ISSN: 1401-5617

Spectral and Hardy Inequalities for some Sub-Elliptic Operators

Lior Aermark

Research Reports in Mathematics
Number 1, 2011

Department of Mathematics
Stockholm University

Electronic versions of this document are available at
http://www.math.su.se/reports/2011/1

Date of publication: January 25, 2011
Keywords: Hardy Inequalities, Sub-Elliptic Operators.

Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.math.su.se/
info@math.su.se

Spectral and Hardy Inequalities for some Sub-Elliptic Operators

Lior Aermark

Spectral and Hardy inequalities for some sub-elliptic operators

Lior Aermark

Dissertation presented to Stockholm University in partial fulfillment of the requirements for the Degree of Licentiate of Philosophy (Filosofie licentiatexamen), to be presented on February 14, 2011 at 10:00 in Room 306, Building 6, Department of Mathematics, Stockholm University (Kräftriket).

Principal advisor: Ari Laptev
Second advisor: Andrzej Szulkin
Opponent: Torbjörn Kolsrud
Examiner: Sara Maad Sasane

Abstract

This thesis consists of two parts:
In the first part we introduce a version of the Ahronov-Bohm magnetic field for a Grushin sub-elliptic operator and then show that its quadratic form satisfies an improved Hardy inequality.
In the second part we obtain Lieb-Thirring inequalities for 3D Schrödinger type operators where instead of the usual Laplacian we have the Heisenberg Laplacian.

Acknowledgements

I would like to express my deepest gratitude to the following:

Professor Ari Laptev, my principal advisor, for the inspiration he has given me and for not letting me give up.

My second advisor, professor Andrzej Szulkin, for being available at any time and for never stopping believing in me.

Tom Britton and Kerstin königsson from Stockholm university that have made things, that seemed impossible before, possible for me.

The Erwin Schrödinger Institute in Vienna, and especially Isabella Miedl and professor Thomas Hoffmann-Ostenhof for their generosity.

Family Birgerson-Irving for giving me a home where I can feel safe in Sweden.

Dr. Nadia Lord, Dr. Daphne Enstam, Vivianne Nisell, Amol Sasane and Lars-Johan Norrby. Without their support I would not be here today.

Delilah, Scaramouche, Boom-Boom, and also Uri, Farrokh, Tuli, Joakim Ben-Tovim Drechsler, Dvash, Annelie Ba'a Me'ahava, Koshka, Boris, Kæreste, Karl-Johan, Magnum, Poona, Wilhelmina, Anton-Sobachka, Lilla, Sivan, Agas and Tamar for their unconditional love.

And to my father, who is the best Abba in the universe. You mean the world to me!

Contents

Chapter 1

Introduction

1.1 Brief historical background of classical mechanics

Mechanics in its classical form was already studied in ancient Greece and was explored through the middle ages and modern age. It was however Sir Isaac Newton that was the first one to develop mechanics as it is known today.
Classical mechanics aims to determine the position of a particle at any given time $x(t)$. Knowing $x(t)$ enables us to find out different dynamical variables of the particle, like the velocity $v = \frac{dx}{dt}$, its momentum $p = mv$ and the kinetic energy $T = \frac{p^2}{2m}$. Newton formulated the laws of motion, that connect between a potential $V(x)$ acting on the particle and its motion accordingly:

$$F = ma = \frac{dp}{dt} = \frac{dV(x)}{dx},$$

where $p = m\frac{dx}{dt}$.
In order to determine $x(t)$, one has to solve the Newton equation

$$m\frac{d^2x}{dt^2} = \frac{dV(x)}{dx}$$

with the appropriate initial conditions.

In 1788 Joseph Louis Lagrange reformulated classical mechanics into what is known as *Lagrangian mechanics*. In 1833 William Rowan Hamilton developed the Lagrangian mechanics into *Hamiltonian mechanics*. According to this method, one considers the *Hamiltonian*, defined by

$$H = \frac{p^2}{2m} + V(x). \tag{1.1}$$

The Hamiltonian equals the sum of the kinetic and potential energies of the system, in the form

$$H = T + V. \tag{1.2}$$

The Hamiltonian mechanics gives insight into quantum mechanics. In this case we are looking for the wave function $\psi(x,t)$ of a particle, which is the solution of the*Schrödinger equation*.

1.2 The Schrödinger operator

Schrödinger replaced p in (1.1) by $i\hbar\nabla$ to construct a momentum operator, where ∇ is the gradient operator, i corresponds to the imaginary unit and $\hbar$ is the (reduced) Planck constant with dimensions energy$\times$time. As T is defined by

$$T = \frac{(i\hbar\nabla)^2}{2m} = -\frac{\hbar^2}{2m}\Delta,$$

the expression in (1.2) becomes

$$H = -\frac{\hbar^2}{2m}\Delta + V(x). \tag{1.3}$$

Here H is the Hamiltonian for a non-relativistic charged particle moving in an electric field.
The time-dependent Schrödinger equation corresponding to it is given by

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\Delta\Psi}{\partial x^2} + V(x)\Psi(x,t) \equiv H\Psi(x,t),$$

where the unknown Ψ describes the wave function, which is a function from a space that maps the possible states of the system into the complex numbers.

A classical problem in the research of the quantum mechanical Hamiltonian is the study of bound states of a given potential V and their energies, that is, the number of negative eigenvalues of the Schrödinger operator $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$, $d \geq 1$, $\lambda_1 \leq \lambda_2 \leq ... < 0$ denoting the negative eigenvalues of the Hamiltonian (if there are any) under appropriate conditions on V. Here the Lieb-Thirring inequalities play a crucial role (see section 3.2).

1.3 Magnetic Schrödinger operator

The Hamiltonian for a nonrelativistic charged particle in an electromagnetic field is given by

$$H = \frac{(i\hbar\nabla - A(x))^2}{2m} + V(x).$$

The operator $-\frac{\hbar^2}{2m}\Delta$ in (1.3) is replaced here by $\frac{(i\hbar\nabla - A(x))^2}{2m}$; $V : \mathbb{R}^d \to \mathbb{R}$ describes the electric (or scalar) potential and $A : \mathbb{R}^d \to \mathbb{R}^d$ is the magnetic (or vector) potential.
The Schrödinger equation is now

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{(i\hbar\nabla - A(x))^2}{2m}\Psi(x,t) + V(x)\Psi(x,t) \equiv H\Psi(x,t). \tag{1.4}$$

The vector potential $A = (A_1, A_2, ..., A_d)$ is a source for the magnetic field $B = \text{curl}A$, where $\text{curl}A$ is the $d \times d$ skew-symmetric matrix with entries $B_{jk} = \partial_j A_k - \partial_k A_j$. A could be also understood as a differential 1-form, and then B could be seen as the 2-form, given by $B = dA$, or alternatively

$$A = \sum_{j=1}^{d} A_j dx^j, \quad B = dA = \sum_{j<k} B_{jk} dx^j \wedge dx^k.$$

If $d = 3$, then $\text{curl}A$ has the usual representation as a vector in $\mathbb{R}^3$. If $\text{curl}\tilde{A} = B = \text{curl}A$, then it can be shown that $A(x) = \tilde{A}(x) + \nabla\varphi(x)$ for some φ.

Let $\tilde{\Psi} = e^{i\varphi}\Psi$. An explicit computation shows that Ψ satisfies (1.4) if and only if $\tilde{\Psi}$ satisfies (1.4), with A replaced by $\tilde{A}$. This is called *gauge invariance* and says that the important physical quantity is B and not A. This remains true if $\mathbb{R}^d$ is replaced by a simply connected domain Ω. For non-simply connected domains this is in general false (see the discussion of the Aharonov-Bohm effect in section 1.9).

1.4 Hardy's inequality

The Hardy inequality in the Euclidien space has an important role in the study of linear and nonlinear partial differential equations. It was first stated in the 1920's by G. H. Hardy in an attempt to find a simpler proof for Hilbert's inequality for double series.
The Hardy inequality enables us to obtain lower bounds on the spectrum of elliptic operators satisfying Dirichlet boundary conditions.

1.4.1 The classical Hardy inequality

If $d \geq 3$, then for any function u such that $u \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \geq \left(\frac{d-2}{2}\right)^2 \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} dx. \tag{1.5}$$

Proof. Let u be a complex function, $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} 0 &\leq \int \left| \nabla u + \alpha \frac{\nabla |x|}{|x|} u \right|^2 \\ &= \int |\nabla u|^2 + \alpha \int \nabla u \cdot \frac{\nabla |x|}{|x|} \bar{u} + \alpha \int \frac{\nabla |x|}{|x|} u \cdot \nabla \bar{u} + \alpha^2 \int \left| \frac{\nabla |x|}{|x|} \right|^2 |u|^2 \\ &= \int |\nabla u|^2 + \alpha \int (\nabla u \cdot \frac{x}{|x|^2}) \bar{u} + \alpha \int u \frac{|x|}{|x|^2} \cdot \nabla \bar{u} + \alpha^2 \int \frac{|u|^2}{|x|^2} \\ &= \int |\nabla u|^2 + \alpha \int \nabla(|u|^2) \cdot \frac{x}{|x|^2} + \alpha^2 \int \frac{|u|^2}{|x|^2}. \end{aligned}$$

Since

$$\int \nabla(|u|^2) \cdot \frac{x}{|x|^2} = -\int (\nabla \cdot \frac{x}{|x|^2})|u|^2 = -(d-2)\int \frac{|u|^2}{|x|^2},$$

$$0 \leq \int |\nabla u|^2 - (d-2)\alpha \int \frac{u^2}{|x|^2} + \alpha^2 \int \frac{u^2}{|x|^2},$$

or

$$\int |\nabla u|^2 \geq \left((d-2)\alpha - \alpha^2\right) \int \frac{u^2}{|x|^2} \quad \forall \alpha.$$

Taking the maximum with respect to α gives

$$\int |\nabla u|^2 \geq \left(\frac{d-2}{2}\right)^2 \int \frac{u^2}{|x|^2}.$$

$\square$

During the years new versions of the Hardy inequality were given. They differ from one another depending on the relation between the parameters, on the weight functions and on the class to which the functions belong.
It is well known that the constant $(d-2)^2/4$ in (1.5) is sharp but not achieved. The literature concerning different versions of Hardy's inequalities and their applications is extensive. In this paper we mention the classical paper of M. Sh. Birman [B], the article by B. Davies [D] and the book of V. Maz'ya [M].
Among many applications of the inequality (1.5) we would like to mention the *Heisenberg uncertainty principle*.

1.4.2 The Heisenberg uncertainty principle

In its classical form the uncertainty principle was developed by Heisenberg in connection with the study of quantum mechanics. According to this principle the position and momentum of a particle could not be defined exactly simultaneously, but only with some uncertainty.

