ISSN: 1401-5617

Exotic automorphisms of the Schouten algebra on a general smooth manifold

Christine Jost

Research Reports in Mathematics
Number 2, 2010

Department of Mathematics
Stockholm University

Electronic versions of this document are available at
http://www.math.su.se/reports/2010/2

Date of publication: August 17, 2010

Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.math.su.se/
info@math.su.se

Filosofie licentiatavhandling

Exotic automorphisms of the Schouten algebra on a general smooth manifold

Christine Jost

Avhandlingen kommer att presenteras fredagen den 27/8 2010,
kl. 10.00 i rum 306, hus 6, Matematiska institutionen,
Stockholms universitet, Kräftriket.

Abstract

In this thesis, we describe a technique of globalizing L_∞-automorphisms of the Schouten algebra of polyvector fields. From a given local automorphism of the Schouten algebra $T_{\text{poly}}(\mathbb{R}^d)$ on affine space satisfying certain conditions, we construct an associated global automorphism of the Schouten algebra $T_{\text{poly}}(M)$ on a general smooth manifold. Exotic automorphisms of the Schouten algebra on affine space were constructed by Merkulov in [15]. It is very plausible that these automorphisms satisfy the conditions posed in this thesis. If this conjecture holds, these results together yield exotic automorphisms of the Schouten algebra of polyvector fields on a general smooth manifold.

Acknowledgments

I would like to thank Sergei Merkulov for suggesting this topic to me and for much help, without which this thesis would never had been written. I also had very helpful discussions with Johan Alm, Henrik Strohmayer and especially Johan Granåker.
Furthermore, I would like to thank Johan Alm, Daniel Bergh, Johan Granåker, Kathrin Vorwerk and Qimh Xantcha for proofreading, Sandra di Rocco for her enthusiasm, and Tom Britton, Ralf Fröberg, Torbjörn Tambour and especially Yishao Zhou for help with practical issues. Thanks to my friends and family for all of their support. And finally, thanks to Daniel for being there.

Contents

1 Introduction

In this thesis we use a method by Fedosov [8] for solving a new instance of an old problem: Extending a local geometric construction to a global one. The construction that we will globalize is an L_∞-automorphism of the Schouten algebra of polyvector fields. A reader unfamiliar with these concepts can find some motivation and a historical outline in Section 1.1, and a short introduction in Chapter 2.

We continue with a more precise statement of the main result. Let F be an L_∞-automorphism of the Schouten algebra $T_{\text{poly}}(\mathbb{R}^d)$ of polyvector fields on affine space. Assume furthermore that the automorphism satisfies the following condition:

Condition 1. *(i) For $n \geq 2$, F_n vanishes on vector fields. That means*

$$F_n(v_1, \dots, v_n) = 0,$$

for vector fields $v_1, \dots, v_n$.

(ii) F vanishes if one of the inputs is a vector field that is linear in the standard coordinates on $\mathbb{R}^d$. That means

$$F_n(\gamma_1, \dots, l^i(x)\frac{\partial}{\partial x^i}, \dots, \gamma_n) = 0,$$

for arbitrary polyvector fields γ_1 to γ_n and a vector field $l^i(x)\frac{\partial}{\partial x^i}$ where the $l^i(x)$ are linear in the coordinates $x^1, \dots, x^d$ of affine space $\mathbb{R}^d$.

Using the given local automorphism F, we construct a new, globalized morphism F_{glob} such that the following holds:

Main Theorem. *For a smooth d-dimensional manifold M and an L_∞-morphism F of $T_{\text{poly}}(\mathbb{R}^d)$ satisfying Condition 1, the globalized morphism F_{glob} constructed below is an L_∞-morphism of $T_{\text{poly}}(M)$.*

This thesis is based on a result of Merkulov's [15], the construction of a family of exotic L_∞-automorphisms of the Schouten algebra on affine space. The constructed morphisms are non-trivial L_∞-morphisms

$$F = \{F_n : \wedge^n T_{\text{poly}}(\mathbb{R}^n) \to T_{\text{poly}}(\mathbb{R}^n)\}_{n \geq 1}$$

of the form

$$F_n = \begin{cases} \text{id} & n = 1, \\ \sum_\Gamma C_\Gamma \Phi_\Gamma & n \geq 2 \end{cases}$$

where the summation runs over graphs Γ with n vertices and $2n-2$ directed edges. The Φ_Γ are multilinear maps $\bigotimes^n T_{\mathrm{poly}}(\mathbb{R}^n) \to T_{\mathrm{poly}}(\mathbb{R}^n)$ associated to the graph Γ and the C_Γ are real weights given by an integral over a compactified configuration space of a differential form associated to Γ. The weights C_Γ are independent of the choice of coordinates, but the maps Φ_Γ are in general only invariant up to affine transformation.

The result described in this thesis parallels a result of Cattaneo, Felder, and Tomassini [3,4], and Dolgushev [7]: The globalization of Kontsevich's L_∞-quasi-isomorphism from polyvector fields $T_{\mathrm{poly}}(\mathbb{R}^d)$ to polydifferential operators $D_{\mathrm{poly}}(\mathbb{R}^d)$ [12]. Kontsevich's result in turn is based on the Hochschild-Kostant-Rosenberg theorem [10]: The Lie algebra of polyvector fields with the Schouten bracket is isomorphic to the cohomology of the Lie algebra of polydifferential operators with the Gerstenhaber bracket. Kontsevich's L_∞-quasi-isomorphism is a stronger result, which solved the problem of deformation quantization of a general Poisson manifold. Kontsevich proved his result for affine space $\mathbb{R}^d$ and sketched the proof for general manifolds. This was elaborated on by Cattaneo, Felder, and Tomassini [3, 4], and later also Dolgushev [7] for obtaining an equivariant formality theorem.

The result of Merkulov [15], which this thesis builds on, is obtained using methods similar to those of Kontsevich's for constructing an L_∞-automorphisms of polyvector fields $T_{\mathrm{poly}}(\mathbb{R}^d)$ on affine space. It should not be surprising that this thesis uses the methods of Cattaneo, Felder, and Tomassini, and Dolgushev respectively, for the globalization of Merkulov's result to general manifolds. They in turn use a trick invented by Fedosov for the construction of the Fedosov star product [8]. This thesis is mostly based on Dolgushev's application of the well-known technique. Indeed, most of the results are careful checks of Dolgushev's results for our case. We state the correspondence with Dolgushev's work in detail. Lemma 3 here corresponds to Dolgushev's Theorem 2, Proposition 1 to Dolgushev's Theorem 3, Proposition 2 to Dolgushev's Proposition 2 and the Main Theorem to Dolgushev's Theorem 4.

Condition 1 is analogous to two conditions in Kontsevich's work [12]. Kontsevich's L_∞-morphism between polydifferential operators and polyvector fields satisfies certain conditions (P4) and (P5), which was proved in the same work. Merkulov's exotic automorphisms are constructed in an analogous way, which makes it plausible that they satisfy the analogous condition. A proper investigation is very obvious future work.

As an alternative to Condition 1, one can impose another condition on the local automorphism constructed as in [15]. Later, we give an easy proof that Condition 1 is satisfied if the following condition holds:

Condition 2. *If the graph Γ contains a vertex with at most one ingoing and at most one outgoing edge, then the weight C_Γ is zero.*

The outline of the thesis is the following: We start with a historical overview over polyvector fields and polydifferential operators. It followed by a short introduction to polyvector fields, the Schouten bracket, L_∞-algebras and Maurer-Cartan elements. Maurer-Cartan elements are the keys to the twist of Lie algebras and L_∞-morphisms, a method which we will need later. After that, the first step towards proving the main result is the introduction of vertical polyvector fields. They are used to construct a resolution of the Schouten algebra $T_{\mathrm{poly}}(M)$. For obtaining a resolution not only as a $C^\infty(M)$-module, but also as a Lie algebra, the resolution has to be twisted using a chosen covariant derivation. This is called the Fedosov trick. The given local L_∞-morphism acts fiberwise on the obtained resolution as a $C^\infty(M)$-module. After a twist of the fiberwise morphism, we obtain an L_∞-morphism of the resolution as a Lie algebra. Using Condition 1, we see that this twisted, fiberwise morphism is invariant under change of coordinates, which proves the result. We conclude with showing that Condition 2 actually implies Condition 1.

Although the aim of this thesis is a global construction, we are going to work in local coordinates most of the time. If not stated otherwise, the formulas that we are going to use are independent of the choice of local coordinates. Throughout this thesis, we use the Einstein summation convention.

1.1 A short history of polyvector fields

This section is intended for the non-expert reader who wonders why polyvector fields or L_∞-algebras are something worth investigating. We have no ambition to write a complete treatise and only aim to tell a story. There are nice and careful expositions such as [17] or [5]. The Schouten algebra of polyvector fields, L_∞-algebras and Maurer-Cartan elements will be defined in Chapter 2, Basic notions. However, we give short definitions here that should suffice for this historical overview.

Let M be a smooth d-dimensional manifold. The Schouten algebra of polyvector fields $T_{\mathrm{poly}}(M)$ on M is the $C^\infty(M)$-module of sections of the exterior algebra of the tangent bundle, i.e.,

$$T_{\mathrm{poly}}(M) = \bigoplus_{n=0}^{\infty} T^n_{\mathrm{poly}}(M),$$

where

$$T^n_{\mathrm{poly}}(M) = \Gamma(\textstyle\bigwedge^n TM).$$

Via the graded Leibniz rule, the Lie bracket on vector fields can be extended to polyvector fields. The resulting Lie bracket is called the Schouten bracket.

