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On eigenvalues of the Schrödinger operator with a complex-valued polynomial potential

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On eigenvalues of the Schrödinger equation

Per Alexandersson

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Abstract

In this thesis, we generalize a recent result of A. Eremenko and A. Gabrielov on irreducibility of the spectral discriminant for the Schroedinger equation with quartic potentials.

In the first paper, we consider the eigenvalue problem with a complex-valued polynomial potential of arbitrary degree d and show that the spectral determinant of this problem is connected and irreducible. In other words, every eigenvalue can be reached from any other by analytic continuation. We also prove connectedness of the parameter spaces of the potentials that admit eigenfunctions satisfying $k > 2$ boundary conditions, except for the case d is even and $k = d/2$. In the latter case, connected components of the parameter space are distinguished by the number of zeros of the eigenfunctions.

In the second paper, we only consider even polynomial potentials, and show that the spectral determinant for the eigenvalue problem consists of two irreducible components. A similar result to that of paper I is proved for k boundary conditions.

Acknowledgments

Thanks to Boris Shapiro for introducing me to this fascinating subject. Many thanks to Andrei Gabrielov, for the collaboration with the first paper and the great help with the difficult parts of the second paper, and the numerous suggestions for improvements. I would also thank the Purdue University, where most of the thesis was written.

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Contents

INTRODUCTION TO THESIS

1. THE BACKGROUND IN QUANTUM MECHANICS

The one-dimensional oscillator

$$\left[-\hbar^2 \frac{d^2}{dz^2} + V(x)\right] Y(x) = EY(x) \tag{1}$$

is of great interest in modern quantum mechanics. It describes the connection between a wave function Y, the potential V and the energy state E. In particular, it describes the different types of waves in one dimension that might occur, and their associated energy. For example, the case with zero potential yields the sine, cosine and exponential functions as solutions.

The most studied case is when the potential V is a general fourth degree polynomial, in which case (1) is called *a quartic oscillator*. The associated boundary condition is $\lim_{x\to\pm\infty} Y(x) = 0$.

One is interested in the different energy levels, E that appears as eigenvalues to the wave function Y, under the boundary conditions. A normalization of (1) puts the equation in the following form:

$$\left[-\frac{d^2}{dz^2} + P(z)\right] y = \lambda y \tag{2}$$

Given a *monic polynomial* potential $P(z)$ of degree d, it is well-known that any solution y of (2) is either increasing or decreasing in each open sector S_j, where

$$S_j = \{z \in \mathbf{C} \setminus \{0\} : |\arg z - 2\pi j/(d+2)| < \pi/(d+2)\}, \quad j = 0, 1, 2, \dots, d+1.$$

That is, for a given j, $y \to 0$ or $y \to \infty$ along each ray in S_j, starting at the origin. Moreover, y cannot decrease in two neighboring sectors. These sectors are called the Stokes sectors of (2).

This circumstance gives rise to the more general boundary condition that

$$y \to 0 \text{ in } S_{j_1}, S_{j_2}, \dots, S_{j_k} \tag{3}$$

for non-adjacent sectors, that is $|j_p - j_q| > 1$ for all $p \neq q$.

If the coefficients of the polynomial potential $P_\alpha(z)$ depends on a (multi)parameter $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{d-1})$,

$$P_\alpha(z) = z^d + \alpha_{d-1} z^{d-1} + \alpha_{d-2} z^{d-2} + \cdots + \alpha_1 z$$

then there exists an entire function, F, with the property that

$$F(\alpha, \lambda) = F(\alpha_1, \alpha_2, \dots, \alpha_n, \lambda) = 0$$

if and only if (2) has a solution satisfyin (3), see [Sib75]. This F is called the spectral determinant of the family P_α.

2. THE HISTORY OF THE PROBLEM

Many properties of $F(\alpha, \lambda)$ were studied over the years. In particular, the problem of determine the irreducibility of F was considered for the first time for the Mathieu equation in 1954, [MS54]. In that particular problem, the potential is not a polynomial. However, the history starts much earlier than that.

In the 1930's, Nevanlinna studied [Nev32, Nev53] meromorphic functions f, with polynomial Schwartzian derivative, S_f. The connection to (2), is that if $f = y_1/y_2$ is a quotient of two linearly independent solutions of (2), then f satisfy $S_f = -2(P - \lambda)$, and the converse holds. The properties of f is therefore closely related to properties of y.

Bender and Wu studied analytic continuation of λ into the complex λ-plane, [BW69] in 1969. They used the so called WKB method, to analyse the asymptotics of y. The WKB method is a main tool for analysing this kind of questions. For the equation

$$\left[-\frac{d^2}{dz^2} + (\beta z^4 + z^2)\right] y = \lambda y \tag{4}$$

one have a discrete spectrum of real eigenvalues $\lambda_1 < \lambda_2 \cdots \to \infty$, if $\beta > 0$. Computer experiments by Bender and Wu indicated that for m, n of the same parity, there is a path in the complex $\beta-$plane such that λ_m is interchanged with λ_n. This corresponds to a spectral determinant with two irreducible connected components, and an intruiging conjecture was now formed.

Moreover, they examined the Riemann surfaces of the functions λ_n in the β-plane. They discovered that the ramification points of the surfaces are algebraic in $\mathbf{C} \setminus \{0\}$, and that there is no analytic continuation of any λ_n to 0, since the ramification points accumulate in a way that makes this impossible.

The book [Sib75], by Sibuya from 1975 is a good reference for different properties related to (2). Among other things, Sibuya interpreted a result by Nevanlinna, and shows that given $d + 2$ points $c_0, c_1, \ldots, c_{d+1}$ points on the Riemann sphere, such that $c_j \neq c_{j+1}$ for $j = 0, \ldots, d + 1$ (index considered modulo $d + 2$), and with at least three distinct c_j, then there exists a polynomial potential P of degree d, such that (2) has two linearly independent solutions y_1, y_2 such that $y_1/y_2 \to c_j$ as $z \to \infty$ in the Stokes sector S_j. (This theorem explains the condition that $|j_p - j_q| > 1$ in (3).)

The next milestone in the history of the quartic oscillator is the paper by Voros [Vor83]. In the paper from 1983, Voros gives a long review of the current status of the various problems related to this subject.

In 1997, Delabaere and Pham published the paper [DP97]. The paper investigates the ramification properties of λ, as the coeficcients in a general fourth degree polynomial are varied in the complex domain. Their approach yielded a more robust algorithm for computing the ramification points, and their results agreed with the conjecture posed by Bender and Wu 30 years earlier.

With the help of the theory developed by Nevanlinna, and later Sibuya, a rigorous proof of the hypothesis posed by Bender and Wu was finally given by Gabrielov and Eremenko in 2008 in [EG09a]. The technique they used turned out to be fruitful, in [EG09b], a variety of spectral determinants can be studied with this method.

3. Some other families of equations

Another reason for extending (1) into the complex domain is due to Zinn-Justin. He considered the PT-symmetric cubic, given by $V(z) = iz^3$ and with wthe boundary condition $y \to 0$ as $z \to \pm\infty$, and conjectured that the spectrum for this problem is real. This was later proved by Dorey and Tateo.

A generalized case with $V(z) = iz^3 + i\alpha z$ was studied by Delabaere and Trinh, [DT00], and they found a branch point structure similar to that of Bender and Wu.

It is proved [EG09b] that the spectral determinant $F(\alpha, \lambda)$ for this problem is irreducible. In the same paper, similar results have been found for other classes of potentials using similar methods as in [EG09a].

4. Results of the thesis

4.1. **Paper I.** In the first paper, we generalize the method in [EG09a], and tread the Schroedinger equation (2) with a polynomial potential of arbitrary degree.

By using the theory from Nevanlinna and Sibuya, the problem is reformulated in a graph-theoretical setting via cell decompositions.

We show that every eigenvalue to the problem (2), (3) can be reached from any other by analytic continuation of the coefficients of the potential, unless the degree of P is even and $y \to 0$ in every other Stokes sector.

This implies that (2), (3) has an irreducible spectral determinant, if $y \to \infty$ in S_j and S_{j+1} for some j in (3).

4.2. **Paper II.** Using the results from the first paper, we may treat the case of an even polynomial potential in a similar fashion as a general polynomial.

The result is another generalization of [EG09a], and we show that (2), (3) has a spectral determinant consisting of two irreducible connected components, if $y \to \infty$ in S_j and S_{j+1} for some j in (3).

5. Further directions of research

The method used in Paper I, II could be generalized to graphs with a higher rotational symmetry, in which case the spectral determinant is irreducible. However, there is no obvious family of polynomials that gives rise to this kind of graphs, in contrast to the connection between even polynomial potentials and centrally symmetric graphs.

There seems to be a strong connection between the branch points of the spectral determinant, and certain graphs presented in paper I, II.

Also, a general theory about the connection between coefficients of P and the set of asymptotic values of f is still in its cradle.

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ON EIGENVALUES OF THE SCHRÖDINGER OPERATOR WITH A COMPLEX-VALUED POLYNOMIAL POTENTIAL

PER ALEXANDERSSON AND ANDREI GABRIELOV

ABSTRACT. In this paper, we generalize a recent result of A. Eremenko and A. Gabrielov on irreducibility of the spectral discriminant for the Schrödinger equation with quartic potentials. We consider the eigenvalue problem with a complex-valued polynomial potential of arbitrary degree d and show that the spectral determinant of this problem is connected and irreducible. In other words, every eigenvalue can be reached from any other by analytic continuation.

We also prove connectedness of the parameter spaces of the potentials that admit eigenfunctions satisfying $k > 2$ boundary conditions, except for the case d is even and $k = d/2$. In the latter case, connected components of the parameter space are distinguished by the number of zeros of the eigenfunctions.

CONTENTS

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1. INTRODUCTION

In this paper we study analytic continuation of eigenvalues of the Schröodinger operator with a complex-valued polynomial potential. In other words, we are interested in the analytic continuation of eigenvalues $\lambda = \lambda(\alpha)$ of the boundary value problem for the differential equation

$$-y'' + P_\alpha(z)y = \lambda y, \tag{1}$$

where

$$P_\alpha(z) = z^d + \alpha_{d-1}z^{d-1} + \cdots + \alpha_1 z \text{ where } \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{d-1}),\ d \geq 2.$$

The boundrary conditions are given by either (2) or (3) below. Namely, set $n = d + 2$ and divide the plane into n disjoint open sectors of the form:

$$S_j = \{z \in \mathbf{C} \setminus \{0\} : |\arg z - 2\pi j/n| < \pi/n\}, \quad j = 0, 1, 2, \ldots, n - 1.$$

These sectors are called the *Stokes sectors* of the equation (1). It is well-known that any solution y of (1) is either increasing or decreasing in each open Stokes sector S_j, i.e. $y(z) \to 0$ or $y(z) \to \infty$ as $z \to \infty$ along each ray from the origin in S_j, see [Sib75]. In the first case, we say that y is *subdominant*, and in the second case, *dominant* in S_j. We impose the boundary conditions that for two *non-adjacent* sectors S_j and S_k, i.e. for $j \neq k \pm 1 \mod n$:

$$y \text{ is subdominant in } S_j \text{ and } S_k. \tag{2}$$

For example, $y(\infty) = y(-\infty) = 0$ on the real axis, the boundary conditions usually imposed in physics for even potentials, correspond to y being subdominant in S_0 and $S_{n/2}$. The existence of analytic continuation is a classical fact, see e.g. references in [EG09a].

The main results of this paper are:

Theorem 1. *For any eigenvalue $\lambda_k(\alpha)$ of equation* (1) *and boundary condition* (2)*, there is an analytic continuation in the α-plane to any other eigenvalue $\lambda_m(\alpha)$.*

We also prove some stronger results in the case where y is subdominant in more than two sectors:

Theorem 2. *Given $k < n/2$ non-adjacent Stokes sectors $S_{j_1}, \ldots, S_{j_k}$, the set of all $(\alpha, \lambda) \in \mathbf{C}^d$ for which the equation $-y'' + (P_\alpha - \lambda)y = 0$ has a solution with*

$$y \text{ subdominant in } S_{j_1}, \ldots, S_{j_k} \tag{3}$$

is connected.

Theorem 3. *For n even and $k = n/2$, the set of all $(\alpha, \lambda) \in \mathbf{C}^d$ for which $-y'' + (P_\alpha - \lambda)y = 0$ has a solution with*

$$y \text{ subdominant in } S_0, S_2, \ldots, S_{n-2} \tag{4}$$

is disconnected. Additionally, the solutions to (1), (3) *have finitely many zeros, and the set of α corresponding to given number of zeros is a connected component of the former set.*

The method we use is based on the "Nevanlinna parameterization" of the spectral locus introduced in [EG09a] (see also [EG09b] and [EG10]).

1.1. **Some previous results.** In the foundational paper [BW69], C. Bender and T. Wu studied analytic continuation of λ in the complex β-plane for the problem

$$-y'' + (\beta z^4 + z^2)y = \lambda y, \quad y(-\infty) = y(\infty) = 0.$$

Based on numerical computations, they conjectured for the first time the connectivity of the sets of odd and even eigenvalues. This paper generated considerable further research in both physics and mathematics literature. See e.g. [Sim70] for early mathematically rigorous results in this direction.

In [EG09a], which is the motivation of the present paper, the even quartic potential $P_a(z) = z^4 + az^2$ and the boundary value problem

$$-y'' + (z^4 + az^2)y = \lambda_a y, \quad y(\infty) = y(-\infty) = 0$$

was considered. It is known that the problem has discrete real spectrum for real a, with $\lambda_1 < \lambda_2 < \cdots \to +\infty$. There are two families of eigenvalues, those with even index and those with odd. The main result of [EG09a] is that if λ_j and λ_k are two eigenvalues in the same family, then λ_k can be obtained from λ_j by analytic continuation in the complex α-plane. Similar results have been obtained for other potentials, such as the PT-symmetric cubic, where $P_\alpha(z) = (iz^3 + i\alpha z)$, with $y(z) \to 0$, as $z \to \pm\infty$ on the real line. See for example [EG09b].

Remark 4. *After this project was finished, the authors found out that a result similar to Theorem 2 was proved in a hardly ever quoted Ph.D thesis,* [Hab52], *page 36. On the other hand, this result is formulated in the setting of Nevanlinna theory, with no connection to properties of* (1).

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2. PRELIMINARIES

First, we recall some basic notions from Nevanlinna theory.

