ISSN: 1401-5617

EXTENSION OF POSITIVE CURRENTS WITH SPECIAL PROPERTIES OF MONGE-AMPÈRE OPERATORS

AHMAD K. AL ABDULAALI

Research Reports in Mathematics
Number 4, 2010

Department of Mathematics
Stockholm University

Electronic versions of this document are available at
http://www.math.su.se/reports/2010/4

Date of publication: November 16, 2010
2000 Mathematics Subject Classification: Primary 32U40.
Keywords: Closed positive currents, pluripolar sets, plurisubharmonic functions, plurisubharmonic positive currents, strictly k-convex functions.

Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.math.su.se/
info@math.su.se

EXTENSION OF POSITIVE CURRENTS WITH SPECIAL PROPERTIES OF MONGE-AMPÈRE OPERATORS

AHMAD K. AL ABDULAALI

Abstract. In this paper we study the extension of currents across small obstacles. Our main results are:

- Let A be a closed complete pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T a negative current of bidimension (p,p) on $\Omega\backslash A$ such that $dd^cT \geq -S$ on $\Omega\backslash A$ for some positive plurisubharmonic currents S on Ω. Assume that the Hausdorff measure $\mathcal{H}_{2p}(A\cap \overline{SuppT}) = 0$. Then $\widetilde{T}$ exists. Furthermore the current $R = \widetilde{dd^cT} - dd^c\widetilde{T}$ is negative supported in A.
- Let u be a positive strictly k-convex function on an open subset Ω of $\mathbb{C}^n$ and set $A = \{z \in \Omega : u(z) = 0\}$. Let T be a negative current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^cT \geq -S$ on $\Omega \setminus A$ for some positive plurisubharmonic (or dd^c-negative) currents S on Ω. If $p \geq k+1$, then $\widetilde{T}$ exists. If $p \geq k+2$, $dd^cS \leq 0$ and u of class $\mathcal{C}^2$, then $\widetilde{dd^cT}$ exists and $\widetilde{dd^cT} = dd^c\widetilde{T}$.

1. Introduction

In this paper we continue the work in [2]. So throughout this paper, we suppose that A is a special type of closed subset of an open subset Ω of $\mathbb{C}^n$ and T is a positive (resp. negative) current of bidimension (p,p) of $\Omega \setminus A$ such that $dd^cT \leq S$ (resp. $dd^cT \geq -S$) on $\Omega \setminus A$ for some currents S on Ω. Our main issue is finding the sufficient conditions on S that guarantee the existence of $\widetilde{T}$ and $\widetilde{dd^cT}$, and afterword studying the features of these extensions and the relations between it. In the literature, this kind of problems have been studied before. For instance, the studies in [5], [6], [11], [13], [15], [17], [21], [22] and [23], were basically based on the case when T is closed positive current. The case when $S = 0$ considered by Dabbek, Elkhadhra and El Mir. In 2009, Dabbek and Noureddine [9] discussed the case when S is closed and positive. As we see the closeness takes its place in this kind of study, so it is natural to ask about the sharpness of the closeness of S specially in [9]. Now, you can feel the theme in this paper which is a generalization of the work in [9] by replacing the closeness of S by other conditions.

The paper is divided into three sections. In the first section we gave definitions, basic properties and some facts about currents.

2000 *Mathematics Subject Classification.* 32U40.
Key words and phrases. Closed positive currents, pluripolar sets, plurisubharmonic functions, plurisubharmonic positive currents, strictly k-convex functions.

In the second one, we considered the case where A is a closed complete pluripolar set and proved our first main result.

1st Main Theorem (Theorem 3.3) Let A be a closed complete pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T a negative current of bidimension (p,p) on $\Omega\setminus A$ such that $dd^cT \geq -S$ on $\Omega\setminus A$ for some positive plurisubharmonic currents S on Ω. Assume that $\mathcal{H}_{2p}(A\cap\overline{SuppT})=0$. Then $\widetilde{T}$ exists. Furthermore the current $R=\widetilde{dd^cT}-dd^c\widetilde{T}$ is negative supported in A. If $dd^cS\leq 0$, then $\widetilde{T}$ has the same properties of T.

Using the above result, we obtained a version of Chern-Levine-Nirenberg inequality.

Theorem (Theorem 3.5) Let A be a closed complete pluripolar subset of an open set Ω of $\mathbb{C}^n$ and T be a positive current of bidimension (p,p) on $\Omega\setminus A$ such that $dd^cT\leq S$ on $\Omega\setminus A$ for some positive plurisubharmonic (resp. dd^c-negative) currents S on Ω. Let K and L compact set in Ω with $L\subset\subset K$. Assume that $\mathcal{H}_{2p}(A\cap\overline{SuppT})=0$, then there exists a constant $C_{K,L}>0$ such that for all u smooth bounded plurisubharmonic on Ω we have the following estimate

$$\int_{L\setminus A} T\wedge du\wedge d^cu\wedge\beta^{p-1}\leq C_{K,L}\|u\|^2_{\mathcal{L}^\infty(K)}(\|\widetilde{T}\|_K+\|\widetilde{dd^cT}\|_K)$$

In the third section, we started with the case where A is a zero set of strictly k-convex function and included our second main result.

2nd Main Theorem (Theorem 4.7) Let Ω be an open subset of $\mathbb{C}^n$ and u be a positive strictly k-convex function on Ω. Set $A=\{z\in\Omega : u(z)=0\}$ and T be a positive current of bidimension (p,p) on $\Omega\setminus A$ such that $dd^cT\leq S$ on $\Omega\setminus A$ for some positive and plurisubharmonic (or dd^c-negative) currents S on Ω. If $p\geq k+1$, then $\widetilde{T}$ exists. If $p\geq k+2$, $dd^cS\leq 0$ and u is of class $\mathcal{C}^2$, then $\widetilde{dd^cT}$ exists and $\widetilde{dd^cT}=dd^c\widetilde{T}$.

We ended this paper by assuming that A is a closed set and proving the following.

Theorem (Theorem 4.10) Let A be a closed subset of an open subset Ω of $\mathbb{C}^n$ and T be a negative current of bidimension (p,p) on $\Omega\setminus A$ such that $dd^cT\geq -S$ on $\Omega\setminus A$ for some positive currents S on Ω. Assume that $\mathcal{H}_{2p-2}(\overline{SuppT}\cap A)$ is locally finite. Then $\widetilde{T}$ exists. If $dd^cS\leq 0$, then $\widetilde{dd^cT}$ exists and $R=\widetilde{dd^cT}-dd^c\widetilde{T}$ is negative current supported in A.

In 1972, Harvey proved the previous result for the closed positive current T when $\mathcal{H}_{2p-1}(\overline{SuppT}\cap A)=0$. The case where $S=0$ was considered in [10]. In the inspirer article of this work [9], the authors proved the case when S is closed positive current.

It is my pleasure to acknowledge several helpful conversations with professors Hassine El Mir and Jan-Erik Björk.

