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Infinite-dimensional homology and multibump solutions

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Abstract

We start by introducing a Čech homology with compact supports which we then use in order to construct an infinite-dimensional homology theory. Next we show that under appropriate conditions on the nonlinearity there exists a ground state solution for a semilinear Schrödinger equation with strongly indefinite linear part. To this solution there corresponds a nontrivial critical group, defined in terms of the infinite-dimensional homology mentioned above. Finally we employ this fact in order to construct solutions of multibump type. Although our main purpose is to survey certain homological methods in critical point theory, we also include some new results.

Keywords: Čech homology, infinite-dimensional homology, critical group, Schrödinger equation, ground state solution, multibump solution.

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1 Introduction

The first topic we consider in this paper is a homology theory of Čech type which satisfies all the Eilenberg-Steenrod axioms and the strong excision property. Such a construction, although known to algebraic topologists, see [24], does not seem to be well known to analysts. The advantages of this theory are that homology is often considered as more geometric and intuitive than cohomology and that strong excision is very convenient in applications – in fact the weaker excision property which holds for singular homology is a source of certain technical difficulties in critical point theory

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in infinite-dimensional spaces. Unlike for the Čech (or Alexander-Spanier) cohomology, the original Čech *homology* construction leads to a theory which is not exact unless it is restricted to compact pairs and coefficients in a field [16]. As we shall see, this deficiency can be removed by introducing a theory with compact supports.

Next we construct an infinite-dimensional homology theory which is suitable for so-called strongly indefinite problems and parallels the cohomology of [19]. An infinite-dimensional cohomology which (like our theory) satisfies all the Eilenberg-Steenrod axioms except the dimension axiom has been first time introduced by Gęba and Granas, see [17] and the references there.

Then we turn our attention to the Schrödinger equation

$$-\Delta u + V(x)u = f(x,u), \quad u \in H^1(\mathbb{R}^N),$$

where V and f are periodic in $x_1, \dots, x_N$ and 0 is in a gap of the spectrum of $-\Delta + V$ in $L^2(\mathbb{R}^N)$. Under appropriate assumptions on f it has been proved in [29] that this equation has a ground state solution u_0. We show that this solution, if isolated, must necessarily have a nontrivial critical group (in the sense of Morse theory). This gives rise to the existence of so-called multibump solutions which are obtained by gluing together translates of u_0 in a suitable way (see Section 6 for a more rigorous definition). The idea of using variational methods in order to find such solutions goes back to the work of Séré [27, 28] and Coti Zelati and Rabinowitz [12, 13].

The paper is organized as follows. In Section 2 we introduce the Čech homology $\check{H}_*(P,Q)$ for compact metric pairs, with coefficients in a field. Our approach is different from the usual one and – we hope – more appealing to geometric intuition. Instead of taking inverse limits of the simplicial homology for nerves of coverings we take inverse limits of the singular homology for neighborhoods of (P,Q) in an ambient space. Then we define a homology with compact supports $\check{H}^c_*(X,A)$ for all metric pairs as the direct limits of $\check{H}_*(P,Q)$ with respect to all compact $(P,Q) \subset (X,A)$. As we have already mentioned, the theories $\check{H}_*$ and $\check{H}^c_*$ satisfy all the Eilenberg-Steenrod axioms including the strong excision, see Theorems 2.6 and 2.9. The reader who wishes to do so may omit the details of the construction and the proofs. In Section 3 we introduce an infinite-dimensional homology, define the notion of critical group (in terms of this homology) and summarize its pertinent properties. The proofs are essentially the same as in [19] and are omitted or briefly outlined. However, since the arguments in [19] concern cohomology, some simple adaptation (mainly "reversing the arrows") is necessary. In Section 4 we consider the Schrödinger equation mentioned above and sketch the procedure which has been employed in [29] in order to obtain a ground state for f of subcritical growth (i.e., $|f(x,u)| \leqslant a(1+|u|^{p-1})$, $2 < p < 2^*$, where $2^* := 2N/(N-2)$ is the critical Sobolev exponent). Then, combining a result in [8] with the method of [29], we show that the ground state exists also when $f(x,u) = |u|^{2^*-2}u$. Section 5 is concerned with the proof that the ground state solution has a nontrivial critical group. In Section 6 we discuss multibump solutions. Since the details of the multibump construction are rather tedious, we first describe the main ideas which are in fact rather simple, and then in Section 7 we provide the technical details.

Although our primary goal in this paper is to survey certain homological methods in critical point theory for strongly indefinite functionals, we also include some results which have not been published earlier. In particular, in Theorems 5.3 and 5.4 an infinite-dimensional homology computation is performed for a strongly indefinite functional which does not satisfy the Palais-Smale condition, and in Theorem 6.3 multibump solutions are found for a Schrödinger equation with strongly indefinite linear part and critical Sobolev exponent. We also believe that our approach to the Čech homology in Section 2 may be of independent interest.

Notation: In what follows $B_\rho(x)$, $\overline{B}_\rho(x)$ and $S_\rho(x)$ will respectively denote the open ball, the closed ball and the sphere centered at x and having radius ρ. The symbol "$\rightharpoonup$" denotes weak convergence, "int" and "cl" are respectively the interior and the closure of a set, and $\mathbb{R}^+ := [0, +\infty)$. We also use the customary notation $\Phi^c := \{u \in E : \Phi(u) \leqslant c\}$, where Φ is a functional defined in a Banach space E and $c \in \mathbb{R}$.

2 Čech homology with compact supports

In order to construct an infinite-dimensional homology theory we shall need an appropriate ordinary homology. The most commonly used singular homology theory $H^s_*(\cdot, \cdot; G)$ (with coefficients in an abelian group G) is defined for arbitrary topological pairs and satisfies all the Eilenberg-Steenrod axioms. However, in singular theory there exist certain pathological examples which may not be desirable. For instance, there are connected spaces (like the so-called Warsaw circle, see Remark 2.7 below) which admit homotopically nontrivial maps into the circle but have trivial 1-dimensional singular homology, and there are compact subsets of $\mathbb{R}^3$ having nontrivial homology groups in arbitrarily high dimensions (see e.g. [6]). Moreover, the singular homology does *not* satisfy the strong excision axiom but only the weaker one saying that given a topological pair (X, B) and a set $A \subset X$ such that $\operatorname{int} A \cup \operatorname{int} B = X$, the inclusion $j : (A, A \cap B) \to (X, B)$ induces an isomorphism $j_* : H^s_*(A, A \cap B; G) \to H^s_*(X, B; G)$.

In [16, Chap. IX], the Čech homology $\check{H}_*(X, A; G)$ with coefficients in an abelian group G for an arbitrary pair (X, A) of topological pairs has been defined. However, in this general situation the exactness axiom is not always satisfied. It holds if G is a field (or a compact group) and (X, A) a compact pair. In this latter case the setting presented in [16] is complete and satisfactory although not intuitive from the geometric viewpoint. Therefore we propose a different approach.

Let (P, Q), where $Q \subset P$, be a pair of compact metric spaces. It is well known that P can be emebedded in the Hilbert cube and hence in any infinite-dimensional normed space E (in other words, we can consider P and Q as compact subsets of E).

Let $\mathcal{F}$ be a fixed field and $\check{H}_*(P, Q) := \{\check{H}_q(P, Q)\}_{q \in \mathbb{Z}}$ the graded vector space defined by

$$\check{H}_q(P, Q) := \varprojlim \{H^s_q(U, V) : (U, V) \in \mathcal{U}\}, \quad q \in \mathbb{Z}, \tag{2.1}$$

where $\mathcal{U}$ stands for the family of all (pairs of) neighborhoods of (P, Q) in E directed by the inverse inclusion and $H^s_*(U, V) = H^s_*(U, V; \mathcal{F})$ denotes the singular homology of $(U, V) \in \mathcal{U}$ with coefficients in $\mathcal{F}$. In other words, $\check{H}_q(P, Q)$ is the inverse limit of the inverse system $\{H^s_q(U, V)\}_{(U,V) \in \mathcal{U}}$ of vector spaces together with homomorphisms induced by the inclusions $(U, V) \hookrightarrow (U', V')$. In what follows we shall make repeated use of standard properties of inverse and direct limits, see e.g. [24, Appendix].

Remark 2.1 Observe that if a family $\mathcal{U}'$ of neighborhoods of the pair (P, Q) is cofinal in $\mathcal{U}$, then

$$\check{H}_*(P, Q) = \varprojlim\{H_*^s(U, V) : (U, V) \in \mathcal{U}'\}.$$

In particular, $\mathcal{U}'$ may consist of all pairs of open neighborhoods of (P, Q) in E.

Lemma 2.2 *Definition (2.1) is correct, i.e., it does not depend on the choice of a normed space E.*

Proof To see this, suppose that E_1, E_2 are normed spaces and let $Q_1 \subset P_1 \subset E_1$, $Q_2 \subset P_2 \subset E_2$ be homeomorphic copies of the pair (P, Q) embedded into E_1 and E_2, respectively. Thus there is a homeomorphism $h' : (P_1, Q_1) \to (P_2, Q_2)$. Let $E := E_1 \times E_2$ and let $j_i : E_i \to E$, $i = 1, 2$, be inclusions given by $j_1(x_1) = (x_1, 0)$ and $j_2(x_2) = (0, x_2)$ for $x_1 \in E_1$ and $x_2 \in E_2$. In view of Lemma (2.4) from [18], there is a homeomorphism $h : E \to E$ such that $hj_1|_{P_1} = j_2h'$.

For $i = 1, 2$, let $\mathcal{U}_i$ be the family of all neighborhoods of the pair (P_i, Q_i) in E_i and let $\widetilde{\mathcal{U}}_i$ be the family of all neighborhoods of the pair $(j_i(P_i), j_i(Q_i))$ in E. Since h is a homeomorphism, we easily see that $\widetilde{\mathcal{U}}_2 = \{(h(U), h(V)) : (U, V) \in \widetilde{\mathcal{U}}_1\}$. This shows that

$$\varprojlim\{H_*^s(U, V) : (U, V) \in \widetilde{\mathcal{U}}_1\} = \varprojlim\{H_*^s(U, V) : (U, V) \in \widetilde{\mathcal{U}}_2\}.$$

On the other hand, given a neighborhood (U, V) of the pair $(j_1(P_1), j_1(Q_1))$ in E, there are $(U_1, V_1) \in \mathcal{U}_1$ and an integer $n \geqslant 1$ such that $U_1 \times B_{1/n}(0) \subset U$ and $V_1 \times B_{1/n}(0) \subset V$, where $B_{1/n}(0)$ is a ball in E_2. In other words, the family $\{(U_1, V_1) \times B_{1/n}(0) : (U_1, V_1) \in \mathcal{U}_1,\ n \geqslant 1\}$ is cofinal in $\widetilde{\mathcal{U}}_1$. Since $H_*^s((U_1, V_1) \times B_{1/n}(0)) \cong H_*^s(U_1, V_1)$, we see that

$$\varprojlim\{H_*^s(U_1, V_1) : (U_1, V_1) \in \mathcal{U}_1\} \cong \varprojlim\{H_*^s(U, V) : (U, V) \in \widetilde{\mathcal{U}}_1\}.$$

Similarly one shows that

$$\varprojlim\{H_*^s(U_2, V_2) : (U_2, V_2) \in \mathcal{U}_2\} \cong \varprojlim\{H_*^s(U, V) : (U, V) \in \widetilde{\mathcal{U}}_2\}.$$

□

Let $f : (P, Q) \to (P', Q')$ be a continuous map of compact metric pairs and assume that $P \subset E$, $P' \subset E'$, where E and E' are normed spaces. In order to define the induced homomorphism $f_* : \check{H}_*(P, Q) \to \check{H}_*(P', Q')$, let $f' : E \to E'$ be an arbitrary (continuous) extension of f which exists in view of the Dugundji theorem. For each $(U', V') \in \mathcal{U}'$, where $\mathcal{U}'$ stands for the family of all neighborhoods of the pair (P', Q') in E', consider the homomorphism $f'_* : H_*^s(f'^{-1}(U'), f'^{-1}(V')) \to H_*^s(U', V')$ induced by $f'|_{f'^{-1}(U)}$. It is easy to see that the family $\{f'_* : (U', V') \in \mathcal{U}'\}$ forms a transformation of the inverse system $\{H_*^s(U, V) : (U, V) \in \mathcal{U}\}$ into the inverse system $\{H_*^s(U', V') : (U', V') \in \mathcal{U}'\}$ and therefore determines a homomorphism

$$f_* := \varprojlim\{f'_* : (U', V') \in \mathcal{U}'\} : \check{H}_*(P, Q) \to \check{H}_*(P', Q'). \tag{2.2}$$

Lemma 2.3 *Definition (2.2) is correct, i.e., it does not depend on the choice of an extension f' of f.*

Proof Suppose that $f'' : E \to E'$ is another extension of f. Define a map $h : (E \times \{0,1\}) \cup (P \times [0,1]) \to E'$ by

$$h(x,t) := \begin{cases} f'(x) & \text{for} \quad x \in E,\ t = 0; \\ f(x) & \text{for} \quad x \in P,\ t \in [0,1]; \\ f''(x) & \text{for} \quad x \in E,\ t = 1. \end{cases}$$

As before, h admits an extension $h' : E \times [0,1] \to E'$. If $(U', V') \in \mathcal{U}'$, then there is a pair (U,V) of neighborhoods of P and Q respectively, such that $U \times [0,1] \subset h'^{-1}(U')$ and $V \times [0,1] \subset h'^{-1}(V')$; in particular, $U \subset f'^{-1}(U') \cap f''^{-1}(U')$ and $V \subset f'^{-1}(V') \cap f''^{-1}(V')$. Since the maps $f'|_{(U,V)}$ and $f''|_{(U,V)}$ are homotopic in (U', V'), they induce the same homomorphism $H^s_*(U,V) \to H^s_*(U',V')$. This completes the proof. □

In order to define the boundary operator $\partial : \check{H}_*(P,Q) \to \check{H}_{*-1}(Q)$ and show the exactness of the homology sequence of (P,Q) we need some preparations. First observe that given a compact subset P of a normed space E and a neighborhood U of P in E, there is an integer $n \geqslant 1$ such that, for any $x \in P$, $B_{1/n}(x) \subset U$. Since P is compact, there are points $x_1, \ldots, x_k \in P$ such that $P \subset \bigcup_{i=1}^k B_{1/n}(x_i) \subset U$.

In other words, we see that the family $\mathcal{U}_0$ of pairs of open sets (U,V), where U (resp. V) is the finite union of balls centered at points of P (resp. Q) and of radius $1/n$, where $n \geqslant 1$, is cofinal in the family of all neighborhoods of (P,Q).

Lemma 2.4 *If $X \subset E$ is a finite union of open convex sets, then for any $q \in \mathbb{Z}$, the vector space $H^s_q(X)$ is finite-dimensional.*

Proof We shall proceed by induction on the number of open convex sets covering X. If X is open convex, then $H^s_0(X) = \mathcal{F}$ and $H^s_q(X) = 0$ for $q \neq 0$. Suppose that, for any $q \in \mathbb{Z}$, $\dim H^s_q(X') < \infty$ whenever X' is the union of k open convex sets. Let $X = \bigcup_{i=1}^{k+1} C_i$ where C_i is open convex in E, $i = 1, ..., k+1$, and put $X' := \bigcup_{i=1}^k C_i$; then $X = X' \cup C_{i+1}$.

