is an R-bimodule and maps are defined as above, to the category of k-modules.

In this case, the pairs of the form (R, R) do not form a subcategory, since there is no map from (R, R) to (R', R'). Instead one may look at pairs of the form (R, R^*), where $R^* = \mathrm{Hom}_k(R, k)$. We get a functor $\mathrm{H}^n(R, R^*)$ from k-algebras to k-modules. Observe that

$$C^n(R, R^*) = \mathrm{Hom}_k(R^{\otimes n}, R^*) = \mathrm{Hom}_k(R^{\otimes n+1}, k)$$

The groups $\mathrm{H}^n(R, R^*)$ are called the Hochschild cohomology of R and are denoted $\mathrm{HH}^n(R)$.

Even if $\mathrm{H}^n(R, R)$ is not a functor, these groups are of great interest in deformation theory.

If M is an R-bimodule, we have for all $n \geq 0$ natural isomorphisms

$$\mathrm{Hom}_k(M \otimes R^{\otimes n}, k) \cong \mathrm{Hom}_k(R^{\otimes n}, \mathrm{Hom}_k(M, k))$$

The isomorphisms are compatible with the differentials in the standard complexes and hence we get an isomorphism of complexes. In case k is a field, this gives

$$\mathrm{H}_n(R, M)^* \cong \mathrm{H}^n(R, M^*) \tag{1}$$

2.4 A resolution

We will now define a resolution of R as an R-bimodule, which is called the twosided bar-resolution and is denoted $B(R, R)$. In homological degree n it is $R^{\otimes n+2}$. The differential is denoted b' and is similar to b with the difference that the last term is missing. We have the following picture

$$B(R, R): \quad \ldots \xrightarrow{b'} R \otimes R \otimes R \otimes R \xrightarrow{b'} R \otimes R \otimes R \xrightarrow{b'} R \otimes R$$

and e.g.,

$$b'(a_1, a_2, a_3, a_4) = (a_1 a_2, a_3, a_4) - (a_1, a_2 a_3, a_4) + (a_1, a_2, a_3 a_4)$$

The fact that $b' \circ b' = 0$ is again proved in the same way as before. Here we have that b' is a map of R-bimodules, where R is considered to act on the left and right side of a tensor. The complex is augmented by the multiplication map $\mu : R \otimes R \to R$. This map is also a map of R-bimodules, but it is not a map of algebras unless R is commutative. Next we define a "homotopy" s, i.e., a map which raises the homological degree by 1 such that $sb' + b's = id$. From this it follows that cycles are boundaries, since if $b'x = 0$, then $b'sx = x$.

Let $s(a_1, \ldots, a_n) = (1, a_1, \ldots, a_n)$ for $n \geq 1$. It is easy to check that the equation $sb' + b's = id$ holds. (Moreover, we observe that s is a map of right R-modules.) Now suppose R is free (or projective) as a k-module. Then it follows that $R \otimes R^{\otimes n} \otimes R$ is a free (or projective) left R^e-module ($R^e = R \otimes R^{op}$) and hence we have a free (or projective) resolution of R as a left R^e-module. If M is an R-bimodule, then M may be considered as a right R^e-module (by defining $m(r \otimes s) = smr$). If we apply the functor $(M \otimes_{R^e} \cdot)$ to the resolution, we get back the complex $C_*(R, M)$ defined in section 2.3 and hence we get

$$\mathrm{H}_n(R, M) = \mathrm{Tor}_n^{R^e}(M, R)$$

Using results from homological algebra, this formula shows that we may use any projective (or even flat) resolution of R as a left R^e-module or a projective (flat) resolution of M as a right R^e-module to compute $\mathrm{H}_n(R, M)$.

The use of R^e above has the advantage that we may apply the standard homology theory of modules over a ring. Sometimes it is however better for the understanding to stay in the terminology of bimodules. If M and N are R-bimodules we may, as we have noted above, consider M as a right R^e-module and N as a left R^e-module and form the tensor product $M \otimes_{R^e} N$. Examining the definition of the tensor product this may be seen as the quotient of $M \otimes N$ by the k-submodule generated by $mr \otimes n - m \otimes rn$ and $sm \otimes n - m \otimes ns$ for all $r, s \in R$, $m \in M$ and $n \in N$. Thus the tensor product of two bimodules may be defined without referring to R^e and therefore we sometimes will use the notation $M \otimes_{R-R} N$ instead of $M \otimes_{R^e} N$. In the same way we will often write $\mathrm{Hom}_{R-R}(M, N)$ for the set of R-bimodule homomorphisms from M to N, but we will also use the notation $\mathrm{Hom}_{R^e}(M, N)$ for the same set (where M and N are considered as left R^e-modules).

The cohomology groups $\mathrm{H}^n(R, M)$ may in a similar way be defined in terms of the Ext-functor in the case that R is projective as a k-module. Let M be an R-bimodule and consider M as a left R^e-module. Apply the functor $\mathrm{Ext}_{R^e}(\cdot, M)$ to the resolution above. Observing that

$$\mathrm{Hom}_{R^e}(R^{\otimes n+2}, M) = \mathrm{Hom}_k(R^{\otimes n}, M)$$

we see that we get back the standard complex used in section 2.3. Hence

$$\mathrm{H}^n(R, M) = \mathrm{Ext}_{R^e}(R, M)$$

It is possible to handle also the case when R is not k-projective. In this case one has to use so called relative homological algebra and the relative functors Tor and Ext for the ring homomorphism $k \to R$. Instead of ordinary projective resolutions one considers complexes which are split as k-complexes (i.e., there is a k-linear homotopy) and consist of modules of the form $R \otimes V$ where V is a k-module. See Weibel 8.7.1 for more details.

When R is augmented, $R = k \oplus I$, the standard resolution can be made somewhat smaller. The new complex is called the normalized twosided bar-resolution and is denoted $\overline{B}(R, R)$. We have

$$\overline{B}(R, R): \quad \ldots \xrightarrow{b'} R \otimes I \otimes I \otimes R \xrightarrow{b'} R \otimes I \otimes R \xrightarrow{b'} R \otimes R$$

The differential b' and the homotopy s are defined exactly as before. When R is projective as a k-module this is still a projective R^e-resolution of R. Hence

$\mathrm{H}_n(R, M)$ may be computed as the homology of $(M \otimes I^{\otimes n}, b)$ and $\mathrm{H}^n(R, M)$ may be computed as the cohomology of $(\mathrm{Hom}_k(I^{\otimes n}, M), \beta)$ where b and β have the same form as before.

It is possible to normalize the bar-resolution even if R is not augmented. Instead of I one uses $\mathrm{coker}(k \to R)$. However, one has to be careful, since the differential is no longer a sum of well-defined terms.

When k is a field, it is possible to get $\mathrm{Tor}^R(M, N)$ and $\mathrm{Ext}_R(M, N)$ as special cases of Hochschild homology and cohomology (which in turn are special cases of Tor and Ext!). If M is a right R-module and N a left R-module, we may form the R-bimodule $M \otimes N$ and if M, N are left R-modules we may form the R-bimodule $\mathrm{Hom}_k(M, N)$. When k is a field we have

$$\mathrm{H}_n(R, M \otimes N) = \mathrm{Tor}_n^R(M, N)$$
$$\mathrm{H}^n(R, \mathrm{Hom}_k(M, N)) = \mathrm{Ext}_R^n(M, N)$$

To proof this, we know that $\mathrm{H}_n(R, M \otimes N)$ may be computed by tensoring the twosided bar-resolution with the R-bimodule $M \otimes N$. This can be carried out in two steps. First we tensor the resolution to the right with the left R-module N. The homotopy s is still defined on the new complex, which hence is a resolution of N and it is also a free left R-resolution, since N and R are k-free. The next step is to tensor this resolution to the left with the right R-module M and take homology. By definition this will give us $\mathrm{Tor}^R(M, N)$.

To prove the second formula, we first tensor to the right with M to get a free resolution of the left R-module M. Next, the functor $\mathrm{Hom}_k(\cdot, N)$ is applied and the resulting homology is $\mathrm{Ext}_R(M, N)$. The formula

$$\mathrm{Hom}_k(P \otimes M, N) = \mathrm{Hom}_k(P, \mathrm{Hom}_k(M, N))$$

shows that these steps may be carried out by directly tensoring the twosided bar-resolution with the R-bimodule $\mathrm{Hom}_k(M, N)$.

2.5 Homology and cohomology in degree zero

We have already given the Hochschild homology and cohomology in degree zero. The result follows directly from the standard complexes. We have

$$\mathrm{H}_0(R, M) = M/[R, M]$$
$$\mathrm{H}^0(R, M) = \{m \in M;\ rm = mr\}$$

where $[R, M] =$ the k-submodule of M generated by $\{rm - mr; r \in R, m \in M\}$. Observe that $\mathrm{H}_0(R, M)$ and $\mathrm{H}^0(R, M)$ are not R-modules in general unless R is commutative.

Example. Let $R = T(V)$ – the tensor algebra on V (= the free associative algebra on V). Then $[R, R] =$ the k-module generated by $x_1 x_2 \cdots x_n - x_n x_1 \cdots x_{n-1}$, for $n \geq 2$ and $x_i \in V$ for all i. It follows that for all $a, b \in R$ we have $ab = ba$ in $\mathrm{HH}_0(T(V)) = R/[R, R]$, but still $R/[R, R] \ncong S(V)$ (the symmetric algebra on V). As an example, we have for $x, y \in V$ that $xyxy = yxyx \neq x^2 y^2$ in $\mathrm{HH}_0(T(V))$.