An L_∞-algebra is a generalization of a Lie algebra which consists not only of a binary operation, the bracket, but of a (possibly vanishing) n-ary operation for each positive integer n. The operations suffice a countable number of compatibility relations, among others a Jacobi identity up to homotopy. A Maurer-Cartan element is an element of the L_∞-algebra that can be used to twist the L_∞-algebra, that is, to construct a new L_∞-algebra structure from the old one with help of the Maurer-Cartan element.

We continue by introducing the $C^\infty(M)$-module of polydifferential operators $D_{\mathrm{poly}}(M)$. On a local patch U of the manifold M where we denote coordinates by $x^1, \dots, x^d$, a poly-differential operator is a map

$$C^\infty(U) \otimes \ldots \otimes C^\infty(U) \to C^\infty(U)$$

of the form

$$\Delta_U = \Delta_U^{I_1 \ldots I_n}(x) \frac{\partial^{|I_1|}}{\partial x^{i_1^{(1)}} \ldots \partial x^{i_{k_1}^{(1)}}} \otimes \ldots \otimes \frac{\partial^{|I_n|}}{\partial x^{i_1^{(n)}} \ldots \partial x^{i_{k_n}^{(n)}}}, \tag{1.1}$$

where the $\Delta_U^{I_1 \ldots I_n}(x)$ are smooth functions, the I_j are multi-indices

$$I_j = (i_1^{(j)}, \ldots, i_{k_j}^{(j)})$$

and $|I_j| = k_j$. On smooth functions $a_1, \ldots, a_n$, they act as

$$\Delta(a_1, \ldots, a_n) = \Delta^{I_1 \ldots I_n} \frac{\partial^{|I_1|} a_1}{\partial x^{i_1^{(1)}} \ldots \partial x^{i_{k_1}^{(1)}}} \cdot \ldots \cdot \frac{\partial^{|I_n|} a_n}{\partial x^{i_1^{(n)}} \ldots \partial x^{i_{k_n}^{(n)}}}.$$

Having described polydifferential operators locally, we define polydifferential operators on the manifold M as maps

$$\Delta : C^\infty(M) \otimes \ldots \otimes C^\infty(M) \to C^\infty(M)$$

that are locally of the form (1.1). More precisely, there is a covering of M with open subsets U_i such that Δ restricted to $C^\infty(U_i)^{\otimes n}$ is of the form (1.1).

The polydifferential operators form a complex with the grading

$$D_{\text{poly}}(M) = \bigoplus_{n=0}^{\infty} D_{\text{poly}}^n(M),$$

where $D_{\text{poly}}^n(M)$ consists of the polydifferential operators from $C^\infty(M)^{\otimes n}$ to $C^\infty(M)$. It is a subcomplex of the Hochschild cochain complex $C^\bullet(A, A)$ for the associative algebra $A := C^\infty(M)$, where

$$C^n(A, A) = \mathrm{Hom}_{\mathbb{R}}(A^{\otimes n+1}, A).$$

The differential in this complex is given by $d : C^n(A, A) \to C^{n+1}(A, A)$ with

$$(d\Delta)(a_0, \ldots a_n) =$$
$$a_0 \Delta(a_1, \ldots, a_n) \; + \; \sum_{i=1}^{n} (-1)^i \Delta(a_0, \ldots, a_{i-1}a_i, \ldots, a_n) \; + \; \Delta(a_0, \ldots, a_{n-1})a_n.$$

The cohomology of the Hochschild cochain complex is denoted by $\mathrm{HH}^*(A, A)$, the cohomology of the subcomplex $D_{\text{poly}}(M)$ is denoted by $\mathrm{HH}^*_{\text{diff}}(M)$. We mention also that the polydifferential operators form a Lie algebra with respect to the so-called Gerstenhaber bracket, which induces a bracket on cohomology. So both the space of polydifferential operators and its cohomology are Lie algebras. They are the topic of a famous theorem:

Hochschild-Kostant-Rosenberg-Theorem. *[10] The cohomology of the differential Hochschild complex of polydifferential operators is isomorphic as a Lie algebra to the Lie algebra of polyvector fields with the Schouten bracket:*

$$(\mathrm{HH}^*_{\text{diff}}, [-,-]_G) = (T_{\text{poly}}(M), [-,-]_S).$$

Why, however, should one be interested in the Hochschild cochain complex of $C^\infty(M)$? One answer is given by Gerstenhaber in the work [9]: The Hochschild cohomology groups of an associative algebra control the deformations of this algebra. Here, we consider a special deformation problem, deformation quantization, posed by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer in [1] and [2].

Let A be a commutative algebra over a field $\mathbb{K}$. We introduce a formal parameter v and deform the algebra A to an associative algebra $A[[v]]$ over $\mathbb{K}[[v]]$. The problem of deformation quantization is to find a so-called star product, i.e., an associative product

$$\star : A[[v]] \otimes_{\mathbb{K}[[v]]} A[[v]] \to A[[v]].$$

It can be written in the form

$$a \star b = ab + vB_1(a,b) + v^2 B_2(a,b) + \dots .$$

Consider the important special case when A is the algebra $C^\infty(M)$ of smooth functions on a manifold M. From the associativity of the star product, it follows that $\{-,-\}$ defined by

$$\{a,b\} = \frac{B_1(a,b) - B_1(b,a)}{2}$$

makes $A[[v]]$ into a Poisson algebra, i.e., a special Lie algebra. If there exists a Poisson algebra structure on $A = C^\infty(M)$, then M is called a Poisson manifold. One can prove that the Poisson algebra structures on $A[[v]]$ determine the star products on $A[[v]]$. A more general formulation of the problem is to construct a star product for a given Poisson algebra structure.

For the mathematician not educated in physics, it is difficult to understand what these products have to do with quantization. An attempt to give some physical intuition behind this formula is to interpret the algebra A as the algebra of classical observables and the deformed associative algebra as the algebra of quantum observables. The formal parameter v is often set to $\frac{i\hbar}{2}$ where $\hbar$ is the Planck constant.

Now L_∞-algebras slowly come into the picture. The solution to the deformation quantization problem for symplectic manifolds, i.e., nondegenerate Poisson manifolds, was given by Fedosov [8]. Its main idea is the Fedosov construction, which this thesis uses heavily. The solution for general Poisson structures is due to Kontsevich [11,12]. It consists of the proof of the Formality Theorem (stated as Formality Conjecture in [11]), which is a stronger version of the Hochschild-Kostant-Rosenberg Theorem.

Formality Theorem. *[12] There exists an L_∞-quasi-isomorphism between the Lie algebras $T_{\mathrm{poly}}(M)$ and $D_{\mathrm{poly}}(M)$.*

The connection to deformation quantization is roughly the following: If the Formality Conjecture is true, one can relate the Maurer-Cartan elements of $D_{\mathrm{poly}}(M)$ and $T_{\mathrm{poly}}(M)$. The Maurer-Cartan elements of $T_{\mathrm{poly}}(M)$ are the Poisson structures, and the Maurer-Cartan elements of $D_{\mathrm{poly}}(M)$ can be identified with deformations of the usual multiplication. Hence, the existence of a star product as above follows from the Formality Conjecture. For more details we refer to [11], [16] and the introduction [5].

Kontsevich succeeded in proving the Formality Theorem [12]. Actually, he proved it only for affine space $\mathbb{R}^d$, but sketched a proof of a globalization to general manifolds. As already written in the beginning of the introduction, this allowed Cattaneo, Felder and Tomassini [3, 4] as well as Dolgushev [7] to establish the globalization of Kontsevich's result.

Now the non-expert reader should have the background to read the beginning of the introduction again.

2 Basic notions

2.1 The Schouten algebra of polyvector fields

Let M be a smooth manifold of dimension d with the $\mathbb{R}$-algebra of smooth functions $C^\infty(M)$ and tangent bundle TM. The Schouten algebra of polyvector fields $T_{\text{poly}}(M)$ on M is the $C^\infty(M)$-module of sections of the exterior algebra of the tangent bundle, i.e.,

$$T_{\text{poly}}(M) = \bigoplus_{n=0}^{\infty} T^n_{\text{poly}}(M),$$

where

$$T^n_{\text{poly}}(M) = \Gamma(\bigwedge{}^n TM)$$

for $n \geq 1$. We set $T^0_{\text{poly}}(M) = C^\infty(M)$. We obtain a grading of the Schouten algebra by saying that an element of $T^n_{\text{poly}}(M) = \Gamma(\bigwedge^n TM)$ has degree n. The degree of a homogeneous polyvector field f is denoted by $|f|$. Usual vector fields are polyvector fields of degree 1.

In the chosen grading, the Lie bracket for vector fields has degree -1. It can be extended to an odd Lie bracket on polyvector fields via the graded Leibniz rule: Let f, g and h be homogeneous polyvector fields, then

$$[f, g \wedge h] = [f, g] \wedge h + (-1)^{(|f|-1)|g|} g \wedge [f, h],$$

the -1 in the exponent coming from the degree of the bracket. We recall the properties of odd Lie brackets:

- Skew-symmetry: $[f, g] = -(-1)^{(|f|+1)(|g|+1)}[g, f]$, and
- Jacobi identity: $[f, [g, h]] = [[f, g], h] + (-1)^{(|f|+1)(|g|+1)}[g, [f, h]]$.