Since the pair $\{X', C_{i+1}\}$ is excisive, the Mayer-Vietoris sequence

$$\begin{aligned} ... \longrightarrow H^s_q(X' \cap C_{i+1}) \xrightarrow{\alpha_1} H^s_q(X') \oplus H^s_q(C_{i+1}) \xrightarrow{\alpha_2} H^s_q(X' \cup C_{i+1}) \\ \xrightarrow{\alpha_3} H^s_{q-1}(X' \cap C_{i+1}) \xrightarrow{\alpha_4} H^s_{q-1}(X') \oplus H^s_{q-1}(C_{i+1}) \longrightarrow ... \end{aligned}$$

is exact. In view of the induction hypothesis, the spaces $H^s_q(X' \cap C_{i+1})$ and $H^s_q(X') \oplus H^s_q(C_{i+1})$ are finite-dimensional for all q (note that $X' \cap C_{i+1}$ is the union of at most k convex open sets). Passing to subspaces and quotient spaces we see that the following sequence

$$0 \longrightarrow \operatorname{coker}(\alpha_1) \xrightarrow{\alpha_2'} H^s_q(X' \cup C_{i+1}) \xrightarrow{\alpha_3} \ker(\alpha_4) \longrightarrow 0$$

is exact. Since the above terms are vector spaces, this sequence is split and

$$H^s_q(X) = H^s_q(X' \cup C_{i+1}) = \operatorname{coker}(\alpha_1) \oplus \ker(\alpha_4);$$

hence $H^s_q(X)$ is finite-dimensional. □

The above lemma implies that if $(U,V) \in \mathcal{U}_0$, then the vector spaces $H^s_q(U)$, $H^s_q(V)$ are finite-dimensional for all q, and therefore so are $H^s_q(U,V)$ in view of the homology sequence of (U,V).

Let (P,Q) be a compact pair in a normed space E. Since the family $\mathcal{U}_0$ is cofinal in the family of all neighborhoods of (P,Q), we have

$$\check{H}_*(P,Q) = \varprojlim\{H^s_*(U,V) : (U,V) \in \mathcal{U}_0\}.$$

Let $q \in \mathbb{Z}$ and consider the inverse system of exact sequences

$$\ldots \longrightarrow H^s_q(V) \longrightarrow H^s_q(U) \longrightarrow H^s_q(U,V) \xrightarrow{\partial} H^s_{q-1}(V) \longrightarrow H^s_{q-1}(U) \longrightarrow \ldots,$$

where $(U,V) \in \mathcal{U}_0$ ($\partial : H^s_q(U,V) \to H^s_{q-1}(V)$ is the connecting homomorphism). Since all terms in these sequences belong to the category of *finite-dimensional vector spaces*, in view of Theorem VIII.5.7 from [16], we conclude that the limit sequence

$$\ldots \longrightarrow \check{H}_q(Q) \longrightarrow \check{H}_q(P) \longrightarrow \check{H}_q(P,Q) \xrightarrow{\partial} \check{H}_{q-1}(Q) \longrightarrow \check{H}_{q-1}(U) \longrightarrow \ldots$$

is exact.

Remark 2.5 It is important that we deal with *finite-dimensional vector spaces* since in general the inverse limit of an inverse system of exact sequences consisting of arbitrary groups is not exact; this is the reason why the Čech homology for arbitrary topological pairs with coefficients in an arbitrary group is not exact.

It is easy to see that the inverse limit homomorphism $\partial := \varprojlim\{\partial : H^s_q(U,V) \to H^s_{q-1}(V) : (U,V) \in \mathcal{U}_0\})$ is in fact the desired boundary operator for the pair (P,Q): if $f : (P,Q) \to (P',Q')$ is a continuous map of compact pairs, then $\partial f_* = (f|_Q)_*\partial$ because inverse limits preserve commutativity.

Theorem 2.6 *The above defined Čech homology $\check{H}_*$ is a functor on the category of compact metric pairs which satisfies the following Eilenberg-Steenrod axioms:*

(i) (Functoriality) *If* id *is the identity map on a compact metric pair (P,Q), then* id_* *is the identity on $\check{H}_*(P,Q)$; if $f : (P,Q) \to (P',Q')$ and $g : (P',Q') \to (P'',Q'')$ are continuous maps of compact metric pairs, then $(g \circ f)_* = g_* \circ f_*$.*

(ii) (Naturality of ∂) *If $f : (P,Q) \to (P',Q')$ is a continuous map of compact metric pairs, then*

$$\begin{tikzcd}
\check{H}_q(P,Q) \arrow[r, "\partial"] \arrow[d, "f_*"'] & \check{H}_{q-1}(Q) \arrow[d, "(f|_Q)_*"] \\
\check{H}_q(P',Q') \arrow[r, "\partial"'] & \check{H}_{q-1}(Q')
\end{tikzcd}$$

(iii) (Exactness) *For a compact metric pair (P,Q), let $i : Q \hookrightarrow P$ and $j : P \hookrightarrow (P,Q)$ be the inclusions. Then, for any $q \in \mathbb{Z}$, the homology sequence*

$$\ldots \longrightarrow \check{H}_q(Q) \xrightarrow{i_*} \check{H}_q(P) \xrightarrow{j_*} \check{H}_q(P,Q) \xrightarrow{\partial} \check{H}_{q-1}(Q) \longrightarrow \ldots$$

is exact.

(iv) (Strong excision) *If P, Q are compact metric spaces, then the inclusion $(P, P \cap Q) \hookrightarrow (P \cup Q, Q)$ induces the excision isomorphism*

$$\check{H}_*(P, P \cap Q) \overset{exc}{\cong} \check{H}_*(P \cup Q, Q).$$

(v) (Homotopy invariance) *If two continuous maps $f, g : (P, Q) \to (P', Q')$ of compact metric pairs are homotopic, then $f_* = g_*$.*

(vi) (Dimension) *If $*$ is a one-point space, then*

$$\check{H}_q(*) = \begin{cases} \mathcal{F} & \text{for } q = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Proof The functoriality of $\check{H}_*$ is easy to see. Using arguments similar to those from the proof of Lemma 2.3, we show the homotopy invariance axiom (this time we extend the homotopy $h : P \times [0, 1] \to P'$ to $h' : E \times [0, 1] \to E'$). It is clear that the dimension axiom is satisfied. The naturality of ∂ and the exactness axiom have been discussed above.

To show the excision property we may assume that P and Q are compact subsets of a normed space E. Given neighborhoods U, V of P and Q respectively, the excision axiom for the singular theory implies that the inclusion $i : (U, U \cap V) \to (U \cup V, V)$ induces an isomorphism $i_* : H^s_*(U, U \cap V) \to H^s_*(U \cup V, V)$. Since the family $\mathcal{U}_1$ (resp. $\mathcal{U}_2$) of neighborhoods of the form $(U, U \cap V)$ (resp. $(U \cup V, V)$), where U is a neighborhood of P and V a neighborhood of Q, is cofinal in the family of all neighborhoods the pair $(P, P \cap Q)$ (resp. $(P \cup Q, Q)$), we obtain that

$$\begin{aligned} \check{H}_*(P, P \cap Q) = \varprojlim\{H^s_*(U, U \cap V) : P \subset U,\ Q \subset V\} \xrightarrow{\cong} \\ \varprojlim\{H^s_*(U \cup V, V) : P \subset U,\ Q \subset V\} = \check{H}_*(P \cup Q, Q), \end{aligned}$$

where the isomorphism is induced as above by the inclusion, i.e., it is given as the inverse limit map

$$\varprojlim\{i_* : H^s_*(U, U \cap V) \to H^s_*(U \cup V, V) : P \subset U,\ Q \subset V\}$$

(remember the that the inverse limit of isomorphisms is an isomorphism). □

Remark 2.7 Recall that the Warsaw circle X is the union of two sets A and B such that A is the closure of $\{(x, y) \in \mathbb{R}^2 : y = \sin(\pi/x),\ 0 < x < 1\}$ and B is an arc in $\mathbb{R}^2$ which meets A only at $(0, -1)$ and $(1, 0)$. It is easy to see that each map $S^1 \to X$ is homotopically trivial, hence $H^s_1(X) = 0$. However, there exists a nested sequence of open annuli $U_n \supset X$ such that $\bigcap_{n=1}^\infty U_n = X$. It follows that $\check{H}_1(X) = \varprojlim H^s_1(U_n) = \mathcal{F}$.

Remark 2.8 (i) Mardesič [22] (see also [23]) shows that our definition (2.1) coincides with the definition of the Čech homology introduced in [16]. Moreover, it can be shown (cf. [22], [23]) that if a compact space P is locally contractible (or – more generally – *homologically locally connected*, see e.g. [24]), then $\check{H}_*(P) \cong H^s_*(P; \mathcal{F})$; therefore, if $Q \subset P$ is also locally contractible, then $\check{H}_*(P, Q) \cong H^s_*(P, Q; \mathcal{F})$. In particular, if (P, Q) is a pair of compact metric absolute neighborhood retracts, then $\check{H}_*(P,Q) \cong H_*^s(P,Q;\mathcal{F})$.

(ii) Another construction of a Čech type homology theory for compact pairs, with coefficients in an arbitrary group G, which satisfies the Eilenberg-Steenrod axioms including the exactness and the strong excision axioms, was provided in Massey [24]. The main advantage of his theory is that it is valid for arbitrary (not necessarily metric) compact pairs and coefficient groups. If G is a field, then the Massey and the Čech theories coincide. The disadvantage of the Massey approach is that it is less intuitive, especially in contrast to the formula (2.1) above.

The Čech homology $\check{H}_*$ introduced above is not sufficient for our purposes because it is defined on compact pairs only. Now we extend it to a more general situation.

Let (X,A) be an arbitrary pair of metric spaces. It easy to see that the family $\mathcal{C}(X,A)$ of all compact pairs $(P,Q) \subset (X,A)$ is directed by inclusion. The family $\{\check{H}_*(P,Q)\}_{(P,Q)\in\mathcal{C}(X,A)}$ together with the family $\check{H}_*(P,Q) \to \check{H}_*(P',Q')$ of homomorphisms induced by inclusions $(P,Q) \subset (P',Q') \in \mathcal{C}(X,A)$ form a direct system of vector spaces. Following [16, Chap. IX, Exercise D] (comp. [24, Chap. 9]), we define the *Čech homology with compact supports* and coefficients in $\mathcal{F}$ by setting

$$\check{H}_*^c(X,A) := \varinjlim_{(P,Q)\in\mathcal{C}(X,A)} \check{H}_*(P,Q).$$

Given a continuous map $f:(X,A) \to (Y,B)$, the family $\{\check{H}_*(P,Q) \to \check{H}_*(f(P),f(Q))\}_{(P,Q)\in\mathcal{C}(X,A)}$ of homomorphisms induced by f determines a map of the system $\{\check{H}_*(P,Q)\}_{(P,Q)\in\mathcal{C}(X,A)}$ to the direct system $\{\check{H}_*(P',Q')\}_{(P',Q')\in\mathcal{C}(Y,B)}$. Therefore f determines a unique (graded) homomorphism $f_*:\check{H}_*^c(X,A) \to \check{H}_*^c(Y,B)$.

In a similar manner one checks that given a pair (X,A), the family $\{\partial:\check{H}_*(P,Q) \to \check{H}_{*-1}(Q)\}$ of boundary homomorphisms in the Čech theory determines a map between the direct systems $\{\check{H}_*(P,Q)\}_{(P,Q)\in\mathcal{C}(X,A)}$ and $\{\check{H}_{*-1}(Q)\}_{Q\in\mathcal{C}(A)}$ and therefore defines the boundary homomorphism $\partial:\check{H}_*^c(X,A) \to \check{H}_{*-1}^c(A)$.

The Čech homology $\check{H}_*^c$ with compact supports and coefficients in $\mathcal{F}$ on the category of arbitrary metric pairs satisfies all the Eilenberg-Steenrod axioms. This follows from the properties of the direct limit and the respective properties of $\check{H}_*$. In particular, the exactness of the sequence

$$\ldots \longrightarrow \check{H}_*^c(A) \xrightarrow{i_*} \check{H}_*^c(X) \xrightarrow{j_*} \check{H}_*^c(X,A) \xrightarrow{\partial} \check{H}_{*-1}^c(A) \longrightarrow \ldots$$

is a consequence of the exactness of the functor of direct limit (see [24, Theorem A.7]). More precisely, the following holds true:

Theorem 2.9 *The Čech homology with compact supports $\check{H}_*^c$ is a functor on the category of metric pairs which satisfies the Eilenberg-Steenrod axioms* (i)-(iii) *and* (v),(vi) *stated in Theorem 2.6. Axiom* (iv) *is satisfied for* closed *pairs, i.e., if $X = A \cup B$, where A,B are closed in X, then the inclusion $(A, A\cap B) \hookrightarrow (X,B)$ induces the isomorphism*

$$\check{H}_*^c(A, A\cap B) \overset{exc}{\cong} \check{H}_*^c(X,B).$$

Only the excision axiom requires some explanation. It is easy to see that the families of pairs $(P, P \cap Q)$ and $(P \cup Q, Q)$, where $P \subset A$, $Q \subset B$ are compact, are cofinal in the families of all compact subsets of respectively $(A, A \cap B)$ and $(A \cup B, B)$. Indeed, if $R \subset A \cup B$ is compact, then so are $R \cap A := P$ and $R \cap B := Q$ (because A, B are closed), and if $R \subset A \cap B$, then we can take $P = Q = R$. Hence passing to the direct limit in the isomorphisms $\check{H}_*(P, P \cap Q) \cong \check{H}_*(P \cup Q, Q)$ gives the conclusion.

Remark 2.10 If a metric pair (X, A) is compact, then $\check{H}^c_*(X, A) = \check{H}_*(X, A)$. In view of the results of [22], if a metric space is locally compact and locally contractible, then $\check{H}^c_*(X) \cong H^s_*(X; \mathcal{F})$. If $A \subset X$ is closed and locally contractible, then $\check{H}^c_*(X, A) \cong H^s_*(X, A; \mathcal{F})$.

3 Infinite-dimensional homology theory

Let E be a Hilbert space and Φ a functional of class C^1. In critical point theory one introduces the notion of critical groups of Φ at an isolated critical point x_0 by setting $c_q(\Phi, x_0) := H^s_q(\Phi^c \cap U, (\Phi^c \cap U) \setminus \{x_0\})$ $(q \in \mathbb{Z})$, where $c = \Phi(x_0)$ and U is a neighborhood of x_0, see e.g. [10]. Suppose $E = E^+ \oplus E^-$ is an orthogonal decomposition and consider the functional $\Phi(x) := \|x^+\|^2 - \|x^-\|^2$, $x^\pm \in E^\pm$. Then 0 is a critical point of Φ and one can show that $c_q(\Phi, 0) = H^s_q(D^-, S^-)$, where D^- and S^- are respectively the unit closed ball and the unit sphere in E^-. So $c_q(\Phi, 0) \neq 0$ if and only if $q = M^-(\Phi''(0))$, the Morse index of $\Phi''(0)$. Hence if $M^-(\Phi''(0)) = \dim E^- = \infty$, $c_q(\Phi, 0) = 0$ for all q. Our purpose in this section is to construct a theory which will give a nontrivial homological information (and a finite Morse index) in this case.

In what follows $(E, \|\cdot\|)$ is a real Banach space. By a *filtration* of E we mean an increasing sequence $(E_n)_{n=1}^\infty$ of closed subspaces of E such that $E = \operatorname{cl} \bigcup_{n=1}^\infty E_n$. Given a filtration (E_n) of E and $X \subset E$, let $X_n := X \cap E_n$. If $A \subset X \subset E$ and $B \subset Y \subset E$, then a (continuous) map $f : (X, A) \to (Y, B)$ is said to be *filtration-preserving* if $f(X_n) \subset E_n$ for almost all n. A homotopy $h : (X, A) \times [0, 1] \to (Y, B)$ is *filtration-preserving* if $h(X_n \times [0, 1]) \subset E_n$ for almost all n.

