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An explicit calculation of the Ronkin function

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Abstract

We calculate the second order derivatives of the Ronkin function in the case of an affine linear polynomial in three variables and give an expression of them in terms of complete elliptic integrals and hypergeometric functions. This gives a semi-explicit expression of the associated Monge-Ampère measure, the Ronkin measure.

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Contents

1 Introduction

Amoebas are certain projections of sets in $\mathbb{C}^n$ to $\mathbb{R}^n$ that are connected to several areas in mathematics such as complex analysis, tropical geometry, real algebraic geometry, special functions and combinatorics to name a few. The term amoeba was first defined by Gelfand, Kapranov and Zelevinsky in [8] and these objects were later studied by several other authors like Mikhalkin, Passare, Rullgård and Tsikh. The Ronkin function of a polynomial is closely connected to the amoeba. The main result in this thesis is an explicit calculation of the second order derivatives of the Ronkin function in the case of an affine linear polynomial L in three dimensions, thus giving an explicit expression of the so called Ronkin measure associated to L.

Assume that f is a Laurent polynomial in n variables over $\mathbb{C}$. This means that

$$f(z) = \sum_{\alpha \in A} a_\alpha z^\alpha$$

for some finite set $A \subset \mathbb{Z}^n$. The convex hull of the points $\alpha \in A$ for which $a_\alpha \neq 0$ is called the Newton polytope of f and is denoted by Δ_f. In order to define the amoeba of f (and the compactified amoeba of f) we need to introduce the two mappings $\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$ and $\nu_f : (\mathbb{C}^*)^n \to \Delta_f$ defined by

$$\begin{aligned} &\mathrm{Log}(z_1, \ldots, z_n) = (\log|z_1|, \ldots, \log|z_n|), \quad \text{and} \\ &\nu(z_1, \ldots, z_n) = \frac{\sum_{\alpha \in A} |z^\alpha| \cdot \alpha}{\sum_{\alpha \in A} |z^\alpha|} \end{aligned}$$

respectively.

Definition 1.1. (Gelfand, Kapranov, Zelevinsky) *Let $f(z)$ be a Laurent polynomial in n variables over $\mathbb{C}$. The* amoeba *of f, denoted by $\mathcal{A}_f$, and the* compactified amoeba *of f, denoted by $\bar{\mathcal{A}}_f$, are defined as the image of the zero set $f^{-1}(0)$ under the maps* Log *and ν respectively.*

We have the following commutative diagram

$$\begin{tikzcd} (\mathbb{C}^*)^n \arrow[r, "\mathrm{Log}"] \arrow[dr, "\nu"'] & \mathbb{R}^n \arrow[d, "\gamma"] \\ & \mathrm{int}(\Delta_f) \end{tikzcd}$$

where

$$\gamma(x) = \frac{\sum_{\alpha \in A} e^{\langle \alpha, x \rangle} \cdot \alpha}{\sum_{\alpha \in A} e^{\langle \alpha, x \rangle}}$$

is a diffeomorphism.

To understand the structure of the amoeba of a Laurent polynomial one needs to take a closer look at the concept of duality between convex subdivisions of convex sets in $\mathbb{R}^n$.

Definition 1.2. *Let K be a convex set in $\mathbb{R}^n$ and let T be a collection of closed convex subsets of K. Then T is said to be a* convex subdivision *if it satisfies all of the following three conditions.*

1. *The union of all sets in K is equal to K.*

2. *A nonempty intersection of two sets in K belongs to K.*

3. *A subset τ of a set σ in K belongs to K if and only if τ is a face of σ.*

For two convex sets σ and τ such that $\tau \subset \sigma$ one can define the convex cone $\text{cone}(\tau, \sigma)$ according to

$$\text{cone}(\tau, \sigma) = \{t(x - y); x \in \sigma, y \in \tau, t \geq 0\}.$$

The dual cone $C^\vee$ of a convex cone C is defined to be

$$C^\vee = \{\xi \in \mathbb{R}^n; \langle \xi, x \rangle \leq 0, \forall x \in C\}.$$

Definition 1.3. *Let T and T' be two convex subdivisions of the sets K and K' respectively. Then T and T' are said to be dual to each other if there exist a bijective mapping from T to T', $\sigma \to \sigma^*$, such that the following two conditions are satisfied for all sets $\tau, \sigma \in T$.*

1. *$\tau \subset \sigma$ if and only if $\sigma^* \subset \tau^*$.*

2. *The cone* $\text{cone}(\tau, \sigma)$ *is dual to* $\text{cone}(\sigma^*, \tau^*)$.

The amoeba of a Laurent polynomial f has the following properties:

1. The connected components of the complement of the amoeba are convex and these complement components are in a bijective correspondence with the different Laurent series expansion of $1/f$

2. The number of complement components is at least equal to the number of vertices in $\Delta_f \cap \mathbb{Z}^n$ and at most equal to the number of points in $\Delta_f \cap \mathbb{Z}^n$

3. The amoeba can be retracted to a subdivision T of $\mathbb{R}^n$ and there exist a triangulation T' of the Newton polytope of f such that T and T' are dual to each other.

The first property is not hard to prove, see for example [18]. The other properties can be proved by a construction of an injective function from the set of connected components of $\mathbb{R}^n \setminus \mathcal{A}_f$ to $\Delta \cap \mathbb{Z}^n$ and a certain tropical polynomial with its tropical hypersurface inside the amoeba. Both of these constructions can be done using the so called Ronkin function first studied by Ronkin, see [22].

Definition 1.4. *Let $f(z)$ be a Laurent polynomial over $\mathbb{C}$. The* Ronkin function *of f, denoted by N_f, is defined by*

$$N_f(x) = \left(\frac{1}{2\pi i}\right)^n \int_{Log^{-1}(x)} \log|f(z)|\frac{dz}{z}$$

The Ronkin function of a product of two polynomials is obviously the sum of the Ronkin function of those two polynomials. It is also easy to see that the Ronkin function of a monomial $az^\alpha \in \mathbb{C}[z_1, \ldots, z_n]$ is an affine linear polynomial in $\mathbb{R}[x_1, \ldots, x_n]$, i.e. if $f(z) = az_1^{\alpha_1} \ldots z_n^{\alpha_n}$ then

$$N_f = \log|a| + \alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_n x_n. \tag{1}$$

The function N_f is convex on $\mathbb{R}^n$ and it is affine linear on an open connected set $\Omega \subset \mathbb{R}^n$ if and only if $\Omega \subset \mathbb{R}^n \setminus \mathcal{A}_f$. In fact the gradient of N_f at a point outside the amoeba is a point in $\Delta \cap \mathbb{Z}^n$, and thus we have a mapping from the set of complement components to the set of points in A. This mapping was proved to be injective in [7]. Moreover, it is easy to see that the amoeba always has components corresponding to the vertices in $\Delta \cap \mathbb{Z}^n$, and thus we get property 2 above.

Example. Let $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots a_n z^n = (z - b_1) \ldots (z - b_n)$ where $a_0 \neq 0$ and $b_1 \leq b_2 < \ldots \leq b_n$. Then for x such that $b_m < e^x < b_{m+1}$ we have

$$\begin{aligned} N_f(x) &= \int_0^{2\pi} \log|f(e^{x+i\phi})|d\phi = \log|a_0| + \sum_{k=1}^{m} \log\left(\frac{e^x}{|b_k|}\right) = \\ &= \log|a_0| - \sum_{k=1}^{m} \log|b_k| + mx. \end{aligned}$$

by Jensen's formula and we see that N_f is a convex piecewise affine linear function, singular at $\log|b_k|, \quad k = 1, \ldots, n$. $\square$

Since we have an injective function from the complement components of the amoeba to $\Delta_f \cap \mathbb{Z}^n$ we can define the concept of an order to every complement component of $\mathcal{A}_f$ in the obvious way. Let $\tilde{A}$ be the subset of $\mathbb{Z}^n$ such that α belongs to $\tilde{A}$ if and only if $\mathcal{A}_f$ has a component of order α. for every α in $\tilde{A}$, define the real number c_α by

$$c_\alpha = N_f(x) - \langle \alpha, x \rangle$$

where x is any point in the complement component of order α and let

$$S(x) = \max_{\alpha \in \tilde{A}} (c_\alpha + \langle \alpha, x \rangle).$$

Then $S(x)$ is a convex piecewise affine linear function that agrees with N_f on the complement of the amoeba. The function $S(x)$ is a so called tropical polynomial and its tropical hypersurface, i.e. the corner locus of the function $S(x)$ is called the spine of the amoeba of f and is denoted by $\mathcal{S}_f$. In [18] the authors prove the following theorem.

Theorem 1.1. (Passare, Rullgård) *The spine $\mathcal{S}_f$ is a deformation retract of $\mathcal{A}_f$ and there exist dual subdivisions T of $\mathbb{R}^n$ and T' of $\tilde{A}$ such that $\mathcal{S}_f$ is the union of the cells in T of dimension less than n. Moreover, the cell of T dual to the point $\alpha \in \tilde{A}$ contains the complement component of order α.*

Another important subset of the amoeba is the so called contour. This set will play an important role in Section 4.

Definition 1.5. *The set of critical values of the mapping* Log *restricted to $f^{-1}(0)$ is called the* contour *of $\mathcal{A}_f$ and is denoted by $\mathcal{C}$.*

The contour is a real analytic hypersurface of $\mathbb{R}^n$ and the boundary of the amoeba is always included in the contour. The following map is closely connected with the contour of an amoeba.

Definition 1.6. *Let Z be an algebraic hypersurface with defining polynomial f. The logarithmic Gauss-map $\gamma : Z \to \mathbb{CP}^{n-1}$ is defined by*

$$\gamma(z_1, \dots z_n) = [z_1 \frac{\partial}{\partial z_1} f(z_1, \dots, z_n) : \dots : z_n \frac{\partial}{\partial z_n} f(z_1, \dots, z_n)].$$

This map can be geometrically described as follows. Take a regular point $z \in Z$. Take a small neighborhood $\mathcal{U}$ of z and map that neighborhood with the complex logarithm and choose a branch. Now take the normal direction of $log(z)$ and you get $\gamma(z)$. Note that this is the Gauss map composed with the complex logarithm, hence the name. The next proposition gives a nice way to describe the contour in terms of the logarithmic Gauss map.

Proposition 1.1. Mikhalkin) *Let f be a Laurent polynomial. The critical points of the map* Log *are exactly the ones that are mapped to the real subspace $\mathbb{RP}^{n-1} \subset \mathbb{CP}^{n-1}$. That is*

$$\mathcal{C} = \mathrm{Log}(\gamma^{-1}(\mathbb{RP}^{n-1}))$$

The proof can be found in [14].

