ISSN: 1401-5617

Some properties of one and two phase quadrature domains

Mahmoudreza Bazarganzadeh

Research Reports in Mathematics
Number 7, 2010

Department of Mathematics
Stockholm University

Electronic versions of this document are available at
http://www.math.su.se/reports/2010/7

Date of publication: December 21, 2010
2000 Mathematics Subject Classification: Primary 35R35, Secondary 35B06, 76D27, 49Q10.
Keywords: Quadrature domain, Two- phase problems, Uniqueness, Free boundary problem, Level set method, Shape optimization.

Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.math.su.se/
info@math.su.se

Filosofie licentiatavhandling

Some properties of one and two phase quadrature domains

Mahmoudreza Bazarganzadeh

Dissertation presented to Stockholm University in partial fulfillment of the requirements for the Degree of Licentiate of Philosophy (*Filosofie licentiatexamen*), to be presented on January 11, 2011 at 11:00 in Room 306, Building 6, Department of Mathematics, Stockholm University (Kräftriket).

Principal advisor: Henrik Shahgholian.
Second advisor: Boris Shapiro.
Opponent: Tomas Sjödin.
Examiner: Anzdrej Szulkin.

ABSTRACT

In recent years, quadrature domains have been encountered in various connections such as inverse problems of Newtonian gravitation, Hele-Shaw flows of viscous fluids and etc. This thesis consists of some properties of two phase quadrature domains and two numerical schemes to approach to one phase subharmonic quadrature domain.

Two phase quadrature domain has been introduced recently by Emamizadeh- Prajapat-Shahgholian. Our goal in the first paper is to investigate general properties of the two-phase quadrature domains. The concept, which is the generalization of the well-known one-phase case, introduces substantial difficulties with interesting and even richer features than its one-phase counterpart. We deal with the following free boundary problem.

For given positive constants $\lambda^{\pm}$ and two bounded and compactly supported measures $\mu^{\pm}$, we investigate the uniqueness of the solution of the following free boundary problem:

$$\begin{cases} \Delta u = (\lambda^+ \chi_{\Omega^+} - \mu^+) - (\lambda^- \chi_{\Omega^-} - \mu^-), & \text{in } \mathbb{R}^N \ \ (N \geq 2), \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega = \Omega^+ \cup \Omega^-$. It is further required that the supports of $\mu^{\pm}$ should be inside $\Omega^{\pm}$.

Along the lines of various properties that we state and prove in Paper A, we also present several conjectures and open problems that we believe should be true.

In the second paper we treat to the one phase subharmonic quadrature domains. It is well known that Ω is a subharmonic quadrature domain with respect to a positive Radon measure μ, if and only if Ω solves the following free boundary problem:

$$\begin{cases} \Delta u = \chi_{\{u>0\}} - \mu, & \text{in } \ \mathbb{R}^N, \\ u \geq 0, & \text{in } \ \mathbb{R}^N, \\ u = 0, & \text{in } \ \mathbb{R}^N \setminus \Omega. \end{cases} \tag{P}$$

Our target is to find an efficient and robust numerical algorithms to approach to the solution of Problem (P). To do this we give two methods.

In the first method by applying the proprieties of given free boundary problem and level set techniques, we derive a method that leads to a fast iterative solver. The iteration procedure is adapted in order to work in the case when topology changes. The second method is based on shape reconstruction to establish an efficient Quasi-Newton-method. Various numerical experiments confirm the efficiency of the derived numerical methods.

ACKNOWLEDGMENT

It is a pleasure to thank those who made this thesis possible. The author wishes to express his gratitude to his supervisor Prof. Henrik Shahgholian for his supervision, advice, and guidance from the very early stage of this research as well as giving extraordinary experiences through out the work. Above all and the most needed, he provided me unflinching encouragement and support in various ways. I am indebted to him more than he knows.

I gratefully acknowledge Prof. Boris Shapiro for his advice, supervision that I would have not finished this thesis without his support.

I am indebted to many of my colleagues to support me, Farid Bozorgnia and Ceni Babaoglu. Farid, I am grateful in your kindness and generous behavior. I hope to keep up our collaboration in the future.

I owe my deepest gratitude to Prof. Björn Gustafsson for fruitful discussion during my studies. I would like to show my gratitude to Prof. Tomas Sjödin for making valuable comments on paper A.

The author would also like to convey thanks Prof. Ralf Fröberg and Prof. Mikael Passare whom gave me this chance to come to Sweden and continue my educations. I would also like to appreciate Prof. Torsten Ekedahl and the director of the postgraduate students, Prof. Yishao Zhou.

I would like to show my gratitude to many of my friends who support me, Martin Strömqvist, Avetik Arakelyan and specially many thanks go to Ayaz Razmjooei and Shiva Samieinia.

I would like also to thank the people in the Mathematics department at Stockholm University whom were important to the successful realization of thesis, as well as expressing my apology that I could not mention personally one by one.

Last but not least, I wish to express my love and gratitude to my beloved family. Words fail me to express my appreciation to my wife Sadna whose dedication, love and persistent confidence in me, has taken the load off my shoulder. Finally my little honey, Viana, thank you to endure me.

To my love:

Sadna

CONTENTS

INTRODUCTION TO THESIS

MAHMOUDREZA BAZARGANZADEH

Contents

1. Notations and Preliminaries

We shall use the following notations in this thesis.

$\mathbb{R}^N$	Euclidean space of dimension N,
μ	an arbitrary measure,
Ω	an open subset of $\mathbb{R}^N$ (generally connected),
$\|\Omega\|$	the volume of Ω,
$L^p(\Omega)$	the usual Lebesgue space with respect to the Lebesgue measure,
$HL^p(\Omega)$	the subspace of $L^p(\Omega)$ that consists of functions harmonic in Ω,
$SL^p(\Omega)$	the subspace of $L^p(\Omega)$ that consists of functions subharmonic in Ω,
χ_Ω	the characteristic function of Ω,
$C^k(\Omega)$	the class of $k-$ times continuously differentiable in Ω,
U^μ	the Newtonian potential of the measure μ,
V	the velocity field,
$\mathbf{n}$	the outward normal vector on the boundary of a level set,
$J(\Omega)$	the shape functional,
$y(\Omega)$	a solution of a boundary value problem defined in Ω.

We shall occasionally use the Sobolev space $W^{m,p}(\Omega)$ of distributions u in Ω such that $\partial^\alpha u \in L^p(\Omega)$ for all multi-indices α with $|\alpha| < m$ and its subspace $W_0^{m,p}(\Omega)$ which is the $\overline{C_0^\infty(\Omega)}$ in $W^{m,p}(\Omega)$, i.e, the infinitely differentiable functions on $\mathbb{R}^N$ whose support is a compact set of Ω. For $p = 2$, we use $H^m(\Omega)$, $H_0^m(\Omega)$ instead of $W^{m,2}(\Omega)$, $W_0^{m,2}(\Omega)$ respectively.

G always denotes the "fundamental solution" for the Laplace operator in $\mathbb{R}^N$. In other words for $x \in \mathbb{R}^N \setminus \{0\}$,

$$G(x) = \begin{cases} \frac{1}{N(N-2)\omega_N}|x|^{2-N}, & \text{for} \quad N \geq 3, \\ -\frac{1}{2\pi}\ln|x|, & \text{for} \quad N = 2, \end{cases}$$

where ω_N is the volume of unit sphere in $\mathbb{R}^N$. It is known that if Ω is open and bounded then for $G(x-y)$ considered as a function of $x \in \Omega$, the following holds (see [11]),

$$\begin{aligned} G(x-y) &\in HL^1(\Omega), \quad \forall y \in \Omega^c, \\ -G(x-y) &\in SL^1(\Omega), \quad \forall y \in \Omega, \\ \pm G_j = \pm\frac{\partial G}{\partial x_j} &\in SL^1(\Omega), \quad \forall y \in \Omega^c, 1 \leq j \leq N. \end{aligned}$$

Moreover, the linear combinations with positive coefficients of the functions

$$\pm G_j(x-y), G(x-y),\ x \in \Omega,\ \forall y \in \Omega^c,$$

$$-G(x-y),\ \forall y \in \mathbb{R}^N,$$

are dense in SL^1, and the linear combination with real coefficients of the functions $G_j(x-y)$ and $G(x-y)$ for $y \in \Omega^c$ are dense in HL^1 (see [11]).

2. Quadrature domains

The English word "quadrature" comes from the Latin word "quadratura". It means "making square shaped" and in general it meant "to divide a land into squares"! In mathematics "quadrature" refer to constructive or numerical methods for determining areas, and recently it is used as a term for computing indefinite integrals in general.

Through this thesis the term "quadrature" has a related meaning. For example, a quadrature identity will typically be an exact formula for the integral of harmonic or analytic functions. The domain of integration is then a quadrature domain. We say a few words of the starting point of quadrature domains theory.

H. S. Shapiro and his group began to extend and generalize the concept of quadrature domains more than thirty years ago. Some basic reference for their efforts are [12] and [29]. For recent contributory, see [32] and [15].

The connection between the Laplacian growth, especially Hele Shaw flow, and quadrature domains has been investigated by Richardson in [27]. Before that these two theory were developing in parallel. For instance, around 1980, the construction of quadrature domains from the potential theoretical point of view ([24] and [25]) and the theory of weak solution for Hele Shaw problem ([13] and [6]) were studied simultaneously and independently. For more information see [15].

2.1. One phase quadrature domains

In this section we give a formal definition of a quadrature domain. First we introduce the Newtonian potential and some of its important properties. The basic sources for theses results are [1], [3] and [16].

Let μ be a measure. By U^μ we mean the Newtonian potential of the measure μ defined by

$$U^\mu(x) := (G * \mu)(x) = \int_{\mathbb{R}^N} G(x-y)d\mu(y), \quad x \in \mathbb{R}^N.$$

Thus, U^{χ_Ω} (from now on U^Ω for simplicity) is the Newtonian potential of Ω considered as a body with density one.

Theorem 2.1. *If μ is a Radon measure with compact support then U^μ and ∇U^μ are defined a.e and are in L^1_{loc}. Moreover, if μ is positive then U^μ is defined everywhere.*

Remark 1. The measure μ is a called Radon measure if it is inner regular and locally finite.

Theorem 2.2. *Suppose that μ is a Radon measure with compact support then one has*

$$-\Delta U^\mu = \mu,$$

in the sense of distributions.

Corollary 2.3. *If μ is a Radon measure with compact support then U^μ is harmonic in the complement of $supp(\mu)$.*

Theorem 2.4. *If μ is a Radon measure with compact support then*

$$|U^\mu(x)| = O(|x|^{2-N}) \to 0 \text{ as } |x| \to \infty \text{ if } N \geq 3,$$

and

$$U^\mu(x) = -\frac{1}{2\pi}\ln|x| \int d\mu + O(|x|^{-1}) \text{ as } |x| \to \infty \text{ if } N = 2.$$

Generally, if $-\Delta u = \mu$ then we can not derive that $u = U^\mu$, since one can add any harmonic function to u. But if u behaves like a potential at infinity we are able to conclude $u = U^\mu$.

Theorem 2.5. *Suppose that μ is a Radon measure with compact support and $-\Delta u = \mu$. If u satisfies*

$$u(x) \to 0 \text{ as } |x| \to \infty \text{ if } N \geq 3,$$

and

$$u(x) = -\frac{1}{2\pi} \ln|x| \int d\mu + O(|x|^{-1}) \text{ as } |x| \to \infty \text{ if } N = 2,$$

then $u = U^{\mu}$.

Now we define a harmonic quadrature domain.

Definition 2.6. Suppose that μ is a measure with compact support. By a *quadrature domain* with respect to μ we mean an open connected set $\Omega \subset \mathbb{R}^N$ such that $supp(\mu) \subset \Omega$ and

$$\int_\Omega h\, dx = \int h\, d\mu, \tag{1.1}$$

holds for all $h \in HL^1(\Omega)$. We will say Ω is a *quadrature domain* (QD) and write $\Omega \in Q(\mu, HL^1)$.

In the simplest case, it is known that discs $D(a;r)$ are the only quadrature domains (see [7]) and the quadrature identity then reduces to the ordinary mean value property for harmonic functions:

$$h(a)|D(a;r)| = \int_{D(a;r)} h\, dx.$$

Generally, if Ω is a bounded domain in $\mathbb{R}^N$ and

$$\int_\Omega h dx = |\Omega| h(x_0), \tag{1.2}$$

holds for all $h \in HL^1(\Omega)$, where x_0 is an arbitrary point, then Ω is a ball centered at x_0, see [7].

Thus a quadrature identity can be thought of as a generalized mean value property. The quadrature identity (1.1) is equivalent to the following identities (see [11]),

$$\begin{cases} U^\Omega = U^\mu, & \text{in } \mathbb{R}^N \setminus \Omega, \\ \nabla U^\Omega = \nabla U^\mu, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.3}$$

It has been explained in [14] and [23] that $\Omega \in Q(\mu, HL^1)$ is equivalent to finding a pair (u, Ω) of solution of the following one-phase free boundary problem:

$$\begin{cases} \Delta u = \chi_\Omega - \mu & \text{in } \mathbb{R}^N, \\ u = \nabla u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.4}$$

where $u = U^\mu - U^\Omega$ is the so-called modified Schwarz potential (MSP) of the pair (μ, Ω).

Remark 2. By a "*free boundary problem*" we mean a boundary value problem in which we deal with solving a partial differential equations in a domain such that a part of the boundary is unknown in advance. That part of the boundary is called the free boundary. In order to solve a free boundary problem we need the standard boundary condition and an additional one which is imposed at the free boundary. One then can determine both the free boundary and the solution of the differential equation. This kind of boundary value problem arise for instance, in fluid dynamics, tumor growth, chemical vapor deposition, image development in electro-photography and financial mathematics. For more information we refer to [4], [9] and [17].

Note that from (1.4) one has $\Delta u = \chi_\Omega$ away from $supp(\mu)$. According to the results on local regularity of solutions of elliptic PDEs, we obtain $u \in W^{2,p}_{\text{loc}}(\Omega)$ for every $1 < p < \infty$. Also $\nabla u \in W^{1,p}_{\text{loc}}(\Omega)$. By Sobolev embedding theorem the first derivatives are therefore Hölder continuous with Hölder exponent $\alpha < 1$.

2.2. Subharmonic quadrature domains

M. Sakai in [25] and [26] realized the importance of subharmonic quadrature domains.

Definition 2.7. Let μ be a measure with compact support. By a *subharmonic quadrature domain* we mean an open connected set $\Omega \subset \mathbb{R}^N$ such that $supp(\mu) \subset \Omega$ and

$$\int_\Omega h\,dx \geq \int h\,d\mu, \tag{1.5}$$

holds for all $h \in SL^1(\Omega)$. We write $\Omega \in Q(\mu, SL^1)$ if (1.5) holds.

For instance, suppose that $\mu = \alpha\delta$ where δ is the Dirac mass at origin and $\alpha > 0$. Then

$$Q(\mu, HL^1) = Q(\mu, SL^1) = \{B(0;r)\},$$

where $r \geq 0$ is determined by $|B(0;r)| = \alpha$, see [11].

Similar discussion shows that $\Omega \in Q(\mu, SL^1)$ if and only if $U^\mu \geq U^\Omega$ in $\mathbb{R}^N$ and $U^\mu = U^\Omega$ in $\mathbb{R}^N \setminus \Omega$. From PDE point of view, $\Omega \in Q(\mu, SL^1)$ is equivalent to the solution of the following free boundary problem, (see [14])

$$\begin{cases} \Delta u = \chi_\Omega - \mu & \text{in} \quad \mathbb{R}^N, \\ u \geq 0, & \text{in} \quad \mathbb{R}^N, \\ u = 0, & \text{in} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.6}$$

It is easy to give examples of quadrature domains that are not subharmonic quadrature domains, see [29].

Example 2.8. Let $\mu = \mu_\alpha = \alpha\,\rho$ where $\alpha > 0$ and ρ is the mass uniformly distributed on the sphere $S = \partial B(0,1)$. Define

$$\Omega_\beta = \{x \in \mathbb{R}^2 \,:\, \beta < \pi |x|^2 < \beta + \alpha\},$$

where $\beta \geq 0$, $\Omega = \Omega_0 \cup \{0\}$. Then $|\Omega_\beta| = \alpha$. Sakai in [25] has proved that for each $0 < \alpha \leq e\pi$ there exists a unique $\beta = \beta_\alpha$ with $\pi - \alpha < \beta_\alpha < \pi$ such that

$$\int_{\Omega_{\beta_\alpha}} G\,dx = \int G\,d\mu_\alpha.$$

For $0 < \alpha \leq \pi$ one can prove (see [11]),

$$Q(\mu_\alpha, HL^1) = Q(\mu_\alpha, SL^1) = \{\Omega_{\beta_\alpha}\},$$

and for all $\alpha > \pi$

$$Q(\mu_\alpha, HL^1) = \{\Omega, \Omega_{\beta_\alpha}\},$$
$$Q(\mu_\alpha, SL^1) = \{\Omega_{\beta_\alpha}\}.$$

2.3. Two-phase quadrature domain

Two-phase quadrature domains has been introduced recently by Emamizadeh, Prajapat and Shahgholian, [5]. They have studied the existence of two-phase quadrature domains with some sign restrictions. Here we generalize one-phase quadrature domain to the two-phase case.

Let Ω be an open and bounded subset of $\mathbb{R}^N$. We define $\widetilde{H}(\Omega)$ by

$$\widetilde{H}(\Omega) = \{U^\eta \,:\, \eta \text{ is a signed Radon measure with compact support and } supp(\eta) \subset \Omega^c\}.$$

It is not difficult to show

- If $h \in \widetilde{H}(\Omega)$ then $h \in L^1_{loc}(\mathbb{R}^N)$.
- All functions in $\widetilde{H}(\Omega)$ are harmonic in Ω.
- For $x \in \Omega^c$ we have $G(x - .) = U^{\delta_x} \in \widetilde{H}(\Omega)$.
- Suppose that h is harmonic on a bounded open set D such that $\Omega \subset\subset D$. There exists a measure ν with compact support such that $supp(\nu) \subset D \setminus \overline{\Omega}$ and $h = U^\nu$.