There are several ways to write down the Schouten bracket explicitly. Here we are going to present two of them. For vector fields $v_0, \ldots, v_k$, $w_0, \ldots, w_l$ and a smooth function a we have that

$$[v_0 \wedge \ldots \wedge v_k, w_0 \wedge \ldots \wedge w_l] =$$
$$\sum_{i=0}^{k} \sum_{j=0}^{l} (-1)^{i+j} [v_i, w_j] \wedge v_0 \wedge \ldots \wedge v_{i-1} \wedge v_{i+1} \wedge \ldots \wedge v_k \wedge w_0 \wedge \ldots \wedge w_{j-1} \wedge w_{j+1} \wedge \ldots \wedge w_l$$

and

$$[v_0 \wedge \ldots \wedge v_k, a] = (-1)^k \sum_{i=0}^{k} (-1)^i v_i(a)\ v_0 \wedge \ldots \wedge v_{i-1} \wedge v_{i+1} \wedge \ldots \wedge v_k.$$

On a local patch U of M with coordinates $x^1, \ldots, x^n$, a polyvector field takes the form

$$f^{i_0 \ldots i_k}(x) \frac{\partial}{\partial x^{i_0}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i_k}}.$$

Here, the $f^{i_0 \ldots i_k}(x)$ are smooth functions on U. Observe that the Einstein summation convention is used in this formula. We write φ_i instead of $\frac{\partial}{\partial x^i}$. Then a polyvector field restricted to the patch U is an element of the graded commutative polynomial ring $C^\infty(U)[\varphi_1, \ldots, \varphi_d]$, where the φ_i are of degree 1. This allows us to write the Schouten bracket in local coordinates simply as

$$[f, g] = -\left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial \varphi_i} + (-1)^{|f|} \frac{\partial f}{\partial \varphi_i} \frac{\partial g}{\partial x^i} \right). \tag{2.1}$$

Here f and g are elements of $C^\infty(U)[\varphi_1, \ldots, \varphi_d]$, i.e. polyvector fields in local coordinates. Observe that $C^\infty(U)[\varphi_1, \ldots, \varphi_d]$ is graded commutative, i.e.,

$$\varphi_i \varphi_j = (-1)^{|\varphi_i||\varphi_j|} \varphi_j \varphi_i = -\varphi_j \varphi_i.$$

Furthermore

$$\frac{\partial}{\partial x^j} \left(f^{i_0 \ldots i_k}(x) \varphi_{i_0} \ldots \varphi_{i_k} \right) = \frac{\partial f^{i_0 \ldots i_k}(x)}{\partial x^j} \varphi_{i_0} \ldots \varphi_{i_k}$$

and

$$\frac{\partial}{\partial \varphi_j} \varphi_i \varphi_j = -\frac{\partial}{\partial \varphi_j} \varphi_j \varphi_i = -\varphi_i.$$

2.2 L_∞-algebras and Maurer-Cartan elements

L_∞-algebras are a generalization of Lie algebras, consisting not only of a binary bracket, but of a set of maps

$$\begin{aligned} &Q_1 : g \to g \\ &Q_2 : g \otimes g \to g \\ &Q_3 : g \otimes g \otimes g \to g \\ &\vdots \end{aligned}$$

where g is a graded vector space. Every Lie algebra is an L_∞-algebra, where all but the binary map are zero. Not every L_∞-algebra is a Lie algebra, but one says that it is a Lie algebra *up to homotopy.* L_∞-algebras are also called *strong homotopy Lie algebras* or *sh Lie algebras.* This introduction is going to be very short. We skip the proofs and specification of signs. For this, we refer to much more elaborate introductions such as [13], [14] or [5].

An L_∞-structure on a $\mathbb{Z}$-graded vector space g is a collection of skew-symmetric maps

$$Q_n : \bigotimes^n g \to g$$

of degree $2-n$ such that

$$\sum_{i+j=n+1} \sum_\sigma \pm Q_i(Q_j(v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(j)}) \otimes v_{\sigma(j+1)} \otimes \ldots \otimes v_{\sigma(n)}) = 0 \tag{2.2}$$

for all $n \geq 1$. The inner sum runs over all permutations σ such that $\sigma(1) < \ldots < \sigma(j)$ and $\sigma(j+1) < \ldots < \sigma(n)$. The sign $\pm$ depends on i, j and the permutation σ. If all the maps except Q_2 are zero, then Equation (2.2) for $n = 3$ is the usual graded Jacobi identity. In general, $n = 3$ yields a Jacobi identity up to higher terms, or up to homotopy.

We are now going to describe another approach to defining L_∞-algebras. Consider the reduced graded symmetric coalgebra

$$\bar{S}(g[1]) = \sum_{i=1}^{\infty} S^n g[1],$$

where S^n denotes the n-th symmetric tensor power and $g[1]$ the vector space g with grading shifted by 1, i.e., $g[1]_n = g_{n+1}$. The coalgebra structure is given by

$$\Delta(v) = v \otimes 1 + 1 \otimes v$$

for an element v in g. It is extended to elements of higher degree such that Δ is an homomorphism of algebras. The algebra structure of $\bar{S}(g[1])$, and hence $\bar{S}(g[1]) \otimes \bar{S}(g[1])$, is given by the tensor product.

As we are going to see, an L_∞-structure is also given by a coalgebra differential and coderivation Q on $\bar{S}(g[1])$, that is, a degree 1 map $Q : \bar{S}(g[1]) \to \bar{S}(g[1])$ such that $Q^2 = 0$ and Q is a coderivation, i.e.,

$$\Delta \circ Q = (Q \otimes Q) \circ \Delta. \tag{2.3}$$

The differential Q is determined by maps $S^n g[1] \to \bar{S}(g[1])$ for every n. Because Q is a coderivation, these maps are in turn determined by their composition with the projection $\bar{S}(g[1]) \to g[1]$. Hence a differential Q on the symmetric coalgebra $\bar{S}(g[1])$ is determined by degree 1 maps

$$Q_n : S^n g[1] \to g[1].$$

They correspond to degree $2-n$ maps

$$Q_n : \bigwedge^n g \to g.$$

The condition $Q^2 = 0$ translates to the conditions (2.2) on the Q_n and vice versa.

Hence, we see that an L_∞-algebra structure on a graded vector space g can be given as skew-symmetric maps

$$Q_n : \bigwedge^n g \to g$$

satisfying condition (2.2), or alternatively as a coderivation Q of the symmetric coalgebra $\bar{S}(g[1])$ satisfying $Q^2 = 0$. In the following, we then say that (g, Q) is an L_∞-algebra.

The second definition makes it easier to define the notion of an L_∞-morphism. Let (g, Q) and (h, R) be L_∞-algebras. Then an L_∞-morphism from (g, Q) to (h, R) is given by a coalgebra morphism

$$\Phi : \bar{S}(g[1]) \to \bar{S}(h[1])$$

that commutes with Q and R, which means that

$$\Phi \circ Q = R \circ \Phi.$$

In a similar way as an L_∞-algebra structure, an L_∞-morphism $\Phi : \bar{S}(g[1]) \to \bar{S}(h[1])$ is uniquely determined by its composition with the projection $\bar{S}(h[1]) \to h[1]$. Hence an L_∞-morphism Φ can also be given by linear maps

$$\Phi_n : \bigwedge\nolimits^n g \to h$$

satisfying compatibility conditions coming from the fact that Φ respects the L_∞-structures Q and R.

We turn to the topic of twisting L_∞-algebras and L_∞-morphisms with a so-called Maurer-Cartan element. For a more detailed introduction, see [6, 18]. In a graded Lie algebra, a Maurer-Cartan element is an element λ such that $[\lambda, \lambda]$ is zero. In an L_∞-algebra (g, Q), a Maurer-Cartan element is an element π of g that satisfies

$$\sum_{i=1}^{\infty} \frac{1}{i!} Q_i(\pi, \ldots, \pi) = 0.$$

Given a Maurer-Cartan element in an L_∞-algebra, one can twist the L_∞-algebra in the following way:

Let (g, Q) and (h, R) be L_∞-algebras as before, π an element of $g[1]$ and $\Phi : (g, Q) \to (h, R)$ an L_∞-morphism. Define $\exp(\pi) : \bar{S}(g[1]) \to \bar{S}(g[1])$ by

$$\exp(\pi)(X) := \sum_{i=0}^{\infty} \frac{1}{i!} \pi^i X$$

for X in $\bar{S}(g[1])$. One checks that $\exp(-\pi) \circ \exp(\pi) = \mathrm{id}$.

Suppose that π is a Maurer-Cartan element. Then the map Q_π defined by

$$Q_\pi = \exp(-\pi) \circ Q \circ \exp(\pi)$$

makes (g, Q_π) into an L_∞-algebra. This is the twisting of (g, Q) with π. It may also be given explicitly by the formula

$$Q_\pi(X) = \sum_{i=0}^{\infty} \frac{1}{i!} Q(\pi^i X).$$

We can also twist the L_∞-morphism $\Phi : (g, Q) \to (h, R)$ with the Maurer-Cartan element π of (g, Q). The first step is to find a corresponding Maurer-Cartan element in (h, R). It is given by

$$\omega = \sum_{i=1}^{\infty} \frac{1}{i!} \Phi_i(\pi^i). \tag{2.4}$$

We can twist (g, Q) with π and (h, R) with ω and get the L_∞-algebras (g, Q_π) and (h, R_ω). The twisted L_∞-morphism Φ_π between them is given by

$$\Phi_\pi = \exp(-\omega) \circ Q \circ \exp(\pi). \tag{2.5}$$

An explicit formula for Φ_π is given by

$$\Phi_\pi(X) = \sum_{i=0}^{\infty} \frac{1}{i!} \Phi(\pi^i X). \tag{2.6}$$

Now the reader should have the necessary prerequisites. The remainder of the thesis consists of the proof of the main result.