Closely related to the Ronkin function is the Mahler measure that was introduced by Mahler in [12]. He made the following definition

Definition 1.7. *Let f be a polynomial in n variables with real or complex coefficients. The number*

$$\mathrm{M}(f) = \begin{cases} \exp\left(\frac{1}{2\pi i}\right)^n \int_{\mathrm{Log}^{-1}(0)} \log |f(z)| \frac{dz}{z} & \textit{if} \quad f \not\equiv 0 \\ 0 & \textit{if} \quad f \equiv 0 \end{cases}$$

is called the Mahler measure *of f.*

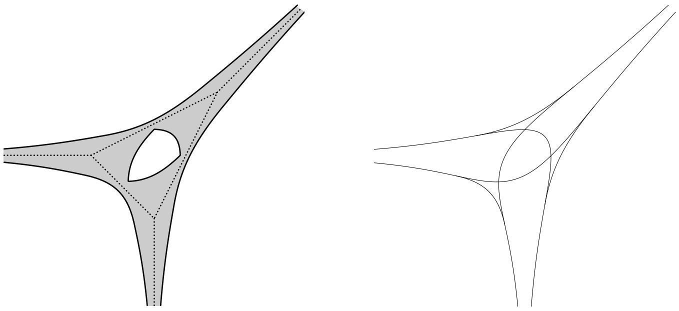

Figure 1: The amoeba and spine of $f(z,w) = 1 + 2zw + z^3 + w^3$
and its contour.

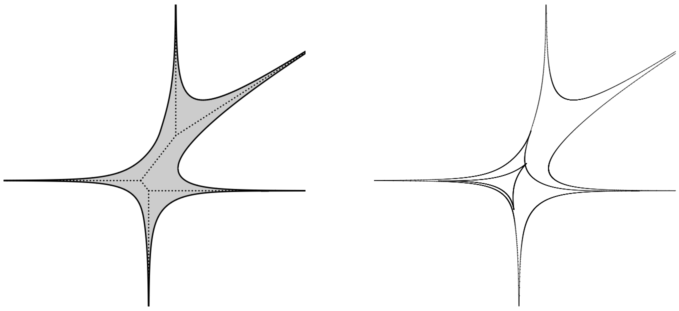

Figure 2: The amoeba and spine of $f(z,w) = 2 + 2z^2 + 2w^2 + 3z^2w + zw^2$
and its contour.

We see that the logarithm of the Mahler measure, $\mathrm{m}(f)$, is the Ronkin function evaluated in the origin. On the other hand if $f(z) = \sum_{\alpha\in A} a_\alpha z^\alpha$ then we have the following equality:

$$N_f(x) = \mathrm{m}\left(\sum_{\alpha\in A} a_\alpha e^{\langle\alpha,x\rangle} z^\alpha\right).$$

In particular, if $f(z_1,\ldots,z_n) = 1 + z_1 + \ldots + z_n$ we have

$$N_f(x_1,\ldots,x_n) = \mathrm{m}(1 + e^{x_1}z_1 + \ldots + e^{x_n}z_n).$$

Thus if one can give an explicit expression of the Mahler measure of $f = a_0 + a_1z_1 + \ldots + a_nz_n$ for $a_j > 0$ one also has an explicit expression of the Ronkin function of $f = 1 + z_1 + \ldots + z_n$ and vice versa. In the case of two variables such an expression is known, see [24], but in higher dimensions it is not.

The Mahler measure had been considered before Mahler by Lehmer, [10], in 1933 but then only in the one variable case. Lehmer made the conjecture that if f is a non-cyclotomic polynomial with integer coefficients then

$$\mathrm{M}(f) \geq M(f_0) = 1.17628\ldots$$

where

$$f_0(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1.$$

This is still an open question. In [1] Boyd proved that the Mahler measure of several variable polynomials is the same as a certain limit of a sequence of Mahler measures of one variable polynomials. In this way one could try to find small values by calculating the Mahler measure in say two variables and maybe disprove the conjecture of Lehmer. Such calculations where done numerically by Boyd and the example

$$M((x+1)y^2 + (x^2+x+1)y + x(x+1)) = 1.25542\ldots$$

is still the smallest value of a two variable polynomial one has found. At the same time the first results on explicit expressions of Mahler measures in several variables was obtained by Smyth [24]. One of his formulas take the following form in terms of the Ronkin function.

Theorem 1.2. (Smyth) *Let $f = 1 + z + w$. Then*

$$N_f(0,0) = \frac{3\sqrt{3}}{4\pi}\,\mathrm{L}(\chi_{-3}, 2)$$

where

$$\mathrm{L}(\chi_{-3}, s) = \sum_{k=1}^{\infty} \frac{\chi_{-3}(k)}{k^s} \quad \textit{and} \quad \chi_{-3}(k) = \begin{cases} 1 & \textit{if} \quad k \equiv 1 \mod 3 \\ -1 & \textit{if} \quad k \equiv -1 \mod 3 \\ 0 & \textit{if} \quad k \equiv 0 \mod 3 \end{cases}$$

Almost 20 years later Maillot generalized the theorem of Smyth by giving an explicit expression for the Ronkin function at every point in $\mathbb{R}^2$, see [13]. The expression involves the so called Block-Wigner dilogarithm, denoted $D(z)$ and defined as

$$D(z) = \mathrm{Im}(\mathrm{Li}_2(z) + \log|z|\log(1-z))$$

for $z \in \mathbb{C}^n \setminus \{0, 1\}$.

Theorem 1.3. (Maillot) *Let $f = 1 + z + w$. Then*

$$N_f(x,y) = \begin{cases} \frac{\alpha}{\pi}x + \frac{\beta}{\pi}y + \frac{1}{\pi}D(e^{-x+i\beta}) & \textit{if } (x,y) \in \mathcal{A}_f \\ \pi \log max\{1, e^x, e^y\} & \textit{otherwise} \end{cases}$$

where α and β are defined in Figure 3 below.

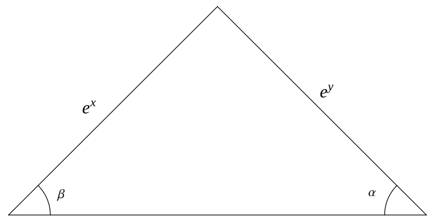

Figure 3:

Interestingly, the partial derivatives of this Ronkin function are very easy to describe.

Lemma 1.1. *Let $f(z,w) = 1 + z + w$. Then*

$$\frac{\partial}{\partial x} N_f = \frac{\alpha}{\pi}, \qquad \frac{\partial}{\partial y} N_f = \frac{\beta}{\pi}$$

where α and β is described in Figure 3.

Proof. A differentiation under the integral sign gives

$$\begin{aligned}
\frac{\partial}{\partial x} N_f(x,y) &= \frac{\partial}{\partial x}\left(\frac{1}{2\pi i}\right)^2 \int_{\mathrm{Log}^{-1}(x,y)} \log|1+z+w| \frac{dz}{z}\frac{dw}{w} = \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\mathrm{Log}^{-1}(x,y)} \frac{dz}{(1+z+w)}\frac{dw}{w} = \\
&= \left(\frac{1}{2\pi i}\right) \int_{|w|=e^y} \left(\left(\frac{1}{2\pi i}\right) \int_{|z|=e^x} \frac{dz}{z-(-1-w)}\right)\frac{dw}{w}.
\end{aligned}$$

Now, the inner integral is 1 when $|z| = e^x < |1+w|$ and 0 when $|z| = e^x > |1+w|$ and since dw/w is the volume measure on the torus $|w| = e^y$ we have that N_f equals the ratio

$$\frac{\lambda\left(\{\phi \in [0,2\pi]; e^x < |1+e^{y+i\phi}|\}\right)}{\lambda\left([0,2\pi]\right)},$$

where λ is the Lebesgue measure, and this expression is obviously equal to α/π. The second part is proved analogously. $\square$

For more about the Mahler measure, see [11].

The Ronkin function of f will give rise to a measure, called the Ronkin measure, with support on the amoeba of f. For every smooth convex function f on $\mathbb{R}^n$, the Hessian matrix of f

$$\mathrm{Hess}(f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$$

is a positive definite matrix. The determinant of the Hessian times the Lebesgue measure is called the real Monge-Ampère measure of f and is denoted by $\mathrm{M}(f)$. If we define the gradient at a point x_0 of a convex function defined in a domain Ω as

$$\operatorname{grad} f(x_0) = \{y \in \mathbb{R}^n; f(x) - f(x_0) \geq \langle y, x - x_0 \rangle, \forall x \in \Omega\}$$

then the Monge-Ampère measure can be extended to all convex functions f by letting

$$\mathrm{M}(f)(E) = \lambda(\operatorname{grad} f(E))$$

where λ is the Lebesgue measure and

$$\operatorname{grad} f(E) = \bigcup_{x \in E} \operatorname{grad} u(x).$$

The Monge-Ampère measure is a positive Borel measure for any convex function, see [20]. We are now ready to define the Ronkin measure.

Definition 1.8. *Let f be a Laurent polynomial. The real Monge-Ampère measure of N_f is called the* Ronkin measure *associated to f and is denoted μ_f.*

Since N_f is affine linear outside the amoeba of f we have that μ_f has support on the amoeba. Moreover, Passare and Rullgård proved that μ_f has finite total mass and that the total mass equals the volume of the Newton polytope of f [18]. They also proved the following theorem about the area of amoebas in two variables.

Theorem 1.4. (Passare, Rullgård) *In the two variable case the area of the amoeba of f is bounded by π^2 times the area of the Newton polytope of f.*

There is no hope of finding a similar theorem in more than two variables because in that situation the volume of the Newton polytope is almost always infinite (see the example on page 11). The inequality in Theorem 1.4 is sharp in the sense that there is a set of Laurent polynomials that have amoebas with maximal area, i.e. with area equal to π^2 times the area of the Newton polytope. This set turns out to be equal to the ones defining so called simple Harnack curves. This especially means that the real part of the zero-set of f have the maximal number of component $(1 + g)$ where g is equal to the number of points in A. More about this can be read in [15].

Theorem 1.5. (Mikhalkin, Rullgård) *The polynomials in two variables associated to amoebas with maximal area are exactly those that define Harnack curves. In other words, the polynomials such that there exist non-zero complex numbers ϵ_j such that $\epsilon_0 f(\epsilon_1 z, \epsilon_2 w)$ has real coefficients and for each $x \in \mathcal{A}_f$ the real torus $\operatorname{Log}^{-1}(x)$ intersects the zero locus of f in at most two points.*

Remark 1.1. Figures 1 and 2 appeared in [19] and were included here by kind permission of the authors of that paper.

2 Hyperplane amoebas

The Newton polytope of a hyperplane amoeba in n variables has $n+1$ integer points and all of them are vertices. This implies that the amoeba has exactly $n+1$ complement components according to property 3 on page 3. By Theorem 1.1 we also know that the amoeba is solid, i.e. has no bounded complement component. The compactified hyperplane amoebas turns out to be particularly easy to express. They are in fact polytopes.

Proposition 2.1. (Forsberg, Passare, Tsikh) *Let f be the affine linear polynomial $a_0 + a_1z_1 + a_2z_2 + \ldots + a_nz_n$ and assume that $|a_j| + |a_k| \neq 0$ for all j and k. Then $\bar{\mathcal{A}}_f$ is the convex hull of the points $v_{jk} = (t_1, \ldots, t_n)$, $j \neq k$, where either*

$$t_j = \frac{|a_0|}{|a_j| + |a_0|}, \quad t_l = 0 \quad \textit{for} \quad l \neq j \quad , \quad \textit{or}$$
$$t_j = \frac{|a_k|}{|a_j| + |a_k|}, \quad t_k = \frac{|a_j|}{|a_k| + |a_j|}, \quad t_l = 0 \quad \textit{for } l \neq j, k.$$

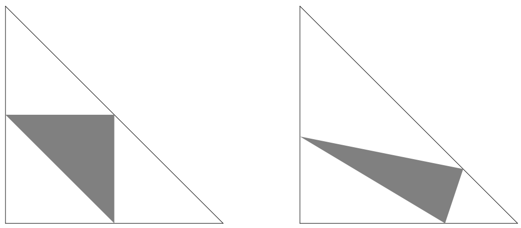

Figure 4: The compactified amoebas of $f(z,w) = 1 + z + w$ and $f(z,w) = 2 + z + 3w$

The fact that hyperplane amoebas are solid makes the spine rather easy to express explicitly.