These useful properties of $\widetilde{H}(\Omega)$ lead us to have the following definition of two-phase quadrature domain.

Definition 2.9. Let $\Omega^\pm$ be two open, disjoint and connected subsets of $\mathbb{R}^N$ and $\mu^\pm$ be two positive Radon measures with compact supports. Moreover, suppose that $\lambda^\pm$ are two positive constants. We say that $\Omega = \Omega^+ \cup \Omega^-$ is a *two-phase quadrature domain*, with respect to $\mu^\pm, \lambda^\pm$ and $\widetilde{H}(\Omega)$, if $supp(\mu^\pm) \subset \Omega^\pm$, and

$$\text{(1.7)} \qquad \int_{\Omega^+} \lambda^+ h \; - \int_{\Omega^-} \lambda^- h = \int h\,(d\mu^+ - d\mu^-), \quad \forall h \in \widetilde{H}(\Omega).$$

We then write $\Omega^{\pm} \in Q(\mu^{\pm}, \widetilde{H}(\Omega))$ or $\Omega \in Q(\mu, \widetilde{H}(\Omega))$ where $\mu = \mu^{+} - \mu^{-}$.

From the potential theory point of view, let us choose $f = \lambda^{+}\chi_{\Omega^{+}} - \lambda^{-}\chi_{\Omega^{-}}$, $\mu = \mu^{+} - \mu^{-}$. Suppose that $y \in \Omega^{c}$ then $h(x) = h_y(x) = G(x-y) \in \widetilde{H}(\Omega)$ and consequently (1.7) yields

$$U^{f} = U^{\mu} \text{ in } \mathbb{R}^{N} \setminus \Omega.$$

Also there is a strong connection between free boundary theory and two phase quadrature domains that we have studied in the first paper. We have showed that $\Omega \in \widetilde{H}(\Omega)$ is and only if (u, Ω) be a solution of the following free boundary problem

$$\begin{cases} \Delta u = (\lambda^{+}\chi_{\Omega^{+}} - \mu^{+}) - (\lambda^{-}\chi_{\Omega^{-}} - \mu^{-}), & \text{in } \mathbb{R}^{N}, \\ u = 0, & \text{in } \mathbb{R}^{N} \setminus \Omega, \end{cases} \tag{1.8}$$

with $supp(\mu^{\pm}) \subset \Omega^{\pm}$. This free boundary problem is a two-phase version of (1.4).

Similarly to the one phase case some natural questions arise . The problem of existence and uniqueness of two phase quadrature domains are more complicated. As far as we know the only literature [5] and [10] deal with the existence problem in two phase case.

In the first paper we investigate some general properties of two phase harmonic and subharmonic quadrature domains. By considering some sign assumptions on $\Omega^{\pm}$ we prove uniqueness for (1.8).

2.4. An application (Hele Shaw flow)

The class of growth processes, in which the dynamics of a moving front (an interface) between two distinct phases is driven by a harmonic scalar field is known under the name "Laplacian growth". The most known examples of Laplacian growth are, viscous fluids in the Hele-Shaw cell, filtration processes in porous media, electrodeposition. For instance, see [2, 22]. In this subsection we study Hele Shaw problem.

In the hydrodynamic interpretation, one imagines that the inner domain is filled with a non-viscous fluid, say air, and the outer domain with a viscous one, say oil. Air is supposed to be injected at the origin and there is an oil drain at infinity. The pressure p, in the air domain is constant and set to zero by convention. In the oil domain the pressure satisfies the Laplace equation $\Delta p = 0$. If we neglect the surface tension, then the pressure vanishes on the boundary curve and the model is equivalent to the Laplacian growth, [18].

Suppose that some incompressible fluid is confined between two parallel plates and we inject more fluid to it with moderate velocity. Therefore, the fluid between plates will occupy more space. We are interested in to study the behavior of its free boundary. Richardson has formulated this problem as follows, see [22].

Suppose that μ is a positive, finite and non zero measure with compact support and $supp(\mu) \subseteq D$ where D is an open subset of $\mathbb{R}^N$ with C^1-boundary. Moreover, consider that the origin is in the $supp(\mu)$. Let p_D be the super harmonic function such that

$$\begin{cases} -\Delta p_D = \mu & \text{in } D, \\ p_D = 0 & \text{on } \partial D. \end{cases} \tag{1.9}$$

We are looking for a family of regions D_t for $t \geq 0$, such that ∂D_t moves with the velocity $-\nabla p_{D_t}$ where p_{D_t} is the unique solution of (1.9).

2.4.1. The Weak solution of the Hele Shaw problem

Let D_0 and μ be as above and I be an open interval. A map $I \ni t \to D_t \subset \mathbb{R}^N$ is a *weak solution* of the free boundary problem if the function $u_t \in H^1(\mathbb{R}^N)$ defined by

$$\chi_{D_t} - \chi_{D_0} = \Delta u_t + t\mu, \tag{1.10}$$

satisfies:

$$u_t \geq 0,$$
$$< u_t, 1 - \chi_{D_t} >= 0,$$

where $< \cdot, \cdot >$ is the duality between H_0^1 and its dual space H^{-1}. For more details see [13].

Theorem 2.10. [13] *Suppose that μ and D_0 be as before and $T > 0$. Then there exists a weak solution*

$$[0, T] \ni t \to D_t \subset \mathbb{R}^N,$$

for the problem which is unique and if u_t be the function appearing above then u_t is also unique and

$$u_t = \int_0^t p_{D_\tau} \, d\tau.$$

Moreover, D_t can be chosen to be

$$D_t = D_0 \cup \{z : u_t(z) > 0\}.$$

In what follows we give simple examples of the Hele Shaw problem.

Example 2.11. Find $p(x,t), T(t)$ such that

$$\begin{cases} \frac{\partial^2 p}{\partial x^2} = 0, & 0 < x < T(t),\ t > 0, \\ p(T(t), t) = 0, & t \geq 0, \\ \frac{\partial p}{\partial x}(T(t), t) = -T', & t > 0, \\ p(0, t) = A > 0, & t \geq 0, \\ T(0) = x_0. \end{cases}$$

Here $T(t)$ is the free boundary and p is interpreted as the pressure. By imposing the first condition one has

$$p(x,t) = K_1(t)x + K_2(t). \tag{1.11}$$

According to the assumptions one can write $K_2(t) = A$ and $K_1(t) = -\frac{A}{T(t)}$, and hence

$$p(x,t) = -\frac{A}{T(t)}x + A = A\Big(1 - \frac{x}{T(t)}\Big).$$

The fixed boundary condition gives us a simple ordinary differential equation, $T'(t)T(t) = A$ and by considering the last condition we have $T(t) = \sqrt{2At + x_0^2}$. It means that

$$p(x,t) = A\Big(1 - \frac{x}{\sqrt{2At + x_0^2}}\Big).$$

Hence $D_t = [-T(t), T(t)]$ and by integrating $p(x,t)$ with respect to t on the interval $[0,t]$, we can find the corresponding $u(x,t)$, see [8].

Example 2.12. We continue our examples by considering the radially symmetric case of the Hele Shaw flow and generalize it. In this case our free boundary is a sphere in $\mathbb{R}^N$, $N \geq 2$ and we consider that the boundary of the initial domain has an equation like $|x| = r_o$.

Find $p(x,t) = p(|x|,t)$ and $T(t)$ such that

$$\begin{cases} -\Delta p(x,t) = 0, & r_0 < |x| < T(t), t > 0, \\ p(T(t),t) = 0, & |x| = T(t), t > 0, \\ \frac{\partial p}{\partial n}(T(t),t) = -T', & |x| = T(t), t > 0, \\ p(x,t) = A, & |x| = r_0, t \geq 0, \\ T(0) = x_0. \end{cases}$$

The solution of the above problem can be calculated as follows.

We know that p is the fundamental solution of Laplacian operator in $r_0 < |x| < T(t)$, i.e,

$$p(x,t) = \begin{cases} -\frac{1}{2\pi}K_1(t)\ln|x| + K_2(t), & N = 2, r_0 < |x| < T(t), \\ \frac{1}{(N-2)|S^{N-1}|}\cdot\frac{K_3(t)}{|x|^{N-2}} + K_4(t), & N \geq 3, r_0 < |x| < T(t), \end{cases} \tag{1.12}$$

where $|S^{N-1}|$ is the area of the unit sphere $S^{N-1} \subset \mathbb{R}^N$. We consider two cases.

- Case $N = 2$: By continuity condition we have

$$p(x,t) = -\frac{1}{2\pi}K_1(t)(\ln|x| - \ln r_0) + A.$$

By imposing the fixed boundary condition $p(x,t) = A$ if $|x| = r_0$ and the continuity conditions at the free boundary $p(x,t) = 0$ if

$|x| = T(t)$, and finally imposing the third condition, we obtain

$$A = T'T \ln \frac{T}{r_0}.$$

Integrating of this ordinary differential equation over $(0,t)$, one obtains an algebraic equation

$$At = \frac{T^2}{2}\left(\ln \frac{T}{r_0} - \frac{1}{2}\right) + \frac{x_0^2}{4}.$$

The solution of this equation is the free boundary, see [8].

- Case $N > 2$: We can compute the solution as follows:
  Set $\frac{1}{(N-2)|S^{N-1}|} = a$, so by the first condition we have

$$p(x,t) = aK_3(t)(|x|^{2-N} - r_0^{2-N}).$$

We impose the fixed and boundary conditions, and derive

$$A(2-N) = T'(t)T(t) - r_0^{2-N}T'(t)T^{N-1}(t).$$

Integrating over $(0,t)$ and with respect to the last condition one gets an algebraic equation which gives us the free boundary, see [8],

$$A(2-N)t = \frac{1}{2}T^2 + \frac{x_0^N}{Nr_0^{N-2}} - \frac{T^N}{Nr_0^{N-2}} - \frac{x_0^2}{2}.$$

3. Level set method and shape optimization

In this section we provide some ingredients related to the second paper.

3.1. Level set method

The main numerical technique to track the evolution of interface is the level set method. The Osher-Sethian level set method tracks the motion of an interface by embedding the interface as the zero level set of the signed distance function which is defined by

$$d_\Omega(x) = \begin{cases} d(x,\Omega^c), & \text{if } x \in \Omega, \\ -d(x,\Omega), & \text{if } x \in \Omega^c, \end{cases}$$

where $d(x,\Omega) = \inf_{y\in\Omega} |x-y|$. If Ω is a subset of the Euclidean space $\mathbb{R}^N$ with a piecewise-smooth boundary, the signed distance function is differentiable almost everywhere, and its gradient satisfies $|\nabla d_\Omega| = 1$. For general information about the level set method see [20, 28, 21].

The key point of the level set approach is to represent domains and their boundaries as level sets of a continuous function ϕ without considering boundary parametrization. For tracking the motion of an open set

$\Omega(t)$, $t \in \mathbb{R}^+$, one can define a function $\phi_t : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ such that

$$\Omega(t) = \{\phi(x,t) < 0 \,:\, x \in \mathbb{R}^N\},$$

and the zero level set will be represented by

$$\Gamma_t = \partial\Omega(t) = \{\phi(x,t) = 0 \,:\, x \in \mathbb{R}^N\}.$$

If the evolution of the shape is determined by a flow $x(t) = \alpha(t, x(0))$ such that

$$\frac{dx}{dt}(t) = \mathbf{V}(x(t), t),$$

then the corresponding level set function ϕ is determined by the first-order Hamilton-Jacobi equation

$$\frac{\partial \phi}{\partial t} + \mathbf{V} \cdot \nabla\phi = 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^+.$$

Now let $F = \mathbf{V} \cdot \mathbf{n}$ where $\mathbf{n}$ is the outward normal vector on Γ and

$$\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|}.$$

Therefore we are able to compute the level set functions by

$$\frac{\partial \phi}{\partial t} + F|\nabla\phi| = 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^+. \tag{2.1}$$

Note that we have to extend the velocity field in the whole $\mathbb{R}^N$ and solve the equation. In this thesis we restrict our attention to the case (2.1) where ϕ is considered as the sign distance function. Therefore, the level set equation (2.1) turns to be

$$\frac{\partial \phi}{\partial t} + F = 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^+.$$

Moreover, we solve a boundary value problem to get F in every iteration.

3.2. Shape optimization

Shape optimization is a indispensable tool in the design and construction of industrial structures. For example, air craft and spacecraft have to satisfy, at the same time, very strict criteria on mechanical performance while weighing as little as possible. The shape optimization problem for such a structure consists of finding a geometry of the structure which minimizes a given functional and yet satisfies specific constraints (like thickness, strain energy or displacement bounds). From mathematics point of view, in shape sensitivity we analyze how the solution of a PDE changes when the domain is changing with a velocity field. This subsection is mainly based on [31].

Through this thesis any shape functional is denoted by $J(\Omega), J : \Omega \to J(\Omega) \in \mathbb{R}$, where Ω is a domain of class C^k for $k \geq 1$. Some examples of the domain functionals are:

$$J_1(\Omega) = \int_\Omega dx, \qquad J_2(\Omega) = \int_{\partial\Omega} ds.$$

In many shape optimization problems the following situations occur.

A shape functional $J(\Omega)$ depends on the domain Ω via the solution, $y(\Omega)$, to a boundary value problem defined in Ω. For instance, in our problem we consider the following free boundary problem

$$\text{(P)} \qquad \begin{cases} \Delta u = \chi_{\{u>0\}} - \mu, & \text{in} \quad \mathbb{R}^N, \\ u \geq 0, & \text{in} \quad \mathbb{R}^N, \\ u = 0, & \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases}$$

for given measure $\mu \geq 0$. It is well known that the minimizer of

$$J(v,\Omega) = \int_\Omega \frac{1}{2}|\nabla v|^2\, dx + \int_\Omega (1-\mu)v^+\, dx, \tag{2.2}$$

for $v \in H_0^1$, is the solution of Problem (P) and vise versa, see [14].

Let $x \in \mathbb{R}^N$, and $\mathbf{V}(t,x)$ be a velocity field (vector field) defined in a domain say D, and $\mathbf{V} \in C^k(D;\mathbb{R}^N), \mathbf{V}|_{\partial D} = 0$. Let t be artificial time. Moreover, assume that $\Sigma \subseteq D$. We define a transformation

$$T_t(\mathbf{V})x = X(t,x), \quad x \in \Sigma,$$

with a velocity field $\mathbf{V}$ by differential equations

$$\frac{\partial X}{\partial t}(t,x) = \mathbf{V}(t,x), \quad X(0,x) = x.$$

We denote the image of $\Sigma \subset \Omega$ under T_t by Σ_t.

Definition 3.1. Let Σ be a measurable subset of D. For any vector field $\mathbf{V} \in C^k(D;\mathbb{R}^N)$ the Eulerian derivative of the domain functional $J(\Sigma)$ at Σ in the direction of the vector field V is defined as the limit

$$\lim_{t\to 0} \frac{J(\Sigma_t) - J(\Sigma)}{t} := dJ(\Sigma,\mathbf{V}),$$

where

$$\Sigma_t = T_t(\mathbf{V})(\Sigma).$$

Example 3.2. Consider the functional $J_1(\Sigma) = \int_\Sigma dx$. Therefore

$$J_1(\Sigma_t) = \int_{\Sigma_t} dx.$$

Transforming the integral to an integral over Σ leads to

$$J_1(\Sigma) = \int_\Sigma \gamma(t)\, dx,$$

where $\gamma(t) = \det(DT_t)$ is the Jacobian of the transformation $T_t(\mathbf{V})$. From proposition 2.44 in [31] it follows that $\gamma(0) = 1$, $\gamma'(0) = div\,\mathbf{V}(0)$, thus

$$\begin{aligned} dJ_1(\Sigma, \mathbf{V}) &= \lim_{t\to 0} \frac{J_1(\Sigma_t) - J_1(\Sigma)}{t} \\ &= \int_\Sigma \lim_{t\to 0} \frac{\gamma(t) - \gamma(0)}{t}\, dx \\ &= \int_\Sigma \gamma'(0) dx \\ &= \int_\Sigma div\,\mathbf{V}(0)\, dx. \end{aligned}$$

By applying the Gauss theorem one can see that

$$dJ_1(\Sigma, \mathbf{V}) = \int_{\partial\Sigma} \mathbf{V}(0) \cdot \mathbf{n}\, ds.$$

Definition 3.3. For a function $y(\Sigma)$, $\Sigma \in C^k, k \geq 1$, we define its *material derivative* as a limit

$$\dot{y}(\Sigma; \mathbf{V})(x) := \lim_{t\to 0} \frac{y(\Sigma_t) \circ T_t(\mathbf{V}) - y(\Sigma)}{t}.$$

Also the *shape derivative* of $y(\Sigma)$ in the direction $\mathbf{V}$ is the element $y'(\Sigma; \mathbf{V})$ defined by

$$y'(\Sigma; \mathbf{V}) := \dot{y}(\Sigma; \mathbf{V}) - \nabla y(\Sigma) \cdot \mathbf{V}(0).$$

The shape derivative represents the change of a function y with respect to the geometry. The following example shows the relation of these two aspects.

Example 3.4. Let

$$J(\Omega) = \int_\Omega y(\Omega)\, dx \qquad \text{with} \quad y(\Omega) : \Omega \to \mathbb{R},$$

and use the change of variables $x = T_t(\mathbf{V})(X)$ the integral defined on Ω_t is transformed to the domain Ω, hence

$$J(\Omega_t) = \int_{\Omega_t} y(\Omega_t)\, dx = \int_\Omega (y(\Omega_t) \circ T_t(\mathbf{V}))\gamma(t)\, dx,$$

where $\gamma(t) = \det(DT_t)$ is the Jacobian of the transformation $T_t(\mathbf{V})$. By definition

$$dJ(\Omega, \mathbf{V}) = \int_\Omega \lim_{t\to 0} \frac{(y(\Omega_t) \circ T_t(\mathbf{V}))\gamma(t) - (y(\Omega) \circ T_t(\mathbf{V}))\gamma(0)}{t}\, dx,$$

and consequently,

$$\begin{aligned} dJ(\Omega, \mathbf{V}) &= \int_{\Omega} \left(\dot{y}(\Omega; \mathbf{V}) + y(\Omega) div \mathbf{V}(0) \right) dx \\ &= \int_{\Omega} \left(\dot{y}(\Omega; \mathbf{V}) - \nabla y(\Omega) \cdot \mathbf{V}(0) + div(y(\Omega)\mathbf{V}(0)) \right) dx \\ &= \int_{\Omega} y'(\Omega, \mathbf{V})\, dx + \int_{\partial\Omega} y(\Omega)\mathbf{V}(0) \cdot \mathbf{n}\, ds. \end{aligned}$$

References

[1] D. Armitage, S. Gardiner, Classical potential theory, Springer Monographs in Mathematics, 2001.