3 The construction of the globalized morphism

3.1 Vertical polyvector fields

In this section, we define vertical polyvector fields and differential forms with values in them. The idea is to work in the module of sections of a large vector bundle, with fibers at any point on the manifold being isomorphic as $\mathbb{R}$-vector spaces to $T_{\mathrm{poly}}(\mathbb{R}^d)$. The Schouten bracket on $T_{\mathrm{poly}}(\mathbb{R}^d)$ extends fiberwise to a Lie bracket on the vertical polyvector fields, the vertical Schouten bracket. In the same way, an L_∞-morphism of $T_{\mathrm{poly}}(\mathbb{R}^d)$ will extend to an L_∞-morphism on the vertical polyvector fields. In practice, this idea has to be modified in two ways: We have to consider polyvector fields on the "thick point" $\mathbb{R}^d_{\mathrm{formal}}$, the formal completion of $\mathbb{R}^d$ along 0, instead of polyvector fields on $\mathbb{R}^d$. Furthermore, as already said, we will not only need polyvector fields, but also differential forms with values in them. Summarizing, we are going to construct a $C^\infty(M)$-module and Lie algebra whose elements in local coordinates are of the form

$$\sum_{k\geq 0}\sum_{l,m\geq 0} f^{b_1\dots b_l}_{a_1\dots a_k,c_1\dots c_m}(x) y^{a_1}\dots y^{a_k}\frac{\partial}{\partial y^{b_1}}\wedge\dots\wedge\frac{\partial}{\partial y^{b_l}}\,\mathrm{d}x^{c_1}\wedge\dots\wedge\mathrm{d}x^{c_m}. \tag{3.1}$$

As first step, consider $\mathbb{R}^d_{\mathrm{formal}}$, the formal completion of $\mathbb{R}^d$ along 0. Roughly speaking, this is the point 0 together with an infinitesimal neighborhood in $\mathbb{R}^d$. We work in local coordinates which we denote by x^1 to x^d. The "smooth functions", i.e., the global sections of the structure sheaf, of $\mathbb{R}^d_{\mathrm{formal}}$ are the power series in d coordinates, $\mathbb{R}[[x^1,\dots,x^d]]$. The tangent vectors, i.e., global sections of the tangent sheaf, are the derivations $f^i(x)\frac{\partial}{\partial x^i}$ where $f^i(x)$ is a formal power series with respect to the $x^1,\dots,x^d$. Hence, the Schouten algebra $T_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ on $\mathbb{R}^d_{\mathrm{formal}}$ is given by

$$T_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}}) = \left\{\sum_{n=0}^{d} f^{i_1\dots i_n}(x)\frac{\partial}{\partial x^{i_1}}\wedge\dots\wedge\frac{\partial}{\partial x^{i_n}} \mid f^{i_1\dots i_n}(x)\in \mathrm{R}[[x^1,\dots,x^d]]\right\},$$

with the Schouten bracket given by the same formula (2.1) as the usual Schouten bracket.

As next step, we are going to construct vertical polyvector fields. Let, as before, M be a manifold of dimension d. The total space of the tangent bundle TM is a manifold of dimension $2d$. When we choose local coordinates on this manifold, we will denote them by $x^1,\dots,x^d,y^1,\dots,y^d$, where the x^i are coordinates on the original manifold M and the y^i coordinates on the fibers of the tangent bundle. We define the sheaf $\hat{\mathcal{T}}$ on TM as the sections

$$f^i(x,y)\frac{\partial}{\partial x^i} + g^i(x,y)\frac{\partial}{\partial y^i}$$

of the iterated tangent bundle $T(TM)$ such that $f^i(x,y)$ and $g^i(x,y)$ are smooth with respect to the x and formal power series with respect to the y. Observe that this definition is independent of the choice of coordinates, as long as one chooses one set of coordinates denoted by x for the original manifold M and one set denoted by y for the fiber at the point x in M. The sheaf $\hat{\mathcal{T}}$ is a sheaf of modules over the formal structure sheaf $\hat{C}$ defined by

$$\begin{aligned}&\hat{C}(U) := \\ &\quad \{f : U \to \mathbb{R} \mid f \text{ smooth with respect to } x, \text{ formal power series with respect to } y\}.\end{aligned}$$

Consider the projection $\pi : TM \to M$. It induces a differential

$$\mathrm{d}\pi : \hat{\mathcal{T}}(TM) \to \pi^*\mathcal{T}M,$$

where $\pi^*\mathcal{T}M$ is the pullback of the tangent sheaf $\mathcal{T}M$. In local coordinates, $\mathrm{d}\pi$ is given by

$$\mathrm{d}\pi\left(f^i(x,y)\frac{\partial}{\partial x^i} + g^i(x,y)\frac{\partial}{\partial y^i}\right) = f^i(x,y)\frac{\partial}{\partial x^i}.$$

Hence the kernel of $\mathrm{d}\pi$ consists of those vector fields taking the form $g^i(x,y)\frac{\partial}{\partial y^i}$ with g^i smooth with respect to the x and formal power series with respect to the y. We define

$$T^{\mathrm{vert}}(M) := \ker \mathrm{d}\pi(M).$$

Elements of $T^{\mathrm{vert}}(M)$ are called vertical vector fields on the manifold M. The vertical vector fields $T^{\mathrm{vert}}(M)$ form both a $C^\infty(M)$-module and a $\hat{C}(TM)$-module. The elements of the exterior algebra of $T^{\mathrm{vert}}(M)$ over $\hat{C}(TM)$ are the vertical polyvector fields. We define

$$T^{\mathrm{vert}}_{\mathrm{poly}}(M) := \bigoplus_{q=0}^{d} \bigwedge\nolimits^q T^{\mathrm{vert}}(M),$$

where we let $\bigwedge^0 T^{\mathrm{vert}}(M)$ be $\hat{C}(TM)$ and the exterior algebra is over $\hat{C}(TM)$. In local coordinates, a vertical polyvector field is of the form

$$g^{i_1\cdots i_n}(x,y)\frac{\partial}{\partial y^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial y^{i_n}},$$

where the $g^{i_1\cdots i_n}(x,y)$ are smooth with respect to the x and formal power series with respect to the y. On a local patch U, we can hence identify $T^{\mathrm{vert}}_{\mathrm{poly}}(U)$ with the graded commutative polynomial ring and formal power series

$$C^\infty(U)[[y^1,\ldots,y^d]][\psi_1,\ldots,\psi_d],$$

where we set $\psi_i = \frac{\partial}{\partial y^i}$. The y^i are of degree 0 and the ψ_i of degree 1.

Tensoring $T^{\mathrm{vert}}_{\mathrm{poly}}(M)$ with the de Rham algebra ΩM over M, we get differential forms over M with values in vertical polyvector fields:

$$\Omega(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M)) := T^{\mathrm{vert}}_{\mathrm{poly}}(M) \otimes_{C^\infty(M)} \Omega M.$$

In local coordinates, the elements of $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$ are of the form (3.1). On a local patch U, we can hence identify $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$ with the graded commutative polynomial ring and formal power series

$$C^\infty(U)[[y^1, \dots, y^d]][\psi_1, \dots, \psi_d, \eta^1, \dots, \eta^d] \tag{3.2}$$

where we set $\eta^i = \mathrm{d}x^i$ with degree 1, and the $\psi_i = \frac{\partial}{\partial y^i}$ as before. Hence $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$ is graded in the following way:

$$\Omega(M, T_{\text{poly}}^{\text{vert}}(M)) = \bigoplus_{r=0}^{d} \Omega^r(M, T_{\text{poly}}^{\text{vert}}(M)),$$

where

$$\Omega^r(M, T_{\text{poly}}^{\text{vert}}(M)) = \bigoplus_{p+q=r} T_{\text{poly}}^{\text{vert,p}}(M) \otimes \Gamma(\textstyle\bigwedge^q(T^*M))$$

and the elements of $T_{\text{poly}}^{\text{vert,p}}(M)$ have degree p with respect to the $\frac{\partial}{\partial y^i}$.