In order to introduce a homology theory of spaces with filtration we will need some preliminaries. Let $(\mathcal{G}_n)_{n=1}^\infty$ be a sequence of abelian groups. We define the *asymptotic* group $[(\mathcal{G}_n)_{n=1}^\infty]$ by the formula

$$[(\mathcal{G}_n)_{n=1}^\infty] := \prod_{n=1}^\infty \mathcal{G}_n \Big/ \bigoplus_{n=1}^\infty \mathcal{G}_n.$$

In other words, in the direct product $\prod_{n=1}^\infty \mathcal{G}_n := \{(\xi_n)_{n=1}^\infty : \xi_n \in \mathcal{G}_n\}$ we introduce an equivalence relation: $(\xi_n) \sim (\eta_n)$ if and only if $\xi_n = \eta_n$ for almost all $n \geqslant 1$. So

$$[(\mathcal{G}_n)_{n=1}^\infty] = \prod_{n=1}^\infty \mathcal{G}_n / \sim .$$

The equivalence class of $(\xi_n) \in \prod_{n=1}^\infty \mathcal{G}_n$ is denoted by $[(\xi_n)_{n=1}^\infty]$. If $\mathcal{G}_n = \mathcal{G}$ for almost all n, then we write $\mathcal{G}$ instead of $[(\mathcal{G}_n)_{n=1}^\infty]$. It is clear that the above construction of asymptotic groups generalizes immediately to modules and vector spaces.

Let $(E_n)_{n=1}^{\infty}$ be a filtration of E. Suppose that a sequence $(d_n)_{n=1}^{\infty}$ of nonnegative integers is given and let $\mathcal{E} := (E_n, d_n)_{n=1}^{\infty}$. If $\mathcal{F}$ is a field and (X, A) a pair of subsets of E, then for any integer $q \in \mathbb{Z}$, we define the q-th $\mathcal{E}$-homology group with coefficients in $\mathcal{F}$ by the formula

$$H_q^{\mathcal{E}}(X, A) := [(\check{H}_{q+d_n}^c(X_n, A_n))_{n=1}^{\infty}].$$

Consequently, $H_*^{\mathcal{E}}(X, A) := \{H_q^{\mathcal{E}}(X, A)\}_{q\in\mathbb{Z}}$. Note that unlike in the usual homology, $H_q^{\mathcal{E}}(X, A)$ need not be 0 if $q < 0$.

As admissible morphisms in the category of pairs in E we take all continuous maps $f : (X, A) \to (Y, B)$ which preserve the filtration $(E_n)_{n=1}^{\infty}$, and as admissible homotopies we take those which preserve this filtration. It is clear that each admissible f induces a (graded group) homomorphism $f_* : H_*^{\mathcal{E}}(X, A) \to H_*^{\mathcal{E}}(Y, B)$ given by the formula $f_* = [f_{n*}]$ or, more precisely,

$$f_*[(\xi_n)_{n=1}^{\infty}] := [(f_{n*}(\xi_n))_{n=1}^{\infty}],$$

where $f_n := f|_{(X_n,A_n)} : (X_n, A_n) \to (Y_n, B_n)$ and $\xi_n \in \check{H}_{*+d_n}^c(X_n, A_n)$.

The boundary homomorphism $\partial = \partial^{(X,A)} : H_*^{\mathcal{E}}(X, A) \to H_{*-1}^{\mathcal{E}}(A)$ is defined by setting $\partial := [\partial_n]$, i.e.,

$$\partial[(\xi_n)_{n=1}^{\infty}] = [(\partial_n(\xi_n))_{n=1}^{\infty}],$$

where $\partial_n : \check{H}_{*+d_n}^c(X_n, A_n) \to \check{H}_{*+d_n-1}^c(A_n)$ is the boundary homomorphism for $\check{H}_*^c$ and $\xi_n \in \check{H}_{*+d_n}^c(X_n, A_n)$.

It is easy to see that $H_*^{\mathcal{E}}$ is a functor from the category of pairs of subsets of E together with admissible maps of such pairs into the category of vector spaces over $\mathcal{F}$. More precisely, we have the following:

Proposition 3.1 (i) (Functoriality) *If* id *is the identity map on* (X, A)*, then* id_* *is the identity on* $H_*^{\mathcal{E}}(X, A)$*; if the maps* $f : (X, A) \to (Y, B)$ *and* $g : (Y, B) \to (Z, C)$ *are admissible, then* $(g \circ f)_* = g_* \circ f_*$.

(ii) (Naturality of ∂) *If* $f : (X, A) \to (Y, B)$ *is admissible, then* $\partial f_* = (f|_A)_*\partial$.

(iii) (Exactness) *For each pair* (X, A) *in* E*, let* $i : A \hookrightarrow X$ *and* $j : X \hookrightarrow (X, A)$ *be the inclusions. Then, for any* $q \in \mathbb{Z}$*, the homology sequence*

$$\ldots \longrightarrow H_q^{\mathcal{E}}(A) \xrightarrow{i_*} H_q^{\mathcal{E}}(X) \xrightarrow{j_*} H_q^{\mathcal{E}}(X, A) \xrightarrow{\partial^{(X,A)}} H_{q-1}^{\mathcal{E}}(A) \longrightarrow \ldots$$

is exact.

(iv) (Strong excision) *If* A, B *are closed subsets of* $X \subset E$ *such that* $A \cup B = X$*, then the inclusion* $(A, A \cap B) \hookrightarrow (X, B)$ *induces the excision isomorphism*

$$H_*^{\mathcal{E}}(A, A \cap B) \overset{exc}{\cong} H_*^{\mathcal{E}}(X, B).$$

(v) (Homotopy invariance) *If* $f, g : (X, A) \to (Y, B)$ *are homotopic by an admissible homotopy, then* $f_* = g_*$.

(vi) (Exact homology sequence of a triple) *For a triple* (X, A, B) *in* E*, i.e.* $B \subset A \subset X \subset E$ *and* *inclusions $i : (A,B) \hookrightarrow (X,B)$, $j : (X,B) \hookrightarrow (X,A)$, there is a homomorphism $\Delta : H_*^{\mathcal{E}}(X,A) \to H_{*-1}^{\mathcal{E}}(A,B)$ such that, for each $q \in \mathbb{Z}$, the sequence*

$$\dots \longrightarrow H_q^{\mathcal{E}}(A,B) \xrightarrow{i_*} H_q^{\mathcal{E}}(X,B) \xrightarrow{j_*} H_q^{\mathcal{E}}(X,A) \xrightarrow{\Delta} H_{q-1}^{\mathcal{E}}(A,B) \longrightarrow \dots$$

is exact.

The proofs of (i)–(v) follow immediately from the definition $H_*^{\mathcal{E}}$ and the respective properties of the homology $\check{H}_*^c$. Property (vi) follows from (iii) upon taking $\Delta = k_* \circ \partial^{(X,A)}$, where k_* is induced by the inclusion $k : A \hookrightarrow (A,B)$, see e.g. [15, III.3.4 and III.3.5].

Proposition 3.1 states that $H_*^{\mathcal{E}}$ satisfies all of the Eilenberg-Steenrod axioms for homology theory except for the dimension axiom which is satisfied only in the trivial case $E_n = E$ and $d_n = 0$ for almost all $n \geqslant 1$. Instead of the dimension axiom we have the following basic example:

Example 3.2 Suppose that F is a closed subspace of E and $k_n := \dim(F \cap E_n)$. Suppose that $d := \lim_{n\to\infty}(k_n - d_n)$ exists, $d \in \mathbb{Z} \cup \{\pm\infty\}$. Given $p \in F$ and $r \geqslant \|p\|$, let $D := \overline{B}_r(p) \cap F$ and $S := S_r(p) \cap F$. For each $n \geqslant 1$, $D_n := D \cap E_n$ is a closed ball with boundary $S_n := S \cap E_n$ and $\dim D_n = k_n$. If $d = \pm\infty$, then $\check{H}_{q+d_n}^c(D_n, S_n) = 0$ for large n; hence $H_q^{\mathcal{E}}(D,S) = 0$ for all $q \in \mathbb{Z}$. If $d \neq \pm\infty$, then $q + d_n = q + k_n - d$ for almost all n and

$$H_q^{\mathcal{E}}(D,S) = \begin{cases} \mathcal{F} & \text{for } q = d \\ 0 & \text{otherwise.} \end{cases}$$

Since $H_*^{\mathcal{E}}(D) = 0$ for all $q \in \mathbb{Z}$, by the exactness of the homology sequence for (D,S) we infer that

$$H_q^{\mathcal{E}}(S) = \begin{cases} \mathcal{F} & \text{for } q = d-1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $(E_n)_{n=1}^\infty$ be a filtration of E and suppose that $\Phi \in C^1(E,\mathbb{R})$. For each $n \in \mathbb{N}$, let $\Phi_n := \Phi|_{E_n}$. It is clear that $\Phi_n \in C^1(E_n,\mathbb{R})$ and $\Phi_n'(x) \in E_n^*$ for $x \in E_n$. Moreover, for $u \in E_n$, $\langle \Phi_n'(x), u\rangle = \langle \Phi'(x), u\rangle$; here and below $\langle \cdot\,, \cdot\rangle$ denotes the duality pairing in E_n (or in E).

Let $N \subset E$. We say that a sequence $(y_j)_{j=1}^\infty$ in N is a $(PS)^*$-*sequence* (with respect to (E_n)) if $y_j \in E_{n_j}$, where $n_j \to \infty$, $(\Phi(y_j))$ is bounded and $\|\Phi_{n_j}'(y_j)\| \to 0$ as $j \to \infty$. If every $(PS)^*$-sequence in N has a convergent subsequence, then Φ is said to verify the $(PS)^*$-*condition* (with respect to (E_n)) on N.

The $(PS)^*$-condition (in a slightly different form) has been introduced by Bahri and Berestycki [4], [5], and Li and Liu [21]. Note that if Φ satisfies the $(PS)^*$-condition on N, then each convergent $(PS)^*$-sequence (y_j) in N tends to a critical point of Φ. Indeed, suppose that $y_j \to y$ and take $\varepsilon > 0$. For large j, $\|\Phi_{n_j}'(y_j)\| < \varepsilon$ and $\|\Phi'(y_j) - \Phi'(y)\| < \varepsilon$. Let $u \in \bigcup_{n=1}^\infty E_n$, $\|u\| \leqslant 1$. Then, for large j, we have $u \in E_{n_j}$ and

$$|\langle \Phi'(y), u\rangle| \leqslant |\langle \Phi'(y) - \Phi'(y_j), u\rangle| + |\langle \Phi_{n_j}'(y_j), u\rangle| < 2\varepsilon.$$

Therefore $\Phi'(y) = 0$.

If Φ satisfies $(PS)^*$, then Φ satisfies the usual (PS)-condition as well. Given a sequence (x_j) such that $(\Phi(x_j))$ is bounded and $\Phi'(x_j) \to 0$, then for each j there is $y_j \in E_{n_j}$ such that $|\Phi(x_j) - \Phi(y_j)| < 1$, $\|x_j - y_j\| < j^{-1}$ and $\|\Phi'(x_j) - \Phi'(y_j)\| < j^{-1}$; moreover, we may asume $n_j < n_{j+1}$. So if $u \in E_{n_j}$ and $\|u\| \leqslant 1$, then

$$|\langle \Phi'_{n_j}(y_j), u\rangle| = |\langle \Phi'(y_j), u\rangle| \leqslant \|\Phi'(y_j)\|;$$

hence (y_j) is a $(PS)^*$ sequence. Thus (y_j), and therefore also (x_j), has a convergent subsequence.

Definition 3.3 Let $N \subset E \setminus K$, where K is the critical set of Φ. A map $V : N \to E$ is called a *gradient-like vector field for* Φ *on* N if:

(i) V is locally Lipschitz continuous;

(ii) $\|V(x)\| \leqslant 1$ for all $x \in N$;

(iii) there is a function $\beta : N \to \mathbb{R}_+$ such that $\langle \Phi'(x), V(x)\rangle \geqslant \beta(x)$ on N and if $Z \subset N$ is bounded away from K and $\sup_Z |\Phi| < \infty$, then $\inf_{z\in Z} \beta(z) > 0$.

We say that a gradient-like vector field V for Φ on N is *related to* (E_n) provided $V|_Z$ preserves this filtration on any set $Z \subset N$ which is bounded away from K and such that $\sup_Z |\Phi| < \infty$.

Lemma 3.4 *Let $N \subset E$ be open. If Φ satisfies the $(PS)^*$-condition on N, then there exists a gradient-like vector field V for Φ on $N \setminus K$ related to (E_n).*

For the proof, see [19, Lemma 2.2]. In [19] it was assumed that E is a Hilbert space; however, it is easy to see that the argument goes through for Banach spaces as well. The same remark also applies to Proposition 3.6 below.

Next we define the notion of admissible pair. It is a suitable adaptation of a Gromoll-Meyer pair to our situation (an extra requirement we need here is that there exists a vector field V satisfying the conditions of Definition 3.3). A detailed study of the classical Gromoll-Meyer theory may be found e.g. in [10].

Definition 3.5 Let A be an isolated compact subset of K. A pair (W, W^-) of closed subsets of E is said to be an *admissible pair for* Φ *and* A *with respect to* (E_n) provided:

(i) W is bounded away from $K \setminus A$, $W^- \subset \partial W$, $A \subset \operatorname{int} W$ and $\Phi|_W$ is bounded;

(ii) there is a neighborhood N of W and a gradient-like vector field V for Φ on $N \setminus A$, related to (E_n);

(iii) W^- is the union of finitely many (possibly intersecting) closed sets each of which lies on a C^1-manifold of codimension 1, V is transversal to each of these manifolds at points of W^-, the flow η of $-V$ can leave W only via W^- and if $x \in W^-$, then $\eta(t, x) \notin W$ for any $t > 0$.

A gradient-like vector field V, corresponding to (W, W^-) in the above sense, will be called an *admissible field.*

In what follows we will usually omit the expressions 'related to the filtration' and 'with respect to the filtration'. In view of [19, Proposition 2.6], for each isolated critical point p of Φ there exists an admissible pair (W, W^-). More precisely, the following holds true:

Proposition 3.6 *Suppose that $\Phi \in C^1(E,\mathbb{R})$ satisfies the $(PS)^*$-condition in a neighborhood N of an isolated critical point p of Φ. For each open neighborhood $U \subset N$ of p, there exists an admissible pair (W, W^-) for Φ and p such that $W \subset U$ and $\Phi|_{W^-} < c := \Phi(p)$. Moreover, there is a $\delta_1 > 0$ such that $\overline{B}_{\delta_1}(p) \subset \operatorname{int} W$ and if $x \in S_{\delta_1}(p) \cap \Phi^c$, then $\eta(t,x) \in W^-$ for some $t > 0$.*

For the reader's convenience we sketch the construction of (W, W^-). Without loss of generality we may assume that $N \cap K = \{p\}$. Let $\delta > 0$ be such that $\overline{B}_\delta(p) \subset U$ and $\Phi(x) > c - \varepsilon$ for all $x \in \overline{B}_\delta(p)$ ($\varepsilon > 0$ small enough). Let V be a gradient-like field for Φ on $N \setminus \{p\}$ and consider the Cauchy problem

$$\frac{d\sigma}{dt} = -\omega(\sigma)V(\sigma), \quad \sigma(0,x) = x,$$

where ω is a cutoff function such that $\omega = 0$ in a neighborhood of p and $\omega = 1$ in $N \setminus B_{\delta_1}(p)$. Then it can be shown that the pair (W, W^-) defined by

$$W = \{\sigma(t,x) : t \geqslant 0,\ x \in \overline{B}_\delta(p),\ \Phi(\sigma(t,x)) \geqslant c - \varepsilon\}, \quad W^- = W \cap \Phi^{-1}(c-\varepsilon)$$

satisfies the properties stated in Proposition 3.6.