[2] D. Crowdy, Quadrature domains and fluid dynamics. Quadrature domains and their applications, 113-129, Oper. Theory Adv. Appl., 156, Birkhäuser, Basel, 2005.

[3] J. Doob, Classical potential theory and its probabilistic counterpart, Springer, 2001.

[4] J.I. Diaz, J. L. Vazquez, M. A. Herrero, A. Linan, Free boundary problems: theory and applications, Longman Scientific & Technical, 219 pages, 1995.

[5] B. Emamizadeh, J. Prajapat, H. Shahgholian , A two-phase free boundary problem related to quadrature domains, Potential Anal. Springer Science, 2010.

[6] C. M. Elliott, V. Janovsky, A variational inequality approach to Hele Shaw flow with a moving boundary, Proc. Royal Soc. Edinburgh, 88A (1981), 93-107.

[7] B. Epestin, M. Shiffer, On the mean value property of harmonic functions, J. d́analyse Math.14, pp. 109-111, 1965.

[8] Q. Fernando, V. Juan Luis, Asymptotic convergence of the Stefan problem to Hele-Shaw, Trans. Amer. Math. Soc. 353 (2001), no. 2, 609-634.

[9] A. Fridman, Variational Principles and Free-Boundary Problems, Wiley-Interscience Publication, Wiley, New York, 1982.

[10] S. Gardiner, T. Sjödin, Two phase quadrature domains, Journal d'Analyse Mathématique, To appear.

[11] B. Gustafsson , On quadrature domains and an inverse problem in potential theory, J.Analyse Math. 55 (1990), 172-216.

[12] B. Gustafsson, Quadrature identities and the Schottky double, Acta Appl. Math. 1 (1983), 209-240.

[13] B. Gustafsson, Applications of variational inequalities to a moving boundary problem for Hele-Shaw flows, SIAM J. Math. Anal. 16 (1985), 279-300.

[14] B. Gustafsson, H. Shahgholian, Existence and geometric properties of solutions of a free boundary problem in potential theory, J. Reine Angew. Math. 473 (1996), 137-179.

[15] B. Gustafsson, H. Shapiro, What is a quadrature domain?, 125, Oper. Theory Adv. Appl., 156, Birkhäuser, Basel, 2005.

[16] O.D. Kellogg, Foundations of Potential Theory, Dover, 1953.

[17] D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Academic Press, New York, 1980.

[18] H. Lama, Hydrodynamics, 6th ed., Dover, New York, 1932.

[19] L. Niemeyer, L. Pietronero and H. Wiesmann, Fractal dimension of dielectric breakdown, Phys. Rev. Lett. 52 (1984), 10-33.

[20] S. Osher, R. P. Fedkiw, Level set methods and dynamic implicit surfaces, Springer, 2003.

[21] S. Osher, J.A Sethian, Fronts Propagating with Curvature Dependent speed: Algorithms Based on Hamilton-Jacobi Formulation, Journal of Computational Physics, Vol. 79, pp. 12-49, 1988.

[22] S. Richardson, Hele Shaw flows with a free boundary produced by the injection of fluid into a narrow channel, J. Fluid Mech., 56 (1972), pp. 609-618.

[23] M. Sakai , Applications of variational inequalities to the existence theorem on quadrature domains, Trans. Am. Math. soc. 276 (1983), 267-279.

[24] M. Sakai, A moment problem on Jordan domains, Proc. Amer. Math. Soc. 70 (1978), 35-38.

[25] M. Sakai, Quadrature Domains, Lect. Notes Math. 934, Springer-Verlag, Berlin-Heidelberg 1982.

[26] M. Sakai, The submeanvalue property of subharmonic functions and its application to the estimation of the Gaussian curvature of the span metric, Hiroshima Math. J. 9 (1979), 555-593.

[27] S. Richardson, Hele Shaw flows with a free boundary produced by the injection of fluid into a narrow channel, J. Fluid Mech. 56 (1972), 609-618.

[28] J.A. Sethian, Level set methods and fast marching methods, Cambridge University Press, 378 pages, 1999.

[29] H.S. Shapiro, The Schwarz function and its generalization to higher dimensions, Uni. of Arkansas Lect. Notes Math. Vol. 9, Wiley, New York, 1992.

[30] E. Sharon, M. Moore, W. McCormick, H. Swinney, Coarsening of fractal viscous fingering patterns, Phys. Rev. Lett. 91 (2003).

[31] J. Sokolowski, J. Zolesio, Introduction to shape optimization, Springer, 1992.

[32] Quadrature Domains and Applications, A Harold S. Shapiro Anniversary Volume, (eds. P. Ebenfelt, B. Gustafsson, D. Khavinson, M. Putinar), Birkhäuser, 2005.

Department of Mathematics, Stockholm University, S-10691, Stockholm, Sweden

E-mail address: mahmoudreza@math.su.se

SOME PROPERTIES OF TWO-PHASE QUADRATURE DOMAINS

CENI BABAOGLU AND MAHMOUDREZA BAZARGANZADEH

ABSTRACT. In this paper, we investigate general properties of the two-phase quadrature domains, which recently has been introduced by Emamizadeh-Prajapat-Shahgholian. The concept, which is the generalization of the well-known one-phase case, introduces substantial difficulties with interesting and even richer features than its one-phase counterpart.

For given positive constants $\lambda^{\pm}$ and two bounded and compactly supported measures $\mu^{\pm}$, we investigate the uniqueness of the solution of the following free boundary problem

$$\begin{cases} \Delta u = (\lambda^+ \chi_{\Omega^+} - \mu^+) - (\lambda^- \chi_{\Omega^-} - \mu^-), & \text{in } \mathbb{R}^N \ (N \geq 2), \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega = \Omega^+ \cup \Omega^-$. It is further required that the supports of $\mu^{\pm}$ should be inside $\Omega^{\pm}$; this in general may fail and give rise to non-existence of solutions.

Along the lines of various properties that we state and prove here, we also present several conjectures and open problems that we believe should be true.

CONTENTS

2000 *Mathematics Subject Classification.* Primary: 35R35, 35B06.
Key words and phrases. quadrature domain, two-phase problems, uniqueness.
C. Babaoglu thanks the Göran Gustafsson Foundation for supporting several visits to KTH.

1. Introduction

The concept of quadrature domains is well known in modern potential theory and concerns generalized form of (sub)mean-value property for (sub)harmonic functions.

The main idea in this paper is to deal with a two-phase version of this concept, introduced in [7]. Our main result concerns uniqueness for two-phase quadrature domains when certain restrictions are made on the sign(s) of the solution function.

This paper is organized as follows. Section 2 contains some background in one-phase case and some fundamental concepts in potential theory. In section 3 we then move to the two-phase case scenario and extract its PDEs formulation and introduce quadrature inequalities and take some examples. In section 4 we note some recently result on existence theory for two phase free boundary problem and finally in the last section we study the uniqueness and prove our main result just by considering some conditions. Also we make some conjectures.

2. One-phase case

The definition of a quadrature domain is as follows.

Definition 2.1. Let μ be a Radon measure with compact support in $\mathbb{R}^N$. An open connected domain $\Omega \subset \mathbb{R}^N$, $(N \geq 2)$ is called *quadrature domain* with respect to μ if

$$\int_\Omega h\,dx = \int h\,d\mu, \qquad \forall h \in HL^1(\Omega), \quad supp(\mu) \subset \Omega, \tag{2.1}$$

where $HL^1(\Omega)$ is the space of harmonic functions in $L^1(\Omega)$.

We denote by $Q(\mu, HL^1)$ the class of all nonempty domains satisfying (2.1) and we write $\Omega \in Q(\mu, HL^1)$.

A simple example of a quadrature domain (in one-phase case) corresponding to the Dirac measure $\mu = \delta_a$ is the appropriate ball $B(a, r)$, (for instance, let $N = 2$, $a = 0, r = \frac{1}{\sqrt{\pi}}$). The mean value theorem for harmonic functions implies

$$\int_{B(a,r)} h\,dx = h(a) = \int h\,d\mu.$$

The quadrature identity (2.1) is equivalent to the following identities (see [13]),

$$\begin{cases} U^{\chi_\Omega} = U^\mu, & \text{in}\, \mathbb{R}^N \setminus \Omega, \\ \nabla U^{\chi_\Omega} = \nabla U^\mu, & \text{in}\, \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.2}$$

where U^μ denotes the Newtonian potential of the measure μ defined by

$$U^\mu(x) := (G * \mu)(x) = \int_{\mathbb{R}^N} G(x-y)d\mu(y), \ x \in \mathbb{R}^N.$$

Here,

$$G(x) = \begin{cases} \frac{1}{N(N-2)\omega_N}|x|^{2-N}, & \text{for } N \geq 3, \\ -\frac{1}{2\pi}\ln|x|, & \text{for } N = 2, \end{cases}$$

denotes the fundamental solution to the Laplace operator and ω_N is the volume of unit sphere in $\mathbb{R}^N$. Thus, U^{χ_Ω} (from now on U^Ω for simplicity) is the Newtonian potential of Ω considered as a body with density one. The second equality in (2.2) is a consequence of the first one except possibly at certain points on $\partial\Omega$. Also we can prove that $-\Delta U^\mu = \mu$ in the sense of distributions (see [5], [1]).

It has been explained in [13] and [16] that this problem is equivalent to finding a pair (u, Ω) of the following one-phase free boundary problem:

$$\begin{cases} \Delta u = \chi_\Omega - \mu, & \text{in } \ \mathbb{R}^N, \\ u = \nabla u = 0, & \text{in } \ \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2.3}$$

where $u = U^\mu - U^\Omega$ is the so-called modified Schwarz potential (MSP) of the pair (μ, Ω).

We also can replace the following inequality in (2.1) for the class of subharmonic functions $SL^1(\Omega)$,

$$\int h\, d\mu \leq \int_\Omega h\, dx, \quad \forall h \in SL^1(\Omega), \tag{2.4}$$

and get a quadrature domain for subharmonic functions. In this case, we call Ω a *subharmonic quadrature domain* with respect to μ and write $\Omega \in Q(\mu, SL^1)$. The authors in [11] and [13] have showed that (2.4) is equivalent to

$$\begin{cases} U^\Omega \leq U^\mu, & \text{in } \ \mathbb{R}^N, \\ U^\Omega = U^\mu, & \text{in } \ \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.5}$$

which is equivalent to

$$\begin{cases} \Delta u = \chi_\Omega - \mu, & \text{in } \ \mathbb{R}^N, \\ u \geq 0, & \text{in } \ \mathbb{R}^N, \\ u = 0, & \text{in } \ \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.6}$$

where $u = U^\mu - U^\Omega$. We note that in (2.6) it is not generally true that $u > 0$ in Ω (see [7]). For more details about quadrature domains, [6], [12] and [15] are basic references.

Moreover, if we also introduce the class $Q(\mu, AL^1)$ by saying that $\Omega \in Q(\mu, AL^1)$ if and only if $\nabla U^\Omega = \nabla U^\mu$ in Ω^c then

$$Q(\mu, SL^1) \subseteq Q(\mu, HL^1) \subseteq Q(\mu, AL^1).$$

For instance, if $\mu = \delta_0$ then all these classes are equal to $\{B(0, r)\}$, see [11]. The existence and uniqueness theorems in one-phase quadrature domains are established in [15] for class SL^1.

3. Two-phase Case

In this section our objective is to define two phase quadrature domain and investigate its PDE formulation.

3.1. Definition and basic properties

Let Ω is an open and bounded subset of $\mathbb{R}^N$. We define $\widetilde{H}(\Omega)$ by

$$\widetilde{H}(\Omega) = \{U^\eta \,:\, \eta \text{ is a signed Radon measure with compact support and } supp(\eta) \subset \Omega^c\}.$$

Next lemma leads us to have a definition of the two-phase quadrature domain and quadrature identity.

Lemma 3.1. *Let Ω and $\widetilde{H}(\Omega)$ be as above.*

1. *If $h \in \widetilde{H}(\Omega)$ then $h \in L^1_{loc}(\mathbb{R}^N)$.*
2. *All functions in $\widetilde{H}(\Omega)$ are harmonic in Ω.*
3. *For $x \in \Omega^c$ we have $G(x - .) = U^{\delta_x} \in \widetilde{H}(\Omega)$.*
4. *Suppose that h is harmonic on a bounded open set D such that $\Omega \subset\subset D$. There exists a measure ν with compact support such that $supp(\nu) \subset D \setminus \overline{\Omega}$ and $h = U^\nu$.*

Proof. The items (1), (2) and (3) are immediately verified by the definition of $\widetilde{H}(\Omega)$. To prove the last one, suppose that h is a harmonic function on a bounded open set $D \subset \mathbb{R}^N$ such that $\Omega \subset\subset D$. Let $\xi \in C_c^\infty(\mathbb{R}^N)$ such that $supp(\xi) \subset D$. Moreover, we choose $\xi = 1$ in a neighborhood closed enough to Ω. For $x \in \Omega$ one obtains

$$\begin{aligned} h(x) = (h\xi)(x) &= \int \delta_x(h\xi)(y)\,dy \\ &= \int -\Delta G(x-y)(h\xi)(y)\,dy \\ &= \int G(x-y)(-\Delta(h\xi))(y)dy. \end{aligned}$$

Now if one sets $d\nu = (-\Delta h\xi)(y)dy$, then $h = U^\nu$. □

Then we have the following definition.

Definition 3.2. Let $\Omega^{\pm}$ be two open, disjoint and connected subsets of $\mathbb{R}^N$ and $\mu^{\pm}$ be two positive Radon measures with compact supports. Moreover, suppose that $\lambda^{\pm}$ are two positive constants. We say that $\Omega = \Omega^+ \cup \Omega^-$ is a *two-phase quadrature domain*, with respect to $\mu^{\pm}, \lambda^{\pm}$ and $\widetilde{H}(\Omega)$ if $supp(\mu^{\pm}) \subseteq \Omega^{\pm}$, and

$$(3.1) \qquad \int_{\Omega^+} \lambda^+ h \ - \int_{\Omega^-} \lambda^- h = \int h \ (d\mu^+ - d\mu^-), \quad \forall h \in \widetilde{H}(\Omega).$$

We then write $\Omega^{\pm} \in Q(\mu^{\pm}, \widetilde{H})$ or $\Omega \in Q(\mu, \widetilde{H})$ where $\mu = \mu^+ - \mu^-$.

To reach a potential theory interpretation of the two phase quadrature domain let us choose $h(x) = h_y(x) = G(x-y)$ in (3.1), as a harmonic function for $y \in \mathbb{R}^N \setminus \Omega$. Then, we have

$$(3.2) \qquad U^f = U^\mu \text{ in } \mathbb{R}^N \setminus \Omega,$$

where $f = \lambda^+ \chi_{\Omega^+} - \lambda^- \chi_{\Omega^-}$, and $\mu = \mu^+ - \mu^-$.

We deal with the following question.

Main question: Whether we can claim that Ω is the unique domain satisfies (3.1)?

It turns out that the uniqueness problems in this case are much more involved than in the one-phase case.

A different formulation (or a different starting point) of our problem would come from the well-known potential theoretic formulation of analyzing gravi-equivalent bodies. Indeed, suppose there are non-empty bounded domains $D = D^+ \cup D^-$ and $\Omega = \Omega^+ \cup \Omega^-$, where

$$D^+ \cap D^- = \Omega^+ \cap \Omega^- = \emptyset,$$

satisfying

$$(3.3) \qquad \int_{\Omega^+} \lambda^+ h \ - \int_{\Omega^-} \lambda^- h = \int_{D^+} \lambda^+ h \ - \int_{D^-} \lambda^- h, \quad \forall h \in \widetilde{H}(\Omega \cup D).$$

Then, we would like to see whether $\Omega^{\pm} = D^{\pm}$, or alternatively what kind of properties such domains would possess. The first property that can be derived from the above integral identity is the following lemma where its idea comes from [11].

Lemma 3.3. *Suppose that* $\Omega = \Omega^+ \cup \Omega^-$ *and* $D = D^+ \cup D^-$. *If (3.3) holds, then for a measure* ν,

$$\Omega, D \in Q(\nu, \widetilde{H}),$$

with $supp(\nu) \subseteq \overline{\Omega \cap D}$.

Observe that here, the support of the measure can not be expected stay in the set $\overline{\Omega^{\pm} \cap D^{\pm}}$, as will be seen from the argument in the proof.

Proof. Define

$$U^\Omega = G * (\lambda^+ \chi_{\Omega^+} - \lambda^- \chi_{\Omega^-}),$$

and U^D correspondingly. Hence, we can define a new function U on $\mathbb{R}^N$ by

$$U = \begin{cases} U^\Omega, & \text{in } \Omega^c, \\ U^D, & \text{in } D^c, \\ \text{“}arbitrary\text{”}, & \text{in } \Omega \cap D, \end{cases} \tag{3.4}$$

where “arbitrary” means a suitable function and as smooth as possible. The definition of U on $\Omega \cap D$ can be chosen such that $-\Delta U \in L^\infty(\mathbb{R}^N)$. Let $\nu = -\Delta U$, we have $U = U^\nu$ (because U behaves like a potential at infinity) and it follows that

$$U^\Omega = U^\nu, \quad \text{in } \Omega^c,$$
$$U^D = U^\nu, \quad \text{in } D^c.$$

Then by (3.3), $U^\Omega = U^D$ in $\mathbb{R}^N \backslash (\Omega \cup D)$. This proves the lemma with respect to (3.2). □

Remark 1. Observe that $supp(\nu) \subseteq \overline{\Omega \cap D}$.