2. Preliminaries and notations

Let Ω be an open subset of $\mathbb{C}^n$. Let $\mathcal{D}_{p,q}(\Omega,k)$ be the space $\mathcal{C}^k$ compactly supported differential forms of bidegree (p,q) on Ω. A form $\varphi\in\mathcal{D}_{p,p}(\Omega,k)$ is said to be strongly positive form if φ can be written as

$$\varphi(z)=\sum_{j=1}^{N}\gamma_j(z)\ i\alpha_{1,j}\wedge\overline{\alpha}_{1,j}\wedge...\wedge i\alpha_{p,j}\wedge\overline{\alpha}_{p,j}$$

where $\gamma_j\geq 0$ and $\alpha_{s,j}\in\mathcal{D}_{0,1}(\Omega,k)$. Then $\mathcal{D}_{p,p}(\Omega,k)$ admits a basis consisting of strongly positive forms. The dual space $\mathcal{D}'_{p,q}(\Omega,k)$ is the space of currents of bidimension (p,q) or bidegree $(n-p,n-q)$ and of order k. A current $T\in\mathcal{D}'_{p,p}(\Omega,k)$ is said to be positive if $\langle T,\varphi\rangle\geq 0$ for all forms $\varphi\in\mathcal{D}_{p,p}(\Omega,k)$ that are strongly positive. If $T\in\mathcal{D}'_{p,p}(\Omega,k)$ then it can be written as

$$T=i^{(n-p)^2}\sum_{|I|=|J|=n-p}T_{I,J}dz_I\wedge d\overline{z}_J$$

where $T_{I,J}$ are distributions on Ω. For the positive current $T\in\mathcal{D}'_{p,p}(\Omega,k)$ the mass of T is denoted by $\|T\|$ and defined by $\sum|T_{I,J}|$ where $|T_{I,J}|$ are the total variations of the measures $T_{I,J}$. Let $\beta=dd^c|z|^2$ be the Khäler form on $\mathbb{C}^n$ (where $d=$ and $d^c=i(-\partial+\overline{\partial})$ thus $dd^c=2i\partial\overline{\partial}$), then there exists a constant $C>0$ depends only on n and p such that

$$T\wedge\frac{\beta^p}{2^p p!}\leq\|T\|\leq C\ T\wedge\beta^p$$

Along the way a current T is said to be plurisubharmonic if dd^cT is positive current. Let (χ_n) be a smooth bounded sequence which vanishes on a neighborhood of closed subset $A\subset\Omega$ and χ_n converges to $1_{\Omega\setminus A}$, and T be a current defined on $\Omega\setminus A$. If χ_nT has a limit which does not depend on (χ_n), this limit is the trivial extension of T by zero across A noted by $\widetilde{T}$. Thus, $\widetilde{T}$ exists if and only if $\|T\|$ has locally finite mass on Ω.

A current T is said to be $\mathbb{C}$-normal if T and dd^cT are of locally finite mass. We recall that T is $\mathbb{C}$-flat current if $T=F+\partial H+\overline{\partial}S+\partial\overline{\partial}R$, where R,H,S and R are currents with locally integrable coefficients. On this class of currents, the support theorem says that for $\mathbb{C}$-flat current T of bidimension (p,p) if $\mathcal{H}_{2p}(SuppT)=0$, then $T=0$. Let $k\leq p$ and $T\in\mathcal{D}'_{p,q}(\Omega)$ with locally integrable coefficients. Set $\pi:\mathbb{C}^n\to\mathbb{C}^k$, $\pi(z',z'')=z'$ and $i_{z'}:\mathbb{C}^{n-k}\to\mathbb{C}^n$, $i_{z'}(z'')=(z',z'')$. Then the slice $\langle T,\pi,z'\rangle$ which is defined by

$$\langle T,\pi,z'\rangle(\varphi)=\int_{z''\in\pi^{-1}(z')}i^*_{z'}T(z'')\wedge i^*_{z'}\varphi(z''),\quad\forall\varphi\in\mathcal{D}_{p-k,p-k}(\Omega)$$

is well defined $(p-q,p-q)$-current for a.e z', and supported in $\pi^{-1}(z')$. Notice that, by the pull back assumptions we obtain

$$dd^c\langle T,\pi,z'\rangle=\langle dd^cT,\pi,z'\rangle,\ d^c\langle T,\pi,z'\rangle=\langle d^cT,\pi,z'\rangle,\ d\langle T,\pi,z'\rangle=\langle dT,\pi,z'\rangle$$

So, we deduce that the slice $\langle T, \pi, z' \rangle$ is well defined for a.e z' and for every $\mathbb{C}$-flat current T. Moreover, we have the slicing formula

$$\int_{\Omega} T \wedge \varphi \wedge \pi^* \beta'^k = \int_{z' \in \pi(\Omega)} \langle T, \pi, z' \rangle(\varphi) \beta'^k$$

This formula is helpful in many cases. Actually, By this formula we can prove the properties of T by testing them for the slice of T.

We end this section by giving the following two theorems. The first one is called Chern-Levine-Nirenberg inequality and the second is a modification for that inequality proved by Al Ameer [3].

Theorem 2.1. *Let Ω be an open subset of $\mathbb{C}^n$ and T be a closed positive current of bidimension (p, p). Let $u_1, ..., u_p$ are locally bounded plurisubharmonic functions on Ω. For all compact subsets K, L of Ω with $L \subset\subset K$, there exists s constant $C_{K,L} \geq 0$ such that*

$$\|T \wedge dd^c u_1 \wedge ... \wedge dd^c u_q\|_L \leq C_{K,L} \|T\|_K \|u_1\|_{\mathcal{L}^\infty(K)} ... \|u_q\|_{\mathcal{L}^\infty(K)}$$

Theorem 2.2. *Let Ω be an open subset of $\mathbb{C}^n$. Let K and L compact sets in Ω with $L \subset\subset K$. Assume that $T \in \mathcal{D}'_{p,p}(\Omega)$ is positive and dd^cT is of order zero, then there exists a constant $C_{K,L} > 0$ such that for all plurisubharmonic function u on Ω of class $\mathcal{C}^2$ we have the following estimate*

$$\|T \wedge dd^c u\|_L \leq C_{K,L} \|u\|_{\mathcal{L}^\infty(K)} (\|T\|_K + \|dd^c T\|_K)$$

3. The case when A is closed pluripolar set

In this section we show our first main result. For closed currents T the result was done by El Mir and Feki [15]. The case when $S = 0$ was proved in [10] by Dabbek, Elkhadhra and El Mir. Recently, Dabbek and Noureddine [9] have shown the result when S is positive closed current. The proof of our main result will pass through several steps. So let us first give an inequality which is a very useful tool in our study.

Lemma 3.1. *Let Ω be an open subset of $\mathbb{C}^n$. Let K and L compact sets in Ω with $L \subset\subset K$. Assume that T is positive and plurisubharmonic (resp. dd^c-negative) (p, p) current, then there exists a constant $C_{K,L} > 0$ such that for all plurisubharmonic functions $u_1, ..., u_q$ of class $\mathcal{C}^2$ we have*

$$\|T \wedge dd^c u_1 \wedge ... \wedge dd^c u_q\|_L \leq C_{K,L} \prod_{j=1}^{q} \|u_j\|_{\mathcal{L}^\infty(K)} (\|T\|_K + \|dd^c T\|_K)$$

Proof. By induction, the case when $q = 1$ follows from [12]. To show the case $q = 2$, let us take L_1 compact subset such that L_1 contains L and relatively compact in K. Since $T \wedge dd^c u_1$ is positive and plurisubharmonic (resp. dd^c-negative), then we get

$$\begin{aligned} &\|T \wedge dd^c u_1 \wedge dd^c u_2\|_L \\ &\quad \leq C^{(1)}_{K,L} \|u_2\|_{\mathcal{L}^\infty(K)} (\|T \wedge dd^c u_1\|_{L_1} + \|dd^c T \wedge dd^c u_1\|_{L_1}) \end{aligned}$$