For the rest of this section suppose that a sequence (d_n) of integers is given and let $\mathcal{E} = (E_n, d_n)_{n=1}^\infty$. Let p be an isolated critical point of a functional $\Phi \in C^1(E,\mathbb{R})$ satisfying the $(PS)^*$-condition in a neighborhood N of p and let (W, W^-) be an admissible pair for Φ and p such that $W \subset N$. For any $q \in \mathbb{Z}$ we define the q-th *critical group* of Φ at p with respect to $\mathcal{E}$ by setting

$$c_q^{\mathcal{E}}(\Phi, p) := H_q^{\mathcal{E}}(W, W^-).$$

Proposition 3.6 asserts the existence of an admissible pair (W, W^-) for Φ and p in N. Exactly as in [19, Proposition 2.7] one shows that the critical groups $c_q^{\mathcal{E}}(\Phi, p)$ are well-defined, i.e., they do not depend on the choice of an admissible pair: if the pairs (W_i, W_i^-), $i = 1, 2$, are admissible for Φ and p, then the groups $H_q^{\mathcal{E}}(W_1, W_1^-)$ and $H_q^{\mathcal{E}}(W_2, W_2^-)$ are isomorphic for all $q \in \mathbb{Z}$.

The proof of this fact is rather technical and employs the strong excision property in an essential way. The rough idea is to find an admissible pair (W_0, W_0^-) such that $W_0 \subset \operatorname{int} W_1 \cap \operatorname{int} W_2$ (which exists by Proposition 3.6) and construct a number of deformations by cutting off the flow σ constructed above. These deformations and excision show that $H_*^{\mathcal{E}}(W_0, W_0^-) \cong H_*^{\mathcal{E}}(W_i, W_i^-)$.

Let now E be a Hilbert space with an inner product $\langle .\,, . \rangle$, let $\Phi \in C^2(U,\mathbb{R})$, where $U \subset E$ is a neighborhood of an isolated critical point p, and set $L := \Phi''(p)$. In what follows we assume via duality that $\Phi'(x) \in E$ for all x and $L \in \mathcal{L}(E,E)$.

Suppose that $L = A + B$, where $A \in \mathcal{L}(E,E)$ is a self-adjoint Fredholm operator of index 0 such that $A(E_n) \subset E_n$ for almost all n and $B \in \mathcal{L}(E,E)$ is self-adjoint compact. It is then clear that L is a self-adjoint Fredholm operator of index 0, and in particular, $\dim N(L) < \infty$ and $N(L) \oplus R(L) = E$. Hence any point $x \in E$ admits a unique representation $x = p + z + y$, where $z \in N(L)$ and $y \in R(L)$. Assume further that there exists $k \in \mathbb{Z}$ such that the Morse index $M^-(A|_{E_n}) = d_n + k$ and let $Q_n : R(L) \to R(L) \cap E_n$ be the orthogonal projection of $R(L)$ on $R(L) \cap E_n$. One shows (see Corollary 4.4 in [19]) that, for each $x \in R(L)$, $Q_n x \to x$; hence

$(R(L)\cap E_n)_{n=1}^{\infty}$ is a filtration of $R(L)$. Moreover, in view of Proposition 5.2 in [19], the *$\mathcal{E}$-Morse index*

$$M_{\mathcal{E}}^{-}(L) := \lim_{n\to\infty}(M^{-}(Q_nL|_{R(L)\cap E_n}) - d_n)$$

is well-defined and finite.

Since Φ is a C^2-functional, we have the representation

$$\Phi(x) = \Phi(p) + \frac{1}{2}\langle Ly, y\rangle + \psi(x),$$

where $\psi \in C^2(U,\mathbb{R})$, $\psi(p)=0$, $\psi'(p)=0$ and $\psi''(p)=0$. Denote the orthogonal projection of E on $R(L)$ by Q. Then

$$\Phi'(p+z+y) = Ly + \psi'(p+z+y). \tag{3.1}$$

The invertibility of $L|_{R(L)}$ and the implicit function theorem imply that there is $\delta>0$ and a C^1-function $y=\alpha(z) : B_\delta(0)\cap N(L)\to R(L)$ such that $B_\delta(p)\subset U$, $\alpha(0)=0$, $\alpha'(0)=0$ and

$$Q\Phi'(p+z+\alpha(z)) = 0 \quad\text{for}\quad z\in B_\delta(0)\cap N(L). \tag{3.2}$$

Define $\varphi : B_\delta(0)\cap N(L)\to\mathbb{R}$ by

$$\varphi(z) := \Phi(p+z+\alpha(z)) - \Phi(p) = \frac{1}{2}\langle L\alpha(z),\alpha(z)\rangle + \psi(p+z+\alpha(z)),\quad \|z\|<\delta. \tag{3.3}$$

It is clear that 0 is an isolated critical point of φ. The next result shows the relationship between the critical groups $c_*^{\mathcal{E}}(\Phi,p)$ and $c_*(\varphi,0) := \check{H}_*^c(\widetilde{W},\widetilde{W}^-)$, where $(\widetilde{W},\widetilde{W}^-)$ is an admissible pair for φ and 0 in $N(L)$ (with respect to the trivial filtration of $N(L)$).

Theorem 3.7 (cf. Theorem 5.4 in [19]) *Under the above assumptions, for all $q\in\mathbb{Z}$, $c_q^{\mathcal{E}}(\Phi,p) = c_{q-M_{\mathcal{E}}^-(L)}(\varphi,0)$.*

We sketch the argument which is rather tedious and consists of several steps. By the continuity property of critical groups [19, Corollary 2.10], we may assume without loss of generality that $p=0$ and $\Phi(p)=0$. Next one shows using a certain homotopy which goes back to [14] that $c_*^{\mathcal{E}}(\Phi,0) = c_*^{\mathcal{E}}(\Phi_1,0)$, where $\Phi_1(z+y) = \frac{1}{2}\langle Ly,y\rangle + \varphi(z)$, $z+y\in(N(L)\oplus R(L))\cap B_\delta(0)$. Then one constructs admissible pairs, respectively (W_1,W_1^-) for $\langle Ly,y\rangle$ in $R(L)$ and $(\widetilde{W},\widetilde{W}^-)$ for φ in $B_\delta(0)\cap N(L)$, and shows that $(W_1,W_1^-)\times(\widetilde{W},\widetilde{W}^-)$ is topologically equivalent to an admissible pair (W,W^-) for Φ_1 and 0 (here we use the customary notation $(A,B)\times(C,D) = (A\times B,(A\times D)\cup(B\times C))$. The pair $(W_1\cap E_n, W_1^-\cap E_n)$ turns out to be homotopy equivalent to $(B,\partial B)$, where B is the closed unit ball of dimension $m_n := M_{\mathcal{E}}^-(L)+d_n$ (n large). We may assume $(\widetilde{W},\widetilde{W}^-)$ is a pair of compact ANR's [14] (in [14] this is shown for Gromoll-Meyer and not for admissible pairs; however, on finite-dimensional spaces these two notions coincide). Hence by the Künneth formula in singular homology [15, Corollary VI.12.12],

$$\begin{aligned}
&\check{H}^c_{q+d_n}((W_1\cap E_n, W_1^-\cap E_n)\times(\widetilde{W},\widetilde{W}^-)) \cong \check{H}_{q+d_n}((B,\partial B)\times(\widetilde{W},\widetilde{W}^-))\\
&\quad\cong \Big(\check{H}_*(B,\partial B)\otimes\check{H}_*(\widetilde{W},\widetilde{W}^-)\Big)_{q+d_n} \cong \check{H}_{q+d_n-m_n}(\widetilde{W},\widetilde{W}^-) = \check{H}_{q-M_{\mathcal{E}}^-(L)}(\widetilde{W},\widetilde{W}^-).
\end{aligned}$$

Here we have used that $\check{H}^c_* = \check{H}_* = H^s_*$ and the excision requirement of the Künneth formula for H^s_* holds on pairs of compact ANR's. The first fact is a consequence of Remarks 2.8, 2.10 and the second one (which is well known) can be immediately deduced from the strong excision axiom for $\check{H}_*$ as long as $\check{H}_* = H^s_*$. It follows that $H^{\mathcal{E}}_q(W, W^-) = \check{H}_{q-M^-_{\mathcal{E}}(L)}(\widetilde{W}, \widetilde{W^-})$ and the proof is complete.

4 Ground states for a Schrödinger equation

In this section we sketch a proof of the existence of a ground state for the Schrödinger equation

$$(4.1) \qquad -\Delta u + V(x)u = f(x,u), \quad u \in H^1(\mathbb{R}^N),$$

where as usual, $H^1(\mathbb{R}^N)$ denotes the Sobolev space of functions $u \in L^2(\mathbb{R}^N)$ such that $\nabla u \in L^2(\mathbb{R}^N, \mathbb{R}^N)$. Denote the spectrum of $-\Delta + V$ in $L^2(\mathbb{R}^N)$ by $\sigma(-\Delta+V)$, let $F(x,u) := \int_0^u f(x,s)\,ds$ and $2^* := 2N/(N-2)$ if $N \geqslant 3$, $2^* := +\infty$ if $N = 1$ or 2. We make the following assumptions on V and f:

- (S_1) V is continuous, 1-periodic in $x_1, \dots, x_N$, $\sigma(-\Delta+V) \cap (-\infty, 0) \neq \emptyset$ and $0 \notin \sigma(-\Delta+V)$,
- (S_2) f is continuous, 1-periodic in $x_1, \dots, x_N$ and $|f(x,u)| \leqslant a(1+|u|^{p-1})$ for some $a > 0$ and $p \in (2, 2^*)$,
- (S_3) $f(x,u) = o(u)$ uniformly in x as $u \to 0$,
- (S_4) $F(x,u)/u^2 \to \infty$ uniformly in x as $|u| \to \infty$,
- (S_5) $u \mapsto f(x,u)/|u|$ is strictly increasing on $(-\infty, 0)$ and on $(0, \infty)$.

Although the results of this section also hold if $\sigma(-\Delta+V) \subset (0, \infty)$ (see the discussion in [29]), we only consider the more difficult case where $\sigma(-\Delta+V) \cap (-\infty, 0) \neq \emptyset$. Recall (see e.g. [20]) that periodicity of V implies $\sigma(-\Delta+V)$ is absolutely continuous, bounded below but not above, and consists of a finite number ($\geqslant 1$) of disjoint closed intervals. We also remark that in Sections 5–7 condition (S_2) will be replaced by a stronger condition (S_2').

Let

$$(4.2) \qquad \Phi(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx.$$

It is well known [30, 31] that $\Phi \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and $\Phi'(u) = 0$ if and only if u is a weak solution of (4.1). Since $0 \notin \sigma(-\Delta+V)$, the quadratic form in (4.2) is non-degenerate, so there exist an equivalent inner product $\langle .\,, . \rangle$ and a corresponding norm $\|.\|$ in $E := H^1(\mathbb{R}^N)$ such that

$$(4.3) \qquad \Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}^N} F(x,u)\,dx.$$

Here $u = u^+ + u^-$, $u^\pm \in E^\pm$, $E = E^+ \oplus E^-$ and $E^\pm$ are the orthogonal invariant subspaces corresponding to the positive and the negative part of the spectrum of $-\Delta+V$. By the absolute continuity of $\sigma(-\Delta+V)$ and (S_1), $\dim E^\pm = +\infty$.

Let

$$\mathcal{M} := \{u \in E \setminus E^- : \langle \Phi'(u), u\rangle = \langle \Phi'(u), v\rangle = 0 \text{ for all } v \in E^-\}.$$

Recall from Section 3 that we identify (via duality) $\Phi'(u) \in E^*$ with an element of E. The set $\mathcal{M}$ has been introduced by Pankov in [25]. He has proved that under stronger conditions $\mathcal{M}$ is a C^1-manifold, see also a comment at the end of Section 6. Here we shall outline an argument showing that under our hypotheses there is a natural homeomorphism between the unit sphere in E^+ and $\mathcal{M}$. Note that if $\Phi'(u) = 0$ and $u \neq 0$, then $u \in \mathcal{M}$.

For $u \in E \setminus E^-$, we define

$$E(u) := E^- \oplus \mathbb{R}u \equiv E^- \oplus \mathbb{R}u^+ \text{ and } \widehat{E}(u) := E^- \oplus \mathbb{R}^+ u. \tag{4.4}$$

Theorem 4.1 *Suppose (S_1)–(S_5) are satisfied and let $c := \inf_{u\in\mathcal{M}} \Phi(u)$. Then c is attained, $c > 0$ and if $u_0 \in \mathcal{M}$ satisfies $\Phi(u_0) = c$, then u_0 is a solution of (4.1).*

Since c is the lowest level of Φ at which there exists a nontrivial solution, u_0 will be called a *ground state*. We shall also show that c has the following minimax characterization:

$$c = \inf_{w\in E^+\setminus\{0\}} \max_{u\in\widehat{E}(w)} \Phi(u). \tag{4.5}$$

We sketch the main steps in the proof of Theorem 4.1 and refer to [29] for the details.

Proposition 4.2 *If $u \in \mathcal{M}$, then $\Phi(u + w) < \Phi(u)$ for all $w \neq 0$ such that $u + w \in \widehat{E}(u)$. In other words, if $u \in \mathcal{M}$, then u is the unique global maximum of $\Phi|_{\widehat{E}(u)}$.*

Note that $u+w \in \widehat{E}(u)$ if and only if $w = su+v$, where $s \geqslant -1$ and $v \in E^-$. The key step in the proof of this proposition is the following inequality which is a consequence of (S_5): Let $u, s, v \in \mathbb{R}$, $s \geqslant -1$ and $w := su + v \neq 0$. Then

$$f(x,u)\,[s(s/2+1)u + (1+s)v] + F(x,u) - F(x,u+w) < 0.$$

The proof, although elementary, is not straightforward, see [29].

Proposition 4.3 *For each $u \in E \setminus E^-$, the set $\mathcal{M} \cap \widehat{E}(u)$ consists of exactly one point which is the (unique) global maximum of $\Phi|_{\widehat{E}(u)}$.*

Proof (outline) Since $\widehat{E}(u) = \widehat{E}(u^+/\|u^+\|)$, we may assume without loss of generality that $u \in S^+ := E^+ \cap S_1(0)$. By (S_3), there exist $\alpha, \rho > 0$ (independent of $u \in S^+$) such that $\Phi(\rho u) \geqslant \alpha$. It follows from (S_4) that $\Phi \leqslant 0$ on $\widehat{E}(u) \setminus B_R(0)$ for a sufficiently large $R > 0$ (R depends on the choice of u). Hence $\alpha \leqslant \sup_{\widehat{E}(u)} \Phi < \infty$. Since $\widehat{E}(u) \cap E^+ = \mathbb{R}^+ u$ and $F \geqslant 0$ (by (S_5)), it is easy to see from (4.3) and Fatou's lemma that $\Phi|_{\widehat{E}(u)}$ is weakly upper semicontinuous. So $\Phi|_{\widehat{E}(u)}$ attains its supremum at some $\bar{u} \neq 0$. Clearly, $\Phi \leqslant 0$ on E^- and therefore $\bar{u}$ is a critical point of $\Phi|_{\widehat{E}(u)}$.