Corollary 3.4. *For $\Omega = \Omega^+ \cup \Omega^-$ and $D = D^+ \cup D^-$ admitting the quadrature identity (3.3) we have the intersection $\overline{\Omega} \cap \overline{D}$ is non-void.*

Proof. Suppose that $\overline{\Omega} \cap \overline{D} = \emptyset$ and consider the function U defined by (3.4) in Lemma 3.3. Hence we find that U is harmonic in $\mathbb{R}^N$, i.e,

$$\Delta U = 0, \quad \text{in } \mathbb{R}^N. \tag{3.5}$$

On the other hand for an arbitrary Radon measure μ one can describe the behavior of potential U^μ as follows (see [14])

$$|U^\mu(x)| = O(|x|^{2-N}) \to 0 \text{ as } |x| \to \infty \text{ if } N \geq 3, \tag{3.6}$$

and

$$U^\mu(x) = -\frac{1}{2\pi} \ln |x| \int d\mu + O(|x|^{-1}) \text{ as } |x| \to \infty \text{ if } N = 2. \tag{3.7}$$

Now with respect to theses properties, we deduce that in the case $N \geq 3$, U is bounded and has logarithmic growth for $N = 2$. By considering (3.5), Liouville's theorem states that $U = c$ where c is a constant. To get a contradiction suppose that B_R is a ball such that $\Omega \subset B_R$. we know that U^Ω is a super solution in B_R and $U = U^\Omega = c$ in $B_R \setminus \Omega$. The strong minimum principle gives us

$$U^\Omega = c, \quad \text{in} \quad B_R,$$

and consequently $\Delta U^\Omega = 0$ in B_R which is a contradiction to $\Delta U^\Omega = -1$ in Ω. □

Remark 2. It remains an open question whether in the above corollary we can conclude that both intersections $\Omega^\pm \cap D^\pm$ are non-void.

Remark 3. If one takes $h = 1$ in (3.3), then

$$\lambda^+|\Omega^+| \ - \lambda^-|\Omega^-| = \lambda^+|D^+| - \lambda^-|D^-|,$$

where $|\Omega|$ denotes the volume of Ω. We shall use this simple property in the proof of some results later.

3.2. PDE formulation

For $\Omega = \Omega^+ \cup \Omega^- \in Q(\mu^\pm, \widetilde{H})$, we can define $u = U^\mu - U^{\lambda^+\chi_{\Omega^+} - \lambda^-\chi_{\Omega^-}}$. Then by (3.1) with $u = 0$ in $\mathbb{R}^N \backslash \Omega$, we have the following free boundary problem

$$\begin{cases} \Delta u = (\lambda^+\chi_{\Omega^+} - \mu^+) - (\lambda^-\chi_{\Omega^-} - \mu^-), & \text{in } \mathbb{R}^N, \\ u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega \ \text{ and } \ supp(\mu^\pm) \subset \Omega^\pm, \end{cases} \tag{3.8}$$

which is a two-phase version of (2.3).

The next theorem verifies the connection between the potential theory formula and the PDE formulation.

Theorem 3.5. *The quadrature identity (3.1) and the potential theory interpretation (3.2) and PDE formulation (3.8) are equivalent.*

Proof. (3.1) $\Rightarrow$ (3.2)$\Rightarrow$ (3.8): This is clear.

(3.8) $\Rightarrow$ (3.1): Suppose that (3.8) is given. For all $h = U^\eta \in \widetilde{H}(\Omega)$ and $\nu = (\lambda^+\chi_{\Omega^+} - \mu^+) - (\lambda^-\chi_{\Omega^-} - \mu^-)$, Fubini's theorem gives

$$\int U^\eta \, d\nu = \int U^\nu \, d\eta = \int_\Omega U^\nu \, d\eta + \int_{\Omega^c} U^\nu \, d\eta. \tag{3.9}$$

We prove that U^ν vanishes in Ω^c and consequently the second term of (3.9) is zero.

Suppose that $y \in \Omega^c$. Let $R > 0$ and B_R be a ball such that $y \in B_R, \Omega \subset B_R$. Then by assumption on ν

$$\begin{aligned} U^\nu(y) &= \int_{B_R} G(x-y) d\nu(x) = \int_{B_R} G(x-y) \, \Delta u dx \\ &= \int_{B_R} \Big(G(x-y) \, \Delta u - \Delta G(x-y) u \Big) \, dx + \int_{B_R} \Delta G(x-y) u \, dx \\ &= \int_{\partial B_R} \left(\frac{\partial G}{\partial n} u - \frac{\partial u}{\partial n} G \right) ds + \int_{B_R} \delta_y(x) u(x) \, dx \\ &= u(y) \\ &= 0. \end{aligned}$$

On the other hand $supp(\eta) \subset \Omega^c$ then the first term of (3.9) is also zero. Therefore we have

$$\int U^\eta \, d\nu = 0, \quad \text{for all } U^\eta \in \widetilde{H}(\Omega),$$

which is (3.1). □

3.3. Quadrature inequalities

The corresponding quadrature inequality (2.4) is more subtle in two-phase case. To derive such an inequality suppose that η is a signed Radon measure with compact support. We define:

$$S^+(B) = \{U^\eta \,:\, \eta|_B \leq 0\},$$
$$S^-(B) = \{U^\eta \,:\, \eta|_B \geq 0\}.$$

In other words all functions in $S^+(B)$ and $S^-(B)$ are subharmonic and super harmonic on B respectively.

Suppose that $\Omega^\pm \subset \{u^\pm \geq 0\}$. Let

$$\widetilde{S}(\Omega) := S^+(\Omega^+) \cap S^-(\Omega^-) = \{U^\eta \,:\, \eta|_{\Omega^+} \leq 0\,,\, \eta|_{\Omega^-} \geq 0\},$$

and consequently for all $h = U^\eta \in \widetilde{S}(\Omega)$ one gets

$$\int_\Omega u\Delta h = \int_{\Omega^+} u^+(-\eta) + \int_{\Omega^-} u^-(-\eta) \geq 0, \tag{3.10}$$

where $u = u^+ - u^-$.

Now again suppose that $\nu = \Delta u = (\lambda^+\chi_{\Omega^+} - \mu^+) - (\lambda^-\chi_{\Omega^-} - \mu^-)$ and $h = U^\eta \in \widetilde{S}(\Omega)$. We claim that

$$\int U^\eta \, d\nu = \int U^\nu \, d\eta \geq 0. \tag{3.11}$$

To prove this let B_R be a ball contains Ω. We apply Green's formula and get

$$\begin{aligned}
\int_{B_R} U^\eta \, d\nu &= \int_{B_R} h\Delta u \\
&= \int_{B_R} \Big(h\Delta u - u\Delta h\Big) + \int_{B_R} u\Delta h \\
&= \int_{\partial B_R} \Big(h\frac{\partial u}{\partial n} - u\frac{\partial h}{\partial n}\Big) + \int_{B_R} u\Delta h \qquad (u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial B_R) \\
&= \int_{B_R} u\Delta h = \int_\Omega u\Delta h \geq 0. \qquad\qquad (\text{by } (3.10))
\end{aligned}$$

Set now $h_y(x) = -|x-y|^{2-N}$ for $y \in \Omega^+, N > 2$ and $h_y(x) = -\ln|x-y|$ for $y \in \Omega^+, N = 2$. Then, it is clear that $h_y(x)$ is subharmonic in Ω^+ and it is harmonic in Ω^-, and consequently (3.11) reads

$$U^f(y) \leq U^\mu(y) \qquad \text{in } \Omega^+. \tag{3.12}$$

Similarly, if we choose $-h_y(x)$, $y \in \Omega^-$ as a test function in the inequality (3.11), then

$$U^f(y) \geq U^\mu(y) \qquad \text{in } \Omega^-. \tag{3.13}$$

Now we are able to make a reasonable definition.

Definition 3.6. Suppose that $\Omega, \mu^\pm, \lambda^\pm$ are the same as in the definition 3.2 and let $f = \lambda^+\chi_{\Omega^+} - \lambda^-\chi_{\Omega^-}$, and $\mu = \mu^+ - \mu^-$, such that

$$\begin{cases} U^f \leq U^\mu, & \text{in } \mathbb{R}^N \setminus \Omega^-, \\ U^f \geq U^\mu, & \text{in } \mathbb{R}^N \setminus \Omega^+, \end{cases}$$

then we say that Ω is a *two-phase quadrature domain* for the class $\widetilde{S}(\Omega)$ and we write $\Omega \in Q(\mu, \widetilde{S})$. It is immediately verified that $Q(\mu, \widetilde{S}) \subset Q(\mu, \widetilde{H})$.

Furthermore, by $\Omega \in Q(\mu^\pm, \widetilde{A})$ we mean $\nabla U^f = \nabla U^\mu$ in $\Omega^c \setminus (\partial\Omega^+ \cap \partial\Omega^-)$ and consequently one has

$$Q(\mu, \widetilde{S}) \subseteq Q(\mu, \widetilde{H}) \subseteq Q(\mu, \widetilde{A}).$$

Remark 4. It is clear that (3.10) is still valid, if one chooses $h \in S^+(\Omega^+) \cap \widetilde{H}(\Omega^-)$ and it reads

$$U^f \leq U^\mu, \qquad \text{in } \mathbb{R}^N \setminus \Omega^-.$$

Similarly, if $h \in S^-(\Omega^-) \cap \widetilde{H}(\Omega^+)$, then

$$U^f \geq U^\mu, \qquad \text{in } \mathbb{R}^N \setminus \Omega^+.$$

Proposition 3.7. *Consider two non-negative bounded Radon measures $\mu^\pm$ with compact supports and two positive constants $\lambda^\pm$. Moreover, suppose that U^μ is the Newtonian potential of $\mu = \mu^+ - \mu^-$ and $f = \lambda^+\chi_{\Omega^+} - \lambda^-\chi_{\Omega^-}$. Then, the following statements are equivalent:*

(1) $\int_{\Omega^+} \lambda^+ h \; - \int_{\Omega^-} \lambda^- h \geq \int h d\mu^+ - \int h d\mu^-, \quad \forall h \in \widetilde{S}(\Omega).$

(2) $\Omega \in Q(\mu, \widetilde{S}(\Omega))$.

(3) *If* $u = U^\mu - U^f$, *then*

$$\begin{cases} \Delta u = (\lambda^+\chi_{\Omega^+} - \mu^+) - (\lambda^-\chi_{\Omega^-} - \mu^-), & \text{in } \mathbb{R}^N, \\ \Omega^\pm \subset \{\pm u \geq 0\}. \end{cases} \tag{3.14}$$

Proof. (1) $\Rightarrow$ (2): It is clear by considering the equations (3.12) and (3.13).

(2) $\Rightarrow$ (3): It is an immediate consequence of Definition 3.6 and the fact that $-\Delta U^f = f$, $-\Delta U^\mu = \mu$.

(3) $\Rightarrow$ (1): By considering (3.11) one obtains

$$\int_\Omega (\lambda^+\chi_{\Omega^+} - \lambda^-\chi_{\Omega^-} - (\mu^+ - \mu^-))h \quad = \int_\Omega h\Delta u \geq 0,$$

which is equivalent to (1). □

Remark 5. Obviously, by taking $\widetilde{\Omega}^\pm = \{u^\pm > 0\}$, the free boundary problem (3.14) can be written as

$$(3.15)\quad \begin{cases} \Delta u = (\lambda^+\chi_{\{u>0\}} - \mu^+) - (\lambda^-\chi_{\{u<0\}} - \mu^-), & \text{in } \mathbb{R}^N, \\ \widetilde{\Omega}^\pm = \{u^\pm \geq 0\}, \end{cases}$$

provided $supp(\mu^\pm) \subset \widetilde{\Omega}^\pm$. This free boundary problem have been studied in [7].

3.4. Some examples

In the special case of (3.8) with $\mu^\pm = 0$, $\lambda^\pm = 1$ one can show that the function $u = \frac{(x_1^+)^2}{2} - \frac{(x_1^-)^2}{2}$, where $x_1^\pm := \max(\pm x_1, 0)$, is a solution of

$$\Delta u = \chi_{\{u>0\}} - \chi_{\{u<0\}}, \text{ in } \mathbb{R}^N.$$

Now suppose that $\mu^- = 0$, $\mu^+ \neq 0$ and $\Omega^\pm = \{x : \pm u(x) > 0\}$ then consequently the PDE formulations (3.8) turns

$$(3.16)\quad \begin{cases} \Delta u = \lambda^+\chi_{\{u>0\}} - \mu^+ - \lambda^-\chi_{\{u<0\}}, & \text{in } \mathbb{R}^N, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

If $\partial\Omega^- \neq \emptyset$ then in Ω^-

$$\begin{cases} \Delta u = -\lambda^- \leq 0 & \text{in } \mathbb{R}^N \setminus \Omega^+, \\ u < 0, & \text{in } \Omega^-, \\ u = 0, & \text{on } \partial\Omega^-, \end{cases}$$

which is an obvious contradiction according to the minimum principle. Therefore (3.16) has no solution.

Example 3.8. $(N = 1)$ Suppose that $\mu^\pm = a^\pm\delta_{x^\pm}$, $x^+ > 0, x^- = -x^+$, $a^\pm > 0$. Hence for $r_1, r_2 > 0$ one has $\Omega^+ = (x^+ - r_1, x^+ + r_1)$ and $\Omega^- = (x^- - r_2, x^- + r_2)$ and they meets each other at $x^+ - r_1 = x^- + r_2$. In other words, $2x^+ = r_1 + r_2$. Regarding (3.8) and the continuity conditions one gets $r_1 = \frac{2a^-x^+}{a^-+a^+}$, $r_2 = \frac{2a^+x^+}{a^-+a^+}$ and

$$u(x) = \begin{cases} \frac{(x-x^+)^2}{2} - a^+(x - x^+)H(x - x^+) + a^-r_2 - \frac{1}{2}r_1^2, & \text{in } \Omega^+, \\ -\frac{(x-x^-)^2}{2} + a^-(x - x^-)H(x - x^-) + \frac{1}{2}r_2^2, & \text{in } \Omega^-, \end{cases}$$

where $H(x)$ is the Heaviside function.

Example 3.9. Suppose that m denotes the Lebesgue measure and $\mu^{\pm}$ are two uniformly distributed surface measure on $|x| = 2, 4$ respectively such that $d\mu^{\pm} = \rho^{\pm} dm$ for some $\rho^{\pm} > 0$ to be decided below. Let

$$u_i^{\pm} = \pm|x|^2/2N + a_i^{\pm}|x|^{2-N} + b_i^{\pm} \text{ for } i = 1, 2.$$

Now one can choose $a_i^{\pm}$,$b_i^{\pm}$ and $\rho^{\pm}$ in a such way so that the function u defined as

$$u = \begin{cases} u_1^+, & \text{in } 1 < |x| < 2, \\ u_2^+, & \text{in } 2 \leq |x| < 3, \\ u_1^-, & \text{in } 3 \leq |x| < 4, \\ u_2^-, & \text{in } 4 \leq |x| < 5, \end{cases}$$

is continuous in $1 \leq |x| \leq 5$ and satisfies the two phase free boundary equation (3.8). Therefore $\Omega = \Omega^+ \cup \Omega^- = \{1 < |x| < 3\} \cup \{3 < |x| < 5\}$ is a two phase quadrature domain with respect to $\mu^{\pm}$. Here, the densities are given by the difference of the normal derivatives of the left- and right-hand sides limits. For existence of these quadrature domains see [15].

4. Discussion on existence theory

In general an existence result of two-phase quadrature domains, is not so easy to obtain. It seems that one needs rather strong assumptions on the densities $\lambda^{\pm}$ as well as the measures $\mu^{\pm}$ to ensure the existence of a solution. For example, in the simpler one phase case the crucial assumption is that the measure should be non-negative and sufficiently concentrated, (see [11]).

In other words to ensure the existence of a solution for (3.15), one has to make a balance between measures. However making such balance conditions are a challenging problem and is under research. As far as we know, the Sakai's concentration condition together with estimates of the one phase solutions of $\mu^{\pm}$ is a sufficient condition (see [7]). For more existence result see the recent article [9].

In the two-phase case, it is far from obvious that such an assumption would be sufficient to guarantee the existence of a solution. Indeed, if one of the measures $\mu^{\pm}$ is so large that it would "eat up" the other one, i.e, large concentration of one of the two measures, force the support of the other to shrink. This can already be seen in one dimension. For instance, see Example 1.1 in [7].

Our objective in this section is to present some known existence result for the problem

$$\Delta u = (\lambda^+ \chi_{\Omega^+} - \mu^+) - (\lambda^- \chi_{\Omega^-} - \mu^-), \tag{4.1}$$

with the crucial sign properties $\Omega^{\pm} = \{\pm u > 0\}$. One of the few paper discussing existence results in a simpler case is [7]. The authors of [7] apply the minimization technique to show the existence of solution of

$$\Delta u = (\lambda^+ - \mu^+)\chi_{\{u>0\}} - (\lambda^- - \mu^-)\chi_{\{u<0\}}, \quad \text{in } \mathbb{R}^N, \tag{4.2}$$

which implies a weaker form of the two-phase problem (4.1) with the sign assumptions. Remarkably, it is not so easy to find appropriate conditions to obtain (4.1) by considering (4.2). In other words, it is a challenging problem to find conditions such that

$$\mu^\pm = \mu^\pm \chi_{\{\pm u>0\}},$$

i.e., *supp* $(\mu^\pm) \subset$ *supp*$(\pm\, u)$. In the one phase case, the authors of [13] have established some conditions to guarantee *supp*$(\mu) \subset$ *supp*(u), but for the two-phase case the problem is almost completely open.

One can easily show the Euler-Lagrange equation for the functional

$$J_\Omega(u) = \int_\Omega \big(\frac{1}{2}|\nabla u|^2 - g(x)u^+ + h(x)u^-\big)\ dx, \tag{4.3}$$

coincides in the following two-phase free boundary problem:

$$-\Delta u = g(x)\chi_{\{u>0\}} - h(x)\chi_{\{u<0\}}. \tag{4.4}$$

The existence of a minimizer for (4.3) in appropriate functional space depends on the existence of the minimizers for the two functionals in one phase case

$$J_+(u) = \int_\Omega \big(\frac{1}{2}|\nabla u|^2 - g(x)u^+\big)\ dx,\ J_-(u) = \int_\Omega \big(\frac{1}{2}|\nabla u|^2 + h(x)u^-\big)\ dx,$$

on the sets $W^\pm = \{u \in W_0^{1,2}(\Omega),\ \pm u \geq 0\}$ respectively.