But $dd^cT \wedge dd^cu_1$ closed and positive (resp. negative) so by the first step and Chern-Levine-Nirenberg inequality we have

$$\begin{aligned}&\|T \wedge dd^cu_1 \wedge dd^cu_2\|_L \\ &\quad \leq C^{(1)}_{K,L}\|u_2\|_{\mathcal{L}^\infty(K)}\left[C^{(2)}_{K,L}\|u_1\|_{\mathcal{L}^\infty(K)}(\|T\|_K + 2\|dd^cT\|_K)\right] \\ &\quad \leq C_{K,L}\|u_1\|_{\mathcal{L}^\infty(K)}\|u_2\|_{\mathcal{L}^\infty(K)}(\|T\|_K + 2\|dd^cT\|_K)\end{aligned}$$

Assume that the inequality holds for $q = k$. We want to show the inequality for $q = k + 1$. Since $T \wedge \bigwedge_{j=1}^k dd^cu_j$ is positive and plurisubharmonic (resp. dd^c-negative) current, then similarly as we have done above we deduce

$$\begin{aligned}&\|T \wedge dd^cu_1 \wedge ... \wedge dd^cu_p\|_L \\ &\leq C^{(1)}_{K,L}\|u_{k+1}\|_{\mathcal{L}^\infty(K)}(\|T \wedge \bigwedge_{j=1}^{k} dd^cu_j\|_{L_1} + \|dd^cT \wedge \bigwedge_{j=1}^{k} dd^cu_j\|_{L_1}) \\ &\leq C^{(2)}_{K,L}\prod_{j=1}^{k+1}\|u_j\|_{\mathcal{L}^\infty(K)}(\|T\|_K + k\|dd^cT\|_K) \\ &\quad + C^{(3)}_{K,L}\prod_{j=1}^{k+1}\|u_j\|_{\mathcal{L}^\infty(K)}\|dd^cT\|_K \\ &\leq C_{K,L}\prod_{j=1}^{k+1}\|u_j\|_{\mathcal{L}^\infty(K)}(\|T\|_K + (k+1)\|dd^cT\|_K)\end{aligned}$$

Proving our lemma. □

Proposition 3.2. *Let A be a closed complete pluripolar subset of an open subset $\Omega \subset \mathbb{C}^n$ and T be a positive current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^cT \leq S$ on $\Omega \setminus A$ for some positive and plurisubharmonic (resp. dd^c-negative) currents S on Ω. Let v be a plurisubharmonic function of class $\mathcal{C}^2$, $v \geq -1$ on Ω such that*

$$\Omega' = \{z \in \Omega : v(z) < 0\}$$

is relatively compact in Ω. Let $K \subset \Omega'$ be a compact subset and let us set

$$c_K = -\sup_{z\in K} v(z)$$

Then there exists a constant $\eta \geq 0$ such that for all integer $1 \leq s \leq p$ and for every plurisubharmonic function u on Ω' of class $\mathcal{C}^2$ satisfying $-1 \leq u < 0$ we have,

$$\begin{aligned}\int_{K\setminus A} T \wedge (dd^cu)^p \leq c_K^{-s}\int_{\Omega'\setminus A} T \wedge (dd^cv)^s \wedge (dd^cu)^{p-s} \\ + \eta(\|S\|_{\Omega'} + \|dd^cS\|_{\Omega'})\end{aligned}$$

This proposition generalizes a result in [10] where the authors considered the case of positive and dd^c-negative current. The case when S is closed positive done in [9].

Proof. We follow the same techniques in [10]. By ([13], Proposition II.2) there exists a negative plurisubharmonic function f on Ω' which is smooth on $\Omega' \setminus A$ such that

$$A \cap \Omega' = \{z \in \Omega' : f(z) = -\infty\}$$

We choose λ, μ such that $0 < \mu < \lambda < c_K$. For $m \in \mathbb{N}$ and ε small enough we set

$$\varphi_m(z) = \mu\, u(z) + \tfrac{f(z)+m}{m+1} \text{ and } \varphi_{m,\varepsilon}(z) = \max\nolimits_\varepsilon\left(v(z)+1, \varphi_m(z)\right)$$

where $\max_\varepsilon$ is the convolution of the function $(x_1, x_2) \mapsto \max(x_1, x_2)$ by a positive regularization kernel on $\mathbb{R}^2$ depending only on $\|(x_1, x_2)\|$. Thus we have $\varphi_{m,\varepsilon}(z) \in psh(\Omega') \cap C^\infty(\Omega')$. Furthermore, $\varphi_{m,\varepsilon}(z) = v(z) + 1$ in a neighborhood of $\partial\Omega' \cup (\Omega' \cap \{f \leq -m\})$. Consider the open subset

$$\Omega'_m = \Omega' \cap \{f > -m\}$$

Then by Stokes formula we have

$$\int_{\Omega'_m} T \wedge (dd^c u)^{p-s} \wedge (dd^c \varphi_{m,\varepsilon})^{s-1} \wedge dd^c(\varphi_{m,\varepsilon} - v - 1)$$
$$\leq \int_{\Omega'_m} (\varphi_{m,\varepsilon} - v - 1) S \wedge (dd^c u)^{p-s} \wedge (dd^c \varphi_{m,\varepsilon})^{s-1}$$

Hence

$$\begin{aligned}
&\int_{\Omega'_m} T \wedge (dd^c u)^{p-s} \wedge (dd^c \varphi_{m,\varepsilon})^{s} \\
&\leq \int_{\Omega'_m} (\varphi_{m,\varepsilon} - v - 1) S \wedge (dd^c u)^{p-s} \wedge (dd^c \varphi_{m,\varepsilon})^{s-1} \\
&\quad + \int_{\Omega'_m} T \wedge (dd^c u)^{p-s} \wedge (dd^c \varphi_{m,\varepsilon})^{s-1} \wedge dd^c v
\end{aligned} \tag{3.1}$$

Let us set

$$S_{k,\varepsilon} := \int_{\Omega'_m} (\varphi_{m,\varepsilon} - v - 1) S \wedge (dd^c u)^{p-s} \wedge (dd^c \varphi_{m,\varepsilon})^{s-1-k} \wedge (dd^c v)^k$$

By iterating the operation in (3.1), we deduce that

$$\int_{\Omega'_m} T \wedge (dd^c u)^{p-s} \wedge (dd^c \varphi_{m,\varepsilon})^{s} \leq \int_{\Omega'_m} T \wedge (dd^c u)^{p-s} \wedge (dd^c v)^{s} + \sum_{k=0}^{s-1} S_{k,\varepsilon}$$

Let $R > 0$ and $K_R = \{z \in K : f(z) \geq -R\}$. For m sufficiently large, $K_R \subset \Omega'_m$ and for any $z \in K_R$,

$$\varphi_m(z) \geq -\mu + \frac{m - R}{m + 1} > 1 - \lambda$$

Moreover, $v \leq -c_K$ on K_R so we get

$$v + 1 \leq 1 - c_K \leq 1 - \lambda$$

then $\varphi_{m,\varepsilon}=\varphi_m$ in a neighborhood of K_R. Therefore, by the above inequality we obtain

$$\int_{K_R} T\wedge(dd^cu)^{p-s}\wedge(dd^c\varphi_m)^s \le \int_{\Omega'_m} T\wedge(dd^cu)^{p-s}\wedge(dd^cv)^s+\sum_{k=0}^{s-1}S_{k,\varepsilon}$$

Notice that $(dd^c\varphi_m)^s\ge\mu^s\,(dd^cu)^s$ because $dd^cf\ge 0$. So

$$\mu^s\int_{K_R} T\wedge(dd^cu)^{p}\le\int_{\Omega'_m} T\wedge(dd^cu)^{p-s}\wedge(dd^cv)^s+\sum_{k=0}^{s-1}S_{k,\varepsilon} \tag{3.2}$$