In particular, $\langle \Phi'(\bar u), \bar u\rangle = \langle \Phi'(\bar u), v\rangle = 0$ for all $v \in E^-$, that is, $\bar u \in \mathcal{M}$. Finally, Proposition 4.2 implies that $\mathcal{M} \cap \widehat{E}(u) = \{\bar u\}$. □

It follows from the above proof that

$$c = \inf_{u\in\mathcal{M}} \Phi(u) \geqslant \inf_{u\in E^+\cap S_\rho(0)} \Phi(u) \geqslant \alpha,$$

so $c > 0$. For $w \in E^+ \setminus \{0\}$, denote the unique point at which $\Phi|_{\widehat{E}(w)}$ attains its maximum by $\widehat{m}(w)$ and set

$$\widehat{\Psi}(w) := \Phi(\widehat{m}(w)).$$

It can be shown that $\widehat{m}$ is continuous and, somewhat surprisingly, $\Psi \in C^1(E^+ \setminus \{0\}, \mathbb{R})$, with

$$\langle \widehat{\Psi}'(w), z\rangle = \frac{\|\widehat{m}(w)^+\|}{\|w\|}\langle \Phi'(\widehat{m}(w)), z\rangle, \quad w \in E^+ \setminus \{0\},\ z \in E^+.$$

Now it is not too difficult to see that $\widehat{m}|_{S^+} : S^+ \to \mathcal{M}$ is a homeomorphism, with the inverse given by $\widehat{m}^{-1}(u) = u^+/\|u^+\|$, and (w_m) is a Palais-Smale sequence for $\Psi := \widehat{\Psi}|_{S^+}$ if and only if $(\widehat{m}(w_m))$ is a Palais-Smale sequence for Φ on $\mathcal{M}$. Moreover, w is a critical point of Ψ if and only if $\widehat{m}(w)$ is a critical point of Φ. Clearly,

$$\inf_{S^+} \Psi = \inf_{\mathcal{M}} \Phi = c,$$

and since $\widehat{E}(w) = \widehat{E}(w/\|w\|)$ for $w \in E^+ \setminus \{0\}$ and $\Phi(\widehat{m}(w)) = \max_{\widehat{E}(w)} \Phi$, c has the minimax characterization (4.5).

Proposition 4.4 *The functional Φ is coercive on $\mathcal{M}$, i.e., $\Phi(u) \to \infty$ as $\|u\| \to \infty$, $u \in \mathcal{M}$.*

Proof We shall show that each sequence (u_m) such that $c \leqslant \Phi(u_m) \leqslant d$ for some $d \geqslant c$ is bounded. Suppose $\|u_m\| \to \infty$ and let $v_m := u_m/\|u_m\|$. Then, after passing to a subsequence, $v_m \rightharpoonup v$ in E and $v_m \to v$ a.e. in $\mathbb{R}^N$. Clearly, $\|v_m^+\|^2 + \|v_m^-\|^2 = 1$ and

$$0 < c \leqslant \Phi(u_m) = \frac{1}{2}(\|u_m^+\|^2 - \|u_m\|^2) - \int_{\mathbb{R}^N} F(x, u_m)\, dx \leqslant \frac{1}{2}(\|u_m^+\|^2 - \|u_m^-\|^2),$$

hence $\|v_m^+\|^2 - \|v_m^-\|^2 > 0$ and therefore $\|v_m^+\|^2 \geqslant \frac{1}{2}$. For any $m \geqslant 1$ there is $y_m \in \mathbb{R}^N$ such that

$$\int_{B_1(y_m)} (v_m^+)^2\, dx = \max_{y\in\mathbb{R}^N} \int_{B_1(y)} (v_m^+)^2\, dx.$$

Since Φ and $\mathcal{M}$ are invariant under translations of the form $u(x) \mapsto u(x - y)$, $y \in \mathbb{Z}^N$ (by the periodicity of V and f), we may assume without loss of generality that the sequence (y_m) is bounded. Suppose

$$\int_{B_1(y_m)} (v_m^+)^2\, dx \to 0. \tag{4.6}$$

Then, according to P.L. Lions' lemma [30, Lemma 1.21], $v_m^+ \to 0$ in $L^p(\mathbb{R}^N)$. Since $sv_m^+ \in \widehat{E}(u_m)$ for each $s \geqslant 0$, it follows that

$$d \geqslant \Phi(u_m) \geqslant \Phi(sv_m^+) = \frac{s^2}{2}\|v_m^+\|^2 - \int_{\mathbb{R}^N} F(x, sv_m^+)\, dx \geqslant \frac{s^2}{4} - \int_{\mathbb{R}^N} F(x, sv_m^+)\, dx \to \frac{s^2}{4}, \tag{4.7}$$

a contradiction if $s > \sqrt{4d}$. Hence the integral in (4.6) is bounded away from 0 and since $v_m^+ \to v^+$ in $L^2_{loc}(\mathbb{R}^N)$, $v^+ \neq 0$ and therefore $v \neq 0$. So by (S_4) and Fatou's lemma,

$$\frac{1}{\|u_m\|^2} \int_{\mathbb{R}^N} F(x, u_m)\, dx = \int_{\mathbb{R}^N} \frac{F(x, u_m)}{u_m^2} v_m^2\, dx \to \infty$$

and thus

$$0 \leqslant \frac{\Phi(u_m)}{\|u_m\|^2} = \frac{1}{2}(\|v_m^+\|^2 - \|v_m^-\|^2) - \frac{1}{\|u_m\|^2} \int_{\mathbb{R}^N} F(x, u_m)\, dx \to -\infty.$$

This contradiction completes the proof. $\square$

Proof of Theorem 4.1 (outline) By Ekeland's variational principle [30], there exists a Palais-Smale sequence $(w_m) \subset S^+$ for Ψ such that $\Psi(w_m) \to c$. Set $u_m := \widehat{m}(w_m)$. Then $(u_m) \subset \mathcal{M}$ is a Palais-Smale sequence for Φ and $\Phi(u_m) \to c$. By Proposition 4.4, Φ is coercive on $\mathcal{M}$. Hence (u_m) is bounded, so $u_m \rightharpoonup u$ in E and $u_m \to u$ in $L^2_{loc}(\mathbb{R}^N)$ after passing to a subsequence. Since $\Phi'(u) = 0$, $u \in \mathcal{M}$ or $u = 0$. Let $(y_m) \subset \mathbb{R}^N$ be a sequence such that

$$\int_{B_1(y_m)} u_m^2\, dx = \max_{y \in \mathbb{R}^N} \int_{B_1(y)} u_m^2\, dx.$$

Using the translation invariance of Φ and $\mathcal{M}$ by elements of $\mathbb{Z}^N$, we may assume as in the preceding proof that the sequence (y_m) is bounded in $\mathbb{R}^N$. If $u = 0$, then

$$\int_{B_1(y_m)} u_m^2\, dx \to 0 \text{ as } m \to \infty \tag{4.8}$$

and it follows from P.L. Lions' lemma [30, Lemma 1.21] again that $u_m \to 0$ in $L^p(\mathbb{R}^N)$. Hence using (S_2), (S_3) and the Hölder and Sobolev inequalities, we obtain

$$o(1) = \langle \Phi'(u_m), u_m^+ \rangle = \|u_m^+\|^2 - \int_{\mathbb{R}^N} f(x, u_m) u_m^+\, dx = \|u_m^+\|^2 + o(1).$$

So $u_m^+ \to 0$ and $c \leqslant 0$ by (4.3). It follows that $u \neq 0$ and $u \in \mathcal{M}$. Obviously, $\Phi(u) \geqslant c$ and it remains to show that the reverse inequality holds. It is easy to see from (S_5) that $\frac{1}{2} f(x, u)u \geqslant F(x, u)$, hence by Fatou's lemma and since (u_m) is bounded,

$$\begin{aligned} c + o(1) = \Phi(u_m) - \frac{1}{2}\langle \Phi'(u_m), u_m \rangle &= \int_{\mathbb{R}^N} \left(\tfrac{1}{2} f(x, u_m) u_m - F(x, u_m) \right) dx \\ &\geqslant \int_{\mathbb{R}^N} \left(\tfrac{1}{2} f(x, u) u - F(x, u) \right) dx + o(1) = \Phi(u) - \frac{1}{2}\langle \Phi'(u), u \rangle + o(1) = \Phi(u) + o(1). \end{aligned} \tag{4.9}$$

So $\Phi(u) \leqslant c$. $\square$

Theorem 4.5 *Suppose $N \geqslant 4$, (S_1) is satisfied and $f(x,u) = |u|^{2^*-2}u$. Then the conclusion of Theorem 4.1 remains valid.*

Let

$$S := \inf_{u \in E \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}.$$

The proof of Theorem 4.5 will follow from the lemma below which is essentially a reformulation of some statements contained in [8]. The main result of [8] asserts that (4.1) has a solution $u_0 \neq 0$; however, no claim has been made there that u_0 is a ground state.

Lemma 4.6 *The following is true under the conditions of Theorem 4.5:*
(i) each Palais-Smale sequence is bounded;
(ii) there exists a Palais-Smale sequence (u_m) such that $\Phi(u_m) \to \widetilde{c} < S^{N/2}/N$;
(iii) each Palais-Smale sequence (u_m) such that $\Phi(u_m) \to \widetilde{c} < S^{N/2}/N$ has a subsequence which, possibly after translation by elements of $\mathbb{Z}^N$, converges weakly to a solution $u \neq 0$ of (4.1) such that $\Phi(u) \leqslant \widetilde{c}$.

Proof (i) is (essentially) Proposition 3.3 in [8]. For (ii), see Propositions 3.2 and 4.2 there. Finally, Proposition 4.1 in [8] implies that (4.8) cannot hold if $\widetilde{c} < S^{N/2}/N$. Hence for a (translated) subsequence we have that $u_m \rightharpoonup u \neq 0$ and $\Phi'(u) = 0$. That $\Phi(u) \leqslant \widetilde{c}$ follows from (4.9) with $f(x,u) = |u|^{2^*-2}u$ and c replaced by $\widetilde{c}$. □

Proof of Theorem 4.5 One sees by inspection of our arguments above and of [29] that $\mathcal{M}$, $\widehat{m}$ and Ψ have the same properties as before except that we do not claim Φ is coercive on $\mathcal{M}$. As in the proof of Theorem 4.1, let (w_m) be a minimizing sequence which we may assume is Palais-Smale. Hence so is (u_m), where $u_m := \widehat{m}(w_m)$. By Lemma 4.6, $c < S^{N/2}/N$ and there exists a solution u_0 with $\Phi(u_0) = c$. □

5 Critical groups for a ground state solution

In Theorems 4.1 and 4.5 it has been shown that (4.1) has a ground state solution u_0. We shall now compute the critical groups of u_0 under the assumption that it is an isolated solution, i.e., an isolated critical point of the functional Φ given by (4.2). We consider subcritical f first and at the end of this section we discuss the case of $f(x,u) = |u|^{2^*-2}u$.

We shall need the following stronger form of (S_2):

(S_2') f and f_u' are continuous and, for some $\bar{a} > 0$ and each $x \in \mathbb{R}^N$, $u \in \mathbb{R}$,

$$|f_u'(x,u)| \leqslant \bar{a}(1 + |u|^{p-2}).$$

Condition (S_2') implies that $\Phi \in C^2(E, \mathbb{R})$; moreover, for any $u, v \in E$,

$$\Phi''(u)v = v^+ - v^- - B(u)v,$$

where

$$\langle B(u)v,w\rangle := \int_{\mathbb{R}^N} f'_u(x,u)vw\,dx.$$

We claim that for each $u \in E$, $B(u): E \to E$ is a compact linear operator. Indeed, suppose $\|w\| \leqslant 1$, $v_j \rightharpoonup 0$ and let $\varepsilon > 0$ be given. By (S'_2) and (S_3), $|f'_u(x,u)| \leqslant \varepsilon + \bar{a}_\varepsilon |u|^{p-2}$. Since u is fixed, there exists $R = R_\varepsilon > 0$ such that, for some constant d independent of ε and w,

$$\int_{|x|\geqslant R} |f'_u(x,u)v_j w|\,dx \leqslant \varepsilon \int_{\mathbb{R}^N} |v_j||w|\,dx + \bar{a}_\varepsilon \int_{|x|\geqslant R} |u|^{p-2}|v_j||w|\,dx \leqslant d\varepsilon \tag{5.1}$$

(we have used the Hölder and Sobolev inequalities). Now the conclusion follows because $v_j \to 0$ in $L^q(B_R(0))$, $1 \leqslant q < 2^*$.

Let $E = E^+ \oplus E^-$ be as in the preceding section and put $L := \Phi''(u_0)$. Then $Lv = v^+ - v^- - B(u_0)v$; thus L is a Fredholm operator of index 0 and $E = N(L) \oplus R(L)$. Choose a filtration (E_m) such that $E_1^\pm := \mathbb{R}u_0^\pm$, $E_m = E_m^+ \oplus E_m^-$, where $E_m^\pm \subset E^\pm$ and $\dim E_m^\pm = m$ (so $\dim E_m = 2m$). Assume without loss of generality that $N(L) \subset E_m$ for almost all m and let $P_m : E \to E_m$ be the orthogonal projection.

Lemma 5.1 *There are $\alpha > 0$ and $m_0 \geqslant 1$ such that, for $m \geqslant m_0$ and $u \in R(L) \cap E_m$, $\|P_m Lu\| \geqslant \alpha\|u\|$.*

Proof Arguing by contradiction, for each j there exist m_j and $v_j \in R(L) \cap E_{m_j}$ such that $m_j \to \infty$, $\|v_j\| = 1$, $v_j \rightharpoonup v$ and

$$P_{m_j} L v_j = v_j^+ - v_j^- - P_{m_j} B(u_0) v_j \to 0$$

(recall $P_{m_j} E^\pm \subset E^\pm$). Since $B(u_0)$ is compact, $P_{m_j} B(u_0) v_j \to B(u_0)v$. Hence $v_j^\pm \to v^\pm$; therefore $v_j \to v$ and $Lv = v^+ - v^- - B(u_0)v = 0$. This is impossible because $v \in R(L)$ and $v \neq 0$. □

Since $\Phi \in C^2(E,\mathbb{R})$, we have the representations

$$\Phi(u) = \Phi(u_0) + \frac{1}{2}\langle L(u-u_0), u-u_0\rangle + \psi(u), \quad \Phi'(u) = L(u-u_0) + \psi'(u),$$

where $\psi \in C^2(E,\mathbb{R})$, $\psi(u_0) = 0$, $\psi'(u_0) = 0$ and $\psi''(u_0) = 0$. Let $Q : E \to R(L)$ be the orthogonal projection. Then any $u \in E$ has a unique representation $u = u_0 + n + v$, where $n \in N(L)$ and $v \in R(L)$. As in Section 3 (comp. (3.1) and (3.2)) we see that there are $\rho > 0$ and a C^1-function $v : B_\rho(0) \cap N(L) \to R(L)$ such that u_0 is the only critical point of Φ in $B_\rho(u_0)$, $v(0) = 0$, $v'(0) = 0$ and, for $n + v \in B_\rho(0)$,

$$Lv + Q\psi'(u_0+n+v) = Q\Phi'(u_0+n+v) = 0 \text{ if and only if } v = v(n). \tag{5.2}$$

Lemma 5.2 *Φ satisfies the $(PS)^*$-condition (with respect to (E_m)) on $B_\rho(u_0)$.*

Proof Let (u_j) be a $(PS)^*$-sequence in $B_\rho(u_0)$, i.e., $u_j \in E_{m_j}$, $m_j \to \infty$ and $P_{m_j}\Phi'(u_j) \to 0$ as $j \to \infty$. Since $\dim N(L) < \infty$, $(I-Q)\Phi'(u_j)$ is strongly convergent to some $z \in N(L)$ after passing to a subsequence, hence also $P_{m_j}(I-Q)\Phi'(u_j) \to z$ (note that $P_m \to I$ uniformly on compact sets as $m \to \infty$). It follows that $P_{m_j} Q\Phi'(u_j) \to -z \in R(L)$. So $z \in N(L) \cap R(L) = \{0\}$, i.e., $z = 0$. Putting $u_j = u_0 + n_j + v_j$, where $n_j \in N(L)$ and $v_j \in R(L)$, we have in particular

$$w_j := P_{m_j} Q\Phi'(u_j) = P_{m_j} Lv_j + P_{m_j} Q\psi'(u_0 + n_j + v_j) \to 0 \text{ as } j \to \infty.$$

Note that $u_0, u_j, n_j \in E_{m_j}$, hence $v_j \in E_{m_j}$ as well. Set $h_j := v_j - v(n_j)$. Then

$$w_j = P_{m_j} Lv(n_j) + P_{m_j} Lh_j + P_{m_j} Q\psi'(u_0 + n_j + v(n_j) + h_j).$$

By (5.2), $Lv(n_j) + Q\psi'(u_0 + n_j + v(n_j)) = 0$; hence

$$w_j = P_{m_j} Lh_j + P_{m_j} Q[\psi'(u_0 + n_j + v(n_j) + h_j) - \psi'(u_0 + n_j + v(n_j))]. \tag{5.3}$$

Passing to a subsequence, $n_j \to n$, and it follows that $P_{m_j} v(n_j) \to v(n)$ and $h_j - P_{m_j} h_j = v(n_j) - P_{m_j} v(n_j) \to 0$. Using this and Lemma 5.1, we have

$$\|P_{m_j} Lh_j\| \geqslant \|P_{m_j} LP_{m_j} h_j\| + o(1) \geqslant \alpha \|P_{m_j} h_j\| + o(1) = \alpha \|h_j\| + o(1)$$

for almost all j. Since $\psi \in C^2$ and $\psi''(u_0) = 0$, taking ρ smaller if necessary we see from (5.3) that

$$\begin{aligned} \alpha\|h_j\| + o(1) &\leqslant \|P_{m_j} Lh_j\| \leqslant \|w_j\| + \|\psi'(u_0 + n_j + v(n_j) + h_j) - \psi'(u_0 + n_j + v(n_j))\| \\ &\leqslant \|w_j\| + \frac{\alpha}{2}\|h_j\|. \end{aligned}$$

Hence $h_j \to 0$ and we see that

$$u_j = u_0 + n_j + v(n_j) + h_j \to u_0 + n + v(n) \text{ as } j \to \infty.$$

This completes the proof. □

For $m \geqslant 1$, let $d_m := m$ and $\mathcal{E} := (E_m, d_m)_{m=1}^{\infty}$.