The cyclic group C_n operates on $V^{\otimes n}$ and the k-module of invariants $(V^{\otimes n})^{C_n}$ is the nth part of the weighted vector space $\mathrm{H}^0(T(V), T(V))$:

$$\mathrm{H}^0(T(V), T(V)) = \mathrm{H}^{0,n}(T(V), T(V)) = \bigoplus_{n \geq 0} (V^{\otimes n})^{C_n}$$

E.g., we have for $x, y \in V$ that

$$xyxy + yxyx \in \mathrm{H}^{0,4}(T(V), T(V))$$

2.6 Cohomology in degree one and derivations

The first cohomology group $\mathrm{H}^1(R, M)$ is also easily obtained from the standard cocomplex. Let f be a cocycle in degree one. This means that for all $r, s \in R$ we have

$$\beta(f)(r, s) = rf(s) - f(rs) + f(r)s = 0$$

i.e., $f : R \to M$ is a derivation. The k-module of all derivations from R to M is denoted $\mathrm{Der}(R, M)$. The coboundaries in degree one are maps f of the form $f(r) = rm - mr$ for some $m \in M$. These derivataions are called inner derivations and we thus have

$$\mathrm{H}^1(R, M) = \mathrm{Der}(R, M)/\{\text{inner derivations}\}$$

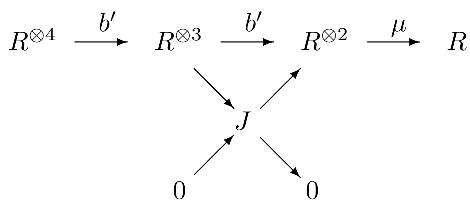
The functor $\mathrm{Der}(R, \cdot)$ from R-bimodules to k-modules is "representable", i.e., there is an R-bimodule J such that

$$\mathrm{Der}(R, M) = \mathrm{Hom}_{R-R}(J, M) \ (= \mathrm{Hom}_{R^e}(J, M))$$

In fact,

$$J = \ker(R \otimes R \xrightarrow{\mu} R) = \mathrm{im}(R^{\otimes 3} \xrightarrow{b'} R \otimes R) = \mathrm{coker}(R^{\otimes 4} \xrightarrow{b'} R^{\otimes 3}) \qquad (2)$$

We have the following picture (with all sequences exact):



(3)

Thus

$$\begin{aligned}\mathrm{Hom}_{R-R}(J, M) &= \ker(\mathrm{Hom}_{R-R}(R^{\otimes 3}, M) \to \mathrm{Hom}_{R-R}(R^{\otimes 4}, M)) \\ &= \ker(\mathrm{Hom}_k(R, M) \xrightarrow{\beta} \mathrm{Hom}_k(R^{\otimes 2}, M)) \\ &= \mathrm{Der}(R, M)\end{aligned}$$

and hence

$$\mathrm{H}^1(R, M) = \mathrm{Hom}_{R-R}(J, M)/\{\text{inner derivations}\}$$

The module J is called the non-commutative R-bimodule of differential forms.

The second description of J in (2) shows that J is generated as an R-bimodule by $b'(1 \otimes r \otimes 1) = r \otimes 1 - 1 \otimes r$ for all $r \in R$. The map $d : R \to J$ given by $d(r) = r \otimes 1 - 1 \otimes r$ is easily seen to be a derivation. It corresponds to $id : J \to J$ under the natural identification $\mathrm{Der}(R, M) = \mathrm{Hom}_{R-R}(J, M)$, which is natural in M. Indeed, given a map $g : J \to M$, the corresponding derivation $f : R \to M$ is obtained as follows

$$r \rightsquigarrow 1 \otimes r \otimes 1 \xrightarrow{b'} r \otimes 1 - 1 \otimes r \xrightarrow{g} M$$

Hence $f = g \circ d$. It follows that d is a universal derivation, in the sense that for any derivation $f : R \to M$, there is a unique R-bimodule homomorphism $g : J \to M$ such that $f = g \circ d$:

$$\begin{array}{ccc} R & \xrightarrow{f} & M \\ & {\scriptstyle d}\searrow \quad \nearrow{\scriptstyle \exists!} & \\ & J & \end{array}$$

By this universal property or by the third description of J in (2), it follows that J has the following presentation as an R-bimodule. The module J is the free R-bimodule on symbols $d(r)$ for $r \in R$, modulo relations which make d to a derivation $R \to J$, namely for all $x, y \in R$ and $\lambda, \mu \in k$,

$$d(\lambda x + \mu y) = \lambda d(x) + \mu d(y) \quad \text{and} \quad d(xy) = xd(y) + d(x)y$$

Example. Let $R = T(V) =$ the tensor algebra on V. Then $J = R \otimes V \otimes R$. This follows from the fact that $\mathrm{Der}(R, M) = \mathrm{Hom}_k(V, M)$. A derivation $f : R \to M$ is uniquely determined by the values $f(v)$ for $v \in V$ and $f : V \to M$ may be any k-linear map. We may also deduce this from a minimal resolution of R as an R-bimodule. We have the following resolution

$$0 \to R \otimes V \otimes R \xrightarrow{b'} R \otimes R \xrightarrow{\mu} R \to 0$$

where $b'(r \otimes v \otimes s) = rv \otimes s - r \otimes vs$. This is a complex and it is exact, since a homotopy s is defined by

$$\begin{aligned} s(r) &= 1 \otimes r \\ s(1 \otimes r) &= 0 \\ s(v_1 v_2 \cdots v_n \otimes x) &= \sum_{i=0}^{n-1} v_1 v_2 \cdots v_i \otimes v_{i+1} \otimes v_{i+2} \cdots v_n x \end{aligned}$$

From the resolution, we see that $\mathrm{H}_n(R, M) = \mathrm{H}^n(R, M) = 0$ for $n > 1$ and

$$\begin{aligned} &\mathrm{H}^1(R, M) = \mathrm{Hom}_k(V, M)/\{\text{inner derivations}\} \\ &\mathrm{H}_1(R, M) = \ker(M \otimes V \to M) \quad \text{where} \quad m \otimes v \mapsto mv - vm \end{aligned}$$

Remark 1. In fact, as we will see later, the resolution in the example is the two-sided Koszul resolution ($T(V)$ is a Koszul algebra and its Koszul dual is $k \oplus V^*$ with zero multiplication on V^*). If $R^!$ is the Koszul dual of R, then there is a differential on $R \otimes (R^!)^* \otimes R$ which defines a resolution of R as an R-bimodule if R is Koszul.

Remark 2. If $R = T(V)/\langle f_1, \ldots, f_n \rangle$, then the R-bimodule of non-commutative differential forms J has the presentation

$$J = R \otimes V \otimes R/(df_1, \ldots, df_n)$$

2.7 Homology in degree one and Kähler differentials

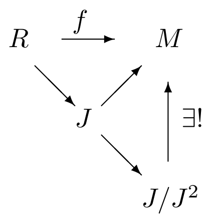
In this section we assume that M is a symmetric R-bimodule, but R may be non-commutative. We will first prove that the functor $\mathrm{Der}(R, \cdot)$ from symmetric R-bimodules to k-modules is representable. Indeed, we will prove that

$$\mathrm{Der}(R, M) = \mathrm{Hom}_{R-R}(J/J^2, M) \tag{4}$$

when M is symmetric. Here $J^2 = J \cdot J$ is computed in $R^e = R \otimes R^{op}$. The result follows from the following two facts.

$$\begin{aligned} &1) \quad J/J^2 \ \text{ is symmetric} \\ &2) \quad f \in \mathrm{Hom}_{R-R}(J, M) \Longrightarrow f(J^2) = 0 \end{aligned}$$

Since, if this is proved, we have a unique factorization of $f \in \mathrm{Der}(R, M)$ as follows



(The map $R \to J/J^2$ will also be denoted by d.)

To prove 1), observe that

$$(r \otimes 1 - 1 \otimes r) \cdot (s \otimes t) = rs \otimes t - s \otimes tr = r(s \otimes t) - (s \otimes t)r$$

Here $r(s \otimes t)$ and $(s \otimes t)r$ are R-bimodule operations and $(r \otimes 1 - 1 \otimes r) \cdot x$ is multiplication in $R \otimes R^{op}$. Hence, if $r \in R$ and $x \in R \otimes R^{op}$, we have

$$rx - xr = (r \otimes 1 - 1 \otimes r) \cdot x$$

In particular, $[R, J] = J^2$ in R^e and it follows that J/J^2 is symmetric. (In this case, we have indeed that $\mathrm{H}_0(R, J) = J/[R, J] = J/J^2$ is an R-module, even if R is non-commutative.)

To prove 2), suppose $f : J \to M$ is R-bilinear. Then, for $r \in R$ and $x \in J$, we have

$$f((r \otimes 1 - 1 \otimes r) \cdot x) = f(rx - xr) = rf(x) - f(x)r = 0$$

Hence $f(J^2) = 0$ and (4) is proved.

