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Abstract

In this thesis, we present some basic research on the coamoeba $\mathcal{A}'_V$ of a complex algebraic variety V and its relation to the corresponding amoeba $\mathcal{A}_V$. The amoeba has proven to be useful in many areas of mathematics, and it is to be expected that its dual companion, the coamoeba, should acquire a similar importance. So far not much has been written about the coamoeba and its position in mathematics is to a large extent yet to be discovered. However, there are already known applications, both within mathematics and also in theoretical physics.

Among the specific results obtained one can mention the following. We provide some general new results about the boundary and closure of the coamoeba, and we also use topological methods to find a minimal extension of $\mathcal{A}'_V$ when V is a hypersurface in $\mathbb{C}^2$. In particular, we study the linear case. Even in this basic setting, where the methods from linear algebra can be efficiently used, there has been very little previous work. Our findings are therefore of a rather fundamental nature. The coamoebas of a line, a hyperplane and a linear space of codimension p in $\mathbb{C}^{2p}$, are particularly closely examined.

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Contents

1 Introduction

The notion of amoebas was introduced 1994 by Gelfand, Kapranov and Zelevinsky and originally used for the study of hypergeometric functions. Since then, amoebas have proven to be of interest in many areas and also in its own right as a bridge between complex analysis and tropical geometry.

In 2005, Passare and Tsikh introduced the concept of coamoebas. It is defined as the set of argument vectors corresponding to points in a variety. Physicians often use the term *algae* instead. Some people who have studied coamoebas are Lisa Nilsson and Mikael Passare at Stockholm University ([12],[11]), Mounir Nisse at Texas University ([13]) and Grigory Mikhalkin at the University of Toronto and Andrei Okounkov at Princeton University ([9]).

The purpose of this thesis is to give a description of the coamoeba of a complex algebraic variety and its relation to the variety. Since the coamoeba has not been an object of study for more than five years, we are still in the stage of very basic research like the study of the extension and boundary. Except for these things, the coamoeba is characterized by its *contour*, that is, the critical values of the argument mapping. Finally we are generally interested of the fiber in the variety of a point in the coamoeba.

In the first chapter, we give the basic definitions and show some basic results. Also, two of the situations where the amoeba and coamoeba are well understood are presented. The first is when the variety is a hyperplane and the second when it is a line.

In the following two chapters, we are concentrating on general questions. The main result concerning the boundary of the coamoeba is Theorem 3.3, a result that has been shown independently by Nisse and Sottile. This result points at the importance of looking at initial forms of the members of the polynomial ideal corresponding to the variety. Just as in the case of amoebas, the initial forms are crucial for the understanding of the coamoeba globally.

Theorems 4.7 and 4.11 concern the extension of the coamoeba on the torus and the fibers of points. Also here the initial forms plays a central role. The main tool used is the *theta variety*. The second half of the chapter is solely about hypersurfaces in $\mathbb{C}^2$.

We return to the affine linear setting in the last chapter. Even the linear situation is not yet completely understood and at the end, we focus on the special case of linear spaces with half the dimension of the space containing them. The baby example of a coamoeba is that of a line V in $\mathbb{C}^2$. Then V is also a hyperplane, and finally a half-dimensional space. Thus it carries three different kind of properties and in fact it becomes clear that we can retrieve different kinds of "traces" of the two-dimensional line as general statements in each of the three special cases.

Almost everything in the thesis is ongoing work. Several results do not yet have its, what we believe, potential application, like Theorem 4.7 and Proposition 5.11. Many concepts we use, like e.g. *theta variety/cone, degenerate*, are not used in any preceding works directly related to amoebas or coamoebas that we know.

2 Definitions and basic results

Throughout this section, V is an algebraic variety in $\mathbb{C}_*^n$ of pure codimension p and f is a Laurent polynomial on $\mathbb{C}^n$. We write

$$f(z) = \sum_{\alpha \in A} a_\alpha z^\alpha$$

where A is the set in $\mathbb{Z}^n \subset \mathbb{R}^n$ for which $a_\alpha \neq 0$, $\alpha \in A$. Here we use multi-index notation:

$$z^\alpha := z_1^{\alpha_1} ... z_n^{\alpha_n}$$

Definition 1. The *Newton polytope* Δ_f of f is the convex hull in $\mathbb{R}^n$ of A.

Let Γ be a face of Δ_f. Then we write $f|_\Gamma$ for the truncation of f to Γ:

$$f|_\Gamma = \sum_{\alpha \in \Gamma} a_\alpha z^\alpha$$

Since the polynomials representing a variety V of codimension 1 have the same index set A up to translation, the Newton polytopes of these polynomials also coincide up to translation and hence it is an invariant of V, which happen to contain a lot of information about the structure of V.

Definition 2. The *amoeba* $\mathcal{A}_V$ of V is the set $\operatorname{Log} V \subseteq \mathbb{R}^n$ where

$$\operatorname{Log} z = (\log |z_1|, ..., \log |z_n|)$$

The *coamoeba* $\mathcal{A}'_V$ of V is the set $\operatorname{Arg} V \subseteq \mathbb{T}^n$ or $\subseteq \mathbb{R}^n$ where

$$\operatorname{Arg} z = (\arg z_1, ..., \arg z_n)$$

The term amoeba alludes to the shape of the set $\operatorname{Log} V$ with its holes and "tentacles". The first picture to have in mind is that of a two-dimensional amoeba $\mathcal{A}$ in $\mathbb{R}^2$. From long distance, $\mathcal{A}$ looks like a fan with a ray at every direction normal to the facet Γ of Δ_f, equalling the amoeba of $f|_\Gamma$. The infinite components of $\mathcal{A}^c$ correspond to regions where monomials at the boundary of Δ_f dominates. In particular, each vertex α of Δ_f corresponds to a cone in $\mathcal{A}^c$ called the *recession cone* , that is bounded by lines normal to the two edges adjacent to α (see [4]). Bounded components of $\mathcal{A}^c$ correspond to regions where a monomial with order in $\operatorname{int} \Delta_f$ dominates. A more careful discussion of this can be found in [14].

We are frequently going to talk about $\operatorname{Exp} \mathcal{A}$, the amoeba lifted to $\mathbb{R}_+^n$ by the coordinatewise mapping $x_j \mapsto e^{x_j}$, rather than $\mathcal{A}$ itself, since this usually can be described more briefly. Note that these sets are homeomorphic. The corresponding lifting $\operatorname{Tan} \mathcal{A}'$ of $\mathcal{A}'$ is that of $\theta_j \mapsto \tan \theta_j$ in each coordinate. However, this mapping is homeomorphic only on a fundamental domain in $\mathbb{R}^n$ of Tan.

Let $P : \mathbb{R}^{2(n-p)} \to V$ be a locally smooth parametrization of V at a regular point z and let $\varphi : V \to \mathbb{R}^n$ be a smooth mapping. If the differential of $\varphi \circ P$ has full rank, i. e. rank $\min(2n - 2p, n)$, at $P^{-1}(z)$, then $\varphi \circ P$ is locally a submersion from $\mathbb{R}^{2n-2p}$ to a smooth manifold of maximal dimension in $\varphi(V)$ and with this motivation, we say that z is a *non-critical point* of φ. If z is a singular point of V or for any parametrization P, the differential of $\varphi \circ P$ does not have full rank at $P^{-1}(z)$, we say that z is a *critical point* of φ.

Theorem 2.1. *The critical points of* Log *and* Arg *on* V *coincide.*

Proof. Let z be a regular point on V. In a small neighborhood U of z, we can choose a branch of the coordinatewise complex logarithm $\log = \operatorname{Log} + i \operatorname{Arg}$ so that it is holomorphic there. Hence $W := U \cap \log V$ is a holomorphic surface and since Re and Im are linear mappings we have for any $w \in W$ that

$$\operatorname{Re} T_W(w) = T_{\operatorname{Re} W}(\operatorname{Re} w), \quad \operatorname{Im} T_W(w) = T_{\operatorname{Im} W}(\operatorname{Im} w)$$

where $T_X(q)$ denotes the tangent space of X at q. Since $\log = \operatorname{Log} + i \operatorname{Arg}$, we are done if we can show that $\operatorname{Re} T_W(z)$ has the same dimension as $\operatorname{Im} T_W(z)$.

Since z is regular on V and log is diffeomorphic on U, $\log z$ is regular on W. Hence $T_W(w)$ has maximal dimension and is defined by p linearly independent forms $\langle c_k, \zeta \rangle = 0$, $c_k = (c_{k1}, ..., c_{kn}) \in \mathbb{C}^n$. Let A, B be the $p \times n$-matrices with $A_{kj} = \operatorname{Re} c_{kj}$, $B_{kj} = \operatorname{Im} c_{kj}$ and let $u, v \in \mathbb{R}^n$. Then $u + iv \in T_W(z)$ if and only if u and v satisfies

$$\operatorname{Re}(\langle c_k, u + iv \rangle) = \operatorname{Im}(\langle c_k, u + iv \rangle) = 0,$$

that is, the real equation systems $Au = Bv$, $Bu = -Av$. Now, we have

$$\begin{pmatrix} A \\ B \end{pmatrix} u = \begin{pmatrix} B \\ -A \end{pmatrix} v \Leftrightarrow$$

$$\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} u = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \begin{pmatrix} B \\ -A \end{pmatrix} v \Leftrightarrow$$

$$\begin{pmatrix} B \\ -A \end{pmatrix} u = \begin{pmatrix} -A \\ -B \end{pmatrix} v$$

So $u+iv \in T_W(w)$ if and only if $-v+iu \in T_W(w)$. In particular, $\operatorname{Re} T_W(w) = \operatorname{Im} T_W(w)$. The theorem follows.

□

We denote the set of critical points of Log and Arg on V by K_V. Let $P\mathbb{R}^{n-1}$ be the real projective space of projective dimension $n-1$. When V is a hypersurface, K_V can be described by the following mapping:

Definition 3. The logarithmic Gauss mapping $\gamma : \operatorname{reg} V \to P^{n-1}$ is given by

$$\gamma(z) = (z_1 \frac{\partial f}{\partial z_1} : ... : z_n \frac{\partial f}{\partial z_n})$$

The vector $\gamma(z)$ is the normal of the tangent space of the manifold $\log V$ at the point $\log z$, where we choose a locally holomorphic branch of log, hence the name of the term. The following theorem was proved by Mikhalkin in [8].

Theorem 2.2. *When* V *is a hypersurface,* $K_V = \gamma^{-1}(\mathbb{R}P^{n-1})$.

Proof. Since V is a hypersurface, the matrices A, B in the proof of Theorem 2.1 are row vectors and since γ is normal to $T_W(w)$, we can choose $A = \operatorname{Re} \gamma$, $B = \operatorname{Im} \gamma$. That is, $u + iv$ is in the tangent space of $\log V$ at $\log z$ precisely when it satisfies the following real equation system:

$$\langle \operatorname{Re} \gamma, u \rangle = \langle \operatorname{Im} \gamma, v \rangle, \quad \langle \operatorname{Re} \gamma, v \rangle = -\langle \operatorname{Im} \gamma, u \rangle \tag{2.1}$$

If $\gamma \notin \mathbb{R}P^{n-1}$), then $\operatorname{Re}\gamma$ and $\operatorname{Im}\gamma$ are linearly independent and for any fixed v there is a solution to (2.1) for u. Hence $z \notin K_V$. Otherwise, $\operatorname{Re}\gamma = \lambda \operatorname{Im}\gamma$ for some $\lambda \in \mathbb{R}$ and we see that a solution (u, v) to (2.1) must satisfy $\langle \operatorname{Re}\gamma, u\rangle = \langle \operatorname{Im}\gamma, u\rangle = 0$. Hence the rank of $\operatorname{Re} T_W$ is not maximal and $z \in K_V$. $\square$

Definition 4. The *contour* $\mathcal{C}^{(\prime)}$ of $\mathcal{A}^{(\prime)}$ is the image of K_V in $\mathcal{A}^{(\prime)}$.

Proposition 2.3. *For an irreducible variety $V \subseteq \mathbb{C}^n$ with complex codimension p, either $V = K_V$ or $\dim(K_V) < \dim(V)$ and*

$$\dim(\mathcal{A}\backslash\mathcal{C}) = \dim(\mathcal{A}'\backslash\mathcal{C}') = \min(2n - 2p, n)$$

In particular, $\mathcal{A}^{(\prime)} = \mathcal{C}^{(\prime)}$ if and only if $V = K_V$.

Proof. By definition, $\dim(\operatorname{Arg}(V\backslash K_V)) = \min(\dim(V\backslash K_V), n)$ Furthermore, K_V is a subvariety of V and hence $\dim(K_V) < \dim(V)$ if $V \neq V_K$. The result follows. $\square$

We finish this section with a simple observation. With real coefficients in the defining polynomials, the coamoeba becomes symmetric.

Proposition 2.4. *If V is a variety cut out by some polynomials f_1, f_2,...,f_3 with real coefficients, then $z \in V$ if and only if $\bar{z} \in V$. In particular, $\theta \in \mathcal{A}'_V$ if and only if $-\theta \in \mathcal{A}'_V$.*

Proof. Since the coefficients are real, $m(\bar{z}) = \bar{m}(z)$ for any monomial in any function f_j. Hence, if $f_j(z) = 0$, then $f_j(\bar{z}) = \bar{f}_j(z) = 0$. $\square$

2.1 Hyperplanes

The case when it is easiest to describe $\mathcal{A}$ and $\mathcal{A}'$ is when V is a hyperplane in $\mathbb{C}^n$. Let us assume that $V = f^{-1}(0)$ for a linear function $f(z) = a_0 + a_1 z_1 + ... + a_n z_n$. We now state a result proved by Forsberg, Passare and Tsikh in [4].

Theorem 2.5. *The set $\operatorname{Exp}\mathcal{A}$ is given by the points $r \in \mathbb{R}^n_+$ that satisfies the following generalized triangle inequalities:*

$$|a_0| \leq \sum_{j=1}^{n} |a_j| r_j$$
$$|a_k| r_k \leq |a_0| + \sum_{j \neq k} |a_j| r_j \quad \forall k = 1, 2, ..., n$$

Clearly $\operatorname{Exp}\mathcal{A}$ is included in the set given by these inequalities since otherwise one of the monomials are dominating over all the others. The other direction will not be discussed here.

