Mathematical Statistics
Stockholm University

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Research Report 2005:3

ISSN 1650-0377

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Mathematical Statistics
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Research Report **2005:3**,
http://www.math.su.se/matstat

A renewal process type expression for the moments of inverse subordinators

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March 2005

Abstract

We define an inverse subordinator as the passage times of a subordinator to increasing levels. It has previously been noted that such processes have many similarities with renewal processes. Here we present an expression for the joint moments of the increments of an inverse subordinator. This is an analogue of a result for renewal processes. The main tool is a theorem on the processes which are both renewal processes and Cox processes.

KEY WORDS: Subordinator; Passage time; Renewal theory; Cox process; Local time
AMS Classification: 60K05; 60G51; 60G55; 60E07

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1 Introduction

Subordinators are nondecreasing processes with independent and stationary increments. The corresponding processes in discrete time are the partial-sum processes with positive, independent and identically distributed summands. Renewal processes can be considered to be passage times of partial-sum processes to increasing levels. Analogously we can define a process by the passage times of a subordinator. We call such a process an inverse subordinator.

The inverse subordinators appear in diverse areas of probability theory: As Bertoin [2] notes, the local times of a large class of well-behaved Markov processes are really inverse subordinators, and any inverse subordinator is the local time of some Markov process. It is well-known, see Karatzas and Shreve [9], that the local time of the Brownian motion is the inverse of a 1/2-stable subordinator. Inverses of α-stable subordinators with $0 < \alpha < 1$ arise as limiting processes of occupation times of Markov processes, see Bingham [4]. Some recent applications of inverse subordinators in stochastic models can be found in [8], [11] and [14]. Kaj and Martin-Löf [8] consider superposition and scaling of inverse subordinators with applications in queueing theory, Kozlova and Salminen [11] uses diffusion local time as input in a so-called storage process and Winkel [14] uses inverse subordinators in financial modelling.

In this paper we study some general distributional properties of inverse subordinators, using renewal theory and some theory about Cox processes. In particular we find an expression for the joint moments of their increments. Other results for inverse subordinators analogous to those in renewal theory has been proved by Bertoin, van Harn and Steutel, see [3] and [7].

Some well-known results on subordinators and infinitely divisible distributions on the positive real line are given in section 2 of this paper. Section 3 introduces the inverse subordinators and hints that they may have properties similar to the renewal processes. In section 4 the main result is given: An expression for the joint moments of the increments of an inverse subordinator. This is proved using a representation of the class of point processes that are both Cox processes and renewal processes. With this representation one can also give an alternative proof of the fact that inverse subordinators can be delayed to be given stationary increments, see [7]. We also provide a bound of the upper tail of the marginal distribution of an inverse subordinator. Finally, section 5 examplifies the results with three types of inverse subordinators.

2 Some basic facts about subordinators

The following results on infinitely divisible distributions and Lévy processes can be found in [13]. Let $\{Y_t\}$ be a Lévy process, i.e. a stochastic process in continuous time with $Y_0 = 0$ and stationary and independent increments. The distribution F of Y_1 is necessarily infinitely divisible, i.e. for all $n \in \mathbb{N}$ there is a distribution F_n such that $F_n^{\star n} = F$. Here $F_n^{\star n}$ is the n-fold convolution of F_n. The converse is also true: Given an infinitely divisible distribution F there is a Lévy process $\{Y_t\}$ such that the distribution of Y_1 is F. Define $F^{\star t}$ for positive, non-integer t by $F^{\star t}(x) = P(Y_t \leq x)$. One recognizes that $F_n = F^{\star 1/n}$.