We will now prove that for M symmetric and R arbitrary, we have

$$\mathrm{H}_1(R, M) = M \otimes_R J/J^2$$

Consider the diagram (3) and apply the funtor $M \otimes_{R-R} \cdot$. Since M is symmetric, we have that

$$1 \otimes b' : \ M \otimes_{R-R} R^{\otimes 3} \to M \otimes_{R-R} R^{\otimes 2}$$

is zero and hence

$$\mathrm{H}_1(R,M) = \mathrm{coker}(M \otimes_{R-R} R^{\otimes 4} \to M \otimes_{R-R} R^{\otimes 3}) = M \otimes_{R-R} J$$

But, again since M is symmetric, we have $M \otimes_{R-R} [R,J] = 0$, since

$$m \otimes (rx - xr) = mr \otimes x - rm \otimes x = 0$$

But we proved above that $[R,J] = J^2$ and hence $M \otimes_{R-R} J^2 = 0$, from which it follows that

$$\mathrm{H}_1(R,M) = M \otimes_{R-R} J = M \otimes_R J/J^2$$

In particular, if R is commutative, we have

$$\mathrm{HH}_1(R) = J/J^2$$

If R is commutative, the R-module J/J^2 is called the module of differential forms or the Kähler differentials and it is also denoted $\Omega^1_{R|k}$. If $R = S(V)/\langle f_1, \dots, f_n\rangle$, where $S(V)$ is the symmetric algebra on V, then

$$\Omega^1_{R|k} = R \otimes V/(df_1, \dots, df_n)$$

2.8 H^2 and extensions

Let $\pi : S \to R$ be a surjective k-split homomorphism of k-algebras such that $M = \ker(\pi)$ has square zero in S. Then M is in a natural way an R-bimodule, by defining rm as sm (and mr as ms), where $\pi(s) = r$. This is well-defined, since $\pi(s) = \pi(s')$ gives $s - s' \in M$ and hence $(s - s')m = m(s - s') = 0$. Since π is k-split, we have that $S = R \oplus M$ as k-modules. Since $M^2 = 0$ and π is a ring homomorphism, we have that multiplication in S has the following form,

$$(r,m)(r',m') = ((r,0)+(0,m))((r',0)+(0,m')) = (rr', rm'+mr'+f(r,r')) \quad (5)$$

where $f : R \times R \to M$ is some k-bilinear map, i.e., f is a 2-cochain in $C^*(R,M)$. We have

$$((r,m)(r',m'))(r'',m'') =$$
$$((rr')r'', (rr')m'' + (rm' + mr' + f(r,r'))r'' + f(rr',r''))$$

$$(r,m)((r',m')(r'',m'')) =$$
$$((r(r'r''), r(r'm'' + m'r'' + f(r',r'')) + m(r'r'') + f(r,r'r''))$$

The associativity law in S gives (using associativity in R and the fact that M is an R-bimodule)

$$f(r,r')r'' + f(rr',r'') = rf(r',r'') + f(r,r'r'')$$

i.e., f is a 2-cocycle. Conversely given a k-algebra R, an R-bimodule M and a 2-cocycle f, the definition (5) defines a k-algebra structure on $R \oplus M$, which we denote by $(R \oplus M, f)$.

Moreover, we say that two extensions $(R\oplus M, f)$ and $(R\oplus M, f')$ are equivalent if there is an isomorphism ϕ of k-algebras, such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & (R\oplus M, f) & \longrightarrow & R & \longrightarrow & 0 \\ & & \| & & \phi\big\downarrow & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & (R\oplus M, f') & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

This means that ϕ in matrix form has the form

$$\phi = \begin{pmatrix} id & 0 \\ g & id \end{pmatrix}$$

where $g : R \to M$. The fact that ϕ is an algebra homomorphism gives the following equation,

$$\phi(rr', rm' + mr' + f(r,r')) = \phi(r,m)\phi(r',m') = (r, g(r)+m)(r', g(r')+m')$$

i.e.,

$$(rr', g(rr') + rm' + mr' + f(r,r')) = (rr', rg(r') + rm' + g(r)r' + mr' + f'(r,r'))$$

Hence

$$f(r,r') - f'(r,r') = rg(r') - g(rr') + g(r)r'$$

i.e., $f - f'$ is a coboundary. Conversely, if $f - f'$ is a coboundary, then $(R\oplus M, f)$ and $(R\oplus M, f')$ are equivalent.

Thus we have proved that $H^2(R,M)$ is in one-to-one correspondence with the equivalence classes of algebra extensions of the form $(R\oplus M) \to R$ with $M^2 = 0$. The extension which corresponds to the zero class is called "trivial" and in this case the map $(R\oplus M) \to R$ is split as algebras.

2.9 Dimension zero

We will in this section consider k-algebras which satisfy $H^1(R,M) = 0$ for all R-bimodules M. They are said to be of dimension zero. More generally a k-algebra is said to be of dimension $\leq n$ if R as a module over R^e has homological dimension $\leq n$. If R is k-projective, this is the same as saying that $H^{n+1}(R,M) = 0$ for all R-bimodules M. A k-algebra R has dimension zero iff R is projective over R^e which happens iff the multiplication map $R\otimes R \to R$ splits as a map of R-bimodules. This means that there are finitely many $x_i, y_i \in R$ such that

$$\sum_i x_i y_i = 1 \quad \text{and} \quad \sum_i r(x_i \otimes y_i) = \sum_i (x_i \otimes y_i) r$$

for all $r \in R$. This condition may also be obtained from the fact that R has dimension zero iff the universal derivation is inner (see exercise 5). It is not enough to have $\mathrm{H}_1(R,M) = 0$ for all M to conclude that R has dimension zero. A counterexample is the field of algebraic numbers over $\mathbb{Q}$ (see exercise 8). Observe that it follows from (1) that in this case J is not of the form M^* for any bimodule M.

That the algebraic numbers over $\mathbb{Q}$ is not zero-dimensional follows from the following lemma by Villamayor and Zelinsky (see also Weibel, Lemma 9.2.12):

Lemma 2.1 *Let R be a zero-dimensional k-algebra which is k-free. Then R is finitely generated as a k-module.*

Proof. We know that there are finitely many $x_i, y_i \in R$ such that $\sum x_i y_i = 1$ and $\sum r x_i \otimes y_i = x_i \otimes y_i r$ for all $r \in R$. Let $\{e_\alpha\}_{\alpha \in I}$ be a k-basis for R. There is a finite subset I_0 of I such that for all i, we have that x_i and y_i are linear combinations of $\{e_\alpha\}_{\alpha\in I_0}$. We have that $\sum x_i \otimes y_i r$ is a linear combination of tensors $e_\alpha \otimes e_\beta$, where $\alpha \in I_0$ and $\beta \in I$. Hence rx_i is a linear combination of $\{e_\alpha\}_{\alpha\in I_0}$ for all i and all $r \in R$. Then, since $r = \sum (rx_i)y_i$ we get that R is generated by $\{e_\alpha e_\beta\}_{\alpha,\beta\in I_0}$. □

Suppose now that k is a field. Since $\mathrm{Ext}^1_R(M,N)$ is a special case of Hochschild cohomology, we get that the global dimension of R is zero if R has (Hochschild) dimension zero. Hence R is semi-simple, i.e., R is a finite product of matrix rings over skewfields. Also, for each field extension $k \subset l$ it is easy to prove that $R \otimes l$ has dimension zero over l and thus is semi-simple. Together with the condition that R is finitely generated, this is also sufficient for an algebra to be of dimension zero. We refer to Weibel for the proof. We will give the version of this fact for the case when R is a field. Recall that a field extension is called separable if every minimal polynomial has only simple roots (in some extension field).

Theorem 2.2 *Suppose F is an extension field of k. Then $\mathrm{H}^1(F, M) = 0$ for all M if and only if F is finite dimensional and separable over k.*

Proof. Suppose first F is finite dimensional and separable over k. Then F is a simple extension of k and hence $F \simeq k[x]/\langle p(x)\rangle$. Let $\overline{k}$ be a splitting field for $p(x)$. Then $F \otimes \overline{k} \simeq \overline{k}[x]/\langle p(x)\rangle \simeq \overline{k} \times \ldots \times \overline{k}$ since $p(x)$ is a product of different linear factors over $\overline{k}$. But a direct product of $\overline{k}$ has dimension zero over $\overline{k}$. This follows most simply directly: We have that $\sum e_i \otimes e_i$ is an element with the right properties (e_i is the standard basis element $(0,\ldots,0,1,0,\ldots,0)$).

Hence $F\otimes\overline{k}$ has dimension 0 as an algebra over $\overline{k}$. Since k is a direct summand of $\overline{k}$ as a k-module, it follows easily that F over k has dimension 0.

Suppose conversely that F has dimension 0 over k. By Lemma 2.1 F is finite dimensional as a vector space over k. Suppose $\alpha \in F$ has minimal polynomial $p(x)$ over k. Let $\overline{k}$ be a splitting field for $p(x)$. If $p(x)$ has multiple zeroes in $\overline{k}$, then $k(\alpha) \otimes \overline{k} \simeq \overline{k}[x]/\langle p(x)\rangle$ has nilpotent elements. But $k(\alpha) \otimes \overline{k}$ is a subring of $F \otimes \overline{k}$, which in that case would have nilpotent elements. But then $F \otimes \overline{k}$ cannot be semi-simple, which we have seen is a consequence of the fact that F has dimension 0. □

3 Cyclic homology

In this section we will introduce cyclic homology in three different ways and prove some general facts.