Proposition 2.2. *Let $f(z) = a_0 + a_1z_1 + \ldots + a_nz_n$. Then $\mathcal{S}(x)$ is the hypersurface of the tropical polynomial*

$$S(x_1, \ldots, x_n) = \max_{j=0,1\ldots,n} (\log |a_j| + x_j).$$

where x_0 is defined to be 0.

An analogous theorem actually holds true for all amoebas that are solid and there are no real differences in the proof. Note that the spine coincide with the tropicalization of f.

Proof. We only need to prove that

$c_{e_j} = \log |a_j|$ where $\{e_1, \ldots, e_n\}$ is the standard basis in $\mathbb{R}^n$ and $e_0 = 0$.

Let x be in the complement component of order e_j, where $j \neq 0$. Then

$$\begin{aligned} c_{e_j} &= N_f(x) - x_j = \left(\frac{1}{2\pi}\right)^n \int_{\mathrm{Log}^{-1}(x)} \log|a_0 + a_1 z_1 + \ldots + a_n z_n| \frac{dz}{z} - x_j = \\ &= \left(\frac{1}{2\pi}\right)^n \int_{\mathrm{Log}^{-1}(x)} \log\left(\frac{|a_0 + a_1 z_1 + \ldots + a_n z_n|}{|z_j|}\right) \frac{dz_1 \ldots dz_n}{z_1 z_2 \ldots z_n}. \end{aligned}$$

Now, since x belongs to the complement component of order e_j we can take the limit when $x_j \to \infty$ and we get that the integral equals $\log|a_j|$. When $j = 0$ we can take the limit when $x_j \to -\infty$ for all $j = 1, \ldots, n$ and we get the result in the proposition. □

In Section 1 we saw that the area of an amoeba in two variables is finite. That is not true in higher dimension as we see in the example below.

Example. Let $f = 1+z+w+t$. According to Proposition 2.2 we have that the spine of $\mathcal{A}_f$ is the corner set of $\max(0, x, y, u)$. Thus the spine contains the ray $(0, 0, t)$ for $t \in [-\infty, 0]$. Actually a whole cylinder containing that ray is contained in the amoeba. This can be seen in the following way. Consider the annulus

$$\mathcal{U} = \{1 + r_1 e^{i\varphi} + r_2 e^{i\theta}; \varphi, \theta \in [0, 2\pi], \frac{2}{3} \leq r_1, r_2 \leq \frac{4}{3}\}.$$

If C is a circle with center at the origin and with radius $r \leq 1$ then it is obvious that $C \cap \mathcal{U} \neq \emptyset$. This means that a point $(x, y, u \in \mathbb{R}^3)$ lies in the amoeba of f if $x, y \in [\log|2/3|, \log|4/3|]$ and $u \in (-\infty, 0]$ thus the amoeba of f contains a set that obviously has infinite volume. □

The affine linear polynomials in two variables are examples of Harnack curves and the Ronkin measures associated to these polynomials consequently have the constant density $1/\pi^2$ on the amoeba by Theorem 1.5. In this case we have such an easy expression of the partial derivatives of N_f that it is easy to verify directly. In the case of three variables this kind of calculation is harder and will be done in Section 4. The following lemma will simplify some of the calculations because it reduces the problem to the case where all the coefficients are equal to 1.

Lemma 2.1. *If*

$$\begin{aligned} f(z) &= 1 + z_1 + \ldots + z_n \quad \textit{and} \\ f_a(z) &= a_0 + a_1 z_1 + \ldots + a_n z_n \end{aligned}$$

Then

$$N_{f_a}(x_1, \ldots x_n) = N_f(x_1 + \log|a_1|, \ldots, x_n + \log|a_n|).$$

Proof. Do the obvious change of variables. □

The contour of the hyperplane amoeba in three variables is easy to describe and it divides the amoeba into eight parts. An easy calculation (or Proposition 1.1) gives that the contour for the amoeba of $f = 1 + z + w + t$ is given by the set of points $(x, y, u) \in \mathbb{R}^3$ that satisfy one of the following equalities:

$$\begin{cases} 1 + e^x = e^y + e^u \\ 1 + e^y = e^x + e^u \\ 1 + e^u = e^x + e^y \\ 1 = e^x + e^y + e^u \\ e^x = 1 + e^y + e^u \\ e^y = 1 + e^x + e^y \\ e^u = 1 + e^x + e^y \end{cases}$$

Corollary 2.1. *Let $f(z, w, t) = 1 + z + w + t$. The compactified amoeba of f is a octahedron and the contour divides it into eight convex chambers. In fact, the part of the contour that is not on the boundary is the union of the three squares naturally defined by the octahedron. See figure 5.*

Proof. The first part is just applying Proposition 2.1. Consider the points on the contour that satisfy $1 + e^x = e^y + e^y$. These points are mapped to the compactified amoeba by the map γ in Definition 1.1 to points

$$\frac{(t, s, 1 + t - s)}{2(1 + t)}.$$

Now, since the sum of the second and third coordinate is equal to $1/2$ we have that the image is equal to the square with vertices in the points $(0, 0, 1/2), (0, 1/2, 0), (1/2, 1/2, 0), (1/2, 0, 1/2)$. The other parts of the contour is done analogously. $\square$

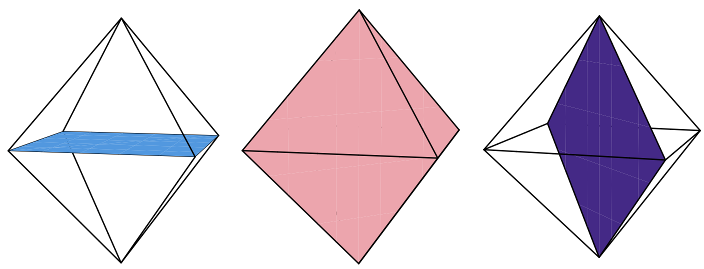

Figure 5: The contour minus the boundary of $\bar{\mathcal{A}}_f$ when
$f = 1 + z + w + t$ is the union of three squares.

Let x be a point in the compactified amoeba that is not on the contour.

Then x satisfies three inequalities, for example

$$\begin{cases} 1+e^x > e^y + e^u \\ 1+e^y > e^x + e^u \\ 1+e^u > e^x + e^y. \end{cases} \tag{2}$$

If the inequality goes in the direction $>$ we associate a $+$ to it and if it goes in the other direction we associate a $-$ to it. In this way we get a triple with minus or plus signs for every point in the amoeba and thus a numbering of the eight chambers. For example, a point x satisfy (2) if and only if x belongs to chamber $(+,+,+)$.

3 Fibers over hyperplane amoebas and linkages

Instead of considering the image of $f^{-1}(0)$ under the map Log it is also natural to look at the image of the argument map Arg defined by

$$\begin{aligned} &\mathrm{Arg} : (\mathbb{C}^*)^n \to \mathbb{T}^n = S^1 \times \ldots \times S^1 \\ &\mathrm{Arg}(z_1, \ldots, z_n) = (\arg(z_1), \ldots, \arg(z_n)). \end{aligned}$$

This image is called the *coamoeba* of f, denoted $\mathcal{A}'_f$ and was first introduced by Passare and Tsikh in 2005. The coamoeba is a set in $\mathbb{T}$ but sometimes it is also viewed as a set in $\mathbb{R}^n$. In the physics literature coamoebas are sometimes called algae. A reference for coamoebas is [17]. In this section we will consider a certain set in the coamoeba of a polynomial f that we call the fiber over a point in the amoeba.

Definition 3.1. *Assume that f is a polynomial in n variables over $\mathbb{C}$ and let $x \in \mathcal{A}_f$. The image in $\mathbb{T}^n$ of $Log^{-1}(x) \cap f^{-1}(0)$ under the map* Arg *is called the fiber over the point x.*

The fibers over the amoeba are closely connected to so called polygon linkages. Consider a closed mechanical linkage, i.e., $n+1$ bars with length l_1 to l_{n+1} attached with revolving joints so that the last bar is connected to the first. Let us ask ourselves what possible shapes the linkage can have. One trivial observation is that it does not matter if we rotate the whole linkage thus we can assume that one of the bars is fixed. Mathematically speaking, given a vector of positive numbers $(l_1, \ldots, l_{n+1}) \in \mathbb{R}^{n+1}$ we are interested in the configuration space M_l defined as

$$\begin{aligned} M_l : \;&=\; \{(p_1, \ldots, p_{n+1}) \in (\mathbb{R}^2)^{n+1}; |p_2 - p_1| = l_1, |p_3 - p_2| = l_2, \\ &\ldots, |p_{n+1} - p_n| = l_n, |p_1 - p_{n+1}| = l_{n+1}, p_1 = (0,0), p_2 = (l_1, 0)\} \cong \\ &\cong\; \{(\varphi_1, \ldots, \varphi_n) \in \mathbb{T}^n; l_0 + \sum_{j=1}^{n} l_j e^{i\varphi_n} = 0\}. \end{aligned}$$

Assume that f is on the form $1 + a_1 z_1 + \ldots + a_n z_n$ and pick a point $x = (x_1, \ldots, x_n)$ on the amoeba of f. Then there exist a point $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{T}^n$ such that $f(e^{x+i\varphi}) = 0$. Thus we have a planar polygon linkage with $n+1$ bars $l_0, \ldots, l_n$ where $|l_0| = 1$ and $|a_k| = e^{x_k}$ for $1 \leq k \leq n$. The fibers over a given point x in the amoeba is exactly M_l, where l_0=1 and $l_j = e^{x_j}$ for $1 \leq j \leq n$.

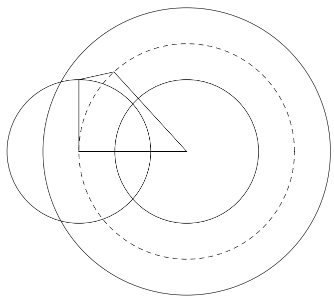

Figure 6: A linkage with four bars with $l_0 = l_1 = l_3 = 1$ and $l_2 = 1/2$ together with the work space of the arm of the bars with length l_1, l_2, l_3.

Several people have worked with spaces like M_l and there is a good understanding of the topology of such spaces. It is our understanding however that nobody has made the connection to amoebas before. Let us therefore say something about the fibers in the general case, when f is an affine linear polynomial in arbitrary dimension, and then take a closer look at the case in three dimensions. In the three dimensional case we will use arguments similar to some that is used in [16].