Lemma 5 (see [Sib75]). *Each solution $y \neq 0$ of* (1) *is an entire function, and the ratio $f = y/y_1$ of any two linearly independent solutions of* (1) *is a meromorphic function, with the following properties:*

(I) For any j, there is a solution y of (1) *subdominant in the Stokes sector S_j. This solution is unique, up to multiplication by a non-zero constant,*

(II) For any Stokes sector S_j, we have $f(z) \to w \in \bar{\mathbf{C}}$ as $z \to \infty$ along any ray in S_j. This value w is called the asymptotic value *of f in S_j.*

(III) For any j, the asymptotic values of f in S_j and S_{j+1} (index taken modulo n) are different. The function f has at least 3 distinct asymptotic values.

(IV) The asymptotic value of f is zero in S_j if and only if y is subdominant in S_j. It is convenient to call such sector subdominant *as well. Note that the boundary conditions in* (2) *imply that the two sectors S_j and S_k are subdominant for f when y is an eigenfunction of* (1), (2).

(V) *f does not have critical points, hence $f : \mathbf{C} \to \bar{\mathbf{C}}$ is unramified outside the asymptotic values.*

(VI) *The Schwartzian derivative S_f of f given by*

$$S_f = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

equals $-2(P_\alpha - \lambda)$. Therefore one can recover P_α and λ from f.

From now on, f always denotes the ratio of two linearly independent solutions of (1), with y being an eigenfunction of the boundary value problem (1), with conditions (2), (3) or (4).

2.1. **Cell decompositions.** Set $n = d + 2$, $d = \deg P$ where P is our polynomial potential and assume that all non-zero asymptotic values of f are distinct and finite. Let w_j be the asymptotic values of f, ordered arbitrarily with the only restriction that $w_j = 0$ if and only if S_j is subdominant. For example, one can denote by w_j the asymptotic value in the Stokes sector S_j. We will later need different orders of the non-zero asymptotic values, see section 2.3.

Consider the cell decomposition Ψ_0 of $\bar{\mathbf{C}}_w$ shown in Fig. 1a. It consists of closed directed loops γ_j starting and ending at ∞, where the index is considered mod n, and γ_j is defined only if $w_j \neq 0$. The loops γ_j only intersect at ∞ and have no self-intersection other than ∞. Each loop γ_j contains a single non-zero asymptotic value w_j of f. For example, the boundary condition $y \to 0$ as $z \to \pm\infty$ for $z \in \mathbf{R}$ for even n implies that $w_0 = w_{n/2} = 0$, so there are no loops γ_0 and $\gamma_{n/2}$. We have a natural cyclic order of the asymptotic values, namely the order in which a small circle around ∞ counterclockwise intersects the associated loops γ_j, see Fig. 1a.

We use the same index for the asymptotic values and the loops, which motivates the following notation:

$j_+ = j + k$ where $k \in \{1, 2\}$ is the smallest integer such that $w_{j+k} \neq 0$.

Thus, γ_{j_+} is the loop around the next to w_j (in the cyclic order mod n) non-zero asymptotic value. Similarly, γ_{j_-} is the loop around the previous non-zero asymptotic value.

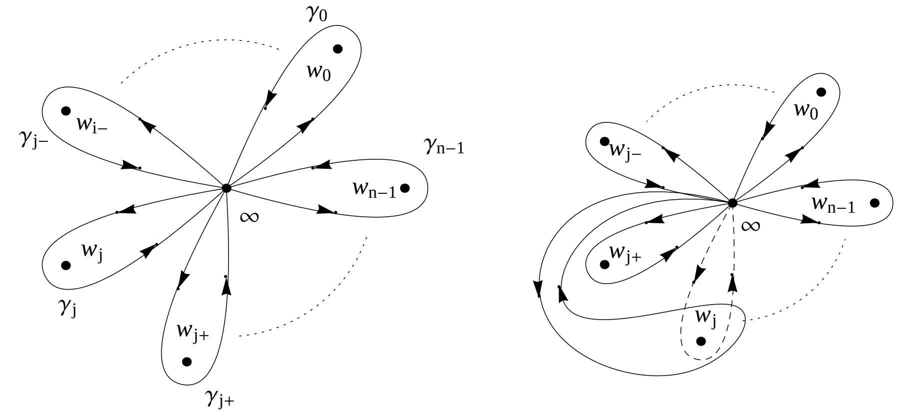
2.2. **From cell decompositions to graphs.** We may simplify our work with cell decompositions with the help of the following:

Lemma 6 (See Section 3 [EG09a]). *Given Ψ_0 as in Fig. 1a, one has the following properties:*

(a) *The preimage $\Phi_0 = f^{-1}(\Psi_0)$ gives a cell decomposition of the plane $\mathbf{C}_z$. Its vertices are the poles of f, and the edges are preimages of the loops γ_j. These edges are labeled by j, and are called j-edges.*

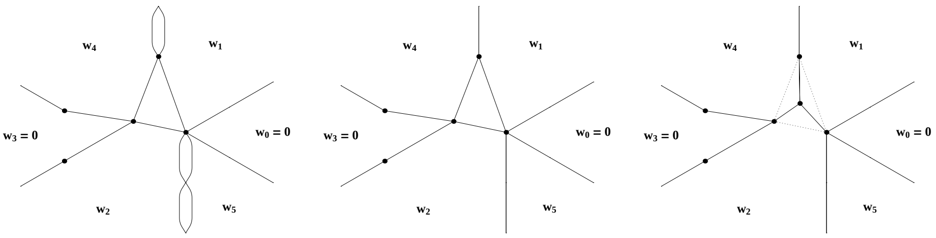
(b) *The edges of Φ_0 are directed, their orientation is induced from the orientation of the loops γ_j. Removing all loops of Φ_0, we obtain an infinite, directed planar graph Γ, without loops.*

(c) *Vertices of Γ are poles of f, each bounded connected component of $\mathbf{C} \setminus \Gamma$ contains one simple zero of f, and each zero of f belongs to one such bounded connected component.*



(a) Ψ_0 (b) $A_j(\Psi_0)$.

Figure 1: Permuting w_j and w_{j_+} in Ψ_0.



Γ T_Γ T^*_Γ

Figure 2: The correspondence between Γ, T_Γ and T^*_Γ.

(d) *There are at most two edges of* Γ *connecting any two of its vertices. Replacing each such pair of edges with a single undirected edge and making all other edges undirected, we obtain an undirected graph* T_Γ.
(e) T_Γ *has no loops or multiple edges, and the transformation from* Φ_0 *to* T_Γ *can be uniquely reversed.*

An example of the transformation from Γ to T_Γ is presented in Fig. 2.

A *junction* is a vertex of Γ (and of T_Γ) at which the degree of T_Γ is at least 3. From now on, Γ refers to both the directed graph without loops and the associated cell decomposition Φ_0.

2.3. **The standard order.** For a potential of degree d, the graph Γ has $d+2 = n$ infinite branches and n unbounded faces corresponding to the Stokes sectors. We defined earlier the ordering $w_0, w_1, \ldots, w_{n-1}$ of the asymptotic values of f.

If each w_j is the asymptotic value in the sector S_j, we say that the asymptotic values have *the standard order* and the corresponding cell decomposition Γ is a *standard graph*.

Lemma 7 (See Prop 6. [EG09a])**.** *If a cell decomposition Γ is a standard graph, the corresponding undirected graph T_Γ is a tree.*

This property is essential in the present paper, and we classify cell decompositions of this type by describing the associated trees.

Below we define the action of the braid group that permute non-zero asymptotic values of Ψ_0. This induces the corresponding action on graphs. Each unbounded face of Γ (and T_Γ) will be labeled by the asymptotic value in the corresponding Stokes sector. For example, labeling an unbounded face corresponding to S_k with w_j or just with the index j, we indicate that w_j is the asymptotic value in S_k.

From the definition of the loops γ_j, a face corresponding to a dominant sector has the same label as any edge bounding that face. The label in a face corresponding to a subdominant sector S_k is always k, since the actions defined below only permute non-zero asymptotic values. We say that an unbounded face of Γ is (sub)dominant if the corresponding Stokes sector is (sub)dominant.

For example, in Fig. 2, the Stokes sectors S_0 and S_3 are subdominant since the corresponding faces have label 0. We do not have the standard order for Γ, since w_2 is the asymptotic value for S_4, and w_4 is the asymptotic value for S_2. The associated graph T_Γ is not a tree.

2.4. **Properties of graphs and their face labeling.**

Lemma 8 (see [EG09a])**.** *The following holds:*

- *(I) Two bounded faces of Γ cannot have a common edge, since a j-edge is always at the boundary of an unbounded face labeled j.*
- *(II) The edges of a bounded face of a graph Γ are directed clockwise, and their labels increase in that order. Therefore, a bounded face of T_Γ can only appear if the order of w_j is non-standard.*

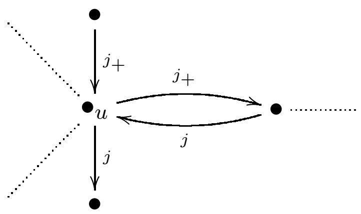
  (As an example, the bounded face in Fig. 2 has the labels $1, 2, 4$ (clockwise) of its boundary edges.)
- *(III) Each label appears at most once in the boundary of any bounded face of Γ.*
- *(IV) Unbounded faces of Γ adjacent to its junction u always have the labels cyclically increasing counterclockwise around u.*
- *(V) To each graph T_Γ, we associate a tree by inserting a new vertex inside each of its bounded faces, connecting it to the vertices of the bounded face and removing the boundrary edges of the original face. Thus we may associate a tree T^*_Γ with* any *cell decomposition, not necessarily with standard order, as in Fig. 2(c). The order of w_j above together with this tree uniquely determines Γ. This is done using the two properties above.*
- *(VI) The boundary of a dominant face labeled j consists of infinitely many directed j-edges, oriented counterclockwise around the face.*
- *(VII) If $w_j = 0$ there are no j-edges.*
- *(VIII) Each vertex of Γ has even degree, since each vertex in $\Phi_0 = f^{-1}(\Psi_0)$ has even degree, and removing loops to obtain Γ preserves this property.*

Following the direction of the j-edges, the first vertex that is connected to an edge labeled j_+ is the vertex where the j-edges and the j_+-edges *meet*. The last such vertex is where they *separate*. These vertices, if they exist, must be junctions.

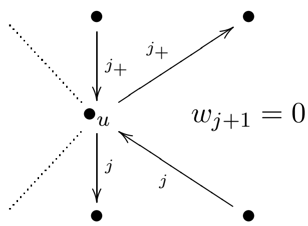
Definition 9. *Let* Γ *be a standard graph, and let* $j \in \Gamma$ *be a junction where the* j*-edges and* j_+*-edges separate. Such junction is called a* j-junction.

There can be at most one j-junction in Γ, the existence of two or more such junctions would violate property (III) of the face labeling. However, the same junction can be a j-junction for different values of j.

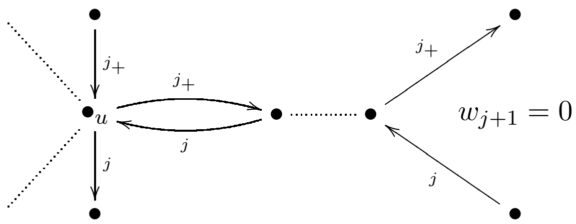
There are three different types of j-junctions, see Fig. 3.



(a) I-structure.



(b) V-structure.



(c) Y-structure.

Figure 3: Different types of j-junctions.

Case (a) only appears when $w_{j+1} \neq 0$. Cases (b) and (c) can only appear when $w_{j+1} = 0$. In (c), the j-edges and j_+-edges meet and separate at different junctions, while in (b), this happens at the same junction.

Definition 10. *Let* Γ *be a standard graph with a* j*-junction* u*. A* structure *at the* j*-junction is the subgraph* Ξ *of* Γ *consisting of the following elements:*

- *The edges labeled* j *that appear before* u *following the* j*-edges.*
- *The edges labeled* j_+ *that appear after* u *following the* j_+*-edges.*
- *All vertices the above edges are connected to.*

If u *is as in Fig. 3a,* Ξ *is called an* I-structure at the j-junction*. If* u *is as in Fig. 3b,* Ξ *is called a* V-structure at the j-junction*. If* u *is as in Fig. 3c,* Ξ *is called a* Y-structure at the j-junction.

Since there can be at most one j-junction, there can be at most one structure at the j-junction.

A graph Γ shown in Fig. 4 has one (dotted) I-structure at the 1-junction v, one (dotted) I-structure at the 4-junction u, one (dashed) V-structure at the 2-junction v and one (dotdashed) Y-structure at the 5-junction u.

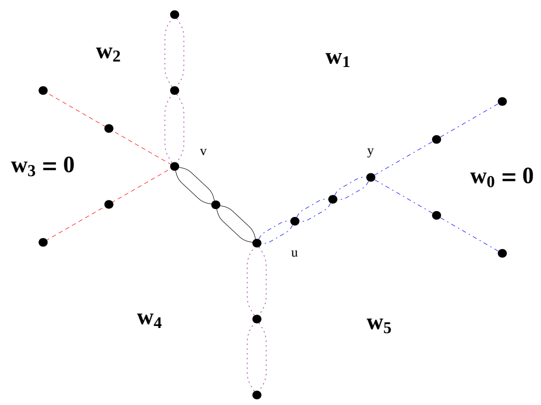


Figure 4: Graph Γ with (dotted) I-structures, a (dashed) Y-structure and a (dotdashed) Y-structure.

Note that the Y-structure is the only kind of structure that contains an additional junction. We refer to such junctions as *Y-junctions*. For example, the junction marked y in Fig. 4 is a Y-junction.

2.5. **Describing trees and junctions.** Let Γ be a graph with n branches, and Λ be the associated tree with all non-junction vertices removed. The dual graph $\hat{\Lambda}$ of Λ, is an n-gon where some non-intersecting chords are present. The junctions of Λ is in one-to-one correspondence with faces of $\hat{\Lambda}$ and vice versa. Two vertices are connected with an edge in $\hat{\Lambda}$ if and only if the corresponding faces are adjacent in Λ.

The extra condition that subdominant faces do not share an edge, implies that there are no chords connecting vertices in $\hat{\Lambda}$ corresponding to subdominant faces. For trees without this condition, we have the following lemma:

Lemma 11. *The number of $n+1$-gons with non-intersecting chords is equal to the number of bracketings of a string with n letters, such that each bracket pair contains at least two symbols.*

Proof. See Theorem 1 in [SS00]. □

The sequence $s(n)$ of bracketings of a string with $n+1$ symbols are called the small Schröder numbers, see [SS00]. The first entries are $s(n)_{n\geq 0} = 1, 1, 3, 11, 45, 197, \dots$.

The condition that chords should not connect vertices corresponding to subdominant faces, translates into a condition on the first and last symbol in some bracket pair.