3.1 Connes' definition and exact sequence

For any k-algebra I, with or without unit, one may form a new augmented k-algebra, $R = k \oplus I$. The reduced standard complex $\bar{C}^*(R,R) = (R \otimes I^{\otimes n}, d)_{n\geq 0}$

decomposes into two pieces,

$$R \otimes I^{\otimes n} = I^{\otimes n} \oplus I^{\otimes n+1}.$$

We will often write the elements in $I^{\otimes n}$ as $(a_1, \ldots, a_n)$ (beginning with index 1) and $(a_0, \ldots, a_n)$ for the elements in $I^{\otimes n+1}$ (beginning with index 0). The differential $d : I^{\otimes n} \oplus I^{\otimes n+1} \to I^{\otimes n-1} \oplus I^{\otimes n}$ may be written as a 2×2-matrix. We have

$$d = \begin{pmatrix} -b' & 0 \\ 1-t & b \end{pmatrix}$$

where

$$\begin{aligned} b'(a_1, \ldots, a_n) &= \sum_{i=1}^{n-1} (-1)^{i+1} (a_1, \ldots, a_i a_{i+1}, \ldots, a_n) \\ t(a_1, \ldots, a_n) &= (-1)^{n-1} (a_n, a_1, \ldots, a_{n-1}) \\ b(a_0, \ldots, a_n) &= \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^n (a_n a_0, a_1, \ldots, a_{n-1}) \end{aligned}$$

Since $d^2 = 0$, we get that $(1-t)b' = b(1-t)$. This means that the map $1-t$ is a map of complexes, $(1-t) : (I^{\otimes n}, b') \to (I^{\otimes n}, b)$,

$$\begin{array}{ccccccccccccc} k & \xleftarrow{0} & I & \xleftarrow{b'} & I^{\otimes 2} & \xleftarrow{b'} & \ldots & \xleftarrow{b'} & I^{\otimes n} & \xleftarrow{b'} & I^{\otimes n+1} & \longleftarrow & \ldots \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle (1-t)} & & & & \downarrow{\scriptstyle (1-t)} & & \downarrow{\scriptstyle (1-t)} & & \\ & & I & \xleftarrow{b} & I^{\otimes 2} & \xleftarrow{b} & \ldots & \xleftarrow{b} & I^{\otimes n} & \xleftarrow{b} & I^{\otimes n+1} & \longleftarrow & \ldots \end{array}$$

In fact, $\bar{C}^*(R, R)$ is the "mapping cone" of the map $1-t$ and also the associated single complex of the above double complex.

The map $(1-t)$ defines a quotient complex and the homology of this is by definition the cyclic homology of I (or the reduced cyclic homology of R), with the homological degree shifted one step,

$$\mathrm{HC}_n(I) = \overline{\mathrm{HC}}_n(R) = \mathrm{H}(I^{\otimes n+1}/(1-t), b)$$

We have $\mathrm{HC}_0(I) = I/[I, I]$ which may be compared with $\mathrm{HH}_0(R) = R/[R, R] = k \oplus I/[I, I]$.

We have that $(I^{\otimes n}, b')$ is the bar-complex for R and module k, whose homology is $\mathrm{Tor}^R(k, k)$ and $(I^{\otimes n}, b)$ is the reduced Hochschild complex for R and bimodule I with homology $\mathrm{H}(R, I)$. A map of complexes gives rise to two spectral sequences converging to the homology of the associated single complex. In our case this homology is $\mathrm{HH}(R)$. One of the spectral sequences gives a long exact sequence connecting the three homologies, $\mathrm{HH}(R)$, $\mathrm{H}(R, I)$ and $\mathrm{Tor}^R(k, k)$.

The second spectral sequence gives a (more interesting) long exact sequence, which is called Connes' exact sequence. To derive this sequence, with the above definition of cyclic homology, we need to assume that the integers are units in k, i.e., $\mathbb{Q} \subset k$.

From spectral sequence theory, we get the following long exact sequence:

$$\begin{aligned} \ldots \to \mathrm{H}_n(\ker(1-t)) \to \mathrm{HH}_n(R) \to \mathrm{H}_{n+1}(\mathrm{coker}(1-t)) \to \\ \mathrm{H}_{n-1}(\ker(1-t)) \to \ldots \end{aligned}$$

We have that $\mathrm{H}_{n+1}(\mathrm{coker}(1-t))$ is by definition $\overline{\mathrm{HC}}_n(R)$. We will now prove that $(\ker(1-t), b')$ and $(\mathrm{coker}(1-t), b)$ are isomorphic as complexes if $\mathbb{Q} \subset k$.

Indeed, define the "norm"-map, $N : I^{\otimes n} \to I^{\otimes n}$ as $N = \sum_{i=0}^{n-1} t^i$. As usual, we use the same symbol N for each n. Then $b'N = Nb$, i.e., N is a map from the complex $(I^{\otimes n}, b)$ to the complex $(I^{\otimes n}, b')$. This may be proved by writing $b = \sum_{i=0}^{n} b_i$, as an operator on $I^{\otimes n+1}$, where

$$\begin{aligned} b_i(a_0, \ldots, a_n) &= (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n) \quad \text{for} \quad 0 \le i < n \\ b_n(a_0, \ldots, a_n) &= (-1)^n (a_n a_0, a_1, \ldots, a_{n-1}) \end{aligned}$$

and using the following commutation rules,

$$\begin{aligned} b_0 t &= b_n \\ b_i t &= t b_{i-1} \quad \text{for} \quad i > 0 \end{aligned}$$

Furthermore, $N(1-t) = (1-t)N = 0$ from which follows that $\mathrm{im}(1-t) \subset \ker(N)$ and $\mathrm{im}(N) \subset \ker(1-t)$. Hence, N induces a map from the complex $(\mathrm{coker}(1-t), b)$ to the complex $(\ker(1-t), b')$. Since division of the polynomial $\sum_{i=0}^{n-1} x^i$ by $x-1$ gives the remainder n, there is a polynomial q such that $q(t)(1-t)+N = n$. Hence, if n is a unit in k, $\ker(1-t) = \mathrm{im}(N)$ and $\mathrm{im}(1-t) = \ker(N)$. Hence N defines an isomorphism from the complex $(\mathrm{coker}(1-t), b)$ to the complex $(\ker(1-t), b')$. The long exact sequence above may hence be written in the following way (we use the notation N also for the induced map in homology).

$$\ldots \to \overline{\mathrm{HC}}_{n-1}(R) \xrightarrow{N} \mathrm{HH}_n(R) \to \overline{\mathrm{HC}}_n(R) \to \overline{\mathrm{HC}}_{n-2}(R) \to \ldots$$

The maps in the sequence above have the following description.

An element in $\overline{\mathrm{HC}}_{n-1}(R)$ is represented by an element $x \in I^{\otimes n}$ such that $bx \in \mathrm{im}(1-t)$. Then $b'Nx = Nbx = 0$ and $(1-t)Nx = 0$ and hence Nx is a cycle in $(I^{\otimes n} \oplus I^{\otimes n+1}, d)$, which gives an element in $\mathrm{HH}_n(R)$.

Starting with a cycle in $(I^{\otimes n} \oplus I^{\otimes n+1}, d)$, one may project onto the second factor. This element x has the property $bx \in \mathrm{im}(1-t)$ and hence it is a cycle in $(I^{\otimes n+1}/(1-t), b)$ and defines an element in $\overline{\mathrm{HC}}_n(R)$.

Again, starting with a cycle x in $(I^{\otimes n+1}/(1-t), b)$ we have $bx = (1-t)y$ for some y. Take $b'y$. We have $(1-t)b'y = b(1-t)y = b^2x = 0$ and hence $b'y \in \ker(1-t)$ and it is a cycle in $(\ker(1-t), b')$. From above we have $N(\frac{1}{n-1}b'y) = b'y$. Hence $\frac{1}{n-1}b'y$ is the corresponding cycle in $(I^{\otimes n-1}/(1-t), b)$ which gives an element in $\overline{\mathrm{HC}}_{n-2}(R)$ (but of course $b'y$ will also work as definition of the map).

3.2 Connes' exact sequence splits

We will now assume that I has a positive weight function, i.e., $I = \oplus_{q\ge 1} I_q$ and $I_q I_r \subset I_{q+r}$. We also assume as in the previous section that $\mathbb{Q} \subset k$. We claim that the differential $d_2 : \overline{\mathrm{HC}}_n(R) \to \overline{\mathrm{HC}}_{n-2}(R)$ in the spectral sequence above is zero in this case and hence Connes' exact sequence splits into short exact sequences,

$$0 \to \overline{\mathrm{HC}}_{n-1}(R) \xrightarrow{N} \mathrm{HH}_n(R) \to \overline{\mathrm{HC}}_n(R) \to 0$$

Moreover, there is a natural splitting of the sequence, which gives a natural decomposition of Hochschild homology,

$$\mathrm{HH}_n(R) = \overline{\mathrm{HC}}_{n-1}(R) \oplus \overline{\mathrm{HC}}_n(R)$$

To prove this we use the fact that there is a derivation $D : I \to I$ defined by $D(x) = \text{weight}(x)x$ (the Euler derivation). In general, any derivation $d : I \to I$ may be extended to a map $L_d : I^{\otimes n+1} \to I^{\otimes n+1}$, where $L_d = \sum_{i=0}^{n} d_i$ and

$$d_i(a_0, \ldots, a_n) = (a_0, \ldots, a_{i-1}, d(a_i), a_{i+1}, \ldots, a_n)$$

The weight function extends to $I^{\otimes n+1}$ and we have $L_D(x) = \text{weight}(x)x$.