In order to get a good understanding of the topology of the configuration space of a planar polygon linkage with $n+1$ bars $l_0, \ldots, l_n$ it is common to introduce a bigger space called the *workspace* of the arm $l_0 + \sum_{j=1}^{n} l_n e^{\varphi_j}$. The workspace is denoted by $\mathcal{W}$ and is defined by

$$\begin{aligned} \mathcal{W} \;:=\; & \{(p_1, \ldots, p_n) \in (\mathbb{R}^2)^n; |p_1| = l_1, |p_2 - p_1| = l_2, \ldots, \\ & |p_n - p_{n-1}| = l_{n-1}, |p_n| = x, x \neq 0\} \end{aligned}$$

Consider the function $\operatorname{dist} : \mathcal{W} \to \mathbb{R}$ given by

$$\operatorname{dist}(p_1, \ldots, p_n) = x.$$

Then $M_l = \operatorname{dist}^{-1}(l_n)$. Now, the point is that one can show that the function dist is a so called Morse function, i.e. the Hessian matrix is non singular. This means that one can use Morse theory to describe the topology of the level sets. More about this can be found in the entertaining paper [23].

Proposition 3.1. *The fiber over a generic point in a hyperplane amoeba in $\mathbb{R}^n$ is a closed smooth manifold of dimension $n-2$. In the non-generic case the fiber is a compact manifold of the same dimension but with finitely many singular points.*

Proof. If x is a generic point, i.e., lies outside $\mathcal{C}$, this follows from the fact that Log is a proper submersion in a neighborhood of $\mathrm{Log}^{-1}(x)$ and the implicit function theorem. For the nonsingular case, see [23]. $\square$

It should be noted that if the polynomial f is not affine linear the conclusion in Proposition 3.1 need not be true. One counterexample is given by the the polynomial $f(z,w) = a + z + w + zw$ where $a < 0$. It was shown in [18] that the preimage of a point x in the amoeba of f consist of two points except for the point $(\log|a|, \log|a|)/2$ where the preimage contains a real curve.

Consider the polynomial $f = 1 + z + w + t$. We are interested in the fibers over points $(x, y, u) \in \mathcal{A}_f$. The workspace of the arm $1 + e^{y+i\beta} + e^{u+i\gamma}$ is an annulus with radius e^y and e^u and center in $(1,0)$. In order for a point $(\beta,\gamma) \in \mathbb{T}^2$ to be in M_l it must satisfy $|1 + e^{y+i\beta} + e^{u+i\gamma}| = l_1$. Figures 7-12 show the workspace and the circle with radius e^x and center in the origin in the generic cases togeher with its chamber, and Figures 13-15 shows the singular cases.

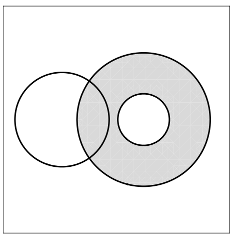

Figure 7:
$(+,+,+)$

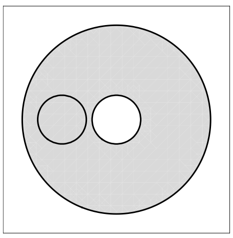

Figure 8:
$(-,+,+)$

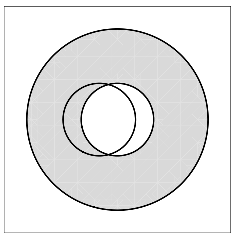

Figure 9:
$(-,-,+), (-,+,-)$

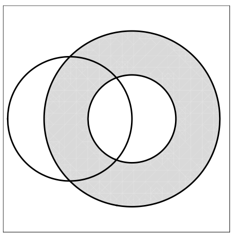

Figure 10:
$(+,-,+), (+,+,-)$

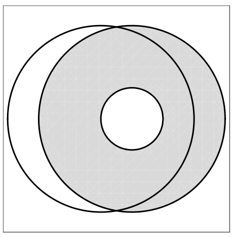

Figure 11:
$(+,-,-)$

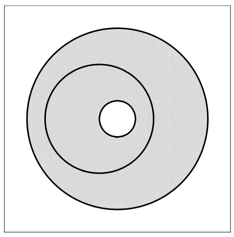

Figure 12:
$(-,-,-)$

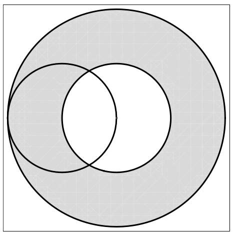

Figure 13:
On exactly one wall

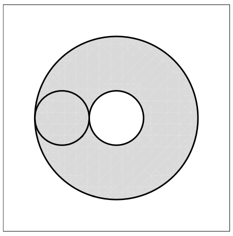

Figure 14:
On exactly two walls

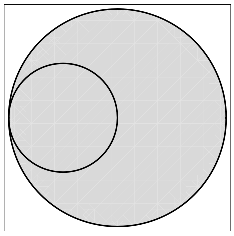

Figure 15:
The origin

Denote by (α, β, γ) the points in in the configuration space of the linkage we get from $f = 1 + z + w + t$. If the intersection $\mathcal{W} \cap \partial D(0, e^x)$ is an arc we have two end points where either $\beta = \gamma$ or $\beta = \pi + \gamma$. In either case the arm $1 + e^{y+i\beta} + e^{u+i\gamma}$ can go from one endpoint to the other in two different ways by bending it in different directions. Since α is uniquely determined by β and γ we have that the possible angles $(\alpha, \beta, \gamma) \in \mathbb{T}$ such that the linkage is closed form a topological circle, i.e., $M_l = S^1$. This means that the fibers in chambers $(+,+,+), (-,-,+), (-,+,-)$ and $(+,-,-)$ topologically are S^1. In chambers $(+,-,+)$ and $(+,+,-)$ we have that $\mathcal{W} \cap \partial D(0, e^x)$ consists of two arcs and the same argument as above gives that the fibers in these chambers topologically are $S^1 \sqcup S^1$. For the chambers $(-,+,+)$ and $(-,-,-)$ we have that $\mathcal{W} \cap \partial D(0, e^x)$ is a full circle. This circle has no point where $\beta = \gamma$ or $\beta = \pi + \gamma$ so every point on the circle gives rise to two points on the fiber that are not connected by any curve in the fiber. This means that the fibers in these chambers topologically are $S^1 \sqcup S^1$.

We have three different singular cases.

1. The point lies on a wall between two different chambers.

2. The point lies on the intersection of two walls.

3. The point lies on the intersection of all three walls, i.e. the origin.

The geometry of the different cases can be seen in Figures 13-15. In the first case $\mathcal{W} \cap \partial D(0, e^x)$ is an arc with one special point ξ such that $\beta = \pi + \gamma$. ξ plays the same role as an end point in the generic cases, and thus the arc from ξ to one of the end points gives rise to a circle and so does the arc from ξ to the other end point. Thus the fiber is two circles with one common point, $(S^1 \sqcup S^1) \sim_1$. In the second case $\mathcal{W} \cap \partial D(0, e^x)$ is a full circle with two special points ξ_1 and ξ_2. There are two arcs from ξ_1 to ξ_2 and each arc gives rise to a circle. In other words, the fiber is $(S^1 \sqcup S^1) \sim_2$. In case 3, the intersection $\mathcal{W} \cap \partial D(0, e^x)$ is a full circle with two special points ξ and $\eta = (1, 0)$. Now,η gives rise to a circle by itself, namely the circle $\{(0, \beta, \pi - \beta) \in \mathbb{T}^3; 0 \leq \beta \leq 2\pi\}$. The two points $(0, 0, \pi)$ and $(0, \pi, 0)$ on that circle can be connected with the point (π, π, π) in two different ways respectively (simply by letting β go from 0 to π or $-\pi$) thus the fiber is $(S^1 \sqcup S^1) \sim_3$. We have now proved the following proposition.

Proposition 3.2. *Outside of $\mathcal{C}$ the fibers over a hyperplane amoeba in 3-space is either a topological circle or two disjoint topological circles. Over points on $\mathcal{C}$ the fiber can be a point (the boundary), two circles with a common point, two circles with two common points or two circles with three common points. Points that lie in the same chamber have fibers with the same topology and different chambers that share a face have fibers with different topology. The different topology of the fiber can be seen in the table below.*

Chamber	Topology of the fibers
$(+,+,+)$	S^1
$(-,+,+)$	$S^1 \sqcup S^1$
$(-,-,+)$	S^1
$(+,-,+)$	$S^1 \sqcup S^1$
$(+,-,-)$	S^1
$(+,+,-)$	$S^1 \sqcup S^1$
$(-,+,-)$	S^1
$(-,-,-)$	$S^1 \sqcup S^1$

In [6] Faber and Schütz proves a useful theorem making it quite easy to decide the topology of the fiber over a given point in the amoeba. In order to state the theorem we need to introduce the notion of short and medium subsets of the set $\{1, 2, \ldots, n\}$.

Definition 3.2. *Given a collection of positive natural numbers $l_1, \ldots l_n$, a subset $J \subset \{1, \ldots, n\}$ is called* short *if*

$$\sum_{i \in J} l_i < \sum_{i \notin J} l_i$$

and median *if the above inequality is an equality.*

Theorem 3.1. *Fix a link with maximal length l_i. Define the numbers a_k and b_k as the number of short and median subsets of $\{1, 2, \ldots, n\}$ with $k+1$ elements containing i. Then the Poincaré polynomial*

$$p(t) = \sum_{k=0}^{n-3} \dim H_k(M_l; \mathbb{Q}) t^k$$

of M_l is equal to the following polynomial

$$q(t) + t^{n-3} q(t^{-1}) + r(t)$$

where

$$q(t) = \sum_{k=0}^{n-3} a_k t^k, \qquad r(t) = \sum_{k=0}^{n-3} b_k t^k.$$

Example. Let $l_1 = 1, l_2 = 1, l_3 = 3, l_4 = 4$. The moduli space M_l will correspond to the fiber over the point $x = (0, \log|3|, \log|4|)$ in the amoeba of $f(z, w, t) = 1 + z + w + t$. The point x belongs to chamber $(-, -, +)$, and thus it has the topology of a circle. Let us prove this fact using Theorem 3.1. The number of short subsets of $\{1, 2, 3, 4\}$ containing 4 with one element is 1 and the number of short subsets containing 4 with two elements is equal to 0 in this case. The number of median sets containing 4 with one and two elements is equal to zero. Thus $a_0 = 1$ and $a_1 = b_0 = b_1 = 0$. According to Theorem 3.1 the Poincaré polynomial is

$$p(t) = 1 + t$$

and thus the fiber is a connected curve with genus 1.
Now let $l_1 = 1, l_2 = 1, l_3 = 1, l_4 = 1$. Then M_l corresponds to the fiber over the origin of the same polynomial as above. This time $a_0 = 1, a_1 = 0, b_0 = 0$ and $b_1 = \binom{3}{2} = 3$. We get the Poincaré polynomial

$$P(t) = 1 + t + 3t = 1 + 4t$$

which is what we expect since M_l topologically is $(S^1 \sqcup S^1) \sim_3$. $\square$

We have seen that the origin is a singular point in the case of a hyperplane amoeba in three variables. In four variables however this is no longer true. The fiber over the origin in that case is a smooth surface of genus 4.

In this thesis we have only considered the case where the linkage is a closed polygon since it corresponds to hyperplane amoebas. Other more complicated linkages have been studied by many people, for example in paper [4], Cruickshank and McLaughlin studied so called series parallel linkages which in the world of the amoebas correspond to fibers over amoebas of certain linear hypersurfaces of higher codimension.