If one restricts F to be a distribution on $\mathbb{R}_+$ then the increments of $\{Y_t\}$ are all non-negative. Lévy processes with non-negative increments are called subordinators. It is well-known that the Laplace-Stieltjes transform of $F^{\star t}$, where F is an infinitely divisible distribution on $\mathbb{R}_+$, can be written

$$\widehat{F^{\star t}}(u) = \int_0^\infty e^{-ux} F^{\star t}(dx) = e^{-t\psi(u)} = \widehat{F}(u)^t,$$

where $\psi(u)$ is called the Lévy exponent. It can be written in the following form

$$\psi(u) = \delta u + \int_0^\infty (1 - e^{-ux})\nu(dx),$$

where $\delta \geq 0$ is called the drift and $\nu(dx)$ is called the Lévy measure. If Y_1 has drift δ then $Y_1 - \delta$ has drift 0. If $\int_0^\infty \nu(dx) < \infty$ then $\{Y_t\}$ is a compound Poisson process, with drift if $\delta > 0$, and thus only makes a finite number of jumps in any finite interval. We call a function π the Lévy density if $\nu(A) = \int_A \pi(x)dx$. If we define $\mu = E[Y_1]$, then $\mu = \delta + \int_0^\infty x\nu(dx)$. Since $\psi'(u) = \delta + \int_0^\infty e^{-ux} x\nu(dx)$, we have

$$\psi'(0) = \mu \text{ and } \psi'(u) \searrow \delta \text{ as } u \nearrow \infty. \tag{1}$$

Some parts of the reasoning in the following sections do not apply to compound Poisson processes without drift. Therefore we will henceforth, albeit somewhat artificially, exclude the compound Poisson processes without drift when referring to subordinators.

3 Inverse subordinators and renewal processes

It is advantageous to recall some results on renewal processes before a more thorough study of subordinators and their inverses. Let $X_2, X_3, \dots$ be a sequence of independent and identically distributed (strictly) positive random variables with distribution F, and X_1 a positive random variable with distribution H, independent of $X_2, X_3, \dots$. Let $S_0 = 0$ and $S_n = \sum_{k=1}^n X_k$, and we call $\{S_n\}$ a partial-sum process. Given a partial-sum process we define the renewal process with interarrival distribution F by $N_t = \min(n \in \mathbb{N} : S_n > t) - 1$. The -1 in the definition comes from the fact that we do not want to count the renewal at the origin, as is sometimes done. If $F = H$ then $\{N_t\}$ is called an ordinary renewal process.

It is well-known that $\{N_t\}$ has stationary increments if and only if $H(x) = \frac{1}{\mu}\int_0^x (1 - F(y))dy$, where $\mu = E[X_2] = \int_0^\infty (1 - F(x))dx < \infty$, see [5]. Then one also has

$$E[X_1] = \frac{E[X_2^2]}{2\mu}, \tag{2}$$

and the Laplace-Stieltjes transform of H is

$$\widehat{H}(s) = \frac{1}{\mu s}(1 - \widehat{F}(s)). \tag{3}$$

We note, as in [3], that subordinators are continuous time analogues of partial-sum processes. A Lévy process sampled at equidistant time points does produce a partial-sum process with infinitely divisible F, e.g. $Y_n = \sum_{k=1}^n (Y_k - Y_{k-1})$, when the time points are the integers. As the renewal processes are integer valued inverses to partial-sum processes, an inverse of a subordinator could be expected to have some properties similar to renewal processes. Given a subordinator $\{Y_t\}$, we define $\tau_t = \inf(\tau > 0 : Y_\tau > t)$, and call the process $\{\tau_t\}_{t\geq 0}$ the inverse subordinator.

The properties of the paths of $\{\tau_t\}$ differ depending on $\{Y_t\}$. When $\{Y_t\}$ is a compound Poisson process with drift $\delta > 0$, then $\{\tau_t\}$ alternates between linear increasing with slope $\frac{1}{\delta}$ for exponential periods of time and being constant for periods of time with lengths drawn from the compounding distribution, with all these periods having independent lengths. When $\{Y_t\}$ is not compound Poisson and the drift is zero, then the trajectories of $\{\tau_t\}$ are continuous singular almost surely.