The construction of L_d may also be applied to define $\text{HH}(R)$ in the case when (R, d) is a differential graded algebra.

The following commutation rules hold (observe that the maps d_i to the left of b_j is operating on $I^{\otimes n}$, while a d_i to the right is operating on $I^{\otimes n+1}$).

$$\begin{aligned} d_i b_j &= b_j d_i \quad \text{for} \quad i < j\,,\ (i,j) \neq (0,n) \\ d_i b_j &= b_j d_{i+1} \quad \text{for} \quad i > j \\ d_i b_i &= b_i d_i + b_i d_{i+1} \\ d_0 b_n &= b_n d_n + b_n d_0 \\ t d_i &= d_{i+1} t \quad \text{for} \quad i < n \\ t d_n &= d_0 t \end{aligned} \tag{6}$$

Given a derivation d, we define a new operation E_d on $I^{\otimes n+1}$ (defined for every n) by

$$E_d = -\sum_{1 \le i \le j \le n} t^{-i} d_j$$

Lemma 3.1 (reformulation of 4.1.8.2 in Loday)

$$\begin{aligned} &a) \quad (1-t)E_d + d_0 N = L_d \\ &b) \quad E_d b = b' E_d + N b_0 d_0 t \end{aligned}$$

where the operators in a) are defined on $I^{\otimes n+1}$. In b) the operators to the left, E_d, b' and N are defined on $I^{\otimes n}$ while the others are defined on $I^{\otimes n+1}$.

Proof. We have $(1-t)\sum_{1 \le i \le j} t^{-i} = (t^{-j} - 1)$. Hence

$$(1-t)E_d = -\sum_{j=1}^{n} t^{-j} d_j + \sum_{j=1}^{n} d_j.$$

Furthermore

$$d_0 N = \sum_{i=1}^{n+1} d_0 t_i = \sum_{i=0}^{n} t d_n t^i = \sum_{i=0}^{n} t^{i+1} d_{n-i} = \sum_{j=0}^{n} t^{n+1-j} d_j = \sum_{j=0}^{n} t^{-j} d_j.$$

Hence

$$(1-t)E_d + d_0 N = d_0 + \sum_{j=1}^{n} d_j = L_d,$$

which proves a).

To prove b), we first choose k such that $1 \le k \le n-1$. For each i, $1 \le i \le n-1$ the commutation rules give

$$\sum_{j=0}^{n} d_i b_j = \sum_{j=i}^{n} b_j d_i + \sum_{j=0}^{i} b_j d_{i+1}.$$

Hence

$$\begin{aligned}
\sum_{\substack{k\le i\le n-1\\0\le j\le n}} d_i b_j &= \sum_{\substack{k\le i\le j\le n\\ i\le n-1}} b_j d_i \quad + \sum_{\substack{k<i\le n\\0\le j<i}} b_j d_i \\
&= \sum_{k\le i\le j\le n} b_j d_i \quad - \quad b_n d_n \quad + \sum_{k\le j<i\le n} b_j d_i \quad + \sum_{0\le j<k<i\le n} b_j d_i \\
&= \sum_{\substack{k\le i\le n\\k\le j\le n}} b_j d_i \quad - \quad b_n d_n \quad + \sum_{0\le j<k<i\le n} b_j d_i\,.
\end{aligned}$$

Observe that the expression is 0 for $k = n$. Since $N = \sum_{k=0}^{n-1} t^k = \sum_{k=1}^{n} t^{-k}$ and $b_n = b_0 t$, we get, after multiplication by t^{-k} and summation of k from 1 to n,

$$\begin{aligned}
-E_d b \quad &= \sum_{\substack{1\le k\le i\le n-1\\0\le j\le n}} t^{-k} d_i b_j \quad = \sum_{\substack{1\le k\le i\le n\\k\le j\le n}} b_{j-k} t^{-k} d_i \quad - \quad N b_0 t d_n \\
&+ \sum_{0\le j<k<i\le n} t^{-k} b_j d_i\,.
\end{aligned}$$

But

$$\begin{aligned}
\sum_{0\le j<k<i\le n} t^{-k} b_j d_i \quad &= \sum_{0\le j<k<i\le n} b_{n+j-k} t^{-k-1} d_i \quad = \sum_{0\le j<k-1<i\le n} b_{n+j-k+1} t^{-k} d_i \\
&= \sum_{\substack{n-k+1\le r<n\\1\le k\le i\le n}} b_r t^{-k} d_i\,.
\end{aligned}$$

Hence

$$-E_d b \quad = \sum_{\substack{0\le r<n\\1\le k\le i\le n}} b_r t^{-k} d_i \quad - \quad N b_0 t d_n \quad = \quad -b' E_d - N b_0 d_0 t\,,$$

and thus *b*) is proved. □

Now we obtain the result mentioned in the beginning of the section.

Theorem 3.2 *Suppose R is an augmented k-algebra such that $\mathbb{Q} \subset k$ and suppose that the augmentation ideal, I, has a positive weight, $I = \oplus_{q\ge 1} I_q$ such that $I_r I_s \subset I_{r+s}$. Then Connes' exact sequence splits into short exact sequences which are naturally split and also decompose into one sequence for each weight,*

$$0 \longrightarrow \overline{\mathrm{HC}}_{n-1,q}(R) \xrightarrow{N} \mathrm{HH}_{n,q}(R) \longrightarrow \overline{\mathrm{HC}}_{n,q}(R) \longrightarrow 0\,.$$

Proof. Observe first that the boundary map b is homogeneous with respect to the weight, i.e., weight(bx) = weight(x) and hence the homology groups and Connes' exact sequence decompose. In order to prove that the map $\mathrm{HH}_n(R) \to \overline{\mathrm{HC}}_n(R)$ in Connes' exact sequence is surjective, we have to prove the following. Take a weight-homogeneous $x \in I^{\otimes n+1}$ such that $bx \in \mathrm{im}(1-t)$. Then there must be a $y \in I^{\otimes n}$ such that $(1-t)y = bx$ and $b'y = 0$. The solution to this is to use the operator E_D for the derivation D on I definied by $Da = \mathrm{weight}(a)a$ and put $y = \frac{1}{\mathrm{weight}(x)}(E_D bx - N b_0 D_0 t x)$. Then, according to *a*) in Lemma 3.1, we have $(1-t)y = bx$, since $Nbx = 0$ and $(1-t)N = 0$. According to *b*) in Lemma 3.1, we have $y = \frac{1}{\mathrm{weight}(x)} b' E_D x$ and hence $b'y = 0$. □

4 Graded theory

We will now generalise the homology theories we have studied to the graded case, i.e., $\mathbb{Z}_2$-graded case. This will be done by introducing a lot of signs in the formulas for the differential operators. In most cases these signs follows the "Koszul rule", which means that when two symbols a, b of degrees $|a|, |b|$ are interchanged, then a sign $(-1)^{|a||b|}$ are introduced. We will also consider the "weighted" case, which does not have any effect on signs. It is used only to divide homology groups into pieces and sometimes it is also important in proving theorems (e.g. Connes' exact sequence is short). The definitions of what it means for an algebra to be graded and weighted are the same, namely that there is a subdivision of the ring (over a semi-group) as a k-module which is compatible with the product in the algebra. It is the use of the term that differ. We use the term "graded" when signs are involved and "weighted" when no signs occur. To get a subdivision of the homology groups for a weighted algebra, one has to check that the differentials are homogeneous with respect to the weight. This is easy to see by inspection for the Hochshild and Cyclic complex.

Suppose now R is $\mathbb{Z}_2$-graded, i.e., $R = R_0 \oplus R_1$ and $R_i R_j \subset R_{i+j \text{mod} 2}$. The elements in R_0 are called even and the elements in R_1 are called odd. We write $|a|$ for the degree of a, i.e., $|a| = 0$ if $a \in R_0$ and $|a| = 1$ if $a \in R_1$.

The graded commutator $[a, b]$ is defined as

$$[a, b] = ab - (-1)^{|a||b|} ba$$

We say that R is graded commutative if $[a, b] = 0$ for all $a, b \in R$. If V_1, V_2 are k-submodules of R, then $[V_1, V_2]$ is defined as the k-submodule of R generated by $[a, b]$ for all $a \in V_1$ and $b \in V_2$.

We make the following definition

$$\mathrm{HH}_0(R) = R/[R, R]$$

This looks the same as in the non-graded case, but now it means quite a different thing. If we apply the forgetful functor to R and get $R^\#$, which is R just as an algebra, then $\mathrm{HH}_0(R^\#)$ is different from $\mathrm{HH}_0(R)$. This phenomenon does not occur, when looking at the functors Ext and Tor. In this case the groups for a graded algebra R are just the groups for $R^\#$ with an extra graded structure.

Example.

Let $R = k\langle x \rangle$ where x is even. Then $R = R_0$ (the elements in k are always even) and $\mathrm{HH}(R) = R$.