4 The Ronkin measure in the case of a hyperplane in three variables

As we saw in Section 1 the Ronkin measure for polynomials in two variables is rather well understood. In particular we saw that the measure of an affine linear polynomial in two variables is identically equal to $1/\pi^2$ times the Lebesgue measure on the amoeba. Not much is known in the case of three variable polynomials. A first step is to look at the case where f is a linear polynomial, i.e., $f = a + bz + cw + dt$ where a, b, c and d are complex numbers. Now, because of Lemma 2.1 we only need to consider the case where a, b, c and d all equal 1. To this end let $N(x, y, u)$ be the Ronkin function of the hyperplane $1 + z + w + t$ in 3-space.

4.1 The derivatives

We have that

$$\begin{aligned}
\frac{\partial N(x,y,u)}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2\pi i}\right)^3 \int_{\mathrm{Log}^{-1}(x,y,u)} \log|1+z+w+t| \frac{dzdwdt}{wt} = \\
&= \left(\frac{1}{2\pi i}\right)^3 \int_{\mathrm{Log}^{-1}(x,y,u)} \frac{1}{1+z+w+t}\frac{dzdwdt}{wt} = \\
&= \left(\frac{1}{2\pi i}\right)^2 \left(\left(\frac{1}{2\pi i}\right) \int_{\mathrm{Log}^{-1}(x,y,u)} \frac{dz}{z-(-1-w-t)}\right) \frac{dzdwdt}{wt}.
\end{aligned}$$

Remember the function dist from Section 3. In this case we have that $\mathrm{dist}(\varphi,\theta) = |1+e^{y+i\varphi}+e^{u+i\theta}|$, thus $(\partial/\partial x)N$ is equal to the area of the set T divided by $(2\pi)^2$ where

$$T = \{(\varphi,\theta) \in \mathbb{T}^2; \mathrm{dist}(\varphi,\theta) < e^x\}.$$

Note that T is equal to the area enclosed by the curve that is the projection of the fiber over the point (x,y,u) onto the $\varphi\theta$ plane.

Proposition 4.1. *Outside the contour we have*

$$\begin{aligned}
&\pi^2 \frac{\partial N(x,y,u)}{\partial x} = \\
&= -\int_{r_0}^{r_1} \arccos\left(\frac{1+r^2-e^{2x}}{2r}\right) \frac{d}{dr} \arccos\left(\frac{r^2-e^{2y}-e^{2u}}{2e^{y+u}}\right) dr \qquad (3)
\end{aligned}$$

where r_0 and r_1 depends on which chamber the point (x,y,u) belongs to according to the following table:

Chamber	r_0	r_1
$(+,+,+)$	$1-e^x$	e^y+e^u
$(-,+,+)$	$1-e^x$	$1+e^x$
$(-,-,+)$	e^u-e^y	$1+e^x$
$(+,-,+)$	e^u-e^y	e^y+e^u
$(+,-,-)$	e^x-1	e^y+e^u
$(+,+,-)$	e^y-e^u	e^y+e^u
$(-,+,-)$	e^y-e^u	$1+e^x$
$(-,-,-)$	e^x-1	$1+e^x$

The chambers are defined at the end of Section 2.

Proof. We need to calculate the area of T and divide by the area of $\mathbb{T}^2$. Let L_γ be the line in the torus defined by $\{\gamma = \varphi-\theta; -\pi < \varphi,\theta < \pi\}$. Consider the function $\mathrm{Arm}_{\varphi,\theta} : \mathcal{A}_f \to \mathbb{C}$ given by

$$\mathrm{Arm}_{\varphi,\theta}(x,y,u) = 1+e^{y+i\varphi}+e^{u+i\theta}.$$

A straight forward calculation gives that the Jacobian of that function is constant along L_γ. This means that we have

$$\begin{aligned}\frac{\text{Length}(L_\gamma \cap A)}{\text{Length}(L_\gamma)} &= \frac{\text{Length}(\text{Arm}_{\varphi,\theta}(L_\gamma) \cap T)}{\text{Length}(\text{Arm}_{\varphi,\theta}(L_\gamma))} = \\ &= \frac{\text{Length}(\partial D(1,r) \cap D(0,e^x))}{\text{Length}(\partial D(1,r))} = \frac{\alpha}{\pi}\end{aligned}$$

where α is the angle that $w+t$ must have to precisely hit $D(0,e^x)$ and where $r = |w+t|$. By integrating this over γ when $0 \le \gamma \le \pi$ we get

$$\frac{\partial N}{\partial x} = \frac{1}{2\pi^2}\int_0^{2\pi} \alpha(\gamma)d\gamma = \frac{1}{\pi^2}\int_0^{\pi} \alpha(\gamma)d\gamma$$

for symmetry reasons. Now, rewrite α and γ in terms of r just by solving the triangles in Figure 16.

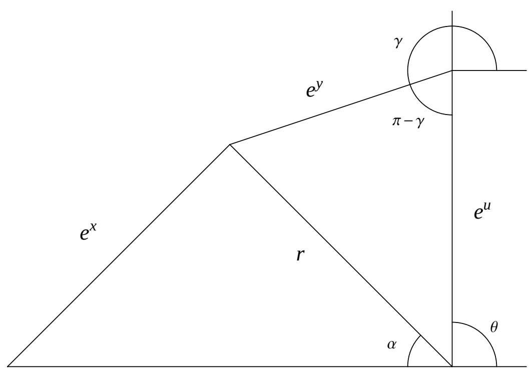

Figure 16:

This gives

$$\begin{aligned}\alpha &= \arccos\left(\frac{1+r^2-e^{2x}}{2r}\right) \qquad \text{and} \\ \gamma &= \arccos\left(\frac{r^2-e^{2y}-e^{2u}}{2e^{y+u}}\right).\end{aligned}$$

The only thing left to do is to figure out what the integration limits should be. This is not hard and is easy to see for example in Figures 7 - 12. Note that the minus sign comes from the fact that the integration limits should change places to get the ones in the theorem. ☐

Remark 4.1. It should be remarked that the integral (3), and hence the derivatives, is continuous inside the amoeba, even at the contour. This can be seen from the fact that the integral in (3) can be written on the form

$$\int_{r_0^2}^{r_1^2} \frac{\alpha(x,y,u,s)}{\sqrt{-(s-(e^y+e^u)^2)(s-(e^y-e^u)^2)}} ds$$

and that this integral is bounded in the closure of each chamber.

Let

$$\phi(r,x,y,u) := \arccos\left(\frac{1+r^2-e^{2x}}{2r}\right) \quad \text{and} \tag{4}$$

$$\psi(r,x,y,u) := \arccos\left(\frac{r^2-e^{2y}-e^{2u}}{2e^{y+u}}\right). \tag{5}$$

Then even though x and y appear in the integration limits r_0 and r_1 we have the following lemma.

Lemma 4.1. *Outside of the contour and for r_0 and r_1 as above we have that*

$$\frac{\partial}{\partial x}\int_{r_0}^{r_1} \phi \frac{d}{dr}\psi dr = \int_{r_0}^{r_1} \frac{\partial}{\partial x}\phi \frac{d}{dr}\psi dr \quad \textit{and}$$

$$\frac{\partial}{\partial y}\int_{r_0}^{r_1} \phi \frac{d}{dr}\psi dr = -\int_{r_0}^{r_1} \frac{\partial}{\partial y}\psi \frac{d}{dr}\phi dr.$$

where ϕ and ψ are defined by (4) and (5).

Proof. We prove the second part of the lemma. The first part is proved along the same lines. If r_1 and r_0 do not depend on y we have nothing to show so we can assume that both depend on y, i.e.

$$r_1 = e^x + e^y, \qquad r_0 = \pm e^y - e^u$$

We first note that

$$e^y \frac{\partial}{\partial y}\phi(r_1) = \left(\frac{d}{dr}\phi\right)(r_1) \quad \text{and} \tag{6}$$

$$-e^y \frac{\partial}{\partial y}\phi(r_0) = \pm\left(\frac{d}{dr}\phi\right)(r_0). \tag{7}$$

We want to prove that

$$\frac{\partial}{\partial y}\int_{r_0}^{r_1} \phi \frac{d}{dr}\psi dr + \int_{r_0}^{r_1} \frac{\partial}{\partial y}\psi \frac{d}{dr}\phi dr = 0$$

By using integration by parts and the definition of derivatives we get

$$\frac{\partial}{\partial y}\left([\phi\psi]_{r_0}^{r_1} - \int_{r_0}^{r_1} \psi \frac{d}{dr}\phi dr\right) + \int_{r_0}^{r_1} \frac{\partial}{\partial y}\psi \frac{d}{dr}\phi dr =$$

$$\frac{\partial}{\partial y}[\psi\phi]_{r_0}^{r_1} - \lim_{h\to 0}\left(\frac{1}{h}\int_{r_0(y+h)}^{r_1(y+h)} \psi(y+h)\frac{d}{dr}\phi dr - \frac{1}{h}\int_{r_0(y)}^{r_1(y)} \psi(y)\frac{d}{dr}\phi dr\right) +$$

$$\int_{r_0(y)}^{r_1(y)} \lim_{h\to 0}\frac{\psi(y+h)-\psi(y)}{h}\frac{d}{dr}\phi dr.$$

By linearity this is equal to

$$\frac{\partial}{\partial y}\left[\phi\psi\right]_{r_0}^{r_1} + \lim_{h\to 0}\frac{1}{h}\int_{r_0(y)}^{r_0(y+h)} \psi(y+h)\frac{d}{dr}\phi dr -$$
$$\lim_{h\to 0}\frac{1}{h}\int_{r_1(y)}^{r_1(y+h)} \psi(y+h)\frac{d}{dr}\phi dr.$$

and since ψ is bounded we get that this is equal to

$$\frac{\partial}{\partial y}\left[\phi\psi\right]_{r_0}^{r_1} + \lim_{h\to 0}\frac{1}{h}\left(r_0(y+h)-r_0(y)\right)\psi\left(\frac{d}{dr}\phi\right)\Big|_{r_0} -$$
$$\lim_{h\to 0}\frac{1}{h}\left(r_1(y+h)-r_1(y)\right)\psi\left(\frac{d}{dr}\phi\right)\Big|_{r_1}.$$

Now, we have that

$$\begin{aligned}&\tfrac{1}{h}(r_1(y+h)-r_1(y)) = \tfrac{1}{h}(e^y(e^h-1)) \to e^y \quad \text{when} \quad h\to 0 \quad \text{and}\\ &\tfrac{1}{h}(r_0(y+h)-r_0(y)) = \pm\tfrac{1}{h}(e^y(e^h-1)) \to \pm e^y \quad \text{when} \quad h\to 0\end{aligned}$$

thus we only need to show that

$$\frac{\partial}{\partial y}\left[\phi\psi\right]_{r_0}^{r_1} \pm e^y\psi\left(\frac{d}{dr}\phi\right)\Big|_{r_0} - e^y\psi\left(\frac{d}{dr}\phi\right)\Big|_{r_1} = 0,$$

but this is true because of (6) and (7) and we are done. □

Lemma 4.1 will be useful to calculate the second order derivatives of N.