Theorem 5.3 *If the ground state u_0 is an isolated critical point of Φ, then $c_1^{\mathcal{E}}(\Phi, u_0) \neq 0$.*

Proof There is $\rho > 0$ such that u_0 is the only critical point of Φ in $B_\rho(u_0)$ and Φ satisfies the $(PS)^*$-condition with respect to the filtration (E_m) on $B_\rho(u_0)$.

Recall from Section 4 that $\widehat{m} : S^+ \to \mathcal{M}$ is a homeomorphism and $\widehat{m}(w) = tw + v$ for some $t \geqslant \alpha > 0$ (α independent of $w \in S^+$) and $v \in E^-$. Furthermore, $\widehat{m}^{-1}(u) = u^+/\|u^+\|$ for any $u \in \mathcal{M}$. Hence $w_0 := \widehat{m}^{-1}(u_0) \in E_m$ for all $m \geqslant 1$ and $E(u_0) \equiv E^- \oplus \mathbb{R}u_0 = E(w_0)$ (cf. (4.4)). Given $\delta > 0$, let

$$E_\delta(w) := \{u \in E(w) : \|u - \widehat{m}(w)\| < \delta\}$$

and

$$U_\delta := \bigcup_{w \in B_\delta(w_0) \cap S^+} E_\delta(w).$$

It is easy to see that if δ is small enough (in particular, $\delta < \rho$), then U_δ is an open neighborhood of u_0 and $U_\delta \subset B_\rho(u_0)$. Hence according to Proposition 3.6, there is an admissible pair (W, W^-)

for Φ and u_0 such that $W \subset U_\delta$ and $\sup_{W^-} \Phi < c = \Phi(u_0)$. Moreover, there is $\delta_1 \in (0, \delta)$ such that $\overline{B}_{\delta_1}(u_0) \subset \operatorname{int} W$ and, for some $t > 0$, $\eta(t, u) \in W^-$ provided $u \in S_{\delta_1}(u_0) \cap \Phi^c$; as before η is the flow of $-V$, where V is an admissible gradient-like vector field corresponding to (W, W^-).

Set

$$S := S_{\delta_1}(u_0) \cap E(u_0), \quad D := \bar{B}_{\delta_1}(u_0) \cap E(u_0)$$

and

$$A := \{\eta(t, u) \in W : t \geqslant 0,\ u \in S\}.$$

In view of Proposition 4.2, $\Phi(u) < c$ for $u \in S$. Hence, for each $u \in A$, $\Phi(u) < c$ and there is a unique $t(u) > 0$ such that $\eta(t(u), u) \in W^-$. According to (iii) of Definition 3.5, $t(u)$ depends continuously on u; thus the map $\gamma : (W^- \cup A) \times [0, 1] \to W^-$ given by

$$\gamma(u, \lambda) := \begin{cases} \eta(\lambda t(u), u) & \text{if } u \in A,\ \lambda \in [0, 1], \\ u & \text{if } u \in W^-,\ \lambda \in [0, 1], \end{cases}$$

provides a filtration-preserving strong deformation retraction of $W_A^- := W^- \cup A$ onto W^-. The exactness of the homology sequence of the triple (W, W_A^-, W^-) implies that

$$H_*^{\mathcal{E}}(W, W^-) \cong H_*^{\mathcal{E}}(W, W_A^-).$$

Since $E(u_0) \cap E_m = E_m^- \oplus E_1^+$, $\dim(D \cap E_m) = m + 1$. In view of Example 3.2,

$$H_q^{\mathcal{E}}(D, S) = H_{q-1}^{\mathcal{E}}(S) = \begin{cases} \mathcal{F} & \text{if } q = 1; \\ 0 & \text{otherwise} \end{cases}$$

and $\partial : H_1^{\mathcal{E}}(D, S) \to H_0^{\mathcal{E}}(S)$ is an isomorphism. Consider the diagram

$$\begin{tikzcd} H_1^{\mathcal{E}}(D,S) \arrow[r, "\partial"] \arrow[d] & H_0^{\mathcal{E}}(S) \arrow[d] \\ H_1^{\mathcal{E}}(W, W_A^-) \arrow[r, "\partial"] & H_0^{\mathcal{E}}(W_A^-) \end{tikzcd}$$

where the vertical arrows are induced by the respective inclusions. It is clear that in order to prove that $H_1^{\mathcal{E}}(W, W^-) \cong H_1^{\mathcal{E}}(W, W_A^-) \neq 0$ it is sufficient to show that the homomorphism $i_* : H_0^{\mathcal{E}}(S) \to H_0^{\mathcal{E}}(W_A^-)$, induced by the inclusion $i : S \hookrightarrow W_A^-$, is nontrivial.

Let $\pi : U_\delta \to E(u_0)$ be the map given by

$$\pi(tw + v) := (t - t(w) + t(w_0))w_0 + (v - v(w) + v(w_0)),$$

where $u = tw + v \in U_\delta$, $w \in B_\delta(w_0) \cap S^+$, $v \in E^-$ and $\widehat{m}(w) = t(w)w + v(w)$, $\widehat{m}(w_0) = t(w_0)w_0 + v(w_0)$. It is easy to see that π is continuous and since $u \in U_\delta \cap E_m$ if and only if $w \in E_m^+$ and $v \in E_m^-$, π is filtration-preserving. Moreover, $\pi|_S$ is the identity map on S and $\pi(u) = u_0$ if and only if $u = \widehat{m}(w) \in \mathcal{M} \cap U_\delta$. Hence $\pi : (W_A^-, S) \to (E(u_0) \setminus \{u_0\}, S)$ and we have the commutative diagram

$$\begin{tikzcd} H_1^{\mathcal{E}}(W_A^-, S) \arrow[r, "\partial"] \arrow[d, "\pi"'] & H_0^{\mathcal{E}}(S) \arrow[r, "i_*"] \arrow[d, "\pi"', "\cong"] & H_0^{\mathcal{E}}(W_A^-) \\ H_1^{\mathcal{E}}(E(u_0) \setminus \{u_0\}, S) \arrow[r, "\partial"] & H_0^{\mathcal{E}}(S) & \end{tikzcd}$$

where the lower left element is 0 because $E(u_0) \setminus \{u_0\}$ can be radially deformation retracted onto S. Therefore $\partial = \partial^{(W_A^-, S)}$ is trivial and it follows that i_* is a monomorphism. In particular, i_* is nontrivial. This completes the proof. □

Theorem 5.4 *Suppose $N \geqslant 4$, (S_1) is satisfied and $f(x,u) = |u|^{2^*-2}u$. Then the conclusion of Theorem 5.3 remains valid.*

Proof Let

$$\langle B(u)v, w\rangle := (2^* - 1)\int_{\mathbb{R}^N} |u|^{2^*-2} vw \, dx.$$

$B(u)$ is no longer a compact operator for all u, however, $B(u_0)$ is compact. Indeed, for R large enough (5.1) still holds; however, in the middle term ε should be replaced by 0, p by 2^* and $\bar{a}_\varepsilon$ by $2^* - 1$. Since $u_0 \in L^\infty(\mathbb{R}^N)$ (see [9]) and $v_j \to 0$ in $L^2(B_R(0))$, $\int_{B_R(0)} |u_0|^{2^*-2} v_j w \, dx \to 0$ uniformly in w, $\|w\| \leqslant 1$. Taking this into account, the arguments of Lemmas 5.1 and 5.2 go through unchanged and so does the argument of Theorem 5.3. □

6 Multibump solutions

Let $\theta = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$ and

$$(\theta * u)(x) := u(x - \theta).$$

If $u \in E \equiv H^1(\mathbb{R}^N)$ is a solution of (4.1), then so is $\theta * u$ for any $\theta \in \mathbb{Z}^N$ as follows from the periodicity of V and f. Suppose now u_0 is a minimizer of Φ on $\mathcal{M}$. Then u_0 solves (4.1) and we will be interested in solutions which are of the form

$$\bar{u} = \theta_1 * u_0 + \cdots + \theta_k * u_0 + v,$$

where $\theta_j \in \mathbb{Z}^N$, $|\theta_i - \theta_j|$ are large enough for $i \neq j$ and v is suitably small. Such $\bar{u}$ will be called a *k-bump solution.*

Theorem 6.1 *Suppose the hypotheses (S_1), (S_2'), (S_3)-(S_5) are satisfied and u_0 is a ground state solution of (4.1), isolated in the set of critical points of Φ. For each $k \geqslant 2$ and $\delta_0 > 0$ there exists $a \in \mathbb{N}$ with the property that if $\theta_1, \theta_2, \dots, \theta_k \in \mathbb{Z}^N$ and $|\theta_i - \theta_j| \geqslant a$ for all $i \neq j$, then there is $v \in E$ such that $\|v\| \leqslant \delta_0$ and $\bar{u} = \theta_1 * u_0 + \cdots + \theta_k * u_0 + v$ is a solution of (4.1).*

Remark 6.2 (i) A similar result, with a independent of k, has been obtained in [3] (see Theorem 6.1 there). However, while here 0 is in a gap of the spectrum of $-\Delta + V$, in [3] $\sigma(-\Delta + V) \subset (0, \infty)$. Also the assumptions on f are somewhat different. Although we believe that under the assumptions above it should still be possible to obtain a k-independent lower bound for a, there are some technical difficulties which we make no attempt to resolve.

(ii) In two recent papers [1, 11] results similar to our Theorem 6.1 have been proved. In [1] the assumptions corresponding to (S_2') and (S_4) are stronger ((S_4) is replaced by the Ambrosetti-Rabinowitz superlinearity condition). Using a version of (S_5), the functional is reduced to another one (on E^+) which has the mountain pass geometry, and a degree-theoretical argument is then employed in order to construct multibumps. In [11] no assumption like (S_5) has been made. On the other hand, the condition corresponding to (S_2') is somewhat stronger there and the Ambrosetti-Rabinowitz condition replaces our (S_4). Moreover, our argument is more direct (it does not use periodic approximations like in [11]).

(iii) In Theorem 6.3 below we formulate an analogue of Theorem 6.1 for the critical Sobolev exponent. This result seems to be new.

Theorem 6.3 *Suppose $N \geqslant 4$, (S_1) is satisfied and $f(x,u) = |u|^{2^*-2}u$. Then the conclusion of Theorem 6.1 remains valid.*

Below we describe the main ideas of the proofs of Theorems 6.1 and 6.3 and postpone the technical details to the next section. We assume that the kernel $N(L)$ of $L = \Phi''(u_0)$ is nontrivial. If $N(L) = \{0\}$, then u_0 is a nondegenerate critical point of Φ and the argument becomes considerably simpler (cf. Remark 2.13 in [3]). Moreover, a stronger conclusion is then known to hold [2].

Let $v = v(n)$, $n \in N(L)$, be as in (5.2) and set $\varphi(n) := \Phi(u_0 + n + v(n)) - \Phi(u_0)$, $\|n\| \leqslant \delta$ ($\delta > 0$ small enough), cf. (3.3). Then $\varphi'(n) = 0$ if and only if $\Phi'(u_0 + n + v(n)) = 0$ and we may assume choosing a smaller δ if necessary that $\varphi'(n) = 0$ if and only if $n = 0$. By Theorems 5.3 and 5.4, $c_1^{\mathcal{E}}(\Phi, u_0) \neq 0$, hence $c_r(\varphi, 0) \neq 0$ for some $r \geqslant 0$ according to Theorem 3.7.

Given $a > 0$, let

$$\Theta_k^a := \{\theta = (\theta_1, \ldots, \theta_k) \in \mathbb{Z}^{Nk} : |\theta_i - \theta_j| \geqslant a \text{ if } i \neq j\}$$

and

$$\|u\|^2 := \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\, dx. \tag{6.1}$$

We emphasize that here we use the original $H^1(\mathbb{R}^N)$-norm and not the one introduced in Section 4. Hence in particular, (4.3) does not hold. The reason for choosing this norm is that it has certain local properties which will be needed in Section 7.

Let $\omega \in C_0^\infty(\mathbb{R}, [0,1])$ be a (cutoff) function such that $\omega(t) = 1$ for $|t| \leqslant 1/8$ and $\omega(t) = 0$ for $|t| \geqslant 1/4$. Put

$$u^a(x) := \omega(|x|/a)u(x) \quad (a > 0),$$

and for $\theta_0 \in \mathbb{Z}^N$ and a set $S \subset E$,

$$S^a := \{u^a : u \in S\}, \quad \theta_0 * S := \{\theta_0 * u : u \in S\}.$$

We see that if $\theta \in \Theta_k^a$, then $\theta_i * u^a$ and $\theta_j * u^a$ have disjoint supports unless $i = j$.

Let $\theta = (\theta_1, \ldots, \theta_k) \in \Theta_k^a$ and

$$V_a = V_a(\theta) := \left(\bigoplus_{i=1}^k \theta_i * N(L)^a \right)^\perp, \qquad P_a = P_a(\theta) : E \to V_a,$$

where P_a is the orthogonal projection on V_a. The direct sum above is indeed well defined since the functions corresponding to different indices i have disjoint supports. Put $z = (n_1, \ldots, n_k) \in$

$N(L)^k$. Using some technical estimates (see Lemmas 7.1, 7.2) and the contraction mapping principle (Lemma 7.3) it is shown in Corollary 7.4 that if δ is sufficiently small, $a_1 \in \mathbb{N}$ sufficiently large, $a \in \mathbb{N}$, $a \geqslant a_1$ and $\|n_i\| \leqslant \delta$, then there is a unique $w = w(\theta, z) \in V_a$ such that $\|w(\theta, z)\| \leqslant \delta$ and

$$P_a \Phi' \left(\sum_{i=1}^{k} \theta_i * (u_0 + n_i + v(n_i))^a + w(\theta, z) \right) = 0. \tag{6.2}$$

Here a_1 depends on δ but not on the particular choice of θ and z. Set $\zeta := \sum_{i=1}^k \theta_i * (u_0 + n_i + v(n_i))^a + w(\theta, z)$ and

$$\widetilde{\varphi}(z) := \Phi(\zeta). \tag{6.3}$$

Using (6.2) one shows (Lemma 7.5) that $\widetilde{\varphi}'(z) = 0$ if and only if $\Phi'(\zeta) = 0$ provided δ is small enough. If z is a critical point of $\widetilde{\varphi}$, then $\bar{u} = \zeta$ is a solution of (4.1) which has the required form, i.e.,

$$\bar{u} = \sum_{i=1}^{k} \theta_i * (u_0 + n_i + v(n_i))^a + w(\theta, z) \equiv \sum_{i=1}^{k} \theta_i * u_0 + v,$$

where

$$v = \sum_{i=1}^{k} \theta_i * (u_0^a - u_0) + \sum_{i=1}^{k} \theta_i * (n_i + v(n_i))^a + w(\theta, z).$$

Since $u_0^a \to u_0$ in E as $a \to \infty$, the first sum above can be made as small as we wish. The same is true of the second sum because $\|n_i\| \leqslant \delta$, $v(0) = 0$ and $\|u^a\| \leqslant c\|u\|$ for all u, where the constant c is independent of $u \in E$ and $a \geqslant a_1$. Finally, $\|w(\theta, z)\| \leqslant \delta$ and it follows that $\|v\| \leqslant \delta_0$ provided δ is small and a_1 large enough.