The coamoeba of V is even easier to compute. The theory that applies is discussed in chapter 4 and 5, but we will give a flavour already now. First look at the case $n = 2$. We can assume that V is the zero set of $f = 1 + az_1 + bz_2$ where $\operatorname{Arg} a = \alpha$, $\operatorname{Arg} b = \beta$. Consider the lines

$$\theta_1 = \pi + \alpha, \quad \theta_2 = \pi + \beta, \quad \theta_2 = \pi + \beta - \alpha + \theta_1$$

on the torus. It is easy to check that they correspond to the coamoebas of $1 + az_1 = 0$, $1 + bz_2 = 0$ and $az_1 + bz_1 = 0$ respectively. Any line is orthogonal

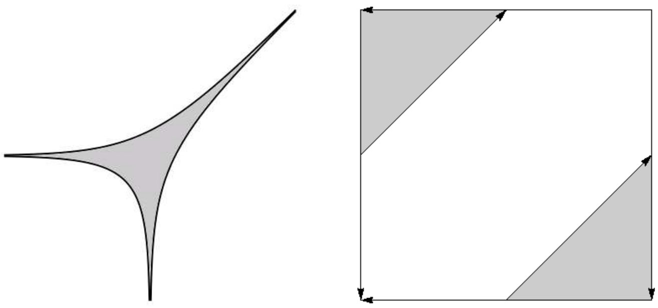


Figure 1: The amoeba and coamoeba for $f = 1 + z_1 + z_2$. The interior of $\mathcal{A}'$ is given by the oriented cells.

to one of the three facets of the unit simplex Δ_f and we orient them outwards from this polygon. Then $\mathcal{A}'$ is given by the interior of the oriented cells that appears on $\mathbb{T}$, plus the three intersection points of the three lines. For arbitrary dimensions, the following theorem now gives us the coamoeba.

Theorem 2.6. *Let S be the set of one dimensional faces of Δ. Then*

$$\mathcal{A}' = \bigcup_{\Gamma \in S} \mathcal{A}'_\Gamma$$

This is a special case of Corollary 5.3 and we postpone the proof.

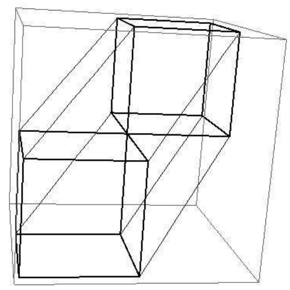


Figure 2: The complement of the coamoeba of $f = 1 + z_1 + z_2 + z_3$ on the fundamental domain $]-\pi, \pi]^3$ is the convex hull of two cubes. Origo is at the center.

2.2 Lines

The other case when $\mathcal{A}$ and $\mathcal{A}'$ are well understood is when V is a line. Then we can, and will, describe the amoeba, the coamoeba and their contours by very explicit calculations. Throughout this chapter, we let $V \subset \mathbb{C}^n$ be given by the parametrization

$$t \mapsto (t, d_2 + e_2 t, ..., d_n + e_n t), \quad t = x + iy, \; (x, y) \in \mathbb{R}^2$$

Then V is cut out e.g. by the hyperplanes defined by

$$f_k(z) = e_k z_1 + d_k - z_k, \;\; k = 2, 3, ..., n \tag{2.2}$$

Definition 5. The line V is said to be *real* if

$$(\frac{d_2}{e_2} : \frac{d_3}{e_3} : ... : \frac{d_n}{e_n}) \in P\mathbb{R}^{n-2}$$

Note that the following assertion is equivalent:

$$\frac{d_j e_k}{d_k e_j} \in \mathbb{R} \;\; \forall\, j, k = 2, 3, ..., n$$

We get the useful equation

$$\begin{aligned} 0 = \mathrm{Im}\,(d_k e_j \bar{d}_j \bar{e}_k) = \mathrm{Re}\,(e_j \bar{d}_j)\mathrm{Im}\,(d_k \bar{e}_k) + \mathrm{Re}\,(d_k \bar{e}_k)\mathrm{Im}\,(e_j \bar{d}_j) = \\ = \mathrm{Re}\,(d_j \bar{e}_j)\mathrm{Im}\,(d_k \bar{e}_k) - \mathrm{Re}\,(d_k \bar{e}_k)\mathrm{Im}\,(d_j \bar{e}_j) \end{aligned} \tag{2.3}$$

for every $j, k = 2, 3, ..., n$. Note that by this notion, the class of real lines is strictly bigger than the class of lines that can be parametrized by a linear mapping with real coefficients. A motivation of this broader definition is the following theorem, for the parts concerning amoebas first proved by Kuzvesov in [7].

Theorem 2.7. *Let $n \geq 3$. If a line $V \subset \mathbb{C}^n$ is real, then* $\mathrm{Tan}\,\mathcal{C}'$ *consists of the single point*

$$(\frac{\mathrm{Im}\,(d_j \bar{e}_j)}{\mathrm{Re}\,(d_j \bar{e}_j)}, \frac{\mathrm{Im}\, d_2}{\mathrm{Re}\, d_2}, ..., \frac{\mathrm{Im}\, d_n}{\mathrm{Re}\, d_n}) \tag{2.4}$$

where j can be any number between 2 and n. Furthermore, $\mathrm{Log}^{-1}(x) \cap V$ *consists of one point if $x \in \mathcal{C}$ and two points otherwise.*

If V is not real, then $\mathcal{C} = \mathcal{C}' = \emptyset$ and both $\mathcal{A}$ and $\mathcal{A}'$ are homeomorphic to the Riemann sphere minus k points where $3 \leq k \leq n+1$ and generically $k = n+1$.

Proof. To decide wether a point $z \in V$ is in K_V, it suffices to see that the rank of the $n \times 2$-matrix $A = \mathrm{Jac}(\mathrm{Tan} \circ \mathrm{Arg})_V$ equals one at z. Let

$$\tau_j = \tan\arg z_j = \mathrm{Im}\, z_j / \mathrm{Re}\, z_j$$

By our parametrization, the first row of A is

$$(\frac{\partial \tau_1}{\partial x}, \frac{\partial \tau_1}{\partial y}) = (-\frac{y}{x^2}, \frac{1}{x})$$

while the j:th row, $2 \leq j \leq n$, equals

$$(\frac{\partial \tau_j}{\partial x}, \frac{\partial \tau_j}{\partial y}) = (\frac{\mathrm{Im}\,(\bar{d}_j e_j) - y|e_j|^2}{(\mathrm{Re}\, d_j + x\mathrm{Re}\, e_j - y\mathrm{Im}\, e_j)^2}, \frac{\mathrm{Re}\,(\bar{d}_j e_j) + x|e_j|^2}{(\mathrm{Re}\, d_j + x\mathrm{Re}\, e_j - y\mathrm{Im}\, e_j)^2})$$

The two columns of A are linearly dependent when every minor of A vanishes, that is $y\mathrm{Re}\,(d_j e_j) - x\mathrm{Im}\,(\bar{d}_j e_j) = 0$. This means that

$$\tau_1 = y/x = -\frac{\mathrm{Im}\,(\bar{d}_j e_j)}{\mathrm{Re}\,(\bar{d}_j e_j)} = \frac{\mathrm{Im}\,(d_j \bar{e}_j)}{\mathrm{Re}\,(d_j \bar{e}_j)}$$

and by (2.3), this equality is possible for every j if and only if V is real. From this we compute τ_k by repeated use of (2.3):

$$\tau_k = \frac{\mathrm{Im}\, d_k + y\mathrm{Re}\, e_k + x\mathrm{Im}\, e_k}{\mathrm{Re}\, d_k + x\mathrm{Re}\, e_k - y\mathrm{Im}\, e_k} =$$

$$\frac{\mathrm{Im}\, d_k\mathrm{Re}\,(d_j\bar{e}_j) + x\mathrm{Re}\, e_k\mathrm{Im}\,(d_j\bar{e}_j) + x\mathrm{Im}\, e_k\mathrm{Re}\,(d_j\bar{e}_j)}{\mathrm{Re}\, d_k\mathrm{Re}\,(d_j\bar{e}_j) + x\mathrm{Re}\, e_k\mathrm{Re}\,(d_j\bar{e}_j) - x\mathrm{Im}\, e_k\mathrm{Im}\,(d_j\bar{e}_j)} \cdot 1 \cdot 1 =$$

$$\frac{\mathrm{Im}\, d_k\mathrm{Re}\,(d_j\bar{e}_j) + x\mathrm{Im}\,(e_kd_j\bar{e}_j)}{\mathrm{Re}\, d_k\mathrm{Re}\,(d_j\bar{e}_j) + x\mathrm{Re}\,(e_kd_j\bar{e}_j)} \cdot \frac{\mathrm{Re}\, d_k\mathrm{Re}\,(d_j\bar{e}_j) - x\mathrm{Re}\,(e_kd_j\bar{e}_j)}{\mathrm{Re}\, d_k\mathrm{Re}\,(d_j\bar{e}_j) - x\mathrm{Re}\,(e_kd_j\bar{e}_j)} \cdot 1 =$$

$$\frac{\mathrm{Im}\, d_k\mathrm{Re}\, d_k\mathrm{Re}^2(d_j\bar{e}_j) - x^2\mathrm{Re}\,(e_kd_j\bar{e}_j)\mathrm{Im}\,(e_kd_j\bar{e}_j)}{\mathrm{Re}^2 d_k\mathrm{Re}^2(d_j\bar{e}_j) - x^2\mathrm{Re}^2(e_kd_j\bar{e}_j)} \cdot \frac{\mathrm{Re}\, d_k}{\mathrm{Re}\, d_k} =$$

$$\frac{\mathrm{Im}\, d_k\mathrm{Re}\, d_k\mathrm{Re}^2(d_j\bar{e}_j)\mathrm{Re}\, d_k - x^2\mathrm{Re}^2(e_kd_j\bar{e}_j)\mathrm{Im}\, d_k}{(\mathrm{Re}^2 d_k\mathrm{Re}^2(d_j\bar{e}_j) - x^2\mathrm{Re}^2(e_kd_j\bar{e}_j))\mathrm{Re}\, d_k} = \frac{\mathrm{Im}\, d_k}{\mathrm{Re}\, d_k}$$

Assume that t is such that $z(t) \notin K_V$. Let $\theta = \mathrm{Arg}\, t$. For a fixed j, $|d_j + a| = |d_j + b|$ for $a \neq b$ with $|a| = |b|$ if and only if b is the reflection of a in the line through d_j and origo, that is, $\arg b = \arg d_j - \arg a$. Hence, letting $\theta' = \arg d_j - (\arg e_j + \theta)$ and $t' = |t|e^{i\theta}$, we have that $|z_j(t)| = |z_j(t')|$. Since V is real, $\arg d_j - \arg e_j$ coincide for every $j = 2, 3, ..., n$ and hence $\mathrm{Log}\, z(t) = \mathrm{Log}\, z(t')$. There are no other points in V with the same value of Log, since $|z_1(t)| = |z_1(t')|$ whenever $|t| \neq |t'|$. Note that $z(t) = z(t')$ if and only if $\arg d_j = \pm(\arg e_j + \theta)$ for $j = 2, 3, ..., n$. But by (2.4) this is exactly when $\mathrm{Arg}\, z(t) \in \mathcal{C}'$, and hence $\mathrm{Log}\, z(t) \in \mathcal{C}$.

If on the other hand V is not real, then we have seen that $\mathcal{A}$ and $\mathcal{A}'$ have no contour, meaning that Log and Arg are local diffeomorphisms. We also check that Log and Arg are injections from V to $\mathcal{A}$ and $\mathcal{A}'$, respectively, and it follows that they are diffeomorphism. Since V is parametrized by $\mathbb{P}\backslash\{0, p_2, ..., p_n, \infty\}$ where $p_j = -d_j/e_j$, and since $-d_j/e_j = 0$ or ∞ for every j implies that V is real, the theorem follows. $\square$

Figure 3: The amoeba of a real line (left) and a line that is not real (right).

We are now going to study $\mathcal{A}$ closer in the real case or rather the lifting $\mathrm{Exp}\, \mathcal{A} \subset \mathbb{R}^n_+$ of $\mathcal{A}$. Hence, we consider $r_j := |z_j|$ rather than $\log |z_j|$.

Setting $\mathrm{Re}\, s = x$ and $\mathrm{Im}\, s = y$ we have that

$$\begin{aligned} r_j^2 &= (\mathrm{Re}\, d_j + x\mathrm{Re}\, e_j - y\mathrm{Im}\, e_j)^2 + (\mathrm{Im}\, d_j + x\mathrm{Im}\, e_j + y\mathrm{Re}\, e_j)^2 = \\ &= |d_j|^2 + 2\mathrm{Re}\,(d_j\bar{e}_j)x + 2\mathrm{Im}\,(d_j\bar{e}_j)y + |e_j|^2 r_1^2 \end{aligned} \tag{2.5}$$

Using the proportionality of (2.3) we hence have that for every $j, k = 3, 4, ..., n$ there is a $\lambda_{jk} \in \mathbb{R}$ such that

$$r_j^2 + \lambda_{jk} r_k^2 - |d_j|^2 - \lambda_{jk}|d_k|^2 - (|e_j|^2 + \lambda_{jk}|e_k|^2) r_1^2 = 0 \tag{2.6}$$

Of these equations we clearly can choose $n-2$ that are algebraically independent and we see that $\mathrm{Exp}\,\mathcal{A}$ must lie on a quadratic surface Z of real dimension 2.

However, the whole Z does not correspond to $\mathcal{A}$. Each polynomial f_{j-1} in (2.2) provides the inequalities

$$(|d_j| - |e_j|r_1)^2 \leq r_j^2 \leq (|d_j| + |e_j|r_1)^2 \tag{2.7}$$

By Theorem 2.5, $z \in \partial\mathcal{A}_{f_j}$ whenever the absolute value of one monomial equals the sum of absolute values of the two other monomials. This is precisely when $\arg z_1 \equiv \arg(d_j/e_j)$ and $\arg z_j \equiv \arg d_j$ modulo π and $f(z) = 0$. This in turn is the case when $\theta \in \mathcal{C}'_V = \partial\mathcal{A}'$. We conclude that

$$\partial\mathcal{A}_V \subset \partial\mathcal{A}_{f_j}$$

Hence, $\mathrm{Exp}\,\mathcal{A}_V = Z \cap \mathrm{Exp}\,\mathcal{A}_{f_j}$. Note that $\{f_2, f_3, ..., f_n\}$ is a basis for the linear polynomials that vanish on V. Hence for any such polynomial, the inequalities in Theorem 2.5 are enough to determine $\mathrm{Exp}\,\mathcal{A}$ given Z.

The following theorem sums up the previous discussion.

Theorem 2.8. *Let V be a real line. Given the algebraic 2-surface Z containing* $\mathrm{Exp}\,\mathcal{A}$ *and any linear polynomial f that vanishes on V,*

$$\mathrm{Exp}\,\mathcal{A} = Z \cap \mathrm{Exp}\,\mathcal{A}_f$$

Next, we study the coamoeba of a line. Analogously to the case with amoebas, we will consider $\mathrm{Tan}\,\mathcal{A}'$ and find the surface and inequalities defining it. To start with we consider any line, not just the real ones. If we set $\tau_j := \tan\theta_j$, then we have $\tau_1 = y/x$ and so, for $k \geq 2$:

$$\tau_j = \frac{\mathrm{Im}\,(d_j + e_j t)}{\mathrm{Re}\,(d_j + e_j t)} = \frac{\mathrm{Im}\, d_j + \tau_1 x\mathrm{Re}\, e_j + x\mathrm{Im}\, e_j}{\mathrm{Re}\, d_j + x\mathrm{Re}\, e_k - \tau_1 x\mathrm{Im}\, e_j}, \quad \forall j \tag{2.8}$$

By this, we get for a fixed k,

$$x = \frac{\mathrm{Im}\,(d_k) - \tau_k\mathrm{Re}\, d_k}{\tau_1\tau_k\mathrm{Im}\, e_k + \tau_k\mathrm{Re}\, e_k - \tau_1\mathrm{Re}\, e_k + \mathrm{Im}\, e_k} \tag{2.9}$$

By exchanging x in (2.8) by the expression given in (2.9), we get $n-2$ algebraically independent cubic equations in $\tau_1, ..., \tau_n$ cutting out a real 2-dimensional surface in $(\mathbb{R} \cup \{\infty\})^n$, quite analogous to the case of the amoeba. We note that these equations become homogeneous of degree two precisely when all coefficients d_j, e_j are real (but hence not necessarily when V is real).