3. Actions on graphs

3.1. **Definitions.** Let us now return to the cell decomposition Ψ_0 in Fig. 1a. Let w_j be a non-zero asymptotic value of f. Choose non-intersecting paths $\beta_j(t)$ and $\beta_{j_+}(t)$ in $\bar{\mathbf{C}}_w$ with $\beta_j(0) = w_j,\ \beta_j(1) = w_{j_+}$ and $\beta_{j_+}(0) = w_{j_+}$,

$\beta_{j_+}(1) = w_j$ so that they do not intersect γ_k for $k \neq j, j_+$ and such that the union of these paths is a simple contractible loop oriented counterclockwise. These paths define a continuous deformation of the loops γ_j and γ_{j_+} such that the two deformed loops contain $\beta_j(t)$ and $\beta_{j_+}(t)$, respectively, and do not intersect any other loops during the deformation (except at ∞). We denote the action on Ψ_0 given by $\beta_j(t)$ and $\beta_{j_+}(t)$ by A_j. Basic properties of the fundamental group of a punctured plane, allows one to express the new loops in terms of the old ones:

$$A_j(\gamma_k) = \begin{cases} \gamma_j \gamma_{j_+} \gamma_j^{-1} \text{ if } k = j \\ \gamma_j \text{ if } k = j_+ \\ \gamma_k \text{ otherwise} \end{cases}, \quad A_j^{-1}(\gamma_k) = \begin{cases} \gamma_{j_+} \text{ if } k = j \\ \gamma_{j_+}^{-1} \gamma_j \gamma_{j_+} \text{ if } k = j_+ \\ \gamma_k \text{ otherwise} \end{cases}$$

Let f_t be a deformation of f. Since a continuous deformation does not change the graph, the deformed graph corresponding to $f_1^{-1}(A_j(\Psi_0))$ is the same as Γ. Let Γ' be this deformed graph with labels j and j_+ exchanged. Then the j-edges of Γ' are $f_1^{-1}(A_j(\gamma_{j_+})) = f_1^{-1}(\gamma_j)$, hence they are the same as the j-edges of $A_j(\Gamma)$. The j_+-edges of Γ' are $f_1^{-1}(A_j(\gamma_j))$. Since $\gamma_{j_+} = \gamma_j^{-1} A_j(\gamma_j)\gamma_j$, (reading left to right) this means that a j_+-edge of $A_j(\Gamma)$ is obtained by moving backwards along a j-edge of Γ', then along a j_+-edge of Γ', followed by a j-edge of Γ'.

These actions, together with their inverses, generate the Hurwitz (or sphere) braid group $\mathcal{H}_m$, where m is the number of non-zero asymptotic values. For a definition of this group, see [LZ04]. The action A_j on the loops in Ψ_0 is presented in Fig. 1b.

The property (V) of the eigenfunctions implies that each A_j induces a monodromy transformation of the cell decomposition Φ_0, and of the associated directed graph Γ.

Reading the action *right to left* gives the new edges in terms of the old ones, as follows:

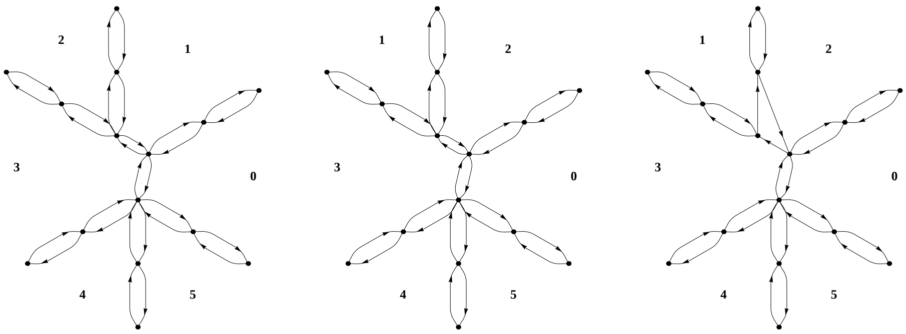
Applying A_j to Γ can be realized by first interchanging the labels j and j_+. This gives an intermediate graph Γ'. A j-edge of $A_j(\Gamma)$ starting at the vertex v ends at a vertex obtained by moving from v following first the j-edge of Γ' backwards, then the j_+-edge of Γ', and finally the j-edge of Γ'. If any of these edges does not exist, we just do not move. If we end up at the same vertex v, there is no j-edge of $A_j(\Gamma)$ starting at v. All k-edges of $A_j(\Gamma)$ for $k \neq j$ are the same as k-edges of Γ'.

An example of the action A_1 is presented in Fig. 5. Note that A_j^2 preserves the standard cyclic order.

3.2. **Properties of the actions.**

Lemma 12. *Let Γ be a standard graph with no j-junction. Then $A_j^2(\Gamma) = \Gamma$.*

Proof. Since we assume $d > 2$, lemma 8 implies that the boundaries of the faces of Γ labeled j and j_+ do not have a common vertex. From the definition of the actions in subsection 3, the graphs Γ and $A_j(\Gamma)$ are the same, except that the labels j and j_+ are permuted. Applying the same argument again gives $A_j^2(\Gamma) = \Gamma$. □



Γ Γ' $A_1(\Gamma)$

Figure 5: The action A_1. All sectors are dominant.

Theorem 13. *Let Γ be a standard graph with a j-junction u. Then $A_j^2(\Gamma) \neq \Gamma$, and the structure at the j-junction is moved one step in the direction of the j-edges under A_j^2. The inverse of A_j^2 moves the structure at the j-junction one step backwards along the j_+-edges.*

Proof. There are three cases to consider, namely I-structures, V-structures and Y-structures resp.
Case 1: The structure at the j-junction is an I-structure and Γ is as in Fig. 6a. The action A_j first permutes the asymptotic values w_j and w_{j_+}, then transforms the new j- and j_+-edges, as defined in subsection 3. The resulting graph $A_j(\Gamma)$ is shown in Fig. 6b. Applying A_j to $A_j(\Gamma)$ yields the graph shown in Fig. 6c.
Case 2: The structure at the j-junction is a V-structure and Γ is as in Fig. 7a. The graphs $A_j(\Gamma)$ and $A_j^2(\Gamma)$ are as in Fig. 7b and in Fig. 7c respectively.
Case 3: The structure at the j-junction is a Y-structure and Γ is as in Fig. 8a. The graphs $A_j(\Gamma)$ and $A_j^2(\Gamma)$ are as in Fig. 8b and in Fig. 8c respectively. The statement for A_j^{-2} is proved similarly. □

Examples of the actions are given in Appendix, Figs. 16, 17 and 18.

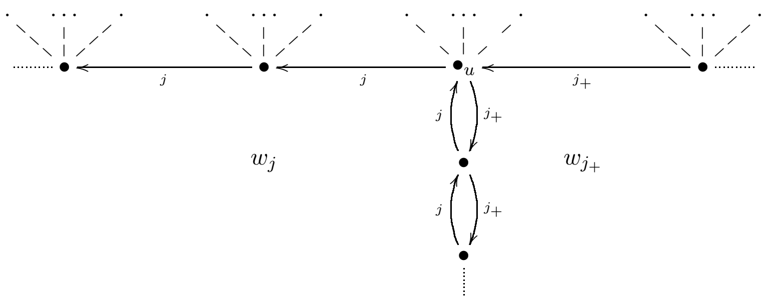
3.3. **Contraction theorems.**

Definition 14. *Let Γ be a standard graph and let u_0 be a junction of Γ. The u_0-*metric *of Γ, denoted $|\Gamma|_{u_0}$ is defined as*

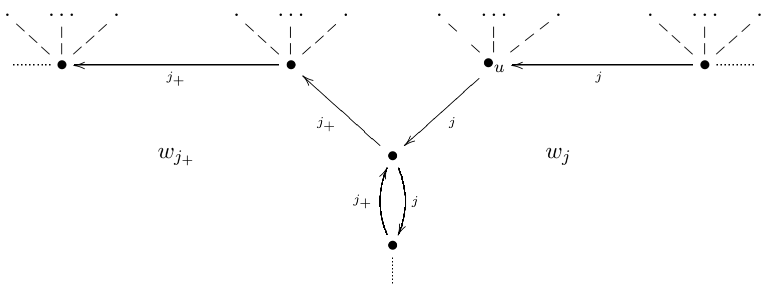
$$|\Gamma|_{u_0} = \sum_v (\deg(v) - 2)\, |v - u_0|$$

where the sum is taken over all vertices v of T_Γ. Here $deg(v)$ is the total degree of the vertex v in T_Γ and $|v - u_0|$ is the length of the shortest path from v to u_0 in T_Γ. (Note that the sum in the right hand side is finite, since only junctions make non-zero contributions.)

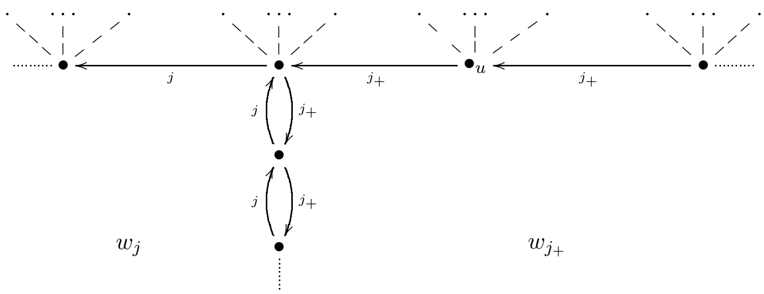
Definition 15. *A standard graph Γ is in* ivy form *if Γ is the union of the structures connected to a junction u. Such junction is called a* root junction.



(a) Graph Γ with an I-structure



(b) Graph $A_j(\Gamma)$



(c) Graph $A_j^2(\Gamma)$

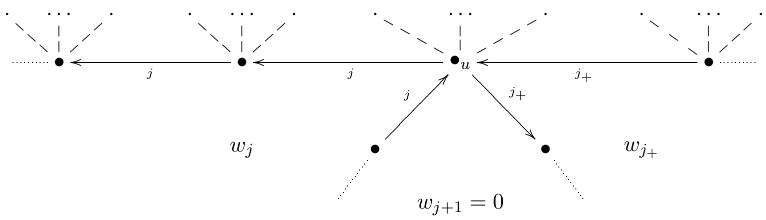
Figure 6: Case 1, moving an I-structure.

Lemma 16. *The graph Γ is in ivy form if and only if all but one of its junctions are Y-junctions.*

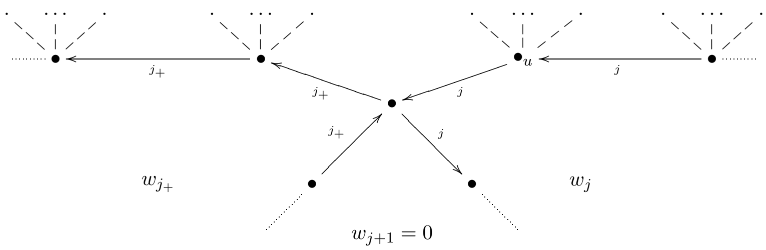
Proof. This follows from the definitions of the structures. □

Theorem 17. *Let Γ be a standard graph. Then there is a sequence of actions $A^* = A_{j_1}^{\pm 2}, A_{j_2}^{\pm 2}, \dots$, such that $A^*(\Gamma)$ is in* ivy form.

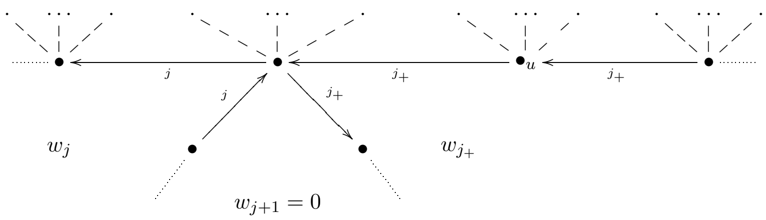
Proof. Assume that Γ is not in ivy form. Let U be the set of junctions in Γ that are not Y-junctions. Since Γ is not in ivy form, $|U| \geq 2$. Let $u_0 \neq u_1$ be two junctions in U such that $|u_0 - u_1|$ is maximal. Let p be the path from u_0 to u_1 in T_Γ. It is unique since T_Γ is a tree. Let v be the vertex immediately preceeding u_1 on the path p. The edge from v to u_1 in T_Γ is adjacent to at least one dominant face with label j such that $w_j \neq 0$. Therefore, there exists a j-edge between v and u_1 in Γ. Suppose first that this j-edge is directed



(a) Graph Γ with a V-structure



(b) Graph $A_j(\Gamma)$



(c) Graph $A_j^2(\Gamma)$

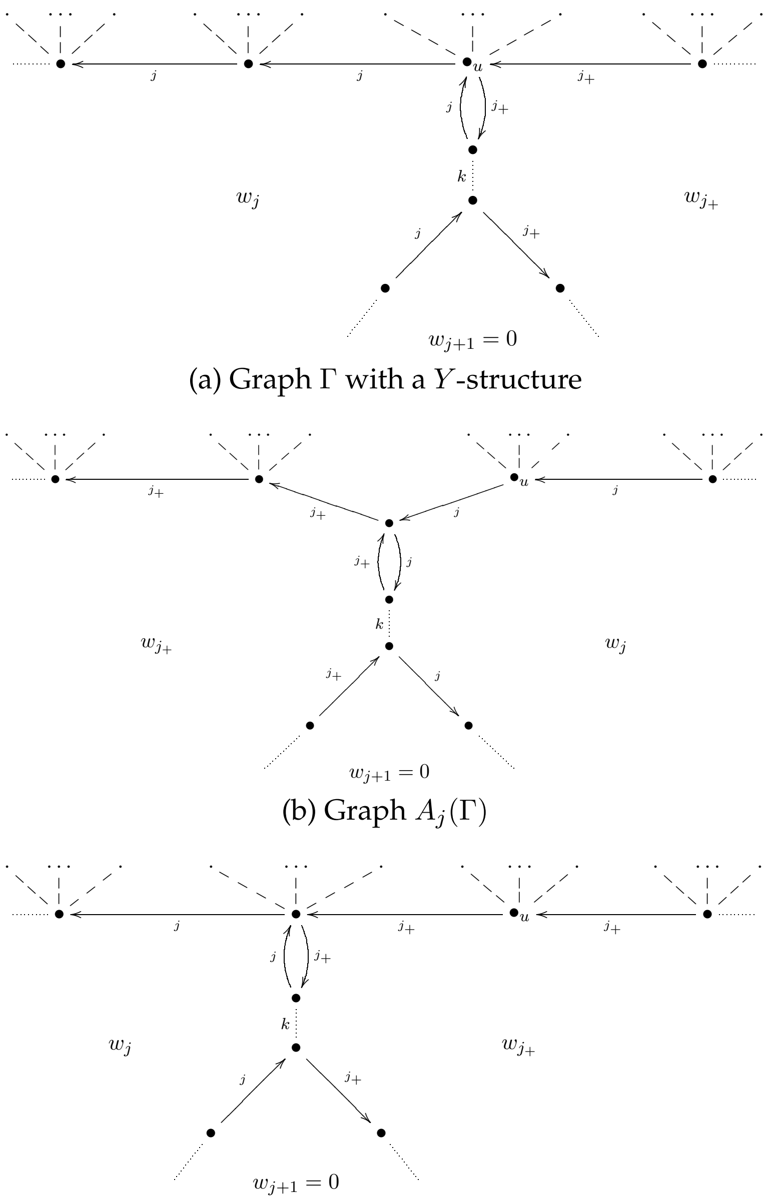
Figure 7: Case 2, moving a V-structure.

from u_1 to v. Let us show that in this case u_1 must be a j-junction, i.e., the dominant face labeled j_+ is adjacent to u_1.