On the Euclidien space $\mathbb{R}^d$ the uncertainty inequality states that

$$\left(\int_{\mathbb{R}^d} |x|^2|u(x)|^2dx\right)\left(\int_{\mathbb{R}^d} |\nabla u(x)|^2dx\right) \geq \left(\frac{d-2}{2}\right)^2 \left(\int_{\mathbb{R}^d} |u(x)|^2dx\right)^2. \tag{1.6}$$

To show this, we start with the Hardy inequality

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2dx \geq \left(\frac{d-2}{2}\right)^2 \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^2}dx. \tag{1.7}$$

Schwarz's inequality applied to (1.7) yields

$$\begin{aligned}
&\left(\frac{d-2}{2}\right)\int_{\mathbb{R}^d} |u(x)|^2\frac{1}{|x|}|x|dx \\
&\quad \leq \left(\frac{d-2}{2}\right)\left(\int_{\mathbb{R}^d} |u(x)|^2|x|^2dx\right)^{1/2}\left(\frac{|u(x)|^2}{|x|^2}dx\right)^{1/2} \\
&\qquad \leq \left(\int_{\mathbb{R}^d} |u(x)|^2|x|^2dx\right)^{1/2}\left(\int_{\mathbb{R}^d} |\nabla u(x)|^2dx\right)^{1/2}.
\end{aligned}$$

This gives (1.6). By applying Parseval formula for the Fourier transform $\hat{u}$ of the function u in the second integral of the left hand side the inequality (1.6) takes a particularly symmetrical form

$$(2\pi)^d\left(\int_{\mathbb{R}^d} |x|^2u(x)|^2dx\right)\left(\int_{\mathbb{R}^d} |\xi|^2|\hat{u}(\xi)|^2d\xi\right) \geq \left(\frac{d-2}{2}\right)^2\left(\int_{\mathbb{R}^d} |u(x)|^2dx\right)^2,$$

where the Fourier transform of the function u is defined by

$$\hat{u}(\xi) = (2\pi)^{-d/2}\int e^{-ix\xi}u(x)dx.$$

This inequality expresses the Heisenberg uncertainty principle which states that a non-trivial L^2-function and its Fourier transform cannot simultaneously be very small near the origin.
Hardy's inequalities were also studied for some sub-elliptic operators, see for example papers [G], [GL], [A1], [A2], [DGN], [NCH], [Ko]. The Heisenberg group $\mathbb{H}$ is the prime example of non-commutative harmonic analysis.

1.5 The Heisenberg group

The Heisenberg group and its Lie algebra have first attained their official names in the 1970's, although they were already studied long before that.
The Heisenberg Lie algebra attained its name due to its structure of equations, which is the Heisenberg canonical commutation relations in quantum mechanics. Yet these relations are the quantized version of the Poisson bracket relations for canonical coordinates in Hamiltonian mechanics.
The Heisenberg group is of great importance in many fields in mathematics, such as representation theory, partial differential equations, harmonic analysis and quantum mechanics. It is due to its crucial contribution in a variety of areas that the Heisenberg group can be constructed in two fundamental but different settings. The first one plays an important role in understanding of several problems in complex function theory of the unit ball.
In quantum mechanics the Heisenberg group can be realized as a group of unitary operators generated by the exponentials of the position and momentum operators. The relation between the pair of operators: x and $(2\pi i)^{-1}\dfrac{d}{dx}$, which can be understood as the generators of the pseudo-differential operators, is dominated by the commutation relations which are so characteristic of the Heisenberg group. As the state of a given particle at a given time t is determined by its position vector $q \in \mathbb{R}^3$ and its momentum vector $p \in \mathbb{R}^3$, Heisenberg's crucial idea that lead to quantum mechanics was to take the components of these vectors to be an operator on the Hilbert space $\mathcal{H}$, satisfying the commutation relation

$$[Q_i, Q_j] = 0, \quad [P_i, P_j] = 0, \quad [P_i, Q_j] = -i\hbar\delta_{i,j}$$

for $i, j = 1, 2, 3$, where $Q_j = x_j$ and $P_j = (2\pi i)^{-1}\dfrac{\partial}{\partial x_j}$.
The Heisenberg group consists of the set

$$\mathbb{C} \times \mathbb{R} = \{[z, t] : z \in \mathbb{C}, t \in \mathbb{R}\}$$

with the multiplication law

$$[z, t] \cdot [u, s] = [z + u, \quad t + s + 2\mathrm{Im}(z \cdot \bar{u})]. \tag{1.8}$$

1.6 The Heisenberg Laplacian

We shall mention briefly the essential attributes of the Laplacian: It is invariant under translations and rotations and is homogeneous of degree two.
Define as $\mathbb{H}$ the sub-Laplacian on the Heisenberg group.
Instead of usual dilations $x \to rx$, we have on the Heisenberg group that $\delta_r(z,t) = (rz, r^2z)$ (δ_r is an automorphism of the Heisenberg group).
The left translation L_g on the Heisenberg group, $g \in \mathbb{R}^d$ is defined by

$$L_g f(h) = f(g^{-1}h), \quad h \in \mathbb{H}^d,$$

and the rotation R_σ is defined by

$$R_\sigma f(z,t) = f(\sigma z, t), \sigma \in \mathcal{U}^d,$$

where $\mathcal{U}^d$ represents the upper half space.
An operator P on $\mathbb{H}^d$ is said to be left invariant if it commutes with L_g, rotation invariant if it commutes with R_σ and homogeneous of degree α if $P(f(\delta_r g)) = r^\alpha P f(\delta_r g)$.

1.7 The Heisenberg Laplacian

The sub-Laplacian is homogeneous of degree two, with unique left invariance and rotation invariance up to a constant multiple.
Let us realize $\mathbb{H}$ as $\mathbb{R}^3$ with coordinates (x,y,t) and the (non-commutative) multiplication
$(x,y,t) \circ (x',y',t') = (x+x', y+y', t+t' - 2(xy' - yx'))$. The vector fields

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$$

are left-invariant and the sub-Laplacian on $\mathbb{H}$ is given by

$$H = -X^2 - Y^2 = -\left(\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}\right)^2 - \left(\frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}\right)^2. \qquad (1.9)$$

The quadratic form of the operator H is defined on functions u from the Sobolev class $\mathcal{H}^1(\mathbb{R}^3)$

$$h[u] = \int_{\mathbb{R}^3} \left(|Xu|^2 + |Yu|^2\right) dzdt. \qquad (1.10)$$

Let $z = (x, y)$, $|z| = \sqrt{x^2 + y^2}$, and let us consider the so-called Kaplan distance function [Ka] from (z, t) to the origin

$$(|z|^4 + t^2)^{1/4}.$$

The function d is positively homogeneous of degree 2 with respect to the dilations

$$d(\lambda z, \lambda^2 t) = \lambda d(z, t), \quad \lambda > 0$$

and has a singularity at zero.
The sub-Laplacian fails to be elliptic, but it still satisfies the conditions of subellipticity.

1.8 The Grushin plane and Grushin operator

1.8.1 Grushin plane

In 1970 V. V. Grushin [Gr] studied a class of operators which were non-elliptic, but still satisfied the requirements of hypoellipticity according to Hörmander's conditions. The first example of such operators is the Grushin operator.

Hörmander's condition

Let X and Y be two vector fields in an open set Ω of $\mathbb{R}^d$. The Lie bracket, denoted by $[X, Y]$ is defined by

$$[X, Y]f = X[Yf] - Y[Xf]$$

(note that $[X, Y]$ is a new vector field).

We say that the Hörmander condition [H] is satisfied at x_0, if there exists $r(x_0) \geq 1$ such that the vector space generated by the iterated brackets $[X, X_k]^\alpha$ at x_0 with $|\alpha| \leq r(x_0) - 1$ is $\mathbb{R}^d$.

When $r(x_0) = 1$, the operator is said to be elliptic. A typical non-trivial example is the Heisenberg group, where $d = 3, r = 2, X = \partial_x + 2y\partial_t, Y = \partial_y - 2x\partial_t$ and $[X, Y] = \partial_t$. Another example is the Grushin operator in dimension 2. In our case $r(0) = 2$.

1.8.2 The Grushin operator in dimension 3

The Grushin operator in the Heisenberg representation is

$$G = -\Delta_z - 4|z|^2\partial_t^2. \tag{1.11}$$

It gives another example of a sub-elliptic operator. Its quadratic form equals

$$g[u] = \int_{\mathbb{R}^3} (|\nabla_z u|^2 + 4|z|^2|\partial_t u|^2) dz dt. \tag{1.12}$$

For the forms (1.10) and (1.12) the following sharp Hardy inequalities were discussed in detail in [G] and [GL]:

$$h[u] = \int_{\mathbb{R}^3} \left(|Xu|^2 + |Yu|^2\right) dzdt \geq \int_{\mathbb{R}^3} \frac{|z|^2}{d^4}|u|^2 dzdt, \tag{1.13}$$

and

$$g[u] = \int_{\mathbb{R}^3} (|\nabla_z u|^2 + 4|z|^2|\partial_t u|^2) dzdt \geq \int_{\mathbb{R}^3} \frac{|z|^2}{d^4}|u|^2 dzdt. \tag{1.14}$$

Inequalities (1.13) and (1.14) are related. The operator H defined in (1.9) could be rewritten in the form

$$Hu = -\Delta_z u - 4|z|^2\partial_t^2 u - 4\partial_t Tu = Gu - 4\partial_t Tu, \tag{1.15}$$

where $T = y\partial_x - x\partial_y$ is the rotation operator. In particular, if $u(z,t) = u(|z|,t)$, then $Tu = 0$, and on this subclass of functions the inequalities (1.11) and (1.12) coincide.

1.9 The Aharonov-Bohm potential

In 1959 Yakir Aharonov and David Bohm observed the phenomenon, where a charged particle is affected by electromagnetic fields, despite being confined to regions where both the magnetic field and the electric field are zero (such effects may arise in both electric and magnetic fields, but the latter is easier to study). An important consequence of this effect is that understanding of the classical electromagnetic field acting locally on a particle is not enough in order to predict the quantum mechanical behaviour of a particle. Assume $d = 3$. By $\mathcal{A}$ we denote the Aharonov-Bohm vector potential, given by

$$\mathcal{A}(x) := \left(-\frac{x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}, 0\right) \tag{1.16}$$

and defined in

$$\Omega = \{x = (x_1, x_2, x_3) : x_1^2 + x_2^2 > 0\}.$$

Note that curl $\mathcal{A} = 0$ outside the x_3-axis and

$$\int_\gamma \mathcal{A} = 2\pi, \tag{1.17}$$

where γ is a properly oriented closed curve which encloses the x_3 axis (this is in contrast to the case, where Ω is simply connected, and then $\int_\gamma \mathcal{A} = 0$ for any smooth $\mathcal{A}$ and any closed curve $\gamma \subset \Omega$ by Stokes' theorem). The integral (1.17) represents the magnetic flux, which describes the magnetic potential on a charged quantum mechanical particle, moving in a region where the magnetic field is 0. This case could be realized as the particle being confined to the outside of an infinitely long solenoid extending along the x_3-axis with a radius that tends to zero.
According to (1.16), $\mathcal{A} = \nabla\theta$, where $\theta = \theta(x)$ is the polar angle of (x_1, x_2).
We note that θ is well defined only locally (globally, it is well defined up to an integer multiple of 2π). So unlike the situation in section 1.3, although curl$\mathcal{A} = 0$, $\mathcal{A}$ is the gradient of a function which is well defined, but only locally. The reason for this is the fact that Ω is not simply connected.
One can also consider the Aharonov-Bohm potential as a vector field in $\mathbb{R}^2$:

$$\mathcal{A}(x) = \left(-\frac{x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right).$$

We will do this below.