The points on $\mathbb{P}$ which we exclude when we parametrize V are $0, \infty$ and for every j, $-d_j/e_j$. If V is real, then by using (2.3), we get

$$\frac{\mathrm{Im}\,(d_k + e_k(-d_j/e_j))}{\mathrm{Re}\,(d_k + e_k(-d_j/e_j))} = \frac{\mathrm{Im}\,d_k}{\mathrm{Re}\,d_k}, \quad j \neq k \tag{2.10}$$

Since $\mathrm{Tan}\,\mathcal{C}'$ is just one point, the boundary of $\mathrm{Tan}\,\mathcal{A}'$ must therefore be given by the union of the following lines:

$$(\lambda, \frac{\mathrm{Im}\,d_2}{\mathrm{Re}\,d_2}, ..., \frac{\mathrm{Im}\,d_n}{\mathrm{Re}\,d_n}) \tag{2.11}$$

$$(\frac{\mathrm{Im}\,d_j\bar{e}_j}{\mathrm{Re}\,d_2\bar{e}_j}, \frac{\mathrm{Im}\,d_2}{\mathrm{Re}\,d_2}, ..., \frac{\mathrm{Im}\,d_{j-1}}{\mathrm{Re}\,d_{j-1}}, \lambda, \frac{\mathrm{Im}\,d_{j+1}}{\mathrm{Re}\,d_{j+1}}, ..., \frac{\mathrm{Im}\,d_n}{\mathrm{Re}\,d_n}), \quad j = 2, 3, ..., n \tag{2.12}$$

$$(\lambda, \frac{\mathrm{Im}\,e_2 + \lambda \mathrm{Re}\,e_2}{\mathrm{Re}\,e_2 - \lambda \mathrm{Im}\,e_2}, ..., \frac{\mathrm{Im}\,e_n + \lambda \mathrm{Re}\,e_n}{\mathrm{Re}\,e_n - \lambda \mathrm{Im}\,e_n}) \tag{2.13}$$

for $\lambda \in \mathbb{R} \cup \{\infty\}$. On the other hand, by using (2.10) again, we see that $\partial\,\mathrm{Tan}\,\mathcal{A}'_{f_j}$ is given by the hyperplanes

$$(\tau_1, \tau_2, ..., \tau_{j-1}, \frac{\mathrm{Im}\,d_j}{\mathrm{Re}\,d_j}, \tau_{j+1}, ..., \tau_n)$$
$$(\frac{\mathrm{Im}\,d_j\bar{e}_j}{\mathrm{Re}\,d_j\bar{e}_j}, \tau_2, ..., \tau_n)$$
$$(\lambda, \tau_j, ..., \tau_{j-1}, \frac{\mathrm{Im}\,e_j + \lambda \mathrm{Re}\,e_j}{\mathrm{Re}\,e_j - \lambda \mathrm{Im}\,e_j}, \tau_{j+1}, ..., \tau_n)$$

We see that $\partial\,\mathrm{Tan}\,\mathcal{A}'_V \subset \partial\,\mathrm{Tan}\,\mathcal{A}'_{f_j}$. Hence we get an analogue for coamoebas of Theorem 2.8.

Theorem 2.9. *Let V be a real line. Given the algebraic 2-surface Z containing $\mathrm{Tan}\,\mathcal{A}'$ and any linear polynomial f that vanishes on V,*

$$\mathrm{Tan}\,\mathcal{A}' = Z \cap \mathrm{Tan}\,\mathcal{A}'_f = Z \backslash (\partial\,\mathrm{Tan}\,\mathcal{A}'_f \backslash \mathrm{Tan}\,\mathcal{C}'_f)$$

Proof. The first equality follows from the discussion preceding the theorem. For the second equality, first assume that Δ_f is of dimension n. Write

$$V_f = \{z \in \mathbb{C}^n_*; z_n = \langle a, (1, z_1, ..., z_{n-1}) \rangle\}, \quad a \in \mathbb{C}^n$$

and set $z_j = x_j + iy_j$. Then the equation system $y_j/x_j = \tau_j$, $1 \leq j \leq n$ in $x_1, ..., x_{n-1}, y_1, ..., y_{n-1}$ is solvable in $(\mathbb{R}^2_*)^n$ for almost every vector $(\tau_1, ..., \tau_n) \in (\mathbb{R} \cup \{\infty\})^n$. Thus, $(\mathrm{Tan}\,\mathcal{A}'_f)^c$ must be included in $\partial\,\mathrm{Tan}\,\mathcal{A}'_f$. By change of variables, we get the same thing for any dimension of Δ_f. Hence we have shown the direction

$$Z \cap \mathrm{Tan}\,\mathcal{A}'_f \subseteq Z \backslash (\partial\,\mathrm{Tan}\,\mathcal{A}'_f \backslash \mathrm{Tan}\,\mathcal{C}'_f)$$

By straightforward calculations we observe that $\mathcal{C}'$ equals the pairwise intersection of the boundary lines of $\mathrm{Tan}\,\mathcal{A}'$ given by (2.11). We will return to this in Theorem 5.7, which together with Theorem 3.3 gives the converse inclusion in Theorem 2.9.

□

2.3 Hypersurfaces

In this chapter, we let $V = f^{-1}(0)$ for a polynomial f on $\mathbb{C}^n$.

While the amoeba is closed, the coamoeba is generally not, since sequences $\{z_j\} \subset V$ with $|z_j| \to \infty$ might correspond to convergent sequences on $\mathbb{T}^n$. However, for hypersurfaces we have the following:

Lemma 2.10. *Let f be a polynomial on $\mathbb{C}^n$. Then $\overline{\mathcal{A}'_f} \subseteq \bigcup_{\Gamma \in S} \mathcal{A}'_{f_\Gamma}$, where S is the set of faces of Δ_f of all dimensions.*

Proof. Assume that $\theta \in \partial \mathcal{A}'_f \backslash \mathcal{A}'_f$. Then we can choose a sequence $\{z_j\}$ in V such that $\arg z_j \to \theta$. Since the support of f is a finite set $A = \{\alpha_1, ..., \alpha_p\}$, we can choose the sequence such that, for the right choice of indexing of A,

$$|z_j^{\alpha_1}| \geq |z_j^{\alpha_2}| \geq ... \geq |z_j^{\alpha_p}|, \ \forall j$$

and, since $[0,1]$ is compact, we can also assume that

$$\lim_{j \to \infty} |z_j^{\alpha_k}| / |z_j^{\alpha_1}| \to d_k$$

for some $d_k \in [0,1]$. Let m be the number such that $d_k > 0$ if and only if $k \leq m$. Since $z_j \in V$, $m \geq 2$. Furthermore, since $\lim z_j \notin V$ and V is closed, the sequence $\{|z_j|\}$ converges to the boundary of $\mathbb{C}^n_*$. That is, $m < p$. In fact, we are now going to show that $\{\alpha_1, ..., \alpha_m\} = \Gamma \cap A$ for some strict subface Γ of Δ_f.

Let $x_j = \operatorname{Log} z_j$ and

$$K_j = \{\xi \in \mathbb{R}^n \,|\, \log d_m - 1 \leq \langle \xi - \alpha_1, x_j \rangle \leq 0\}$$

Then $\alpha_1, ..., \alpha_m$ is contained in K_j for j big enough. Since $|x_j| \to \infty$, K_j is flattening out to a hyperplane as $j \to \infty$. We conclude that

$$\{\alpha_1, ..., \alpha_m\} = A \cap \bigcap_j K_j = A \cap \Gamma$$

for a strict subface Γ of Δ_f.

Choose a hyperplane containing Γ and let μ be its unit normal. Then $\langle \alpha_k, \mu \rangle$ equals some $c \in \mathbb{R}$ for all k and hence

$$\langle \alpha_k, \frac{\langle \alpha_1, \operatorname{Log} z_j \rangle}{c} \mu \rangle = \langle \alpha_1, \operatorname{Log} z_j \rangle$$

for all $k = 1, 2, ..., m$ and j. Denoting the second argument on the left hand side by y_j, we hence have $\langle \alpha_k, \operatorname{Log} z_j - y_j \rangle \to \log d_k$, so $\lim \operatorname{Log} z_j - y_j \in \mathbb{R}^n$. This means that $z_j e^{-y_j} \to w \in (\mathbb{C}^*)^n$. Since $\arg w = \theta$ and $f_\Gamma(w) = \lim f(z_j) e^{-\langle y_j, \alpha_1 \rangle} = 0$, we have shown that $\theta \in \mathcal{A}'_{f|_\Gamma}$. □

We will see later that in fact equality holds in the lemma. The natural question is to ask if something similar is true for amoebas of varieties in arbitrary codimension. The main theorem in Chapter 3 is an affirmative answer to this question.

3 General varieties

For a general description of the coamoebas for varieties of arbitrary codimension, the following result is essential.

Theorem 3.1. *For an algebraic variety V in $\mathbb{C}^n$,*

$$\mathcal{A}_V = \bigcap_{f \in I(V)} \mathcal{A}_f$$

and

$$\mathcal{A}'_V = \bigcap_{f \in I(V)} \mathcal{A}'_f$$

Proof. The inclusions $\mathcal{A}^{(\prime)}_V \subseteq \bigcap_{f \in I(V)} \mathcal{A}^{(\prime)}_f$ are trivial. For the other direction, let $f_1, ..., f_p$ be polynomials cutting out V,

$$f_j(z) = \sum_\alpha a_{j\alpha} z^\alpha$$

Assume that $\theta \in \mathbb{T}^n \backslash \mathcal{A}'_V$. We need to find some f generated by these polynomials such that $\theta \notin \mathcal{A}'_f$. To this end, set

$$g_j(z) = \sum_\alpha \bar{a}_{j\alpha} (e^{-2i\theta_j} z)^\alpha$$

Since g_j is a polynomial, setting $f = \sum_j f_j g_j$ we have that $f \in I(V)$. Since for any $r \in \mathbb{R}^n_+$,

$$f(re^{i\theta}) = \sum_j f_j(re^{i\theta}) g_j(re^{i\theta}) = \sum_j |f_j(re^{i\theta})|^2 > 0$$

we have shown that $\bigcap_{f \in I(V)} \mathcal{A}'_f \subseteq \mathcal{A}'_V$.

For amoebas, the proof is analogous. Assume that $r \in \mathbb{R}^n_+ \setminus \operatorname{Exp} \mathcal{A}_V$ and set

$$h_j(z) = \sum_\alpha \bar{a}_{j\alpha} (r_j^2 / z)^\alpha$$

Setting $f = \sum_j f_j g_j$ we have just as above that $f \in I(V)$ and $r \notin \operatorname{Exp} \mathcal{A}'_f$. Clearly $\bigcap_{f \in I(V)} \mathcal{A}_f \subseteq \mathcal{A}_V$. □

From the proof it is clear that it is actually enough to intersect the amoebas or coamoebas of the polynomials of degree $\leq 2 \max_{k \in [p]} (\deg f_k)$ to get $\mathcal{A}$ or $\mathcal{A}'$ respectively, given defining polynomials $f_1, ..., f_p$ of V. However, if for example V is linear, it is not true that $\mathcal{A}$ is given as the intersection of amoebas of hyperplanes containing V. Look for example at the line V in $\mathbb{C}^3$ given by.

$$t \mapsto (t, 1 + t, 2 - t)$$

Clearly, there is no $z \in V$ with $|z| = (1, 2, 3)$. Furthermore, if $f(z)$ vanishes on V, then

$$f(z) = (a + 2b) + (a - b) z_1 - a z_2 - b z_3, \quad a, b \in \mathbb{C}$$

We check that every positive number $|a + 2b|$, $|a - b|$, $2|a|$, $3|b|$ is less than or equal to the sum of the others. Hence, by Theorem 2.5, $(1, 2, 3) \in \mathcal{A}_f$.

We are now going to take a closer look at the boundary of general coamoebas. To begin with, we need some definitions to generalise the face coamoebas $\mathcal{A}'_\Gamma$ of hyperplanes to what we will call directed coamoebas for arbitrary varieties.

Definition 6. Let f be a polynomial on $\mathbb{C}^n$ and $\omega \in \mathbb{R}^n$. Then the *initial form* $f|_\omega$ is the sum of terms $a_\alpha z^\alpha$ of f such that $\alpha \cdot \omega$ is maximal.

Clearly $f|_\omega = f|_\Gamma$ for the face Γ of Δ_f of highest dimension whose directed normal space includes ω. Note also that $f|_0 = f$. By the definiton of Minkowski sum, we see immediately that the following property holds.

Proposition 3.2. *If x is a normal directed outwards from the face Γ of Δ_{fg}, then*

$$(fg)|_\Gamma = f|_x g|_x$$

The next definition is central for the chapter.

Definition 7. Let I be a polynomial ideal over $\mathbb{C}^n$ and $\omega \in \mathbb{R}^n$. Then we set the *initial ideal* of I at ω to be $I_\omega := \langle f_\omega ; f \in I \rangle$. If furthermore V is determined by I, we say that the variety V_ω determined by I_ω is the *initial variety* of V at ω. If $\mathcal{A}'$ is the coamoeba of V, we say that the coamoeba of V_ω is the *initial coamoeba* of $\mathcal{A}'$ at ω and denote it by $\mathcal{A}'_\omega$.

We can now state the main theorem of the chapter. An equivalent theorem has been proven independently by Nisse and Sottile.

Theorem 3.3. *For a variety V in $\mathbb{C}^n$,*

$$\overline{\mathcal{A}'_V} = \bigcup_\omega \mathcal{A}'_\omega$$

To prove this, the following generalisation to n-variate functions of Rouché's theorem will be useful.

Theorem 3.4. *Let f and g be holomorphic mappings from some bounded open $U \subset \mathbb{C}^m$ to $\mathbb{C}^m$ such that*

$$|g(z)| < |f(z)|, \ z \in \partial U$$

Then f and $f+g$ have the same number of zeros in U, counted with multiplicity.

The theorem is a consequence of classical results of Poincaré' and Bol and the proof will not be discussed here. A good reference is [1] p. 18-23.