Let $R' = k\langle x \rangle$ where x is odd. Then $[x, x] = 2x^2$ and more generally $[x^n, x] = 2x^{n+1}$ for all odd n. Hence

$$\mathrm{HH}(R') = k \oplus kx \oplus kx^3 \oplus \cdots$$

In order to define $H_n(R, M)$ in general, we will give the graded version of the two-sided bar-resolution $B(R, R)$ with differential b' (see setion 2.4). The modules in the resolution should be graded and the differential b' should be homogeneous of degree 1 (it has homological degree -1). This implies that b' should be R-bilinear in the following sense.

$$b'(rx) = (-1)^{|r|} r d(x) \quad \text{and} \quad b'(xr) = b'(x) r$$

The resolution is built from R-bimodules of the form $P^n = R \otimes R^{\otimes n} \otimes R$ as before. These modules are graded by considering the elements from R in the n tensors in the middle to be shifted in degree. We will use the operator s to indicate degree shift, i.e., $|sr| = |r| + 1$. The elements in P^n are written

$$a_0(sa_1, sa_2, \ldots, sa_n)a_{n+1} \quad \text{for} \quad n \geq 1 \quad \text{and} \quad a_0()a_1 \quad \text{for} \quad n = 0$$

Since b' is R-bilnear (in the sense above), we need only give the definition of b' when $a_0 = a_{n+1} = 1$. We have

$$\begin{aligned} b'(sa_1, sa_2, \ldots, sa_n) =& a_1(sa_2, \ldots, sa_n) \\ &+ \sum_{i=1}^{n-1} (-1)^{\epsilon(i)}(sa_1, \ldots, s(a_i a_{i+1}), \ldots, sa_n) \\ &- (-1)^{\epsilon(n-1)}(sa_1, \ldots, sa_{n-1})a_n \end{aligned}$$

where $\epsilon(i) = i + \sum_{r=1}^{i} |a_r|$. The signs may be obtained by using the homotopy σ which maps $a_0(sa_1, sa_2, \ldots, sa_n)a_{n+1}$ to $(sa_0, sa_1, sa_2, \ldots, sa_n)a_{n+1}$ and the inductive equation $d\sigma = 1 - \sigma d$ together with $b'(rx) = (-1)^{|r|} r b'(x)$.

Let M be a graded R-bimodule. Then we get the following graded version of the differential b on the standard complex $M \otimes R^n$:

$$\begin{aligned} b(m, sa_1, \ldots, sa_n) =& (-1)^{|m|}(ma_1, sa_2, \ldots, sa_n) \\ &+ \sum_{i=1}^{n-1} (-1)^{\epsilon(i)+|m|}(m, sa_1, \ldots, s(a_i a_{i+1}), \ldots, sa_n) \\ &+ (-1)^{\epsilon}(a_n m, sa_1, \ldots, sa_{n-1}) \end{aligned}$$

where

$$\epsilon = 1 + (|a_n| + 1)(|m| + |sa_1| + \ldots + |sa_{n-1}|)$$

A graded derivation d on a graded algebra R satifies the following

$$d(ab) = d(a)b + (-1)^{|d||a|} a d(b)$$

and the extension L_d to R^{n+1}, which was defined in section 3.2, is defined as $L_d = \sum_{i=0}^{n} d_i$ where

$$d_i(a_0, \ldots, a_n) = (-1)^{|d| \sum_{r=0}^{i-1}(|a_r|+1)}(a_0, \ldots, a_{i-1}, d(a_i), a_{i+1}, \ldots, a_n)$$

The commutation rules (6) in section 3.2 still hold with a factor $(-1)^{|d|}$ added to the right hand side in the first four equations.

5 Non-commutative models

In this section we will introduce models as a tool to define homology. Instead of resolutions one uses differential algebras whose underlying algebra is free. This is particular useful when one has no modules and no additive functor to work with, which is the case for the cyclic homology. One may think of the reduced cyclic homology as the derived funtors of the functor which to an augmented k-algebra $R = k \oplus I$ associates the vector space $I/[I, I]$. The category of $k$algebras is not abelian so we cannot use the ordinary procedure with projective resolutions. The situation is similar for the functors $\mathrm{Tor}^R(k,k)$. They may be seen as the derived functors of the functor which to an augmented k-algebra $R = k \oplus I$ associates the vector space I/I^2. In this case however it is also possible to view the functor as a special case of the additive functor $\mathrm{Tor}^R(M,N)$ defined on an abelian catogory.

In order to define models, one has to introduce differential graded algebras, which are $\mathbb{Z}_2$-graded and weighted in the sense of the previous section.

A differential graded k-algebra (A,d) (DG-algebra for short) is an augmented k-algebra, $A = k \oplus I$, where k is a field and A is $\mathbb{Z}_2 \times \mathbb{N} \times \mathbb{N}^k$-graded. Here the first degree is the "sign degree", which is denoted $|a|$, the second degree is the homological degree and the third degree is the "weight"(which could be defined over any semi-group, but $\mathbb{N}^k$ is most common). The differential d is a graded derivation on A which is of degree $(1,-1,0)$, i.e., the sign degree is 1, the homological degree is -1 ($d = 0$ in homological degree zero) and d preserves weight.

An ordinary algebra A without any gradings may be seen as a DG-algebra concentrated in degree zero (for all gradings) and with differential zero.

The homology $\mathrm{H}(A)$ of a DG-algebra (A,d) is a multiply weighted algebra.

A DG-algebra is said to be connected if the only elements of weight $(0,0,\dots,0)$ are the elements in k.

A $\mathbb{Z}_2 \times \mathbb{N}$-graded algebra may be seen as a DG-algebra in two ways:

- A is concentrated in homological degree zero and the $\mathbb{N}$-grading is considered as a weight.
- The $\mathbb{N}$-grading is considered as the homological degree (and the weight is zero).

A homomorphism $f : (A, d_A) \to (B, d_B)$ is an algebra homomorphism which preserves all degrees and $f \circ d_A = d_B \circ f$.

A quasi-homomorphism (quism for short) is a homomorphism such that $\mathrm{H}(f)$ is an isomorphism.

A surjective quism $f : (T(V), d) \to (A, d_A)$ is said to be a model for (A, d_A).

A model $(T(V), d)$ is also called a free DG-algebra and it is called minimal if the image of d restricted to V is contained in $\oplus_{n\geq 2} V^{\otimes n}$.

We will prove that any connected algebra concentrated in homological degree zero has a minimal model (it is true more generally if the quism is not required to be surjective, see Baues-Lemaire, Minimal models in homotopy theory, Math. Annalen, 225, pp. 219–242).

For two augmented k-algebras A and B, a new algebra $A * B$ is defined (free product or coproduct). Suppose $\{a_i\}$ is a k-basis for A^+ (the augmentation ideal of A) and $\{b_i\}$ a k-basis for B^+. Then a k-basis for $(A * B)^+$ is given by $(c_0, c_1, \dots, c_n)$, $n \geq 0$, where each c is alternating an a or a b. Multiplication is given by concatenation followed, if necessary, by multiplication of two adjacent a or b. We have that $T(V_1 \oplus V_2) = T(V_1) * T(V_2)$.

If (A, d_A) and (B, d_B) are DG-algebras, then the free product of them is defined as $(A * B, d)$, where d is the unique derivation which extends d_A and d_B. By induction over the word length, it is easily proven that $d^2 = 0$. The degree of a "word" in $A * B$ is defined by adding the degrees of the "letters".

A DG-algebra (R, d) is called a free extension of (A, d_A) if R as an algebra is $A * T(V)$ and the restriction of d to A is d_A. Such a DG-algebra is written $(A\langle V\rangle, d)$. Observe that this is not in general the free product of (A, d_A) and a free DG-algebra.

There is also the notion of a relative model for a homomorphism $f : (A, d_A) \to (B, d_B)$. This is a free extension $(A\langle V\rangle, d)$ of (A, d_A) together with a map $(A\langle V\rangle, d) \to (B, d_B)$ which is a surjective quism and the restriction to A is f.

There is a general theory called "Model categories" (see Quillen, Homotopical algebra, SLN 43), which has DG-algebras as one application. In the theory there are three groups of maps, called "fibrations", "cofibrations" and "weak equivalences" which satisfy certain axioms. In our situation "fibrations" are surjections, "cofibrations" are free extensions and "weak equivalences" are quisms. We will not go through all details in the theory, but only prove what is necessary for us.

We will now prove the existence of a minimal model in the case we are interested in. We begin with a lemma, called Nakayama's lemma in the commutative case.

Lemma 5.1 *Suppose $A = k \oplus I$ is a connected $\mathbb{N}$-weighted k-algebra and M a $\mathbb{Z}$-weighted A-module with $M_n = 0$ for $n << 0$* or *(A, I) is a commutative local noetherian ring with maximal ideal I and M a finiteley generated A-module. Then*

1) $IM = M \Rightarrow M = 0$
2) *Suppose $f : M \to N$ is a A-module homomorphism (N has the same properties as M) and that the induced map $\overline{f} : M/IM \to N/IN$ is an isomorphism. Then f is surjective and* $\ker(f) \subset IM$.