It remains to show that $\widetilde{\varphi}$ indeed has a critical point. If δ is small enough, then 0 is the only critical point of φ in the set $\|n\| \leqslant \delta$ and we can find an admissible pair (W, W^-) for φ and 0. Moreover, we may assume (W, W^-) is a pair of ANR's, see [14] and the end of Section 3. Let

$$(\widetilde{W}, \widetilde{W}^-) := (W, W^-)^k = \underbrace{(W, W^-) \times \cdots \times (W, W^-)}_{k \text{ times}}$$

(recall that $(A, C) \times (B, D) := (A \times B, A \times D \cup C \times B)$). We shall show in Lemma 7.6 that choosing a larger a_1 if necessary, $(\widetilde{W}, \widetilde{W}^-)$ is an admissible pair for $\widetilde{\varphi}$ and the (possibly empty) set K of critical points contained in the interior of $\widetilde{W}$. By Künneth's formula [15, Corollary VI.12.12] (cf. the argument at the end of Section 3),

$$H_*^s(\widetilde{W}, \widetilde{W}^-) = H_*^s(W, W^-) \otimes \cdots \otimes H_*^s(W, W^-).$$

Since $c_r(\varphi, 0) \neq 0$, $H_{kr}^s(\widetilde{W}, \widetilde{W}^-) \neq 0$ and it follows that $K \neq \emptyset$ (otherwise $\widetilde{W}^-$ is a strong deformation retract of $\widetilde{W}$ by a standard argument, hence $H_*^s(\widetilde{W}, \widetilde{W}^-) = 0$).

We remark that if (S_5) is replaced by the somewhat stronger condition $f'(u)u^2 > f(u)u > 0$ for all $u \neq 0$ (which is certainly satisfied if $f(x, u) = |u|^{2^*-2}u$), then it can be shown that $\mathcal{M} \in C^1$ and it is easy to see that $N(L) \subset T_{u_0}\mathcal{M}$ and $M_{\mathcal{E}}^-(L) = 1$. So $c_0(\varphi, 0) \neq 0$, hence 0 is the minimum of φ and $W^- = \emptyset$. Since we make no use of this fact, we leave out the details.

7 Details of proofs of Theorems 6.1 and 6.3

The arguments we provide below are taken from the proof of Theorem 1.1 in [3] but are simpler because we allow a to be dependent of k. We also make use of some ideas which may be found in [7]. Recall that the norm in E we use here is given by (6.1). We first consider f satisfying (S_2)-(S_5) and at the end of the section we point out what needs to be changed if $f(x,u) = |u|^{2^*-2}u$.

Let $\widetilde{L} : E \to E$ be the operator defined by

$$\langle \widetilde{L}w, v\rangle = \int_{\mathbb{R}^N} (\nabla w \cdot \nabla v + V(x)wv)\, dx$$

and note that

$$\langle Lw, v\rangle = \langle \Phi''(u_0)w, v\rangle = \langle \widetilde{L}w, v\rangle - \int_{\mathbb{R}^N} f'_u(x, u_0)wv\, dx.$$

For notational convenience we let $k = 2$; the case of $k > 2$ is treated in the same way.

Lemma 7.1 *There exist $c > 0$, $\delta > 0$ and $a_0 \in \mathbb{N}$ such that*

$$\|P_a\Phi''(\theta_1 * u_0^a + \theta_2 * u_0^a + u)w\| \geqslant c\|w\| \tag{7.1}$$

for all $a \in \mathbb{N}$, $a \geqslant a_0$, $\theta \in \Theta_2^a$, $w \in V_a$ and $u \in E$, $\|u\| \leqslant \delta$.

Proof We first show that

$$\|\Phi''(\theta_1 * u_0^a + \theta_2 * u_0^a + u)w\| \geqslant c\|w\|. \tag{7.2}$$

Arguing by contradiction, we can find $a_m \to \infty$, $\theta^m = (\theta_1^m, \theta_2^m) \in \Theta_2^{a_m}$, $u_m \in E$ and $w_m \in V_{a_m}$ such that $u_m \to 0$, $\|w_m\| = 1$ and

$$\Phi''(\theta_1^m * u_0^{a_m} + \theta_2^m * u_0^{a_m} + u_m)w_m \to 0 \text{ as } m \to \infty. \tag{7.3}$$

By the $\mathbb{Z}^N$-invariance of Φ we may assume $\theta_1^m = 0$ for all m. Then V_{a_m} is orthogonal to $N(L)^{a_m}$, so passing to a subsequence, $w_m \rightharpoonup w$ and

$$0 = \langle w_m, z^{a_m}\rangle \to \langle w, z\rangle \text{ for all } z \in N(L).$$

Thus $w \in R(L)$. Since

$$\langle \Phi''(u_0^{a_m} + \theta_2^m * u_0^{a_m} + u_m)w_m,\, v\rangle = \langle \widetilde{L}w_m, v\rangle - \int_{\mathbb{R}^N} f'_u(x, u_0^{a_m} + \theta_2^m * u_0^{a_m} + u_m)w_m v\, dx, \tag{7.4}$$

$|\theta_2^m| \to \infty$ and $u_m \to 0$, we see letting $m \to \infty$ that

$$\langle Lw, v\rangle = \langle \widetilde{L}w, v\rangle - \int_{\mathbb{R}^N} f'_u(x, u_0)wv\, dx = 0 \text{ for all } v \in C_0^\infty(\mathbb{R}^N).$$

Hence $w = 0$ and $w_m \rightharpoonup 0$. Replacing w_m by $-\theta_2^m * w_m$ we see passing to a subsequence once more that also $-\theta_2^m * w_m \rightharpoonup 0$. Next we show that

$$\int_{\mathbb{R}^N} f'_u(x, u_0^{a_m} + \theta_2^m * u_0^{a_m} + u_m)w_m v_m\, dx \to 0, \tag{7.5}$$

where $v_m := \widetilde{L}w_m$. Since $w_m \rightharpoonup 0$ and $-\theta_2^m * w_m \rightharpoonup 0$ in E, $w_m \to 0$ and $-\theta_2^m * w_m \to 0$ in $L^q(B_R(0))$ for any $R > 0$ and $1 \leqslant q < 2^*$. It follows that

$$\int_{B_R(0)\cup B_R(\theta_2^m)} f'_u(x, u_0^{a_m} + \theta_2^m * u_0^{a_m} + u_m) w_m v_m \, dx \to 0.$$

Let $Q := \mathbb{R}^N \setminus (B_R(0) \cup B_R(\theta_2^m))$. It is well known that $u_0(x) \to 0$ as $|x| \to \infty$ (in fact exponentially, see e.g. [26]). By (S_2') and (S_3), for each $\varepsilon_0 > 0$ we can find $\bar{a}_{\varepsilon_0}$ such that $|f'_u(x,u)| \leqslant \varepsilon_0 + \bar{a}_{\varepsilon_0}|u|^{p-2}$ (cf. (5.1)). Hence we see from the Hölder and Sobolev inequalities that given $\varepsilon > 0$, there exists $R > 0$ for which

$$\begin{aligned}
&\int_Q |f'_u(x, u_0^{a_m} + \theta_2^m * u_0^{a_m} + u_m) w_m v_m| \, dx \\
&\qquad \leqslant \int_Q (\varepsilon_0 + \bar{a}_{\varepsilon_0} |u_0^{a_m} + \theta_2^m * u_0^{a_m} + u_m|^{p-2}) |w_m| \, |v_m| \, dx \leqslant \varepsilon \|w_m\|^2 = \varepsilon.
\end{aligned}$$

Now by (7.3)-(7.5) and since $\|\widetilde{L}w_m\| \geqslant \widetilde{c}\|w_m\|$ for some $\widetilde{c} > 0$ (recall $\widetilde{L}$ is invertible), we obtain

$$\widetilde{c}^2 = \widetilde{c}^2 \|w_m\|^2 \leqslant \langle \widetilde{L}w_m, \widetilde{L}w_m \rangle \to 0,$$

a contradiction. Hence (7.2) is satisfied.

Since c may be replaced by $2c$ in (7.2), the conclusion will follow once we prove that if a_0 is large and δ small enough, then

$$\|(I - P_a)\Phi''(\theta_1 * u_0^a + \theta_2 * u_0^a + u)w\| \leqslant c\|w\| \text{ for all } w \in E.$$

Let $\|w\| = 1$ and set $(I - P_a)\Phi''(\theta_1 * u_0^a + \theta_2 * u_0^a + u)w =: z = z_1^a + z_2^a$, where $z_i \in \theta_i * N(L)$, $i = 1, 2$. Assume without loss of generality that $\theta_1 = 0$. Then

$$\begin{aligned}
(7.6) \qquad \|z_1^a\|^2 = \langle z, z_1^a \rangle &= \left\langle \left(\Phi''(u_0^a + \theta_2 * u_0^a + u) - \Phi''(u_0)\right) w,\ z_1^a \right\rangle + \langle \Phi''(u_0)w, z_1^a \rangle \\
&= -\int_{\mathbb{R}^N} \left(f'_u(x, u_0^a + \theta_2 * u_0^a + u) - f'_u(x, u_0)\right) w z_1^a \, dx + \langle \Phi''(u_0)w, z_1^a \rangle.
\end{aligned}$$

Let $\varepsilon > 0$ be given. Since $\Phi''(u_0)z_1 = 0$, the second term on the right-hand side above can be made $\leqslant \varepsilon\|z_1^a\|$ by letting a be sufficiently large. Since $\operatorname{supp}(\theta_2 * u_0^a) \cap \operatorname{supp} z_1^a = \emptyset$,

$$\left| \int_{\mathbb{R}^N} \left(f'_u(x, u_0^a + \theta_2 * u_0^a + u) - f'_u(x, u_0)\right) w z_1^a \, dx \right| \leqslant \int_{\mathbb{R}^N} \left| f'_u(x, u_0^a + u) - f'_u(x, u_0) \right| |w||z_1^a| \, dx.$$

The function f'_u is uniformly continuous on sets of the form $\{(x,u) : |u| \leqslant A\}$, hence choosing ε_0 sufficiently small and a sufficiently large, we obtain

$$\int_{|u|\leqslant\varepsilon_0} \left| f'_u(x, u_0^a + u) - f'_u(x, u_0) \right| |w||z_1^a| \, dx \leqslant \varepsilon \|z_1^a\|.$$

Furthermore, using (S_2'), the fact that $u_0(x) \to 0$ as $|x| \to \infty$ and the Hölder and Sobolev inequalities, we have

$$\begin{aligned}
\int_{|u|>\varepsilon_0} \left| f'_u(x, u_0^a + u) - f'_u(x, u_0) \right| |w||z_1^a| \, dx &\leqslant c_1 \int_{|u|>\varepsilon_0} (1 + |u_0|^{p-2} + |u|^{p-2}) |w||z_1^a| \, dx \\
&\leqslant c_2 \|z_1^a\| \, \mu(|u| > \varepsilon_0)^{(2^*-p)/2^*},
\end{aligned}$$

where μ denotes the measure. Since $\mu(|u| > \varepsilon_0) \to 0$ as $\|u\| \to 0$, the right-hand side above is $\leqslant \varepsilon \|z_1^a\|$ whenever $\|u\| \leqslant \delta$ and δ is small enough. It follows therefore from (7.6) that $\|z_1^a\| \leqslant 3\varepsilon$. Since the same argument applies to z_2^a, we obtain the conclusion. □

Denote the space of bounded linear operators on E by $\mathcal{L}(E)$.

Lemma 7.2 *For each $k \geqslant 2$ the map $P_a\Phi'' : E \to \mathcal{L}(E)$ is uniformly continuous and uniformly bounded on bounded sets. Moreover, the modulus of continuity and the uniform bound are independent of a.*

Proof We have

$$|\langle(\Phi''(u) - \Phi''(\widetilde{u}))w, v\rangle| \leqslant \int_{\mathbb{R}^N} |f'_u(x,u) - f'_u(x,\widetilde{u})|\, |w||v|\, dx. \tag{7.7}$$

Suppose $\|u\|, \|\widetilde{u}\| \leqslant c_1$, $\|w\|, \|v\| \leqslant 1$ and let $\varepsilon > 0$ be given. By (S_2') and the Hölder and Sobolev inequalities,

$$\begin{aligned} \int_{|u|>A} |f'_u(x,u) - f'_u(x,\widetilde{u})|\, |w||v|\, dx &\leqslant c_3 \int_{|u|>A} (1 + |u|^{p-2} + |\widetilde{u}|^{p-2})|w||v|\, dx \\ &\leqslant c_4\, \mu(|u| > A)^{(2^*-p)/2^*}, \end{aligned} \tag{7.8}$$

where c_3, c_4 are independent of A. Hence the right-hand side above can be made $\leqslant \varepsilon$ by taking A large enough, and the same inequality holds on the set $|\widetilde{u}| > A$. By the uniform continuity of f'_u, there exists $\delta_0 > 0$ such that the integral on the right-hand side of (7.7), taken over the set $|u|, |\widetilde{u}| \leqslant A$, $|u - \widetilde{u}| \leqslant \delta_0$, is $\leqslant \varepsilon$. (7.8) still holds with $|u| > A$ replaced by $|u|, |\widetilde{u}| \leqslant A$, $|u - \widetilde{u}| > \delta_0$ and the right-hand side will be $\leqslant \varepsilon$ if $\|u - \widetilde{u}\| \leqslant \delta$ and δ is small enough (because $\mu(|u - \widetilde{u}| > \delta_0) \to 0$ as $\|u - \widetilde{u}\| \to 0$). Hence $\|\Phi''(u) - \Phi''(\widetilde{u})\|_{\mathcal{L}(E)} \leqslant 4\varepsilon$ whenever $\|u - \widetilde{u}\| \leqslant \delta$ and Φ'' is uniformly continuous on bounded sets. That Φ'' is uniformly bounded can now be easily seen by considering $\langle(\Phi''(u) - \Phi''(0))w, v\rangle$.