As last step, we will show that the Schouten bracket on $T_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$ induces a Lie algebra structure on both $T_{\text{poly}}^{\text{vert}}(M)$ and $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$. Consider the vector bundle $\hat{S}(T^*M) \otimes \bigwedge TM$, where $\hat{S}(T^*M)$ is the completed symmetric algebra over the cotangent bundle T^*M,

$$\hat{S}(T^*M) = \prod_{i=0}^{\infty} S^i(T^*M)$$

and $\bigwedge TM$ the exterior algebra over the tangent bundle TM. It holds that

$$\Gamma\left(M, \hat{S}(T^*M) \otimes \bigwedge TM\right) \simeq T_{\text{poly}}^{\text{vert}}(M)$$

as $C^\infty(M)$-modules, by the identification

$$\sum_{k,l \geq 0} f_{a_1 \dots a_k}^{b_1 \dots b_l}(x) \mathrm{d}x^{a_1} \dots \mathrm{d}x^{a_k} \frac{\partial}{\partial x^{b_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{b_l}} \mapsto$$

$$\mapsto \sum_{k,l \geq 0} f_{a_1 \dots a_k}^{b_1 \dots b_l}(x) y^{a_1} \dots y^{a_k} \frac{\partial}{\partial y^{b_1}} \wedge \dots \wedge \frac{\partial}{\partial y^{b_l}}.$$

Hence the $C^\infty(M)$-module of the global sections of $\hat{S}(T^*M) \otimes \bigwedge TM$ is isomorphic to $T_{\text{poly}}^{\text{vert}}(M)$. The fibers of the bundle $\hat{S}(T^*M) \otimes \bigwedge TM$ are isomorphic to $T_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$ as $\mathbb{R}$-vector spaces. This induces a fiberwise Lie algebra structure on $T_{\text{poly}}^{\text{vert}}(M)$. Explicitly and in local coordinates, it is given by

$$[f, g]^{\text{vert}} = -\left(\frac{\partial f}{\partial y^i} \frac{\partial g}{\partial \psi_i} + (-1)^{|f|} \frac{\partial f}{\partial \psi_i} \frac{\partial g}{\partial y^i}\right). \tag{3.3}$$

Here f and g are elements of $C^\infty(U)[[y^1, \dots, y^n]][\psi_1, \dots, \psi_n]$, i.e., vertical polyvector fields on a local patch U. We call this bracket the vertical Schouten bracket. It can be extended to $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$ and is given explicitly by the same formula (3.3) where the f and g are elements of $C^\infty(U)[[y^1, \dots, y^d]][\psi_1, \dots, \psi_d, \eta^1, \dots, \eta^d]$, i.e., differential forms with values in the vertical polyvector fields on a local patch U.

In a similar fashion as the construction of the vertical Schouten bracket, an L_∞-morphism on $T_{\text{poly}}^{\text{vert}}(M)$ induces a fiberwise L_∞-morphism on $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$, see Section 3.4.

We will also need a subalgebra of $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$, the Lie algebra $T_{\text{poly}}^{\text{vert}}(M)|_{y=0}$ of vertical polyvector fields which are constant with respect to the y. Its elements are of the form

$$g^{i_1 \dots i_n}(x) \frac{\partial}{\partial y^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial y^{i_n}}.$$

It is not hard to see that $T_{\text{poly}}^{\text{vert}}(M)|_{y=0}$ is isomorphic to $T_{\text{poly}}(M)$ as $C^\infty(M)$-module (but not as Lie algebra). So $T_{\text{poly}}(M)$ can be seen as submodule of $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$, a fact that we are going to use soon.

Summarizing, we have proved the following lemma:

Lemma 1. *The $C^\infty(M)$-module $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$ is a Lie algebra together with the vertical Schouten bracket. An element of $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$ can locally be written in the form*

$$\sum_{k \geq 0} \sum_{l,m \geq 0} f(x)^{b_1 \dots b_l}_{a_1 \dots a_k, c_1 \dots c_m} y^{a_1} \dots y^{a_k} \frac{\partial}{\partial y^{b_1}} \wedge \dots \wedge \frac{\partial}{\partial y^{b_l}} \, \mathrm{d}x^{c_1} \wedge \dots \wedge \mathrm{d}x^{c_m}. \tag{3.4}$$

In local coordinates, $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$ can be identified with $C^\infty(U)[[y^1, \dots, y^n]][\psi_1, \dots, \psi_n]$ and the vertical Schouten bracket is given by

$$[f, g]^{\text{vert}} = -\left(\frac{\partial f}{\partial y^i} \frac{\partial g}{\partial \psi_i} + (-1)^{|f|} \frac{\partial f}{\partial \psi_i} \frac{\partial g}{\partial y^i} \right). \tag{3.5}$$

The submodules $T_{\text{poly}}^{\text{vert}}(M)$ and $T_{\text{poly}}^{\text{vert}}(M)|_{y=0}$ are subalgebras of $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$.

3.2 Resolution of $T_{\text{poly}}(M)$ via the Poincaré lemma

Using the Poincaré lemma, differential forms with values in the vertical polyvector fields provide us with a resolution of $T_{\text{poly}}(M)$ as a $C^\infty(M)$-module. Consider the following differential on $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$:

$$\delta : \Omega^r(M, T_{\text{poly}}^{\text{vert}}(M)) \to \Omega^{r+1}(M, T_{\text{poly}}^{\text{vert}}(M))$$

given in local coordinates by

$$\delta(f) := [\eta^i \psi_i, f]^{\text{vert}} = \mathrm{d}x^i \frac{\partial f}{\partial y^i}.$$

This is well-defined since $\eta^i \psi_i = \mathrm{d}x^i \frac{\partial}{\partial y^i}$ is invariant under change of coordinates on M. The differential δ is similar to the de Rham differential. In fact, we are going to use a version of the proof of the Poincaré lemma in order to prove the following lemma:

Lemma 2. *As $C^\infty(M)$-module, the cohomology of $\Omega(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M))$ with respect to δ is given by*

$$H^n(\Omega(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M)), \delta) \cong \begin{cases} T_{\mathrm{poly}}(M) & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We use a contracting homotopy δ^*. It is defined by

$$\delta^* : \Omega^r(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M)) \to \Omega^{r-1}(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M))$$

with

$$\delta^*(f_{I,J}(x,\psi) y^I \eta^J) = \frac{1}{p+q} y^a \frac{\partial}{\partial \eta^a} f(x,\psi) y^I \eta^J,$$

for f non-constant with respect to the y and η, where I and J are multi-indices of degree p and q, respectively. For f constant with respect to the y and η, the contracting homotopy δ^* is given by $\delta^*(f) = 0$. Furthermore, there is a projection map

$$\sigma : \Omega(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M)) \to T^{\mathrm{vert}}_{\mathrm{poly}}(M)|_{y=0} \ \subset \Omega(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M))$$

given by

$$\begin{aligned} &\sigma(\sum_{k\geq 0} \sum_{l,m\geq 0} f^{b_1 \dots b_l}_{a_1 \dots a_k, c_1 \dots c_m}(x) y^{a_1} \dots y^{a_k} \frac{\partial}{\partial y^{b_1}} \wedge \dots \wedge \frac{\partial}{\partial y^{b_l}} \, \mathrm{d}x^{c_1} \wedge \dots \wedge \mathrm{d}x^{c_m} \ ) \\ &= \sum_{l\geq 0} f^{b_1 \dots b_l}(x) \frac{\partial}{\partial y^{b_1}} \wedge \dots \wedge \frac{\partial}{\partial y^{b_l}}. \end{aligned}$$

One checks that

$$f = \sigma f + \delta\delta^* f + \delta^* \delta f. \tag{3.6}$$

Hence we have proved that the cohomology of $\Omega(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M))$ is $T^{\mathrm{vert}}_{\mathrm{poly}}(M)|_{y=0}$. But $T^{\mathrm{vert}}_{\mathrm{poly}}(M)|_{y=0}$ is isomorphic to $T_{\mathrm{poly}}(M)$ as $C^\infty(M)$-module via $\frac{\partial}{\partial y^i} \mapsto \frac{\partial}{\partial x^i}$. This concludes the proof. □

3.3 The Fedosov resolution

The Poincaré resolution is a resolution of $T_{\mathrm{poly}}(M)$ as a $C^\infty(M)$-module, but not as a Lie algebra. Recall that $T_{\mathrm{poly}}(M)$ is endowed with the Schouten bracket and $\Omega(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M))$ with the vertical Schouten bracket. However, the resolution can be transformed into a resolution of Lie algebras by the Fedosov trick, first applied in [8].

Choose a torsion-free connection on M and denote its Christoffel symbols by $\Gamma^k_{ij}(x)$. Then we are given a derivation ∇ of $\Omega(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M))$ by

$$\nabla f = df + [\Gamma, f]^{\mathrm{vert}}$$

where

$$d = \mathrm{d}x^i \frac{\partial}{\partial x^i} \qquad \text{and}$$

$$\Gamma = -\eta^i \Gamma_{ij}^k(x) y^j \psi_k = -\mathrm{d}x^i \Gamma_{ij}^k(x) y^j \frac{\partial}{\partial y^k}.$$

In general, ∇ is not a differential. Instead we have

$$\nabla^2 f = [R, f]^{\mathrm{vert}},$$

where

$$R = -\frac{1}{2}\eta^i \eta^j R_{kij}^l y^k \psi_l = -\frac{1}{2}\mathrm{d}x^i \mathrm{d}x^j R_{kij}^l y^k \frac{\partial}{\partial y^l}$$

is given by the Riemann curvature tensor of the connection. It is not hard to check that

$$\delta\nabla + \nabla\delta = 0.$$

We are going to twist the differential δ with ∇. As ∇ is not a differential, $\delta + \nabla$ is not a differential either. However, it can be made into one by adding an extra term. This is the content of the following lemma:

Lemma 3. *There exists an element A in $\Omega(M, T_{\mathrm{poly}}^{\mathrm{vert}}(M))$ such that $\delta^* A = 0$ and*

$$D := \nabla - \delta + [A, \bullet]^{\mathrm{vert}}$$

is a differential. The element A has the form

$$A = \sum_{p=2}^{\infty} \eta^k A_{k,i_1\dots i_p}^j(x) y^{i_1} \dots y^{i_p} \psi_j = \sum_{p=2}^{\infty} \mathrm{d}x^k A_{k,i_1\dots i_p}^j(x) y^{i_1} \dots y^{i_p} \frac{\partial}{\partial y^j}.$$

Proof. The first observation is that $D^2 = 0$ follows from

$$R + \nabla A + \frac{1}{2}[A, A]^{\mathrm{vert}} = \delta A. \tag{3.7}$$

Furthermore, observe that the we are looking for an A such that $\delta^* A = 0$ and $\sigma A = 0$. Hence, by using Equation (3.6),

$$A = \sigma A + \delta\delta^* A + \delta^*\delta A = \delta^*\delta A.$$

Together with (3.7) we get that A should satisfy

$$A = \delta^* R + \delta^* \left(\nabla A + \frac{1}{2}[A, A]^{\mathrm{vert}} \right).$$

This is a recurrence formula for A. Convergence follows from the fact that δ^* increases the degree with respect to the y. We now prove that D^2 actually vanishes using the A obtained from the recurrence. We write

$$C := R + \nabla A + \frac{1}{2}[A, A]^{\mathrm{vert}} - \delta A$$

and prove that $C = 0$.