1.10 Hardy's inequality in two dimensions with Aharonov-Bohm vector potential

The classical Hardy inequality (1.5) becomes trivial for the two-dimensional case. In an article by Laptev and Weidl [LW] the authors have noticed that for some magnetic forms in two dimensions the Hardy inequality holds its classical form. More precisely, consider the Aharonov-Bohm magnetic potential

$$\beta A = \beta\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right), \quad \beta \in \mathbb{R}$$

and the quadratic form

$$\begin{aligned} I = \int_{\mathbb{R}^2} |(\nabla + i\beta A)u|^2 dxdy \\ = \int_{\mathbb{R}^2} \left|\left(\partial_x - \beta i\frac{y}{x^2+y^2}\right)u\right|^2 dxdy \\ + \int_{\mathbb{R}^2} \left|\left(\partial_y + \beta i\frac{x}{x^2+y^2}\right)u\right|^2 dxdy. \end{aligned} \tag{1.18}$$

By introducing polar coordinates we get

$$r = \sqrt{x^2+y^2}; \qquad \frac{x}{r} = \cos\varphi, \quad \frac{y}{r} = \sin\varphi$$

and

$$\frac{\partial\varphi}{\partial x} = -\frac{y}{r^2}, \quad \frac{\partial\varphi}{\partial y} = \frac{x}{r^2}, \quad \partial_x = \cos\varphi\frac{\partial}{\partial r} - \frac{y}{r^2}\frac{\partial}{\partial\varphi}, \quad \partial_y = \sin\varphi\frac{\partial}{\partial r} + \frac{x}{r^2}\frac{\partial}{\partial\varphi}.$$

Hence (1.18) becomes

$$\begin{aligned} I = \int \left|\left(\cos\varphi\frac{\partial}{\partial r} - \frac{y}{r^2}\frac{\partial}{\partial\varphi} - i\beta\frac{\sin\varphi}{r}\right)u\right|^2 rdrd\varphi \\ + \int \left|\left(\sin\varphi\frac{\partial}{\partial r} + \frac{x}{r^2}\frac{\partial}{\partial\varphi} + i\beta\frac{\cos\varphi}{r}\right)u\right|^2 rdrd\varphi \\ = \int \left(\cos^2\varphi|u'_r|^2 + \frac{(\sin\varphi)^2}{r^2}|(\partial_\varphi u + i\beta u)|^2\right) rdrd\varphi \\ + \int \left(\sin^2\varphi|u'_r|^2 + \frac{(\cos\varphi)^2}{r^2}|(\partial_\varphi u + i\beta u)|^2\right) rdrd\varphi \\ = \int \left(|u'_r|^2 + \frac{1}{r^2}|(\partial_\varphi u + i\beta u)|^2\right) rdrd\varphi. \end{aligned} \tag{1.19}$$

Expanding u into Fourier series with respect to φ

$$u = \sum_{k=-\infty}^{\infty} u_k(r)\frac{e^{ik\varphi}}{\sqrt{2\pi}}$$

enables us to rewrite (1.19) as

$$\int \left(|u'_r|^2 + \frac{1}{r^2} |(\partial_\varphi u + i\beta u)|^2 \right) r dr d\varphi$$
$$\geq \int \frac{1}{r^2} \left| \sum (ik + i\beta) u_k(r) \frac{e^{ik\varphi}}{\sqrt{2\pi}} \right|^2 r dr d\varphi$$
$$\geq \int \frac{1}{r^2} \sum |ik + i\beta|^2 |u_k(r)|^2 r dr$$
$$\geq \min_k |k + \beta|^2 \int \frac{1}{r^2} \sum |u_k(r)|^2 r dr$$
$$\geq \min_k |k + \beta|^2 \int \int \frac{1}{r^2} \left| \sum u_k(r) \frac{e^{ik\varphi}}{\sqrt{2\pi}} \right|^2 r dr d\varphi$$
$$= \min_k |k + \beta|^2 \int \int \frac{1}{r^2} |u(r, \varphi)|^2 r dr d\varphi$$
$$= \min_k |k + \beta|^2 \int \frac{1}{x^2 + y^2} |u(x, y)|^2 dx dy.$$

Hence

$$\int_{\mathbb{R}^2} |(\nabla + i\beta A)u|^2 dx dy \geq \min_k |k + \beta|^2 \int_{\mathbb{R}^2} \frac{|u|^2}{x^2 + y^2} dx dy.$$

Here the form in the left hand side is considered on the function class $\mathcal{H}^1(\mathbb{R}^2)$, obtained by the completion of the class $C_0^\infty(\mathbb{R}^2 \setminus 0)$ with respect to the metric defined by the form

$$\int_{\mathbb{R}^2} (|\nabla u|^2 + |x|^{-2} |u|^2) \, dx$$

and β can naturally be interpreted as the magnetic flux through a disc with the center at the origin.

Chapter 2

Hardy inequality for a magnetic Grushin operator

In this section we introduce a suitable notion of an Aharonov-Bohm type vector field for the Grushin operator defined in (1.12) and obtain an improvement of the Hardy inequality in (1.14).
Let us first define a Grushin type vector field

$$\nabla_G = (\partial_x, \partial_y, 2x\partial_t, 2y\partial_t);$$

Clearly

$$G = -|\nabla_G|^2.$$

We introduce now an Aharonov-Bohm type magnetic potential

$$\mathcal{A} = \left(-\frac{\partial_y d_H}{d_H}, \frac{\partial_x d_H}{d_H}, -2y\frac{\partial_t d_H}{d_H}, 2x\frac{\partial_t d_H}{d_H} \right),$$

where d_H is the *Kaplan distance*

$$d_H = ((x^2 + y^2)^2 + t^2)^{1/4}.$$

The magnetic Grushin operator with the magnetic potential $\mathcal{A}$ and with the "flux" β could then be defined as

$$G_{\mathcal{A}} = -(\nabla_G + i\beta\mathcal{A})^2. \tag{2.1}$$

Our main result of this section is the following theorem.

Theorem 1. *Assume that $-1/2 \leq \beta \leq 1/2$. Then for the quadratic form of the magnetic Grushin operator* (2.1) *we have the following Hardy inequality*

$$\int_{\mathbb{R}^3} (|(\nabla_{G_0} + i\beta\mathcal{A})u|^2) dzdt \geq (1+\beta^2) \int_{\mathbb{R}^3} \frac{|z|^2}{d^4} |u|^2 dzdt. \tag{2.2}$$

We shall prove it in section 2.2.

2.1 Simple proofs of Hardy's inequality for Heisenberg and Grushin operators

Before proving (2.2), we present here simple proofs of the inequalities (1.13) and (1.14).

Proposition 1. *For any function u for which $h[u] < \infty$ the following inequality holds true:*

$$\int_{\mathbb{R}^3} \left(|Xu|^2 + |Yu|^2\right) dzdt \geq \int_{\mathbb{R}^3} \frac{|z|^2}{d^4} |u|^2 dzdt. \tag{2.3}$$

Proof. It is enough to prove (2.3) for functions $u \in C_0^\infty(\mathbb{R}^3 \setminus 0)$. Let us consider the following non-negative expression

$$I = \int_{\mathbb{R}^3} \left| \left(X + \alpha \frac{Xd_H}{d_H} \right) u \right|^2 dzdt + \int_{\mathbb{R}^3} \left| \left(Y + \alpha \frac{Yd_H}{d_H} \right) u \right|^2 dzdt,$$

where $\alpha \in \mathbb{R}$.
Clearly

$$d(z,t)^{-1} X d(z,t) = \frac{x|z|^2 + yt}{d^4(z,t)}, \quad d(z,t)^{-1} Y d(z,t) = \frac{y|z|^2 - xt}{d^4(z,t)}.$$

We look at

$$\begin{aligned} 0 \leq I = \int_{\mathbb{R}^3} & \left(X + \alpha \frac{Xd_H}{d_H} \right) u \left(X + \alpha \frac{Xd_H}{d_H} \right) \bar{u} \\ & + \left(Y + \alpha \frac{Yd_H}{d_H} \right) u \left(Y + \alpha \frac{Yd_H}{d_H} \right) \bar{u} dxdydt. \end{aligned}$$

$$
\begin{aligned}
I = \int_{\mathbb{R}^3} & \left(|Xu|^2 + |Yu|^2\right) dxdydt \\
& + \int_{\mathbb{R}^3} Xu\alpha \frac{Xd_H}{d_H}\bar{u}\, dxdydt + \int_{\mathbb{R}^3} \alpha \frac{Xd_H}{d_H} uX\bar{u}\, dxdydt \\
& + \int_{\mathbb{R}^3} Yu\alpha \frac{Yd_H}{d_H}\bar{u}\, dxdydt + \int_{\mathbb{R}^3} \alpha \frac{Yd_H}{d_H} Y\bar{u}\, dxdydt \\
& + \alpha^2 \int_{\mathbb{R}^3} \left| \frac{Xd_H}{d_H} u \right|^2 dxdydt + \alpha^2 \int_{\mathbb{R}^3} \left| \frac{Yd_H}{d_H} u \right|^2 dxdydt. \quad (2.4)
\end{aligned}
$$

Since

$$
\begin{aligned}
& \int_{\mathbb{R}^3} Xu\alpha \frac{Xd_H}{d_H}\bar{u}\, dxdydt + \int_{\mathbb{R}^3} \alpha \frac{Xd_H}{d_H} uX\bar{u}\, dxdydt \\
& \qquad + \int_{\mathbb{R}^3} Yu\alpha \frac{Yd_H}{d_H}\bar{u}\, dxdydt + \int_{\mathbb{R}^3} \alpha \frac{Yd_H}{d_H} Y\bar{u}\, dxdydt \\
& = -\alpha \int_{\mathbb{R}^3} \left[u \left(X \left(\frac{Xd_H}{d_H} \right) \right) \bar{u} + u \frac{Xd_H}{d_H} X\bar{u} \right] dxdydt + \alpha \int_{\mathbb{R}^3} \frac{Xd_H}{d_H} uX\bar{u}\, dxdydt \\
& + \alpha \int_{\mathbb{R}^3} \left[u \left(Y \left(\frac{Yd_H}{d_H} \right) \right) \bar{u} + u \frac{Yd_H}{d_H} Y\bar{u} \right] dxdydt + \alpha \int_{\mathbb{R}^3} \frac{Yd_H}{d_H} uY\bar{u}\, dxdydt \\
& \quad = -\alpha \int_{\mathbb{R}^3} X \left(\frac{Xd_H}{d_H} \right) |u|^2\, dxdydt - \alpha \int_{\mathbb{R}^3} Y \left(\frac{Yd_H}{d_H} \right) |u|^2\, dxdydt,
\end{aligned}
$$

(2.4) becomes

$$
\begin{aligned}
I = \int_{\mathbb{R}^3} (|Xu|^2 + |Yu|^2)\, dzdt - \alpha \int_{\mathbb{R}^3} \left(X \frac{Xd_H}{d_H} + Y \frac{Yd_H}{d_H} \right) |u|^2\, dzdt \\
+ \alpha^2 \int_{\mathbb{R}^3} \left(\left(\frac{Xd_H}{d_H} \right)^2 + \left(\frac{Yd_H}{d_H} \right)^2 \right) |u|^2\, dzdt \geq 0.
\end{aligned}
$$