Proposition 3.5. *Let I be an ideal generated by a class J of polynomials of uniformly bounded degree. Then*

$$\bigcap_{f \in I} \bigcup_{\omega \in \mathbb{R}^n} \mathcal{A}'_{f|_\omega} \subseteq \bigcup_{\omega \in \mathbb{R}^n} \bigcap_{f \in J} \mathcal{A}'_{f|_\omega}$$

Proof. Choose $\theta \in T^{n-1}$ and assume that

$$\theta \notin \bigcup_\omega \bigcap_{f \in J} \mathcal{A}'_{f|_\omega} \tag{3.1}$$

It suffices to show that there is a $P \in I$ such that θ is not contained in $\mathcal{A}'_{P|_\omega}$ for any $\omega \in \mathbb{R}^n$, that is, θ is not contained in $\mathcal{A}'_{P|_\Gamma}$ for any $\Gamma \in \Delta_P$. To this end, first note that there are only finitely many possible Newton polytopes for

$f \in J$ and hence a finite number, say m, of faces. In order for (3.1) to be true, there must hence be a finite set of polynomials $J' = \{f_1, ..., f_p\}$, $p \leq m$ such that (3.1) does not hold when replacing J with J'.

To construct the desired polynomial P, we now use a similar technique as in the proof of Theorem 3.1. First, for every f_j, we find a g_j as in that proof such that $f_j g_j(z)$ is nonnegative for every z with $\operatorname{Arg} z = \theta$. Next, we introduce p polynomials k_j such that $\Delta_{k_j} = \Delta_{f_j g_j}$ and with coefficient $r_{j\alpha} e^{-i\alpha\cdot\theta}$ for the monomial with exponent α, $r_{j\alpha} > 0$. Now, setting

$$h_j = k_1 k_2 ... k_{j-1} f_j g_j k_{j+1} ... k_p$$

we have that Δ_{h_j} is identically equal to some Δ for every $j \in \{1, 2, ..., p\}$. Now consider the polynomial $P = \sum_j h_j$. Obviously, $\Delta_P = \Delta$ for the right choices of $r_{j\alpha}$. Choose any face Γ of Δ and let ω be a normal of Γ directed outwards from Δ. Now by Proposition 3.2,

$$P|_\Gamma(z) = \sum h_j|_\Gamma(z) = \sum_j (f_j g_j)|_\omega(z) \prod_i k_i|_\omega(z)$$

Assume that $\operatorname{Arg} z = \theta$. Then the products on the right hand side are all strictly positive numbers. Furthermore, by our assumption at least one product $(f_j g_j)(z)$ is strictly positive, while the others are nonnegative real numbers. Hence P is the polynomial we needed and the proposition is proved. □

Proof of Theorem 3.3. Applying Theorem 3.1 and Lemma 2.10, we get

$$\overline{\mathcal{A}'_V} = \overline{\bigcap_{f \in I(V)} \mathcal{A}'_f} \subseteq \bigcap_{f \in I(V)} \overline{\mathcal{A}'_f} = \bigcap_{f \in I(V)} \bigcup_{\omega} \mathcal{A}'_{f|_\omega} \tag{3.2}$$

Let J be a set of polynomials of high enough uniform bound of the degree as in Theorem 3.1 and let S be the finite set of faces of the Newton polytopes of these polynomials. By Proposition 3.5, we have that

$$\bigcap_{f \in I(V)} \bigcup_{\omega \in \mathbb{R}^n} \mathcal{A}'_{f|_\omega} \subseteq \bigcup_{\omega \in \mathbb{R}^n} \bigcap_{f \in J} \mathcal{A}'_{f|_\omega} = \bigcup_{\omega \in \mathbb{R}^n} \bigcap_{f \in I(V)} \mathcal{A}'_{f|_\omega}$$

(the last equality is trivial). Applying Theorem 3.1 one more time, we have shown the inclusion $\overline{\mathcal{A}'_V} \subseteq \bigcup_\omega \mathcal{A}'_{V\omega}$

For the other inclusion, set $V = V(f)$, $f = (f_1, ..., f_p)$, $1 \leq p < n$, fix $\omega \in \mathbb{R}^n$ such that $\mathcal{A}'_\omega$ is nonempty and fix $\theta \in \mathcal{A}'_\omega$. Let $\tilde{z} \in V_\omega$ be such that $\operatorname{Arg} \tilde{z} = \theta$. Choose $\varepsilon > 0$ so that $|\tilde{z} - z| < \varepsilon$ implies that $|\theta - \arg z|$ is as small as desired.

For every t, $\tilde{z}_t = (\tilde{z_1} e^{t\omega_1}, ..., \tilde{z_n} e^{t\omega_n})$ is a zero of $f|_\omega$ with argument θ. Now let B be a p-ball with radius ε centered at $\tilde{z}$ with $\partial B \cap V_\omega = \emptyset$. Then $\partial B_t \cap V_\omega = \emptyset$ where B_t is the translation of B around $\tilde{z}_t$. Hence, if $t \in \mathbb{R}$ is big enough, then $|f - f|_\omega|$ is strictly less than $|f|_\omega|$ on ∂B_t. Composed with a parametrization of the p-plane containing B_t, $f|_\omega$ and $f - f|_\omega$ are holomorphic mappings from $\mathbb{C}^p$ to $\mathbb{C}^p$. Now theorem 3.4 implies that also f has a zero z in B_t. But then $|\tilde{z} - z e^{-t\omega}| < \varepsilon$, and since $\arg(z e^{-t\omega}) = \arg z$, we conclude that $\theta \in \overline{\mathcal{A}'}$. □

We really did not use any concepts or results that was not directly related to the formulation of Theorem 3.3, to prove the direction $\overline{\mathcal{A}'} \subseteq \bigcup_\omega \mathcal{A}'_\omega$. However, it is possible to get this direction in an easier way by introducing some new concepts. We will do this in next chapter.

4 The theta variety

In this chapter, we will consider V as the zero-set of a polynomial mapping f and for fixed argument vectors θ study the variety of values of f, the *theta variety* of f. By a compactification of this variety, we get a sense of the topology of V at infinity, which is essential when studying the coamoeba.

The varieties to first have in mind when studying the theta variety (or theta cone as i Chapter 5), is those of dimension $n/2$ since then the theta variety of V has the same dimension as V itself. The main benefits from the present chapter comes in 4.2, where we consider this case.

4.1 Definintions and general facts

Let V be a variety in $\mathbb{C}^n$ defined by the polynomials f_k, $1 \leq k \leq p$ or equivalently as the zero-set of the mapping $f := (f_1, f_2, ..., f_p)$. Here we do not require that $f_1, ..., f_p$ are algebraically independent: hence V_p might be of codimension less than p and possibly $p > n$. Write $f_k(z) = \sum_\alpha m_{k\alpha}$ where $m_{k\alpha}$ is the monomial with exponent α. We can assume that $m_{k0} = 1$ for every k. For a fixed $\theta \in \mathbb{T}^n$, define the mappings $\phi_{k\theta} : \mathbb{R}^n \to \mathbb{C}$, $F_{k\theta} : \mathbb{R}^n \to D$,

$$\phi_{k\theta}(x) = f_k(e^{x+i\theta}), \tag{4.1}$$

$$F_{k\theta}(x) = (f_k / \sum |m_{k\alpha}|)(e^{x+i\theta}). \tag{4.2}$$

Here, D is the complex unit disc. Set $\phi_\theta = (\phi_{1\theta}, ..., \phi_{p\theta})$, $F = (F_{1\theta}, ..., F_{p\theta})$.

Definition 8. The *theta variety* $\mathcal{M}_f(\theta) = \mathcal{M}(\theta)$ of f at θ is the set

$$\phi_\theta(\mathbb{R}^n) \subset \mathbb{C}^p.$$

The *compactified theta variety* $\mathcal{K}_f(\theta) = \mathcal{K}(\theta)$ of f at θ is the set

$$F_\theta(\mathbb{R}^n) \subset D^p.$$

Some important properties of $\mathcal{M}_f$ and $\mathcal{K}_f$ are determined by V. To start with, the following of course holds regardless of the choice of f:

Proposition 4.1.

$$\theta \in \mathcal{A}'_V \Leftrightarrow 0 \in \mathcal{M}_f(\theta) \Leftrightarrow 0 \in \mathcal{K}_f(\theta)$$

Rather immediate is also the next result.

Proposition 4.2. *Let* $\dim V = p = n/2$. *If* 0 *is a singular value of* ϕ_θ, *then* $\theta \in \mathcal{C}'_V$.

Proof. If 0 is a singular value of ϕ_θ, then there is an $x \in \mathbb{R}^n$ such that $\phi_\theta(x) = 0$ and $\operatorname{Jac} \phi_\theta = 0$ at x, but this means that $x \in \mathcal{C}_V$ and hence by Theorem 2.1, $\theta \in \mathcal{C}'_V$. □

For any $\omega \in \mathbb{R}^n$, let Γ_k be the face of Δ_{f_k} such that $F_k|_{\Gamma_k} = F_k|_\omega$ and $\phi_k|_{\Gamma_k} = \phi_k|_\omega$. Set $F|_\omega := (F_1|_\omega, ..., F_n|_\omega)$ and define ϕ_ω analogously. If $\omega \neq 0$ we can translate Γ_k so that they are all contained in a hyperplane P orthogonal to ω. It is easy to verify that $F|_\omega(\mathbb{R}^n) = F|_\omega(P)$ and hence we have the following:

Proposition 4.3. *When $\omega \neq 0$, $\dim \mathcal{K}_\omega \leq n-1$.*

In view of to Proposition 4.1, we want to study $\mathcal{K}$ as closely as possible. But of course, the lower dimension the faces of $\Delta_1, \Delta_2, ...$ corresponding to ω have, the easier it is to describe $\mathcal{K}_\omega$. Set $\mathcal{K}_\infty := \bigcup_{\omega \neq 0} \mathcal{K}_\omega$.

Proposition 4.4. *For any $\theta \in \mathbb{T}^n$, $F_\theta(rS^n) \to \mathcal{K}_\infty(\theta)$ in the Hausdorff metric as $r \to \infty$. In particular,*

$$\overline{\mathcal{K}} = \bigcup_{\omega \in \mathbb{R}^n} \mathcal{K}_\omega$$

Note that $\mathcal{K}$ is contained in the union on the right hand side since $\mathcal{K}_0 = \mathcal{K}$.

Proof. If $z = F|_\omega(x)$ for some $\omega \in S^n$, $x \in \mathbb{R}^n$, then

$$z = \lim_{r \to \infty} F(x + r\omega)$$

and hence $z \in \overline{\mathcal{K}}$. Since $\mathcal{K}_\infty$ is compact, the convergence is uniform, that is

$$\lim_{r \to \infty} \sup_{z \in \mathcal{K}_\infty} |z - F(x + r\omega)| \to 0 \tag{4.3}$$

On the other hand, assume that $z \in \partial\mathcal{K} \backslash \mathcal{K}$. Then $z = \lim_{j \to \infty} F(x_j)$ for a sequence $\{x_j\} \subseteq \mathbb{R}^n$ such that $|x_j| \to \infty$. Since S^n is compact, we can choose a subsequence $\{x_{j_k}\}$ such that $x_{j_k}/|x_{j_k}|$ converges to some $\omega \in S^n$. Now

$$F|_\omega(\omega) = \lim F|_\omega(x_j) = \lim F(x_j) = z$$

and hence $z \in \mathcal{K}_\omega$. By (4.3), $F(rS^n) \to \mathcal{K}_\infty$ in the Hausdorff metric as $r \to \infty$. The proposition follows. □

A *Gröbner basis* $\mathcal{G}$ of a polynomial ideal I with respect to the weight $\omega \in \mathbb{R}^n$ can be defined as a subset of I such that $I_\omega = \{f|_\omega; f \in \mathcal{G}\}$. A *universal* Gröbner basis $\mathcal{U}$ of I is a subset of I such that $I_\omega = \{f|_\omega; f \in \mathcal{G}\}$ for any weight $\omega \in \mathbb{R}^n$. For any I there is a finite universal Gröbner basis. To read more about this, see [16] pag. 1-2.

In regard of this, we get without much effort the hardest direction of the main result in Chapter 3 as a corollary of Proposition 4.4.

Corollary 4.5. *For a variety V in $\mathbb{C}^n$,*

$$\overline{\mathcal{A}'_V} \subseteq \bigcup_\omega \mathcal{A}'_{V_\omega}$$

Proof. Let $\{f_1, ..., f_p\}$ be a finite universal Gröbner basis of $I(V)$ and set $f = (f_1, ..., f_p)$. Then $f|_\omega$ cuts out V_ω for every $\omega \in \mathbb{R}^n$. Let $\mathcal{K} = \mathcal{K}_f$ and consider the set

$$B = \{(x, \theta) \in D^p \times \mathbb{T}^n; x \in \bigcup_{\omega \in \mathbb{R}^n} \mathcal{K}_\omega(\theta)\}$$

Since F_θ depends continuously on θ and $\bigcup_{\mathbb{R}^n} \mathcal{K}_\omega(\theta)$ is closed by Proposition 4.4, B is a closed set. But in view of Proposition 4.1, $\bigcup_\omega \mathcal{A}'_{V_\omega}$ is the projection of $B \cap \{x = 0\}$ on $\mathbb{T}^n$ and is hence closed. The result follows. $\square$

4.2 The case p=n/2

Throughout this chapter we will use well-known theory and results from algebraic topology. For a more careful discussion of these matters, see for example [10] or [6].

Let X be a topological space with a subspace Y. Consider the quotient $C_\bullet(X, Y) := C_\bullet(X)/C_\bullet(Y)$ of the chains on X and Y respectively. The *relative homology group* $H_k(X, Y)$ is the k-th homology group of $C_\bullet(X, Y)$,

$$H_k(X, Y) = \mathrm{Ker}\,_\partial C_k(X, Y)/\mathrm{Im}\,_\partial C_{k+1}(X, Y)$$

Here ∂ is the boundary operator on $C_\bullet(X)$; clearly $\partial^2 = 0$ also on the quotient complex. We see that an element of $H_k(X, Y)$ is represented by a *relative cycle*: a k-chain σ over X such that $\partial\sigma \in C_{k-1}(Y)$. Hence $\partial :\ H_k(X, Y) \to H(Y)$. In fact, we have the following:

Theorem 4.6. *There is an exact sequence*

$$... \xrightarrow{\partial} H_k(Y) \to H_k(X) \to H_k(X, Y) \xrightarrow{\partial} H_{k-1}(Y) \to ... \to H_0(X, Y) \to 0$$

This type of exact sequence has the property of *naturality.* This means that given two spaces X_1, X_2 with subspaces Y_1 and Y_2 and a continuous mapping $\varphi : X_1 \to X_2$ for which $\varphi(Y_1) \subseteq Y_2$, the following diagram is commutative:

$$\begin{array}{ccccccccc}
... & \longrightarrow & H_k(X_1) & \longrightarrow & H_k(X_1, Y_1) & \xrightarrow{\partial} & H_{k-1}((Y_1) & \longrightarrow & ... \\
 & & \Big\downarrow \varphi_* & & \Big\downarrow \varphi_* & & \Big\downarrow \varphi_* & & \\
... & \longrightarrow & H_k(X_2) & \longrightarrow & H_k(X_2, Y_2) & \xrightarrow{\partial} & H_{k-1}(Y_2) & \longrightarrow & ...
\end{array} \tag{4.4}$$

Here φ_* is the lifting of φ to the current homology group.

Assume that n is even and $p = n/2 = \dim V$. We state the main result of this chapter.