From the Bianchi identities for the Riemann curvature tensor we get

$$\delta R = \nabla R = 0.$$

A consequence of this is that C lies in the kernel of D:

$$DC = \nabla C - \delta C + [A, C]^{\mathrm{vert}} = 0. \tag{3.8}$$

It holds that $\sigma(C) = 0$ and $\delta^* C = 0$ by

$$\delta^*(C) = \delta^* \left(R + \nabla A + \frac{1}{2}[A, A]^{\mathrm{vert}} \right) - \delta^* \delta A = A - \delta^* \delta A = 0,$$

because $A = \delta^* \delta A$. Using the contracting homotopy from the Poincaré lemma and Equation (3.8) it follows that

$$\begin{aligned} C &= \sigma(C) + \delta\delta^* C + \delta^* \delta C \\ &= \delta^*(\nabla C + [A, C]^{\mathrm{vert}}). \end{aligned}$$

This recursion equation for C has the unique solution 0, because δ^* increases the degree with respect to the y. Hence $C = 0$, which proves the claim. □

Twisting the differential does not change cohomology. Indeed, the following holds:

Proposition 1. *As $C^\infty(M)$-module, the cohomology of $\Omega(M, T_{\mathrm{poly}}^{\mathrm{vert}}(M))$ with respect to D is given by*

$$H^n(\Omega(M, T_{\mathrm{poly}}^{\mathrm{vert}}(M)), D) \cong \begin{cases} T_{\mathrm{poly}}(M) & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We use recursion equations as in the foregoing proof.

At first, we consider elements f of $\Omega^r(M, T_{\mathrm{poly}}^{\mathrm{vert}}(M))$ with $r \geq 1$. Let $Df = 0$. We try to find an element g in $\Omega^{r-1}(M, T_{\mathrm{poly}}^{\mathrm{vert}}(M))$ with $Dg = f$ and $\delta^* g = \sigma(g) = 0$. Then g satisfies

$$g = \sigma g + \delta\delta^* g + \delta^* \delta g = \delta^* \delta g$$

and $Dg = \nabla g - \delta g + [A, g]^{\mathrm{vert}} = f$, so

$$\delta g = -f + \nabla g + [A, g]^{\mathrm{vert}}.$$

Hence g has to satisfy

$$g = -\delta^* f + \delta^*(\nabla g + [A, g]^{\mathrm{vert}}).$$

This is a recurrence equation for g. It converges and yields g such that $\sigma g = 0$ and $\delta^* g = 0$. We prove that in fact $Dg = f$.

Consider the element $h := Dg - f$. It holds that

$$\delta^* h = \delta^*(Dg) - \delta^* f = g - \delta^* \delta g = 0$$

because of the recurrence equation, and $\sigma h = 0$ because h lies in $\Omega^r(M, T_{\text{poly}}^{\text{vert}}(M))$, $r \geq 1$. By applying the contracting homotopy to $h = Dg - f$, and by the fact that $Dh = 0$, it follows that

$$h = \sigma h + \delta\delta^* h + \delta^* \delta h = \delta^*(\nabla h + [A, h]^{\text{vert}}) \tag{3.9}$$

because $\sigma h = \delta^* h = 0$. Since δ^* increases the degree with respect to the y, zero is the only solution of the recursion equation (3.9). Hence $h = Df - g = 0$ and $(\Omega(M, T_{\text{poly}}^{\text{vert}}(M)), D)$ is acyclic.

The second step is to find a bijection τ between $T_{\text{poly}}^{\text{vert}}|_{y=0}$ and the elements of $\Omega^0(T_{\text{poly}}^{\text{vert}}(M))$ that lie in $\ker(D)$. Let f_0 be an element of $T_{\text{poly}}^{\text{vert}}|_{y=0}$. The aim is to find an element f in $\Omega^0(T_{\text{poly}}^{\text{vert}}(M))$ that satisfies

$$\begin{aligned} Df &= 0, \\ \sigma f &= f_0 \end{aligned}$$

and is unique with these properties. Observe that $f \in \Omega^0(T_{\text{poly}}^{\text{vert}})$ implies that

$$\delta^* f = 0.$$

Hence, such an f satisfies

$$\begin{aligned} f &= \sigma f + \delta\delta^* f + \delta^* \delta f \\ &= f_0 + \delta^* \delta f \\ &= f_0 + \delta^*(\nabla f + [A, f]^{\text{vert}}), \end{aligned}$$

the latter because we want $Df = 0$. As δ^* increases the degree with respect to the y, this equation has a unique solution. We check that actually $Df = 0$. That $\sigma f = f_0$ is clear. Denote Df by u. Then $\sigma u = 0$ because u lies in the image of D and

$$\delta^* u = -\delta^* \delta f + \delta^*(\nabla f + [A, f]^{\text{vert}}) = -\delta^* \delta f + f - f_0 = \delta\delta^* f = 0$$

because $\delta^* f = 0$. Hence

$$\begin{aligned} u &= \sigma u + \delta\delta^* u + \delta^* \delta u \\ &= \delta^*(\nabla u + [A, u]^{\text{vert}}), \end{aligned} \tag{3.10}$$

because $Du = 0$. Equation (3.10) has $u = 0$ as only solution, thus $Df = 0$.

This proves that $H^0(\Omega(M, T_{\text{poly}}^{\text{vert}}(M)), D)$ is isomorphic as a $C^\infty(M)$-module to $T_{\text{poly}}^{\text{vert}}(M)|_{y=0}$. That it is isomorphic to $T_{\text{poly}}(M)$ is clear from before. This concludes the proof. □

The vertical Schouten bracket on $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$ commutes with the differential D, that is,

$$D[f, g]^{\text{vert}} = [Df, g] + (-1)^{|f|}[f, Dg]^{\text{vert}}.$$

Indeed, D can be written as

$$D = \mathrm{d}x^i \frac{\partial}{\partial x^i} + [\mathrm{d}x^i \frac{\partial}{\partial y^i} - \Gamma + A, \bullet]^{\mathrm{vert}}.$$

One checks easily that $\mathrm{d}x^i \frac{\partial}{\partial x^i}$ commutes with the bracket. From the Jacobi identity for odd Lie brackets, it follows that $[\mathrm{d}x^i \frac{\partial}{\partial y^i} - \Gamma + A, \bullet]^{\mathrm{vert}}$ commutes with the vertical Schouten bracket.

Hence, the vertical Schouten bracket induces a Lie bracket on the cohomology of $\Omega(M, T_{\mathrm{poly}}^{\mathrm{vert}}(M))$ with respect to D, that means on $T_{\mathrm{poly}}(M)$. We show that the induced bracket coincides with the usual Schouten bracket on $T_{\mathrm{poly}}(M)$.

Proposition 2. *Consider the Lie algebra $T_{\mathrm{poly}}(M)$ equipped with the Schouten bracket, and the Lie algebra*

$$H^0(\Omega(M, T_{\mathrm{poly}}^{\mathrm{vert}}(M)), D)$$

equipped with the induced vertical Schouten bracket. The $C^\infty(M)$-module isomorphism

$$T_{\mathrm{poly}}(M) \to H^0(\Omega(M, T_{\mathrm{poly}}^{\mathrm{vert}}(M)), D)$$

given by Proposition 1 is an isomorphism of Lie algebras.

Proof. We introduce the following notation: σ' denotes the composition

$$\Omega^0(M, T_{\mathrm{poly}}^{\mathrm{vert}}(M)) \xrightarrow{\sigma} T_{\mathrm{poly}}^{\mathrm{vert}}(M)|_{y=0} \xrightarrow{\sim} T_{\mathrm{poly}}(M)$$

and τ' denotes the composition

$$\Omega^0(M, T_{\mathrm{poly}}^{\mathrm{vert}}(M)) \xleftarrow{\tau} T_{\mathrm{poly}}^{\mathrm{vert}}(M)|_{y=0} \xleftarrow{\sim} T_{\mathrm{poly}}(M).$$

The morphism τ was defined in the proof of Proposition 1.

We have to show that

$$\tau'[f_0, g_0] = [\tau' f_0, \tau' g_0]^{\mathrm{vert}} \tag{3.11}$$

for $f_0, g_0 \in T_{\mathrm{poly}}(M)$. We denote $\tau' f_0$ by f and $\tau' g_0$ by g. By the definition of τ', the equation (3.11) for $f_0, g_0 \in T_{\mathrm{poly}}(M)$ is equivalent to

$$[\sigma' f, \sigma' g] = \sigma'[f, g]^{\mathrm{vert}}$$

for $f, g \in \Omega^0(T_{\mathrm{poly}}^{\mathrm{vert}}(M))$ with $Df = Dg = 0$.