Splitting the computation into three parts gives

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left(\left| \frac{X d_H}{d_H} u \right|^2 + \left| \frac{Y d_H}{d_H} u \right|^2 \right) dxdydt \\
&= \int_{\mathbb{R}^3} \left(\left(\frac{((x^2+y^2)x+yt)^2 + ((x^2+y^2)y-xt)^2}{((x^2+y^2)^2+t^2)^2} \right) |u|^2 \right) dxdydt \\
&= \int_{\mathbb{R}^3} \left(\left(\frac{((x^2+y^2)^2(x^2+y^2) + t^2(x^2+y^2)}{((x^2+y^2)^2+t^2)^2} \right) |u|^2 \right) dxdydt \\
&= \int_{\mathbb{R}^3} \left(\frac{(x^2+y^2)((x^2+y^2)^2+t^2)}{((x^2+y^2)^2+t^2)^2} |u|^2 \right) dxdydt \\
&= \int_{\mathbb{R}^3} \left(\frac{x^2+y^2}{(x^2+y^2)^2+t^2} |u|^2 \right) dxdydt;
\end{aligned}$$

$$\begin{aligned}
-\alpha &\int_{\mathbb{R}^3} \left(X \frac{X d_H}{d_H} + Y \frac{Y d_H}{d_H} \right) |u|^2 \, dzdt \\
&= -\alpha \int \left((\partial_x + 2y\partial_t) \left(\frac{(x^2+y^2)x+yt}{(x^2+y^2)^2+t^2} \right) |u|^2 \right) dxdydt \\
&\qquad - \alpha \int \left((\partial_y - 2x\partial_t) \left(\frac{(x^2+y^2)y-xt}{(x^2+y^2)^2+t^2} \right) |u|^2 \right) dxdydt
\end{aligned}$$

where

$$\begin{aligned}
&-\partial_x \left(\frac{(x^2+y^2)x+yt}{(x^2+y^2)^2+t^2} \right) - \partial_y \left(\frac{(x^2+y^2)y-xt}{(x^2+y^2)^2+t^2} \right) \\
&\qquad = -\frac{t^2(x^2+3y^2+y^2+3x^2)}{((x^2+y^2)^2+t^2)^2} = -\frac{t^2(4x^2+4y^2)}{((x^2+y^2)^2+t^2)^2}
\end{aligned}$$

and

$$
\begin{aligned}
&-2y\partial_t\left(\frac{(x^2+y^2)x+yt}{(x^2+y^2)^2+t^2}\right)+2x\partial_t\left(\frac{(x^2+y^2)y-xt}{(x^2+y^2)^2+t^2}\right)\\
&\qquad=\frac{-2y((x^2+y^2)+yt^2)-2tx(x^2+y^2)-2t^2y}{((x^2+y^2)^2+t^2)^2}\\
&\qquad+\frac{2x(-x(x^2+y^2)-xt^2)-2ty(x^2+y^2)+2t^2x)}{((x^2+y^2)^2+t^2)^2}\\
&=\frac{-2y^2(x^2+y^2)-2x^2(x^2+y^2)-2y^2t^2-2x^2t^2+4t^2y^2+4t^2x^2}{((x^2+y^2)^2+t^2)^2}\\
&\qquad=\frac{-2(x^2+y^2)(x^2+y^2)^2+2t^2(x^2+y^2)}{((x^2+y^2)^2+t^2)^2}.
\end{aligned}
$$

It holds then that

$$
\begin{aligned}
&-\alpha\int(\partial_x+2y\partial_t)\left(\frac{(x^2+y^2)x+yt}{(x^2+y^2)^2+t^2}\right)\\
&\qquad+(\partial_y-2x\partial_t)\left(\frac{(x^2+y^2)y-xt}{(x^2+y^2)^2+t^2}\right)|u|^2dxdydt\\
&=-\alpha\int\frac{2(x^2+y^2)(x^2+y^2)^2+2t^2(x^2+y^2)+4t^2(x^2+y^2)}{((x^2+y^2)^2+t^2)^2}|u|^2dxdydt\\
&\qquad=-2\alpha\int\frac{x^2+y^2}{(x^2+y^2)^2+t^2}|u|^2dxdydt.
\end{aligned}
$$

In conclusion

$$
\begin{aligned}
0\le&\int\left[\left(X+\alpha\frac{Xd_H}{dh}\right)u\left(X+\alpha\frac{Xd_H}{dh}\right)\bar{u}\right.\\
&\qquad\left.+\left(Y+\alpha\frac{Yd_H}{dh}\right)u\left(Y+\alpha\frac{Yd_H}{dh}\right)\bar{u}\right]dxdydt\\
=&\int\left(|Xu|^2+|Yu|^2+\alpha^2\frac{x^2+y^2}{(x^2+y^2)^2+t^2}|u|^2-2\alpha\frac{x^2+y^2}{(x^2+y^2)^2+t^2}|u|^2\right)dxdydt,
\end{aligned}
$$

and by choosing $\alpha = 1$ we get that

$$\int \left(|Xu|^2 + |Yu|^2 - \frac{x^2+y^2}{(x^2+y^2)^2+t^2}|u|^2 \right) dxdydt$$
$$= \int \left(|Xu|^2 + |Yu|^2 - \frac{|z|^2}{d^4}|u|^2 \right) dxdydt \geq 0.$$

Hence, for (1.10), the sharp Hardy inequality is

$$h[u] = \int_{\mathbb{R}^3} \left(|Xu|^2 + |Yu|^2\right) dzdt \geq \int_{\mathbb{R}^3} \frac{|z|^2}{d^4}|u|^2 dzdt.$$

□

Proposition 2. *For any function u such that $g[u] < \infty$ we have*

$$\int_{\mathbb{R}^3} (|\nabla_z u|^2 + 4|z|^2|\partial_t u|^2) dzdt \geq \int_{\mathbb{R}^3} \frac{|z|^2}{d^4}|u(z,t)|^2 dzdt. \tag{2.5}$$

Proof. By introducing polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, $r = |z|$ we obtain

$$\int_{\mathbb{R}^3} (|\nabla_z u|^2 + 4|z|^2|\partial_t u|^2)\, dzdt =$$
$$\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} (|\partial_r u|^2 + r^{-2}|\partial_\varphi u|^2 + 4r^2|\partial_t u|^2)\, r\, dr\, d\varphi\, dt$$
$$\geq \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} (|\partial_r u|^2 + 4r^2|\partial_t u|^2)\, r\, dr\, d\varphi\, dt.$$

So the proof is may be reduced to the inequality

$$0 \leq \int_{-\infty}^{\infty}\int_0^{\infty} (|\partial_r u|^2 + 4r^2|\partial_t u|^2)\, r\, drdt \geq \int_{-\infty}^{\infty}\int_0^{\infty} \frac{r^2}{r^4+t^2}|u|^2\, r\, drdt.$$

Let $d = d(r,t) = (r^4+t^2)^{1/4}$ and $\alpha \in \mathbb{R}$. Then a simple computation and integration by parts gives

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\left|\left(\partial_r+\alpha\frac{\partial_r d}{d}\right)u\right|^2+4r^2\left|\left(\partial_t+\alpha\frac{\partial_t d}{d}\right)u\right|^2\right)r\,drdt$$

$$=\int_{-\infty}^{\infty}\int_{0}^{\infty}(|\partial_r u|^2+4r^2|\partial_t u|^2)\,r\,drdt$$

$$-\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(6\alpha\frac{r^2}{d^4}-4\alpha\frac{r^6+r^2t^2}{d^8}-\alpha^2\frac{r^6+r^2t^2}{d^8}\right)|u|^2r\,drdt$$

$$=\int_{-\infty}^{\infty}\int_{0}^{\infty}(|\partial_r u|^2+4r^2|\partial_t u|^2)r\,drdt-\int_{-\infty}^{\infty}\int_{0}^{\infty}(2\alpha-\alpha^2)\frac{r^2}{d^4}\,|u|^2r\,drdt.$$

We now complete the proof by taking $\alpha=1$. $\square$

2.2 Proof of Theorem 1

Let us now consider

$$\int_{\mathbb{R}^3} (|\nabla_G + i\beta\mathcal{A})u|^2\, dzdt$$
$$= \int_{\mathbb{R}^3} \left(\left| \left(\partial_x - i\beta \frac{\partial_y d_H}{d_H} \right) u \right|^2 + \left| \left(\partial_y + i\beta \frac{\partial_x d_H}{d_H} \right) u \right|^2 \right) dxdydt$$
$$+ \int_{\mathbb{R}^3} \left(\left| \left(2x\partial_t - 2i\beta y \frac{\partial_t d_H}{d_H} \right) u \right|^2 + \left| \left(2y\partial_t + 2i\beta x \frac{\partial_t d_H}{d_H} \right) u \right|^2 \right) dxdydt.$$

We introduce polar coordinates for the z-plane:

$$r = \sqrt{x^2 + y^2}; \qquad \frac{x}{r} = \cos\varphi, \quad \frac{y}{r} = \sin\varphi$$

so that

$$\frac{\partial\varphi}{\partial x} = -\frac{y}{r^2}, \quad \frac{\partial\varphi}{\partial y} = \frac{x}{r^2}, \quad \partial_x = \cos\varphi \frac{\partial}{\partial r} - \frac{y}{r^2}\frac{\partial}{\partial\varphi}, \quad \partial_y = \sin\varphi \frac{\partial}{\partial r} - \frac{x}{r^2}\frac{\partial}{\partial\varphi}.$$

As before, the distance function is defined by $d = (r^4 + t^2)^{1/4}$.
We also have that

$$\frac{\partial_y d_H}{d_H} = \frac{r^3 \sin\varphi}{r^4 + t^2}, \quad \frac{\partial_x d_H}{d_H} = \frac{r^3 \cos\varphi}{r^4 + t^2}$$

and

$$2y \frac{\partial_t d_H}{d_H} = \frac{yt}{r^4 + t^2}, \quad 2x \frac{\partial_t d_H}{d_H} = \frac{xt}{r^4 + t^2}.$$

We shall consider then

$$\int_{\mathbb{R}^3} (|\nabla_G + i\beta\mathcal{A})u|^2\, dzdt = I_1 + I_2,$$

where

$$I_1 = \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \left(\left| \left(\cos\varphi\partial_r - \frac{\sin\varphi}{r}\partial_\varphi - i\beta \frac{r^3 \sin\varphi}{r^4 + t^2} \right) u \right|^2 \right.$$
$$\left. + \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \left| \left(\sin\varphi\partial_r + \frac{\cos\varphi}{r}\partial_\varphi + i\beta \frac{r^3 \cos\varphi}{r^4 + t^2} \right) \right|^2 \right) rdrd\varphi dt \quad (2.6)$$

and

$$I_2 = \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}\left(\left|\left(2r\cos\varphi\partial_t - i\beta\sin\varphi\frac{rt}{r^4+t^2}\right)u\right|^2\right) rdrd\varphi dt$$
$$+\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}\left(\left|\left(2r\sin\varphi\partial_t + i\beta\cos\varphi\frac{rt}{r^4+t^2}\right)u\right|^2\right) rdrd\varphi dt. \tag{2.7}$$

Computation of (2.6) gives

$$I_1 = \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}\left(\left|\left(\cos\varphi\partial_r - \frac{\sin\varphi}{r}\left(\partial_\varphi + i\beta\frac{r^4}{r^4+t^2}\right)\right)u\right|^2\right.$$
$$+\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}\left(\left|\left(\sin\varphi\partial_r + \frac{\cos\varphi}{r}\left(\partial_\varphi + i\beta\frac{r^4}{r^4+t^2}\right)\right)u\right|^2\right.$$
$$= \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}\left(|\partial_r u|^2 + \frac{1}{r^2}\left|\partial_\varphi u + i\beta\frac{r^4}{r^4+t^2}u\right|^2\right) rdrd\varphi dt.$$