Since u_1 is not a Y-junction, there is a dominant face adjacent to u_1 with a label $k \neq j, j_+$. Hence no vertices of p, except possibly u_1 may be adjacent to j_+-edges. If u_1 is not a j-junction, there are no j_+-edges adjacent to u_1. This implies that any vertex of Γ adjacent to a j_+-edge is further away from u_0 that u_1.

Let u_2 be the closest to u_1 vertex of Γ adjacent to a j_+-edge. Then u_2 should be a junction of T_Γ, since there are two j_+-edges adjacent to u_2 in Γ and at least one more vertex (on the path from u_1 to u_2) which is connected to u_2 by edges with labels other than j_+. Since u_2 is further away from u_0 than u_1 and the path p is maximal, u_2 must be a Y-junction. If the j-edges and j_+-edges would meet at u_2, u_1 would be a j-junction. Otherwise, a subdominant face labeled $j+1$ would be adjacent to both u_1 and u_2, while a subdominant face adjacent to a Y-junction cannot be adjacent to any other junctions.

Hence u_1 must be a j-junction. By Theorem 13, the action A_j^2 moves the structure at the j-junction u_1 one step closer to u_0 along the path p, decreasing $|\Gamma|_{u_0}$ at least by 1.



(a) Graph Γ with a Y-structure

(b) Graph $A_j(\Gamma)$

(c) Graph $A_j^2(\Gamma)$

Figure 8: Case 3, moving a Y-structure.

The case when the j-edge is directed from v to u_1 is treated similarly. In that case, u_1 must be a j_--junction, and the action $A_{j_-}^{-2}$ moves the structure at the j_--junction u_1 one step closer to u_0 along the path p.

We have proved that if $|U| > 1$ then $|\Gamma|_{u_0}$ can be reduced. Since it is a non-negative integer, after finitely many steps we must reach a stage where $|U| = 1$, hence the graph is in ivy form. $\square$

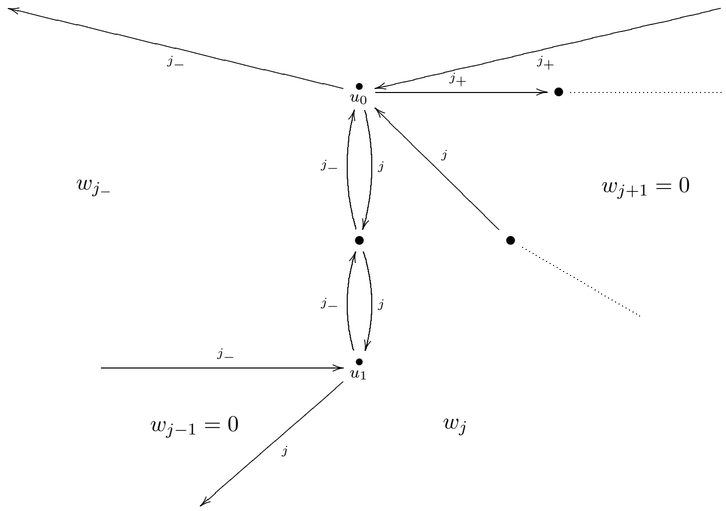


Figure 9: Adjacent Y- and V-structures.

Remark 18. *The outcome of the algorithm is in general non-unique, and might yield different final values of* $|A^*(\Gamma)|_{u_0}$.

Lemma 19. *Let* Γ *be a standard graph with a junction* u_0 *such that* u_0 *is both a* j_-*-junction and a* j*-junction. Assume that the corresponding structures are of types* Y *and* V*, in any order. Then there is a sequence of actions from the set* $\{A_j^2, A_{j_-}^2, A_j^{-2}, A_{j_-}^{-2}\}$ *that interchanges the* Y*-structure and the* V*-structure.*

Proof. We may assume that the Y- and V-structures are attached to u_0 counterclockwise around u_0, as in Fig. 9, otherwise we reverse the actions. By Theorem 13, the action A_j^{2k} moves the V-structure k steps in the direction of the j-edges. Choose k so that the V-structure is moved all the way to u_1, as in Fig. 10. Then u_1 becomes both a j_--junction and j-junction, with two V-structures attached. Proceed by applying $A_{j_-}^{2k}$ to move the V-structure at the j_--junction u_1 up to u_0, as in Fig. 11. □

Lemma 20. *Let* Γ *be a standard graph with a junction* u_0*, such that* u_0 *is both a* j_-*-junction and a* j*-junction, with the corresponding structures of type* I *and* Y*, in any order. Then there is a sequence of actions from the set* $\{A_j^2, A_{j_-}^2, A_j^{-2}, A_{j_-}^{-2}\}$ *converting the* Y*-structures to a* V*-structure.*

Proof. We may assume that the I- and Y-structures are attached to u_0 counterclockwise around u_0, as in Fig. 12, otherwise, we just reverse the actions. By Theorem 13, we can apply $A_{j_-}^{-2}$ several times to move the I-structure down to u_1. *(For example, in Fig. 12, we need to do this twice. This gives the configuration shown in Fig. 13.)* Now u_1 becomes a j_--junction and a j-structure, with the I- and V-structures attached. Applying A_j^{2k}, we can move the V-structure at u_1 up to u_0. *(In our example, this final configuration is presented in Fig. 14.)* Thus the Y-structure has been transformed to a V-structure. □

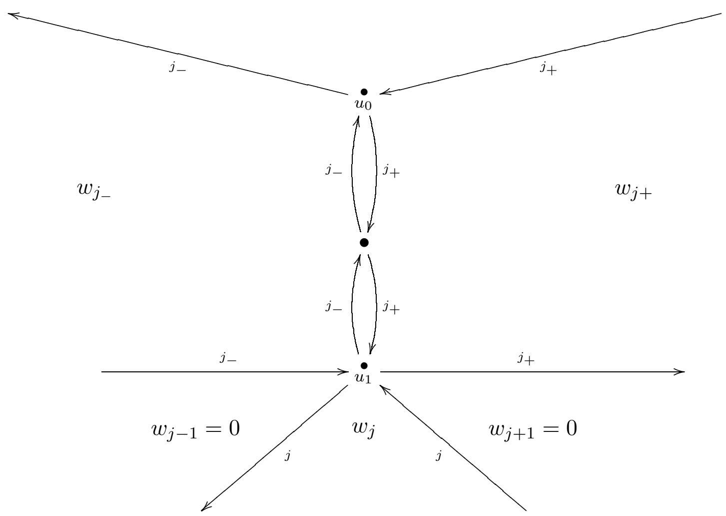


Figure 10: Intermediate configuration: two adjacent V-structures.

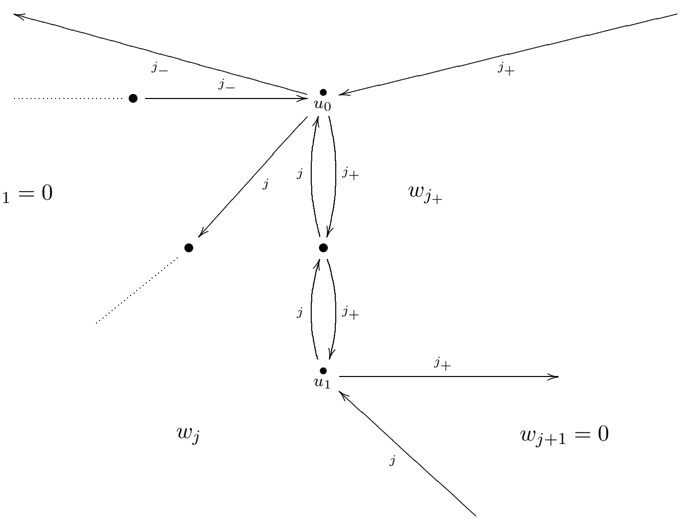


Figure 11: Y- and V-structures exchanged.

Theorem 21. *Let Γ be a standard graph with at least two adjacent dominant faces. Then there exists a sequence of actions $A^* = A_{j_1}^{\pm 2} A_{j_2}^{\pm 2} \ldots$ such that $A^*(\Gamma)$ have only* one *junction.*

Proof. By Theorem 17 we may assume that Γ is a graph in ivy form with the root junction u_0. The existence of two adjacent dominant faces implies the existence of an I-structure. If there are only I-structures and V-structures, then u_0 is the only junction of Γ. Assume that there is at least one Y-structure. By Lemma 19, we may move a Y-structure so that it is counterclockwise next to an I-structure. By Lemma 20, the Y-structure can be transformed to a V-structure, and the Y-junction removed. This can be repeated, eventually removing all junctions of Γ except u_0. $\square$

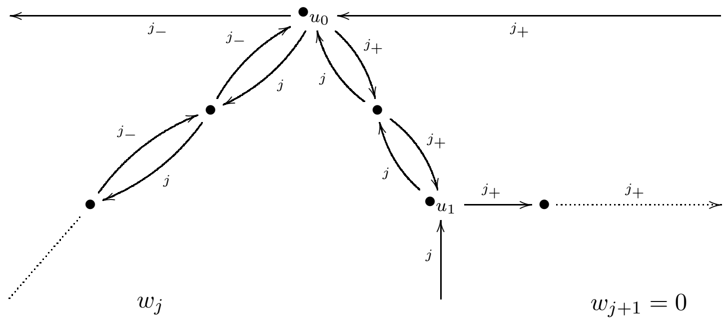


Figure 12: Adjacent I- and Y-structures

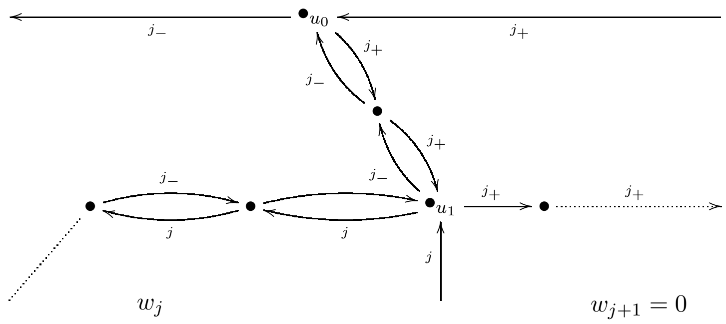


Figure 13: Moving the I-structure to u_1

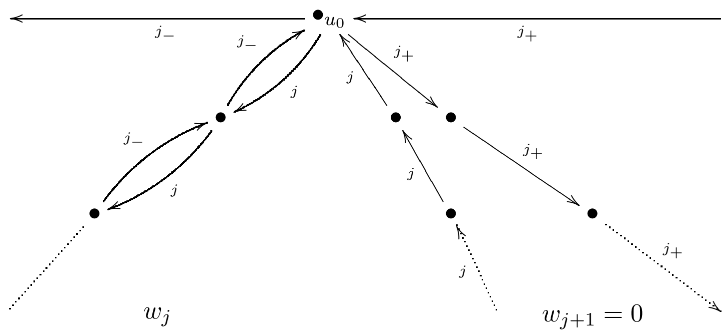


Figure 14: Moving the V-structure to u_0

Lemma 22. *Let Γ be a standard graph with a junction u_0, such that u_0 is both a j_--junction and a j-junction, with two adjacent Y-structures attached. Then there is a sequence of actions from the set $\{A_j^2, A_{j_-}^2, A_j^{-2}, A_{j_-}^{-2}\}$ converting one of the Y-structures to a V-structure.*

Proof. This can be proved by the arguments similar to those in the proof of Theorem 21. □

Theorem 23. *Let Γ be a standard graph such that no two dominant faces are adjacent. Then there exists a sequence of actions $A^* = A_{j_1}^{\pm 2}, A_{j_2}^{\pm 2}, \dots,$ such that $A^*(\Gamma)$ is in ivy form, with at most one Y-structure.*

Proof. One may assume by Theorem 17 that Γ is in ivy form, with the root junction u_0. Since no two dominant faces are adjacent, there are only V- and Y-structures attached to u_0. If there are at least two Y-structures, we may assume, by Lemma 19, that two Y-structures are adjacent. By Lemma 22, two adjacent Y-structures can be converted to a V-structure and a Y-structure. This can be repeated until at most one Y-structure remains in Γ. □

Lemma 24. *Let Γ be a standard graph such that no two dominant faces are adjacent. Then the number of bounded faces of Γ is finite and does not change after any action A_j^2.*

Proof. The bounded faces of Γ correspond to the edges of T_Γ separating two dominant faces. Since no two dominant faces are adjacent, any two dominant faces have a finite common boundary in T_Γ. Hence the number of bounded faces of Γ is finite. Lemma 12 and Theorem 13 imply that this number does not change after any action A_j^2. □

4. Irreducibility and connectivity of the spectral locus

In this section, we prove the main results stated in the introduction. We start with the following statements.

Lemma 25. *Let Σ be the space of all $(\alpha, \lambda) \in \mathbf{C}^d$ such that equation (1) admits a solution subdominant in non-adjacent Stokes sectors $S_{j_1}, \dots, S_{j_k}$, $k \le (d+2)/2$. Then Σ is a smooth complex analytic submanifold of $\mathbf{C}^d$ of the codimension $k-1$.*

Proof. Let f be a ratio of two linearly independent solutions of (1), and let $w = (w_0, \dots, w_{d+1})$ be the set of asymptotic values of f in the Stokes sectors $S_0, \dots, S_{d+1}$. Then w belongs to the subset Z of $\bar{\mathbf{C}}^{d+2}$ where the values w_j in adjacent Stokes sectors are distinct and there are at least three distinct values among w_j. The group G of fractional-linear transformations of $\bar{\mathbf{C}}$ acts on Z diagonally, and the quotient Z/G is a $(d-1)$-dimensional complex manifold.