Notice that each $S_{k,\varepsilon}$ is bounded independently of ε. Indeed, since S is positive plurisubharmonic current and $\varphi_{m,\varepsilon}-v-1=0$ on $\partial\Omega'_m$, then by the previous lemma there exists $\eta_k\ge 0$ such that

$$S_{k,\varepsilon}\le\eta_k\|u\|^{p-s}_{\mathcal{L}^\infty(\Omega')}\|\varphi_{m,\varepsilon}\|^{s-k-1}_{\mathcal{L}^\infty(\Omega')}\|v\|^{k}_{\mathcal{L}^\infty(\Omega')}(\|S\|_{\Omega'}+\|dd^cS\|_{\Omega'}) \tag{3.3}$$

Therefore there exists $\eta\ge 0$ making (3.2) as follows

$$\begin{aligned}\mu^s\int_{K_R} T\wedge(dd^cu)^{p}\le&\int_{\Omega'_m} T\wedge(dd^cu)^{p-s}\wedge(dd^cv)^s\\ &+\eta(\|S\|_{\Omega'}+(p-1)\|dd^cS\|_{\Omega'})\end{aligned}$$

We finished the proof by letting first $m\to\infty$ and secondly $R\to\infty$. □

Now we will prove our first main theorem using the same technique in [10] and Proposition 3.2.

Theorem 3.3. *Let A be a closed complete pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T a negative current of bidimension (p,p) on $\Omega\setminus A$ such that $dd^cT\ge -S$ on $\Omega\setminus A$ for some positive plurisubharmonic currents S on Ω. Assume that $\mathcal{H}_{2p}(A\cap\overline{SuppT})=0$. Then $\widetilde{T}$ exists. Furthermore the current $R=\widetilde{dd^cT}-dd^c\widetilde{T}$ is negative supported in A. If $dd^cS\le 0$, then $\widetilde{T}$ has the same properties of T.*

Proof. Let us first assume that $\widetilde{T}$ exists. Then by [12] the extension $\widetilde{dd^cT}$ exists and R is negative current. If S is dd^c-negative current then by ([10], Proposition 2) the current $\widetilde{-S}$ is negative plurisubharmonic. Implies that $\widetilde{T}$ is negative and $dd^c\widetilde{T}\ge\widetilde{dd^cT}\ge\widetilde{-S}$. In other word, $\widetilde{T}$ and T are of the same class.

In order to show the existence of $\widetilde{T}$ we will proceed as in [10]. Since the problem is local, we will show that T is of lacally finite mass near every point z_0 in A. Without loss of generality, one can assume that z_0 is the origin. Since $\mathcal{H}_{2p}(A\cap\overline{SuppT})=0$, there exists a system of coordinates and a polydisk $\triangle^p\times\triangle^{n-p}\subset\mathbb{C}^p\times\mathbb{C}^{n-p}$ such that $(A\cap\overline{SuppT})\cap(\triangle^p\times\partial\triangle^{n-p})=\phi$. Moreover the projection map $\pi:(A\cap\overline{SuppT})\cap(\triangle^p\times\triangle^{n-p})\to\triangle^p$ is proper, and as $\pi(A\cap\overline{SuppT})$ is closed with a zero Lebesgue measure in $\triangle^p$ one can find an open subset $O\subset\triangle^p\setminus\pi(A\cap\overline{SuppT})$. Therefore the current has locally finite on $O\times\triangle^{n-p}$. Let $0<\delta<1$ such that

$(A\cap \overline{SuppT})\cap(\triangle^p\times\triangle^{n-p}(1-\delta,1+\delta))=\phi$ and fix a and t two reals in $(\delta,1)$ such that $a<t$. Set

$$\rho_\varepsilon=\max\nolimits_\varepsilon\left(\pi^*\rho,\tfrac{1}{t^2-a^2}(|z''|^2-t^2)\right)$$

where ρ is a plurisubharmonic function on $\triangle^p$ such that $(dd^c\rho)^p$ supported in O. We have $-1\le\rho_\varepsilon<0$ in $t\triangle^n$ and $\rho_\varepsilon=\pi^*\rho$ on $|z''|\le a$, and we obtain

$$\int_{(t\triangle^n)\setminus A}T\wedge(dd^c\rho_\varepsilon)^p=\int_{(t\triangle^p)\times\{|z''|<a\}\setminus A}T\wedge(dd^c(\pi^*\rho))^p+\int_{(t\triangle^p)\times\{a\le|z''|<t\}}T\wedge(dd^c\rho_\varepsilon)^p$$

since $(dd^c\pi^*\rho)^p$ supported in $O\times\triangle^{n-p}$ then both integrals of the right hand side are finite. By applying Proposition 3.2 on $-T$, we deduce that $\widetilde{T}$ exists. □

Corollary 3.4. *Let A be a closed complete pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T a positive current of bidimension (p,p) on $\Omega\setminus A$ such that $dd^cT\ge -S$ on $\Omega\setminus A$ for some positive dd^c-negative currents S on Ω. Assume that $\mathcal{H}_{2p-2}(A\cap\overline{SuppT})=0$. Then $\widetilde{T}$ exists. Furthermore the current $\widetilde{dd^cT}=dd^c\widetilde{T}$.*

This result has been studied in many different cases before. Actually, the authors in [10] considered the case when $S=0$. The case when $\widetilde{dd^cT}$ exists and $\mathcal{H}_{2p}(A\cap\overline{SuppT})=0$ done by Dabbek in [Da]. Dabbek proved that in this case the residual current is positive and closed by using the same technique in [10] with the local potential of a positive closed current given in [5]. In [2], the result was proved for the positive closed current S.

Proof. Applying ([10], Theorem 1) for the current dd^cT+S, the extension $\widetilde{dd^cT}$ exists. Now, the result follows from Theorem 5 in [10]. □

We end this section by the following theorem which is a version of Chern-Levine-Nirenberg inequality.

Theorem 3.5. *Let A be a closed complete pluripolar subset of an open set Ω of $\mathbb{C}^n$ and T be a positive current of bidimension (p,p) on $\Omega\setminus A$ such that $dd^cT\le S$ on $\Omega\setminus A$ for some positive plurisubharmonic (resp. dd^c-negative) currents S on Ω. Let K and L compact set in Ω with $L\subset\subset K$. Assume that $\mathcal{H}_{2p}(A\cap\overline{SuppT})=0$, then there exists a constant $C_{K,L}>0$ such that for all u plurisubharmonic function on Ω of class $\mathcal{C}^2$ we have the following estimate*

$$\int_{L\setminus A}T\wedge dd^cu\wedge\beta^{p-1}\le C_{K,L}\|u\|_{\mathcal{L}^\infty(K)}(\|\widetilde{T}\|_K+\|\widetilde{dd^cT}\|_K)$$

Proof. From Theorem 3.3, the extensions $\widetilde{T}$ and $\widetilde{dd^cT}$ exist. Moreover, $\widetilde{T}$ is positive and $\widetilde{dd^cT}$ is of order zero. Hence the result follows from Theorem 2.2. □

Application of Theorem 3.3. Let A be a closed complete pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T be a closed positive current of bidimension (p,p) on $\Omega \setminus A$. Assume that $\mathcal{H}_{2p-2}(A) = 0$. Now, suppose that g is plurisubharmonic function on Ω which is smooth on $\Omega \setminus A$ and (g_j) is decreasing sequence of smooth plurisubharmonic functions on Ω converging pointwise to g on $\Omega \setminus A$. In this case by using ([10], Theorem 1), we can find the extensions $\widetilde{g_jT}$ and $\widetilde{gT}$. But we can't use that theorem to find $\widetilde{g^2T}$, since we don't know weather g^2 is plurisubharmonic or not. Despite this, we can extend g^2T over A. In fact, the current g^2T is positive. We may assume that locally $g \leq 0$, so simple computation shows that

$$\begin{aligned} dd^c(g^2T) &= 2dg \wedge d^cg \wedge T + 2gdd^cg \wedge T \\ &\leq 2dg \wedge d^cg \wedge T \end{aligned}$$

Now, set $S = 2dg \wedge d^cg \wedge T$, then S is positive dd^c-negative current on $\Omega \setminus A$. Applying [10], the current $\widetilde{S}$ exists and is positive dd^c-negative on Ω. Hence by Theorem 3.3, $\widetilde{g^2T}$ and $\widetilde{g_j^2T}$ exist for all j. Moreover, $\widetilde{g_j^2T}$ converges to $\widetilde{g^2T}$.