Let us represent u via Fourier series

$$u(r,\varphi,t) = \sum_{k=-\infty}^{\infty} u_k(r,t)\frac{e^{ik\varphi}}{\sqrt{2\pi}}$$

and thus

$$\partial_\varphi u(r,\varphi,t) = \sum_{k=-\infty}^{\infty} iku_k(r,t)\frac{e^{ik\varphi}}{\sqrt{2\pi}}.$$

Then, since $-1/2 \le \beta \le 1/2$, we find that

$$\frac{1}{r^2}\int_0^{2\pi}\left|\partial_\varphi u + i\beta\,\frac{r^4}{r^4+t^2}\,u\right|^2 d\varphi = \frac{2\pi}{r^2}\sum_k\left(k+\beta\,\frac{r^4}{r^4+t^2}\right)^2|u_k|^2$$
$$\ge \frac{2\pi}{r^2}\min_k\left(k+\beta\,\frac{r^4}{r^4+t^2}\right)^2\sum_k|u_k|^2$$
$$= \frac{1}{r^2}\min_k\left(k+\beta\,\frac{r^4}{r^4+t^2}\right)^2\int_0^{2\pi}|u|^2\,d\varphi$$
$$= \beta^2\,\frac{r^6}{(r^4+t^2)^2}\int_0^{2\pi}|u|^2\,d\varphi,$$

because the minimum is reached when $k = 0$. Hence

$$I_1 \geq \beta^2 \frac{r^6}{(r^4+t^2)^2} \int_0^{2\pi} |u|^2 d\varphi.$$

Computing (2.7) gives that

$$I_2 = \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \left(4r^2 |\partial_t u|^2 + \beta^2 \frac{r^2 t^2}{(r^4+t^2)^2} |u|^2 \right) r dr d\varphi dt.$$

Putting I_1 and I_2 together gives

$$\begin{aligned}
\int_{\mathbb{R}^3} & \left(|(\nabla_G + i\beta\mathcal{A})u|^2 \right) dz dt \\
&\geq \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \left(|\partial_r u|^2 + 4r^2 |\partial_t u|^2 \right) r dr d\varphi dt \\
&\qquad + \beta^2 \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \frac{r^2(r^4+t^2)}{(r^4+t^2)^2} |u|^2 r dr d\varphi dt
\end{aligned}$$

which yields then

$$\begin{aligned}
\int_{\mathbb{R}^3} & \left(|(\nabla_G + i\beta\mathcal{A})u|^2 \right) dz dt \\
&\geq \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \left(|\partial_r u|^2 + 4r^2 |\partial_t u|^2 \right) r dr d\varphi dt \\
&\qquad + \beta^2 \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \frac{r^2}{r^4+t^2} |u|^2 r dr d\varphi dt.
\end{aligned}$$

Applying Proposition 2 to the first integral of the right hand side gives that

$$\int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \left(|\partial_r u|^2 + 4r^2 |\partial_t u|^2 \right) r dr d\varphi dt \geq \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \frac{r^2 |u|^2}{r^4+t^2} r dr d\varphi dt,$$

which leads to the final conclusion

$$\int_{\mathbb{R}^3} \left(|(\nabla_{G_0} + i\beta\mathcal{A})u|^2 \right) dz dt \geq (1+\beta^2) \int_{\mathbb{R}^3} \frac{|z|^2}{z^4+t^2} dz dt,$$

and that completes the proof.

Chapter 3

Lieb-Thirring inequalities for a class of sub-elliptic operators

The Lieb-Thirring inequalities are eminent in the study of the stability of matter in quantum mechanics. Define m and $\hbar$ as in section 1.1. Let V be a smooth bounded non-positive potential on $\mathbb{R}^d$. Then, by denoting the finite sequence of all negative eigenvalues of the Schrödinger operator (1.3) by

$$\lambda_1(v) < \lambda_2(V) \leq \lambda_3(V) \leq ... \leq \lambda_N(V) < 0,$$

one may for any N find a bound for the sum of eigenvalues $\sum_{i=1}^{N} |\lambda_i(V)|^\gamma$ in terms of $||V||_{L^{\gamma+d/2}}$.
The inequality

$$\sum_{i=1}^{N} |\lambda_i(V)|^\gamma \leq C(\gamma, d) \int_{\mathbb{R}^d} |V|^{\gamma+d/2} dx$$

is the celebrated *Lieb-Thirring inequality.* $C(\gamma, d)$ is the smallest constant possible independent of V.

3.1 The Birman-Schwinger operator and the Birman-Schwinger principle

In 1961 M. Sh. Birman [B] and J. Schwinger [Sch] presented independently a method for controlling the number of negative eigenvalues of Schrödinger operators, such that the problem could be rewritten as a compact operator. This operator is called *the Birman-Schwinger operator*. In its classical form the principle states that given a strictly positive operator H_0 and a non-negative compact operator V on the Hilbert space $\mathcal{H}$, the number of negative eigenvalues of the operator $H = H_0 - V$ coincides with the number of eigenvalues greater than one of the Birman-Schwinger operator $V^{1/2}H_0^{-1}V^{1/2}$. The Birman-Schwinger principle is an important device in the process of proving many different inequalities.

3.1.1 The Birman-Schwinger operator

Let $H_0 = -\Delta$. Consider the operator $H = H_0 - V$, and look at the eigenvalue problem

$$(H_0 - V)u = -\lambda u \tag{3.1}$$

with $\lambda > 0$ and $V \geq 0$. V is the multiplication operator, denoting the multiplication by a function $V(x)$.
By rewriting (3.1) as

$$(-\delta + \lambda)u = Vu = \sqrt{V}\sqrt{V}u,$$

we come to the Birman-Schwinger operator

$$K_\lambda = V^{1/2}(-\Delta + \lambda)^{-1}V^{1/2}$$

with an integral kernel $V^{1/2}(x)(-\Delta + \lambda)^{-1}(x, y)V^{1/2}(y)$ (if λ not in the spectrum). $(-\Delta + \lambda)(x, y)$ is the kernel of Green's function of $(-\Delta + \lambda)$. By writing $\sqrt{V}\sqrt{V}u = g$, we also get that

$$u = ((-\delta + \lambda)^{-1}g.$$

The operator is well defined, since for every fixed $g \in \mathcal{H}^{-1}(\mathbb{R}^d)$ there exists a unique $u \in \mathcal{H}^1(\mathbb{R}^d)$, such that

$$\int \nabla u \cdot \nabla f + \lambda \int uf = \int gf$$

holds for every $f \in \mathcal{H}^1(\mathbb{R}^d)$, this is due to the Riesz representation theorem. Furthermore, the functions g and u are linearly related functions. Moreover,

$$||u||^2_{\mathcal{H}^1(\mathbb{R}^d)} \sim \int_{\mathbb{R}^d} ((-\delta+\lambda)u \cdot u dx = \int_{\mathbb{R}^d} gu\, dx \leq ||g||_{\mathcal{H}^{-1}(\mathbb{R}^d)} ||u||_{\mathcal{H}^1(\mathbb{R}^d)},$$

where $||u||_{\mathcal{H}^1(\mathbb{R}^d)} \leq C(\lambda)||g||_{\mathcal{H}^{-1}(\mathbb{R}^d)}$.
This shows that $g \mapsto u$ is a bounded operator from $\mathcal{H}^{-1}(\mathbb{R}^d)$ to $\mathcal{H}^1(\mathbb{R}^d)$.
In one dimension the kernel of this operator is given by

$$(-\Delta+\lambda)^{-1}(x,y) = \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}|x-y|},$$

$(H_0 = -\Delta)$, and in three dimensions by

$$(-\Delta+\lambda)^{-1}(x,y) = \frac{1}{4\pi} \frac{e^{-\sqrt{\lambda}|x-y|}}{|x-y|}.$$

3.1.2 The Birman-Schwinger principle

We assume a potential V is of class $L^{d/2} + L^\infty$ and vanishes at infinity. Hence, the Birman-Schwinger operator is bounded on $L^2(\mathbb{R}^d)$. One can also see from the Sobolev inequality

$$\int V(x)|\psi(x)|^2 dx \leq \alpha \int |\nabla\psi(x)|^2 dx + \beta||\psi||_2^2,$$

where α and β are constants, that $V^{1/2}$ is bounded from $\mathcal{H}^1(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ as a multiplication operator, i.e.

$$\int V|\psi(x)|^2 dx \leq \left(\int V^{d/2} dx\right)^{2/d} \left(\int |\psi|^{\frac{2d}{d-2}}\right)^{\frac{d-2}{d}} \leq \left(\int |\nabla\psi|^2\right)\left(\int V^{d/2} dx\right)^{2/d}.$$

Lemma 1. *(The Birman-Schwinger principle) Consider the self-adjoint operator $-\Delta - V(x)$, where $V(x)$ is relatively bounded with respect to $-\Delta$ with bound less than 1.*
A number $-\lambda < 0$ is an eigenvalue of $-\Delta - V(x)$ if and only if 1 is an eigenvalue of the bounded positive operator

$$K_\lambda := V^{1/2}(-\Delta+\lambda)^{-1}V^{1/2}.$$

It is important to note, that the eigenvalues of K_λ are monotonically decreasing continuous functions of λ and go towards zero as λ goes to infinity. Therefore, if we are able to locate an eigenvalue that is greater than 1, we can be sure that there is an eigenvalue that equals to 1 for some larger value of λ. Hence the original problem that deals with an inequality has been reduced to a problem about the asymptotics of K_λ as λ tends to zero.