Theorem 4.1. *(Proposition 2.1 in* [7] *) Assume that Ω is a bounded domain. The functional J_Ω has a minimizer u in the space $W_0^{1,2}$ and it satisfies the following inequality*

$$U^- \leq u \leq U^+,$$

where $U^\pm$ are the minimizers of $J_\pm$.

Using Theorem (4.1) with $g = \mu^+ - \lambda^+,\ h = \lambda^- - \mu^-$, we get the existence of solution for (4.2), see [7].

5. Uniqueness results

In this section, we try to prove the uniqueness for (3.8) in some specific cases. To be more clear the problem, from the point of view of the potential theory, we can rewrite the main question.

By a *solid* domain we mean a domain U such that it is bounded, $U = (\overline{U})^\circ$ and the complement of $\overline{U}$, i.e, $(\overline{U})^c$ is connected.

Question: Suppose that μ is a positive measure with compact support. Can $Q(\mu, \widetilde{H})$ contain two distinct domains $\Omega = \Omega^+ \cup \Omega^-$, $D = D^+ \cup D^-$ for solid domains $\Omega^\pm$ and $D^\pm$?

If one does not consider "solid" assumption on the domains , uniqueness can fail. For instance, in [11] and [15] one can find examples which indicate a non-uniqueness for the one-phase case without such assumptions.

It should be remarked that uniqueness in one-phase case is already a challenging problem and there are studies on it such as [17] and [18]. The following theorem provides uniqueness under the special sign assumptions.

Theorem 5.1. *Let u, v be two solutions of (3.8) and suppose that*

$$\Omega^\pm := \{\pm u > 0\}, \quad D^\pm := \{\pm v > 0\}.$$

Then, $\Omega^\pm = D^\pm$ and $u \equiv v$.

Proof. Set $w := u - v$ in $\Omega^+ \cup D^-$. Then, in $\Omega^+ \cup D^-$ we have

$$\begin{aligned} \Delta w = \Delta u - \Delta v &= (\lambda^+ \chi_{\Omega^+} - \lambda^- \chi_{\Omega^-}) - (\lambda^+ \chi_{D^+} - \lambda^- \chi_{D^-}) \\ &= \lambda^+ (\chi_{\{\Omega^+ \setminus D^+\}} - \chi_{\{D^+ \setminus \Omega^+\}}) + \lambda^- (\chi_{\{D^- \setminus \Omega^-\}} - \chi_{\{\Omega^- \setminus D^-\}}) \\ &= \lambda^+ \chi_{\{\Omega^+ \setminus D^+\}} + \lambda^- \chi_{\{D^- \setminus \Omega^-\}} \geq 0. \end{aligned} \tag{4.1}$$

For the boundary of the union one has

$$\partial(\Omega^+ \cup D^-) = (\partial \Omega^+ \setminus D^-) \cup (\partial D^- \setminus \Omega^+) := L_1 \cup L_2. \tag{4.2}$$

Now, we have

$$w = u - v = -v \leq 0, \quad \text{on} \quad L_1, \tag{4.3}$$

since $v \geq 0$ outside D^-. Similarly

$$w = u - v = u \leq 0, \quad \text{on} \quad L_2, \tag{4.4}$$

since $u \leq 0$ outside Ω^+. Totally we get

$$w = u - v \leq 0, \quad \text{on} \quad \partial(\Omega^+ \cup D^-). \tag{4.5}$$

Then, by the maximum principle

$$u \leq v, \quad \text{in} \quad \Omega^+ \cup D^-.$$

Suppose that $|\cdot|$ denotes the volume of a set. In Ω^+, we have $0 < u \leq v$ which gives $\Omega^+ \subset D^+$ and $|\Omega^+| < |D^+|$, unless $D^+ = \Omega^+$. In D^-, we have $u \leq v < 0$ which gives $D^- \subset \Omega^-$ and $|D^-| < |\Omega^-|$, unless $D^- = \Omega^-$.

Then, we get

$$\lambda^+ |\Omega^+| - \lambda^- |\Omega^-| < \lambda^+ |D^+| - \lambda^- |D^-|.$$

The latter inequality contradicts Remark 3, otherwise $D^\pm = \Omega^\pm$. This proves the theorem. □

Remark 6. Theorem (5.1) it is the theorem 4.7(b) of [9] if $\lambda^\pm = 1$.

For simplicity, a general assumption will be made, that is, all domains $\Omega^{\pm}$ and $D^{\pm}$ in the next theorem will be assumed solid. Now, we present a generalization of the previous theorem.

Theorem 5.2. *If u, v are two solutions of (3.8) and $\Omega^{\pm}$, $D^{\pm}$ are the corresponding regions respectively satisfy*

$$\Omega^- \subseteq \{u < 0\} \quad \textit{and} \quad D^+ \subseteq \{v > 0\}, \tag{4.6}$$

then $\Omega^{\pm} = D^{\pm}$ with $u \equiv v$.

Proof. Set $w := u - v$ then in $\Omega^+ \cup D^-$, we have (4.1). Here Ω^+ and D^- do not necessarily have the sign property, but still we can conclude that $v \geq 0$ outside D^- and that $u \leq 0$ outside Ω^+. This shows that the equations (4.2)-(4.5) in Theorem 5.1 are still valid. Then, again by using the maximum principle we obtain

$$w \leq 0, \quad \text{in} \quad \Omega^+ \cup D^-. \tag{4.7}$$

By assumption (4.6) and (4.7) one concludes that

$$w \leq 0, \qquad \text{in } \mathbb{R}^N. \tag{4.8}$$

Let $L = B_R \setminus \overline{[(D^+ \cup \Omega^-) \setminus (\Omega^+ \cup D^-)]}$, where $\Omega \cup D \subset\subset B_R$. Then,

$$\begin{cases} \Delta w \geq 0, & \text{in } L, \\ w \leq 0, & \text{on } \partial L. \end{cases}$$

The strong maximum principle for w in L states that either $w < 0$ in L, or $w = 0$ in L. But $w = 0$ in $(D \cup \Omega)^c \subset L$. Hence $w = 0$ in L. For L^c we have

$$\begin{cases} \Delta w \leq 0, & \text{in } L^c, \\ w = 0, & \text{on } \partial L^c = \partial L. \end{cases}$$

Then the inequality (4.8) along with the strong minimum principle imply that $w = 0$ in L^c. Therefore, $w \equiv 0$ in B_R, and hence

$$u \equiv v, \text{ in } B_R, \tag{4.9}$$

and finally $\Omega^{\pm} = D^{\pm}$. □

The next proposition states that with no sign assumption, there are always stationary points $\{\nabla u = 0\}$ in $\Omega^{\pm}$ provided $\partial\Omega^{\pm}$ are locally $C^{1,\alpha}$ away from the so called branch points, (see [17]).

We say that a domain Ω satisfy the exterior sphere condition if for every $x \in \partial\Omega$, there exists a ball of radius r, centered at $y \in \Omega$, such that $B(y, r) \cap \overline{\Omega} = \{x\}$. This is a sufficient condition to use Hopf's lemma (see [8]).

Proposition 5.3. *(Special Points) Suppose that u, v are two solutions of (3.8) and $\Omega^{\pm}$, $D^{\pm}$ are the corresponding regions respectively. Moreover, suppose that $\partial\Omega^{\pm}$, $\partial D^{\pm}$ satisfy the exterior sphere condition. Then at least one of the following holds.*

(1) *v attains its minimum (maximum) in $D^+ \setminus \Omega^+$ $(\Omega^+ \setminus D^+)$.*
(2) *u attains its maximum (minimum) in $\Omega^- \setminus D^-$ $(D^- \setminus \Omega^-)$.*

Proof. Consider $L = B_R \setminus \overline{[(D^+ \cup \Omega^-) \setminus (\Omega^+ \cup D^-)]}$ with $\Omega \cup D \subset\subset B_R$, so $w := u - v$ is a subsolution in L and, consequently w attains its maximum, say at x_0, on ∂L. It is clear that

$$w\lfloor_{\partial L} = \begin{cases} u, & \text{on} \quad (\partial D^+ \cup \partial D^-) \cap \partial L, \\ -v, & \text{on} \quad (\partial \Omega^+ \cup \partial \Omega^-) \cap \partial L, \\ 0, & \text{on} \quad \partial B_R. \end{cases} \tag{4.10}$$

If $\max w = 0$ then by considering the maximum principle on L we derive that $u \equiv v$ on L and (3.8) yields

$$\lambda^+ = \Delta u = \Delta v = -\lambda^-, \quad \text{on } (\Omega^+ \cup D^-) \setminus (supp\, \mu^+ \cup supp\, \mu^-).$$

This is a contradiction to the positivity assumptions of $\lambda^\pm$ and hence

$$\max w \neq 0. \tag{4.11}$$

By considering (4.11) one has either $x_0 \in (\partial \Omega^+) \cap D^+$ or $x_0 \in (\partial D^-) \cap \Omega^-$. Therefore we have two cases.

- $x_0 \in (\partial \Omega^+) \cap D^+$ and $\max_{\Omega^+ \cup D^-} w = -v(x_0)$.

  Now Hopf's lemma gives $\partial_\nu w(x_0) = -\partial_\nu v(x_0) > 0$ where ν is the outer normal vector on $\partial \Omega^+$ pointing into $D^+ \setminus \Omega^+$. It means that v decreases in $D^+ \setminus \Omega^+$ so we should have $y_0 \in D^+ \setminus \Omega^+$ such that

  $$v(y_0) = \min_{D^+ \setminus \Omega +} v \qquad \text{and} \qquad \nabla v(y_0) = 0.$$

- $x_0 \in (\partial D^-) \cap \Omega^-$ and $\max_{\Omega^+ \cup D^-} w = u(x_0)$.

  Similar discussion shows that there exists $y_0 \in \Omega^- \setminus D^-$ such that

  $$u(y_0) = \max_{\Omega^- \setminus D^-} u \qquad \text{and} \qquad \nabla u(y_0) = 0.$$

One can follow this recipe for w in $L = B_R \setminus \overline{[(D^- \cup \Omega^+) \setminus (\Omega^- \cup D^+)]}$ and obtain a similar result. □

The conclusion is that even in the case of non-uniqueness we have special points in $\Omega^\pm$ and $D^\pm$.

6. Conjectures

Conjecture 6.1. Theorem 5.2, should still be valid if either $\Omega^- := \{u < 0\}$ or $D^+ := \{v > 0\}$.

Conjecture 6.2. The uniqueness for the solution of (3.8) can be obtained by considering only $\Omega^{\pm} = \{\pm u > 0\}$ without sign properties for $D^{\pm}$.

Conjecture 6.3. It would be an interesting problem to generalize Theorem (5.2) for the p-Laplacian operator, i.e, $\Delta_p u = div(|\nabla u|^{p-2} \nabla u)$ for $1 < p < \infty$. According to the comparison principle for p-Laplacian (see [4]), it is straightforward to prove the uniqueness theorem for this operator with all sign properties. We guess that one is able to prove our main result (Theorem 5.2) for the p-Laplacian operator. For more information on p-Laplacian properties and its relation with free boundary problems see [2], [3] and [4] for instance.

Acknowledgments: This problem was suggested by Professor Henrik Shahgholian. The authors thank him for fruitful discussions and useful suggestions. We would also like to thank Tomas Sjödin for valuable comments.

References

[1] Armitage D., Gardiner S., Classical potential theory, Springer Monographs in Mathematics, 2001.

[2] Bahrami F., Chademan A., Existence of a uniform bound for quadrature domains associated to p-Laplacian, R. Oldenbourg Verlag Munchen, 19, 319-326, 1999.

[3] Bahrami F., Chademan A., Existence of unbounded quadrature domains for the p-Laplacian operator, Bulletin of Iranian Mathematical Society, Vol. 24, No. 2, pp 1-13, 1998.

[4] Bahrami F., Shahgholian H., Uniqueness for an overdetermined boundary value problem for the p-Laplacian, Proceeding of AMS, V.126, N. 3, pages 745-750, 1998.

[5] Doob J., Classical potential theory and its probabilistic counterpart, Springer, 2001.

[6] Ebenfelt P., Gustafsson B., Khavinson D., Putinar M., Quadrature Domains and Applications, a Harold S. Shapiro Anniversary Volume, Birkhäuser, 2005.

[7] Emamizadeh B., Prajapat J., Shahgholian H., A two-phase free boundary problem related to quadrature domains, Potential Anal, Springer Science, 2010.

[8] Evans, Lawrence C. Partial differential equations. Graduate Studies in Mathematics 19., American Mathematical Society, Providence, RI., 662 pp, 1998.

[9] Gardiner S., Sjödin T., Two phase quadrature domains, Journal d'Analyse Mathématique, To appear.

[10] Gardiner S., Sjödin T., Convexity and the exterior inverse problem of potential theory, Proc. Amer. Math. Soc. 136 (2008), 1699-1703.

[11] Gustafsson B., On quadrature domains and an inverse problem in potential theory, J.Analyse Math. 55 (1990), 172-216.

[12] Gustafsson B., Putinar M., Selected topics on quadrature domains. Physica D 235, 90-100 (2007).

[13] Gustafsson B., Shahgholian H., Existence and geometric properties of solutions of a free boundary problem in potential theory, J. Reine Angew. Math. 473 (1996), 137-179.

[14] Kellogg O.D., Foundations of Potential Theory, Dover, 1953.

[15] Sakai M., Quadrature Domains, Lect. Notes Math. Vol. 934, Berlin-Heidelberg: Springer-Verlag, 1982.

[16] Sakai M., Applications of variational inequalities to the existence theorem on quadrature domains, Trans. Am.Math. Soc. 276 (1983), 267-279.

[17] Shahgholian H., On quadrature domains and the Schwarz potential, J. Math. Anal. and Appl. 171 (1992), 61-78.

[18] Shahgholian H., Convexity and uniqueness in an inverse problem of potential theory, Proc. Amer. Math. Soc. 116 (1992), 1097-1100.

[19] Widman K., Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations, Math.Scand. 21, (1968), 17-37.

Department of Mathematics, Faculty of Science and Letters, Istanbul Technical University, 34469 Maslak-Istanbul, Turkey

E-mail address: ceni@itu.edu.tr

Department of Mathematics, Stockholm University, S-10691, Stockholm, Sweden

E-mail address: mahmoudreza@math.su.se

NUMERICAL APPROXIMATION OF ONE PHASE QUADRATURE DOMAINS

MAHMOUDREZA BAZARGANZADEH AND FARID BOZORGNIA

ABSTRACT. In this work, we present two numerical schemes for a free boundary problem called one phase quadrature domain. In the first method by applying the proprieties of given free boundary problem, we derive a method that leads to a fast iterative solver. The iteration procedure is adapted in order to work in the case when topology changes. The second method is based on shape reconstruction to establish an efficient Shape-Quasi-Newton-Method. Various numerical experiments confirm the efficiency of the derived numerical methods.

CONTENTS

Key words and phrases. Quadrature domain, Free boundary problem, Level set method, Shape optimization.

M. Bazarganzadeh thanks Stockholm University for supporting to visit IST, Lisbon.

F. Bozorgnia was supported by the UT Austin-Portugal partnership through the FCT post-doctoral fellowship SFRH/BPD/33962/2009 and grants PTDC/MAT/114397/2009, UT Austin/MAT/0057/2008.

1. Introduction.

In this paper we shall consider general mathematical approaches to solve the free boundary problems of type

$$A(u, \Omega) = 0, \tag{1.1}$$

$$B(u, \Omega) = 0. \tag{1.2}$$

Here A corresponds to a well posed elliptic boundary value problem in an unknown domain $\Omega = \{x : u(x) > 0\} = \{u > 0\}$ and B operates on the functions supported at the free boundary $\Gamma = \partial\Omega$. It is supposed that function u can be solved from equation (1.1) for any given suitable domain Ω. More precisely, in this paper we consider the following problem:

$$\text{(P)} \qquad \begin{cases} \Delta u = \chi_\Omega - \mu, & \text{in} \quad \mathbb{R}^N, \\ u \geq 0, & \text{in} \quad \mathbb{R}^N, \\ u = 0, & \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases}$$

where μ is a given measure. Our aim in this work is to study systematic and efficient ways to solve Problem (P) numerically.

The outline of the paper is as follows: In section 2, we present some basic facts and mathematical background of quadrature domains. In section 3, we investigate one of the applications of quadrature domains, Hele Shaw flow. Section 4 is devoted to derive a numerical scheme which is based on the properties of the free boundary especially blow up techniques. In section 5 we construct a numerical scheme for Problem (P) based on shape reconstruction formulation. Finally, in last section we investigate some numerical examples which show the efficency of the numerical algorithms.

2. Notations and mathematical background of quadrature domains.

Let us review some notations that we use here. By Ω we mean an open subset of $\mathbb{R}^N$ and $L^p(\Omega)$ the usual Lebesgue space with respect to the Lebesgue measure. $HL^p(\Omega)$ denote the subspace of $L^p(\Omega)$ that consists of harmonic functions and $SL^p(\Omega)$ for the subspace of $L^p(\Omega)$ that consists of subharmonic functions. We also show the characteristic function of Ω by χ_Ω and G always denotes the usual "fundamental solution" for the Laplace operator in $\mathbb{R}^N$. In other words, for $x \in \mathbb{R}^N \setminus \{0\}$

$$G(x) = \begin{cases} \frac{1}{N(N-2)\omega_N} |x|^{2-N}, & \text{for } N \geq 3, \\ -\frac{1}{2\pi} \ln |x|, & \text{for } N = 2, \end{cases}$$

where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$.