Lemma 4.2. *For $(x, y, u) \in \mathcal{A}_f \setminus \mathcal{C}$ and with r_0 and r_1 as above we have*

$$\frac{\partial^2 N}{\partial x^2} = \frac{2e^x}{\pi^2}\int_{r_0^2}^{r_1^2} \frac{1}{\sqrt{(s-A)(s-B)(s-C)(s-D))}}ds \quad \textit{and} \tag{8}$$

$$\frac{\partial^2 N}{\partial x\partial y} = \frac{1}{2\pi^2}\int_{r_0^2}^{r_1^2} \frac{s^2+P_1 s+P_2}{s\sqrt{(s-A)(s-B)(s-C)(s-D))}}ds \tag{9}$$

where

$$\begin{aligned}A &= (1+e^x)^2, \quad B = (e^y+e^u)^2,\\ C &= (1-e^x)^2, \quad D = (e^y-e^u)^2\end{aligned}$$

and

$$P_1 = (e^{2x}+e^{2y}-1-e^{2u}), \qquad P_2 = (1+e^x)(1-e^x)(e^y+e^u)(e^u-e^y).$$

Proof. We start with the first equality. By Proposition 4.1 and Lemma 4.1 we have

$$\frac{\partial^2 N}{\partial x^2} = -\frac{1}{\pi^2}\int_{r_0}^{r_1} \frac{\partial}{\partial x}\arccos\left(\frac{1+r^2-e^{2x}}{2r}\right)\frac{d}{dr}\arccos\left(\frac{r^2-e^{2y}-e^{2u}}{2e^{y+u}}\right)dr.$$

An easy calculation shows that we have

$$\begin{aligned}\frac{\partial}{\partial x}\arccos\left(\frac{1+r^2-e^{2x}}{2r}\right) &= -\frac{2e^x}{\sqrt{4r^2-(1+r^2-e^{2x})^2}},\\ \frac{d}{dr}\arccos\left(\frac{r^2-e^{2y}-e^{2u}}{2e^{y+u}}\right) &= \frac{2r}{\sqrt{4e^{2(y+u)}-(r^2-e^{2y}-e^{2u})^2}}.\end{aligned}$$

Now, do the change of variables $s = r^2$ and make use of the formula

$$\begin{aligned}4a^2b^2-(c^2-a^2-b^2)^2 &= -(a^2-(b+c)^2)(a^2-(b-c)^2)\\ &= -(b^2-(a+c)^2)(b^2-(a-c)^2)\\ &= -(c^2-(a+b)^2)(c^2-(a-b)^2)\end{aligned}$$

that is valid for all a and b. The first equation in the lemma is thereby proved. The second equation is proved in a similar way. $\square$

Note that r_1^2 is either A or B and r_0^2 is either C or D. We see that integrals in (8) and (9) depend on x, y and u in a smooth manner except at the singular points where $A = B, C = D, B = C$ and possibly when $r_0 = 0$, i.e when $1 = e^x$ or when $e^y = e^u$. The good thing is that $P_2 = 0$ at the points where $1 = e^x$ or when $e^y = e^u$ thus there might be that the integral converges anyway. That is actually the case. To see this it is enough to realize that

$$\lim_{\epsilon\to 0}\int_{\epsilon}^{M}\frac{\epsilon}{s\sqrt{s-\epsilon}}ds = 0.$$

for some constant $M \neq 0$. But that is true because

$$\lim_{\epsilon\to 0}\int_{\epsilon}^{M}\frac{\epsilon}{s\sqrt{s-\epsilon}}ds = \lim_{\epsilon\to 0}\sqrt{\epsilon}\int_{1}^{M/\epsilon}\frac{1}{s\sqrt{s-1}}ds.$$

Now, a similar argument give that $\partial^2 N/\partial x\partial y$ not only is continuous but also smooth at the points where $1 = e^x$ and $e^y = e^u$. Note that the equations $B = C$ is true exactly on the boundary of the amoeba and that the equations $A = B$ and $C = D$ are true exactly on the other part of the contour. We therefore have the following proposition.

Proposition 4.2. *Let $f = 1 + z + w + t$. Then μ_f is smooth outside the contour of the amoeba of f.*

4.2 Connections to elliptic integrals

Elliptic integrals naturally comes up in many situations. For example when calculating the arc length of an ellipse (hence the name). Lemma 4.2 says that the second order derivatives of the three dimensional Ronkin function in the affine linear case are complete elliptic integrals.

Definition 4.1. *An* elliptic integral *is an integral on the form* $\int R(s, \sqrt{P(s)})$ *where* P *is a polynomial of degree* 3 *or* 4 *with no multiple roots and* R *is a rational function of* s *and* $\sqrt{P}$*. It is always possible to express the elliptic integrals as linear combinations in terms of elementary functions and the following three integrals.*

$$\begin{aligned}
\mathrm{K}(\varphi,k) &:= \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^t \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}},\\
\mathrm{E}(\varphi,k) &:= \int_0^{\varphi} \sqrt{1-k^2\sin^2\theta}\,d\theta = \int_0^t \sqrt{\frac{1-k^2s^2}{1-s^2}}ds,\\
\Pi(\varphi,\alpha^2,k) &:= \int_0^{\varphi} \frac{d\theta}{(1-\alpha^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} =\\
&= \int_0^t \frac{ds}{(1-\alpha^2s^2)\sqrt{(1-s^2)(1-k^2s^2)}}.
\end{aligned}$$

The integrals above are said to be on normal form or on Legendre form. If $\varphi = \frac{\pi}{2}$ *we say that the integrals are* complete *and we denote the three complete integrals on normal form by* $\mathrm{K}(k)$*,* $\mathrm{E}(k)$ *and* $\Pi(\alpha^2,k)$ *respectively.*

Lemma 4.3. *Assume* $a > b > c > d$*. Then* $\int_c^b \frac{s^j ds}{\sqrt{(s-a)(s-b)(s-c)(s-d)}}$*,* $j = -1, 0, 1$ *transforms into the following complete elliptic integrals on normal form:*

$$\begin{array}{ll}
g\,\mathrm{K}(k) & \textit{if } j=0\\
dg\,\mathrm{K}(k) + g(c-d)\Pi(\alpha^2,k) & \textit{if } j=1\\
\frac{g}{d}\mathrm{K}(k) + g(\frac{1}{c}-\frac{1}{d})\Pi(\alpha^2\frac{d}{c},k) & \textit{if } j=-1
\end{array}$$

where

$$k^2 = \frac{(b-c)(a-d)}{(a-c)(b-d)}, \qquad \alpha^2 = \frac{b-c}{b-d}, \qquad g = \frac{2}{\sqrt{(a-c)(b-d)}}.$$

These results are well-known, see for example [3], but we will give a proof of the case when $j = 0$ to show the general idea.

Proof. Let k be as in the lemma and define the functions $h(t)$ and $u(t)$ by

$$h(t) = \sqrt{\frac{(b-d)(t-c)}{(b-c)(t-d)}} \quad \text{and}$$

$$u(t) = \int_0^{h(t)} \frac{d\tau}{\sqrt{1-\tau^2}\sqrt{1-k^2\tau^2}}.$$

Note that $h(b) = 1$ and $h(c) = 0$. A straight forward calculation shows that

$$\frac{du}{dt} = \frac{\sqrt{(a-c)(b-d)}}{2\sqrt{(t-a)(t-b)(t-c)(t-d)}}.$$

Thus

$$\int_0^1 \frac{d\tau}{\sqrt{1-\tau^2}\sqrt{1-k^2\tau^2}} = \int_c^b \frac{du}{dt}dt = g^{-1}\int_c^b \frac{dt}{\sqrt{(t-a)(t-b)(t-c)(t-d)}}.$$

□

Lemmas 4.2 and 4.3 make it possible to express the second order derivatives of N in terms of complete elliptic integrals of the first and third kind. The only thing one has to do is to determine how A, B, C, D in Lemma 4.2 are ordered. In chamber $(+,+,+)$ we have $A > B > C > D$ for example. Doing this gives us the following expressions of the second order derivatives in the different chambers.

Proposition 4.3. *Let $f = 1+z+e+t$. The second order derivatives of the Ronkin function N can be expressed in terms of complete elliptic integrals of the first and third kind in the following way.*

$$\begin{aligned} \frac{\partial^2 N_f}{\partial x^2} &= \frac{2ge^x}{\pi^2}\,\mathrm{K}(k), \\ \frac{\partial^2 N_f}{\partial x \partial y} &= \frac{g}{2\pi^2}\left(Q_1\,\mathrm{K}(k) + Q_2\Pi(\alpha_1, k) + Q_3\Pi(\alpha_2, k)\right) \end{aligned}$$

where $k^2, \alpha_1^2\alpha_2^2, g^2, Q_1, Q_2$ and Q_3 are rational functions in e^x, e^y and e^u and depend on what chamber (x, y, u) lies in. With the quantity ξ defined as

$$(1+e^x+e^y-e^u)(1+e^x-e^y+e^u)(1-e^x+e^y+e^u)(-1+e^x+e^y+e^u)$$

these functions will take the form according to the following.

In the chambers $(+,+,+)$ *and* $(+,-,-)$*:*

$$
\begin{aligned}
g &= \frac{1}{2\sqrt{e^{x+y+u}}} \\
k^2 &= \frac{\xi}{16e^{x+y+u}}, \\
Q_1 &= 2\frac{e^y(e^{2x}+e^{2y}+e^{2u}-1-2e^{y+u})}{(e^y-e^u)}, \\
Q_2 &= (1-e^x+e^y-e^u)(1-e^x-e^y+e^u), \\
Q_3 &= \frac{(e^u+e^y)(1-e^x+e^y-e^u)(1+e^x)(1-e^x-e^y+e^u)}{(e^u-e^y)(e^x-1)}, \\
\alpha_1^2 &= \frac{(1-e^x+e^y+e^u)(-1+e^x+e^y+e^u)}{4e^{y+u}}, \\
\alpha_2^2 &= \alpha_1\frac{(e^y-e^u)^2}{(1-e^x)^2}.
\end{aligned}
$$

In the chambers $(-,+,+)$ *and* $(-,-,-)$*:*

$$
\begin{aligned}
g &= \frac{2}{\sqrt{\xi}} \\
k^2 &= \frac{16e^{x+y+u}}{\xi}, \\
Q_1 &= 2\frac{e^y(e^{2x}+e^{2y}+e^{2u}-1-2e^{y+u})}{(e^y-e^u)}, \\
Q_2 &= (1-e^x+e^y-e^u)(1-e^x-e^y+e^u), \\
Q_3 &= \frac{(e^u+e^y)(1-e^x+e^y-e^u)(1+e^x)(1-e^x-e^y+e^u)}{(e^u-e^y)(e^x-1)}, \\
\alpha_1^2 &= \frac{4e^x}{(1+e^x+e^y-e^u)(1+e^x-e^y+e^u)}, \\
\alpha_2^2 &= \alpha_1\frac{(e^y-e^u)^2}{(1-e^x)^2}.
\end{aligned}
$$