Proof. Consider ϕ to be the solution of the Schrödinger equation, that is, for all $f \in \mathcal{H}^1(\mathbb{R}^d)$ it holds that

$$\int \nabla\phi \cdot \nabla f + \lambda \int \phi f = \int V \phi f.$$

As we asserted before, the operator $V^{1/2}$ is bounded from $\mathcal{H}^1(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$, and therefore $V\phi \in \mathcal{H}^{-1}(\mathbb{R}^d)$ and thus

$$\phi = (-\Delta + \lambda)^{-1} V \phi$$

and

$$V^{1/2}\phi = V^{1/2}(-\Delta + \lambda)^{-1} V^{1/2} V^{1/2} \phi.$$

This shows that the eigenvalue of the Birman-Schwinger operator $K_\lambda(V)$ is 1, with the corresponding eigenfunction $V^{1/2}\phi$.
On the other hand, if

$$\phi = (-\Delta + \lambda)^{-1} V^{1/2} \psi,$$

where ψ satisfies

$$\psi = V^{1/2}(-\Delta + \lambda)^{-1} V^{1/2} \psi,$$

we see the $\phi \in \mathcal{H}^1(\mathbb{R}^d)$, that is ϕ satisfies

$$\int \nabla\phi \nabla f + \lambda \int \phi f = \int V^{1/2} \psi f \quad \forall f \in \mathcal{H}^1(\mathbb{R}^d)$$

for all $f \in \mathcal{H}^1(\mathbb{R}^d)$. But $V^{1/2}\psi \in \mathcal{H}^{-1}(\mathbb{R}^d)$, and moreover from the eigenvalue relation we see that

$$V^{1/2}\psi = V\phi.$$

$\square$

3.2 Classical Lieb-Thirring inequalities

The Lieb-Thirring inequalities give a sharp upper bound for the L^p-norm of a function, which is the pointwise sum of the squares of a finite orthonormal sequence of functions that are elements of a suitable Sobolev space.
It was originally proven for functions on the whole d-dimensional Euclidean space, where the proof later was extended to bounded domains and to suborthogonal sequences of functions.
Consider the Schrödinger operator $H = -\Delta + V$ in $L^2(\mathbb{R}^d)$, $V_\pm = \dfrac{V_\pm + |V|}{2}$. When the potential V is non-negative there are no non-negative eigenvalues. When V consists of both a positive and a negative parts, i.e. $V = V_+ + V_-$, then by the variational principle the eigenvalues of $-\Delta + V$ can be bounded from below by the eigenvalues of $-\Delta - V_-$. In this case, the Lieb-Thirring bound comes into use. Although both parts affect the negative eigenvalues, the Lieb-Thirring bound ignores the positive part of the vector potential: Only V_- has significance in the estimate of the negative eigenvalues of $-\Delta - V_-$. For this operator the negative eigenvalues satisfy the Lieb-Thirring inequalities of the form

$$\sum_{j\geq 1} |\lambda_j(-\Delta - V)|^\gamma \leq C(\gamma, d) \int V_-(x)^{\gamma + \frac{d}{2}} dx \tag{3.2}$$

for a fixed $\gamma > 0$ and a finite constant $C(\gamma, d)$; $\sum\limits_{j\geq 1} \lambda_j^\gamma$ is the Riesz-means of negative eigenvalues.
When $\gamma = 1$, the Riesz-mean gives the sum of absolute values of negative eigenvalues. When $\gamma = 0$ the Riesz-mean gives the number of negative eigenvalues.
It is known that the inequality (3.2) holds for $\gamma = \dfrac{1}{2}, d = 1$ [W] and $\gamma > \dfrac{1}{2}, d = 1$ [LTh], such that $\gamma \geq \dfrac{1}{2}$ if $d = 1$, $\gamma > 0, d \geq 2$ [LTh], and for $\gamma = 0, d \geq 3$ [C], [L], [Roz].
We shall give the proof of Lieb-Thirring inequalities, which was given in the paper [LTh].

3.2.1 Lieb-Thirring inequality

The Lieb-Thirring inequlity says, that If $d = 1, \gamma > 1/2$ and $d \geq 2, \gamma > 0$, then (3.2) holds.

3.2.2 Proof of the Lieb-Thirring inequality

In order to start the proof we need a useful inequality [Sim]:

Lemma 2. *Consider the operators $A, B \geq 0$. Then for any real number $m \geq 2$ it holds that*

$$\|AB\|_m \leq \left\|A^{m/2}B^{m/2}\right\|_2^{2/m} = \left\|A^{m/2}B^m A^{m/2}\right\|_2^{1/m}. \tag{3.3}$$

Proof. Set $C = A^{m/2}, D = B^{m/2}$, and consider $f(\theta) = ||C^\theta D^\theta||_{2\theta}$ where $0 < \theta \leq 1$ and $f(0) = 1$. Then f is continuous and (3.3) indicates that

$$f\left(\theta_0 = \frac{2}{p}\right) \leq f(1)^{\theta_0} f(0)^{1-\theta_0}. \tag{3.4}$$

In order to show that (3.4) golds, we need to show that $\log f$ is convex. Therefore we will prove that

$$f\left(\frac{\theta_1 + \theta_2}{2}\right) \leq f(\theta_1)^{1/2} f(\theta_2)^{1/2} \tag{3.5}$$

holds.

Let $\theta_1 \leq \theta \leq \theta_2$ and $\theta = \frac{1}{2}(\theta_1 + \theta_2)$. Then

$$f(\theta)^2 = ||C^\theta D^{2\theta} C^\theta||_{1/\theta}. \tag{3.6}$$

$$\leq ||C^{\theta_1} D^{\theta_1} D^{\theta_2} C^{\theta_2}||_{1/\theta} \tag{3.7}$$

$$\leq ||C^{\theta_1} D^{\theta_1}||_{2/\theta_1} ||D^{\theta_2} C^{\theta_2}||_{2/\theta_2} \tag{3.8}$$

$$= f(\theta_1) f(\theta_2). \tag{3.9}$$

The first inequality in (3.7) is due to the Golden-Thompson inequality [Gol], [Tho]; the inequality in (3.8) follows from Hölder's inequality; (3.9) follows from the fact that the operators are self-adjoint on the Hilbert space, i.e. $||T^*||_p = ||T||_p$.

□

Using the Birman-Schwinger principle we give a bound on the number of bound states that are less than $-\lambda$, $\lambda > 0$. We shall denote by μ_k are eigenvalues of K_λ. Start with a small λ, so that some of the eigenvalues of K_λ are big. These values decrease as μ grows, and every time one of them hits the value 1 the (negative) value of μ is an eigenvalue of the Schrödinger equation. If μ reaches λ, the number of these intersections equals the number of eigenvalues of $K_\lambda(V)$ that are greater or equal to 1.
Consider

$$H_0 + V + \lambda = H_0 + \alpha\lambda + (V + (1-\alpha)\lambda), \tag{3.10}$$

where $\alpha, \quad 0 < \alpha < 1$, is a constant. The new eigenvalue counting functions attained after the process will depend on α.
According to the variational principle the eigenvalue of $H_0 + V + \lambda$ can be bounded from below by

$$H_0 + \alpha\lambda - (V + (1-\alpha)\lambda)_-.$$

Let K_λ be the Birman Schwinger operator

$$K_\lambda = \sqrt{(V + (1-\alpha)\lambda)_-}(H_0 + \alpha\lambda)^{-1}\sqrt{(V + (1-\alpha)\lambda)_-}. \tag{3.11}$$

We see that $v = \sqrt{(V - (1-\alpha)\lambda)_+}u = K_\lambda v$, which indicates that whenever $-\lambda$ is a negative eigenvalue of H, then 1 is an eigenvalue of K_λ. Moreover, when $v \in L^2$, there is a one-to-one correspondence between the negative eigenvalues $-\lambda$ and the eigenvalue 1 of K_λ, hence the corresponding eigenspaces have the same dimension.
Let N_λ represents the number of eigenvalues of the Schrödinger problem that are less than $-\lambda$. This number is given by the number of eigenvalues which are greater or equal to 1 of the Birman-Schwinger operator $K_\lambda(V)$.
Note that $N(\lambda)$ is a step function that cannot be differentiated, but as a distribution it is the sum of a δ-function, and by integrating it, one attains the function's value at the point:

$$\int \underbrace{f(x)}_{\lambda^\gamma}\delta(x-a)dx = f(a), \quad \frac{\partial}{\partial\lambda}N_\lambda = \sum \delta(-\lambda + \lambda_j).$$

We have therefore

$$\sum_j |\lambda_j|^\gamma = \int_0^\infty \lambda^\gamma \sum \delta(\lambda_j+\lambda) d\lambda = \int_0^\infty \lambda^\gamma dN(\lambda) = \gamma \int_0^\infty \lambda^{\gamma-1} N_\lambda d\lambda, \tag{3.12}$$

where

$$\lambda_j = \int_0^\infty \lambda\delta(-\lambda+\lambda_j) d\lambda.$$

We consider the following (estimating N_λ was also discussed in [RozSol]:
Let μ_j be the eigenvalues of the operator K_λ. Then

$$N_\lambda = \#\{j : \lambda_j < -\lambda\} = \#\{`j : \mu_j \geq 1\}$$

and therefore

$$\begin{aligned} Tr\sqrt{(V+(1-\alpha)\lambda)_-}(H_0+\alpha\lambda)^{-1}\sqrt{(V+(1-\alpha)\lambda)_-} \\ \geq \#\{j : \mu \geq 1\} = \#\{j : \lambda_j < -\lambda\}. \end{aligned} \tag{3.13}$$

and the fact that the most obvious upper bound would be $Tr(K_\lambda)$, the following Lemma holds:

Lemma 3. *Using the Birman-Schwinger principle it holds that for every $\lambda \geq 0, m \geq 1$ and $\alpha \in [0,1]$ it holds that*

$$N_\lambda \leq K_\lambda. \tag{3.14}$$

Proof. $(-\Delta+\alpha\lambda)^{-m}$ is an integral operator of the kernel

$$\frac{1}{(2\pi)^d}\int e^{i(x-y)\xi}\frac{1}{(|\xi|^2+\alpha\lambda)^m}d\xi.$$

We have

$$\begin{aligned} N_\lambda &\leq Tr(K_\lambda) \leq Tr(K_\lambda^m) \\ &\leq Tr\left(\sqrt{(V+(1-\alpha)\lambda)_-}\frac{1}{H_0+\alpha\lambda}\sqrt{(V+(1-\alpha)\lambda)_-}\right)^m \\ &\leq Tr\left(\sqrt{(V+(1-\alpha)\lambda)_-}\right)^m (H_0+\alpha\lambda)^{-m}\left(\sqrt{(V+(1-\alpha)\lambda)_-}\right)^m \\ &= \int\int (V(x)+(1-\alpha)\lambda)_-^m \cdot \frac{1}{(|\xi|^2+\alpha\lambda)^m} dx d\xi, \quad x,\xi \in \mathbb{R}^d. \end{aligned}$$

Using (3.12) we may write

$$\sum_j |\lambda_j|^\gamma \leq \gamma \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \lambda^{\gamma-1} \frac{(V(x)+(1-\alpha)\lambda)_-^m}{(|\xi|^2+\alpha\lambda)^m} dx d\xi d\lambda$$

$$= \gamma \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \lambda^{\gamma-1} \frac{(-V_(x)+(1-\alpha)\lambda)_-^m}{(|\xi|^2+\alpha\lambda)^m} dx d\xi d\lambda$$

$$= \gamma \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \lambda^{\gamma-1} \frac{V_-^m(x)(-1+\frac{1-\alpha}{V_-(x)}\lambda)_-^m}{(|\xi|^2+\alpha\lambda)^m} dx d\xi d\lambda. \quad (3.15)$$

Change of variables

$$\frac{1-\alpha}{V_-(x)}\lambda = \mu,$$

allows us to continue and rewrite (3.15) as

$$\gamma \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \lambda^{\gamma-1} \frac{V_-^m(x)(-1+\frac{1-\alpha}{V_-(x)}\lambda)_-^m}{(|\xi|^2+\alpha\lambda)^m} dx d\xi d\lambda$$

$$= \gamma \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 \left(\frac{V_(x)}{1-\alpha}\right)^{\gamma-1} \mu^{\gamma-1} V_-(x)^m \frac{(-1+\mu)_-^m}{(|\xi|^2+\frac{\alpha V_-(x)}{1-\alpha}\mu)^m} \left(\frac{V_-(x)}{1-\alpha}\right) d\mu d\xi dx$$

$$= \gamma \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 \left(\frac{V_-(x)}{1-\alpha}\right)^{\gamma} V_-(x)^m \frac{\mu^{\gamma-1}(1-\mu)_+^m}{\left(\frac{\alpha V_-(x)}{1-\alpha}\right)^m \left(\frac{|\xi|^2(1-\alpha)}{\alpha V_-(x)}+\mu\right)^m} d\mu d\xi dx.$$