Now we will show that $\{\tau_t\}$ can be arbitrarily closely approximated by a scaled renewal process. For any $c > 0$, let $\{Y_t^c\}$ be defined by $Y_t^c = Y_{t/c}$. Note that $\{Y_t^c\}$ is a subordinator with $Y_1^c \sim F^{\star 1/c}$. Also define the renewal process $N_t^c = \min(n \in \mathbb{N} : Y_n^c > t) - 1$. Since

$$\begin{aligned} c\tau_t &= c \inf(\tau > 0 : Y_\tau > t) \\ &= \inf(\tau > 0 : Y_{\tau/c} > t) \\ &= \inf(\tau > 0 : Y_\tau^c > t) \\ &\geq \min(n \in \mathbb{N} : Y_n^c > t) - 1 \\ &\geq \inf(\tau > 0 : Y_\tau^c > t) - 1 \\ &= c\tau_t - 1, \end{aligned}$$

$$\tau_t = \tfrac{1}{c} N_t^c + r_t, \text{ where } 0 \leq r_t \leq \tfrac{1}{c},$$

and the approximation becomes arbitrarily good as $c \to \infty$. This result suggests that the inverse subordinators may have some properties similar to renewal processes. That this is in fact true will be shown in the following section.

An important function in the theory of renewal processes is the so called renewal function $V(t) = E[N_t]$. We note that for an ordinary renewal process $V(t) = \sum_{k=1}^{\infty} F^{\star k}(t)$, and for a stationary renewal process $V(t) = \frac{t}{\mu}$. If there is a function v such that $V(t) = \int_0^t v(s)ds$, then v is called the renewal density. If the renewal process would have been defined to also count the renewal at the origin, then the renewal function would be $V(t)+1$. One can also define a renewal function for the inverse subordinator. Given an inverse subordinator $\{\tau_t\}$, we define its renewal function U by $U(t) = E[\tau_t]$. The renewal function can be expressed as follows:

$$U(t) = E[\tau_t] = \int_0^\infty P(\tau_t > x)dx = \int_0^\infty P(Y_x \leq t)dx = \int_0^\infty F^{\star x}(t)dx$$

The expression on the right hand side might be hard to evaluate, but its Laplace-Stieltjes transform is easily calculated:

$$\begin{aligned} \widehat{U}(s) &= \int_0^\infty e^{-st} \int_0^\infty F^{\star x}(dt)dx = \int_0^\infty \widehat{F}(s)^x dx \\ &= \int_0^\infty e^{-x\psi(s)}dx = \frac{1}{\psi(s)} \end{aligned} \tag{4}$$

Thus there is a one-to-one correspondence between the renewal function and the distribution of $\{\tau_t\}$. This also correlates with the similar result for ordinary renewal processes and their renewal functions. Define the factorial power $n^{[k]}$ for $n, k \in \mathbb{N}$ by:

$$n^{[k]} = \begin{cases} n(n-1)\cdots(n-k+1) & \text{for } n \geq k \geq 1 \\ 1 & \text{for } k = 0 \\ 0 & \text{for } n < k, k \geq 1. \end{cases}$$

Note that $n^{[k]} = \#\{(a_1, \ldots, a_k) \in \{1, \ldots, n\}^k; a_r \neq a_s, r \neq s\}$, i.e. the number of k-tuples with integer values from 1 to n such that no coordinates are the same. We will call expressions such as $E[N^{[k]}]$ factorial moments instead of the ordinary moments $E[N^k]$, for integer valued random variables N. Given a renewal process and its renewal function, moments of all orders can be calculated as stated in the following proposition, see [5].

Proposition 1 Let $\{N_t\}$ be a renewal process with interarrival distribution F and let $V(t) = \sum_{k=1}^{\infty} F^{\star k}(t)$. If $\{N_t\}$ is an ordinary renewal process then, for $0 \leq s_1 < t_1 \leq s_2 < \cdots < t_n$ and $k_1, \ldots, k_n \in \mathbb{N} \setminus \{0\}$ such that $k_1 + \cdots + k_n = k$,

$$E\Big[\prod_{i=1}^{n} (N_{t_i} - N_{s_i})^{[k_i]}\Big] = \prod_{i=1}^{n} k_i! \cdot \int\limits_{C} \prod_{j=1}^{k} V(dx_j - x_{j-1}), \tag{5}$$

where $C = \{x_0, \ldots, x_k; x_0 = 0, s_i < x_{k_0+\cdots+k_{i-1}+1} < \cdots < x_{k_0+\cdots+k_i} \leq t_i, i = 1, \ldots, n, k_0 = 0\}$. If $\{N_t\}$ is stationary, then the proposition also holds with the first factor of the rightmost product in equation (5) replaced by $\frac{dx_1}{\mu}$.