In the chambers $(-,-,+)$ *and* $(-,+,-)$*:*

$$
\begin{aligned}
g &= \frac{1}{2\sqrt{e^{x+y+u}}} \\
k^2 &= \frac{\xi}{16e^{x+y+u}}, \\
Q_1 &= 2\frac{e^x(e^{2x}+e^{2y}-e^{2u}+1-2e^x)}{(e^x-1)}, \\
Q_2 &= -(1-e^x+e^y-e^u)(1-e^x-e^y+e^u), \\
Q_3 &= -\frac{(e^u+e^y)(1-e^x+e^y-e^u)(1+e^x)(1-e^x-e^y+e^u)}{(e^u-e^y)(e^x-1)}, \\
\alpha_1^2 &= \frac{(1+e^x-e^y+e^u)(1+e^x+e^y-e^u)}{4e^x}, \\
\alpha_2^2 &= \alpha_1\frac{(e^x-1)^2}{(e^y-e^u)^2}.
\end{aligned}
$$

In the chambers $(+,-,+)$ *and* $(+,+,-)$*:*

$$
\begin{aligned}
g &= \frac{2}{\sqrt{\xi}} \\
k^2 &= \frac{16e^{x+y+u}}{\xi}, \\
Q_1 &= -2\frac{e^x(e^{2x}+e^{2y}-e^{2u}+1-2e^x)}{(1-e^x)}, \\
Q_2 &= -(1-e^x+e^y-e^u)(1-e^x-e^y+e^u), \\
Q_3 &= -\frac{(e^u+e^y)(1-e^x+e^y-e^u)(1+e^x)(1-e^x-e^y+e^u)}{(e^u-e^y)(e^x-1)}, \\
\alpha_1^2 &= \frac{4e^{y+u}}{(1-e^x+e^y+e^u)(-1+e^x+e^y+e^u)}, \\
\alpha_2^2 &= \alpha_1\frac{(1-e^x)^2}{(e^y-e^u)^2}.
\end{aligned}
$$

Even though it appears that the mixed second order derivative of N is singular at the points $(x,y,u) \in \mathbb{R}^3$ where $e^x = 1$ or $e^y = e^u$ we saw that P_2 in Lemma 4.2 vanishes at those points. This means that $Q_3 = 0$ and that Q_1 take the form of $g(1+e^x)(1-e^x)$, and thus is not singular.

A priori we know that the Hessian matrix will be symmetric in every chamber. This gives us several relations between elliptic integrals of the first and third kind that as far as we know can not be explained by the known relations that are to be found in the literature. There might thus be some interesting hidden relations in the following equation that we get by considering the case of chamber $(+,+,+)$.

For $a,b,c>0$ that satisfy the inequalities $1+a>b+c, \quad 1+b>a+c, \quad 1+c>a+b$ we have that

$$\begin{array}{rcl}
2\dfrac{(1+a+b-c)(1-a-b+c)(a-b)c}{(a-c)(c-b)} & \mathrm{K}(k) & + \\
+(1-a-b+c)(1-a+b-c) & \Pi\left(\alpha_1^2,k\right) & + \\
+(1-a-b+c)(1+a-b-c) & \Pi\left(\alpha_3^2,k\right) & + \\
+\dfrac{(1+a)(b+c)(1-a-b+c)(1-a+b-c)}{(1-a)(b-c)} & \Pi\left(\alpha_2^2,k\right) & + \\
+\dfrac{(1+b)(a+c)(1-a-b+c)(1+a-b-c)}{(1-b)(a-c)} & \Pi\left(\alpha_4^2,k\right) & \equiv 0
\end{array}$$

with

$$\begin{aligned}
k^2 &= \frac{(1+a+b-c)(1+a-b+c)(1-a+b+c)(-1+a+b+c)}{16abc}, \\
\alpha_1^2 &= \frac{(1-a+b+c)(-1+a+b+c)}{4bc}, \\
\alpha_2^2 &= \frac{(1+a-b+c)(-1+a+b+c)}{4ac}, \\
\alpha_3^2 &= \frac{(1-a+b-c)(-1+a+b+c)(b-c)^2}{4bc(1-a)^2}, \\
\alpha_4^2 &= \frac{(1+a-b+c)(-1+a+b+c)(a-c)^2}{4ac(1-b)^2}.
\end{aligned}$$

4.3 Connections to hypergeometric functions

The elliptic integrals are connected to so called hypergeometric functions. In fact the former are special cases of the latter. Hypegeometric functions are very important in the field of special functions and mathematical physics.

The Gauss hypergeometric function ${}_2\mathrm{F}_1$ is defined by the series

$$ {}_2\mathrm{F}_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \tag{10}$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined as $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$ and Γ is the gamma function. The parameter c is assumed not to be a non-positive integer. The radius of convergence for ${}_2\mathrm{F}_1$ is 1 if not a or b is a non-positive integer. In that case the radius of convergence is infinite (the series is finite). The Gauss hypergeometric function is a solution to the following linear differential equation

$$z(z-1)\frac{d^2y}{dz^2} + ((a+b+1)z-c)\frac{dy}{dz} + aby = 0. \tag{11}$$

The equation (11) has regular singularities at the points $0, 1$ and ∞. In fact, every second order linear differential equation with three regular singularities can be reduced to (11) by a change of variables.

The series (10) can be written as the integral (see for example [5])

$$ {}_2\mathrm{F}_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt \quad (c > b > 0). $$

Now, if we choose $a = b = 1/2$ and $c = 1$ the equation above becomes

$$ \begin{aligned} {}_2\mathrm{F}_1(1/2, 1/2; 1; z) &= \frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{t}\sqrt{1-t}\sqrt{1-tz}} dt = \\ &= \frac{2}{\pi} \int_0^1 \frac{1}{\sqrt{1-s^2}\sqrt{1-s^2 z}} ds. \end{aligned} $$

Thus in view of Definition 4.1 we got the following relation between the elliptic integrals of the first kind and Gauss hypergeometric function.

$$ \mathrm{K}(k) = \frac{\pi}{2} {}_2\mathrm{F}_1(1/2, 1/2; 1; k^2). \tag{12} $$

There are many ways to generalize the function ${}_2\mathrm{F}_1$. One way is to simply allow more Pochhammer symbols in the series (10). We then get the functions ${}_p\mathrm{F}_q$ defined by taking p Pochhammer symbols in the numerator and q of them in the denominator. Another way is to try to generalize the hypergeometric functions to several variables. Appell was one of the first who did this. He defined four series in two variables that are known as the Appell's double hypergeometric functions.

Definition 4.2. *The four functions* $\mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3$ *and* F_4 *is defined by*

$$ \begin{aligned} \mathrm{F}_1(a, b, b'; c; z; w) &= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{z^m w^n}{m!n!}, \\ \mathrm{F}_2(a, b, b'; c, c'; z; w) &= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n} \frac{z^m w^n}{m!n!}, \\ \mathrm{F}_3(a, a', b, b'; c, ; z; w) &= \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}} \frac{z^m w^n}{m!n!}, \\ \mathrm{F}_4(a, b, ; c, c'; z; w) &= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n} \frac{z^m w^n}{m!n!}. \end{aligned} $$

The series F_1 and F_3 converges for $|z| < 1$ and $|w| < 1$, F_2 converges for $|z| + |w| < 1$ and the series F_4 converges for $\sqrt{z} + \sqrt{w} < 1$. Appell's functions have integral formulas as Gauss' function has. In addition to some two variable integrals Picard gave the following representation of the Appell double hypergeometric function F_1, see [5].

$$\begin{aligned} & \mathrm{F}_1(a,b,b';c;z,w) = \\ = \quad & \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tz)^{-b}(1-tw)^{-b'}dt, \end{aligned}$$

for $\mathrm{Re}(a), \mathrm{Re}(c-a) > 0$. If we choose the parameters $a = b' = 1/2$ and $b = c = 1$ the above integral representation gives

$$\begin{aligned} \mathrm{F}_1(1/2,1,1/2;1;z,w) &= \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{t}\sqrt{(1-t)(1-tz)}\sqrt{(1-tw)}}dt = \\ &= \frac{2}{\pi}\int_0^1 \frac{1}{\sqrt{(1-s^2)(1-s^2z)}\sqrt{(1-s^2w)}}ds. \end{aligned}$$

Thus we get the following relation between the elliptic integrals of the third kind and Appell's double hypergeometric series F_1.

$$\Pi(\alpha^2,k) = \frac{\pi}{2}\mathrm{F}_1(1/2;1,1/2;1;\alpha^2,k^2). \tag{13}$$

Gelfand, Kapranov and Zelevinsky revolutionized the theory of hypergeometric functions by considering a system of differential equations in several variables [9]. The solutions to that specific system, called GKZ-system, have certain homogeneities and they are defined to be A-hypergeometric or GKZ-hypergeometric functions. By dehomogenizing these functions one can get ${}_2\mathrm{F}_1$ and Appell's functions and many other generalizations of Gauss' function.

Following [17], given a $(n \times N)$-matrix A on the form

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha^1 & \alpha^2 & \dots & \alpha^N \end{pmatrix}$$

such that the maximal minors are relatively prime we consider the $(N \times N - n)$-matrix B such that $AB = 0$. Moreover, B should be such that the rows in B span $\mathbb{Z}^{N-n}$ and such that it is on the form $(B', E_m)^{tr}$ where E_m is the unit $(N-n \times N-n)$-matrix. Let $\mathbb{C}^A$ be the vector space consisting of vectors $(a_\alpha)_{\alpha\in A}$ and write $a = (a_1, \dots, a_N)$. Let $b^1, \dots, b^{N-n}$ be the columns in B. The differential operators $\Box_i$ and $\mathcal{E}_i$ on $\mathbb{C}^A$ are defined by

$$\Box_i = \prod_{j:b^i_j>0} (\partial/\partial a_j)^{b^i_j} - \prod_{j:b^i_j<0} (\partial/\partial a_j)^{-b^i_j} \tag{14}$$

and

$$\mathcal{E}_i = \sum_{j=1}^N \alpha^j_i a_j(\partial/\partial a_j), \quad i = 1, \dots, n. \tag{15}$$

where α_j^i is the entry in A on row i and column j.