Another change of variables

$$\xi \left(\frac{1-\alpha}{\alpha V_-(x)}\right)^{1/2} = \eta,$$

leads us to

$$\gamma \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 \left(\frac{V_-(x)}{1-\alpha}\right)^{\gamma} V_-(x)^m \frac{\mu^{\gamma-1}(1-\mu)_+^m}{\left(\frac{\alpha V_-(x)}{1-\alpha}\right)^m \left(\frac{|\xi|^2(1-\alpha)}{\alpha V_-(x)}+\mu\right)^m} d\mu d\xi dx$$

$$= \gamma \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 \frac{V_-(x)^\gamma}{(1-\alpha)^\gamma} \frac{\alpha^m}{(1-\alpha)^m} \frac{\mu^{\gamma-1}(1-\mu)_+^m}{(|\eta|^2+\mu)^m} \left(\frac{\alpha V_-(x)}{1-\alpha}\right)^{d/2} d\mu d\eta dx.$$

Hence (3.12) can be estimated in the following way:

$$\sum_j |\lambda_j|^\gamma \leq \gamma \int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} V_-(x)^{\gamma+d/2} \frac{\alpha^{m+d/2}}{(1-\alpha)^{\gamma+m+d/2}} \frac{\mu^{\gamma-1}(1-\mu)_+^m}{|\xi|^2+\mu)^m} d\mu d\eta dx$$
$$= \int_{\mathbb{R}^d} V_-(x)^{\gamma+d/2} dx \cdot C(\gamma, d, m, \alpha),$$

where

$$C(\gamma, d, m, \alpha) = \frac{\alpha^{m+d/2}}{(1-\alpha)^{\gamma+m+d/2}} \int_{\mathbb{R}^d} \int_0^1 \frac{\mu^{\gamma-1}(1-\mu)_+^m}{|\xi|^2+\mu)^m} d\mu d\eta.$$

$C(\gamma, d, m, \alpha)$ is finite if $\dfrac{d}{2} < m < \dfrac{d}{2} + 1$.
Minimizing $C(\gamma, d, m, \alpha)$ with respect to m and α as it is done in [LTh], we find that

$$C(\gamma, d, m, \alpha) = \gamma(4\pi)^{-d/2} \alpha^{-m+d/2} (1-\alpha)^{m-\gamma-d/2} \frac{\Gamma(\gamma - m + d/2)\Gamma(m - n/2)}{\Gamma(\gamma + 1 + d/2)} m.$$

$\square$

Chapter 4

Lieb-Thirring inequalities for the Heisenberg Laplacian

In this chapter we consider the eigenvalue problem

$$-X^2 - Y^2 + Vu = \lambda u.$$

The main result of this chapter is the following thereom:

Theorem 2. *Let $\{\lambda_k\}_{k=1}^{\infty}$ denote the negative eigenvalues of the operator*
$H = H_0 - V$. For any $\gamma > 0$ if $V \in L^{2+\gamma}(\mathbb{R}^3)$, then

$$\sum_j |\lambda_k|^\gamma \le C(\gamma) \int V_-(x)^{\gamma+2} dx, \quad \gamma > 0.$$

4.1 Main result

4.1.1 Spectral decompositions

Spectral decomposition is the factorization of a matrix into a canonical form, whereby the matrix is represented in terms of its eigenvalues and eigenfunctions.
It is convenient to introduce Fourier Transform

$$\mathcal{F}u(x_1, x_2, \xi_3) = \int_{-\infty}^{\infty} u(x_1, x_2, x_3) e^{-ix_3\xi_3} dx_3.$$

Then

$$
\begin{aligned}
H_0 &= \mathcal{F}^{-1}\left[(i\partial_{x_1}+x_2\xi_3)^2+(i\partial_{x_2}-x_1\xi_3)^2\right]\mathcal{F} \\
&= \mathcal{F}^{-1}_{-i\partial_{x_3}\to\xi_3} H_0 \mathcal{F}_{-i\partial_{x_3}\to\xi_3} = (i\partial_{x_1}+x_2\xi_3)^2+(i\partial_{x_2}-x_1\xi_3)^2 \\
&= \mathcal{F}^{-1}\sum_{k=0}^{\infty}|\xi_3|(2k+1)P_k^{(|\xi_3|)}\mathcal{F},
\end{aligned}
\tag{4.1}
$$

see [RW].
Therefore H_0 has a spectrum $|\xi_3|(2k+1), k=0,1,......$
P_k are integral operators in $(L^2(R^2))$, $z=(x_1,x_2), z'=(x_1',x_2')$ that are orthogonal properties. It is also well known, see for example [Fra Lp] that $P_k^{|\xi_3|}$ are integral operators with integral kernel

$$
P_k^{|\xi_3|}(z,z') = \frac{B}{2\pi}e^{-iB(z\times z')/2-B|z-z'|^2/4}L_{k-1}\left(B|z-z'|^2/2\right), \quad B\to|\xi_3|.
\tag{4.2}
$$

Here L_{k-1} represemt the Laguerre polynomial of degree $k-1$, normalized by $L_{k-1}(0)=1$.
The kernel of the operator $\mathcal{F}^{-1}\mathcal{H}_0^{-m}\mathcal{F}$ is equal to

$$
\sum\frac{1}{(|\xi_3|(2k+1))^m}P_k^{|\xi_3|}(z,z').
$$

4.1.2 Proof of theorem 2

According to Lemma 2 , if A, B are self adjoint and $A, B\geq 0$, then

$$
\left[Tr\left(A^{1/2}BA^{1/2}\right)^m\right]^{1/m}\leq\left(TrA^{m/2}B^mA^{m/2}\right)^{1/m} = Tr\left(B^mA^m\right)^{1/m}.
$$

Applying this to the Birman-Schwinger Kernel yields

$$
Tr(K_\lambda^m)\leq Tr\left(V+(1-\alpha)\lambda\right)_-^m\left(H_0+\alpha\lambda\right)^{-m} =
$$

$$
=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(V+(1-\alpha)\lambda)_-^m\sum\frac{1}{(|\xi_3|(2k+1)+\alpha\lambda)^m}P_k(z,z,|\xi_3|)dzdx_3
$$

$$
=\sum_{k=0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(V+(1-\alpha)\lambda)_-^m}{(|\xi_3|(2k+1)+\alpha\lambda)^m}P_k(z,z,|\xi_3|)dzdx_3.
$$

Since $P(z,z,|\xi_3|) = \dfrac{\xi_3}{2\pi}$, using (4.2), we have that

$$I = \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(V+(1-\alpha)\lambda)_-^m}{(|\xi_3|^m(2k+1)+\alpha\lambda)^m} \frac{|\xi_3|}{2\pi} d\xi_3 dx_3.$$

By applying (3.12), using homogeneity with respect to λ and $|\xi_3|$ and introducing change of variables we obtain

$$\begin{aligned}\sum_{j=0}^{\infty} |\lambda_j|^\gamma &\leq \gamma \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_0^\infty \lambda^{\gamma-1} \sum_{k=0}^{\infty} \frac{(V-(1-\alpha)\lambda)_+^m}{(|\xi_3|(2k+1)+\alpha\lambda)^m} \frac{|\xi_3|}{2\pi} d\lambda d\xi dx_3 dx_2 dx_1 \\ &= 2\gamma \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_0^\infty V^{\gamma+2} \sum_{k=0}^{\infty} \frac{t^{\gamma-1}\,(1-(1-\alpha)t)_+^m}{(|\eta|(2k+1)+\alpha t)^m} \frac{\eta}{2\pi} d\eta dt dx_3,\end{aligned} \tag{4.3}$$

and hence (4.3) becomes

$$\sum_{j=0}^{\infty} |\lambda_j|^\gamma \leq \int_0^\infty V_-^{\gamma+2} dx \frac{\gamma}{(2-m)(1-m)} \frac{\pi}{8} \int_0^\infty \alpha^{2-m} t^{1-m+\gamma} (1-(1-\alpha)t)_+^m dt. \tag{4.4}$$

Set $(1-\alpha)t = u$, such that $t = \dfrac{u}{1-\alpha}$.
The second integral on the right hand side of (4.4) becomes

$$\begin{aligned}&\int_0^1 \alpha^{2-m} \left(\frac{1}{(1-\alpha)}\right)^{1-m+\gamma} u^{1-m+\gamma} (1-u)_+^m \frac{du}{(1-\alpha)} \\ &= \frac{\alpha^{2-m}}{(1-\alpha)^{2-m+\gamma}} \int_0^1 u^{1-m+\gamma}(1-u)_+^m du.\end{aligned}$$

Recalling that $m > 2$ and in order to be able to integrate (4.4) with respect to t avoiding singularity requires $1-m+\gamma > -1$.
This shows that $\gamma > 0$.

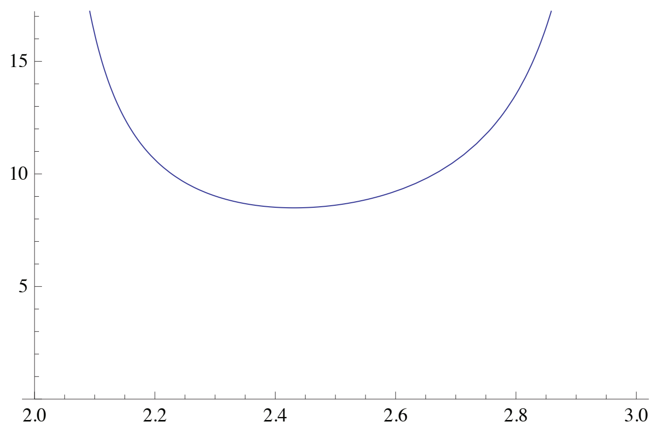
Minimizing α for some fixed m gives

$$
\begin{aligned}
&\frac{d}{d\alpha}\left(\alpha^{2-m}(1-\alpha)^{-2-\gamma+m}\right) \\
&\quad = (2-m)\alpha^{1-m}(1-\alpha)^{-2-\gamma+m} - \alpha^{2-m}(-2-\gamma+m)(1-\alpha)^{-3-\gamma+m} \\
&\quad = \alpha^{1-m}(1-\alpha)^{-3-\gamma+m}\left((2-m)(1-\alpha) - \alpha(-2-\gamma+m)\right) \\
&\quad = (2-m)(1-\alpha) - \alpha(-2-\gamma+m) \\
&\quad = 2-m+\alpha\gamma = 0.
\end{aligned}
$$

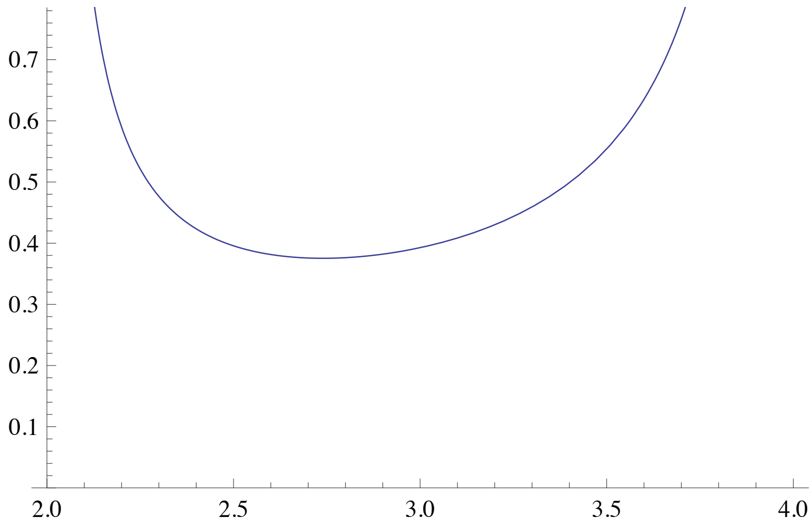
and hence $\alpha = \dfrac{m-2}{\gamma}$.