Definition 2.1. Let μ be a measure with compact support. By a *quadrature domain* we mean an open connected set $\Omega \subset \mathbb{R}^N$ such that $supp(\mu) \subset \Omega$ and

$$\int_\Omega h\,dx \geq \int h\,d\mu, \tag{2.1}$$

holds for all $h \in SL^1(\Omega)$. We write $\Omega \in Q(\mu, SL^1)$ if (2.1) holds and $\mu(\Omega) < \infty$.

If one consider $\int_\Omega h\,dx = \int h\,d\mu$ for all $h \in HL^1(\Omega)$ then Ω is a quadrature domain and we write $\Omega \in Q(\mu, HL^1)$.

The simplest quadrature domain is a circular disc. Suppose that $\mu = \alpha\delta$ where δ is a Dirac mass at origin and $\alpha > 0$. Then

$$Q(\mu, HL^1) = Q(\mu, SL^1) = \{B(0;r)\},$$

where $r \geq 0$ is determined by $|B(0;r)| = \alpha$, (see [6]). Generally if Ω is a bounded domain in $\mathbb{R}^N$ and

$$\int_\Omega hdx = |\Omega|\,h(x_0), \tag{2.2}$$

holds for all $h \in HL^1(\Omega)$, where x_0 is an arbitrary point, then Ω is a ball centered at x_0.

Let U^μ, the Newtonian potential of the measure μ which is defined by

$$U^\mu(x) := (G * \mu)(x) = \int_{\mathbb{R}^N} G(x-y)d\mu(y), \;\; x \in \mathbb{R}^N,$$

and it satisfies the Poisson's equation $-\Delta U^\mu = \mu$ in the distribution sense. For the sake of simplicity, we shall use U^Ω instead of U^{χ_Ω}. It is immediately verified that if Ω is open and bounded then as function of $x \in \Omega$,

$$G(x-y) \in HL^1(\Omega), \quad \forall y \in \Omega^c \quad \text{and} \quad -G(x-y) \in SL^1(\Omega), \quad \forall y \in \Omega.$$

Gustafsson in [6] has showed that $\Omega \in Q(\mu, SL^1)$ if and only if

$$\begin{cases} U^\Omega \leq U^\mu, & \text{in}\,\mathbb{R}^N, \\ U^\Omega = U^\mu, & \text{in}\,\mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.3}$$

Also if one considers $u = U^\mu - U^\Omega \geq 0$, then

$$\Delta u = \chi_\Omega - \mu \quad \text{in} \quad \mathbb{R}^N. \tag{2.4}$$

Note that from (2.4) one has $\Delta u = \chi_\Omega$ away from $supp(\mu)$ and according to results in local regularity of solutions of elliptic PDEs, we obtain $u \in W^{2,p}_{\text{loc}}(\Omega)$ and for every $1 < p < \infty$. Also $\nabla u \in W^{1,p}_{\text{loc}}(\Omega)$. By the Sobolev embedding theorem the first derivatives are therefore Hölder continuous with Hölder exponent $\alpha < 1$.

Sakai in [13] has proved that the definition of quadrature domain is equivalent to the well-known one-phase free boundary problem in distribution scenes. More precisely, from PDE point of view, $\Omega \in Q(\mu, SL^1)$ is equivalent to

$$\begin{cases} \Delta u = \chi_\Omega - \mu, & \text{in} \quad \mathbb{R}^N, \\ u \geq 0, & \text{in} \quad \mathbb{R}^N. \end{cases} \tag{2.5}$$

Remark 1. Suppose that m denotes the Lebesgue measure. By (2.3) we know that $u = \nabla u = 0$ in $\mathbb{R}^N \setminus \Omega$. Now taking integration of (2.4) gives

$$0 = \int_{\partial\Omega} \frac{\partial u}{\partial n}\, ds = \int_\Omega \Delta u \, dx \; = \; m(\Omega) - \mu(supp(\mu)).$$

This fact is also a consequence of (2.1). In other words, we know the volume of the solution priori.

Example 2.2. As an other example of one phase quadrature domain, suppose that $x_0 \in \mathbb{R}^N$ and $a > 0$, $M > 1$. Let $B_1 = B_1(x_0, a)$ and $\mu = M\chi_{B_1}$. Then (2.5) reads

$$\begin{cases} \Delta u = \chi_\Omega - M\chi_{B_1}, & \text{in } \Omega, \\ u = \nabla u = 0, & \text{in } \Omega^c. \end{cases} \tag{2.6}$$

The spherical symmetry of the problem shows that the we have to find a radial solution $u = u(|x|)$ for (2.6). Consequently we suppose that $\Omega = B_2 = B_2(x_0, r)$ for some $r > a$. To make more easier we consider that $u = u_1$ on B_1 and $u = u_2$ on $B_2 \setminus B_1$. We desire to patch u_1 and u_2 on ∂B_1 without loosing regularity. Therefore our problem is

$$\Delta u = \begin{cases} 1 - M, & \text{on } B_1, \\ 1, & \text{on } B_2 \setminus B_1, \end{cases} \tag{2.7}$$

with the following conditions

$$\begin{cases} u_1 = u_2, & \text{on } B_1, \\ \nabla u_1 = \nabla u_2, & \text{on } B_1, \\ u_2 = 0, & \text{on } (B_2)^c, \\ \nabla u_2 = 0, & \text{on } (B_2)^c. \end{cases} \tag{2.8}$$

By some calculations and considering the fundamental solution for Laplacian operator one has

$$u(x) = \begin{cases} (1 - M)\frac{|x - x_0|^2}{2N} + A_1, & \text{on } B_1, \\ \frac{|x - x_0|^2}{2N} + A_2|x - x_0|^{2-N} + A_3, & \text{on } B_2 \setminus B_1, \\ 0, & \text{on } (B_2)^c, \end{cases} \tag{2.9}$$

where A_1, A_2, A_3 are appropriate constants which are computed with respect to (2.8). Now we obtain

$$u(x) = \begin{cases} (1-M)\frac{|x-x_0|^2}{2N} + \frac{r^2 - Ma}{2(2-N)}, & \text{on } B_1, \\ \frac{|x-x_0|^2}{2N} - \frac{r^N |x-x_0|^{2-N}}{N(N-2)} + \frac{r^2}{2(2-N)}, & \text{on } B_2 \setminus B_1, \\ 0, & \text{on } (B_2)^c, \end{cases} \tag{2.10}$$

where $r = M^{\frac{1}{N}} a$.

If $N = 2$ we can replace $\log|x - x_0|$ instead of $|x - x|^{2-N}$ in (2.10) and we derive that

$$u(x) = \begin{cases} (1-M)\frac{|x-x_0|^2}{4} + \frac{a^2}{4}M + \frac{r^2}{2}(\log\frac{a}{r} - \frac{1}{2}), & \text{on } B_1, \\ \frac{|x-x_0|^2}{4} + \frac{r^2}{2}\log|x-x_0| - \frac{r^2}{2}\big(\log r + \frac{1}{2}\big), & \text{on } B_2 \setminus B_1, \\ 0, & \text{on } (B_2)^c, \end{cases} \tag{2.11}$$

with $r = M^{\frac{1}{2}} a$.

Remark 2. Suppose that $N = 2$, $\mu = \delta_0$. Let $B(0, \epsilon)$ be an approximation of *supp*(μ) with $M = \frac{1}{\pi\epsilon^2}$ for ϵ enough small. Then one can obtain $r = \frac{1}{\sqrt{\pi}}$.

2.1. An estimate of quadrature domain.

In our problem the domain Ω is part of the solution, so in order to generate a mesh, one needs to find a domain which contains Ω. To do this we find a bigger domain such that Ω is embedded in it as follows.

By $r(\mu)$ we mean a positive number corresponding to the positive measure μ such that

$$|B_{r(\mu)}| = m(B_{r(\mu)}) = \int_{\mathbb{R}^N} d\mu = \mu(\mathbb{R}^N),$$

where m denotes Lebesgue measure in $\mathbb{R}^N$. The following theorem is due to Sakai, see [14].

Theorem 2.3. [14] *Let μ be a finite positive measure with support in the closed ball $\overline{B_R}$, $R > 0$. Then every quadrature domain Ω of μ for subharmonic functions satisfies*

$$\Omega \subset B_{r(\mu)+R}. \tag{2.12}$$

Furthermore, if $r(\mu) > 2R$ then

$$B_{r(\mu)-R} \subset \Omega.$$

For instance let $\mu = g(x)\chi_{B_1}$, where g is a positive function with $M = \sup_{B_1} g(x)$, then $\Omega \subset B_{\sqrt{M}+1}$.

For more information about one phase quadrature domain see [6, 7, 8, 13, 15].

3. An application (Hele Shaw flow).

One application of the problem appears in Laplacian growth like Hele-Shaw flow which comes up in flow's dynamic. One imagines that a domain like D_0 is filled with a fluid, say water, and the outer domain with another fluid, say oil. Water is supposed to be injected at the origin and there is an oil drain at infinity. The pressure p, satisfies the Laplace equation

$$\Delta p = 0.$$

We neglect the surface tension and suppose that pressure vanishes on the boundary. More precisely and from a mathematical point of view, suppose that some incompressible fluid has been confined between two parallel plate and we inject more fluid by moderate velocity to it. Therefore the fluid between plates begin to occupy more space. We are interested in to study the behavior of the boundary of the fill space.

This problem was introduced by S. Richardson [12]. Suppose that ν is a positive, finite and non zero measure with compact support and $supp(\nu) \subseteq D$ where D is an open subset of $\mathbb{R}^N$ by a C^1 boundary. Let p_D the super harmonic function such that

$$\begin{cases} -\Delta p_D = \nu & \text{in } D, \\ p_D = 0 & \text{on } \partial D. \end{cases} \tag{3.1}$$

We are looking for a family of regions D_t for $t \geq 0$, such that ∂D_t moves with the velocity $-\nabla p_{D_t}$.

Definition 3.1. Suppose that I is an interval in $\mathbb{R}$. Let $\mu = \chi_{D_0} + t\nu$, $t \in I$. A map $t \to D_t \subset \mathbb{R}^N$ is a *weak solution* of the free boundary problem if the function $u_t \in H^1(\mathbb{R}^N)$ defined by

$$\Delta u_t = \chi_{D_t} - \mu, \tag{3.2}$$

satisfies the following conditions:

- $u_t \geq 0$,
- $\int u_t(1 - \chi_{D_t})\, dx = 0$.

Last condition guarantee that $u_t = 0$ in $\mathbb{R}^N \setminus D_t$, see [5].

Remark 3. PDE (3.2) together with the above conditions are a special case of Problem (P).

Next theorem states the corresponding quadrature domain of the solution of the Hele-Shaw problem.

Theorem 3.2. [5] *Suppose that μ and D_0 are as before and $T > 0$. Then there exists a weak solution*

$$[0, T] \ni t \to D_t \subset \mathbb{R}^N,$$

for Hele Shaw problem which is unique and if u_t be the function appearing in (3.2), then u_t is also unique and

$$u_t = \int_0^t p_{D_\tau}\, d\tau.$$

Moreover, D_t can be chosen to be

$$D_t = D_0 \cup \{z : u_t(z) > 0\}.$$

For more information about Hele shaw see [5], [12], [9] and references therein.

4. First numerical method to approximate the solution of Problem (P).

In this part using the properties of given free boundary problem, we construct an algorithm that leads to a fast iterative solver. The level set method is next applied to evolve the interface in the direction of the normal velocity field.

Consider Problem (P) in dimension one. Our motivation for the first method is based on the fact that for any x outside of the supp(μ) one has

$$u'(x) = \pm\sqrt{2u}.$$

To be more precise, one has

$$\Delta u = 1, \quad \text{in } \{x : u(x) > 0\} \setminus supp(\mu). \tag{4.1}$$

Let x_f be a free boundary point. Multiply (4.1) by u' and integrate over $[x\,,\, x_f]$ to find that

$$\frac{1}{2}(u')^2(x) = u(x).$$

Let $[c\,,\, d]$ be an initial guess for $\{x : u(x) > 0\}$ which contains the support of measure μ. Next we solve the following boundary value problem

$$\begin{cases} u'' = 1 - \mu & \text{in} \quad [c\,,\, d], \\ u'(c) = \sqrt{2u(c)}, & u'(d) = -\sqrt{2u(d)}. \end{cases} \tag{4.2}$$

Then to get the free boundary points, we move the points c, d in the normal direction with speeds $\sqrt{2u(c)}$ and $\sqrt{2u(d)}$, i.e,

$$d_f = d - \sqrt{2u(d)}, \quad c_f = c + \sqrt{2u(c)},$$

where c_f and d_f are free boundary points. Note that in this case we need only one iteration, see section 4.3.

Remark 4. To have existence for the boundary value problem (4.2) one has to choose $[c, d]$ enough closed to supp(μ).

4.1. Blow up techniques and the main idea.

In higher dimensions we shall prove that when we are enough close to the free boundary still the quotient $\frac{|\nabla u(x)|}{\sqrt{2u(x)}}$ goes to one. First, we recall some known properties and lemmas that have been proved in [11], which we shall use in the proof of Theorem 4.6.

The following lemma shows the growth of u away from the free boundary Γ.

Lemma 4.1. [11] *Let $u \in L^\infty_{loc}(\Omega)$, $\Omega = \{u > 0\}$ be a solution of Problem (P). If $x_0 \in \Gamma$ then*

$$\sup_{B_r(x_0)} u \leq Cr^2,$$

where $C = C(N)$.

Corollary 4.2. *Let u be as in Lemma 4.1. Then*

$$u(x) \leq C dist(x, \partial\Omega)^2.$$

Also we need the following **Non degeneracy** property of the solutions.

Lemma 4.3. [11] *Let u be a solution of given free boundary problem, then we have the inequality*

$$\sup_{\partial B_r(x_0)} u \geq \frac{r^2}{8N}, \qquad \text{for any } x_0 \in \Gamma.$$

Definition 4.4. (Local solutions) For given $R, M > 0$, and $x_0 \in \Gamma$, let $P_R(x_0, M)$ be the class of $C^{1,1}$ solutions u of Problem (P) in $B_R(x_0)$ such that

$$|Du(x) - Du(y)| \leq M|x - y| \qquad \forall x, y \in \mathbb{R}^N.$$

In the case $x_0 = 0$ we also set $P_R(M) = P_R(0, M)$.

In the above definition if $R = \infty$ then we get solutions in the entire space $\mathbb{R}^N$ and grow quadratically at infinity, which are called global solutions.

If $u \in P_R(x_0, M)$ and $\lambda > 0$, then the proper re-scaling of u at x_0 is defined by

$$u_{x_0,\lambda}(x) = \frac{u(x_0 + \lambda x) - u(x_0)}{\lambda^2}.$$

Note that by using non degeneracy, Lemma 4.3, and quadratic growth properties, Lemma 4.1, it can be shown that when $\lambda \to 0$ then

$$u_{x_0,\lambda} \to u_0 \quad \text{in } C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N) \text{ for any } 0 < \alpha < 1,$$

where $u_0 \in C^{1,1}_{\text{loc}}(\mathbb{R}^N)$. This u_0 is called a *blowup* of u with fixed center x_0 and also u_0 is a global solution, i.e, $u_0 \in P_\infty(M)$. For more details see [11].

Theorem 4.5. [11] *(Blow up with fixed center). Let $u \in P_R(x_0, M)$ be a solution of Problem (P). Suppose that*

$$u_0(x) = \lim_{j\to\infty} u_{x_0,\lambda_j}(x), \quad x \in \mathbb{R}^N,$$

for some sequence $\lambda_j \to 0$ *as* $j \to \infty$. *Then* u_0 *is homogeneous of degree two with respect to the origin, i.e.*

$$u_0(\lambda x) = \lambda^2 u_0(x), \quad \text{for any } x \in \mathbb{R}^N \text{ and } \lambda > 0.$$

In the proof of next theorem we will use the concept of *regular points.* $x_0 \in \Gamma$ is a regular point if every blow up of u at x_0 is a half plane solution. Precisely, there is two category of blowup for a solution of the Problem (P). Let u_0 be a blowup with a fixed center then it has one of the following forms (see [11]):

- Polynomial solution: $u_0(x) = \frac{1}{2}(x \cdot Ax)$, $x \in \mathbb{R}^N$. Here A is an $n \times n$ symmetric matrix with $Tr(A) = 1$.
- Half plane solutions: $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$, $x \in \mathbb{R}^N$ where e is a unit vector.

Theorem 4.6. *Let* x_0 *be a free boundary point and* $x \in \{u > 0\}$ *then*

$$\limsup_{x \to x_0} = \liminf_{x \to x_0} \frac{|\nabla u(x)|}{\sqrt{2u(x)}} = 1.$$

Proof. By Theorem 4.5, blowup solutions at fixed point $x_0 \in \Gamma$ is a global homogeneous solution of degree two. Let u be a homogeneous global solution. Then by above discussion, u has the following form

$$u_0(x) = \frac{1}{2}(x \cdot e)_+^2, \; x \in \mathbb{R}^N \text{ where } e \text{ is a unit vector.}$$

Without loss of generality assume that $x_0 = 0$ then we know that

$$\tfrac{u(rx)}{r^2} \to \tfrac{(x_1)_+^2}{2} \text{ in } C^{1,\alpha},$$

which means

$$\left| \frac{u(rx)}{r^2} - \frac{(x_1)_+^2}{2} \right| \to 0,$$

and consequently,

$$\left| \frac{\nabla u(rx)}{r} - x_1 e_1 \right| \to 0.$$

From above one can get

$$u(rx) = \frac{(rx_1)_+^2}{2} + cr^2, \text{ where } c \text{ is an arbitrary small constant and}$$

$$\nabla u(rx) = (rx_1)e_1 + O(r^\alpha), \quad \alpha < 1.$$

Using above expression for $u(rx)$ and $|\nabla u(rx)|$ and taking the quotient implies the limit. □

4.2. Level set formulation.

The level set method was introduced by Osher and Sethian for implicitly tracking dynamic surfaces and curves, see [10, 16]. The main idea behind this method is to embed an interface Γ, which lies in $\mathbb{R}^{N-1}$ into a surface in dimension $\mathbb{R}^N$. We can do this embedding by defining a proper function ϕ such that Γ is the zero level set of ϕ, i.e,

$$\Gamma = \partial\Omega = \{x \in \mathbb{R}^N;\ \phi(x) = 0\}.$$

Suppose that Γ divides $\mathbb{R}^N$ into multiple connected components then one can recognize the inside of one component from its exterior when the sign of ϕ changes.