4. The case when A is a zero set of a strictly k-convex function

In this section we include our second main result. The result was considered before in several case. In 1984, El Mir [13] studied the case when T is positive closed current and A is a zero set of exhaustion strictly plurisubharmonic function. For positive dd^c-negative currents T the result was proved in [10]. In [9] the authors obtained the result when S is closed positive current. Al Abdulaali [2] considered the case when S is positive and A is a zero set of exhaustion strictly plurisubharmonic function.

Let us start this section with the definition of k-convex functions followed by a lemma which is given in [14].

Definition 4.1. *Let u be a continuous real function defined on an open subset Ω of $\mathbb{C}^n$. we say that u is strictly k-convex if there exists a continuous $(1,1)$-form γ defined on Ω which admits $(n-k)$-positive eigenvalues at each point, and such that the current $dd^cu - \gamma$ is positive on Ω.*

Lemma 4.2. *Let u be a strictly k-convex function on an open subset Ω of $\mathbb{C}^n$ and let $\gamma \geq 0$ be a continuous (1,1)-form on Ω. Then for all $z \in \Omega$, there exists a neighborhood V_z of z and a smooth strictly plurisubharmonic function f on V_z such that*

$$dd^cu \wedge (dd^cf)^k - \gamma^{k+1} \quad \text{is positive on } V_z$$

Proposition 4.3. *Let Ω be an open subset of $\mathbb{C}^n$ and u be a strictly k-convex function on Ω. For $c \in \mathbb{R}$, we set $\Omega_c = \{z \in \Omega : u(z) \leq c\}$. Let T be a positive current of bidimension (p,p) on $\Omega \setminus \Omega_c$ such that $dd^cT \leq S$ on*

$\Omega \setminus \Omega_c$ for some positive and plurisubharmonic (resp. dd^c-negative) currents S on Ω. If $p \geq k+1$, then T is of finite mass near Ω_c.

Proof. As in [10] we can assume that $u \in \mathcal{C}^\infty(\Omega \setminus A)$. Since the problem is local, all what we need is to show that for every $z \in u^{-1}\{c\}$, there exists $\omega \subset\subset \Omega$ contains z such that

$$\int_{\omega \setminus \Omega_{c+\frac{2}{n}}} T \wedge \beta^p < \infty$$

independently of n. Since u is strictly k-convex function then there exists a system of coordinates on $\mathbb{C}^n$ and an open neighborhood V of z and $\lambda > 0$ such that

$$dd^c u + \frac{\lambda}{2}\beta' - 2\beta''$$

is a positive current on V, where $\beta' = dd^c|z'|^2$, $z' \in \mathbb{C}^k$ and $\beta'' = dd^c|z''|^2$, $z'' \in \mathbb{C}^{n-k}$. Let $r > 0$ such that $B(z,r) \subset V$, and χ be a smooth function satisfying $\chi = 0$ on $\overline{B}(z,r)$ and $\chi = -1$ on $\Omega \setminus B(z, \frac{2}{3}r)$. For a sufficiently small $\delta > 0$, we set $v = u + \delta\chi$ and denote by φ_ε a regularization kernel on $\mathbb{C}^n$ depending only on $|z|$. Choose ε_n small enough so that $v_n = v * \varphi_{\varepsilon_n}$ satisfies $0 < v - v_n < \frac{1}{n}$ and

$$dd^c v_n + \lambda\beta' - \beta''$$

is a positive form for all n. By Lemma 4.2, if $\alpha = dd^c f$ and $n \in \mathbb{N}$, then we find that $T \wedge \beta^p \leq T \wedge dd^c v_n \wedge \alpha^{p-1}$ on $V \setminus \Omega_c$. Now let $(h_n)_n$ be a sequence of increasing convex positive functions such that

$$0 \leq \sup(t-c, 0) - h_n(t) \leq \frac{1}{n}, \forall n \in \mathbb{N}, \forall t \in \mathbb{R}$$

and

$$h_n'(t) = 1 \text{ for } t \geq c + \tfrac{1}{n}.$$

If we set $u_n = h_n \circ v_n$, then clearly we have

$$dd^c u_n \wedge \alpha^{p-1} = (h_n' \circ v_n) dd^c v_n \wedge \alpha^{p-1} + (h_n'' \circ v_n) i\partial v_v \wedge \overline{\partial} v_n \wedge \alpha^{p-1}$$

From the above equality and the hypotheses of $(h_n)_n$, it follows that $dd^c \wedge \alpha^{p-1} \geq \beta^p$ on $B(z, \frac{r}{2}) \setminus \Omega_{c+\frac{2}{n}}$. Indeed, $\chi = 0$ on $B(z, \frac{r}{2})$. So when $u > c + \frac{2}{n}$ we have

$$v_n \geq v - \frac{1}{n} = u - \frac{1}{n} > c + \frac{1}{n}$$

Therefore $h_n' \circ v_n = 1$ and $h_n'' \circ v_n = 0$. Implies that $dd^c u_n \wedge \alpha^{p-1} = dd^c v_n \wedge \alpha^{p-1}$ on $B(z, \frac{r}{2}) \setminus \Omega_{c+\frac{2}{n}}$. Moreover, (u_n) vanishes in a neighborhood of Ω_c, depending on n. Let g be a smooth function with compact support belonging to $\Omega \setminus \Omega_c$, $g = 1$ in a neighborhood of $\partial B(z,r)$, $0 \leq g \leq 1$ and vanishes on a neighborhood of $(\Omega \setminus \Omega_c) \cap B(z, \frac{2}{3}r)$. Let $T_{\varepsilon_k} = T * \varphi_{\varepsilon_k}$ be a smoothing of T which is of course convergent weakly* to T. Let us set $B_r = B(z,r)$ and $\omega = B_{\frac{r}{2}}$, hence

$$\int_{\omega\setminus\Omega_{c+\frac{2}{n}}} T\wedge\beta^p \le \lim_{\varepsilon_k\to 0}\int_{B_r} T_{\varepsilon_k}\wedge dd^c u_n\wedge\alpha^{p-1} \tag{4.1}$$