Theorem 4.7. *For a generic $\theta \in \mathbb{T}^n$, letting d be the degree of the cycle $\mathcal{K}_{f^\infty}(\theta)$ on the punctured polydisc $D^{n/2}\backslash\{0\}$ we have that*

$$|\,\mathrm{Arg}^{-1}(\theta) \cap V| \geq |d|$$

The more precise condition on θ that we use in the proof of this result, is that $\theta \notin \mathcal{C}' \cup \bigcup_{\omega \neq 0} \mathcal{A}'_{f_\omega}$. Note here that by the Rank Theorem (see e.g. [3] p. 47) $\dim \mathcal{C}' < n$ and by Proposition 4.3, $\dim \mathcal{A}'_{f_\omega} < n$.

To prove Theorem 4.7, we need the following basic result in homological algebra (see for example [2] p. 192).

Theorem 4.8. *Let $\varphi : S^n \to S^n$ be differentiable and $y \in S^n$ a regular value with $\varphi^{-1}(x) = \{x_1, ..., x_m\}$. Then*

$$\sum_{j=1}^{m} \operatorname{sgn} \operatorname{Jac}_\varphi(x_j) = \deg(\varphi)$$

.

Proof of Theorem 4.7. First note that d equals the degree of $f_\theta(x)$ on $\mathbb{C}^{n/2} \backslash \{0\}$ since only the radius of each coordinate in f_θ and F_θ differs.

If $\theta \notin \mathcal{C}'$, then $\operatorname{Log} \operatorname{Arg}^{-1}(\theta)$ is finite and hence contained in a big enough ball. Choose a possibly even bigger ball $U \subset \mathbb{R}^n$ to assure that $F_\theta(\partial U)$ is homotopy equivalent to $\mathcal{K}_\infty$ with respect to $D^{n/2} \backslash \{0\}$; this is possible by Proposition 4.4. We have that

$$U \cong f_\theta(U) \cong B^n, \quad \partial U \cong f_\theta(\partial U) \cong S^{n-1}$$

The k-th homology of a ball is trivial for $k \neq 0$ since the set is contractible. Hence, if we set $X_1 = U$, $X_2 = f_\theta(U)$, $Y_1 = \partial U$, $Y_2 = f_\theta(\partial U)$ in (4.4), we get as a part the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_n(U, \partial U) & \xrightarrow{\partial} & H_{n-1}(\partial U) & \longrightarrow & 0 \\ & & \Big\downarrow f_\theta^* & & \Big\downarrow f_\theta^* & & \\ 0 & \longrightarrow & H_n(f_\theta(U), f_\theta(\partial U)) & \xrightarrow{\partial} & H_{n-1}(f_\theta(\partial U)) & \longrightarrow & 0 \end{array} \tag{4.5}$$

By basic homology theory we have

$$H_n(B^n, S^n) = H_n(S^n) = \mathbb{Z}$$

and since ϕ_θ is continuous, the diagram commutes. Furthermore, the vertical mapping to the right equals d by our assumption. Hence, so does the vertical mapping to the left. By Lemma 4.2, origo is a regular value and hence by Theorem 4.8, there are at least d points $x_1, ..., x_d$ in U such that $\phi_\theta(x_j) = 0$. This means that $f(e^{x_j + i\theta}) = 0$ and we are done. ☐

Before we discuss this bound in details, we will see how this local result can help us find similar bounds globally. However, we have so far only worked this out properly for the case when $n = 2$.

4.3 The case $n = 2$

Consider a polynomial f on $\mathbb{C}^2$ with 2-dimensional Newton polytope Δ. We will write f as

$$f(z) = \sum_{\alpha \notin \operatorname{vert} \Delta} m_\alpha + \sum_{j=1}^{m} m_j, \ m_\alpha = a_\alpha z^\alpha, \ m_j = a_j z^{\alpha_j}$$

where the indexes α_j are the vertices of Δ and numbered anticlockwise. For a multiargument $\theta \in \mathbb{T}^2$, let

$$v_j = v_j(\theta) = \frac{m_j}{|m_j|}(e^{(x,y)+i\theta})$$

Now number the faces Γ_k of Δ anticlockwise and assume that α_k is the endpoint of Γ_k with the lowest index so that α_{k+1} becomes the other endpoint, $k < m$, while Γ_m starts at α_m and ends at α_1.

Let $|\Gamma|$ be the *integer length* of the facet Γ, that is, the number of points in $\mathbb{Z}^2$ contained in Γ minus one.

Proposition 4.9. *The coamoeba $\mathcal{A}'_k$ of f truncated to Γ_k is a union of maximally, and generically, $|\Gamma_k|$ parallel lines on $\mathbb{T}^2$ orthogonal to Γ_k.*

Proof. If $\alpha_k = (a, b)$, $\alpha_{k+1} = (c, d)$, let l be the largest common divisor of $c - a$ and $d - b$. Clearly, $l = |\Gamma_k|$ and letting f_k be the truncation of f to Γ_k, we have that

$$\mathcal{A}'_k = \mathcal{A}'_{f_k} = \mathcal{A}'_{f_k/z_1^a z_2^b}$$

By the coordinate change

$$w = z_1^{\frac{c-a}{l}} z_2^{\frac{d-b}{l}}$$

we have that $f_k/z_1^a z_2^b$ is of degree l in w and hence can be written as $f(w) = (a_1 - w)(a_2 - w)...(a_l - w)$ for complex numbers $a_1, ..., a_L$. Hence, $\mathcal{A}'_k$ consists of the lines

$$\frac{c-a}{l} \arg z_1 + \frac{d-b}{l} \arg z_2 = \arg w = \pi + \arg a_j$$

which obviously are orthogonal to Γ_k. $\square$

By this result we may construct a weighted directed graph on $\mathbb{T}^2$ in the following way. Let $(a_1 - w)...(a_l - w)$ be the factorization of f_k as in the proof of proposition 4.9 and for $L \subseteq \mathcal{A}'_k$, assume that d of these factors have coamoeba equal to L, that is, L has *multiplicity* $d \geq 1$ in $\mathcal{A}'_k$. Either L is not contained in any other facet coamoeba $\mathcal{A}'_j$ and then we set $w(L) = d$ and orient L outward from the facet Γ_k of Δ. Otherwise there is a facet Γ_j parallel to Γ_k, $j \neq k$ containing the line L with the multiplicity of L in $\mathcal{A}'_j$ equalling $c \geq 1$. If $c < d$, we orient L outwards from the face Γ_k of Δ and set $w(L) = d - c$. If $c > d$ we do the opposite. If $c = d$ we set $w(L) = 0$.

Definition 9. Let $H = H_f$ be the weighted directed graph on $\mathbb{T}^2$ obtained by taking the union of the directed lines L in the facet coamoebas as above with $w(L) \neq 0$, and giving each line L the weight $w(L)$.

Compare this definition with the discussion of hyperplane arrangements in [13].

Proposition 4.10. *The graph H is a balanced, that is, any closed curve on $\mathbb{T}^2$ "crosses" H equally many times from the right and from the left, where a crossing of a line $L \subset H$ counts as many times as the weight of L.*

The proof is straight-forward but a little lengthy. However, it is also a corollary to Theorem 4.11.

Recall the definition in (4.2) of $F_\theta : \mathbb{R}^2 \to D$ for a polynomial f and an argument vector $\theta \in \mathbb{T}^2$ and denote the corresponding compactification for the polynomial f_k at θ by $F_k = F_{\theta k}$. We see that F_k will map $\mathbb{R}^2$ to some curve in D connecting the two boundary points v_k and v_{k+1}. In the special case where

f_k is a binomial, this curve will in fact be a straight line segment, since one has then for every $\theta \in \mathbb{R}^2$,

$$F_k(\theta) = \frac{m_k + m_{k+1}}{|m_k| + |m_{k+1}|} = tv_k + (1-t)v_{k+1}$$

where $t = |m_k|/(|m_k| + |m_{k+1}|)$. Orient the curve $F_k(\mathbb{R}^2)$ from v_k to v_{k+1} and denote it $\mathcal{K}_k(\theta)$. The graph $\mathcal{K}_\infty := \bigcup \mathcal{K}_k$ is now an oriented cycle that could be considered as a deformation of the anticlockwise oriented Newton polygon Δ.

Define an integer-valued function $W = W_f$ on $\mathbb{T}^2$ by letting $W(\theta)$ be the winding number of $\mathcal{K}_\infty$ with respect to the origin.

Theorem 4.11. *Let $\beta, \beta' \in \mathbb{T}^2$ be connected by a path γ oriented from β to β' crossing the lines $L_1, ..., L_l$. We set $\sigma_j = 1$ if γ crosses L_j from the right and $\sigma_j = -1$ otherwise. If*

$$\sum_{j=1}^{l} \sigma_j w(L_j) = d$$

then $W(\beta) = W(\beta') - d$. In particular, if β and β' are in the same complement component of H, $W(\beta) = W(\beta')$.

As an immediate consequence of Theorem 4.11, we get Proposition 4.10, since we in particular can choose γ to be a cycle.

Definition 10. The *multiplicity* of an argument vector θ with respect to f, is the number of roots z to $f(z) = 0$, counted with ordinary multiplicity, with $\operatorname{Arg} z = \theta$.

Note that if $W(\theta)$ is known for one particular θ, then Theorem 4.11 determines W on the whole torus $\mathbb{T}^2$. To check $W(\theta)$ for a fixed θ is often straightforward. Hence, in view of Theorem 4.7, Theorem 4.11 gives a minimum of the multiplicity of θ in V for any $\theta \in \mathbb{T}^2$. In particular, it approximates the extension of $\mathcal{A}'$.

Lemma 4.12. *Let L be any non-trivial oriented circle on $\mathbb{T}$, that is, $L = \{(n_1t, n_2t)\}$ for some $n_1, n_2 \in \mathbb{Z}$. Let d be the integer length of $[\alpha_k, \alpha_{k+1}]$ and assume that v_{k+1}/v_k moves anticlockwize around S^1 when θ moves along L. Orient $\mathcal{K}_k$ from v_k to v_{k+1}. Assume that for every $\theta \in L$, $G(\theta)$ is a cycle depending continuously on θ such that $G \supseteq \mathcal{K}_k(\theta)$ and $0 \notin G \backslash \mathcal{K}_g(\theta)$. Then, letting θ increase from zero to 2π, $\mathcal{K}_k(\theta)$ will cross the origin exactly d times counted with multiplicity and the winding number of G with respect to the origin will decrease with every crossing.*

Proof. After a coordinate change as in the proof of Proposition 4.9, we can assume that $f|_{\Gamma_k} = a_0 + a_1 z + ... + a_d z^d$ so that $v_k = a_0/|a_0|$ and $v_{k+1} = a_d z^d/|a_d z^d|$. Since $\mathcal{K}_k(\theta + 2\pi) = \mathcal{K}_k(\theta)$ and the transforming along $[\theta, \theta + 2\pi]$ is continuous, we have that $G(\theta + 2\pi) = G(\theta) - dS^1$ where S^1 is oriented anticlockwise, and hence the winding number of the origin by $G(\theta + 2\pi)$ equals the winding number of the origin by $G(\theta + 2\pi)$ minus d.

On the other hand, since $\deg g = d$, there are d roots of g, that is, d arguments $\varphi \in]-\pi, \pi]$ for which the origin is contained in $\mathcal{K}_k(\varphi)$, counted with multiplicity. Since G crosses the origin if and only if $\mathcal{K}_k$ does, the lemma follows. $\square$

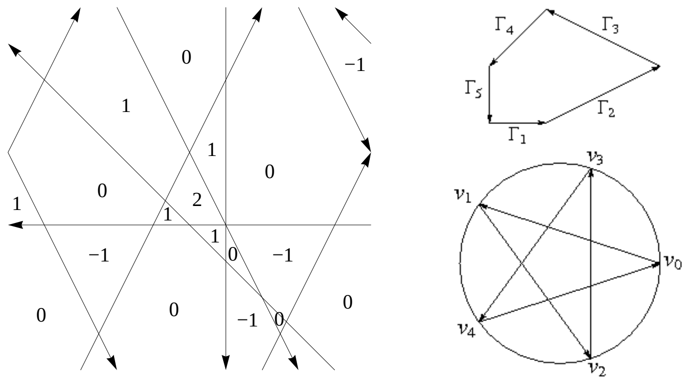
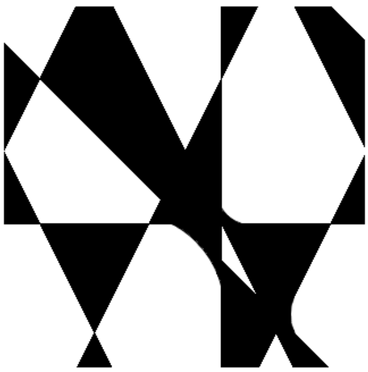


Figure 4: The graphs H_f (to the left), Δ_f (top right) and $\mathcal{K}_{f\infty}(0,0)$ for
$f(z,w) = 1 + ze^{i4\pi/5} + z^3we^{i8\pi/5} + zw^2e^{i2\pi/5} + we^{i6\pi/5}$
with the winding number of origo by $\mathcal{K}_{f\infty}$ written out for each complement component of H_f. In the pictures of $\mathcal{K}_{f\infty}$ and H_f, origo is at the center. Below is the coamoeba of f.



We are now ready to prove Theorem 4.11.

Proof of Theorem 4.11. By Proposition 4.1, we have that $\theta \in \mathcal{A}'_k$ precisely when $0 \in \mathcal{K}_k(\theta)$ and hence W is constant on every complement component of H.

Now assume that β and β' are in adjacent complement components of H, separated by a line L. We can assume that the line segment $[\beta, \beta']$ directed from β to β', intersects L from the right.

If $\mathcal{A}'_k$ is the only facet coamoeba containing L, then $w(L)$ is the multiplicity of any $\theta \in L$ with respect to f_k. Furthermore, setting $G(\theta) = \mathcal{K}_\infty(\theta)$ for $\theta \in [\beta, \beta']$, G is as in Lemma 4.12. By this lemma, W can only decrease when $\mathcal{K}_\infty(\theta)$, that is $\mathcal{K}_k(\theta)$, crosses origo by letting θ moves from β to β' and we conclude that $W(\beta') = W(\beta) - w(L)$.

If there is a unique $j \neq k$ such that L is contained also in $\mathcal{A}'_j$, then let S_j be the anticlockwise oriented circle segment from v_j to v_{j+1} and S_j^{-1} the same segment oriented clockwise. Now, $\mathcal{K}_\infty = G_1 \cup G_2$ where G_1, G_2 are the cycles given by

$$G_1 = \mathcal{K}_j \cup S_j, \quad G_2 = \mathcal{K}_\infty \backslash \mathcal{K}_j \cup S_j^{-1}$$

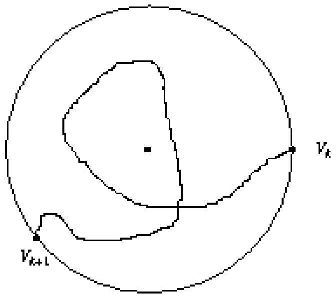


Figure 5: The graph $\mathcal{K}_k$ for a fixed θ.