Regarding to Theorem 2.3 let $\mathcal{T}$ be a given rectangle such that $\Omega \subset B_{r(\mu)+R} \subset \mathcal{T}$ for appropriate $R > 0$. To apply the level set method for Problem (P), we need ϕ be positive in $\mathcal{T} \setminus \Omega$ and negative in Ω. By this way the outward normal vector of Ω is given by

$$\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|}.$$

We note that Problem (P) is stationary and the level set formulation requires a time evolution so we define the parameter t and introduce a family of boundaries $\Omega(t)$ for $t > 0$ as the level sets by

$$\partial\Omega(t) = \{x \in \mathbb{R}^N;\ \phi(x,t) = 0\},$$

for unknown function $\phi : \mathcal{T} \times \mathbb{R}^+ \to \mathbb{R}$. By chain rule

$$\phi_t + \nabla\phi(x(t), t) \cdot x'(t) = 0.$$

Let $F = x'(t) \cdot \mathbf{n}$ which means that F is speed in outward normal direction. Then the level set equation will be as follows

$$\begin{cases} \phi_t + F|\nabla\phi| = 0, \\ \phi(x, t = 0) \text{ is given.} \end{cases}$$

In this paper we restrict our attention to the case that ϕ is considered as the sign distance function and therefore $|\nabla\phi| = 1$. Hence the level set equation turns to

$$\frac{\partial\phi}{\partial t} + F = 0 \quad \text{in} \quad \mathcal{T} \times \mathbb{R}^+. \tag{4.3}$$

4.3. A mixed boundary value problem and first algorithm.

Assume (Ω, u) be a smooth solution of Problem (P). Our aim is to build a sequence (Ω_k, u_k) of solutions of an approximate quadrature domain problem which converges towards (Ω, u). Assume that $p_k \in \partial\Omega_k$. Let $\mathbf{n}_k$ be the normal outward vector on $\partial\Omega_k$. By Taylor formula, one can write

$$u(p_k + d_k\mathbf{n}_k) \simeq u(p_k) + d_k\nabla u(p_k) \cdot \mathbf{n}_k + \frac{d_k^2}{2}\,\mathbf{n}_k^T \cdot D^2u(p_k) \cdot \mathbf{n}_k.$$

We wish to have $u(p_k+\mathbf{n}_k d_k)=0$, so if we put the $\nabla u(p_k)\cdot\mathbf{n}_k=-\sqrt{2u(p_k)}$ and use the approximation $D^2u\simeq\frac{1}{2}(\Delta u)I$, then one gets $d_k=\sqrt{2u(p_k)}$. It means that if $\Gamma_k=\partial\Omega_k$ then $\{\Gamma_k+d_k\cdot\mathbf{n}_k\}$ converges to Γ.

To construct an algorithm let U be an initial guess of Ω which contains $supp(\mu)$. Consider the following boundary value problem which has a vital role in the numerical scheme

$$\begin{cases}\Delta u=1-\mu, & \text{in } U,\\ \frac{\partial u}{\partial n}=-\sqrt{2u}, & \text{on } \partial U.\end{cases}\tag{4.4}$$

Remark 5. We note that (4.4) is not stable at the points close to the free boundary, therefore alternatively we solve the following problem to have more efficient and robust scheme,

$$\begin{cases}\Delta u_k=1-\mu, & \text{in } \Omega_k,\\ \frac{\partial u_k}{\partial n_k}=-\theta u_k, & \text{on } \partial\Omega_k.\end{cases}\tag{4.5}$$

We desire that θu_k behaves like $\sqrt{2u_k}$, therefore one is able to choose

$$\theta=\sqrt{\frac{2}{u_{k-1}}}.$$

The existence of (4.5) is based on minimization techniques and is a special case of the next lemma.

Lemma 4.7. [3] *Assume* $\beta:\mathbb{R}\to\mathbb{R}$ *is smooth, with*

$$0<a\le\beta'(z)\le b\quad(z\in\mathbb{R}),$$

for constants a,b. *Let* $f\in L^2(U)$, $U\subset\mathbb{R}^N$ *is a bounded, open set with smooth boundary. For*

$$\begin{cases}-\Delta u=f, & \textit{in } U,\\ \frac{\partial u}{\partial n}+\beta(u)=0, & \textit{on } \partial U,\end{cases}\tag{4.6}$$

there exists a unique weak solution.

Now consider the following boundary value problem

$$\begin{cases}\Delta u(t)=1-\mu, & \text{in}\quad \Omega(t),\\ \frac{\partial u(t)}{\partial n}=-\sqrt{\frac{2}{u(t-1)}}u(t), & \text{on}\quad \partial\Omega(t).\end{cases}\tag{4.7}$$

We choose the quantity $\sqrt{2u(t)}$ as the speed which decreases in $\Omega(t)\backslash supp(\mu)$ and goes to zero when $\Omega(t)$ approaches to the free boundary. Regarding to (4.3), the displacement of the boundary $\Omega(t)$ can be obtained by considering the following equation :

$$\frac{\partial\phi}{\partial t}+\sqrt{2u(t)}=0,\quad \text{on } \partial\Omega(t).\tag{4.8}$$

Now let $\mathcal{T}$ be the rectangle in section 4.2. The extension of the previous equation to whole domain $\mathcal{T}$, is one of the important issue in the level set approach. To do this we solve the problem:

$$\begin{cases} \Delta v(t) = 1, & \text{in} \quad \Omega(t) \setminus supp(\mu), \\ \Delta v(t) = 0, & \text{in} \quad \mathcal{T} \setminus \Omega(t), \\ v(t) = 0, & \text{on} \quad \partial(supp(\mu)) \cup \partial\mathcal{T}, \\ v(t) = \sqrt{2u(t)}, & \text{on} \quad \partial\Omega(t). \end{cases} \tag{4.9}$$

We now extend equation (4.8) to $\mathcal{T}$ by

$$\frac{\partial \phi}{\partial t} + v(t) = 0, \quad \text{in } \mathcal{T} \setminus supp(\mu). \tag{4.10}$$

For more information on velocity extension see [4, 10].

4.3.1. First algorithm for Problem (P).

Choose a tolerance, TOL<< 1.

(1) Set $k = 0$, choose an initial domain Ω_0 with $\Gamma_0 = \partial\Omega_0$ such that

$$supp(\mu) \subset \Omega_0 \subset B_{r(\mu)+R}.$$

(2) Compute u_k on Ω_k which is the solution of the following elliptic boundary value problem

$$(\star) \qquad \begin{cases} \Delta u_k = 1 - \mu, & \text{in } \Omega_k, \\ \frac{\partial u_k}{\partial n_k} = -\theta u_k, & \text{on } \partial\Omega_k. \end{cases}$$

(3) Solve (4.9) and obtain v.
(4) Update the level set function ϕ from (4.10) to get Ω_{k+1}.
(5) Solve $(\star)$ in Ω_{k+1} and get u_{k+1}.
(6) If $|u_{k+1}| < \text{TOL}$, then stop else set $k = k + 1$ and go to (2).

5. Second numerical method to approach to the solution of Problem (P) based on shape optimization.

The shape sensitivity analysis is used to define a velocity field, which allows us to update the surface while decreasing a given cost function. The solution of an elliptic boundary value problem usually depends highly nonlinearly on the geometry of the given domain. Thus the geometry can not be solved straightforward from a linear equation.

In shape optimization approach, we rewrite the free boundary problem such that the minimum of some cost functional is attained at the solution of free boundary. The solution of Problem (P) minimizes the functional

$$E(u, \Omega) = \int_\Omega \frac{1}{2} |\nabla u|^2 dx + \int_\Omega (1 - \mu)\, u\, dx, \tag{5.1}$$

over $u \in H^1(\Omega)$ where $\Omega = \{u > 0\}$. Note that we get $u = 0$ on $\partial\Omega$.

In the following we discuss the shape sensitivity analysis for the above shape functional related to Problem (P). At first, we briefly recall some basic facts related to shape calculus [20].

In shape sensitivity we analyze how the solution of a PDE changes when the domain is changing with a velocity field. Let $x \in \mathbb{R}^N$, and $\mathbf{V}(t,x)$ be a velocity field (vector field) defined in $D, \mathbf{V} \in C^k(D;\mathbb{R}^N), \mathbf{V}|_{\partial D} = 0$. Let t be artificial time. Assume that $\Sigma \subseteq D$. It is natural to define transformation $T_t(\mathbf{V})x = X(t,x)$ with a velocity field $\mathbf{V}$ by differential equations

$$\frac{\partial X}{\partial t}(t,x) = \mathbf{V}(t,x), \quad X(0,x) = x, \ x \in \Sigma.$$

One can see that this transformation is quite close to a perturbation of the identity in [20, 1], where the transformation was defined by

$$T_t(\mathbf{V}) = I + t\mathbf{V}(x).$$

For small perturbations these two transformations are close (see [21]). The image of $\Sigma \subset \Omega$ under T_t is Σ_t.

Let J be a domain functional $J : \Sigma \longmapsto \mathbb{R}$. We say that the functional has a directional shape derivative to direction $\mathbf{V}$ at Σ if the limit

$$\lim_{t \to 0} \frac{J(\Sigma_t) - J(\Sigma)}{t} := dJ(\Sigma, \mathbf{V}),$$

exists. If further $dJ(\Sigma, \mathbf{V})$ is linear and continuous with respect to $\mathbf{V}$ and it exists for all directions $\mathbf{V}$, we say that J is shape differentiable at Σ. By Hadamard's structure theorem, $dJ(\Sigma, \mathbf{V})$ depends only on the normal component of $\mathbf{V}$ on the boundary of Σ, see [22, 23].

We use the notations u_Ω or $u(\Omega)$ to show the dependence of solution of a given PDE with respect to the domain Ω. For a function $v(\Sigma)$ and $\Sigma \in C^k$, $k \geq 1$, we define material derivative as a limit

$$\dot{v}(\Sigma; \mathbf{V})(x) := \lim_{t \to 0} \frac{v(\Sigma_t) \circ T_t(\mathbf{V}) - v(\Sigma)}{t}.$$

This limit may exist either in a weak or a strong sense, and the material derivative is called a weak or strong material derivative respectively, see [20].

The shape derivative of $v(\Sigma)$ in the direction $\mathbf{V}$ is the element $v'(\Sigma; \mathbf{V})$ defined by

$$v'(\Sigma; \mathbf{V}) := \dot{v}(\Sigma; \mathbf{V}) - \nabla v(\Sigma) \cdot \mathbf{V}(0),$$

whenever it exists either in a weak or a strong sense. For simplicity's sake we shall utilize v'_Σ instead of $v'(\Sigma; \mathbf{V})$.

Shape derivative represents the change of function v with respect to the geometry. Equivalently, shape derivative is the variation of the state variable with respect to the shape change.

The following lemmas represent the basic formulas for shape differentiation of integrals. In the following we assume that Ω is bounded.

Lemma 5.1. [20] *Let $f(\Omega_t) \in L^1(\Omega_t)$ be shape differentiable and $f'(\Omega_t) \in L^1(\Omega_t)$, $t \in [0,T]$ and $T > 0$. If Ω_t is a $C^{0,1}$-domain, then*

$$\left(\frac{d}{dt}\int_{\Omega_t} f(\Omega_t)dx\right)\Big|_{t=0} = \int_{\Omega} f'(\Omega)dx + \int_{\partial\Omega} f(\Omega) < \mathbf{V}, \mathbf{n} > ds. \tag{5.2}$$

5.1. Shape optimization techniques for Problem (P) and second algorithm.

First ingredient is the shape derivative of the function u_Ω.

Lemma 5.2. *The shape derivative of u_Ω in the normal direction $\mathbf{V}$, is given by the function u'_Ω, satisfies*

$$\begin{cases} \Delta u'_\Omega = 0, & in \quad \Omega, \\ u'_\Omega = -\frac{\partial u}{\partial \mathbf{n}} < \mathbf{V}(0), \mathbf{n} >, & on \quad \partial\Omega. \end{cases} \tag{5.3}$$

Proof. The minimizer of the functional in (5.1) satisfies the following equation

$$\Delta u_{\Omega_t} = f = 1 - \mu \quad \text{in } \Omega_t.$$

By multiplying a test function, $\varphi \in H_0^1(\Omega)$, and taking integral one obtains

$$\int_{\Omega_t} \nabla u_{\Omega_t} \cdot \nabla\varphi \; dx = -\int_{\Omega_t} f\,\varphi\, dx. \tag{5.4}$$

Taking the derivative of the above equation respect to t and considering Lemma 5.1 one can see that u'_Ω satisfies

$$\int_{\Omega} \nabla u'_\Omega \cdot \nabla\varphi\, dx = -\int_{\partial\Omega} f'\, dx = 0.$$

That is

$$\Delta u'_\Omega = 0.$$

The boundary condition in (5.3) is verified by equation (3.6), chapter 3 in [20]. □

Remark 6. Let $\Gamma = \partial\Omega$ be the free boundary for the solution of Problem (P). Then

$$u'_\Omega = 0 \qquad \text{in } \Omega.$$

Let us now to analyze the behavior of the energy near the solution.

Lemma 5.3. *Consider the energy functional (5.1) of Problem (P). Then the shape derivative of E with respect to $\mathbf{V}$ is*

$$dE(\Sigma, \mathbf{V}) \;=\; \int_{\Sigma} div(-\frac{1}{2}|\nabla u|^2\, \mathbf{V})\, dx. \tag{5.5}$$

Proof. By Lemma 5.1 one can see

$$dE(\Sigma, \mathbf{V}) = \int_{\Sigma} \big(\nabla u \cdot \nabla u' + (1-\mu)u'\big)\, dx + \int_{\partial\Sigma} \big(\frac{1}{2}|\nabla u|^2 + (1-\mu)u\big)\mathbf{V}\cdot\mathbf{n}\, ds,$$

where u' is the shape derivative of u into direction $\mathbf{V}$. Our assumption on Problem (P) states that $u\lfloor_{\partial\Sigma}= 0$. Then the shape derivative of E is

$$dE(\Sigma, \mathbf{V}) = \int_{\Sigma} \nabla u \cdot \nabla u'\; dx + \int_{\Sigma} (1-\mu)u' + \int_{\partial\Sigma} \frac{1}{2}|\nabla u|^2\, \mathbf{V}\cdot\mathbf{n}\, ds. \tag{5.6}$$

According to Green's theorem, the first term of (5.6) is

$$\int_{\Sigma} \nabla u \cdot \nabla u'\, dx = -\int_{\Sigma} u'\,\Delta u\, dx + \int_{\partial\Sigma} u' \frac{\partial u}{\partial \mathbf{n}}\, ds,$$

and we get

$$\begin{aligned} dE(\Sigma, \mathbf{V}) &= -\int_{\Sigma} u'\Delta u\, dx + \int_{\partial\Sigma} u'\frac{\partial u}{\partial \mathbf{n}} ds \\ &\quad + \int_{\Sigma} (1-\mu)u'\, dx + \int_{\partial\Sigma} \frac{1}{2}|\nabla u|^2\, \mathbf{V}\cdot\mathbf{n}\, ds \\ &= -\int_{\Sigma} u'(1-\mu)\, dx + \int_{\partial\Sigma} u'\frac{\partial u}{\partial \mathbf{n}} ds \\ &\quad + \int_{\Sigma} (1-\mu)u'\, dx + \int_{\partial\Sigma} \frac{1}{2}|\nabla u|^2\, \mathbf{V}\cdot\mathbf{n}\, ds \\ &= \int_{\partial\Sigma} u'\frac{\partial u}{\partial \mathbf{n}} ds + \int_{\partial\Sigma} \frac{1}{2}|\nabla u|^2\, \mathbf{V}\cdot\mathbf{n}\, ds. \end{aligned}$$

As u is the solution of a Dirichlet problem, Lemma 5.2 gives us $u' = -\frac{\partial u}{\partial \mathbf{n}} < \mathbf{V}, \mathbf{n} >$ on $\partial\Sigma$. Hence we have for $dE(\Sigma, \mathbf{V})$ the expression

$$dE(\Sigma, \mathbf{V}) = -\int_{\partial\Sigma} \frac{1}{2}|\nabla u|^2\, \mathbf{V}\cdot\mathbf{n}\, ds,$$

and by Stock's theorem it turns

$$dE(\Sigma, \mathbf{V}) \;=\; \int_{\Sigma} div(-\frac{1}{2}|\nabla u|^2\, \mathbf{V})\, dx.$$

□

Corollary 5.4. *The solution of Problem (P) is a critical point of the energy functional E.*

Proof. We choose $\mathbf{V}\cdot\mathbf{n} = -\frac{\partial u_\Sigma}{\partial \mathbf{n}}$ on $\partial\Sigma$. If $\Sigma \subset \Omega$ then $\frac{\partial u_\Sigma}{\partial \mathbf{n}} < 0$ so we have $dE(\Sigma, \mathbf{V}) \leq 0$ and it means that E is decreasing respect to V and the solution of free boundary where $\nabla u = 0$, is a critical point of E. □

5.1.1. Second algorithm for Problem (P).

(1) Set $k = 0$, choose an initial domain Σ_0 such that $supp(\mu) \subset \Sigma_0$ and set $\Gamma_0 = \partial\Sigma_0$.
(2) Solve $\Delta u_k = 1$ in $\Sigma_k \setminus supp(\mu)$ with Dirichlet boundary condition $u_k = 0$ on Γ_k,
(3) Compute a normal velocity from (2), i.e.

$$\mathbf{V} \cdot \mathbf{n} = -\nabla u_k \cdot \mathbf{n}_{\Gamma_k}$$

(4) Stop if $\|\nabla u_k\|_{L^2(\Gamma)}$ is sufficiently small.
(5) Given Γ_k, move the free boundary by Quasi-Newton method, i.e,
In dimension one

$$x_{k+1} = x_k - u'(x_k).$$

In dimension two

$$\Gamma_{k+1} = \Gamma_k - \nabla u(x_k) \cdot I.$$

Obtain the new shape Σ_{k+1} with free boundary Γ_{k+1}.
(6) Set $k = k + 1$ and go to (2).