From $Df = 0$ it follows that

$$\mathrm{d}x^i \frac{\partial f}{\partial y^i} = \mathrm{d}x^i \frac{\partial f}{\partial x^i} - \mathrm{d}x^i \Gamma_{ij}^k(x) \psi_k \frac{\partial f}{\partial \psi_j} + \mathrm{d}x^i(\text{terms containing } y).$$

Hence, using that f lies in $\Omega^0(T_{\mathrm{poly}}^{\mathrm{vert}}(M))$, we have

$$\sigma'(\frac{\partial f}{\partial y^i}) = \sigma'(\frac{\partial f}{\partial x^i}) - \sigma'(\Gamma_{ij}^k(x) \psi_k \frac{\partial f}{\partial \psi_j}).$$

From the explicit formula for the vertical Schouten bracket (3.3), we obtain

$$\sigma'[f,g]^{\mathrm{vert}} = [\sigma' f, \sigma' f] - \sigma' S$$

where

$$\begin{aligned} S &= \Gamma_{ij}^k(x)\psi_k \frac{\partial f}{\partial \psi_j}\frac{\partial g}{\partial \psi_i} + (-1)^{|f|}\frac{\partial f}{\partial \psi^i}\Gamma_{ij}^k(x)\psi_k\frac{\partial g}{\partial \psi_j} \\ &= 0 \end{aligned}$$

because Γ_{ij}^k is symmetric in the lower indices. This proves the claim. □

3.4 The twisted automorphism

The Schouten algebra $T_{\mathrm{poly}}(M)$ is isomorphic as a Lie algebra to the cohomology of $\Omega(M, T_{\mathrm{poly}}^{\mathrm{vert}}(M))$ with respect to the Fedosov differential D, that is the content of Proposition 2. It is quite straightforward to construct an L_∞-automorphism of $\Omega(M, T_{\mathrm{poly}}^{\mathrm{vert}}(M))$ given an L_∞-automorphism of $T_{\mathrm{poly}}(\mathbb{R}^n)$: one applies the given automorphism fiberwise. However, we are looking for an L_∞-automorphism of $\Omega(M, T_{\mathrm{poly}}^{\mathrm{vert}}(M))$ that commutes with the differential D, so that it induces an L_∞-automorphism on its cohomology, the Schouten algebra. To accomplish this we have to twist the automorphism. We will see that this twist is independent of the choice of coordinates if the original automorphism of $T_{\mathrm{poly}}(\mathbb{R}^d)$ satisfies Condition 1.

More precisely, the last steps of the proof of the main result are the following: Let F be an L_∞-automorphism of $T_{\mathrm{poly}}(\mathbb{R}^d)$ as constructed in [15]. We construct an L_∞-automorphism F^{vert} of $\Omega(M, T_{\mathrm{poly}}^{\mathrm{vert}}(M))$. Then we twist this automorphism with a certain Maurer-Cartan element B which we define explicitly below and obtain a morphism F_B^{vert} that commutes with D. Finally, we show that this construction is independent of the choice of coordinates if F satisfies Condition 1.

Assume, hence, that F is an L_∞-automorphism of $T_{\mathrm{poly}}(\mathbb{R}^n)$ as in [15]. Its components are defined as

$$F_n = \begin{cases} \mathrm{id} & n = 1, \\ \sum_\Gamma C_\Gamma \Phi_\Gamma & n \geq 2 \end{cases}$$

where the summation runs over graphs Γ with n vertices and $2n-2$ directed edges. The Φ_Γ are polydifferential operators $\bigotimes^n T_{\mathrm{poly}}(\mathbb{R}^n) \to T_{\mathrm{poly}}(\mathbb{R}^n)$. As we are working over $\mathbb{R}^d$, the Schouten algebra $T_{\mathrm{poly}}(\mathbb{R}^d)$ can be identified with the polynomial ring $C^\infty(\mathbb{R}^d)[\varphi^1, \ldots, \varphi^d]$. Hence, Φ_Γ can be seen as a polydifferential operator

$$\bigotimes^n C^\infty(\mathbb{R}^d)[\varphi_1, \ldots, \varphi_d] \to C^\infty(\mathbb{R}^d)[\varphi_1, \ldots, \varphi_n].$$

The Φ_Γ are defined by

$$\Phi_\Gamma(f_1, \ldots, f_n) = \Big[\Big(\prod_{e \in \mathrm{Edges}(\Gamma)} \Delta_e \Big) f_1(\varphi_{(1)}, x_{(1)}) \ldots f_n(\varphi_{(n)}, x_{(n)}) \Big]_{x_{(1)} = \ldots = x_{(n)}, \varphi_{(1)} = \ldots = \varphi_{(n)}}$$

with

$$\Delta_e := \sum_{a=1}^{d} \frac{\partial^2}{\partial x^a_{(j)} \partial \varphi_{(i)a}}$$

where the edge e starts at the vertex labeled by i and ends at the vertex labeled by j. For example, if

$$\Gamma = \begin{array}{c} \bullet^{2} \\ \uparrow \\ \bullet^{1} \end{array} ,$$

then

$$\Phi_\Gamma(f_1(x,\varphi), f_2(x,\varphi)) = \frac{\partial f_1}{\partial x^i}(-1)^{|f_1|}\frac{\partial f_2}{\partial \varphi^i}.$$

We aim to construct a morphism $\Phi^{\mathrm{vert}}_\Gamma$ that works fiberwise on $\Omega(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M))$. At first, we choose a local patch U on M. Restricted to this patch, $\Omega(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M))$ can be identified with

$$C^\infty(U)[[y^1, \ldots, y^d]][\psi_1, \ldots, \psi_d, \eta^1, \ldots, \eta^d],$$

as in (3.2) in Section 3.1. We define $\Phi^{\mathrm{vert}}_\Gamma$ on this patch as

$$\Phi^{\mathrm{vert}}_\Gamma(f_1, \ldots, f_n) = \\ \Bigg[\Big(\prod_{e \in \mathrm{Edges}(\Gamma)} \Delta_e\Big)\ f_1(\psi_{(1)}, y_{(1)}) \ldots f_n(\psi_{(n)}, y_{(n)})\Bigg]_{y_{(1)}=\ldots=y_{(n)}, \psi_{(1)}=\ldots=\psi_{(n)}}$$

with

$$\Delta_e := \sum_{a=1}^{d} \frac{\partial^2}{\partial y^a_{(j)} \partial \psi_{(i)a}}.$$

Thus $\Phi^{\mathrm{vert}}_\Gamma$ differentiates with respect to y^i and ψ_i where Φ_Γ differentiates with respect to x^i and φ_i. The difference is that $\Phi^{\mathrm{vert}}_\Gamma$ is independent of the choice of the coordinates $x^1, \ldots, x^d$, whereas Φ_Γ is not. We define F^{vert} locally by

$$F^{\mathrm{vert}}_n = \sum_\Gamma C_\Gamma \Phi^{\mathrm{vert}}_\Gamma \tag{3.12}$$

where the summation runs over graphs Γ with n vertices and $2n-2$ edges as before. As both the weights C_Γ and $\Phi^{\mathrm{vert}}_\Gamma$ are independent of the choice of coordinates, this formula gives a global F^{vert}, i.e.,

$$F^{\mathrm{vert}}_n : \bigotimes^n \Omega(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M)) \to \Omega(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M)).$$

In [15] it is proved that F is an L_∞-morphism of $T_{\mathrm{poly}}(\mathbb{R}^d)$ equipped with the Schouten bracket. The proof also holds for F^{vert} with the vertical Schouten bracket. Hence we have proved the following lemma:

Lemma 4. *F^{vert} as defined in* (3.12) *is an L_∞-automorphism of $\Omega(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M))$ with respect to the vertical Schouten bracket.*

This automorphism F^{vert} commutes with the differential d, but not with

$$D = d + [\Gamma - \eta^i \psi_i + A, \bullet]^{\mathrm{vert}}.$$

We write

$$B = \Gamma - \eta^i \psi_i + A = -\mathrm{d}x^i \Gamma_{ij}^k(x) y^j \frac{\partial}{\partial y^k} - \mathrm{d}x^i \frac{\partial}{\partial y^i} + \sum_{p=2}^{\infty} \mathrm{d}x^k A^j_{k,i_1 \dots i_p}(x) y^{i_1} \dots y^{i_p} \frac{\partial}{\partial y^j}. \quad (3.13)$$

We reformulate the above statement: F^{vert} commutes with the differential d, but not with $D = d + [B, -]^{\mathrm{vert}}$. This problem will be solved by twisting with the Maurer-Cartan element B, a technique we explained in Section 2.2. That B is a Maurer-Cartan element in the differential graded Lie algebra $\Omega(M, T_{\mathrm{poly}}^{\mathrm{vert}}(M))$ with differential d and the vertical Schouten bracket is most easily seen backwards. Twisting this L_∞-algebra with B, we obtain $\Omega(M, T_{\mathrm{poly}}^{\mathrm{vert}}(M))$ with differential $D = d + [B, -]^{\mathrm{vert}}$ and the vertical Schouten bracket. As this is a differential graded Lie algebra as well, it follows that B is a Maurer-Cartan element. We will see later that B depends on the choice of coordinates. Hence we start working on a local patch of M with fixed coordinate system.