Theorem 7.2, [Bak77] implies that the mapping $W : \mathbf{C}^d \to Z/G$ assigning to (α, λ) the equivalence class of w is submersive. More precisely, W is locally invertible on the subset $\{\alpha_{d-1} = 0\}$ of $\mathbf{C}^d$ and constant on the orbits of the group $\mathbf{C}$ acting on $\mathbf{C}^d$ by translations of the independent variable z. In particular, the preimage $W^{-1}(Y)$ of any smooth submanifold $Y \subset Z/G$ is a smooth submanifold of $\mathbf{C}^d$ of the same codimension as Y.

The set Σ is the preimage of the set $Y \subset Z/G$ defined by the $k-1$ conditions $w_{j_1} = \cdots = w_{j_k}$. Hence Σ is a smooth manifold of codimension $k-1$ in $\mathbf{C}^d$. □

Proposition 26. *Let Σ be the space of all $(\alpha, \lambda) \in \mathbf{C}^d$ such that equation* (1) *admits a solution subdominant in the non-adjacent Stokes sectors $S_{j_1}, \dots, S_{j_k}$. If at least two remaining Stokes sectors are adjacent, then Σ is an irreducible complex analytic manifold.*

Proof. Let Σ_0 be the intersection of Σ with the subspace $\mathbf{C}^{d-1} = \{\alpha_{d-1} = 0\} \subset \mathbf{C}^d$. Then Σ has the structure of a product of Σ_0 and $\mathbf{C}$ induced by translation of the independent variable z. In particular, Σ is irreducible if and only if Σ_0 is irreducible.

Let us choose a point $w = (w_0, \dots, w_{d+1})$ so that $w_{j_1} = \cdots = w_{j_k} = 0$, with all other values w_j distinct, non-zero and finite. Let Ψ_0 be a cell decomposition of $\bar{\mathbf{C}} \setminus \{0\}$ defined by the loops γ_j starting and ending at ∞ and containing non-zero values w_j, as in Section 2.1.

Nevanlinna theory (see [Nev32, Nev53]), implies that, for each standard graph Γ with the properties listed in Lemma 8, there exists $(\alpha, \lambda) \in \mathbf{C}^d$ and a meromorphic function $f(z)$ such that f is the ratio of two linearly independent solutions of (1) with the asymptotic values w_j in the Stokes sectors S_j, and Γ is the graph corresponding to the cell decomposition $\Phi_0 = f^{-1}(\Psi_0)$. This function, and the corresponding point (α, λ) is defined uniquely up to translation of the variable z. We can choose f uniquely if we require that $\alpha_{d-1} = 0$ in (α, λ). Conditions on the asymptotic values w_j imply then that $(\alpha, \lambda) \in \Sigma'$. Let f_Γ be this uniquely selected function, and $(\alpha_\Gamma, \lambda_\Gamma)$ the corresponding point of Σ'.

Let $W : \Sigma' \to Y \subset Z/G$ be as in the proof of Lemma 25. Then Σ' is an unramified covering of Y. Its fiber over the equivalence class of w consists of the points $(\alpha_\Gamma, \lambda_\Gamma)$ for all standard graphs Γ. Each action A_j^2 corresponds to a closed loop in Y starting and ending at w. Since for a given list of subdominant sectors a standard graph with one vertex is unique, Theorem 21 implies that the monodromy action is transitive. Hence Σ' is irreducible as a covering with a transitive monodromy group (see, e.g., [Kho04, §5]). □

This immediately implies Theorem 2, and we may also state the following corollary equivalent to Theorem 1:

Corollary 27. *For every potential P_α of even degree, with $\deg P_\alpha \geq 4$ and with the boundary conditions $y \to 0$ for $z \to \pm\infty$, $z \in \mathbf{R}$, there is an analytic continuation from any eigenvalue λ_m to any other eigenvalue λ_n in the α-plane.*

Proposition 28. *Let Σ be the space of all $(\alpha, \lambda) \in \mathbf{C}^d$, for even d, such that equation* (1) *admits a solution subdominant in the $(d+2)/2$ Stokes sectors $S_0, S_2, \dots, S_d$. Then irreducible components Σ_k, $k = 0, 1, \dots$ of Σ, which are also its connected components, are in one-to-one correspondence with the sets of standard graphs with k bounded faces. The corresponding solution of* (1) *has k zeros and can be represented as $Q(z)e^{\phi(z)}$ where Q is a polynomial of degree k and ϕ a polynomial of degree $(d+2)/2$.*

Proof. Let us choose w and Ψ_0 as in the proof of Proposition 26. Repeating the arguments in the proof of Proposition 26, we obtain an unramified covering $W : \Sigma' \to Y$ such that its fiber over w consists of the points $(\alpha_\Gamma, \lambda_\Gamma)$ for all standard graphs Γ with the properties listed in Lemma 8. Since we have no adjacent dominant sectors, Theorem 23 implies that any standard graph Γ can be transformed by the monodromy action to a graph Γ_0 in ivy form with at most one Y-structure attached at its j-junction, where j is any index such that S_j is a dominant sector. Lemma 24 implies that Γ and Γ_0

have the same number k of bounded faces. If $k = 0$, the graph Γ_0 is unique. If $k > 0$, the graph Γ_0 is completely determined by k and j. Hence for each $k = 0, 1, \dots$ there is a unique orbit of the monodromy group action on the fiber of W over w consisting of all standard graphs Γ with k bounded faces. This implies that Σ' (and Σ) has one irreducible component for each k.

Since Σ is smooth by Lemma 25, its irreducible components are also its connected components.

Finally, let $f_\Gamma = y/y_1$ where y is a solution of (1) subdominant in the Stokes sectors $S_0, S_2, \dots, S_d$. Then the zeros of f and y are the same, each such zero belongs to a bounded domain of Γ, and each bounded domain of Γ contains a single zero. Hence y has exactly k simple zeros. Let Q be a polynomial of degree k with the same zeros as y. Then y/Q is an entire function of finite order without zeros, hence $y/Q = e^\phi$ where ϕ is a polynomial. Since y/Q is subdominant in $(d+2)/2$ sectors, $\deg\phi = (d+2)/2$. □

The above propisition immediately implies Theorem 3.

5. Alternative viewpoint

In this section, we provide an example of the correspondence between the actions on cell decompositions with some subdominant sectors and actions on cell decompositions with no subdominant sectors. This correspondence can be used to simplify calculations with cell decompositions. We will illustrate our results on a cell decomposition with 6 sectors, the general case follows immediately.

Let C_6 be the set of cell decompositions with 6 sectors, none of them subdominant. Let $C_6^{03} \subset C_6$ be the set of cell decompositions such that for any $\Gamma \in C_6^{03}$, the sectors S_0 and S_3 do not share a common edge in the associated undirected graph T_Γ. Define D_6^{03} to be the set of cell decompositions with 6 sectors where S_0 and S_3 are subdominant.

Lemma 29. *There is a bijection between C_6^{03} and D_6^{03}.*

Proof. Let $\Gamma \in C_6^{03}$ be a cell decomposition, and let T_Γ be the associated undirected graph, see section 2.2. Then consider T_Γ as the (unique) undirected graph associated with some cell decomposition $\Delta \in D_6^{03}$. This is possible since the condition that the sectors 0 and 3 do not share a common edge in Γ, ensures that the subdominant sectors in Δ do not share a common edge. Let us denote this map π. Conversely, every cell decomposition $\Delta \in D_6^{03}$ is associated with a cell decomposition $\Gamma \in C_6^{03}$ by the inverse procedure π^{-1}. □

We have previously established that $\mathcal{H}_6$ acts on C_6 and that $\mathcal{H}_4$ acts on D_6^{03}. Let $B_0, B_1, \dots, B_5$ be the actions generating $\mathcal{H}_6$, as described in subsection 3, and let A_1, A_2, A_4, A_5 generate $\mathcal{H}_4$. Let $\mathcal{H}_6^{03} \subset \mathcal{H}_6$ be the subgroup generated by $B_1, B_2B_3B_2^{-1}, B_4, B_5B_0B_5^{-1}$, and their inverses. It is easy to see that $\mathcal{H}_6^{03}$ acts on elements in C_6^{03} and preserves this set.

Lemma 30. *The diagrams in Fig. 15 commute.*

$$\begin{tikzcd} \Gamma \arrow[r, "B_1"] \arrow[d, "\pi"] & B_1(\Gamma) \arrow[d, "\pi"] \\ \Delta \arrow[r, "A_1"] & A_1(\Delta) \end{tikzcd} \qquad \begin{tikzcd} \Gamma \arrow[r, "B_2B_3B_2^{-1}"] \arrow[d, "\pi"] & B_2B_3B_2^{-1}(\Gamma) \arrow[d, "\pi"] \\ \Delta \arrow[r, "A_2"] & A_2(\Delta) \end{tikzcd}$$

$$\begin{tikzcd} \Gamma \arrow[r, "B_4"] \arrow[d, "\pi"] & B_4(\Gamma) \arrow[d, "\pi"] \\ \Delta \arrow[r, "A_4"] & A_4(\Delta) \end{tikzcd} \qquad \begin{tikzcd} \Gamma \arrow[r, "B_5B_0B_5^{-1}"] \arrow[d, "\pi"] & B_5B_0B_5^{-1}(\Gamma) \arrow[d, "\pi"] \\ \Delta \arrow[r, "A_5"] & A_5(\Delta) \end{tikzcd}$$

Figure 15: The commuting actions

Proof. Let (a, b, c, d, e, f) be the 6 loops of a cell decomposition Ψ_0 as in Fig. 1, looping around the asymptotic values $(w_0, \dots, w_5)$. Let Ψ'_0 be the cell decomposition with the four loops (b, c, e, f), such that if $\Gamma \in C_6^{03}$ is the preimage of Ψ_0, then $\pi(\Gamma)$ is the preimage of Ψ'_0. That is, the preimages of the loops a and d in Ψ_0 are removed under π.

B_j acts on Ψ_0 and A_j acts on Ψ'_0. (See subsection 3 for the definition.) We have

$$A_1(b, c, e, f) = (bcb^{-1}, e, f), \ A_4(b, c, d, e) = (b, c, efe^{-1}, e). \tag{5}$$

and

$$\begin{aligned} B_1(a, b, c, d, e, f) &= (a, bcb^{-1}, d, e, f), \\ B_4(a, b, c, d, e, f) &= (a, b, c, efe^{-1}, e, f). \end{aligned} \tag{6}$$

Equation (5) and (6) shows that the left diagrams commute, since applying π to the result from (6) yields (5). We also have that

$$A_2(b, c, e, f) = (b, cec^{-1}, c, f), A_5(b, c, e, f) = (f, c, e, fbf^{-1}). \tag{7}$$

We now compute $B_3^{-1}B_2B_3(a, b, c, d, e, f)$. Observe that we must apply these actions *left to right*:

$$\begin{aligned} B_3^{-1}B_2B_3(a, b, c, d, e, f) &= B_2B_3(a, b, c, e, e^{-1}de, f) \\ &= B_3(a, b, cec^{-1}, c, e^{-1}de, f) \\ &= (a, b, cec^{-1}, c(e^{-1}de)c^{-1}, c, f) \end{aligned} \tag{8}$$

A similar calculation gives

$$B_0^{-1}B_5B_0(a, b, c, d, e, f) = (f(b^{-1}ab)f^{-1}, f, c, d, e, f, b, f^{-1}), \tag{9}$$

and applying π to the results (8) and (9) give (7). $\square$

Remark 31. *Note that* $B_j^{-1}B_{j-1}B_j(\Gamma) = B_{j-1}B_jB_{j-1}^{-1}(\Gamma)$ *for all* $\Gamma \in C_6$*, which follows from basic properties of the braid group.*

The above result can be generalized as follows: Let C_n be the set of cell decompositions with n sectors such that all sectors are dominant. Let $C_n^{\mathbf{l}} \subset C_n,\ \mathbf{l} = \{l_1, l_2, \dots, l_k\}$ be the set of cell decompositions such that for any $\Gamma \in C_n^{\mathbf{l}}$, no two sectors in the set $S_{l_1}, S_{l_2}, \dots, S_{l_k}$ have a common edge in the associated undirected graph T_Γ. Let $D_n^{\mathbf{l}}$ be the set of cell decompositions with n sectors such that the sectors $S_{l_1}, S_{l_2}, \dots, S_{l_k}$ are subdominant. Let $\{A_j\}_{j \notin \mathbf{l}}$ be the $n-k$ actions acting on $C_n^{\mathbf{l}}$ indexed as in subsection 3. Let $\{B_j\}_{j=0}^{n-1}$ be the actions on C_n. Let $\pi : C_n^{\mathbf{s}} \to D_n^{\mathbf{s}}$ be the map similar to the bijection above, where one obtain a cell decomposition in $D_n^{\mathbf{s}}$ by removing edges with a label in $\mathbf{l}$ from a cell decomposition in $C_n^{\mathbf{s}}$. Then

$$\begin{cases} \pi(B_j(\Gamma)) = A_j(\pi(\Gamma)) & \text{if } j, j+1 \notin \mathbf{l}, \\ \pi(B_j^{-1}B_{j-1}B_j(\Gamma)) = A_j(\pi(\Gamma)), & j \notin \mathbf{l}, j+1 \in \mathbf{l}. \end{cases} \tag{10}$$

Remark 32. *There are some advantages with cell decompositions with no subdominant sectors:*

- *An action* A_j *always interchanges the asymptotic values* w_j *and* w_{j+1}.
- *Lemma 8, item II implies* T_Γ *have no bounded faces iff order of the asymptotic values is a cyclic permutation of the standard order.*

6. Appendix

6.1. **Examples of monodromy action.** Below are some specific examples on how the different actions act on trees and non-trees.

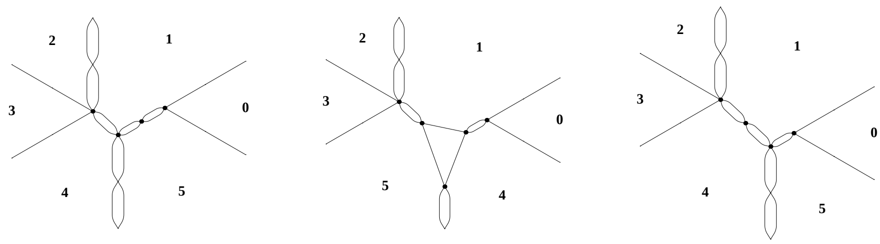


Figure 16: Example action of A_4^{-1} and A_4^{-2} in case 1.