A sketch of a proof:

$$\begin{aligned} E\Big[\prod_{i=1}^{k} N(dx_i)\Big] &= P(N(dx_1) = 1, \ldots, N(dx_k) = 1) \\ &= P(N(dx_{(1)}) = 1) \prod_{i=2}^{k} P(N(dx_{(i)}) = 1 | N(dx_{(i-1)}) = 1) \\ &= P(N(dx_{(1)}) = 1) \prod_{i=2}^{k} V(dx_{(i)} - x_{(i-1)}), \end{aligned}$$

and the first factor equals $V(dx_{(1)})$ and $\frac{dx_{(1)}}{\mu}$ in the ordinary and stationary case, respectively. Let $A_i = \{(y_{i1}, \ldots, y_{ik_i}) \in (s_i, t_i]^{k_i}; y_{ir} \neq y_{is} \text{ for } r \neq s\}$ and $B_i = \{(y_{i1}, \ldots, y_{ik_i}); s_i < y_{i1} < \cdots < y_{ik_i} \leq t_i\}$. Thus, in the ordinary case,

$$\begin{aligned} E\Big[\prod_{i=1}^{n} (N_{t_i} - N_{s_i})^{[k_i]}\Big] &= E\Big[\prod_{i=1}^{n} \int_{A_i} \prod_{j=1}^{k_i} N(dy_{ij})\Big] \\ &= \prod_{i=1}^{n} k_i! \cdot E\Big[\prod_{i=1}^{n} \int_{B_i} \prod_{j=1}^{k_i} N(dy_{ij})\Big] \\ &= \prod_{i=1}^{n} k_i! \cdot \int_{C} \prod_{l=1}^{k} V(dx_l - x_{l-1}). \end{aligned}$$

4 Inverse subordinators and Cox processes

An expression similar to (5) for the moments of $\{\tau_t\}$ can be obtained. First recall the definition of a Cox process. Let $\{N_t^\lambda\}$ be an inhomogeneous Poisson process on $\mathbb{R}_+$ with intensity measure λ. Let Λ be a random measure on $\mathbb{R}_+$. If the point process $\{M_t\}$ has the distribution of $\{N_t^\lambda\}$ conditional on $\Lambda = \lambda$, then $\{M_t\}$ is called a Cox process directed by Λ.

Also define a slight generalisation of the inverse subordinators: Let $\widetilde{Y}_0$ have the distribution G on $\mathbb{R}_+$ and be independent of the subordinator $\{Y_t\}$ with $Y_1 \sim F$. Define the process $\{\widetilde{Y}_t\}$ by $\widetilde{Y}_t = Y_t + \widetilde{Y}_0$. Let $\tau_t = \inf(\tau > 0 : \widetilde{Y}_\tau > t)$, and call the process $\{\tau_t\}_{t\geq 0}$ a general inverse subordinator. If $\widetilde{Y}_0 \equiv 0$ then we call $\{\tau_t\}$ an ordinary inverse subordinator.

We will see in Proposition 4 that $\widetilde{Y}_0$ can be chosen so that the generalised inverse subordinator $\{\tau_t\}$ has stationary increments, if $\mu = E[Y_1] < \infty$. The following proposition is by Kingman [10] and Grandell [6].

Proposition 2 The Cox process $\{M_t\}$ directed by Λ is a renewal process if and only if $\Lambda((s,t]) = \tau_t - \tau_s$ for all $t > s$, where $\{\tau_t\}$ is a general inverse subordinator.

The proof will not be reproduced here, but it is worth noticing that the interarrival distribution in such a Cox process that is also a renewal processes is of compound-exponential type: If Z is the length of an interval in this renewal process and $\varepsilon \sim \text{Exp}(1)$, then $Z \stackrel{d}{=} Y_\varepsilon$ for the subordinator $\{Y_t\}$ corresponding to $\{\tau_t\}$. The Laplace-Stieltjes transform of the distribution of Z is given by

$$\widehat{F}_Z(s) = E[e^{-sZ}] = E[E[e^{-sY_\varepsilon}|\varepsilon]] = E[e^{-\varepsilon\psi(s)}] = \frac{1}{1+\psi(s)}, \tag{6}$$

where $\psi(s)$ is the Lévy exponent of Y_1. We now have the tools to prove the main result:

Theorem 1 Let $\{\tau_t\}$ be an ordinary inverse subordinator with renewal function $U(t)$. Then, for $0 \leq s_1 < t_1 \leq s_2 < \cdots < t_n$ and $k_1, \ldots, k_n \in \mathbb{N} \setminus \{0\}$ such that $k_1 + \cdots + k_n = k$,

$$E\Big[\prod_{i=1}^{n}(\tau_{t_i} - \tau_{s_i})^{k_i}\Big] = \prod_{i=1}^{n} k_i! \cdot \int_C \prod_{j=1}^{k} U(dx_j - x_{j-1}) \tag{7}$$

where C is as in Proposition 1. If $\{\tau_t\}$ is stationary, then the theorem also holds with the change that the first factor of the rightmost product in equation (7) is replaced by $\frac{dx_1}{\mu}$, but with the same U in the remaining factors as the ordinary inverse subordinator.

Proof. Define the random measure Λ on $\mathbb{R}_+$ by $\Lambda((s,t]) = \tau_t - \tau_s$ for all $t > s \in \mathbb{R}_+$, and let $\{M_t\}$ be the Cox process directed by Λ. By Proposition 2, $\{M_t\}$ is also a renewal process. Write $V(t)$ for its renewal function. Then

$$V(t) = E[M_t] = E[E[M_t|\tau_t]] = E[\tau_t] = U(t). \tag{8}$$

Thus one can replace $V(t)$ by $U(t)$ in (5) when calculating the factorial moments of $\{M_t\}$. As noted in [5], the factorial moments of the Cox process coincide with the ordinary moments of its directing measure, and by the construction of the directing measure the stated result follows.

A renewal theorem for the inverse subordinators can also be given following Bertoin [2], Theorem I.21.

Proposition 3 If $\mu < \infty$, then $U(t) \sim \frac{t}{\mu}$ as $t \to \infty$.

Proof. Let $\{M_t\}$ be a Cox process directed by $\{\tau_t\}$ as in Proposition 2, and $V(t)$ its renewal function. By (8), $V(t) = U(t)$. An application of the renewal theorem for renewal processes, see [5], provides the desired result.

Similar to renewal processes, the inverse subordinators can be delayed to become stationary. This has been proved by different methods in [7] and [8]. We state the result and provide a proof based on the connection with Cox processes.

Proposition 4 Let $\{\tau_t\}$ be a general inverse subordinator with $\widetilde{Y}_0 \sim G$ and $Y_1 = \widetilde{Y}_1 - \widetilde{Y}_0 \sim F$ and $\mu = E[Y_1] < \infty$, where

$$\psi(s) = -\log \widehat{F}(s) = \delta s + \int_0^\infty (1 - e^{-sx})\nu(dx) \text{ and}$$

$$G(x) = \begin{cases} \frac{1}{\mu}\left(\delta + \int_0^x \int_y^\infty \nu(dz)dy\right) & \text{for } x \geq 0 \\ 0 & \text{for } x < 0. \end{cases} \tag{9}$$

Then $\{\tau_t\}$ has stationary increments.

Proof. By Theorem 1.4 in [6], a Cox process is stationary if and only if its directing measure Λ has stationary increments. Therefore it suffices to check that the Cox process $\{M_t\}$ directed by $\{\tau_t\}$ is stationary. Its interarrival distribution is F_Z given by (6). The X_1 of $\{M_t\}$ can be decomposed into $X_1 \stackrel{d}{=} \widetilde{Y}_0 + Z$, with $\widetilde{Y}_0$ and Z independent, since the inverse subordinator is delayed a time $\widetilde{Y}_0$ during which it is constant equal to 0. The Laplace-Stieltjes transform of the distribution H of X_1 is $\widehat{H}(s) = \widehat{G}(s)\widehat{F}_Z(s)$, where

$$\begin{aligned}\widehat{G}(s) &= \frac{1}{\mu}\int_0^\infty e^{-sx}\left(\delta + \int_x^\infty \nu(dy)\right)dx = \frac{1}{\mu}\left(\frac{\delta}{s} + \int_0^\infty \int_0^y e^{-sx} dx \nu(dy)\right) \\ &= \frac{1}{\mu s}\left(\delta + \int_0^\infty (1 - e^{-sy})\nu(dy)\right) = \frac{\psi(s)}{\mu s}.\end{aligned} \tag{10}$$

Combining (6) and (10), we get

$$\widehat{H}(s) = \widehat{G}(s)\widehat{F}_Z(s) = \frac{\psi(s)}{\mu s}\frac{1}{1+\psi(s)} = \frac{1}{\mu s}(1-\widehat{F}_Z(s)).$$

By (3), X_1 thus have the right distribution to make $\{M_t\}$ stationary.