Proof. (The graded case.) Let $a \in M_n$ and suppose $M_r = 0$ for $r < n$. Since $a \in IM$ and I has positive weight, we have that $a = \sum i_r m_r$, where $m_r \in \oplus_{k<n} M_k = 0$. Hence $a = 0$. By induction, $M = 0$. (In the local case one uses the fact that a matrix is invertible if the diagonal elements are of the form $1+x$, $x \in I$ while the elements outside the diagonal $\in I$.)

2) follows from 1). We have that $M \to N \to C \to 0$ exact implies that

$$M/IM \to N/IN \to C/IC \to 0$$

is exact. The assumptions give $C/IC = 0$ and by 1), $C = 0$. Furthermore if $f(x) = 0$, then $\overline{f}(\overline{x}) = 0$ and hence, by the assumptions, $\overline{x} = 0$, i.e., $x \in IM$. □

Proposition 5.2 *Let A be a connected k-algebra concentrated in homological degree zero. Then there is a minimal model $\pi : (T(V), d) \to (A, 0)$.*

Proof.
Step 0

Let $A = k \oplus I$ and define $V_0 = I/I^2$ as a space with sign degree, weight and homological degree (=0). Choose a homogeneous basis for I/I^2 and define a map $\pi : V_0 \to I$ by choosing a homogeneous representative in I for each basis element. The map π composed with the projection $I \to I/I^2$ is the identity and π preserves all degrees. Extend π to an algebra homomorphism $T(V_0) \to A$. Also define a differential d on $T(V_0)$ as zero. Then $\pi : (T(V_0), 0) \to (A, 0)$ is a map of DG-algebras. I claim that π is surjective and that $\ker(\pi) \subset \oplus_{n\geq 2} V_0^{\otimes n}$. This is proved in analogous way to the proof of Nakayama's lemma above. We have that π maps k isomorphically to k and hence π is surjective in weight 0. Suppose $n \geq 1$ and π is surjective in weight $< n$ and suppose $a \in A$ is of weight n. Then $a \in I$ and there is $v \in V_0$ such that $\pi(v) - a \in I^2$. Since the elements of I has positive weight, it follows that $\pi(v) - a = \sum x_i y_i$ where the weights of x_i and y_i are $< n$. Hence x_i and y_i are in $\mathrm{im}(\pi)$ and it follows that $\pi(v) - a$ is in $\mathrm{im}(\pi)$ and finally $a \in \mathrm{im}(\pi)$. Hence, by induction, π is surjective. Suppose $a \in \ker(\pi)$. Then a has positive weight and hence $a = v_1 + v_2 + \ldots$ where $v_i \in V_0^{\otimes i}$ for all i. We have that $\pi(v_2 + \ldots) \in I^2$ and hence, since $\pi(a) = 0$, $\pi(v_1) \in I^2$. Hence the projection of $\pi(v_1)$ to I/I^2 gives zero and at the same time it gives v_1. Hence $a \in \oplus_{n\geq 2} V_0^{\otimes n}$ as claimed.

Step 1

Define $M = \ker(\pi)/V_0 \ker(\pi) + \ker(\pi)V_0$ and $V_1 = s(M)$, where s stands for the sign-shift functor, i.e., $V_1 = \{sv; v \in M\}$ and $|sv| = |v| + 1$ and the weight is unchanged. By abuse of language, we use the same symbol s for the functor $V \mapsto s(V)$ and for the natural transformation $V \to s(V)$ defined on V by the map of sign degree 1, $v \mapsto sv$. The homological degree of the elements in V_1 are defined to be 1 and π is defined to be zero on V_1 and is extended as an algebra map to $\pi : T(V_0 \oplus V_1) \to A$. The differential d is defined on V_1 by composing the map $s(M) \to M$ with a homogeneous section to the projection $\ker(\pi) \to M$. This defines a map $d : V_1 \to T(V_0)$ of sign degree 1, homological degree -1 and weight 0 and it is extended as a derivation to $T(V_0 \oplus V_1)$. This extension satisfies $d^2 = 0$, which is proved easily by induction on the homological degree. Observe that it is essential that d has sign degree 1 in this proof. Since the image of d is in $\ker(\pi)$ we have that $\pi : (T(V_0 \oplus V_1), d) \to (A, 0)$ is a map of DG-algebras. Now apply Nakayama's lemma above to the algebra $T(V_0) \otimes T(V_0)^{op}$ and the $T(V_0)$-bimodule map $d : T(V_0) \otimes V_1 \otimes T(V_0) \to \ker(\pi)$ to get that $\mathrm{im}(d) = \ker(\pi)$ and

$$ker(d) \subset (\oplus_{n\geq 1} V_0^{\otimes n}) \otimes V_1 \otimes T(V_0) + T(V_0) \otimes V_1 \otimes (\oplus_{n\geq 1} V_0^{\otimes n}) \subset \oplus_{n\geq 2} (V_0 \oplus V_1)^{\otimes n}$$

Step n

Let $n \geq 2$ and assume inductively $V_0, V_1, \ldots, V_{n-1}$ have been constructed, where V_i has homological degree i, and put $W = \oplus_{i=0}^{n-1} V_i$. Assume also that $\pi : R = (T(W), d) \to (A, 0)$ is a surjective map of DG-algebras, such that $\mathrm{H}_0(\pi)$ is an isomorphism, $\mathrm{H}_i(R) = 0$ for $0 < i \leq n-2$ and the cycles in positive homological degrees $\leq n-1$ are contained in $\oplus_{i\geq 2} W^{\otimes i}$.

We have that $\mathrm{H}_{n-1}(R)$ is a $T(V_0)$-bimodule (since a product of two cycles is a cycle and a product of a cycle and a boundary is a boundary). Put

$$M = \mathrm{H}_{n-1}(R)/(V_0 \mathrm{H}_{n-1}(R) + \mathrm{H}_{n-1}(R) V_0) \quad \text{and} \quad V_n = s(M).$$

The elements of V_n have homological degree n and π is defined to be zero on V_n and is extended as an algebra map to $\pi : T(W) \to A$. The differential d is defined on V_n by composing the map $s(M) \to M$ with a homogeneous section to the projection $\ker(d)_{n-1} \to M$. This defines a map $d : V_n \to T(W)$

of sign degree 1, homological degree -1 and weight 0 and it is extended as a derivation to $T(W \oplus V_n)$. As above, this extension satisfies $d^2 = 0$. We have that $\pi : (T(W \oplus V_n), d) \to (A, 0)$ is a map of DG-algebras. Now apply Nakayama's lemma to the algebra $T(V_0) \otimes T(V_0)^{op}$ and the composition of $T(V_0)$-bimodule maps

$$\alpha : T(V_0) \otimes V_n \otimes T(V_0) \xrightarrow{d} \ker(d)_{n-1} \to \mathrm{H}_{n-1}(R)$$

We get that α is surjective, which implies that $\mathrm{H}_{n-1}(T(W \oplus V_n), d) = 0$. Also, suppose we have a cycle of homological degree n in $(T(W \oplus V_n), d)$. This cycle is of the form $v + x$, where $v \in T(V_0) \otimes V_n \otimes T(V_0)$ and $x \in T(W)$. By degree reason, we have that $x \in \oplus_{i \geq 2} W^{\otimes i}$. Since $v + x$ is a cycle, we have $d(v) = -d(x)$ and hence $d(v)$ is a boundary in R, which implies that $\alpha(v) = 0$. By Nakayama's lemma we get that

$$v \in (\oplus_{i \geq 1} V_0^{\otimes i}) \otimes V_n \otimes T(V_0) + T(V_0) \otimes V_n \otimes (\oplus_{i \geq 1} V_0^{\otimes i}) \subset \oplus_{i \geq 2} (W \oplus V_n)^{\otimes i}$$

Finally, we get that $v + x \in \oplus_{i \geq 2} (W \oplus V_n)^{\otimes i}$. □

Next, we prove that any map of DG-algebras has a relative model.

Proposition 5.3 *Any homomorphism $f : (A, d_A) \to (B, d_B)$ has a relative model.*

Proof. First choose V_0 of homological degree zero and a map $\pi : A\langle V_0 \rangle \to B$ that extends f and is surjective in homological degree zero. The sign degree and weight for the elements in V_0 are defined such that π has sign degree and weight zero. The differential on V_0 is defined as zero. Then $\mathrm{H}_0(\pi)$ is surjective. Suppose inductively that $V_0, V_1, \ldots, V_{n-1}, d$ and π are constructed such that the restriction of d to A is d_A and $\pi : (A\langle \oplus_{i=0}^{n-1} V_i \rangle, d) \to (B, d_B)$ is an extension of f, π is surjective in homological degrees $\leq n-1$, $\mathrm{H}_i(\pi)$ is an isomorphism in homological degrees $< n-1$ and $\mathrm{H}_{n-1}(\pi)$ is surjective. Now define

$$V_n = \{(a, b); a \in (A\langle \oplus_{i=0}^{n-1} V_i \rangle)_{n-1}, b \in B_n, d_B(b) = \pi(a) \text{ and } d(a) = 0\}$$

and put $d(a, b) = a$ and $\pi(a, b) = b$. The homological degree of (a, b) is n. For elements (a, b) which are homogeneous with respect to sign degree and weight, we define $|(a, b)| = |b|$ (or $|a| + 1$ if $b = 0$) and the weight of (a, b) as the weight of a (or b). Then π has degree zero with respect to all degrees and d has sign degree 1, weight zero and homological degree -1. Now extend π as an algebra homomorphism to $A\langle \oplus_{i=0}^{n-1} V_i \rangle \langle V_n \rangle \to B$ and extend d as a derivation on $A\langle \oplus_{i=0}^{n-1} V_i \rangle \langle V_n \rangle$. Then it is easy to prove that $d^2 = 0$ (see also the proof of Proposition 5.2) and that π is a map of DG algebras. It is also easy to prove that π_n and $\mathrm{H}_n(\pi)$ are surjective and that $\mathrm{H}_{n-1}(\pi)$ is injective. □

The proposition may be applied to the case when $A = k$. This gives a model for (B, d_B). The construction in the proof is however far from being minimal.