Now, G_1 and G_2 are as in Lemma 4.12 and $W_{\mathcal{K}}$ is given by $W_{G_1} + W_{G_2}$. The theorem follows. $\square$

Denote by $\mathcal{B}'$ the closure of the union of those complement components of H on which W is non-zero. By Theorem 4.7, $\mathcal{B}' \subseteq \mathcal{A}'$ since the degree of a mapping and the winding number coincide on D. But in general, the bound for $\mathrm{Arg}^{-1}(\theta)$ given by Theorem 4.7 is not high enough on the whole torus. In particular, $\mathcal{A}'$ is usually not equal to $\mathcal{B}'$. We are now going to study some special cases when this is true. First we will state a quite immediate result for the "upper bound" of $\mathcal{A}'$.

Proposition 4.13. *Let $f_{\alpha\beta\gamma}$ be the trinomial given by f restricted to the index set $\{\alpha, \beta, \gamma\}$ and $\mathcal{A}'_{\alpha\beta\gamma}$ the coamoeba of $f_{\alpha,\beta,\gamma}$. Then*

$$\mathcal{A}'_f \subseteq \bigcup \overline{\mathcal{A}'}_{\alpha\beta\gamma} := \mathcal{D}'$$

Proof. If there are no indices α, β, γ such that $\theta \in \mathcal{A}'_{\alpha\beta\gamma}$, then by Proposition 4.1, origo is not contained in $\mathcal{K}_{f_{\alpha\beta\gamma}}(\theta)$ for any θ and any triple α, β, γ and hence not in the convex hull of $\mathcal{K}_\infty(\theta)$ for any θ. But every value of F is a mean value of the monomials of F, so $F(\mathbb{R}^2) \subseteq \mathrm{Conv}(\mathcal{K}_\infty(\theta))$ and the result follows. $\square$

An obvious case where $\overline{\mathcal{A}'} = \mathcal{B}'$ is of course when $\mathcal{B}' = \mathcal{D}'$, that is, when every $\theta \in \mathbb{T}$ either assures that $0 \notin \mathrm{Conv}(G_\theta)$ or assures convexity for $\mathcal{K}(\theta)$. The simplest example of this is when f consists of three monomials so that $\mathcal{K}(\theta)$ is a (possibly degenerated) triangle for every θ and hence always convex. In fact, there are not many other possible polynomials for which the identity $\mathcal{B}' = \mathcal{D}'$ holds.

Theorem 4.14. *For a polynomial f with 2-dimensional Newton polytope, we have the equality $\mathcal{B}' = \overline{\mathcal{A}'} = \mathcal{D}'$ if and only if either f is a trinomial or a tetranomial of the form*

$$f(z) = 1 + az^\alpha + bz^\beta - rabz^{\alpha+\beta}, \quad a,\ b \in \mathbb{C}^*,\ r > 0 \tag{4.6}$$

up to multiplication with a monomial.

Proof. For the case of the coamoeba of a polynomial with four or more monomials, we can assume that $f = 1 + \sum_{j=1}^m m_j$. Let L be the circle on $\mathbb{T}$ given by $v_1 = \pi$. Note that origo is on the boundary of $\mathcal{K}$ whenever $\theta \in L$ and hence $\theta \in \partial \mathcal{B}'$. If $\mathcal{B}' = \mathcal{D}'$, then this means that θ is also on the boundary of $\mathcal{D}'$, so $v_2(\theta), v_3(\theta), ..., v_m(\theta)$ must be on the upper and on the lower unit circle at the same time for every $\theta \in L$. Since v_j depends linearly on θ along L, this means that $v_2(\theta) = v_3(\theta) = ... = v_m(\theta)$ for every $\theta \in L$. Hence, the only possibility is that f is as in (4.6). Clearly, $\mathcal{B}' = \mathcal{D}'$ for this polynomial.

□

So far it is unknown if there are irreducible hypersurfaces in $\mathbb{C}^2$ except the ones mentioned in Theorem 4.14, for which $\overline{\mathcal{A}'}$ equals $\mathcal{B}'$ and has full dimension.

4.4 The case $p < n/2$

Theorem 4.7 concerns only varieties of half the dimension n of the space they live in. However, it implicitly deals with varieties of any dimension less than $n/2$. This because of Proposition 4.4. One expects the compactified theta variety $\mathcal{K}_\omega(\theta)$ to be of dimension $2p$ if the faces Γ_k such that $F_k|_{\Gamma_k} = F_k|_\omega$ are of high enough dimension. In these cases we can use that Theorem 3.3 yields $\operatorname{int} \mathcal{A}'_\omega \subseteq \mathcal{A}'$, in order to describe $\mathcal{A}'$.

As soon as V is not a hypersurface, it is in some sense non-generic that the Newton polytopes of all polynomials f_1,...,f_p have any common normal vector $\neq 0$ for nonzero dimensional faces, and even less generic that they have enough common normal space N for $\bigcup_{\omega \in N} \operatorname{int} \mathcal{A}'_\omega$ to be a good approximation of $\mathcal{A}'$. However, when V is a hypersurface, the approximation might sometimes be good. For linear spaces, $\mathcal{A}'$ is in fact completely described by the initial coamoebas of dimension $n/2$ (see Corollary 5.3). The general result we get in this direction is the following.

Proposition 4.15. *If $V \subset \mathbb{C}^n$ is a hypersurface and $n \geq 2$, then*

$$\mathcal{A}' \supseteq \bigcup_{\Gamma \in S} \operatorname{int} \mathcal{A}'_{f|_\Gamma} \supseteq \bigcup_{\Gamma \in S} \operatorname{int} \mathcal{B}'_{f|_\Gamma}$$

where S is the set of 2-dimensional faces of Δ.

5 Linear spaces and the theta cone

5.1 Preliminaires

In this chapter we study the theta variety for affine linear spaces. Recall that the theta variety $\mathcal{M}(\theta)$ of V is the image of a defining polynomial mapping f of the set $\{\operatorname{Arg} z = \theta\} \subset \mathbb{C}^n$ in $\mathbb{C}^p$. Now, we assume that $V \subset \mathbb{C}^n_*$ is an affine linear variety of codimension p cut out by independent linear polynomials $f_1, ..., f_p$ for some $p \in \{1, ..., n\}$ and as usual, we denote by $\mathcal{A}'$ its coamoeba. We suppose that the polynomials are given by

$$f_k(z) = c_{k0} + \sum_{j=1}^{n} c_{kj} z_j$$

where $c_{kj} = a_{kj} + b_{kj} i$, $a_{kj}, b_{kj} \in \mathbb{R}$, and we let C denote the complex $p \times n$-matrix (c_{kj}).

Definition 11. The variety V is *semi-degenerate* if there are k columns in C for some $k \leq p$ such that the submatrix consisting of these columns has rank $< k$.

The notion of semi-degeneration is related to the stronger condition of *degeneration*, see Definition 12.

Let Δ_n be the n-th unit simplex,

$$\Delta_n = \text{Conv}\{e_0, e_1, ..., e_n\}$$

where e_j is the j:th unit vector, $j \geq 0$ and e_0 is origo. We denote the face spanned by the vertices $e_{l_1}, ..., e_{l_d}$ by $(l_1...l_d)$. Hence, in particular $\mathcal{A}'_{01...n} = \mathcal{A}'$. For a set $I \subseteq \{0, 1, ..., n\}$, we now use the notation g_I for the linear function g restricted to the indices in I and V_I, $\mathcal{A}'_I$ for the variety defined by $f_{1I}, ..., f_{pI}$ and its coamoeba. We will compare this notion with the notion of initial coamoebas. Set

$$\omega_I = \sum_{j \in I} e_j, \quad I \subseteq \{1, 2, ..., n\}$$

$$\omega_I = \sum_{j \in \{1,2,...,n\} \setminus I} -e_j, \quad 0 \in I$$

Proposition 5.1. *If V is not semi-degenerate, then $\mathcal{A}'_I = \mathcal{A}'_{\omega_I}$.*

Proof. Clearly $g_I = g|_{\omega_I}$ for a polynomial g whenever $g_I \neq 0$, so $\mathcal{A}'_I \supseteq \mathcal{A}'_{\omega_I}$. Assume now that $\mathcal{A}'_I \neq \emptyset$ and V is not semi-degenerate. Then $|I| \geq p$ and the restriction C_I of C to the columns with indices in I, has rank p. Hence, for every $g \in I(V)$, $g_I \neq 0$, and hence $g_I = g|_{\omega_I}$. The proposition follows. □

Let us look at an example. If V is the line in $\mathbb{C}^3$ given by $t \mapsto (t, -1-t, -1)$, then

$$C = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

up to linear equivalence. Hence, if $I = \{1, 2\}$, then V_I is the plane $(s, t) \mapsto (s, -s, t)$ while $V_{\omega_I} = V(\{z_1 + z_2 = 0, 1 + z_3 = 0\}) = \{(s, -s, -1)\}$.

Finally some words about classifying linear spaces. The Grassmannian $G(k, n)$ is the space of linear k-subspaces of $\mathbb{C}^n$. There are several ways to equip $G(k, n)$ with coordinates. We will use the *Plücker coordinates*. Our space V has $\binom{n}{p}$ coodinates $q_{j_1,...,j_{n-p}}$ where $q_{j_1,...,j_{n-p}}$ is the maximal minor of C. That V is semi-degenerate hence means precisely that one of the Plücker coordinates for V is zero.

5.2 The theta cone

We change to polar coordinates and fix the arguments $(\theta_1, \theta_2, ..., \theta_n) = \theta$. By separating the real and imaginary part of every equation, we now have a real system of $2p$ linear equations in n variables r_j with the restriction $r_j > 0$. We will write this in a compact way.

Define vectors $m_j = m_j(\theta) \in \mathbb{R}^{2p}$ by setting

$$m_0 = (a_{10}, b_{10}, a_{20}, b_{20}, ..., a_{n0}, b_{n0})$$

and

$$m_j^{(2k-1)} = \mathrm{Re}\,(c_{kj}e^{i\theta_j}) = a_{kj}\cos\theta_j - b_{kj}\sin\theta_j,$$
$$m_j^{(2k)} = \mathrm{Im}\,(c_k^{(j)}e^{i\theta_j}) = b_{kj}\cos\theta_j + a_{kj}\sin\theta_j,\ 1 \le j \le n \tag{5.1}$$

We denote by $M(\theta)$ the real $2p \times (n+1)$-matrix with columns $m_j(\theta)$. If M_1 and M_2 are such matrices of two different linear mappings $f, f' : \mathbb{C}^n \to \mathbb{C}^p$ both cutting out V, we note that M_1 and M_2 are linearly equivalent. In particular V and θ uniquely determine a linear space L in $\mathbb{R}^{2p}$ with generators $m_0, m_1, ..., m_n$. Let $\mathcal{M} = \mathcal{M}(\theta) = \mathcal{M}_V(\theta)$ be the cone in $\mathbb{R}^{2p}$ obtained by intersecting L with the open positive orthant. Recalling the definition of the theta variety in Chapter 4, we see that $\mathcal{M}(\theta)$ is the theta variety of V at θ (see definition 8). Since it is a cone, we call it *the theta cone of V at θ*.

The following proposition is fundamental when using $\mathcal{M}$ for the study of $\mathcal{A}'$.

Proposition 5.2. *If $\theta \in \mathcal{A}'$, $\dim \mathcal{M}(\theta) = d$ and $k \in \{0, 1, ..., n\}$, then there are d monomials $m_{j_1}, ..., m_{j_d}$ and non-negative numbers $t_{j_1}, ..., t_{j_d}$ such that*

$$m_k + \sum_{i=1}^{d} t_{j_i} m_{j_i} = 0$$

Proof. For $n = d$, this follows immediately. Assume that it is true for $n = d + m - 1$. Now let $n = d + m$. Choose $r_j > 0$ such that $m_0 + \sum_{j=1}^{n} r_j m_j = 0$. Without loss of generality, we may assume that $m_0, ..., m_{d-1}$ are linearly independent and $k \notin \{0, 1, ..., d\}$. Now there are real numbers λ_j such that $m_d = \sum_{j=1}^{d} \lambda_j m_j$. For any $\mu \in \mathbb{R}_+$ we hence have

$$\sum_{j=0}^{d-1} (r_j - \mu\lambda_j) m_j + (r_d - \mu) m_d + \sum_{j=d+1}^{n} r_j m_j = 0$$

Choose the minimal μ such that either $r_j - \mu\lambda_j = 0$ for some j or $r_d - \mu = 0$. Then we have that zero is included in the space spanned by $d + m$ monomials, among these m_k, over $\mathbb{R}_+$. Hence by the assumption the assertion in the lemma holds when $n = d + m$. By induction, the proposition follows. □

We get the following corollary.

Corollary 5.3. *If $n \ge 2p$ and $d \le n - 2p$, then $\mathcal{A}' = \bigcup_{|I|=n-d} \mathcal{A}'_I$.*

Proof. The maximal dimension of $\mathcal{M}(\theta)$ is $2p$. Thus if $\theta \in \mathcal{A}'$, then by Proposition 5.2 there is an interval $I \subseteq \{0, 1, ..., n\}$ of length $\le 2p$ such that there are $r_j > 0$, $j \in I$ such that $\sum_I r_j m_j = 0$. Hence, $\theta \in \mathcal{A}'_I$. □

Note that applying this result on a hypersurface, that is letting $p = 1$, we get Theorem 2.6, since obviously $\mathcal{A}'_I = \mathcal{A}'_\Gamma$ if Γ is the face spanned by the points in I (that is, no hypersurface is semi-degenerate).

The next theorem points at the usefulness of the study of $\mathcal{M}$.

Theorem 5.4. *If $\dim \mathcal{M}(\theta) = d$ and $\theta \in \mathcal{A}'$, then the fiber $\mathrm{Arg}^{-1}(\theta) \subseteq V$ is a surface of dimension $n - d$.*

Proof. Choose d linear independent vectors $m_1, ..., m_d$ in $\mathcal{M}$ and let $r \in \mathbb{R}_+^n$ be such that $m_0 + \sum_{j=1}^n r_j m_j = 0$. Then

$$F(t_1, ..., t_d) := \sum_{j=0}^{d} t_j m_j$$

is a bijection between $\mathbb{R}_+^d$ and some open neighborhood U of $-(m_0 + \sum_{j=d+1}^n r_j m_j)$. Hence, there is a $\delta > 0$ such that for every $(t_{d+1}, ..., t_n) \in \mathbb{R}_+^{n-d} \cap B_\delta(r_{d+1}, ..., r_n)$, $-(m_0 + \sum_{j=d+1}^n t_j m_j) \in U$ and hence there is a unique tuple $(t_1, ..., t_d) \in \mathbb{R}$ such that

$$F(t_1, ..., t_d) = -(m_0 + \sum_{j=d+1}^{n} t_j m_j)$$

But this means that $t_1, ..., t_n$ correspond to a z in the fiber of θ. The theorem follows. $\square$

The geometric approach with theta cones is intuitively satisfactory useful for obtaining general results, but things are in general not hard to understand algebraically either. We will now obtain the fiber in V at θ explicitly by means of a linear system of equations.