Let W_t be the excess of the renewal process and Cox process $\{M_t\}$, i.e. the time from t to the next point of the process. When $\{M_t\}$ is stationary, $W_t \stackrel{d}{=} X_1 \stackrel{d}{=} \widetilde{Y}_0 + Z$. The decomposition of the excess can be given the following interpretation: From any given time t the inverse subordinator will remain constant a period which has the distribution G. During this time no points in the Cox process will occur. After that time the inverse subordinator starts anew and the distribution to the next point in the point process is given by F_Z. In the stationary case, we do not have to know G explicitly to calculate $E[\widetilde{Y}_0]$, if we use (2): $E[X_1] = \frac{E[Z^2]}{2E[Z]}$. $E[X_1] = E[\widetilde{Y}_0] + EZ$, and by straightforward calculation, using e.g. (6), $E[Z] = E[Y_1]$ and $E[Z^2] = \mathrm{Var}(Y_1) + 2E[Y_1]^2$. Collecting and rearranging yields $E[\widetilde{Y}_0] = \frac{\mathrm{Var}(Y_1)}{2EY_1}$.

The expression (7) may be hard to use in practice to calculate higher joint moments. Nonetheless the results above show that the covariance of two increments of a stationary inverse subordinator is a simple expression in the renewal function. Let $\{\tau_t\}$ be stationary and let $U(t)$ denote the renewal function of the corresponding ordinary inverse subordinator. Also let $0 < r \le s < t$.

$$\begin{aligned}\mathrm{Cov}(\tau_r, \tau_t - \tau_s) &= E[\tau_r(\tau_t - \tau_s)] - E[\tau_r]E[\tau_t - \tau_s] \\ &= \int_0^r \int_s^t U(dx - y)\frac{dy}{\mu} - \frac{r}{\mu}\frac{t-s}{\mu} \\ &= \frac{1}{\mu}\int_0^r (U(t-y) - U(s-y))dy - \frac{r(t-s)}{\mu^2}.\end{aligned}$$

Now consider the particular case where $r = 1, s = n \ge 1$ and $t = n+1$ and U has a density u, such that $U(t) = \int_0^t u(s)ds$. Also assume, for simplicity, that $\mu = 1$. Then the following approximation can be done:

$$\mathrm{Cov}(\tau_1, \tau_{n+1} - \tau_n) = \int_0^1 (U(n+1-y) - U(n-y))dy - 1 \approx u(n) - 1.$$

Given the distribution of the subordinator $\{Y_t\}$, the distribution of its inverse is given by $P(\tau_t \le x) = P(Y_x > t)$. It may still be hard to find a closed form expression of this distribution function. The tail probabilities for the ordinary inverse subordinator can nonetheless be estimated. Only the case $\delta = 0$ is interesting since if the drift δ is positive then $\{Y_x - \delta x\}$ is non-negative and thus $P(Y_x \leq t) = P(Y_x - \delta x \leq t - \delta x) = 0$ for $x > \frac{t}{\delta}$. Let $s \geq 0$. Then we have that

$$P(\tau_t > x) = P(Y_x \leq t) = P(e^{-sY_x} \geq e^{-st}) \leq \frac{E\left[e^{-sY_x}\right]}{e^{-st}} = e^{st - x\psi(s)}.$$

By (1) the last expression has unique minimum as a function of s. If x is large enough ($x > \frac{t}{\mu}$), the s that minimizes the expression is non-zero and given by $s = \psi'^{-1}(\frac{t}{x})$. Thus, for large enough x,

$$P(\tau_t > x) \leq \exp\left(t\psi'^{-1}(\tfrac{t}{x}) - x\psi(\psi'^{-1}(\tfrac{t}{x}))\right). \tag{11}$$