Definition 4.3. *For every complex vector $\gamma = (\gamma_1, \dots, \gamma_n)$, we define the GKZ-system with parameters γ as the following system of linear differential equations on functions Φ on $\mathbb{C}^A$.*

$$\Box_i \Phi(a) = 0, \qquad \mathcal{E}_j \Phi = \gamma_j \Phi, \quad i = 1, \dots, N-n, \quad j = 1, \dots, n. \tag{16}$$

The holomorphic solutions to the system (16) are called A-hypergeometric functions. A formal explicit solution to the system (16) is given by

$$\Phi(a) = \sum_{k \in \mathbb{Z}^{N-n}} \frac{a^{\gamma + \langle B, k \rangle}}{\prod_{j=1}^{n} \Gamma\left(\gamma_j + \langle B_j, k \rangle + 1\right) k!} \tag{17}$$

where B_j denotes the rows in the matrix B and $\gamma_{n+1}, \dots, \gamma_N = 0$.
Remember the formula

$$\Gamma(s)\Gamma(1-s) = \pi / \sin(\pi s). \tag{18}$$

In the generic case (noninteger parameters) the formula (18) directly gives us the following formula making it possible to move the gamma functions in (17) from the denominator to the numerator.

$$\frac{\Gamma(s+n)}{\Gamma(s)} = (-1)^n \frac{\Gamma(1-s)}{\Gamma(1-n-s)} \tag{19}$$

We can now relate the functions ${}_2\mathrm{F}_1$ and Φ.

$$\begin{aligned}
{}_2\mathrm{F}_1(a,b;c;z) &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(c+n)\Gamma(a)\Gamma(b)} \frac{z^n}{n!} = \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(1-a)\Gamma(1-b)\Gamma(c)}{\Gamma(1-n-a)\Gamma(1-n-b)\Gamma(c+n)} \frac{z^n}{n!} = \\
&= \Gamma(1-a)\Gamma(1-b)\Gamma(c)\Phi(1,1,1,z).
\end{aligned}$$

with

$$\gamma = (-a, -b, c-1) \quad \text{and} \quad B = (-1,-1,1,1)^{tr}.$$

The above equation together with (12) make it possible for us to express the complete elliptic integral of the first kind as an A-hypergeometric function as follows.

$$\mathrm{K}(k) = \frac{\pi^2}{2} \Phi(1,1,1,z). \tag{20}$$

with

$$\gamma = (-1/2, -1/2, 0) \quad \text{and} \quad B = (-1,-1,1,1)^{tr}.$$

We can do the same procedure for the Appell hypergeometric function F_1 but we have to modify the function Φ a bit because we have a non generic parameter in the numerator. We therefore introduce the series $\tilde{\Phi}$ defined by

$$\tilde{\Phi}(a) = \sum_{k \in \mathbb{Z}^{N-n}} \frac{(-1)^{\langle B_1, k \rangle} \Gamma\left(-\gamma_1 - \langle B_1, k \rangle\right) a^{\gamma + \langle B, k \rangle}}{\prod_{j=2}^{n} \Gamma\left(\gamma_j + \langle B_j, k \rangle + 1\right) k!}. \tag{21}$$

The series $\tilde{\Phi}$ should be regarded as a meromorphic function with removable singularities (the $k!$ in the denominator take care of the possible singularities of the gamma function in the numerator). Note that for generic parameters we can use (19) to move the gamma function in the numerator to the denominator and we get

$$\tilde{\Phi} = \Gamma(1 + \gamma_1)\Gamma(-\gamma_1)\Phi.$$

We can now do the same reasoning as in the case of the Gauss hypergeometric function and use (13) to get

$$\Pi(\alpha^2, k) = \frac{\pi^2}{2} \tilde{\Phi}(1, 1, 1, 1, \alpha^2, k^2) \tag{22}$$

with

$$\gamma = (-1, 0, -1/2, -1/2) \quad \text{and} \quad B = \begin{pmatrix} -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{pmatrix}^{tr}.$$

If we combine (20) and (22) with Proposition 4.3 we get an expression of the second order derivative of the Ronkin function of an affine linear polynomial in three variables in terms of A-hypergeometric functions.

Proposition 4.4. *Let $f = 1 + z + e + t$ and set*

$$\gamma_1 = (-1/2, -1/2, 0), \quad \gamma_2 = (-1, 0, -1/2, -1/2), \quad B = (-1, -1, 1, 1)^{tr},$$

$$B_2 = \begin{pmatrix} -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{pmatrix}^{tr}.$$

Then the second order derivatives of the Ronkin function N_f can be expressed in terms of A-hypergeometric functions in the following way.

$$\frac{\partial^2 N_f}{\partial x^2} = g e^x \Phi(1, 1, 1, k^2),$$
$$\frac{\partial^2 N_f}{\partial x \partial y} = \frac{g}{4} \left(Q_1 \Phi(1, 1, 1, k^2) + Q_2 \tilde{\Phi}(1, 1, 1, 1, \alpha_1^2, k^2) + Q_3 \tilde{\Phi}(1, 1, 1, 1, \alpha_2^2, k^2) \right)$$

with parameters γ_1, γ_2 and matrices B_1, B_2. The functions and parameters $k^2, \alpha_1^2 \alpha_2^2, g^2, Q_1, Q_2$ and Q_3 are defined in Proposition 4.3.

5 The logarithmic Mahler measure

Recall the definition of the Mahler measure.

Definition 5.1. *Given a Laurent polynomial f the* logarithmic Mahler measure *of f is defined by*

$$m(f) := \left(\frac{1}{2\pi i}\right)^n \int_{\mathrm{Log}^{-1}(0)} \log|f(z)| \frac{dz}{z}$$

In Section 1 we saw that semi-explicit expressions of the Mahler measure of an affine linear polynomial give us an semi-explicit expression of the Ronkin function of affine linear polynomial and vice versa. This is because of the relation

$$N_f(x_1, \ldots, x_n) = m(1 + e^{x_1} z_1 + \ldots + e^{x_n} z_n)$$

for $f = 1 + z + \ldots + z_n$. In [24] Smyth proved a formula for the affine linear case in the three variables case but this only gives the values of the Ronkin function at points where four of the chambers meet.

Theorem 5.1. (Smyth)

$$m(1 + z + aw + at) = \begin{cases} \frac{2}{\pi^2}\left(\mathrm{Li}_3(a) - \mathrm{Li}_3(-a)\right) & \textit{if } a \leq 1 \\ \log(a) + \frac{2}{\pi^2}\left(\mathrm{Li}_3(a^{-1}) - \mathrm{Li}_3(-a^{-1})\right) & \textit{if } a \geq 1 \end{cases}$$

where Li_3 *is the trilogarithm defined as*

$$\mathrm{Li}_3(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^3}.$$

No more general formula has been proved so far. It is possible that the fact that we can express the second order derivatives of the Ronkin function can be of some help. The theorem by Smyth and formula 3 give us the following formula for $e^x < 1$.

$$\mathrm{Li}_2(-e^x) - \mathrm{Li}_2(e^x) = \int_{1-e^x}^{1+e^x} \arccos\left(\frac{1 + r^2 - e^{2x}}{2r}\right) \frac{d}{dr} \arccos\left(\frac{r^2 - 1 - e^{2x}}{2e^x}\right) dr$$

Maybe there is a similar kind of relation in the more general expression of (3)?

It seems to be of interest to estimate affine linear polynomials in n variables, both for fixed n or when letting n tend to infinity. In [25] the author proves that there exists an analytic function F such that the Mahler measure of the linear form $z_1 + \ldots + z_n$ up to an explicit constant is equal to $F(1/n)$. There is also an recursive expression of that analytic function in terms of Laguerre polynomials and Bessel functions. Note that this corresponds to the Ronkin function evaluated at the origin. In the paper [21] the authors estimate the growth of the Mahler measure in the linear case when the number of variables goes to infinity and also establish a lower and upper bound in terms of the norm of the coefficient vector. The reason for the interest in these kind of estimates is that it is hard numerically to calculate the Mahler measure and numerical calculations are of interest when looking for relations between the Mahler measure and special values of L-functions. Several such relations has been conjectured by Boyd, see [2]. We have not calculated the actual Ronkin function but all the second order derivatives. Note that the Ronkin function of $f = 1 + z + w + t$ is determined by its second order derivatives up to a polynomial on the form $a + b(x + y + u)$.

References

[1] Boyd, D: Speculations concerning the range of Mahler's measure, *Canad. Math. Bull.* **24** (1981), 453-469.

[2] Boyd, D: Mahler's measure and special values of L-functions, *Experiment. Math.* **7** (1998), no. 1, 37-82.

[3] Byrd, P - Friedman, M: *Handbook of elliptic integrals for engineers and physicists (Die Grundlehren der mathematischen Wissenschaften, vol. 67.)*, Springer, Berlin, 1954.

[4] Cruickshank, J - McLaughlin, J: Series parallel linkages, *Preprint, arXiv:0911.5293v1 [math.GT]* (2009).

[5] Exton, H: *Multiple hypergeometric functions and applications*, Ellis Horwood Ltd. Chichester, 1976.

[6] Faber, M - Schütz, D: Homology of planar polygon spaces, *Geom. Dedicata.* **125** (2007), 75-92.

[7] Forsberg, M - Passare, M - Tsikh, A: Laurent determinants and arrangement of hyperplane amoebas, *Adv. Math.* **151** (2000), 45-70.

[8] Gelfand, I - Kapranov, M - Zelevinsky, A: *Discriminants, resultants and multidimensional determinants*, Birkhäuser, Boston, 1994.

[9] Gelfand, I - Kapranov, M - Zelevinsky, A: Equations of hypergeometric type and Newton polyhedra, (Russian) *Dokl. Akad. Nauk SSSR.* **300** (1988), no. 3, 529-534; translation in *Soviet Math. Dokl.* **37** (1988), no. 3, 678-682.

[10] Lehmer, D: Factorization of certain cyclotomic functions, *Ann. of Math.* (2) **34** (1933) no. 3, 461-479.

[11] McKee, J - Smyth, C: Number theory and polynomials. Proceedings of the workshop held at Bristol University, Bristol, April 3–7, 2006. Edited by James McKee and Chris Smyth., *London Math. Soc. Lecture Note Ser.* **352**, Cambridge University Press, Cambridge, 2008.

[12] Mahler, K: On some inequalities for polynomials in several variables, *J. London Math. Soc.* **37** (1962), 341-344.

[13] Maillot, V: Géométrie d'Arakelov des variétés toriques et fibrés en droites intégrables, *Mém. Soc. Math. Fr. (N.S.)* **80** (2000), 129pp.

[14] Mikhalkin, G: Real algebraic curves, the moment map and amoebas, *Ann. Math.* **151** (2000), 309-326.

[15] Mikhalkin, G - Rullgård, H: Amoebas of maximal area, *Internat. Math. Res. Notices.* **9** (2001), 441-451.

[16] Milgram, R - Trinkle, J: The geometry of configuration spaces for closed chains in two and three dimensions, *Homology Homotopy Appl.* **6** no. 1 (2004), 237-267.

[17] Nilsson, L: Amoebas, discriminants, and hypergeometric functions, Doctoral dissertation, Stockholm Univerity, 2009, ISBN: 978-91-7155-889-3

[18] Passare, M - Rullgård, H: Amoebas, Monge-Ampère measures, and triangulations of the newton polytope, *Duke Math. J.* **3** (2004), 481-507.

[19] Passare, M - Tsikh, A: Amoebas: their spines and their contours, *Idempotent mathematics and mathematical physics*, 275-288, *Contemp. Math.* **377**, Amer. Math. Soc., Providence, RI 2005.

[20] Rauch, J - Taylor, A: The Dirichlet problem for the multidimensional Monge-Ampère equation, *Rocky Mountain J. Math.* **7** (1977), 345-364.

[21] Rodriguez-Villegas, F - Toledano, R - Vaaler, J: Estimates for the Mahler's measure of a linear form, *Proc. Edinb. Math. Soc.* (2) **47** (2004), no. 2, 473-494.

[22] Ronkin, L.I: *Introduction to the theory of entire functions of several variables*, American Mathematical Society, Providence, 1974.

[23] Shimamoto, D - Vanderwaart, C: Spaces of polygons in the Plane and Morse Theory, *Amer. Math. Monthly* **112**, (2005), no. 4, 289-310.

[24] Smyth, C.J: On measures of polynomials in several variables, *Bull. Austral. Math. Soc.* **23** (1981), no. 1, 49-63.

[25] Toledano, R: The Mahler measure of linear forms as special values of solution of algebraic differential equations, *Rocky Mountain J. Math.* **39** (2009), no. 4, 1323-1338.