Fixing γ and plotting m as a function of γ yields the following:

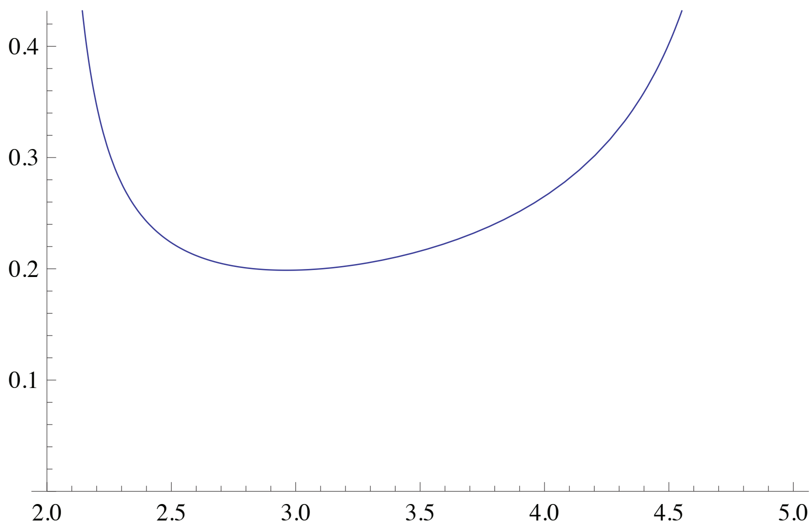
$$\gamma = 1:$$



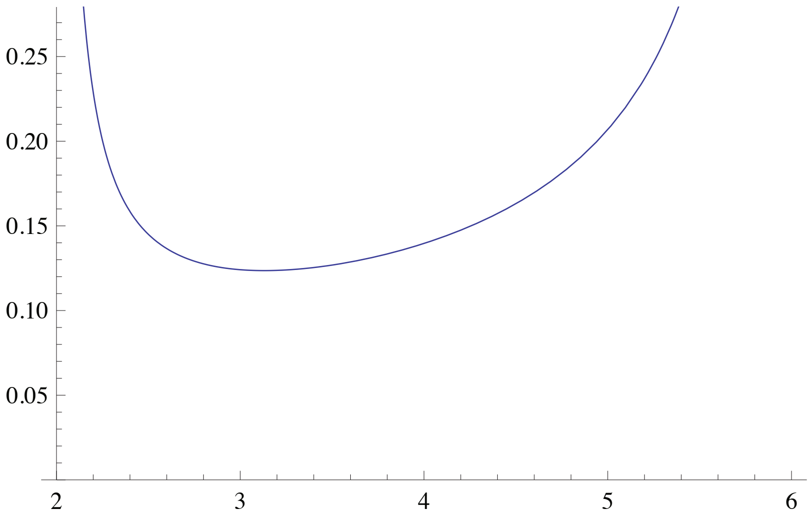
$\gamma = 2:$



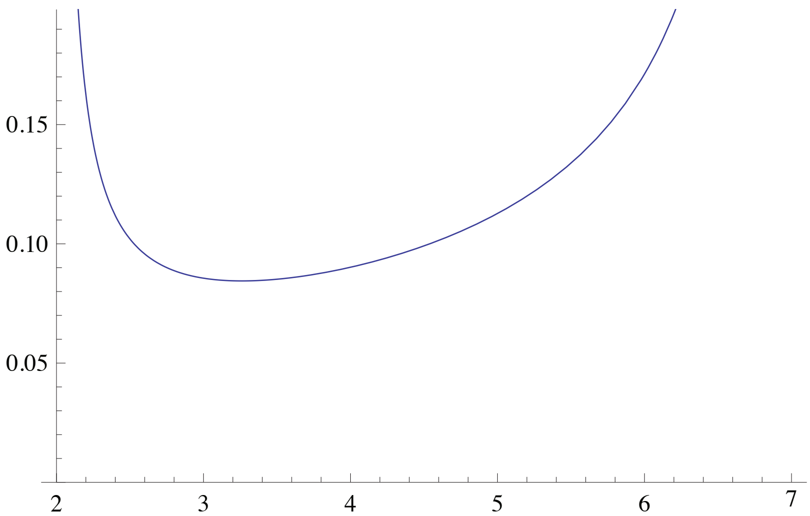
$\gamma = 3:$



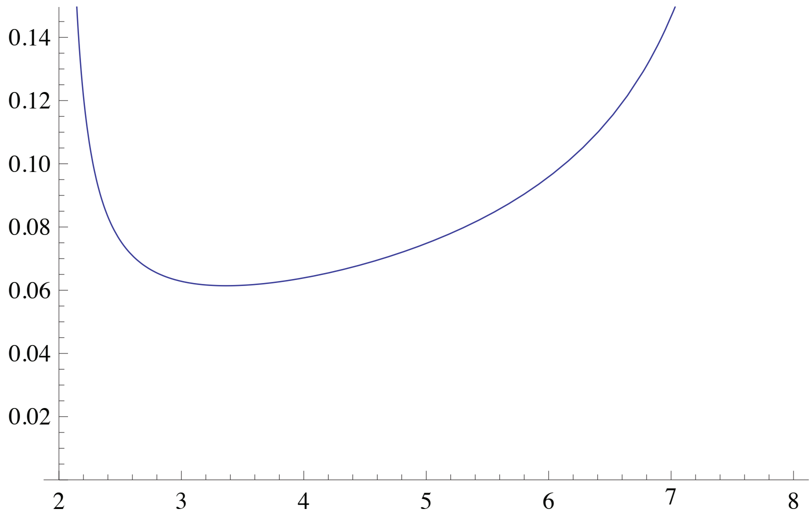
$\gamma = 4:$



$\gamma = 5:$



$\gamma = 6:$



Bibliography

[B] M. Sh. Birman, *On the spectrum of singular boundary-value problems* (in Russian) Mat. Sb. **55** (1961), 125–174; English transl. in Amer. Math. Soc. Trans., **53** (1966), 23–80.

[D] E. B. Davies, *A review of Hardy inequalities*, in Operator Theory: Advances and Applications, Birkhäuser-Verlag Basel, **110** (1999), 55–67.

[A1] L. D'Ambrosio, *Some Hardy inequalities on the Heisenberg group*, Differ. Equ., **40** (2004), no. 4, 552–564.

[A2] L. D'Ambrosio, *Hardy inequalities related to Grushin type operators*, Proc. Amer. Math. Soc., **132** (2004), no. 3, 725–734.

[C] M. Cwikel, *Weak type estimates for singular values and the number of bound states of Schrödinger operators* Ann. Math. **106** (1977), no.1, 93-100.

[DGN] J. Dou, Q. Guo and P. Niu, *Hardy inequalities with remainder terms for the generalized Baouendi-Grushin vector fields*, Math. Inequal. Appl., **13** (2010), no. 3, 555–570.

[G] N. Garofalo, *Unique continuation for a class of elliptic operators which degenerate on a manifold of arbitrary codimension*, J. Differential Equations, **104** (1993), no. 1, 117–146.

[GL] N. Garofallo and E. Lanconelli, *Frequency functions on Heisenberg group, the uncertaincy principle and unique continuation* , Ann. Inst. Fourier, **40** (1990), no 2, 313–356.

[Gol] S. Golden, *Lower bounds for the Helmholtz function*, Phys. Rev., **137** (1965), no. 2, B1127-B1128.

[Gr] V. V. Grushin, *On a class of hypoelliptic operators*, Math. USSR Sbornik, **12** (1970), 458–476.

[H] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147-171.

[Ka] A. Kaplan, *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms*, Trans. Amer. Math. Soc., **258** (1980), no 1, 147-153.

[Ko] I. Kombe, *Hardy, Rellich and uncertainty principle inequalities on Carnot groups*, preprint arXiv:math/0611850 .

[L] E. H. Lieb, *Bounds on the eigenvalues of the Laplace and Schrödinger operators*, Bull. Am. Math. Soc., **82** (1976), 751-753.

[LTh] E. H. Lieb and W.E. Thirring, *Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities*, Stud. math. Phys., Essays Honor Valentine Bargmann, (1976), 269-303.

[LW] A. Laptev and T. Weidl, *Hardy inequalities for magnetic Dirichlet forms*, in Operator Theory: Adv. and Appl., **108** (1999), 299–305.

[M] V. G. Maz'ya, *Sobolev Spaces*, Springer Verlag, Berlin New York, 1985.

[NCH] P. Niu, Y. Chen and Y. Han, *Some Hardy-type inequalities for the generalised Baouendi-Grushin operators*, Glasgow Math. J., **46** (2004), 515–527.

[Roz] G. V. Rozenblum, *Distribution of the discrete spectrum of singular differential operators*, Dokl. Akad. Nauk SSSR **202** (1972), no.5, 1012-1015. Translated in Soviet Math. Dok. , **13** (1972), 245-249. See also Izv. Vyssh. Uchebn. Zaved. Mat., (1976), no. 1, 75-86.

[RozSol] G. V. Rozenblum and M. Solomyak, *CLR-estimate revisited: Lieb's approach with no path integrals*, St. Petersburg Math. J., **9** (1998), no. 6, 1195-1211.

[RW] G. D. Raikov and S. Warzel, *Quasi-classical versus non-classical spectral asymptotics for magnetic Schrdinger operators with decreasing electric potentials.*, Rev. Math. Phys., **14** (2002), No. 10, 1051-1072.

[Sch] J. Schwinger, *On the bound states of a given potential*, Proc. Nat. Acad. Sci. U.S.A., **47** (1961), 122-129.

[Sim] B. Simon, *Trace ideals and their applications. 2nd edition* Mathematical Surveys and Monographs, **120**(2005), Providence, RI: American Mathematical Society (AMS).

[Sol] M. Solomyak, *A remark on the Hardy inequalities*, Integral Equations Oper. Theory **19** (1994), no.1, 120-124.

[Ste] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, **43**. Princeton University Press, Princeton, NJ, 1993.

[Szu] A. Szulkin, *A semilinear Schrödinger equation with Aharonov- Bohm magnetic potential.* Proceedings of the workshop "Progress in Variational Problems", RIMS, Kyoto, June 2010, to appear.

[Tha] S. Thangavelu, *Harmonic analysis on the Heisnberg group*, Progress in Mathematics, Birkhäuser, Bosteon-Basel-Berlin, 1998.

[Tho] C. J. Thompson, *Inequality with applications is statistical mechanics*, J. Math. Phys., **6** (1965), 1812-1813.

[W] T. Weidl, *On the Lieb-Thirring constants $L_{\gamma,1}$ for $\gamma \geq 1/2$*, Commun. Math. Phys., /textbf178 (1996), no.1, 135-146.