The following result on the marginal distribution of $\{\tau_t\}$ also deserves mentioning, see [8] and [12] for details. Let $\{\tau_t\}$ and $\{\widetilde{\tau}_t\}$ be the ordinary and stationary inverse subordinator respectively. Let $\varepsilon_s \sim \mathrm{Exp}(s)$, i.e. $E[\varepsilon_s] = \frac{1}{s}$. Then the Laplace-Stieltjes transforms of the distribution of τ_{ε_s} and $\widetilde{\tau}_{\varepsilon_s}$ are given by:

$$\begin{aligned} E[e^{-u\tau_{\varepsilon_s}}] &= 1 - \frac{u}{\psi(s) + u} \\ E[e^{-u\widetilde{\tau}_{\varepsilon_s}}] &= 1 - \frac{\psi(s)}{\mu s}\frac{u}{\psi(s) + u}. \end{aligned}$$

5 Examples

The α-stable distribution on $\mathbb{R}_+$ has Lévy exponent $\psi(s) = s^\alpha$ with $0 < \alpha < 1$. This gives a renewal density $u(t) = 1/(\Gamma(\alpha)t^{1-\alpha})$ for the corresponding inverse stable subordinator by inverting (4). Theorem 1 thus confirms the moment expressions in [4], e.g. equation (18).

The main obstacle to use Theorem 1 is the possible difficulties in finding an expression for the renewal function. It is possible to find the renewal density not only for the inverse stable subordinator, but also for the inverses of subordinators with inverse gaussian and gamma distributed increments. In these two cases it is also possible to delay the processes to obtain stationary versions, which is not possible in the stable case.

For the inverse gaussian distribution, with probability density

$$f(x) = \frac{\delta}{\sqrt{2\pi x^3}} \exp\left(\delta\gamma - \frac{1}{2}\left(\frac{\delta^2}{x} + \gamma^2 x\right)\right), \delta > 0, \gamma > 0,$$

and Lévy exponent and Lévy density, repectively,

$$\psi(s) = \delta\sqrt{\gamma^2 + 2s} - \delta\gamma$$
$$\pi(x) = \frac{\delta}{\sqrt{2\pi x^3}} \exp\left(-\frac{\gamma^2 x}{2}\right),$$

we get a probability density of the delay $\widetilde{Y}_0$ by integrating π ($\mu = \psi'(0) = \frac{\delta}{\gamma}$)

$$g(t) = \frac{1}{\mu}\int_t^\infty \pi(x)dx = \gamma\sqrt{\frac{2}{\pi t}} \exp\left(-\frac{\gamma^2 t}{2}\right) - \gamma^2 \operatorname{erfc}\left(\gamma\sqrt{\frac{t}{2}}\right)$$

Here erfc is the complementary error function defined by $\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}}\int_t^\infty \exp(-s^2)ds$. We note that the density does not depend on the parameter δ. One obtains the renewal density $u(t)$ from its Laplace transform by rewriting (4):

$$\begin{aligned}
\widehat{u}(s) &= \frac{1}{\psi(s)} = \frac{1}{\delta\sqrt{\gamma^2 + 2s} - \delta\gamma} \\
&= \frac{\gamma}{2\delta s} + \frac{1}{\delta\sqrt{\gamma^2 + 2s}} + \frac{\gamma^2}{2\delta s\sqrt{\gamma^2 + 2s}} \\
&\Rightarrow \{\text{by [1] (29.3.1), (29.3.11) and (29.3.44)}\} \\
u(t) &= \frac{\gamma}{\delta} + \frac{1}{\delta\sqrt{2\pi t}} \exp\left(-\frac{\gamma^2 t}{2}\right) - \frac{\gamma}{2\delta}\operatorname{erfc}\left(\gamma\sqrt{\frac{t}{2}}\right)
\end{aligned}$$

The estimate (11) gives

$$P(\tau_t > x) \leq \exp\left(-\frac{\delta^2 x^2}{2t} + \delta\gamma x - \frac{\gamma^2 t}{2}\right).$$