For every z_j, let $x_j = \operatorname{Re} z_j$, $y_j = \operatorname{Im} z_j$. Let A be the $2p \times 2n$-matrix with columns u_1,...,u_n, v_1,...,v_n where

$$\begin{aligned} u_j^{(2k-1)} &= v_j^{(2k)} = a_{jk} \\ -u_j^{(2k)} &= v_j^{(2k-1)} = b_{jk} \end{aligned} \tag{5.2}$$

Hence,

$$M(\theta) = \begin{pmatrix} m_0 & A \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & \cos\theta_1 & 0 & ... & 0 \\ 0 & 0 & \cos\theta_2 & ... & 0 \\ ... & & & & \\ 0 & 0 & 0 & ... & \cos\theta_n \\ 0 & \sin\theta_1 & 0 & ... & 0 \\ 0 & 0 & \sin\theta_2 & ... & 0 \\ ... & & & & \\ 0 & 0 & 0 & ... & \sin\theta_n \end{pmatrix}$$

By convention, we write $\theta \equiv \theta'$ whenever $\theta_j \equiv \theta_j' \mod \pi$ for every coordinate j. The space $L_A = \{x \in \mathbb{R}^{2n}; Ax = m_0\}$ has dimension $2n - d$ where d is the rank of A and clearly $z \in V$ if and only if $(x_1, ..., x_n, y_1, ..., y_n) \in L_A$. Furthermore $\operatorname{Arg} z \equiv \theta$ for $\theta \in \mathbb{T}^n$ if and only if $(x_1, ..., x_n, y_1, ..., y_n)$ is included in the linear n-subspace

$$L_\theta = \{\sin\theta_j x_j = \cos\theta_j y_j\}$$

Hence, the union of the fibers of the n points $\theta' \equiv \theta$ equals the intersection of L_A and L_θ or alternatively the space $\{A_\theta x = \hat{u}_0\}$ where $\hat{m}_0$ is the $2p+n$-vector obtained by adding n zeroes at the end of m_0 and A_θ is $(2p+n) \times 2n$-matrix obtained by adding the rows $(0, ..., 0, \sin\theta_j, 0, ..., 0, -\cos\theta_j, 0, ..., 0)$ to A, where $\sin\theta_j$ is in the j:th and $-\cos\theta_j$ is in the $n+j$:th column.

To sum up this, we state the following result:

Proposition 5.5. *For $\theta \in \mathbb{T}^n$ the following assertions are equivalent:*

1. There is a unique $\theta' \equiv \theta$ such that $\theta' \in \mathcal{A}'$ $\Leftrightarrow$

2. $2p \geq n$ and there is a $\theta' \equiv \theta$ such that $\theta' \in \mathcal{A}' \backslash \mathcal{C}'$ $\Leftrightarrow$

3. For every j, the matrix M_j obtained by taking away the $j:th$ column of $M(\theta)$ has the same rank as $M(\theta)$, and this rank is n ($\dim \mathcal{M}(\theta) = n$ and each n-tuple of vectors $m_j(\theta)$ spans $\mathcal{M}(\theta)$).

4. The system $A_\theta x = \hat{m}_0$ has a unique solution in $(\mathbb{R}^2)^n_*$.

Proof. We have that $\theta \in \mathcal{A}'$ if and only if there are numbers $r_j \in \mathbb{R}_+$ such that $\sum_{j=1}^n r_j m_j(\theta) = -m_0$. If we could allow r_j to be any real number, we just have to determine when m_0 is contained in the space L spanned by $m_1, ..., m_n$. But by allowing any $\theta' \equiv \theta$, this can be done - a negative r_j corresponds to adding π to the j:th coordinate of θ. We just have to exclude for every j the linear subspace where $r_j = 0$.

Futhermore, if $-m_0 \in L$ and $\dim L < n$, then there is obviously a subspace of positive dimension of solutions $\lambda_1 m_1 + ... + \lambda_n m_n = -m_0$ to L, while the solution is unique when $\dim L = n$. Hence we have 1$\Leftarrow$2$\Leftrightarrow$3. The equivalence 1$\Leftrightarrow$4 was discussed above.

It remains to show 3$\Rightarrow$1. Assume that 3 is not true. Then by Lemma 5.2, there are d monomials $m_{j_1}, ..., m_{j_d}$ and non-negative numbers $t_{j_1}, ..., t_{j_d}$ such that $m_0 + \sum_{i=1}^d t_{j_i} m_{j_i} = 0$. Hence for any $\mu > 1$,

$$0 = m_0 + \sum_{j=1}^n r_j m_j - \mu(m_0 + \sum_{i=1}^d t_{j_i} m_{j_i})$$

$$= (1-\mu) m_0 + \sum_{i=1}^d (r_{j_i} - \mu t_{j_i}) m_{j_i} + \sum_{j \notin \{j_1, ..., j_d\}} r_j m_j$$

Choose μ so that $(r_{j_i} - \mu t_{j_i}) < 0$ for every i. We see now that the expression equals $(1-\mu) f(z)$ for some z with $\arg z \neq \theta$ but $\equiv \theta$. The implication is proved. $\square$

From the same proof, we also obtain the following complementary result:

Proposition 5.6. *For $\theta \in \mathbb{T}^n$ the following assertions are equivalent:*

1. There are several $\theta' \equiv \theta$ such that $\theta' \in \mathcal{A}'$ $\Leftrightarrow$

2. Either $2p < n$ or there is a $\theta' \equiv \theta$ such that $\theta' \in \mathcal{C}'$ $\Leftrightarrow$

3. For every j, the matrix obtained by taking away the $j:th$ column of $M(\theta)$ has the same rank as $M(\theta)$ has, and this rank is $< n$ ($\dim \mathcal{M}(\theta) < n$ and each n-tuple of vectors $m_j(\theta)$ spans $\mathcal{M}(\theta)$).

4. The system $A_\theta x = \hat{m}_0$ has infinitely many solutions in $(\mathbb{R}^2)^n_*$.

The only case that is not covered by these two propositions is clearly the one when $\operatorname{Tan} \theta \notin \operatorname{Tan} \mathcal{A}'$.

A consequence of the Propositions 5.2 and 5.6 is the following useful tool for the study of the contour $\mathcal{C}'$.

Theorem 5.7. *For any $\theta \in \mathbb{T}^n$, the following statements are equivalent:*

1. There is more than one point in V with argument θ.

2. There is a curve in V of positive length with constant argument θ (if the codimension of V is less than or equal to $n/2$, $\theta \in \mathcal{C}'$).

3. There is a subset S of $P(\{0,1,...,n\})$ not including $\{0,1,...,n\}$ such that $\bigcap_S I = \{0,1,...,n\}$ and $\theta \in \bigcap_S \mathcal{A}'_I$.
4. $\theta \in \mathcal{A}'$ and there is an $I \neq \{0,1,...,n\}$ with $\theta \in \mathcal{A}'_I$.
5. $\theta \in \mathcal{A}'$ and $\dim \mathcal{M}(\theta) < n$.

Proof. By Proposition 5.6, we have the equivalences $1 \Leftrightarrow 2 \Leftrightarrow 5$. Furthermore by Proposition 5.2, one also has $5 \Rightarrow 4$. Let $I \neq \emptyset$ be such that $\theta \in \mathcal{A}'_\theta$. For every $k \in I$, we can by Proposition 5.2 choose k such that there is a $J \subset \{0,1,...,n\}$ with $k \in J$ such that $\theta \in \mathcal{A}'_J$, and this yields the implication $5 \Rightarrow 3$. Finally, by addition of equation systems as in the proof of Proposition 5.5, we see that 3 and 4 both imply 1. $\square$

For the case when $p \leq n/2$, Theorem 5.7 implies that Arg is a bijection from the non-critical points of V to $\mathcal{A}' \backslash \mathcal{C}'$. Recall that by Theorem 2.1, the critical points of Arg are the same as the critical points of Log. It follows that we can define a surjective mapping G from $\mathcal{A}' \backslash \mathcal{C}'$ to $\mathcal{A} \backslash \mathcal{C}$ by setting $G = \mathrm{Log} \circ \mathrm{Arg}^{-1}$.

Proposition 5.8. *The mapping G is an isometry, that is, the Jacobian determinant of G equals one.*

Proof. Denote the Jacobian matrix of Log, Arg and Arg^{-1} by J, J' and J'^{-1} respectively. The Jacobian matrix of G then equals JJ'^{-1}. The complex logarithm $\log = \mathrm{Log} + i\,\mathrm{Arg}$ is a holomorphic function. Hence, by the Cauchy-Riemann equations,

$$\frac{\partial \,\mathrm{Log}\, z_k}{\partial x_j} = \frac{\partial \,\mathrm{Arg}\, z_k}{\partial y_j}\ ,\ \frac{\partial \,\mathrm{Log}\, z_k}{\partial y_j} = -\frac{\partial \,\mathrm{Arg}\, z_k}{\partial x_j}$$

Thus, J is obtained from J' by interchanging pairs of rows and changing sign on one row in every pair and then JJ'^{-1} is obtained from the unit matrix $E = J'J'^{-1}$ in the same way. By elementary linear algebra, it follows that

$$|JJ'^{-1}| = |E| = 1$$

$\square$

5.3 Conditions for degeneration

Let us now take a look at the case $2p \leq n$. The maximal, and "expected", dimension of $\mathcal{A}'$ is $2n - 2p$, since there are $2p$ real equations $\mathrm{Re}\, f_k = 0$, $\mathrm{Im}\, f_k = 0$ to be satisfied by $2p$ variables r_j, θ_j. In fact, by Theorem 2.3, on has $\dim(\mathcal{A}' \backslash \mathcal{C}') = 2n - 2p$ whenever $\mathcal{A}' \neq \mathcal{C}'$. More generally,

Proposition 5.9. *If $|I| = d$ and the rank of C_I is q, then* $\dim \mathcal{A}'_I \leq n - 2q + d - 1$.

Proof. For $d = n$, the result follows from the discussion above. Now let $d < n$. By a linear change of coordinates, one get defining equations for V_I in, say, $w_1, w_2, ..., w_{d-1}$. Let W be the space given by these variables. Now V_I is a subspace of W whose coamoeba $\mathcal{A}'_W$ has dimension $\leq 2(d - 1 - q)$. But clearly $\mathcal{A}'_I$ is given by the union of $(n + 1 - d)$-planes in $\mathbb{R}^n$ whose intersection with $\mathrm{Log}\, W$ is a point in $\mathcal{A}'_W$. The result follows. $\square$

Let us study the conditions for $\mathcal{A}'$ not to be equal to $\mathcal{C}'$. By Proposition 5.5, this occurs precisely when there is a $\theta \in \mathcal{A}'$ such that A_θ has rank $2n$. Hence we want to see when this is the case. First, we need some linear algebra.

Let L_j be the plane in $\mathbb{R}^{2p}$ generated by the column vectors u_j and v_j of A. For any $N \subseteq \{0,1,...,n\}$, let L_I be the minimal linear space including all the spaces L_j for $j \in I$. Recall the definition of u_j, v_j in (5.2). For $\varphi \in \mathbb{T}^n$, let $K_{\varphi I}$ be the linear space in $\mathbb{R}^{2p}$ generated by $\cos\varphi_j u_j + \sin\varphi_j v_j$, $j \in I$. Then clearly $K_{\varphi I} \subseteq L_I$. Note also that $K_{\varphi\{0,1,...,n\}}$ is the minimal linear space that contains $\mathcal{M}(\varphi)$.

Lemma 5.10. *Assume that* $\dim L_I \geq |I|$ *for every* $I \subseteq \{0,1,...,n\}$ *and let* $I_1,...,I_d$ *be a partition of* $\{0,1,...,n\}$ *such that the span of the spaces* L_{I_j}, $j \neq l$, *does not intersect* L_{I_k} *for any* $k \in [d]$ *(generically, this partition is trivial). Let* $\dim K_{\varphi I_k} < |I_k|$. *Then there are multiarguments* ψ *such that* $K_{\psi I_k}$ *is of dimension* $|I_k|$, $K_{\varphi I_k} \subset K_{\psi I_k}$ *and* $\bigcup_\psi K_{\psi I_k}$ *generates* L_{I_l}.

Proof. We are going to show this by induction over n. The statement is trivial for $n = 1$. Assume it holds for $n-1$, $n \geq 2$ and show it for n. For contradiction, we want to assume that $\dim K_{\varphi I} < n$ for some I in the partition, but there is no loss of generality in letting $\dim K_{\varphi I} = n-1$.

First assume that every $n-1$-set of generators $\cos\varphi_j u_j + \sin\varphi_j$ of $K_{\varphi I}$, generates the whole $K_{\varphi I}$. Then it is clear that the set of spaces $K_{\psi I}$ where $\psi_j = \varphi_j$, $j \neq k$ for a distinct k, all include $K_{\varphi I}$. Since $\dim L_I \geq |I|$, there are integer sets J_1, J_2 not both empty such that $u_j, v_k \in L_I \backslash K_{\varphi I}$ if and only if $j \in J_1, k \in J_2$. Hence, letting $\alpha^j \in \mathbb{T}^n$ be obtained from ϕ by changing φ_j to 0, $j \in J_1$ and $\beta^k \in \mathbb{T}^n$ by the analogue change of φ_k to $\pi/2$, $k \in J_2$,

$$\dim K_{\alpha^j I} = \dim K_{\beta^k I} = |I|$$

and these spaces generate L_I.

Next assume that there is a k such that the linear space $K_{\varphi I\backslash\{k\}}$ does not equal $K_{\varphi I}$. Then $\dim K_{\varphi I\backslash\{k\}} = n-2$ and by the assumption on I made in the lemma, L_k is not disjoint from $L_{I\backslash\{k\}}$ and hence $\dim L_{I\backslash\{k\}} \geq \dim L_I - 1$. Make a partition P of $\{0,1,...,n\}\backslash\{k\}$ as in the lemma. Then $I\backslash\{k\}$ is the union of $I_1,...,I_d \in P$ and

$$\dim K_{\varphi I\backslash\{k\}} = \sum_{j=1}^{d} \dim K_{\varphi I_j}$$

and hence there is a $p \in [d]$ such that $\dim K_{\varphi I_p} = |I_p| - 1$. By the assumption for dimension $n-1$, there are $\psi \in \mathbb{T}^{|I_p|-1}$ such that $K_{\psi I_p} \supset K_{\varphi I_p}$, $\dim K_{\psi I_p} = |I_p|$ and $K_{\psi I_p}$ generates L_{I_p}. Since $\dim L_{I_p} \geq p$ and $\cos\varphi_k u_k + \sin\varphi_k v_k \notin L_{I_p}$, the last property means that we can choose such a ψ so that $\cos\varphi_k u_k + \sin\varphi_k v_k \notin K_{\psi I_p}$. We let $\psi' \in \mathbb{T}^n$ be given by exchanging every coordinate φ_j of φ with $j \in I_p$, by ψ_j. Now $K_{\psi' I}$ is of dimension $|I|$, includes $K_{\varphi I}$ and all such spaces generates L_I. The result now follows by induction. ☐

Proposition 5.11. *Assume that* $\dim L_I \geq |I|$ *for any* $I \subseteq \{0,1,...,n\}$ *and let* $\dim K_{\varphi\{0,1,...,n\}} < n$, $\varphi \in \mathbb{T}^n$. *Then there is a* $\psi \in \bigcup_I \mathcal{A}'_I$ *such that* $\dim \mathcal{M}(\psi) = n$.