6 Series and Logarithms

In this section we will get a formula for the series of

$$\overline{\mathrm{HC}}_0(T(V)) = T(V)^+ / [T(V), T(V)]$$

for a positively weighted and graded vector space V. The series for a $\mathbb{N}$-weighted vector space $V = \oplus_{i\geq 0} V_i$ over k, where V_i are finite dimensional for all i, is defined as the formal power series

$$V(z) = \sum_{i\geq 0} \dim_k(V_i) z^i$$

The series for a $\mathbb{Z}_2 \times \mathbb{N}$-weighted vector space

$$V = V_0 \oplus V_1 = \oplus_{i\geq 0}(V_{0i} \oplus V_{1i})$$

is defined as

$$V(z, y) = V_0(z) \oplus yV_1(z)$$

considered as an element in $\mathbb{Z}[[z, y]]/(y^2 - 1)$. We have

$$(V_1 \oplus V_2)(z, y) = V_1(z, y) + V_2(z, y)$$
$$(V_1 \otimes V_2)(z, y) = V_1(z, y) \cdot V_2(z, y)$$

Let P denote the additive group of formal power series

$$\sum_{n\geq 1} a_n z^n + y \sum_{n\geq 1} b_n z^n \in \mathbb{Z}[[z, y]]/(y^2 - 1)$$

and let U denote the multiplicative group $1+P$. The inverse of $1 - \sum_{n\geq 1} a_n z^n - y \sum_{n\geq 1} b_n z^n$ is

$$1 + (\sum_{n\geq 1} a_n z^n + y \sum_{n\geq 1} b_n z^n) + (\sum_{n\geq 1} a_n z^n + y \sum_{n\geq 1} b_n z^n)^2 + \dots$$

(observe that any series in P may be inserted in any formal power series in one variable to get a new well-defined series).

A topology on P and U is defined as follows. We say that $f_n \to f$ if, for any r, the coefficients in f_n and f are the same up to z^r and yz^r.

Instead of assuming that the coefficients in the formal power series are integers, we will also consider the case when the coefficients are rational numbers. In this case we write $P^{\mathbb{Q}}$ and $U^{\mathbb{Q}}$ for the corresponding additive and multiplicative groups.

Definition

A continuous group isomorphism $P \to U$ (or $P^{\mathbb{Q}} \to U^{\mathbb{Q}}$) is called an "exponential" and a continuous isomorphism $U \to P$ (or $U^{\mathbb{Q}} \to P^{\mathbb{Q}}$) is called a "logarithm".

Example

$$\exp(X) = \sum_{k\geq 0} \frac{X^k}{k!}, \quad X \in P^{\mathbb{Q}}, \quad \text{is an exponential } P^{\mathbb{Q}} \to U^{\mathbb{Q}}$$
$$\log(X) = -\sum_{k\geq 1} \frac{(1-X)^k}{k}, \quad X \in U^{\mathbb{Q}}, \quad \text{is a logarithm } U^{\mathbb{Q}} \to P^{\mathbb{Q}}$$

Proof

The usual proof of $e^{x+y} = e^x \cdot e^y$ using formal power series depends only on the binomial theorem and that x and y are commuting symbols. Hence also $\exp(X+Y) = \exp(X) \cdot \exp(Y)$, since $P^{\mathbb{Q}}$ is commutative. The identities $\exp(\log(X)) = X$ and $\log(\exp(X)) = X$ hold, since they hold for convergent series with complex coefficients. Hence exp and log are bijective and log is also a homomorphism and clearly they are continuous.

By means of “log” we get a lot of other logarithms:

Example

Let $c = \{c_k\}_{k=1}^{\infty}$ be any sequence of rational numbers with $c_1 \neq 0$. Then

$$\mathrm{Log}_c(X) = \sum_{k=1}^{\infty} c_k \log(X(z^k, (-1)^{k+1} y^k)), \quad X \in U^{\mathbb{Q}}$$

is a logarithm $U^{\mathbb{Q}} \to P^{\mathbb{Q}}$.

The substitution $(-1)^{k+1} y^k$ for y in the formula will be useful later. To get a logarithm we could also have used just y instead. The proof of the homomorphism law is evident. The inverse of Log_c is given by

$$\mathrm{Exp}_{c'}(X) = \prod_{k=1}^{\infty} (\exp(X(z^k, (-1)^{k+1} y^k)))^{c'}$$

where

$$(\sum_{k=1}^{\infty} c'_k z^k) \cdot (\sum_{k=1}^{\infty} c_k z^k) = z$$

The series for the symmetric algebra $S(X)$ of a graded and weighted vector space $X = \oplus_{i \geq 1} X_i$ gives an exponential $P \to U$:

$$S(X) = \prod_{k=1}^{\infty} \frac{(1+yz^k)^{b_k}}{(1-z^k)^{a_k}}, \quad X = \sum_{i=1}^{\infty} a_k z^k + y \sum_{i=1}^{\infty} b_k z^k$$

(By means of the Taylor expansion of $(1+z)^{\alpha}$ the operator S extends to an exponential $P^{\mathbb{Q}} \to U^{\mathbb{Q}}$.)

It is evident that S is a homomorphism and that it is continuous. To prove bijectivity, suppose $f \in U$ and suppose inductively that there are uniquely determined $a_1, b_1, \ldots, a_{k-1}, b_{k-1} \in \mathbb{Z}$ such that

$$\prod_{j=1}^{k-1} \frac{(1+yz^j)^{b_j}}{(1-z^j)^{a_j}} \equiv f(z,y) \mod z^k$$

Hence

$$\prod_{j=1}^{k-1} \frac{(1-z^j)^{a_j}}{(1+yz^j)^{b_j}} f(z,y) = 1 + a_k z^k + b_k y z^k + \text{ higher terms}$$

and hence

$$\frac{(1-z^k)^a}{(1+yz^k)^b} \prod_{j=1}^{k-1} \frac{(1-z^j)^{a_j}}{(1+yz^j)^{b_j}} f(z,y) \equiv 1 \mod z^{k+1} \Leftrightarrow a = a_k \text{ and } b = b_k$$

The inverse of S is a logarithm called Lie, since $\mathrm{Lie}(X)$ gives the series for the Lie algebra whose envelopping algebra has series X. In particular $\mathrm{Lie}(\frac{1}{1-X})$ gives the series for the free Lie algebra on a graded and weighted vector space with series X.

We will now determine c, such that $\mathrm{Lie} = \mathrm{Log}_c$. In fact, it is enough to determine c such that $\mathrm{Lie}(X) = \mathrm{Log}_c(X)$ for $X = 1 - z$ and $X = 1 + yz$. This follows from the fact that S is surjective and both Lie and Log_c are logarithms which commute with the substitution $z \to z^k$. Now $\mathrm{Lie}(\frac{1}{1-z}) = z$ and $\mathrm{Lie}(1 + yz) = yz$ and

$$-\mathrm{Log}_c(1-z) = -\sum_{k=1}^{\infty} c_k \log(1-z^k) = \sum_{j,k=1}^{\infty} c_k \frac{z^{kj}}{j} = \sum_{n=1}^{\infty} (\sum_{k|n} kc_k) \frac{z^n}{n}$$

Hence, $\mathrm{Lie}(1-z) = \mathrm{Log}_c(1-z)$ if $c_1 = 1$ and $\sum_{k|n} kc_k = 0$ for $n > 1$. Möbius inversion formula gives that the solution is given by $kc_k = \mu(k)$. By definition of Log_c we have

$$\begin{aligned}\mathrm{Log}_c(1+yz) &= \sum_{k=1}^{\infty} c_k \log(1 + (-1)^{k+1} y^k z^k) \\ &= \sum_{k=1}^{\infty} c_k \log(1 - (-yz)^k) = \mathrm{Log}_c(1-z)\big|_{z=-yz}\end{aligned}$$

Since also $\mathrm{Lie}(1+yz) = yz = \mathrm{Lie}(1-z)\big|_{z=-yz}$ we get that

$$\mathrm{Lie}(X) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log(X(z^k, (-1)^{k+1} y^k))$$

for all $X \in U$.

From the above it follows that any logarithm L is of the form Log_c for a unique c if the following conditions hold:

- L commutes with the substitution $z \to z^k$ for any k
- $L(1+yz) = L(1-z)\big|_{z=-yz}$

If we apply the formula for Lie on $1 - dz$ we get the well-known series for the free Lie algebra on d even generators:

$$\sum_{n=1}^{\infty} (\frac{1}{n} \sum_{j|n} \mu(\frac{n}{j}) d^j) z^n$$