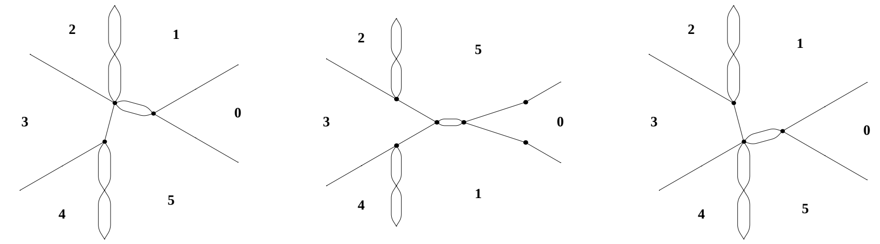


Figure 17: Example action of A_5 and A_5^2 in case 2.

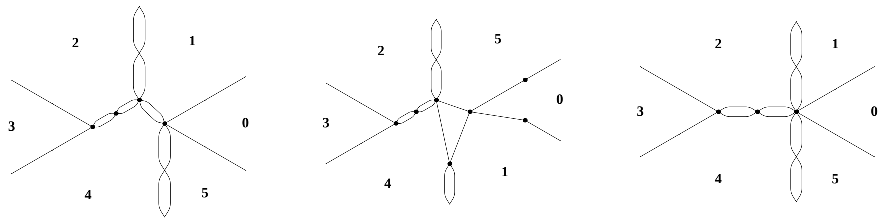


Figure 18: Example action of A_5^{-1} and A_5^{-2} in case 3.

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ON EIGENVALUES OF THE SCHRÖDINGER OPERATOR WITH AN EVEN COMPLEX-VALUED POLYNOMIAL POTENTIAL

PER ALEXANDERSSON

ABSTRACT. In this paper, we generalize several results of the article "Analytic continuation of eigenvalues of a quartic oscillator" of A. Eremenko and A. Gabrielov.

We consider a family of eigenvalue problems for a Schrödinger equation with even polynomial potentials of arbitrary degree d with complex coefficients, and $k < (d+2)/2$ boundary conditions. We show that the spectral determinant in this case consists of two components, containing even and odd eigenvalues respectively.

In the case with $k = (d+2)/2$ boundary conditions, we show that the corresponding parameter space consists of infinitely many connected components.

CONTENTS

1. INTRODUCTION

We study the problem of analytic continuation of eigenvalues of the Schrödinger operator with an even complex-valued polynomial potential,

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that is, analytic continuation of $\lambda = \lambda(\alpha)$ in the differential equation

$$-y'' + P_\alpha(z)y = \lambda y, \tag{1}$$

where $\alpha = (\alpha_2, \alpha_4, \dots, \alpha_{d-2})$ and $P_\alpha(z)$ is the even polynomial

$$P_\alpha(z) = z^d + \alpha_{d-2}z^{d-2} + \dots + \alpha_2 z^2.$$

The boundary conditions for (1) are as follows: Set $n = d + 2$ and divide the plane into n disjoint open sectors

$$S_j = \{z \in \mathbf{C} \setminus \{0\} : |\arg z - 2\pi j/n| < \pi/n\}, \quad j = 0, 1, 2, \dots, n-1.$$

The index j should be considered mod n. These are the *Stokes sectors* of the equation (1). A solution y of (1) satisfies $y(z) \to 0$ or $y(z) \to \infty$ as $z \to \infty$ along each ray from the origin in S_j, see [Sib75]. The solution y is called *subdominant* in the first case, and *dominant* in the second case.

The main result of this paper is as follows:

Theorem 1. *Let $\nu = d/2 + 1$ and let $J = \{j_1, j_2, \dots, j_{2m}\}$ with $j_{k+m} = j_k + \nu$ and $|j_p - j_q| > 1$ for $p \neq q$. Let Σ be the set of all $(\alpha, \lambda) \in \mathbf{C}^\nu$ for which the equation $-y'' + (P_\alpha - \lambda)y = 0$ has a solution with with the boundrary conditions*

$$y \text{ is subdominant in } S_j \text{ for all } j \in J \tag{2}$$

where $P_\alpha(z)$ is an even polynomial of degree d. For $m < \nu/2$, Σ consists of two irreducible connected components. For $m = \nu/2$, which can only happen when $d \equiv 2 \mod 4$, Σ consists of infinitely many connected components, distinguished by the number of zeros of the corresponding solution to (1).

1.1. **Previous results.**

The first study of analytic continuation of λ in the complex β-plane for the problem

$$-y'' + (\beta z^4 + z^2)y = \lambda y, \quad y(-\infty) = y(\infty) = 0$$

was done by Bender and Wu [BW69], They discovered the connectivity of the sets of odd and even eigenvalues, rigorous results was later proved in [Sim70].

In [EG09a], the even quartic potential $P_a(z) = z^4 + az^2$ and the boundary value problem

$$-y'' + (z^4 + az^2)y = \lambda_a y, \quad y(\infty) = y(-\infty) = 0$$

was considered.

The problem has discrete real spectrum for real a, with $\lambda_1 < \lambda_2 < \dots \to +\infty$. There are two families of eigenvalues, those with even index and those with odd. If λ_j and λ_k are two eigenvalues in the same family, then λ_k can be obtained from λ_j by analytic continuation in the complex α-plane. Similar results have been found for other potentials, such as the PT-symmetric cubic, where $P_\alpha(z) = (iz^3 + i\alpha z)$, with $y(z) \to 0$, as $z \to \pm\infty$ on the real line. See for example [EG09b].

1.2. **Acknowledgements.**

The author would like to thank Andrei Gabrielov for the introduction to this area of research, and for enlightening suggestions and improvements to the text. Great thanks to Boris Shapiro, my advisor.

2. Preliminaries on general theory of solutions to the Schroedinger equation

We will review some properties for the Schrödinger equation with a general polynomial potential. In particular, these properties hold for an even polynomial potential. These properties may also be found in [EG09a, AG10].

The general Schroedinger equation is given by

$$-y'' + P_\alpha(z)y = \lambda y, \tag{3}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{d-1})$ and $P_\alpha(z)$ is the polynomial

$$P_\alpha(z) = z^d + \alpha_{d-1}z^{d-1} + \cdots + \alpha_1 z.$$

We have the associated Stokes sectors

$$S_j = \{z \in \mathbf{C} \setminus \{0\} : |\arg z - 2\pi j/n| < \pi/n\}, \quad j = 0, 1, 2, \dots, n-1,$$

where $n = d+2$, and index considered mod n. The boundary conditions to (3) are of the form

$$y \text{ is subdominant in } S_{j_1}, S_{j_2}, \dots, S_{j_k} \tag{4}$$

with $|j_p - j_q| > 1$ for all $p \neq q$.

Notice that any solution $y \neq 0$ of (3) is an entire function, and the ratio $f = y/y_1$ of any two linearly independent solutions of (3) is a meromorphic function with the following properties, (see [Sib75]).

(I) For any j, there is a solution y of (3) subdominant in the Stokes sector S_j, where y is unique up to multiplication by a non-zero constant.

(II) For any Stokes sector S_j, we have $f(z) \to w \in \bar{\mathbf{C}}$ as $z \to \infty$ along any ray in S_j. This value w is called *the asymptotic value* of f in S_j.

(III) For any j, the asymptotic values of f in S_j and S_{j+1} (index still taken modulo n) are distinct. Furthermore, f has at least 3 distinct asymptotic values.

(IV) The asymptotic value of f in S_j is zero if and only if y is subdominant in S_j. We call such sector *subdominant* for f as well. Note that the boundary conditions given in (4) imply that sectors $S_{j_1}, \dots, S_{j_k}$ are subdominant for f when y is an eigenfunction of (3), (4).

(V) f does not have critical points, hence $f : \mathbf{C} \to \bar{\mathbf{C}}$ is unramified outside the asymptotic values.

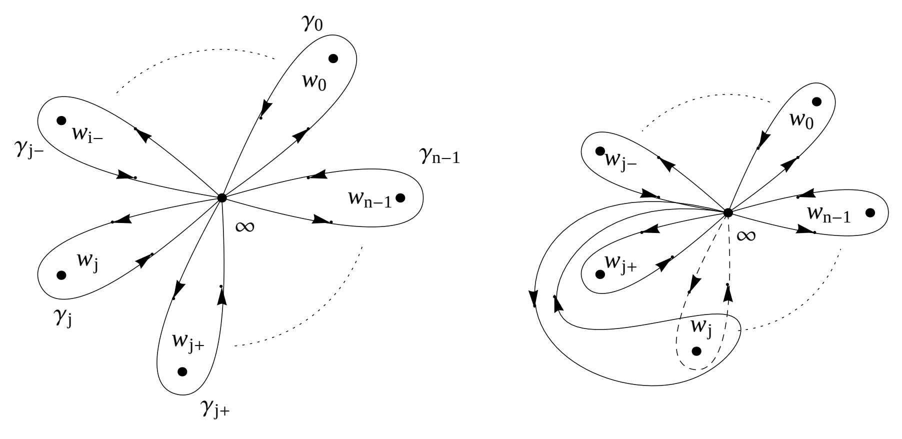
(VI) The Schwartzian derivative S_f of f given by

$$S_f = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

equals $-2(P_\alpha - \lambda)$. Therefore one can recover P_α and λ from f.

From now on, f denotes the ratio of two linearly independent solutions of (3), (4).

2.1. **Cell decompositions.** As above, set $n = \deg P + 2$ where P is our polynomial potential and assume that all non-zero asymptotic values of f are distinct and finite. Let w_j be the asymptotic values of f with an arbitrary ordering satisfying the only restriction that if S_j is subdominant, then $w_j =$



(a) Ψ_0 (b) $A_j(\Psi_0)$.

Figure 1: Permuting w_j and w_{j_+} in Ψ_0.

0. One can denote by w_j the asymptotic value in the Stokes sector S_j, which will be called the *standard order*, see section 2.3.

Consider the cell decomposition Ψ_0 of $\bar{\mathbf{C}}_w$ shown in Fig. 1a. It consists of closed directed loops γ_j starting and ending at ∞, where the index is considered mod n, and γ_j is defined only if $w_j \neq 0$. The loops γ_j only intersect at ∞ and have no self-intersection other than ∞. Each loop γ_j contains a single non-zero asymptotic value w_j of f. For example, for even n, the boundary condition $y \to 0$ as $z \to \pm\infty$ for $z \in \mathbf{R}$ implies that $w_0 = w_{n/2} = 0$, so there are no loops γ_0 and $\gamma_{n/2}$. We have a natural cyclic order of the asymptotic values, namely the order in which a small circle around ∞ traversed counterclockwise intersects the associated loops γ_j, see Fig. 1a.

We use the same index for the asymptotic values and the loops, so define

$$j_+ = j + k \text{ where } k \in \{1,2\} \text{ is the smallest integer such that } w_{j+k} \neq 0.$$

Thus, γ_{j_+} is the loop around the next to w_j (in the cyclic order mod n) non-zero asymptotic value. Similarly, γ_{j_-} is the loop around the previous non-zero asymptotic value.

2.2. **From cell decompositions to graphs.** Proofs of all statements in this subsection can be found in [EG09a].

Given f and Ψ_0 as above, consider the preimage $\Phi_0 = f^{-1}(\Psi_0)$. Then Φ_0 gives a cell decomposition of the plane $\mathbf{C}_z$. Its vertices are the poles of f and the edges are preimages of the loops γ_j. An edge that is a preimage of γ_j is labeled by j and called a j-edge. The edges are directed, their orientation is induced from the orientation of the loops γ_j. Removing all loops of Φ_0, we obtain an infinite, directed planar graph Γ, without loops. Vertices of Γ are poles of f, each bounded connected component of $\mathbf{C} \setminus \Gamma$ contains one simple zero of f, and each zero of f belongs to one such bounded connected component. There are at most two edges of Γ connecting any two of its vertices. Replacing each such pair of edges with a single undirected edge and making all other edges undirected, we obtain an undirected graph T_Γ.

It has no loops or multiple edges, and the transformation from Φ_0 to T_Γ can be uniquely reversed.

A *junction* is a vertex of Γ (and of T_Γ) at which the degree of T_Γ is at least 3. From now on, Γ refers to both the directed graph without loops and the associated cell decomposition Φ_0.

2.3. **The standard order of asymptotic values.** For a potential P of degree d, the graph Γ has $n = d + 2$ infinite branches and n unbounded faces corresponding to the Stokes sectors of P. We fixed earlier the ordering $w_0, w_1, \dots, w_{n-1}$ of the asymptotic values of f.

If each w_j is the asymptotic value in the sector S_j, we say that the asymptotic values have *the standard order* and the corresponding cell decomposition Γ is a *standard graph*.

Lemma 2 (See Prop. 6 [EG09a]). *If a cell decomposition Γ is a standard graph, then the corresponding undirected graph T_Γ is a tree.*

In the next section, we define some actions on Ψ_0 that permute non-zero asymptotic values. Each unbounded face of Γ (and T_Γ) will be labeled by the asymptotic value in the corresponding Stokes sector. For example, labeling an unbounded face corresponding to S_k with w_j or just with the index j, indicates that w_j is the asymptotic value in S_k.

From the definition of the loops γ_j, a face corresponding to a dominant sector has the same label as any edge bounding that face. The label in a face corresponding to a subdominant sector S_k is always k, since the actions defined below only permute non-zero asymptotic values.

An unbounded face of Γ is called (sub)dominant if the corresponding Stokes sector is (sub)dominant.

2.4. **Properties of graphs and their face labeling.**

Lemma 3 (See Section 3 in [EG09a]). *Any graph Γ have the following properties:*

- *(I) Two bounded faces of Γ cannot have a common edge, (since a j-edge is always at the boundary of an unbounded face labeled j.)*
- *(II) The edges of a bounded face of a graph Γ are directed clockwise, and their labels increase in that order. Therefore, a bounded face of T_Γ can only appear if the order of w_j is non-standard.*
- *(III) Each label appears at most once in the boundary of any bounded face of Γ.*
- *(IV) The unbounded faces of Γ adjacent to a junction u, always have the labels cyclically increasing counterclockwise around u.*
- *(V) The boundary of a dominant face labeled j consists of infinitely many directed j-edges, oriented counterclockwise around the face.*
- *(VI) If $w_j = 0$ there are no j-edges.*
- *(VII) Each vertex of Γ has even degree, since each vertex in $\Phi_0 = f^{-1}(\Psi_0)$ has even degree, and removing loops to obtain Γ preserves this property.*

Following the direction of the j-edges, the first vertex that is connected to an edge labeled j_+ is the vertex where the j-edges and the j_+-edges *meet*. The last such vertex is where they *separate*. These vertices, if they exist, must be junctions.