Proof. Let $\varphi \in \mathcal{A}'$. Note that for a partition of $\{0,1,...,n\}$ as in Lemma 5.10, $\dim K_{\varphi\{0,1,...,n\}} = \sum \dim K_{\varphi I_j}$. Hence the lemma says that there is a $\psi \in$

$\mathbb{T}^n$ such that $\dim K_{\psi\{0,1,\dots,n\}} = n$, that is $\dim \mathcal{M}_\psi = n$, and $K_{\psi\{0,1,\dots,n\}} \supset K_{\varphi\{0,1,\dots,n\}}$. The latter means that $\psi' \in \bigcup_I \mathcal{A}'_I$ for some $\psi' \equiv \psi$. $\square$

With these results at hand, we are ready to solve our problem. Recall that C is the matrix with the complex multicoefficients $c_0, c_1, ..., c_n$ of f as columns.

Definition 12. A linear variety V is *degenerate* if there is a $(k \times p)$-submatrix of C with rank $< k/2$, $k \leq n$.

The condition of degeneration is directly related to the Plücker coordinates of V since we can reformulate it in the following way: there are at least $2k+1$ columns in C such that all minors of C containing $\min(k+1, p)$ of these columns must be zero.

In the real setting, degeneration means that the $2p \times 2k$-matrix of k pairs u_j, v_j or the $2p \times (2k-1)$-matrix of $k-1$ pairs u_j, v_j and $\hat{m}_0$ has rank $< k$, that is, for some I, $\dim L_I < k$ and $|I| = k$ or $\dim L_I = k-1$, $|I| = k-1$ and $\hat{m}_0 \in L_I$.

Proposition 5.12. *If $2p \geq n$ and V is degenerate, then $\mathcal{A}' = \mathcal{C}'$.*

Proof. First assume that there are k pairs u_j, v_j, $j \geq 1$, such that the matrix with these vectors as columns has rank $< k$. Then regardless of the last n rows of A_θ, and hence of θ, the rank of the submatrix of A_θ with the same columns, has rank $< 2k$ since the last n columns have zeroes in all but k rows. Hence the rank of A_θ is $< 2k + (2n-2k) = 2n$ for any $\theta \in \mathbb{T}^n$ and there is a subspace in $\mathbb{R}^{2n}$ of solutions to $A_\theta x = \hat{m}_0$. In particular, if one solution is in $(\mathbb{R}^2)^n_*$, that is $\theta \in \mathcal{A}'$, then infinitely many solutions are there, that is $\theta \in \mathcal{C}'$.

The other possibility is that there are k pairs u_j, v_j, $j \geq 1$, such that the $2p \times 2k$-matrix A_I with these vectors as columns has rank k and $\hat{m}_0$ is generated by $u_1, ..., u_k, v_1, ..., v_k$. Let $\theta \in \mathcal{A}'$. Then the restriction $A_{I\theta}$ of A_θ to the columns of A_I has rank $\leq 2k$ and hence there is a solution to $A_{I\theta} x = \hat{m}_0$, that is, $\theta \in \mathcal{A}'_J$ for some $J \subseteq I$. But then by Theorem 5.7, $\theta \in \mathcal{C}'$. $\square$

5.4 The case p=n/2

Proposition 5.11 indicates that the converse of Proposition 5.12 should hold. However, we just prove it in the case $p = n/2$. Since A_θ is quadratic, $A_\theta x = \hat{m}_0$ is solvable in $\mathbb{R}^n$ for every θ so for every $\theta \in \mathbb{T}^n$ there is a $\theta' \equiv \theta$ with $\theta' \in \bigcup_{I \subseteq \{0,1,\dots,n\}} \mathcal{A}_I$. Hence Proposition 5.12 just says that $\dim \mathcal{M}(\theta) = n$ for some $\theta \in \mathbb{T}^{n-1}$.

Theorem 5.13. *If $2p = n$, then $\mathcal{A}' = \mathcal{C}'$ if and only if V is degenerate. Furthermore the dimension of $\mathcal{A}'$ is maximal, that is equal to n, if and only if V is non-degenerate.*

Proof. In view of Proposition 5.12, it suffices to prove that every V with $\mathcal{A}' = \mathcal{C}'$ is degenerate. Assume the former. Then by Proposition 5.11, there is a $\psi \in \mathbb{T}^{n-1}$ such that $\dim \mathcal{M}(\psi) = n$ and hence there is an open set $U \subset \mathbb{T}^{n-1}$ such that $\dim \mathcal{M}(\theta) = n$ for every $\theta \in U$. Assume that there is an I such that $\mathcal{A}'_I$ contains an open subset of U, that is, $\dim \mathcal{A}'_I = n$. Set $d = |I|$ and $q = \operatorname{rank} C_I$. Then by Proposition 5.9,

$$n = \dim \mathcal{A}'_I \leq n - 2q + d - 1$$

that is, $q \le (d-1)/2$ or $\operatorname{rank} C_I < d/2$, which is a contradiction since V is non-degenerate.

Hence there is an open set $U' \subseteq U$ such that for every $\theta \in U'$ there is a $\theta' \equiv \theta$ with $\theta' \subseteq \mathcal{A}'$. The result follows. $\square$

We illustrate what we have done so far with an example. Let V be the plane in $\mathbb{C}^4$ with coefficient matrix

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 0 & 0 \end{pmatrix}$$

Then for $\theta \in \mathbb{T}^3$,

$$\hat{m}_0 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T,$$

$$A_\theta = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ \sin\theta_1 & 0 & 0 & 0 & -\cos\theta_1 & 0 & 0 & 0 \\ 0 & \sin\theta_2 & 0 & 0 & 0 & -\cos\theta_2 & 0 & 0 \\ 0 & 0 & \sin\theta_3 & 0 & 0 & 0 & -\cos\theta_3 & 0 \\ 0 & 0 & 0 & \sin\theta_4 & 0 & 0 & 0 & -\cos\theta_4 \end{pmatrix}$$

The determinant of the 8×8-matrix A_θ is non-zero except if

$$\theta_1 \equiv \theta_2 \text{ or } \theta_3 \equiv \theta_4 \mod \pi \tag{5.3}$$

The initial coamoebas $\mathcal{A}'_j$ of $\mathcal{A}'_V$ can be described as coamoebas $\mathcal{A}'_{W_j}$ of lines $W_j \subset \mathbb{C}^3$ stretched out along the j:th axis of $\mathbb{R}^4$ (where the zeroeth axis is the line $(\lambda, \lambda, \lambda, \lambda)$). However, $\mathcal{A}'_0$, $\mathcal{A}'_1$ and $\mathcal{A}'_2$ are degenerate in the sense that W_0, W_1, W_2 are lines in $\mathbb{C}^2$ embedded in $\mathbb{C}^3$.

The first case in (5.3) corresponds to $\theta \in \mathcal{A}'_{1234}$ and the second to $\theta \in \mathcal{A}'_{34}$. Note that for every $I \neq \{3,4\}$ with $|I| \le 3$, the dimension of I is zero. Hence by Theorem 5.7, the contour $\mathcal{C}'$ is given by

1) The intersection of $\mathcal{A}'_{34}$ with $\mathcal{A}'_{0123}$ and $\mathcal{A}'_{0124}$.
2) The intersection of $\mathcal{A}'_{1234}$ with $\mathcal{A}'_{0134}$ and $\mathcal{A}'_{0234}$.
3) The intersection of $\mathcal{A}'_{1234}$ with $\mathcal{A}'_{0123}$ and $\mathcal{A}'_{0124}$.
4) Some isolated points.

From 1) we get a two-dimensional surface whose projection on the torus $\theta_4 = 0$ is a coamoeba of a line, and with $\theta_4 = \theta_3 + \pi$. From 2) we get a two-dimensional flat surface: the coamoeba of the line $1 + z + w = 0$ in the coordinate plane given by θ_3, θ_4 embedded at $\theta_1 = 0, \theta_2 = 0$, $\theta_1 = \pi, \theta_2 = 0$ and $\theta_1 = \pi, \theta_2 = \pi$. From 3), we get intersections of two three-dimensional surfaces on $\mathbb{T}^3$, which we also expect to have dimension two.

Assume that the dimension of V is $n/2$. If V is non-degenerate, then by Theorem 5.9, $\dim \mathcal{A}'_I < n$ for every $I \subset \{0, 1, ..., n\}$. Since $\bigcup_{I \subseteq \{0,1,...,n\}} \mathcal{A}'_I = \mathbb{T}^n$ and $\dim \mathcal{C}' < n$, Proposition 5.5 gives that for almost every $\theta \in \mathbb{T}^n$, there is a unique $\theta' \equiv \theta$ such that $\theta \in \mathcal{A}'$. We get the following result.

Theorem 5.14. *If V is non-degenerate, the volume of $\mathcal{A}'_V$ equals π^n.*

For the rest of the chapter, we will concentrate on the simplest case of a "middle-dimensional" space that is not a hyperplane and a line at the same time; that is, we let V be a complex plane in $\mathbb{C}^4$.

Assume that the defining polynomials for V are $a_0 + \sum_{j=1}^4 a_j z_j = 0$, $b_0 + \sum_{j=1}^4 b_j z_j = 0$. Then, V satisfy the latter condition in Theorem 5.13 if and only if either a column of C is zero, or three columns of C gives a matrix or rank one. In the first case, V equals some of its initial ideals and hence is a line V' in $\mathbb{C}^3$ stretched out along the axes L corresponding to the coordinate with coefficients zero, and $\mathcal{A}'$ equals the coamoeba of V' stretched out along the corresponding axes in $\mathbb{R}^4$ with each point $\theta \in \mathcal{A}'$ corresponding to a real line in V' through L. Hence, clearly $\mathcal{A}' = \mathcal{C}'$ with $\dim \mathcal{A}' = 3$.

In the second case, we can without loss of generality assume that

$$1 + \sum_{j=1}^{4} a_j z_j = 0, \quad 1 + b_4 z_4 = 0$$

Then z_4 is fixed in V while the other variables only depend on one equation so V is a hypersurface in $\mathbb{C}^3$ embedded in $\mathbb{C}^4$. Hence, $\mathcal{A}'$ is the coamoeba of a hypersurface in $\mathbb{R}^3$ with $G(\theta)$ being a curve in $\mathcal{A}$ for every θ, embedded in $\mathbb{R}^4$ and again, $\mathcal{A}' = \mathcal{C}'$ and $\dim \mathcal{A}' = 3$.

Consider the parametrization

$$(s,t) \mapsto (s, t, c_3 + d_3 s + e_3 t, c_4 + d_4 s + e_4 t)$$

of V, $(s,t) \in \mathbb{C}^2$.

Definition 13. The plane V is *real* if

$$(\frac{c_j}{d_j} : \frac{c_k}{d_k}), (\frac{c_j}{e_j} : \frac{c_k}{e_k}), (\frac{d_j}{e_j} : \frac{d_k}{e_k}) \in P\mathbb{R} \ \forall j,k = 3,4,...,n \tag{5.4}$$

Note that it is enough to assume that two of the three projective points for fixed j and k are real.

We compare with the definition in Chapter 2 of a real line and see that every line on a real plane is not real. For example, if $n = 4$, $c_3 = d_3 = 1$, $e_3 = 2$ and $c_4 = d_4 = e_4 = i$, then V contains the line

$$u \mapsto (u, iu, 1 + (1+2i)u, i + (-1+i)u)$$

Proposition 5.15. *If V is a real plane in $\mathbb{C}^4$, then for every $x \in \mathcal{A}\backslash\mathcal{C}$, the number of points in $G^{-1}(x)$ is 4.*

Proof. Write z_j and coefficients in the parametrization on polar form:

$$c_j = C_j e^{\gamma_j i}, \quad d_j = D_j e^{\delta_j i}, \quad e_j = E_j^{\epsilon_j i}, \quad z_j = r_j e^{\theta_j i}$$

Then by some calculation we get

$$\begin{aligned} r_j^2 = \operatorname{Re}^2(c_j + d_j s + e_j t) + \operatorname{Im}^2(c_j + d_j s + e_j t) = \\ C_j^2 + D_j^2 r_1^2 + E_j^2 r_2^2 + 2C_j D_j r_1 \cos(-\gamma_j + \delta_j + \theta_1) + \\ 2C_j E_j r_2 \cos(-\gamma_j + \epsilon_j + \theta_2) + 2D_j E_j r_1 r_2 \cos(\delta_j - \epsilon_j + \theta_1 - \theta_2) \end{aligned} \tag{5.5}$$

Since V is real, $\delta_j - \gamma_j$, $\epsilon_j - \gamma_j$ and $\delta_j - \epsilon_j$ do not change with j. Set

$$\xi_1 = \cos(\delta_j - \gamma_j + \theta_1)$$

$$\xi_2 = \cos(\epsilon_j - \gamma_j + \theta_2)$$

$$\xi_3 = \cos(\delta_j - \epsilon_j + \theta_1 - \theta_2)$$

Then for $r = (r_1, ..., r_4)$ fixed, (5.5) determines a linear equation system in ξ_1, ξ_2, ξ_3 with 2 equations, and hence a line or a plane L_r. Furthermore, ξ_1, ξ_2, ξ_3 must satisfy

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = 1 + 2\xi_1\xi_2\xi_3, \quad |\xi_1|, |\xi_2|, |\xi_3| \leq 1 \tag{5.6}$$

This describes the boundary of a convex region E, an "inflated" tetrahedron, see Fig. 6. Now, the intersection of E and L determines the points $\theta \in \mathcal{A}'$ for which $G(\theta) = \operatorname{Log} r$. If L is a tangent to E, then there is an r' in every neighborhood of r such that $L_{r'} \cap E = \emptyset$. Hence $\operatorname{Log} r \in \partial\mathcal{A} \subseteq \mathcal{C}$. If L is a plane intersecting E, then clearly $\theta \in \mathcal{C}'$. Otherwise either $L \cap E = \emptyset$ or it consists of two points. Since every point $(\xi_1, \xi_2, \xi_3) \in E$ corresponds to the pair of points $\pm(\arccos \xi_1, \arccos \xi_2) \in \mathcal{A}'$, the theorem follows. $\square$

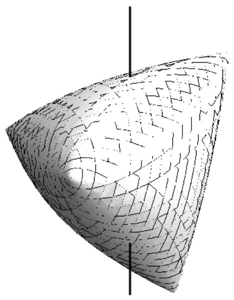


Figure 6: The intersection of the region given by (5.5) and a line is generically empty or consist of two points.

Theorem 5.16. *If V is a real, non-degenerate plane in $\mathbb{C}^4$, then the volume of $\mathcal{A}_V$ is $\pi^4/4$.*

Proof. By Theorem 5.14, the volume of $\mathcal{A}'_V$ is π^4. Hence the result follows from Propositions 5.8 and 5.15. $\square$

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