On the other hand, we have

$$\begin{aligned}\int_{B_r} T_{\varepsilon_k}\wedge dd^c u_n\wedge\alpha^{p-1} &= \int_{B_r} T_{\varepsilon_k}\wedge dd^c(gu_n+(1-g)u_n)\wedge\alpha^{p-1}\\ &= \int_{B_r} T_{\varepsilon_k}\wedge dd^c(gu_n)\wedge\alpha^{p-1}\\ &\quad+\int_{B_r} u_n(1-g)dd^cT_{\varepsilon_k}\wedge\alpha^{p-1}\\ &\le \int_{B_r} T_{\varepsilon_k}\wedge dd^c(gu_n)\wedge\alpha^{p-1}\\ &\quad+\int_{B_r} u_n(1-g)S_{\varepsilon_k}\wedge\alpha^{p-1}\end{aligned} \tag{4.2}$$

The nice choice of g makes the sequence (gu_n) converges uniformly to $(v-c)g$. Moreover, on $Supp\ g\cap Supp u_n$ the positive current T has locally finite mass. So by Lemma 3.1, we obtain that the last right hand side integrals in (4.2) are bounded independently of ε_k and n. In virtue of (4.1) we deduce that T is of finite mass on $\omega\setminus\Omega_c$. □

Remark 4.4. In the case of strictly 0-convex functions, the condition $dd^cS\ge 0$ (or $dd^cS\le 0$) can be omitted. Indeed, in this case we can replace α by β in the proof of last proposition. As S is positive, there exists $C>0$ so that

$$\begin{aligned}&\int_{B_r} T_{\varepsilon_k}\wedge dd^c u_n\wedge\beta^{p-1}\\ &\quad\le \int_{B_r} T_{\varepsilon_k}\wedge dd^c(gu_n)\wedge\beta^{p-1}+\int_{B_r} u_n(1-g)S_{\varepsilon_k}\wedge\beta^{p-1}\\ &\quad\le \int_{B_r} T_{\varepsilon_k}\wedge dd^c(gu_n)\wedge\beta^{p-1}+C\|S_{\varepsilon_k}\|_{B_r}\end{aligned}$$

Corollary 4.5. *Let Ω be an open subset of $\mathbb{C}^n$ and let u be a positive plurisubharmonic function of class $\mathcal{C}^2$ and $0\le s<r$ such that $B_r\{z\in\Omega, u(z)<r\}\subset\subset\Omega$. Let T be a positive current of bidimension (p,p) on $\Omega\setminus B_s$ such that $dd^cT\le S$ on $\Omega\setminus B_s$ for some positive and plurisubharmonic (resp. dd^c-negative) current on Ω. Choose $\delta\in\mathbb{R}$ such that $0<\delta<r-s$ and $B_{r+\delta}\subset\subset\Omega$. Then there exist $C_1>0$ and $C_2>0$ such that*

$$\begin{aligned}\int_{B_r\setminus B_s} T\wedge(dd^cu)^p &\le C_1\int_{C(r-\delta,r+\delta)} T\wedge(dd^cu)^p\\ &\quad+C_2\|u\|^{p-1}_{\mathcal{L}^\infty(L)}(\|S\|_L+(p-1)\|dd^cS\|_L)\end{aligned}$$

where $C(r-\delta,r+\delta)=\{z\in\Omega, r-\delta<u(z)<r+\delta\}$ and $L=\overline{B_{r+\delta}}$

Proof. We set $\varphi_n = \max(u - \frac{1}{n} - s) * \alpha_{\varepsilon_n}$. For ε_n small enough have $dd^c\varphi_n \geq \frac{1}{2}dd^c u$ on $\{u > \frac{2}{n} + s\}$, then

$$\frac{1}{2}\int_{C(\frac{2}{n}+s,r)} T \wedge (dd^c u)^p \leq \lim_{\varepsilon_k \to 0} \int_{B_r} T_{\varepsilon_k} \wedge dd^c\varphi_n \wedge (dd^c u)^{p-1} \tag{4.3}$$

Let g be a smooth function with support in $C(r-\delta, r+\delta)$ such that $0 \leq g \leq 1$ and $g = 1$ on a neighborhood of ∂B_r. The sequence $g\varphi_n$ converges toward $g(u - s)$ in $\mathcal{C}^2$. Then by similar argument as in Proposition 4.3, we have

$$\begin{aligned}\int_{B_r} T_{\varepsilon_k} \wedge dd^c\varphi_n \wedge (dd^c u)^{p-1} &= \int_{B_r} T_{\varepsilon_k} \wedge dd^c(g\varphi_n + (1-g)\varphi_n) \wedge (dd^c u)^{p-1} \\ &= \int_{B_r} T_{\varepsilon_k} \wedge dd^c(g\varphi_n) \wedge (dd^c u)^{p-1} \\ &\quad + \int_{B_r} \varphi_n(1-g) dd^c T_{\varepsilon_k} \wedge (dd^c u)^{p-1} \\ &\leq \int_{B_r} T_{\varepsilon_k} \wedge dd^c(g\varphi_n) \wedge (dd^c u)^{p-1} \\ &\quad + \int_{B_r} \varphi_n(1-g) S_{\varepsilon_k} \wedge (dd^c u)^{p-1}\end{aligned}$$

and by Lemma 3.1, there exist $C_1 > 0$ and $C_2 > 0$ independent of n and ε_k such that

$$\begin{aligned}\lim_{\varepsilon_k \to 0} \int_{B_r} T_{\varepsilon_k} \wedge dd^c\varphi_n \wedge (dd^c u)^{p-1} \leq C_1 &\int_{Supp(g)} T \wedge (dd^c u)^p \\ &+ C_2 \|u\|^{p-1}_{\mathcal{L}^\infty(L)}(\|S\|_L + (p-1)\|dd^c S\|_L)\end{aligned}$$

We end the proof by letting n tends to ∞ in (4.3). □

Remark 4.6. If $u(z) = |z|^2$, then we don't need the plurisubharmonicity of S in Corollary 4.5.

Theorem 4.7. *Let Ω be an open subset of $\mathbb{C}^n$ and u be a positive strictly k-convex function on Ω. Set $A = \{z \in \Omega : u(z) = 0\}$ and T be a positive current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^c T \leq S$ on $\Omega \setminus A$ for some positive and plurisubharmonic (or dd^c-negative) currents S on Ω. If $p \geq k+1$, then $\widetilde{T}$ exists. If $p \geq k+2$, $dd^c S \leq 0$ and u is of class $\mathcal{C}^2$, then $\widetilde{dd^c T}$ exists and $\widetilde{dd^c T} = dd^c\widetilde{T}$.*

Proof. If $p \geq k+1$, then by the previous proposition $\widetilde{T}$ exists. To show the second part we first note that $S - dd^c T$ is positive dd^c-negative $(p-1,p-1)$-current. So if $p-1 \geq k+1$, then $\widetilde{S - dd^c T}$ exists. Implies that $\widetilde{dd^c T}$ exists, and by ([10], Theorem 4) the result follows. □

Corollary 4.8. *Ω be an open subset of $\mathbb{C}^n$ and A be a Cauchy-Riemann variety of class $\mathcal{C}^1$ in Ω with dimension k. Let T be a positive current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^c T \leq S$ on $\Omega \setminus A$ for some for some positive dd^c-negative currents S on Ω. If $p \geq k+1$, then $\widetilde{T}$ exists. If $p \geq k+2$, then $\widetilde{dd^c T}$ exists and $dd^c\widetilde{T} = \widetilde{dd^c T}$.*

Notice that, For strictly 0-convex functions we only need the positivity of S to find $\widetilde{T}$, thanks to Remark 4.4.

Proof. By Theorem III.6 and Theorem II.7 in [13], locally there exists a positive strictly k-convex function u of class $\mathcal{C}^2$ such that $A = u^{-1}(\{0\})$. Then the result follows from Theorem 4.7. □

As we saw in the case of pluripolar sets A, the condition on the Hausdorff measure of A is sharp (see [10], Example 3). But using Proposition 4.3. we can obtain the extension in the case of compact pluripolar sets even if its Hausdorff measure is very high.