Since $\|P_a\|_{\mathcal{L}(E)} = 1$, the same conclusions hold for $P_a\Phi''$. □

Let $z = (n_1, n_2) \in N(L) \times N(L)$, $\theta = (\theta_1, \theta_2) \in \Theta_2^a$, $w \in V_a$ and

$$F(\theta, z, w) := P_a\Phi'\left(\theta_1 * (u_0 + n_1 + v(n_1))^a + \theta_2 * (u_0 + n_2 + v(n_2))^a + w\right),$$

where $v(n_i)$ are given by (5.2). Then $w \mapsto F(\theta, z, w) : V_a \to V_a$ and

$$F_w(\theta, z, 0) = P_a\Phi''\left(\theta_1 * (u_0 + n_1 + v(n_1))^a + \theta_2 * (u_0 + n_2 + v(n_2))^a\right).$$

Since $v(n_i) \to 0$ as $n_i \to 0$ and $F_w(\theta, z, 0)$ is self-adjoint, it follows from Lemma 7.1 that if a is large enough and $\|n_i\|$ small enough, then $F_w(\theta, z, 0)$ is invertible and

$$\|F_w(\theta, z, 0)^{-1}v\| \leqslant c_0\|v\|, \tag{7.9}$$

where c_0 is independent of θ and z. Set

$$R(z, w) := w - F_w(\theta, z, 0)^{-1}F(\theta, z, w); \tag{7.10}$$

then $R(z,w) = w$ if and only if $F(\theta, z, w) = 0$. We shall show that this equation can be uniquely solved for w. Our proof follows the usual argument of the implicit function theorem, however, we include it because we need estimates which are uniform with respect to the choice of θ and z.

Lemma 7.3 *Given $k \geqslant 2$ and $\delta_0 > 0$, there exist $a_1 \in \mathbb{N}$ and $\delta \in (0, \delta_0)$ such that if $\|n_i\| \leqslant \delta$ $(i = 1, 2)$, $a \geqslant a_1$ and $\theta \in \Theta_2^a$, then $R(z, \cdot)$ is a contraction on the ball $\|w\| \leqslant \delta$.*

Proof By Lemma 7.2, if $\|w\| \leqslant \delta$, and δ is small enough, then

$$\|F_w(\theta, z, w) - F_w(\theta, z, 0)\|_{\mathcal{L}(E)} \leqslant \frac{1}{3c_0}. \tag{7.11}$$

Since

$$F(\theta, z, 0) = P_a \Phi' \left(\theta_1 * (u_0 + n_1 + v(n_1))^a\right) + P_a \Phi' \left(\theta_2 * (u_0 + n_2 + v(n_2))^a\right),$$

$V_a = [\theta_1 * N(L)^a]^\perp \cap [\theta_2 * N(L)^a]^\perp$ and $Q\Phi'((u_0 + n + v(n))^a) \to 0$ uniformly in n ($\|n\| \leqslant \delta$) as $a \to \infty$, it is easy to see that $F(\theta, z, 0) \to 0$ uniformly in θ, z as $a \to \infty$. We may therefore choose a_1 so that

$$\|F(\theta, z, 0)\| \leqslant \frac{\delta}{3c_0} \tag{7.12}$$

whenever $a \geqslant a_1$. Since

$$R(z,w) = -F_w(\theta, z, 0)^{-1} F(\theta, z, 0) - F_w(\theta, z, 0)^{-1} \left(F(\theta, z, w) - F(\theta, z, 0) - F_w(\theta, z, 0)w\right),$$

it follows from (7.9) and (7.11), (7.12) that

$$\begin{aligned} \|R(z,w)\| &\leqslant \|F_w(\theta, z, 0)^{-1}\|_{\mathcal{L}(V_a)} \|F(\theta, z, 0)\| \\ &\quad + \|F_w(\theta, z, 0)^{-1}\|_{\mathcal{L}(V_a)} \|F(\theta, z, w) - F(\theta, z, 0) - F_w(\theta, z, 0)w\| \\ &\leqslant \frac{\delta}{3} + c_0 \int_0^1 \|F_w(\theta, z, sw) - F_w(\theta, z, 0)\|_{\mathcal{L}(V_a)} \|w\| \, ds \leqslant \frac{\delta}{3} + c_0 \frac{\delta}{3c_0}. \end{aligned}$$

Hence R maps the ball $\|w\| \leqslant \delta$ into itself. Also, for $w, \widetilde{w}$ in this ball,

$$\begin{aligned} \|R(z,w) - R(z,\widetilde{w})\| &\leqslant \|F_w(\theta, z, 0)^{-1}\|_{\mathcal{L}(V_a)} \|F(\theta, z, w) - F(\theta, z, \widetilde{w}) - F_w(\theta, z, 0)(w - \widetilde{w})\| \\ &\leqslant c_0 \int_0^1 \|F_w(\theta, z, sw + (1-s)\widetilde{w}) - F_w(\theta, z, 0)\|_{\mathcal{L}(V_a)} \|w - \widetilde{w}\| \, ds \\ &\leqslant \frac{1}{3} \|w - \widetilde{w}\|. \end{aligned} \tag{7.13}$$

It follows that $R(z, \cdot)$ is a contraction as claimed. □

Corollary 7.4 *Given $k \geqslant 2$ and $\delta_0 > 0$, there exist $a_1 \in \mathbb{N}$ and $\delta \in (0, \delta_0)$ such that if $\|n_i\| \leqslant \delta$ $(i = 1, 2)$, $a \geqslant a_1$ and $\theta \in \Theta_2^a$, then there is a unique $w = w(\theta, z) \in V_a$ such that $F(\theta, z, w(\theta, z)) = 0$ and $\|w(\theta, z)\| \leqslant \delta$. Moreover, $w(\theta, z)$ is of class C^1 and $w(\theta, z) \to 0$ uniformly in θ, z as $a \to \infty$.*

Proof In view of the preceding lemma, existence and uniqueness of w follow from the contraction mapping principle. Moreover, $F_w(\theta, z, w)$ is invertible according to (7.9) and (7.11), hence $z \mapsto w(\theta, z)$ is of class C^1 by the implicit function theorem.

Let $w = w(\theta, z)$ and $\widetilde{w} = 0$ in (7.13). Then $\|w\| = \|R(z, w)\| \leqslant \|R(z, 0)\| + \frac{1}{3}\|w\|$ and by (7.10),

$$\|w\| \leqslant \frac{3}{2}\|R(z,0)\| \leqslant \frac{3}{2}\|F_w(\theta, z, 0)^{-1}\|_{\mathcal{L}(V_a)}\|F(\theta, z, 0)\| \to 0 \text{ as } a \to \infty.$$

□

Set $D_\delta := \{z = (n_1, n_2) \in N(L) \times N(L) : \|n_i\| \leqslant \delta\}$ and (cf. (6.3))

$$\widetilde{\varphi}(z) := \Phi(\zeta), \text{ where } \zeta := \theta_1 * (u_0 + n_1 + v(n_1))^a + \theta_2 * (u_0 + n_2 + v(n_2))^a + w(\theta, z).$$

Lemma 7.5 *Given $k \geqslant 2$ and $\delta_0 > 0$, there exist $a_1 \in \mathbb{N}$ and $\delta \in (0, \delta_0)$ such that whenever $a \geqslant a_1$, $\theta \in \Theta_2^a$ and $z \in D_\delta$, then $\widetilde{\varphi}'(z) = 0$ if and only if $\Phi'(\zeta) = 0$.*

Proof Since $P_a\Phi'(\zeta) = F(\theta, z, w(\theta, z)) = 0$ and $w'(\theta, z)$ maps $N(L) \times N(L)$ into V_a, for each $y = (m_1, m_2) \in N(L) \times N(L)$ we have

$$\begin{aligned} \widetilde{\varphi}'(z) \cdot y &= \left\langle \Phi'(\zeta),\, \theta_1 * (m_1 + v'(n_1)m_1)^a + \theta_2 * (m_2 + v'(n_2)m_2)^a + w'(\theta, z)y \right\rangle \\ &= \left\langle (I - P_a)\Phi'(\zeta),\, \theta_1 * (m_1 + v'(n_1)m_1)^a + \theta_2 * (m_2 + v'(n_2)m_2)^a \right\rangle. \end{aligned} \tag{7.14}$$

Clearly, if $\Phi'(\zeta) = 0$, then $\widetilde{\varphi}'(z) = 0$.

Suppose $\widetilde{\varphi}'(z) = 0$ and let $(I - P_a)\Phi'(\zeta) = \xi_1^a + \xi_2^a$, where $\xi_i \in \theta_i * N(L)$. Choosing $y = (\xi_1, 0)$ and assuming without loss of generality $\theta_1 = 0$, we obtain

$$0 = \widetilde{\varphi}'(z) \cdot y = \langle \xi_1^a + \xi_2^a,\, (\xi_1 + v'(n_1)\xi_1)^a\rangle = \|\xi_1^a\|^2 + \langle \xi_1^a, (v'(n_1)\xi_1)^a\rangle.$$

Since $v'(n_1)\xi_1 \in R(L)$, $\langle \xi_1^a, (v'(n_1)\xi_1)^a\rangle \geqslant -\frac{1}{2}\|\xi_1^a\|^2$ provided a is large enough. Hence $\xi_1^a = 0$ and similarly, $\xi_2^a = 0$. □

Let Y be a pseudo-gradient vector field for φ and (W, W^-) a corresponding Gromoll-Meyer pair contained in the ball $\|n\| < \delta$ (recall that Gromoll-Meyer and admissible pairs coincide in finite-dimensional spaces).

Lemma 7.6 *There exist $a_2 \geqslant a_1$ and $\delta_1 < \delta_0$ such that if $a \geqslant a_2$ and $0 < \delta < \delta_1$, then $(\widetilde{W}, \widetilde{W}^-) := (W, W^-) \times (W, W^-)$ is a Gromoll-Meyer pair for $\widetilde{\varphi}$ and the (possibly empty) set K of critical points contained in the interior of $\widetilde{W}$.*

Proof Clearly, $\widetilde{W} \subset D_\delta$. Let η be the flow given by

$$\frac{d\eta}{dt} = -\chi(\eta)Y(\eta), \quad \eta(0, n) = n,$$

where $n \in N(L) \cap B_\delta(0)$, $\chi \in C^\infty(B_\delta(0), [0, 1])$ and $\chi = 0$ close to $n = 0$, $\chi = 1$ close to the boundary of W. Further, let $z = (n_1, n_2) \in N(L) \times N(L)$ and $\widetilde{Y}(z) = (\chi(n_1)Y(n_1), \chi(n_2)Y(n_2))$. The flow of $-\widetilde{Y}$ is

$$\widetilde{\eta}(s, z) = (\eta(s, n_1), \eta(s, n_2)).$$

It is clear that $\widetilde{\eta}$ can leave $\widetilde{W}$ only through $\widetilde{W}^-$ and $s \mapsto \eta(s,z)$ is transversal to $\widetilde{W}^-$. Also, $\widetilde{Y}$ is bounded. We shall show that $\widetilde{\varphi}'(z) \cdot \widetilde{Y}(z) \geqslant \varepsilon$ for some $\varepsilon > 0$ if z is close to the boundary of $\widetilde{W}$. This will complete the proof because using partition of unity, Y can be modified in the interior of $\widetilde{W}$ so that it becomes a pseudo-gradient field in $\widetilde{W} \setminus K$.

Let $y = (m_1, 0) \in N(L) \times N(L)$ and assume without loss of generality that $\theta_1 = 0$. Since $P_a\Phi'(\zeta) = 0$ and the supports of $\theta_2 * (n_2 + v(n_2))^a$ and $(m_1 + v'(n_1)m_1)^a$ are disjoint, it follows from (7.14) that

$$\begin{aligned}\widetilde{\varphi}'(z) \cdot y = \langle \Phi'(\zeta), (m_1 + v'(n_1)m_1)^a \rangle &= \langle \Phi'((u_0 + n_1 + v(n_1))^a), (m_1 + v'(n_1)m_1)^a \rangle \\ &+ \langle \Phi'((u_0 + n_1 + v(n_1))^a + w(\theta, z)) - \Phi'((u_0 + n_1 + v(n_1))^a), (m_1 + v'(n_1)m_1)^a \rangle.\end{aligned}$$

Take $m_1 = \chi(n_1)Y(n_1)$. Since $u^a \to u$ uniformly on compact sets as $a \to \infty$, $v'(0) = 0$ and $\varphi(n_1) = \Phi(u_0 + n_1 + v(n_1)) - \Phi(u_0)$, it is easy to see that the first term on the right-hand side above is larger than or equal to $2\varepsilon\chi(n_1)$ for some $\varepsilon > 0$ provided a is large and δ small ($a \geqslant a_2$, $0 < \delta < \delta_1$). Since $w(\theta, z) \to 0$ as $a \to \infty$, the second term can be made smaller than $\varepsilon/2$. The same argument applies to $y = (0, m_2)$, hence

$$\widetilde{\varphi}'(z) \cdot \widetilde{Y}(z) = \widetilde{\varphi}'(z) \cdot (\chi(n_1)Y(n_1), \chi(n_2)Y(n_2)) \geqslant \varepsilon$$

for z close to the boundary of $\widetilde{W}$ because $\chi(n_1) + \chi(n_2) \geqslant 1$ there. □

Finally we describe the changes that need to be made in the arguments above if $f(x,u) = |u|^{2^*-2}u$.

In Lemma 7.1 the arguments of (7.5) and (7.6) require some modifications. Since $u_0^{a_m} + u_m \to u_0$ and $w_m \rightharpoonup 0$ as $m \to \infty$, it is easy to see that

$$\int_{B_R(0)} |u_0^{a_m} + \theta_2^m * u_0^{a_m} + u_m|^{2^*-2} w_m v_m \, dx = \int_{B_R(0)} |u_0^{a_m} + u_m|^{2^*-2} w_m v_m \, dx \to 0$$

and

$$\int_{B_R(\theta_2^m)} |u_0^{a_m} + \theta_2^m * u_0^{a_m} + u_m|^{2^*-2} w_m v_m \, dx = \int_{B_R(0)} |u_0^{a_m} + \widetilde{u}_m|^{2^*-2} \widetilde{w}_m \widetilde{v}_m \, dx \to 0,$$

where tilde denotes translation by $-\theta_2^m$. We complete the proof of (7.5) by noting that

$$\int_Q |u_0^{a_m} + \theta_2^m * u_0^{a_m} + u_m|^{2^*-2} |w_m||v_m| \, dx \leqslant \varepsilon \|w_m\|^2 = \varepsilon$$

for R large enough because $u_0(x) \to 0$ as $|x| \to \infty$ (see [9]). In (7.6) it suffices to show that for each $\varepsilon > 0$ there are δ and a_0 such that

$$\begin{aligned}(2^* - 2) &\int_{\mathbb{R}^N} \left| |u_0^a + \theta_2 * u_0^a + u|^{2^*-2} - |u_0^a|^{2^*-2} \right| |w||z_1^a| \, dx \\ &= (2^* - 2) \int_{\mathbb{R}^N} \left| |u_0^a + u|^{2^*-2} - |u_0^a|^{2^*-2} \right| |w||z_1^a| \, dx \leqslant \varepsilon \|z_1^a\|\end{aligned}$$

whenever $\|u\| \leqslant \delta$ and $a \geqslant a_0$. However, this follows easily from the calculus inequality

$$\left| |u_0^a + u|^{2^*-2} - |u_0^a|^{2^*-2} \right| \leqslant C(1 + |u_0^a| + |u|)^{2^*-2-\alpha} |u|^\alpha$$

which holds for some $\alpha \in (0, 1]$ and $C > 0$ (one can e.g. choose $\alpha = 2^* - 2$ if $2^* < 3$ and $\alpha = 1$ otherwise).

In Lemma 7.2 we must modify the proof of (7.7) which is easily done by applying the inequality above. Indeed,

$$\int_{\mathbb{R}^N} \left| |u|^{2^*-2} - |\widetilde{u}|^{2^*-2} \right| |w||v| \, dx \leqslant C \int_{\mathbb{R}^N} (1 + |u| + |\widetilde{u}|)^{2^*-2-\alpha} |u - \widetilde{u}|^\alpha |w||v| \, dx,$$

and if $\|u\|, \|\widetilde{u}\| \leqslant c_1$, $\|w\|, \|v\| \leqslant 1$, then the right-hand side above can be made arbitrarily small by letting $\|u - \widetilde{u}\|$ be small enough.

Since no other modifications are necessary, this completes the argument.

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