For the gamma distribution we have probability density, Lévy exponent and Lévy density:

$$f(x) = \frac{\alpha^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\alpha x}, \nu > 0, \alpha > 0$$
$$\psi(s) = \nu \log\left(1 + \frac{s}{\alpha}\right)$$
$$\pi(x) = \frac{\nu}{x} e^{-\alpha x}$$

so the density of the delay is

$$g(t) = \alpha \mathrm{E}_1(\alpha t),$$

where E_1 the exponential integral defined by $\mathrm{E}_1(t) = \int_t^\infty \exp(-s)\frac{ds}{s}$. As in the inverse gaussian case the density only depends on one parameter. The renewal density is also in the gamma case most easily obtained by first rewriting (4):

$$\begin{aligned}
\widehat{u}(s) &= \frac{1}{\nu \log(1+\frac{s}{\alpha})} \\
&= \frac{\alpha}{\nu s}\int_0^1 \left(1+\frac{s}{\alpha}\right)^u du \\
&= \frac{\alpha}{\nu}\int_0^1 \left(\frac{1}{s}\frac{1}{(1+\frac{s}{\alpha})^{1-u}} + \frac{1}{\alpha}\frac{1}{(1+\frac{s}{\alpha})^{1-u}}\right) du \\
&\Rightarrow \{\text{by [1], (29.3.11), (29.2.6) and (6.5.2)}\} \\
u(t) &= \frac{\alpha}{\nu}\int_0^1 \frac{du}{\Gamma(u)}\left(\gamma(u,\alpha t) + (\alpha t)^{u-1}e^{-\alpha t}\right),
\end{aligned}$$

where $\gamma(u,t) = \int_0^t s^{u-1}e^{-s}ds$ is the incomplete gamma function. We also have a tail estimate:

$$P(\tau_t > x) \leq \exp\left(-\nu x \log x + \nu\left(1 - \log\frac{\nu}{\alpha t}\right)x - \alpha t\right) = \left(\frac{\alpha t}{\nu x}\right)^{\nu x} e^{\nu x - \alpha t}.$$

6 Acknowledgements

I am grateful to Jan Grandell for discussing Cox processes and renewal processes with me, and also to Thomas Höglund for careful reading of an earlier draft of this paper and pointing out some errors.

References

[1] Abramowitz, M. and Stegun, I. A., Eds. (1992). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Dover Publications Inc., New York.

[2] Bertoin, J. (1996). *Lévy Processes.* Cambridge University Press, Cambridge.

[3] Bertoin, J. and van Harn, K. and Steutel, F. W. (1999). Renewal theory and level passage by subordinators. *Statist. Probab. Lett.* **45,** 65–69.

[4] Bingham, N. H. (1971). Limit theorems for occupation times of Markov processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **17,** 1–22.

[5] Daley, D. J. and Vere-Jones, D. (2003). *An Introduction to the Theory of Point Processes*, vol. 1, 2:nd ed. Springer-Verlag, New York.

[6] Grandell, J. (1976). *Doubly Stochastic Poisson Processes.* Lecture Notes in Mathematics **529**. Springer-Verlag, Berlin.

[7] van Harn, K. and Steutel, F. W. (2001). Stationarity of delayed subordinators. *Stoch. Models* **17,** 369–374.

[8] Kaj, I. and Martin-Löf, A. (2004). Scaling limit results for the sum of many inverse Lévy subordinators. Preprint 2004:13. Department of Mathematics, Uppsala University.

[9] Karatzas, I. and Shreve, S. E. (1988). *Brownian Motion and Stochastic Calculus.* Springer-Verlag, New York.

[10] Kingman, J. F. C. (1964). On the doubly stochastic Poisson processes. *Proc. Camb. Phil. Soc.* **60,** 923–930.

[11] Kozlova, M. and Salminen, P. (2004). Diffusion local time storage. *Stoch. Proc. Appl.* **114,** 211–229.

[12] Sato, K.-i. (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge.

[13] Steutel, F. W. and van Harn, K. (2004). *Infinite Divisibility of Probability Distributions on the Real Line.* Marcel Dekker Inc., New York.

[14] Winkel, M. (2005). Electronic foreign exchange markets and passage events of independent subordinators. To appear in *J. Appl. Prob.*