Definition 4. *Let Γ be a standard graph, and let $j \in \Gamma$ be a junction where the j-edges and j_+-edges separate. Such junction is called a* j-junction.

There can be at most one j-junction in Γ, the existence of two or more such junctions would violate property (III) of the face labeling. However, the same junction can be a j-junction for different values of j.

There are three different types of j-junctions, see Fig. 2.

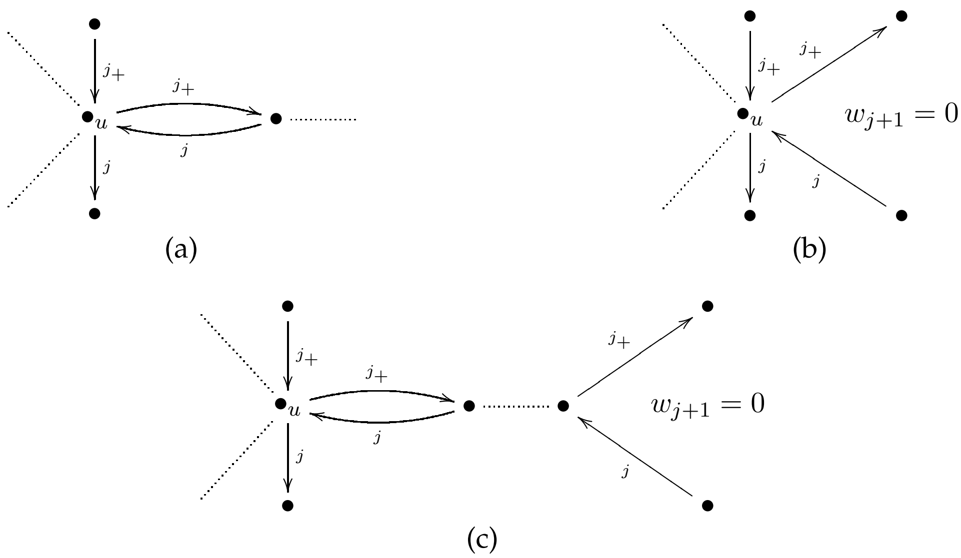


Figure 2: Different types of j-junctions.

Case (a) only appears when $w_{j+1} \neq 0$. Cases (b) and (c) can only appear when $w_{j+1} = 0$. In (c), the j-edges and j_+-edges meet and separate at different junctions, while in (b), this happens at the same junction.

Definition 5. *Let Γ be a standard graph with a j-junction u. A* structure *at the j-junction is the subgraph Ξ of Γ consisting of the following elements:*

- *The edges labeled j that appear before u following the j-edges.*
- *The edges labeled j_+ that appear after u following the j_+-edges.*
- *All vertices the above edges are connected to.*

If u is as in Fig. 2a, Ξ is called an I-structure at the j-junction. *If u is as in Fig. 2b, Ξ is called a* V-structure at the j-junction. *If u is as in Fig. 2c, Ξ is called a* Y-structure at the j-junction.

Since there can be at most one j-junction, there can be at most one structure at the j-junction.

A graph Γ shown in Fig. 3 has one (dotted) I-structure at the 1-junction v, one (dotted) I-structure at the 4-junction u, one (dashed) V-structure at the 2-junction v and one (dotdashed) Y-structure at the 5-junction u.

Note that the Y-structure is the only kind of structure that contains an additional junction. We refer to such additional junctions as *Y-junctions*. For example, the junction marked y in Fig. 3 is a Y-junction.

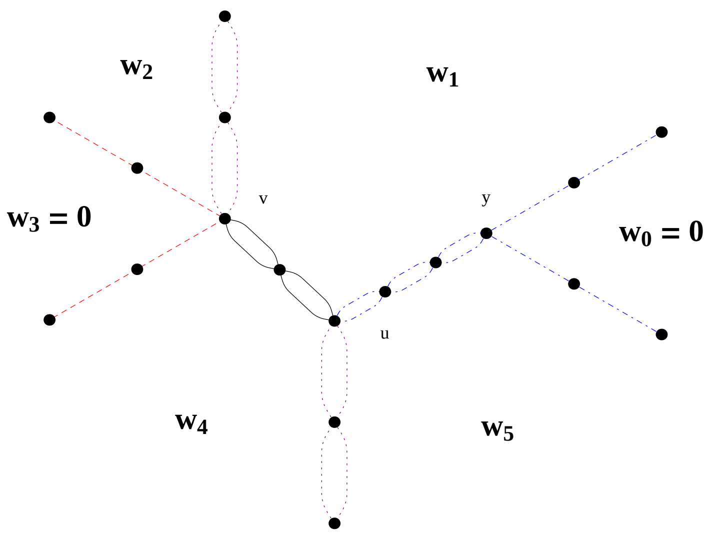


Figure 3: Graph Γ with (dotted) I-structures, a (dashed) Y-structure and a (dotdashed) Y-structure.

2.5. **Braid actions on graphs.** As in [AG10], we define continuous deformations A_j of the loops in Fig. 1a, such that the new loops are given in terms of the old ones by

$$A_j(\gamma_k) = \begin{cases} \gamma_j\gamma_{j+}\gamma_j^{-1} \text{ if } k = j \\ \gamma_j \text{ if } k = j_+ \\ \gamma_k \text{ otherwise} \end{cases} , \quad A_j^{-1}(\gamma_k) = \begin{cases} \gamma_{j+} \text{ if } k = j \\ \gamma_{j+}^{-1}\gamma_j\gamma_{j+} \text{ if } k = j_+ \\ \gamma_k \text{ otherwise} \end{cases}$$

These actions, together with their inverses, generate the Hurwitz (or sphere) braid group $\mathcal{H}_m$, where m is the number of non-zero asymptotic values. (For a definition of this group, see [LZ04].) The action of the generators A_j and A_k commute if $|j - k| \geq 2$.

The property (V) of the eigenfunctions implies that each A_j induces a monodromy transformation of the cell decomposition Φ_0, and of the associated directed graph Γ.

3. Properties of even actions on centrally symmetric graphs

3.1. **Additional properties for even potential.** In addition to the previous properties for general polynomials, these additional properties holds for even polynomial potentials P (see [EG09a]). From now until the end of the article, $\nu = (\deg(P) + 2)/2$.

Each solution y of (1) is either even or odd and we may choose y and y_1 such that $f = y/y_1$ is odd.

If the asymptotic values $w_0, w_1, \ldots, w_{2\nu-1}$ are ordered in the standard order, we have that $w_j = -w_{j+\nu}$.

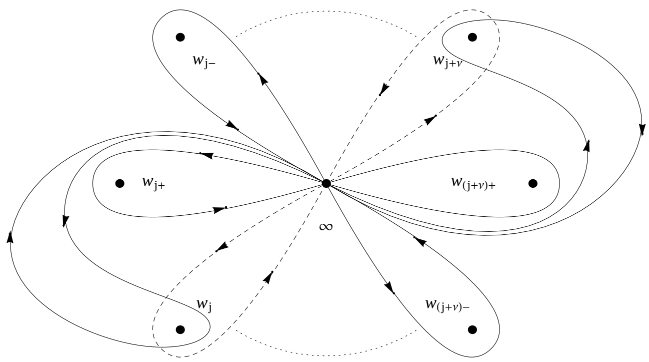


Figure 4: $E_j(\Psi_0)$

We may choose the loops centrally symmetric in Fig. 1a which implies that Φ_0 and Γ are centrally symmetric.

3.2. **Even braid actions.** Define the *even actions* E_j as $E_j = A_j \circ A_{j+\nu}$.

Assume that Γ is a graph with the property that if w_j is the asymptotic value in S_k, then w_{j+n} is the asymptotic value in $S_{k+\nu}$. (For example, all standard graphs have this property, with $j = k$.) It follows from the symmetric property of E_j that E_j preserves this property. To illustrate, we have that $E_j(\Psi_0)$ is given in Fig. 4.

Lemma 6. *If Γ is centrally symmetric, then $E_j(\Gamma)$ and $E_j^{-1}(\Gamma)$ are centrally symmetric graphs.*

Proof. We may choose the deformations of the paths γ_j and $\gamma_{j+\nu}$ being centrally symmetric, which implies that the composition $A_j \circ A_{j+\nu}$ preserves the property of Γ being centrally symmetric, see details in [EG09a]. $\square$

Lemma 7. *Let Γ be a centrally symmetric standard graph with no j-junction. Then $E_j^2(\Gamma) = \Gamma$.*

Proof. Since A_j and A_{j+n} commute, we have that $E_j^2 = A_j^2 A_{j+\nu}^2$, and the statement then follows from [AG10, Lemma 12]. $\square$

Theorem 8. *Let Γ be a centrally symmetric standard graph with a j-junction u. Then $E_j^2(\Gamma) \neq \Gamma$, and the structure at the j-junction is moved one step in the direction of the j-edges under E_j^2. The inverse of E_j^2 moves the structure at the j-junction one step backwards along the j_+-edges.*

Since Γ is centrally symmetric, it also has a $j+\nu$-junction, and the structure at the $j+\nu$-junction is moved one step in the direction of the $j+n$-edges under E_j^2. The inverse of E_j^2 moves the structure at the $j+\nu$-junction one step backwards along the $(j+\nu)_+$-edges.

Proof. Since $E_j^2 = A_j^2 A_{j+\nu}^2$, the result follows from [AG10, Theorem 13]. $\square$

4. PROVING MAIN THEOREM 1

Notice that each centrally symmetric standard graph Γ has either a vertex in its center, or a double edge, connecting two vertices. This property follows from the fact that Γ_T is a centrally symmetric tree.

Lemma 9. *Let Γ be a centrally symmetric graph. Then for every action E_j, Γ has a vertex at the center iff $E_j(\Gamma)$ has a vertex at the center.*

Proof. This is evident from the definition of the actions, since the action only changes the edges, and preserves the vertices. □

Corollary 10. *The spectral determinant has at least two connected components.*

Each centrally symmetric standard graph Γ is of one of two types:

(1) Γ has a central double edge. The vertices of the central double edge are called *root junctions*.
(2) Γ has a junction at its center. This junction is called the *root junction* u_r.

Definition 11. *A centrally symmetric standard graph Γ is in* ivy form *if Γ consists of structures connected to one or two root junctions.*

Definition 12. *Let Γ be a centrally symmetric standard graph.*
The root metric *of Γ, denoted $|\Gamma|_r$ is defined as*

$$|\Gamma|_r = \sum_{v \in \Gamma} (\deg(v) - 2)\,|v - u_r|$$

where the sum is taken over all vertices v of Γ_1. Here $deg(v)$ is the total degree of the vertex v in T_Γ and $|v - u_r|$ is the length of the shortest path from v to the closest root junction u_r in T_Γ.

Lemma 13. *The graph Γ is in ivy form if and only if all but its root junctions are Y-junctions.*

Proof. This follows from the definitions of the structures. □

Theorem 14. *Let Γ be a centrally symmetric standard graph. Then there is a sequence of even actions $E^* = E_{j_1}^{\pm 2}, E_{j_2}^{\pm 2}, \ldots$, such that $E^*(\Gamma)$ is in* ivy form.

Proof. Assume that Γ is not in ivy form.

Let U be the set of junctions in Γ that are not Y-junctions. Since Γ is not in ivy form we have that $|U| \geq 3$. Let $u_r \neq u_1$ be two junctions in U such that $|u_r - u_1|$ is maximal, and u_r is the central junction closest to u_1. Let p be the path from u_r to u_1 in T_Γ. It is unique since T_Γ is a tree. Let v be the vertex preceding u_1 on the path p. The edge from v to u_1 in T_Γ is adjacent to at least one dominant face with label j such that $w_j \neq 0$. Therefore, there exists a j-edge between v and u_1 in Γ. Suppose first that this j-edge is directed from u_1 to v. Let us show that in this case u_1 must be a j-junction, i.e., the dominant face labeled j_+ is adjacent to u_1.

Since u_1 is not a Y-junction, there is a dominant face adjacent to u_1 with a label $k \neq j, j_+$. Hence no vertices of p, except possibly u_1 can be adjacent to j_+-edges. If u_1 is not a j-junction, there are no j_+-edges adjacent to u_1. This implies that any vertex of Γ adjacent to a j_+-edge is further away from u_r than u_1.

Let u_2 be the closest to u_1 vertex of Γ adjacent to a j_+-edge. Then u_2 should be a junction of T_Γ, since there are two j_+-edges adjacent to u_2 in Γ and at least one more vertex (on the path from u_1 to u_2) which is connected to u_2 by edges with labels other than j_+. Since u_2 is further away from u_r that u_1 and the path p is maximal, u_2 must be a Y-junction. If the j-edges and j_+-edges would meet at u_2, u_1 would be a j-junction. Otherwise, a subdominant face labeled $j+1$ would be adjacent to both u_1 and u_2, while a subdominant face adjacent to a Y-junction cannot be adjacent to any other junctions.

Hence u_1 must be a j-junction. By Theorem 8, the action E_j^2 moves the structure at the j-junction u_1 one step closer to u_r along the path p, and similarly happens on the opposite side of Γ, decreasing $|\Gamma|_c$ by at least 2.

The case when the j-edge is directed from v to u_1 is treated similarly. In that case, u_1 must be a j_--junction, and the action $A_{j_-}^{-2}$ moves the structure at the j_--junction u_1 one step closer to u_r along the path p.

We have proved that if $|U| > 1$ then $|\Gamma|_r$ can be reduced. Since it is a non-negative integer, after finitely many steps we must reach a stage where U consists only of the root junctions. Hence $E^*(\Gamma)$ is in ivy form. □

The above Theorem shows that for every centrally symmetric standard graph Γ, there is a sequence of actions that turns Γ into ivy form. A graph in ivy form consists of one or two root junctions, with attached structures. These structures can be ordered counterclockwise around each root junction. These observations motivates the following lemmas:

Lemma 15. *Let Γ be a centrally symmetric standard graph, and let $u_r \in \Gamma$ be a root junction of type j_- and of type j. Let S_1 and S_2 be the corresponding structures attached to u_r.*

(1) If S_1 and S_2 are of type Y resp. V, then there is a sequence of even actions that interchange these structures.
(2) If S_1 and S_2 are of type I resp. Y, then there is a sequence of even actions that converts the type Y structure to a type V structure.
(3) If S_1 and S_2 are both of type Y, then there is a sequence of even actions that converts one of the Y-structures to a V-structure.

Proof. By symmetry, there are identical structures in Γ attached to a root junction of type $n + j_-$ and $n + j$, with attached structures S'_1 and S'_2 of the same type as S_1 resp. S_2.