Theorem 4.9. *Let A be a compact pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T be a positive (p,p) current on $\Omega\backslash A$ such that $dd^cT \leq S$ on $\Omega\backslash A$ for some positive currents S on Ω. If $p \geq 1$, then $\widetilde{T}$ exists and $R = \widetilde{dd^cT} - dd^c\widetilde{T}$ is positive current supported in A.*

Proof. By Proposition II.2. in [13], there exists a strictly pseudoconvex open set Ω' such that $A \subset \Omega' \subset\subset \Omega$, and a negative plurisubharmonic function u on Ω' satisfying $A = \{z \in \Omega', u(z) = -\infty\}$ and such that e^u is continuous. Let φ be an exhaustion continuous strictly plurisubharmonic function on Ω' and set $c =$sup$\{\varphi(z), z \in A\}$. Now consider the following sequence

$$u_n = \sup\left(\varphi - c - \frac{1}{n}, e^{(\frac{1}{n})u+|z|^2} - \frac{1}{n}, 0\right)$$

Since φ is exhaustion, then there exists $\Omega'' \subset\subset \Omega'$ and contains A such that $u_n = \varphi - c - \frac{1}{n}$ on $\Omega' \setminus \Omega''$ for all n. Now consider $A_n = \{z \in \Omega', u_n = 0\}$ and $g \in \mathcal{C}_0^\infty(\Omega \setminus \Omega'')$, $0 \leq g \leq 1$ and $g = 1$ in a neighborhood of $\partial\Omega''$. By similar argument as in Proposition 4.3, one can show that

$$\int_{\Omega'\backslash A_n} T \wedge \beta^p < \infty$$

independently of n. Hence $\widetilde{T}$ exists, and by [12], the current R is positive and supported in A. □

If T is positive closed current, Corollary 4.6 due to El Mir [13]. The case where T positive dd^c-negative current considered in [10], they proved that $dd^c\widetilde{T} = \widetilde{dd^cT}$, if $p \geq 2$. Recently, Dabbek and Noureddine studied the case when T is quasi-plurisubharmonic current.

In what remains in this paper we suppose that A is a closed obstacle.

Theorem 4.10. *Let A be a closed subset of an open subset Ω of $\mathbb{C}^n$ and T be a negative current of bidimension (p,p) on $\Omega\backslash A$ such that $dd^cT \geq -S$ on $\Omega \setminus A$ for some positive currents S on Ω. Assume that $\mathcal{H}_{2p-2}(\overline{SuppT} \cap A)$ is locally finite. Then $\widetilde{T}$ exists. If $dd^cS \leq 0$, then $\widetilde{dd^cT}$ exists and $R = \widetilde{dd^cT} - dd^c\widetilde{T}$ is negative current supported in A.*

The same result obtained by Harvey [17] when T is closed positive current and $\mathcal{H}_{2p-1}(\overline{SuppT}\cap A)=0$. The case when $S=0$ due to Dabbek, Elkhadhra and El Mir [10]. In [9], Dabbek and Noureddine studied the case when T is quasi-plurisubharmonic current.

Proof. Our problem is local. So we may assume that $0\in\overline{SuppT}\cap A$ and our aim now is studying the mass of T in a neighborhood of 0. Since $\mathcal{H}_{2p-1}(\overline{SuppT}\cap A)=0$, there exists a system of coordinates (z',z'') of $\mathbb{C}^{p-1}\times\mathbb{C}^{n-p+1}$ and plydisk $\triangle^{p-1}\times\triangle^{n-p+1}\subset\mathbb{C}^{p-1}\times\mathbb{C}^{n-p+1}$ such that $(A\cap\overline{SuppT})\cap(\triangle^{p-1}\times\partial\triangle^{n-p+1})=\phi$. Moreover, for any projection $\pi_I:\mathbb{C}^n\to\mathbb{C}^{p-1}$ and any strictly multi-index $I=(i_1,...,i_{p-1})$, one has $\pi_I\{0\}\cap(\overline{SuppT}\cap A)=\{0\}$ (cf. [21]). Let $0<t<1$ such that $\triangle^{p-1}\times\{z'',t<|z''|<1\}\cap(\overline{SuppT}\cap A)=\phi$. For each $z'\in\triangle^{p-1}$, we set $A_{z'}=(\overline{SuppT}\cap A)\cap(\{z'\}\times\triangle^{n-p+1})$. Since $\mathcal{H}_{2p-2}(\overline{SuppT}\cap A)$ is locally finite, then by [21] we have that $\mathcal{H}_0(A_{z'})$ is finite. Implies that $A_{z'}$ is a discrete subset for a.e z'. Without lose of generality, we may assume that $A_{z'}$ is reduced to a single point $(z',0)$. On the other hand, T is a $\mathbb{C}$-normal current on $\Omega\setminus A$, so it is $\mathbb{C}$-flat on $\Omega\setminus A$ (cf. [4], pages, 573-574). The slice $\langle T,\pi_I,z'\rangle$ exists for a.e z', and is negative current of bidimension $(1,1)$ on $\Omega\setminus A_{z'}$, supported in $\{z'\}\times\triangle^{n-p+1}$ such that $dd^c\langle T,\pi_I,z'\rangle\geq\langle -S,\pi_I,z'\rangle$ on $\Omega\setminus A_{z'}$. Let K be a compact subset of $\triangle^{p-1}\times\triangle^{n-p+1}$. Since T is negative, it is enough to show that

$$\int_{K\setminus A}-T\wedge\pi_I^*\beta'^{p-1}\wedge\beta<\infty$$

where $\beta'=dd^c|z'|^2$. Applying Remark 4.6 on the current $-T$, we obtain

$$\int_{\triangle^{n-p+1}((z',0),1)\setminus A_{z'}}\langle T,\pi_I,z'\rangle\wedge\beta\leq C_1\int_{\{z''\in\triangle^{n-p+1},|z''|>t\}}\langle T,\pi_I,z'\rangle\wedge\beta + C_2\|\langle S,\pi_I,z'\rangle\|_L$$

where $L=(1+\varepsilon)\overline{\triangle^{n-p+1}}$, for small $\varepsilon>0$. Now, by slice formula we get

$$\begin{aligned}\int_{K\setminus A}-T\wedge\pi_I^*\beta'^{p-1}\wedge\beta &\leq C\int_{z'}\left(\int_{\triangle^{n-p+1}((z',0),1)\setminus A_{z'}}\langle T,\pi_I,z'\rangle\wedge\beta\right)\beta'^{p-1}\\ &\leq C_1'\int_{z'}\left(\int_{\{z''\in\triangle^{n-p+1},|z''|>t\}}\langle T,\pi_I,z'\rangle\wedge\beta\right)\beta'^{p-1}\\ &\quad+C_2'\int_{z'}\left(\int_L\langle S,\pi_I,z'\rangle\right)\beta'^{p-1}\\ &\leq D_1\int_{\triangle^{p-1}\times\{z'',t<|z''|<1\}}-T\wedge\pi_I^*\beta'^{p-1}\wedge\beta\\ &\quad+D_2\int_{\triangle^{p-1}\times L}S\wedge\pi_I^*\beta'^{p-1}\end{aligned}\tag{4.4}$$