5.2. Alternative viewpoint.

One can consider another starting point. We try to determine a shape Ω such that

$$\frac{\partial u_\Omega}{\partial \mathbf{n}} = 0, \qquad \text{on } \Gamma.$$

In order to derive a suitable weak formulation, we multiply the normal derivative by a smooth test function φ and integrate over Γ, i.e. we have

$$\int_\Gamma \frac{\partial u_\Omega}{\partial \mathbf{n}} \varphi \, d\sigma = 0.$$

By Gauss' Theorem together with the Poisson equation for u_Ω we have

$$\int_\Omega (f\varphi + \nabla u_\Omega \cdot \nabla\varphi) \, dx = 0, \qquad \forall \ \varphi \in H_0^1(\Omega).$$

In other words, the first optimality condition for E (with respect to v) reads

$$dE(u;\varphi,\Omega) := dE(u + \varepsilon\,\varphi,\Omega)\lfloor_{\varepsilon=0} = \int_\Omega (f\varphi + \nabla u_\Omega \cdot \nabla\varphi) \, dx = 0,$$

for all $u \in H_0^1(\Omega)$. If one consider

$$J(\varphi,\Omega) = \int_\Omega (f\varphi + \nabla u_\Omega \cdot \nabla\varphi) \, dx,$$

then $J(\Omega, .)$ is a continuous linear functional on $H_0^1(\Omega)$, i.e, it can be interpreted as an element of $H^{-1}(\Omega)$ and we can define an operator $F(\Omega) = J(\Omega, .)$ mapping into $H^{-1}(\Omega)$ such that (5.7) is equivalent to solving

$$F(\Omega) = 0 \qquad \text{in } H^{-1}(\Omega). \tag{5.7}$$

Now we can do all similar calculations for the functional J and deduce same results.

6. Numerical examples.

Example 6.1. In this example the support of the measure μ is a polygon which is shown in Figure 1. We obtain the corresponding quadrature domain. Set $\mu = 1.5\chi_P$. In Figure 1, the initial guess is the circle and this figure shows the solution after first iteration. Figure 2 states the result after four iterations. Figure 3 illustrates the norm of the gradient on the boundary of the solution in forth iteration.

Now let $\mu = 11\chi_P$. Figure 4 shows the solution after first iteration and Figure 5 states the final result which is close to a ball. Figure 7 illustrates the quantity of $\frac{|\nabla u|}{\sqrt{2u}}$ on a cross section line which has been shown in Figure 6. This Figure verifies Theorem 4.6.

Example 6.2. Suppose that $\mu = t(\chi_{B_1} + 2\chi_{B_2})$ is uniformly distributed on two circles $B_1(x_1, 1), B_2(x_2, 1)$ where $x_1 = (-2, 0), x_2 = (\sqrt{8}, 0)$. According to Example 2.2 or Remark 1 we find that if $t = 4$ then B_1 and B_2 touch each other at origin tangentially. Let time increase to $t = 5$ and solve

$$\begin{cases} \Delta u = 1 - t(\chi_{B_1} + 2\chi_{B_2}), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

to get the corresponding quadrature domain. Figure 8 shows the solution at $t = 5$ and Figure 9 illustrates $|\nabla u|$ for $t = 5$. Figure 10 is the solution of similar PDE for $t = 6$.

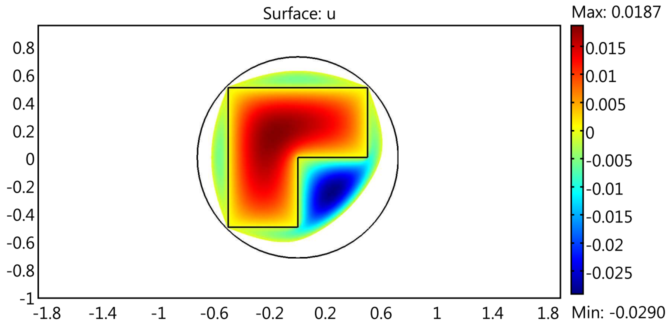

FIGURE 1. The colored part shows the solution Ω_1, after first iteration, where support of μ is the polygon and the initial guess (Ω_0) is a ball.

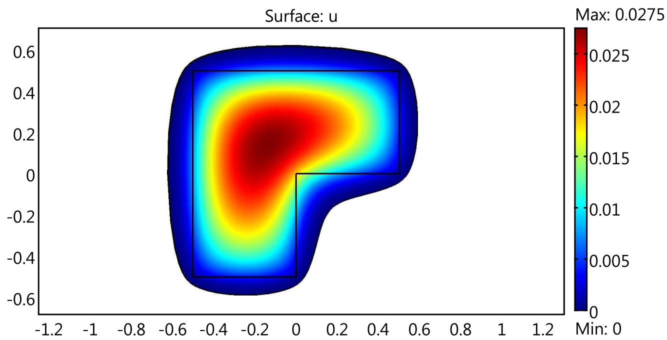

FIGURE 2. Final domain after four iterations when $\mu = 1.5\chi_P$ and where P is the polygon.

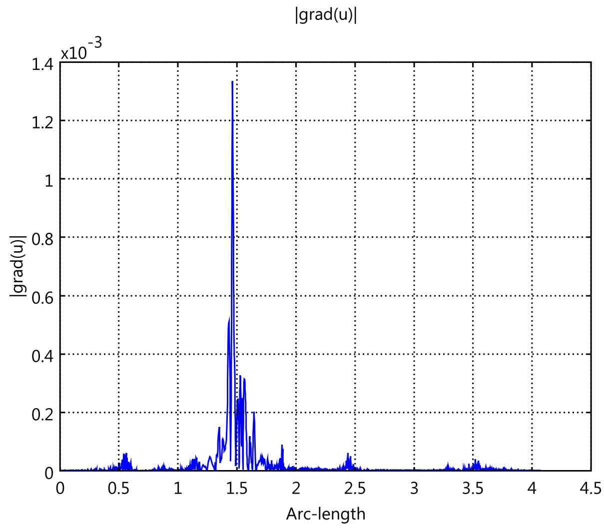

FIGURE 3. The value of $|\nabla u|$ on the boundary of the solution after four iterations.

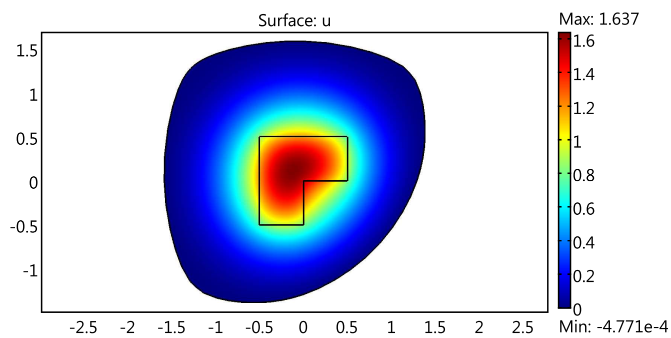

FIGURE 4. The first iteration for $\mu = 11\chi_P$, where P is the polygon. Initial guess is a ball with center at origin.

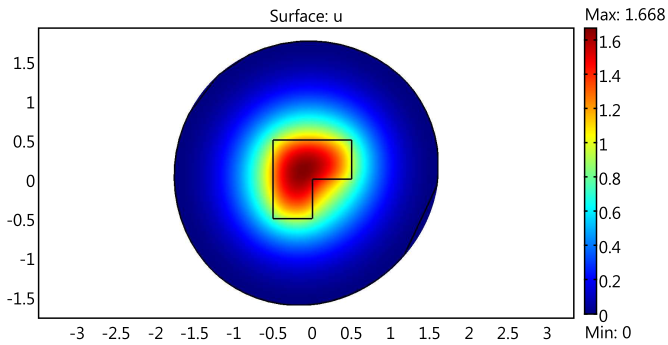

FIGURE 5. Final quadrature domain when $\mu = 11\chi_P$ and where P is the polygon.

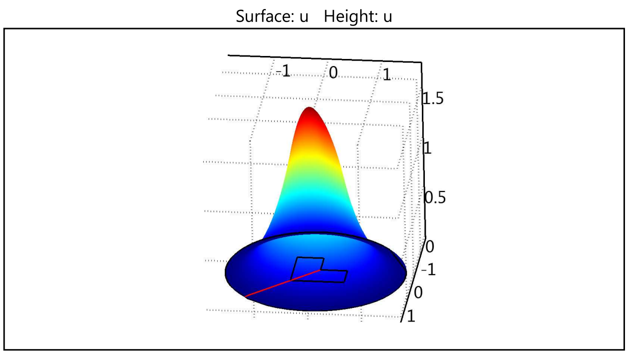

FIGURE 6. The surface of the solution u and a cross section line.

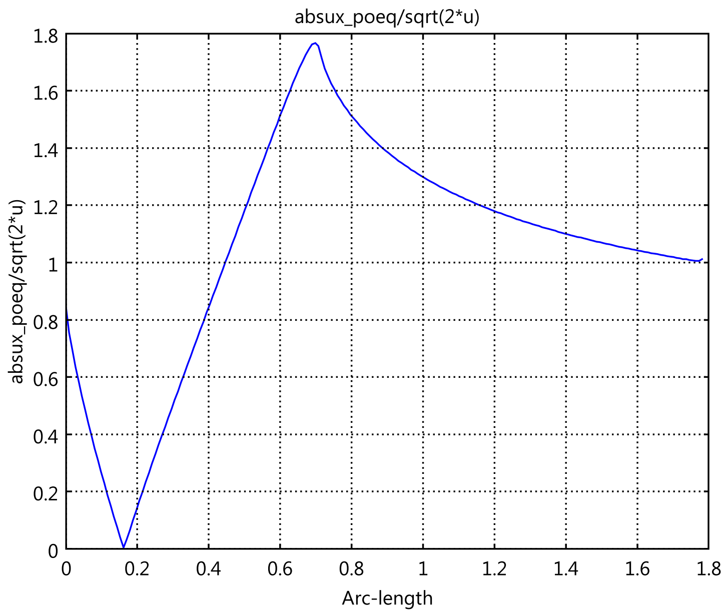

FIGURE 7. The amount of $\frac{|\nabla u|}{\sqrt{2u}}$ on the cross section line in figure (6).

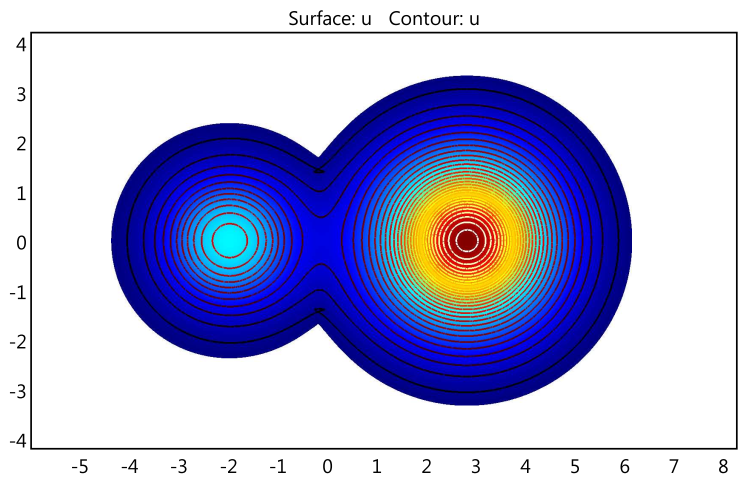

FIGURE 8. The quadrature domain corresponding to the solution of (6.1) for $t = 5$.

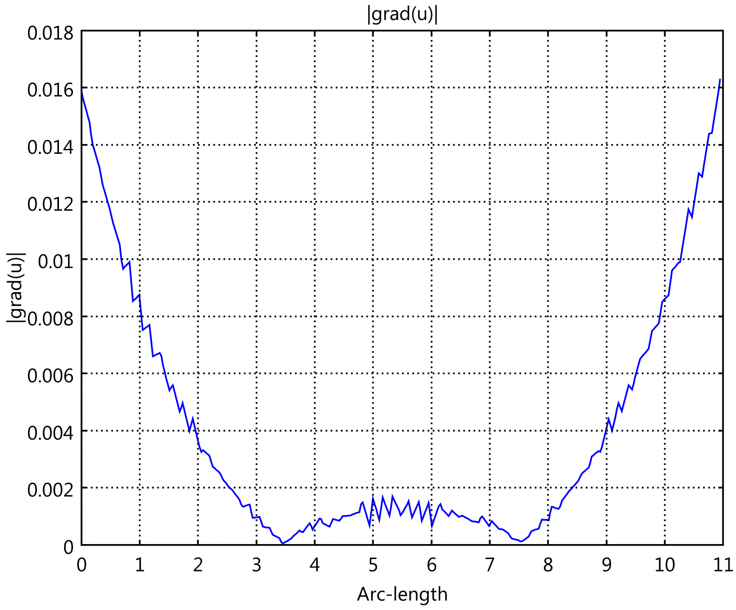

FIGURE 9. The quantity of $|\nabla u|$ on the boundary of the solution of (6.1) for $t = 5$.

Acknowledgments: This problem was suggested by Professor Henrik Shahgholian. We wish to thank him for plentiful discussions and profitable suggestions.

References

[1] Delfour M.C, Shape derivatives and Differentiabilty of Min Max, in "shape optimization and free boundaries, M.C.Delfour and G. Sabidussi (eds)", Kluwer, Dordrecht, 1990, pp 35-339.

[2] Ebenfelt P., Gustafsson B., Khavinson D., Putinar M., Quadrature Domains and Applications, a Harold S. Shapiro Anniversary Volume. , Birkhauser, 2005.

[3] Evans, Lawrence C. Partial differential equations. Graduate Studies in Mathematics 19., American Mathematical Society, Providence, RI., 662 pp, 1998.

[4] Flusher M., Rumpf M., Bernoulli's free-boundary problem, Qualitative theory and numerical approximation, J. Reine Angew. Math., Vol. 486, (1997) 165-204.

[5] Gustafsson B., Applications of variational inequalities to a moving boundary problem for Hele Shaw flows, SIAM J. Math. Anal. 16 (1985), 279-300.

[6] Gustafsson B., On quadrature domains and an inverse problem in potential theory, J.Analyse Math. 55 (1990), 172-216.

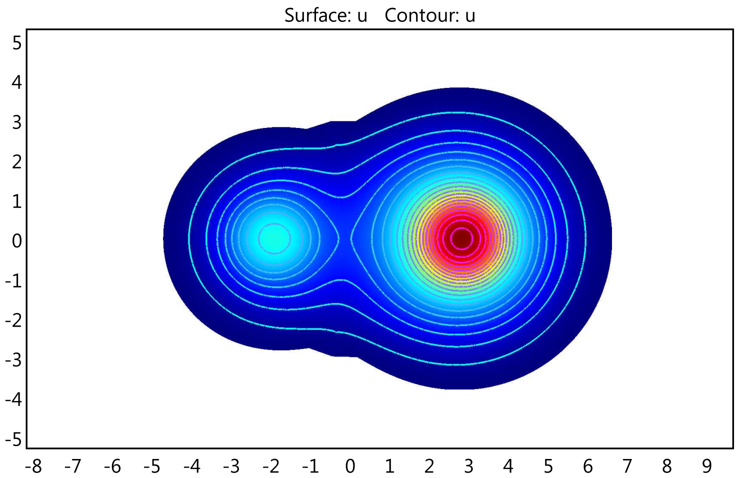

FIGURE 10. The quadrature domain corresponding to the solution of (6.1) for $t = 6$.

[7] Gustafsson B., Putinar M., Selected topics on quadrature domains. Physica D 235, 90-100 (2007).

[8] Gustafsson B., Shahgholian H., Existence and geometric properties of solutions of a free boundary problem in potential theory, J. Reine Angew. Math. 473 (1996), 137-179.

[9] Gustafsson B., Vasilev A., Conformal and potential analysis in Hele-Shaw cells, ISBN 3-7643-7703-8, Birkhäuser Verlag, 2006.

[10] Osher J., Fedkiw R., Level set methods and dynamic implicit surfaces, Springer, 2003.

[11] Petrosyan A., Shahgholian H., Uraltseva N., Regularity of free boundaries in obtacle type problem, book, coming up.

[12] Richardson S., Hele Shaw flows with a free boundary produced by the injection of fluid into a narrow channel, Journal of Fluid Mechanics, 1972 - Cambridge Univ Press.

[13] Sakai M., Quadrature Domains, Lect. Notes Math. Vol. 934, Berlin-Heidelberg: Springer-Verlag, 1982.

[14] Sakai M., Sharp estimates of the distance from a fixed point to the frontier of a Hele-Shaw flow. Potential Anal. 8 (1998), no. 3, 277-302.

[15] Sakai M., Applications of variational inequalities to the existence theorem on quadrature domains, Trans. Am.Math. soc. 276(1983), 267-279.

[16] Sethian J.A, Level set methods and fast marching method, Cambridge University Press, 378 pages, 1993.

[17] Shahgholian H., On quadrature domains and the Schwarz potential, J. Math. Anal. and Appl. 171 (1992), 61-78.

[18] Shahgholian H., Convexity and uniqueness in an inverse problem of potential theory, Proc. Amer. Math. Soc. 116 (1992), 1097-1100.

[19] Shahgholian H., Uraltseva N., Weiss G., The two-phase membrane problem-Regularity of the free boundaries in higher dimensions. Int. Math. Res. Not. IMRN 2007, no. 8.

[20] Sokolowski J., Zolesio J., Introduction to shape optimization: shape sensitivity analysis, Springer, 1992.

[21] Tiihonen T., Shape optimization and trial methods for free boundary problems, Mathematical Modelling and Numerical Analysis, 31:805-825, 1997.

[22] Tiihonen T., Finite element approximation of non local heat radiation problems, Math. Mod. Meth. Appl. Sci, 1998.

[23] Zolesio J., Identification de domaines par deformations, Thèse dètat, Univ. Nic, 1979.

Department of Mathematics, Stockholm University, S-10691, Stockholm, Sweden
E-mail address: mahmoudreza@math.su.se

Department of Mathematics, Instituto Superior Técnico, Lisbon.
E-mail address: bozorg@math.ist.utl.pt