Lemma 19, 20 and 22 in [AG10], gives the existence of a non-even sequence of actions, that only acts on S_1 and S_2 in the desired way.

In all these cases, the sequence is of the form

$$A^* = A_{k_1}^{\pm 2} A_{k_2}^{\pm 2} \dots A_{k_m}^{\pm 2}$$

where $k_1, k_2, \dots k_m \in \{j, j_-\}$. It follows that the action

$$B^* = A_{k_1+\nu}^{\pm 2} A_{k_2+\nu}^{\pm 2} \dots A_{k_m+\nu}^{\pm 2}$$

do the same as A^* but on S'_1 and S'_2.

Now, $E^* = A^* \circ B^*$ is even, since by commutativity[1], it is equal to

$$(A^{\pm 2}_{k_1} A^{\pm 2}_{k_1+\nu})(A^{\pm 2}_{k_2} A^{\pm 2}_{k_2+\nu}) \dots (A^{\pm 2}_{k_m} A^{\pm 2}_{k_m+\nu})$$

which easily may be written in terms of our even actions as

$$E^{\pm 2}_{k_1} E^{\pm 2}_{k_2} \dots E^{\pm 2}_{k_m}.$$

This sequence of actions has the desired property. □

Corollary 16. *Let Γ be a centrally symmetric graph, with two adjacent dominant faces. Then there is a sequence of even actions E^* such that $E^*(\Gamma)$ has either one or two junctions.*

Proof. We may apply even actions to make Γ into a standard graph, and then convert it to ivy form. The condition that we have two dominant faces, is equivalent to existence of I-structures. If there are no Y-structures, then the only junctions of Γ are the root junctions, and we are done. Otherwise, we may the Y- and V-structures, so that a Y-structure appears next to the I-structure. By using the second part of the above lemma, we decrease the number of Y-structures of Γ by two. After a finite number of actions, we arrive at a graph in ivy form without Y-structures. □

Lemma 17. *Let Γ be a centrally symmetric graph, with no adjacent dominant faces. Then there is a sequence of even actions E^* such that $E^*(\Gamma)$ is in ivy form, with at most two Y-structures.*

Proof. By Teorem 14, we may assume that Γ is in ivy form. Since there are no adjacent dominant sectors, the only structures of Γ are of Y and V type. These are attached to the one or two root junctions.

Assume that there are more than two Y-structures present. Two of these must be attached to the same root junction, u_r. By repeatedly applying part one of Lemma 15, we may interchange the Y- and V-structures attached to u_r such that the two Y-structures are adjacent. Applying part three of Lemma 15, we may then convert one of the two Y-structures to a V-structure.

By symmetry, the same change is done on the opposite side of Γ and total number of Y-structures of Γ have therefore been reduced by two. We may repeat this procedure a finite number of times, until the number of Y-structures is less than three. This implies the lemma. □

Lemma 18 (See [AG10]). *Let Γ be a standard graph such that no two dominant faces are adjacent. Then the number of bounded faces of Γ is finite and does not change after any action A^2_j.*

Corollary 19. *The number of bounded faces of Γ does not change under any* even *action $E^2_j = A^2_j A^2_{j+\nu}$.*

Lemma 20. *Let $\nu = n/2 = d/2 + 1$ and let Σ be the space of all $(\alpha, \lambda) \in \mathbf{C}^{\nu-1}$ such that equation* (1) *admits a solution subdominant in non-adjacent Stokes sectors*

$$S_{j_1}, S_{j_2}, \dots, S_{2m} \tag{5}$$

[1]We have at least 4 structures, 2 of them are Y or V structures. Hence $n \geq 6$ and we have commutativity.

with $j_{k+m} = j_k + \nu$ and $1 \leq m \leq \nu/2$. Then Σ is a smooth complex analytic submanifold of $\mathbf{C}^{\nu-1}$ of the codimension m.

Proof. We consider the space $\mathbf{C}^{\nu-1}$ as a subspace of the space $\mathbf{C}^{n-2}$ of all (α, λ) corresponding to the general polynomial potentials in (3), with $\alpha = (\alpha_1, \dots, \alpha_{d-1})$. Let f be a ratio of two linearly independent solutions of (3), and let $w = (w_0, \dots, w_{n-1})$ be the set of the asymptotic values of f in the Stokes sectors $S_0, \dots, S_{n-1}$.

Then w belongs to the subset Z of $\bar{\mathbf{C}}^{n-1}$ where the values w_j in adjacent Stokes sectors are distinct and there are at least three distinct values among w_j. The group G of fractional-linear transformations of $\bar{\mathbf{C}}$ acts on Z diagonally, and the quotient Z/G is a $(n-3)$-dimensional complex manifold.

Theorem 7.2, [Bak77] implies that the mapping $W : \mathbf{C}^{n-2} \to Z/G$ assigning to (α, λ) the equivalence class of w is submersive. More precisely, W is locally invertible on the subset $\{\alpha_{d-1} = 0\}$ of $\mathbf{C}^{n-2}$

For an even potential, there exists an odd function f. The corresponding set of asymptotic values satisfies ν linear conditions $w_{j+\nu} = -w_j$ for $j = 0, \dots, \nu - 1$. For $(\alpha, \lambda) \in \Sigma$, we can assume that $S_{j_1}, \dots, S_{j_m}$ are subdominant sectors for f. This adds m linearly independent conditions $w_{j_1} = \dots = w_{j_m} = 0$. Let Z_0 be the corresponding subset of Z. Its codimension in Z is $\nu + m$. The one-dimensional subgroup $\mathbf{C}^*$ of G consisting of multiplications by non-zero complex numbers preserves Z_0, and $gZ_0 \cap Z_0 = \emptyset$ for each $g \in G \setminus \mathbf{C}^*$. The explaination is as follows:

Since we have at least two subdominant sectors, only fractional linear transforms that preserves 0 are allowed. Furthermore, there exists a sector S_k with the value w_k different from 0 and ∞ (otherwise we would have only two asymptotic values). There is a unique transformation, multiplication by w_k^{-1}, preserving 0 and sending $\pm w_k$ to ± 1. This implies that the only transformation preserving 0 and sending $\pm w_k$ to another pair of opposite numbers is multiplication by a non-zero constant.

Hence GZ_0 is a G-invariant submanifold of Z of codimension $\nu + m - 2$, and its image $Y_0 \subset Y$ is a smooth submanifold of codimension $\nu + m - 2$. Due to Bakken's theorem, $W^{-1}(Y_0)$ intersected with the $(n-3)$-dimensional space of (α, λ) with $\alpha_{d-1} = 0$ is a smooth submanifold of codimension $\nu + m - 2$, dimension $\nu - m - 1$. Accordingly, it is a smooth submanifold of codimension m of the space $\mathbf{C}^{\nu-1}$. □

Proposition 21. *Let Σ be as in Lemma 20. If at least two adjacent Stokes sectors are missing in* (5)*, then Σ consists of two irreducible complex analytic manifolds.*

Proof. Nevanlinna theory (see [Nev32, Nev53]), implies that, for each symmetric standard graph Γ with the properties listed in Lemma 3, there exists $(\alpha, \lambda) \in \mathbf{C}^{n-1}$ and an *odd* meromorphic function $f(z)$ such that f is the ratio of two linearly independent solutions of (1) with the asymptotic values w_j in the Stokes sectors S_j, and Γ is the graph corresponding to the cell decomposition $\Phi_0 = f^{-1}(\Psi_0)$. This function, and the corresponding point (α, λ) is defined uniquely.

Let $W : \Sigma \to Y_0$ be as in the proof of Lemma 20. Then Σ is an unramified covering of Y_0. Its fiber over the equivalence class of $w \in Y_0$ consists of the points $(\alpha_\Gamma, \lambda_\Gamma)$ for all standard graphs Γ. Each action A_j^2 corresponds to a closed loop in Y_0 starting and ending at w. It should be noted that Y_0 is a connected manifold. Since for a given list of subdominant sectors a standard graph with one vertex is unique, Theorem 15 implies that the monodromy group has two orbits; odd and even eigenfunctions cannot be exchanged by any path in Y_0, while any odd (even) can be transferred into any other odd (even) eigenfunction by a sequence of $E_k^{\pm 2}$.

Hence Σ consists of two irreducible connected components (see, e.g., [Kho04]). $\square$

This immediately implies Theorem 1, for $m < \nu/2$. The following propostion implies the case where $m = \nu/2$.

Proposition 22. *Let Σ be the space of all $(\alpha, \lambda) \in \mathbf{C}^{\nu-1}$, for even ν, such that equation* (1) *admits a solution subdominant in every other Stokes sector, that is, in $S_0, S_2, \dots, S_{n-2}$.*

Then irreducible components Σ_k, $k = 0, 1, \dots$ of Σ, which are also its connected components, are in one-to-one correspondence with the sets of centrally symmetric standard graphs with k bounded faces. The corresponding solution of (1) *has k zeros and can be represented as $Q(z)e^{\phi(z)}$ where Q is a polynomial of degree k and ϕ a polynomial of degree $(d+2)/2$.*

Proof. Let us choose w and Ψ_0 as in the proof of Proposition 21. Repeating the arguments in the proof of Proposition 21, we obtain an unramified covering $W : \Sigma \to Y_0$ such that its fiber over w consists of the points $(\alpha_\Gamma, \lambda_\Gamma)$ for all standard graphs Γ with the properties listed in Lemma 3.

Since we have no adjacent dominant sectors, Lemma 17 implies that any standard graph Γ can be transformed by the monodromy action to a graph Γ_0 in ivy form with at most two Y-structures attached at the root junction(s) of type j and $j + \nu$.

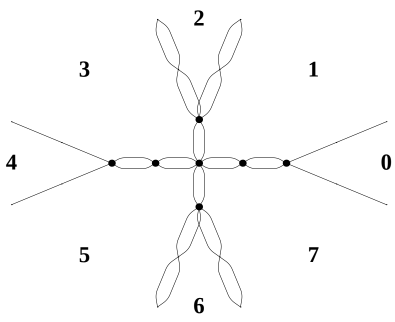
Lemma 18 implies that Γ and Γ_0 have the same number k of bounded faces. If $k = 0$, the graph Γ_0 is unique. If $k > 0$, the graph Γ_0 is completely determined by k. Hence for each $k = 0, 1, \dots$ there is a unique orbit of the monodromy group action on the fiber of W over w consisting of all standard graphs Γ with k bounded faces. This implies that Σ has one irreducible component for each k.

Since Σ is smooth by Lemma 20, its irreducible components are also its connected components.

Finally, let $f_\Gamma = y/y_1$ where y is an odd solution of (1) subdominant in the Stokes sectors $S_0, S_2, \dots, S_{n-2}$. Then the zeros of f and y are the same, each such zero belongs to a bounded domain of Γ, and each bounded domain of Γ contains a single zero. Hence y has exactly k simple zeros. Let Q be a polynomial of degree k with the same zeros as y. Then y/Q is an entire function of finite order without zeros, hence $y/Q = e^\phi$ where ϕ is a polynomial. Since y/Q is subdominant in $(d+2)/2$ sectors, $\deg \phi = (d+2)/2$. $\square$

5. Illustrating example

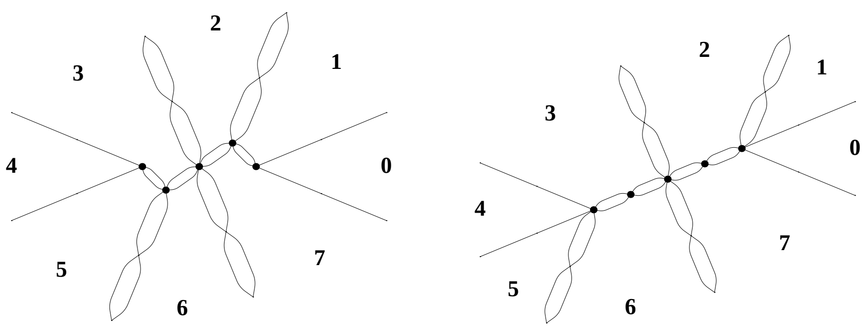
We will now give a small example on how to apply the method given in the previous section, Theorem 14 and Lemma 15. Let Γ be as in Fig. 5a. From subsection 2.4, we have that a dominant face with label j have j-edges as boundaries. Hence the faces 0 and 4 are subdominant. Also, the direction of the edges are directed counterclockwise in each of the dominant faces. Applying E_1^2, moves the I-structure at the 1-junction one step to the right,



(a) (b)

Figure 5: The graphs Γ and $E_1^2(\Gamma)$

following the 1-edges. Similarly, the I-structure at the 5-junction moves one step to the left. Therefore, $E_1^2(\Gamma)$ is given in Fig. 5b. The graph $E_1^2(\Gamma)$ is now in ivy form, it consists of a center junction connected to four I-structures and two Y-structures. We proceed by using the algorithm in Lemma 15, and apply E_1^2 two times more. These steps are given in Fig. 6. The next



(a) (b)

Figure 6: The graphs $E_1^4(\Gamma)$ and $E_1^6(\Gamma)$

step in the lemma is to move the newly created V-structures to the center junction. We therefore apply E_3^2 two times. These final steps are presented in Fig. 7, and we have reached the unique graph with only one junction.

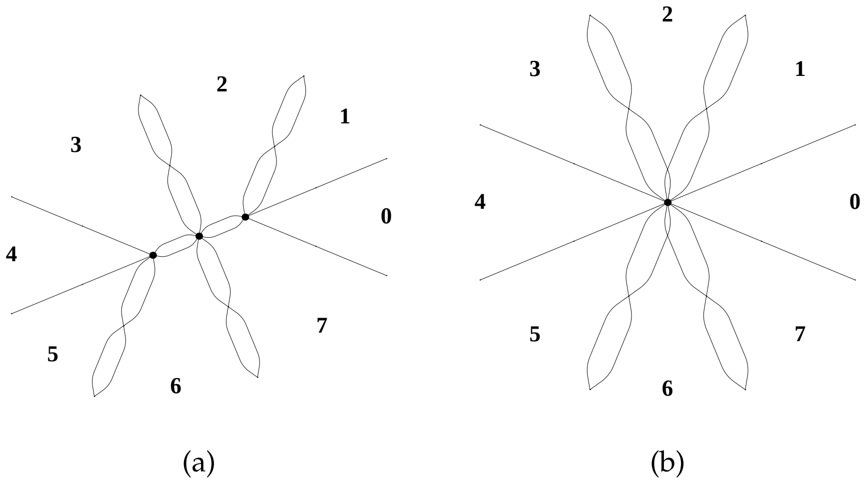


Figure 7: The graphs $E_3^2 E_1^6(\Gamma)$ and $E_3^4 E_1^6(\Gamma)$

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