Lemma 5. *The twisted morphism F_B^{vert} defined by*

$$F_B^{\mathrm{vert}} := \exp(-B) \circ F^{\mathrm{vert}} \circ \exp(B)$$

on a local patch U of M is an L_∞-automorphism of $\Omega(U, T_{\mathrm{poly}}^{\mathrm{vert}}(U))$ with respect to D and the vertical Schouten bracket $[-,-]^{\mathrm{vert}}$.

Proof. We proceed as in Section 2.2. At first, we use (2.4) and compute

$$\sum_{i=1}^{\infty} \frac{1}{i!} F_i(B^i).$$

By the definition of F and the first part of Condition 1, this is B. Hence, in Equation (2.5), we twist with the same Maurer-Cartan element from both sides. We get that $F_B^{\mathrm{vert}} = \exp(-B) \circ F^{\mathrm{vert}} \circ \exp(B)$ is an L_∞-automorphism of $\Omega(M, T_{\mathrm{poly}}^{\mathrm{vert}}(M))$ with respect to D, which concludes the proof. ☐

Lemma 6. *The twisted morphism F_B^{vert} defined in Lemma 5 is independent of the choice of coordinates. Hence it glues together to an L_∞-automorphism of $\Omega(M, T_{\mathrm{poly}}^{\mathrm{vert}}(M))$ with respect to D and the vertical Schouten bracket $[-,-]^{\mathrm{vert}}$.*

Proof. We analyze how B transforms under change of coordinates. The terms $\mathrm{d}x^i \frac{\partial}{\partial y^i}$ and A are invariant under change of coordinates. The transformation of Γ is more complicated due to the presence of the Christoffel symbols. We compute that B transforms as

$$B' = B + \mathrm{d}x^i H_{ij}^k(x) y^j \frac{\partial}{\partial y^k}$$

for some $H^k_{ij}(x)$, where the exact form of $H^k_{ij}(x)$ is not important.[1]

We have a closer look at the explicit formula for Φ^{vert}_B. By Equation (2.6) in Section 2.2, it holds that

$$F^{\mathrm{vert}}_{B,n}(X) = \sum_{i=0}^{\infty} \frac{1}{i!} F^{\mathrm{vert}}_{n+i}(B^i X)$$

for X in $\bigwedge^n \Omega(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M))$. However, F^{vert} is zero on any summand of the form $\mathrm{d}x^i H^k_{ij}(x) y^j \frac{\partial}{\partial y^k}$ by the second part of Condition 1. Hence F^{vert}_B is independent of the choice of coordinates, which concludes the proof. □

Lemma 6 and Proposition 1 together result in the main theorem of this thesis.

Main Theorem. *For a smooth d-dimensional manifold M and an L_∞-morphism F of $T_{\mathrm{poly}}(\mathbb{R}^d)$ satisfying Condition 1, the globalized morphism F_{glob} constructed above is an L_∞-morphism of $T_{\mathrm{poly}}(M)$.*

Proof. Because F^{vert}_B commutes with D, it induces an L_∞-automorphism $\tilde{F}^{\mathrm{vert}}_B$ on cohomology $H^0(\Omega(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M)), D)$. Together with the Lie algebra isomorphism τ' from $T_{\mathrm{poly}}(M)$ to $H^0(\Omega(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M)), D)$ and its inverse τ'^{-1} from $H^0(\Omega(M, T^{\mathrm{vert}}_{\mathrm{poly}}(M)), D)$ to $T_{\mathrm{poly}}(M)$ we get that

$$F_{\mathrm{glob}} = \tau'^{-1} \circ \tilde{F}^{\mathrm{vert}}_B \circ \tau'$$

is an L_∞-automorphism of $T_{\mathrm{poly}}(M)$. □

3.5 The conditions on the local automorphism

In this section, we show that an L_∞-automorphism of $T_{\mathrm{poly}}(\mathbb{R}^d)$ constructed as in [15] that satisfies Condition 2 also satisfies Condition 1. Assume we are given such an automorphism F. Recall that it is constructed as

$$F_n = \begin{cases} \mathrm{id} & n = 1, \\ \sum_\Gamma C_\Gamma \Phi_\Gamma & n \geq 2 \end{cases}$$

where the summation runs over graphs Γ with n vertices and $2n-2$ directed edges. The Φ_Γ are polydifferential operators

$$\bigotimes^n C^\infty(\mathbb{R}^d)[\varphi_1, \ldots, \varphi_d] \to C^\infty(\mathbb{R}^d)[\varphi_1, \ldots, \varphi_n]$$

defined by

$$\Phi_\Gamma(f_1, \ldots, f_n) = \left[\Big(\prod_{e \in \mathrm{Edges}(\Gamma)} \Delta_e \Big) f_1(\varphi_{(1)}, x_{(1)}) \ldots f_n(\varphi_{(n)}, x_{(n)}) \right]_{x_{(1)} = \ldots = x_{(n)}, \varphi_{(1)} = \ldots = \varphi_{(n)}}$$

[1]This step is taken directly from [7], see Equation (58).

with

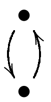
$$\Delta_e := \sum_{a=1}^{d} \frac{\partial^2}{\partial x^a_{(j)} \partial \varphi_{(i)a}}$$

where the edge e starts at the vertex labeled by i and ends at the vertex labeled by j. We assume that F satisfies Condition 2, i.e., $C_\Gamma = 0$ if Γ contains a vertex with at most one outgoing and at most one ingoing edge.

At first, we show that

$$F_n(v_1, \dots, v_n) = 0$$

for vector fields $v_1, \dots, v_n$. The only polydifferential operators Φ_Γ that are nonzero on vector fields correspond to those graphs Γ in which every vertex has at most one outgoing edge. As the number of edges is $2n-2$, this is satisfied only by the graph

$\Gamma =$  .

However, the weight C_Γ for this graph is zero by Condition 2. Hence F is zero on vector fields.

Finally we show that

$$F_n(\gamma_1, \dots, l^i(x)\frac{\partial}{\partial x^i}, \dots, \gamma_n) = 0,$$

for arbitrary polyvector fields γ_1 till γ_n and a vector field $l^i(x)\frac{\partial}{\partial x^i}$ where the $l^i(x)$ are linear in the coordinates $x^1, \dots, x^d$ of $\mathbb{R}^d$. A polydifferential operator Φ_Γ does not vanish on $l^i(x)\frac{\partial}{\partial x^i}$ only if the graph Γ contains vertices with at most one ingoing and at most one outgoing edge. The weight C_Γ for these graphs is zero by Condition 2. Hence F vanishes if one of the inputs is a vector field linear in the coordinates of $\mathbb{R}^d$.

We conclude that Condition 2 implies Condition 1.

Bibliography

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Deformation theory and quantization. I. Deformations of symplectic structures. *Annals of Physics*, 111(1):61–110, 1978.

[2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Deformation theory and quantization. II. Physical applications. *Annals of Physics*, 111(1):111–151, 1978.

[3] A.S. Cattaneo and G. Felder. On the globalization of Kontsevich's star product and the perturbative Poisson sigma model. *Progress of theoretical physics - supplement*, 144:38–53, 2001.

[4] A.S. Cattaneo, G. Felder, and L. Tomassini. From local to global deformation quantization of Poisson manifolds. *Duke Mathematical Journal*, 115(2):329–352, 2002.

[5] A.S. Cattaneo and D. Indelicato. Formality and star products. In *Poisson Geometry, Deformation and Group Representations*, pages 81–144. London Mathematical Society, 2004.

[6] V.A. Dolgushev. *A Proof of Tsygan's Formality Conjecture for an Arbitrary Smooth Manifold*. PhD thesis, Massachusetts Institute of Technology, Arxiv preprint math/0504420, 2005.

[7] V.A. Dolgushev. Covariant and equivariant formality theorems. *Advances in Mathematics*, 191(1):147–177, 2005.

[8] B.V. Fedosov. A simple geometrical construction of deformation quantization. *Journal of Differential Geometry*, 40(2):213–238, 1994.

[9] M. Gerstenhaber. On the deformation of rings and algebras. *Annals of Mathematics*, 79(1):59–103, 1964.

[10] G. Hochschild, B. Kostant, and A. Rosenberg. Differential forms on regular affine algebras. *Transactions of the American Mathematical Society*, 102(3):383–408, 1962.

[11] M. Kontsevich. Formality conjecture. In *Proceedings of the Ascona Meeting*, pages 139–156. Kluwer Academic Publishers, 1996.

[12] M. Kontsevich. Deformation quantization of Poisson manifolds. *Letters in Mathematical Physics*, 66(3):157–216, 2003.

[13] T. Lada and M. Markl. Strongly homotopy Lie algebras. *Communications in Algebra*, 23(6):2147–2161, 1995.

[14] T. Lada and J. Stasheff. Introduction to sh Lie algebras for physicists. *International Journal of Theoretical Physics*, 32(7):1087–1103, 1993.

[15] S.A. Merkulov. Exotic automorphisms of the Schouten algebra of polyvector fields. *Arxiv preprint arXiv:0809.2385*, 2008.

[16] A.A. Voronov. Quantizing Poisson Manifolds. In *Perspectives on quantization: proceedings of the 1996 AMS-IMS-SIAM Joint Summer Research Conference*, pages 189–195. American Mathematical Society, 1998.

[17] S. Waldmann. *Poissongeometrie und Deformationsquantisierung: Eine Einführung.* Springer, 2007.

[18] A. Yekutieli. Continuous and twisted morphisms. *Journal of Pure and Applied Algebra*, 207(3):575–606, 2006.