As $-T$ is of locally finite mass outside A and S is positive, the last right hand side integrals in (4.4) are bounded. Hence, $\widetilde{T}$ exists. Now, assume that $dd^cS \leq 0$. We want to show the existence of $\widetilde{dd^cT}$. As we saw above, for almost every z', the current $\langle T, \pi_I, z'\rangle$ is negative and $dd^c\langle T, \pi_I, z'\rangle \geq \langle -S, \pi_I, z'\rangle$ apart of $A_{z'}$, which is complete pluripolar. So by Theorem 3.3, $\widetilde{\langle T, \pi_I, z'\rangle}$ exists and $\langle R, \pi_I, z'\rangle = \langle \widetilde{dd^cT}, \pi_I, z'\rangle - \langle dd^c\widetilde{T}, \pi_I, z'\rangle$ is negative for a.e z'. By similar argument as in above we find that $\widetilde{dd^cT}$ exists. Indeed, for K compact subset of $\triangle^{p-1} \times \triangle^{n-p+1}$ we have

$$\int_{K\setminus A} (dd^cT + S) \wedge \pi_I^*\beta'^{p-1}$$
$$\leq B \int_{z'} \left(\int_{\triangle^{n-p+1}((z',0),1)\setminus A_{z'}} \langle (dd^cT + S), \pi_I, z'\rangle \right) \beta'^{p-1}$$

As $\langle \widetilde{dd^cT}, \pi_I, z'\rangle$ exists, the right hand side integral in the previous inequality is bounded. So, $\widetilde{dd^cT + S}$ exists, implies that $\widetilde{dd^cT}$ exists. Remains to show that R is negative, so take a positive function $\varphi \in \mathcal{D}(\Omega)$. By slice formula, we have

$$\int R \wedge \pi_I^*\beta'^{p-1} \wedge \varphi = \int_{z'} \langle R, \pi_I, z'\rangle(\varphi)\beta'^{p-1} \leq 0 \tag{4.5}$$

Hence, $R \wedge \pi_I^*\beta'^{p-1} \leq 0$. Since (4.5) true for almost all choice of unitary coordinates (z', z''), the current R is negative and supported in A. □

Remark 4.11. In the previous theorem, the currents T and dd^cT are $\mathbb{C}$-normal on $\Omega\setminus A$, so the extensions $\widetilde{T}$ and $\widetilde{dd^cT}$ are $\mathbb{C}$-flat (cf. [4]). Therefore, by the support theorem $dd^c\widetilde{T} = \widetilde{dd^cT}$ as soon as $\mathcal{H}_{2p-2}(\overline{SuppT} \cap A) = 0$. Moreover, if $\mathcal{H}_{2p-4}(\overline{SuppT} \cap A)$ is locally finite, then by [10], the extension $\widetilde{-S}$ is positive and plurisubharmonic. Implies that in this case $\widetilde{T}$ has the same properties of T.

Corollary 4.12. *Let A be a closed subset of an open subset Ω of $\mathbb{C}^n$ and T be a positive current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^cT \geq -S$ on $\Omega \setminus A$ for some positive and dd^c-negative currents S on Ω. Assume that $\mathcal{H}_{2p-4}(\overline{SuppT} \cap A)$ is locally finite. Then $\widetilde{T}$ exists. Moreover, $\widetilde{dd^cT} = dd^c\widetilde{T}$.*

Proof. As $dd^cT + S$ is a positive and dd^c-negative current of bidimension $(p-1, p-1)$, the extension $\widetilde{dd^cT + S}$ exists (cf. [10], Theorem 6). Implies $\widetilde{dd^cT}$ exists, and the results follows thanks to Theorem 5 in [10]. □

References

[1] Al Abdulaali, Ahmad K., *On the extension of plurisubharmonic currents.* Master thesis. K.F.U. 2008.

[2] Al Abdulaali, Ahmad K., *Extension and embedding of plurisubharmonic currents.* Research reports in department of mathematics, Stockholm University, Number 3, 2010.

[3] Al Ameer. A., *Extension of plurisubharmonic current across the zero set of a k-convex function.* Master thesis K.F.U 2009.

[4] Bassanelli, G., *A cut-off theorem for plurisubharmonic currents.* Forum Math. 6 (1994), no. 5, 567–595.

[5] Ben Messaoud, Hedi; El Mir, Hassine, *Opérateur de Monge-Ampère et formule de tranchage pour un courant positif fermé.* (French) [Monge-Ampère operator and slicing formula for a positive d-closed current] C. R. Acad. Sci. Paris Sér. I Math. 321 (1995), no. 3, 277–282.

[6] Cegrell, U., *Relations between removable singularity sets for plurisubharmonic functions and positive, closed* $(1,1)$*-currents.* Arch. Math. (Basel) 30 (1978), no. 4, 422–426.

[7] Dabbek, K., *Prolongement d'un courant positif plurisousharmonique.* (French) [Extension of a positive plurisubharmonic current] C. R. Math. Acad. Sci. Paris 342 (2006), no. 11, 819–823.

[8] Dabbek, K.; Elkhadhra, F., *Prolongement d'un courant positif à travers une sous-variété non Levi-plate.* (French) [Extension of a positive current across a non-Levi-flat submanifold] C. R. Math. Acad. Sci. Paris 340 (2005), no. 4, 263–268.

[9] Dabbek, K.; Noureddine, G., *Prolongement d'un courant positif quasi-plurisurharmonique.* (French) [Extension of a positive quasi-plurisuperharmonic current] Ann. Math. Blaise Pascal 16 (2009), no. 2, 287–304.

[10] Dabbek K.; Elkhadra F.; El Mir H., *Extension of plurisubharmonic currents* Math Zeitschrift 245 , 455- 481 (2003).

[11] Demailly, J.-P., *Complex Analytic and differential Geometry.* Free accessible book http://www-fourier.ujf-grenoble.fr

[12] Dinh, T.; Sibony, N., *Pull-back currents by holomorphic maps.* Manuscripta Math. 123 (2007), no. 3, 357–371.

[13] El Mir, H., *Sur le prolongement des courants positifs fermés.* (French) [On the extension of closed positive currents] Acta Math. 153 (1984), no. 1-2, 1–45.

[14] El Mir,H., *prolongement des courants positifs fermés et fonctions* k*-convexe.* Complex Analysis and Applications'85, Sofia, 1986.

[15] El Mir, H.; Feki, I., *Prolongement et contrôle d'un courant positif fermé par ses tranches.* (French) [Extension and mass control of a positive d-closed current by its slices] C. R. Acad. Sci. Paris Sér. I Math. 327 (1998), no. 9, 797–802.

[16] Federer, H., *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer-Verlag New York Inc., New York 1969.

[17] Harvey, R., *Removable singularities for positive currents.* Amer. J. Math. 96 (1974), 67–78.

[18] Harvey, R; Wells, R. O., *Holomorphic approximation and hyperfunction theory on a* C^1 *totally real submanifold of a complex manifold.* Math. Ann. 197 (1972), 287–318.

[19] Hörmander, L., *An introduction to complex analysis in several variables.* Third edition. North Holland Publishing Company.

[20] Hörmander, L., *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis.* Springer-Verlag.

[21] Shiffman, B., *On the removal of singularities of analytic sets.* Michigan Math. J. 15 1968 111–120.

[22] Sibony, N, *problèmes de prolongement de courants en analyse complexe.* (French) [Some extension problems for currents in complex analysis] Duke Math. J. 52 (1985), no. 1, 157–197.

[23] Siu, Y.T, *Analyticity of sets associated to Lelong numbers and the extension of closed positive currents.* Invent. Math. 27 (1974), 53–156.

The Department of Mathematics, Stockholm University, 106 91 Stockhlom, Sweden

E-mail address: ahmad@math.su.se