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MINIMIZERS AND SYMMETRIC MINIMIZERS FOR PROBLEMS WITH CRITICAL SOBOLEV EXPONENT

SHOYEB WALIULLAH

ABSTRACT. In this paper we will be concerned with the existence and non-existence of constrained minimizers in Sobolev spaces $D^{k,p}(\mathbb{R}^N)$, where the constraint involves the critical Sobolev exponent. Minimizing sequences are not, in general, relatively compact for the embedding $D^{k,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N, Q)$ when Q is a non-negative, continuous, bounded function. However if Q has certain symmetry properties then all minimizing sequences are relatively compact in the Sobolev space of appropriately symmetric functions. For Q which does not have the required symmetry, we give a condition under which an equivalent norm in $D^{k,p}(\mathbb{R}^N)$ exists so that all minimizing sequences are relatively compact. In fact we give an example of a Q and an equivalent norm in $D^{k,p}(\mathbb{R}^N)$ so that all minimizing sequences are relatively compact.

1. INTRODUCTION

In this paper we will be concerned with the existence and non-existence of constrained minimizers in Sobolev spaces $D^{k,p}(\mathbb{R}^N)$, where $p > 1$ and the constraint involves the critical Sobolev exponent. It is well known that such minimizers correspond to non-trivial solutions of nonlinear elliptic partial differential equations. After the minimization problem has been formulated one can easily state conditions under which non-trivial solutions to the minimization problem will not exist. One can then go on to state conditions under which the problem will have a solution. In general these conditions are not easy to check, but in some cases this can be done.

We would also like to mention that some of the problems we look at here have already been considered by other authors, but the results presented here are improvements of the existing results, and our method is technically somewhat simpler.

The paper is organized as follows. We initially consider the problem of finding a minimizer associated with the embedding $D^{k,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N, Q)$, with the usual norm in $D^{k,p}(\mathbb{R}^N)$. To this end, we use some preliminary results to establish the well known concentration-compactness lemma. We then give a proof of the known result, that minimizers in general do not exist if Q is not constant and $Q \geq 0$, and in this case minimizing sequences concentrate at the maximum of Q. However, such concentration does not take place if Q has certain symmetry properties, which will be defined later on, and provided we can show that a certain inequality is strict. Examples

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show the existence of Q so that the afore mentioned inequality is strict. In section 7 we apply our results to non-linear partial differential equations to show the existence of solutions. There, we derive some more conditions on Q so that solutions to the partial differential equations exist, and give a result which improves a result given in [6].

In the section following that one we obtain results concerning the weighted Sobolev embedding $D^{k,p}(\mathbb{R}^N, H) \hookrightarrow L^{p^*}(\mathbb{R}^N, Q)$, where we choose the weight H to be a continuous bounded positive function such that $\inf_{x\in\mathbb{R}^N} H > 0$. This ensures that $D^{k,p}(\mathbb{R}^N, H)$ is just the space $D^{k,p}(\mathbb{R}^N)$ equipped with an equivalent norm. We proceed by first proving the existence of minimizers, provided a certain condition is satisfied. An example is then provided to verify the existence of functions H and Q so that the above mentioned condition is satisfied. Before ending the section with a treatment of the symmetric case, we give conditions under which minimizers do not exist.

The final section is devoted to problems with singular weights. These problems arise from the well-known Caffarelli-Kohn-Nirenberg inequality. Our work here generalizes the work in [30] and improves a result in [11].

2. Notation and conventions

In order to keep ourselves from repeating let us state here some notation and conventions we will use throughout this papper. Q will denote a continuous, bounded, non-negative function in $\mathbb{R}^N$. $Q_0 := Q(0)$, $Q_\infty := \overline{\lim}_{|x|\to\infty} Q(x)$ and if we write $Q_\infty = \lim_{|x|\to\infty} Q(x)$ we assume that the limit exists. This distinction is made because many of our results do not require the existence of this limit.

We will denote by G any subgroup of $O(N)$, the group of orthogonal transformations, with the property that $\mathrm{Fix}(G) = \{0\}$, where $\mathrm{Fix}(G) := \{x \in \mathbb{R}^N : gx = x \text{ for all } g \in G\}$ is the fixed point set of the action of G on $\mathbb{R}^N$.

As usual, $D^{k,p}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|\nabla^k u\|_p := (\sum_{|\alpha|=k} \int_{\mathbb{R}^N} |D^\alpha u|^p dx)^{1/p} \tag{2.1}$$

and $|\nabla^k u|^p := \sum_{|\alpha|=k} |D^\alpha u|^p$.

The equivalent norm for $1 < p < \infty$ which will be useful is the following one:

$$\begin{aligned} &\|(-\Delta)^{k/2} u\|_p \quad \text{if } k \text{ is even} \\ &\|\nabla(-\Delta)^{(k-1)/2} u\|_p \quad \text{if } k \text{ is odd.} \end{aligned} \tag{2.2}$$

This is a consequence of the inequality $\|\nabla^2 u\|_p \leq C\|\Delta u\|_p$, which can be found in [15, 22, 24]. Since many of our results are independent of the norm used in $D^{k,p}(\mathbb{R}^N)$, we will denote both of them by $\|u\|_{k,p}$. Where necessary we will specify which norm is being used.

For the sake of convenience we will write $L^p(\mathbb{R}^N, Q) = L^p(Qdx)$ where the norm is denoted by $\|u\|_{p,Q} = (\int |u|^p Q dx)^{1/p}$. Also we wil usually write $\int_\Omega u$ instead of $\int_\Omega u(x)\, dx$, and if no region of integration is mentioned then the integration is to be taken over $\mathbb{R}^N$. Further, following the notations used in distribution theory, we will use the notation $\mu(\phi)$ to mean $\int_{\mathbb{R}^N} \phi d\mu$.

3. Preliminary remarks

We will begin by considering the following minimization problem:

$$\bar{S} = \inf\{\|u\|_{k,p}^p : u \in D^{k,p}(\mathbb{R}^N), \int Q|u|^{p^*} = 1\}, \tag{3.1}$$

where $p^* := \frac{Np}{N-kp}$ is the critical Sobolev exponent and $pk < N$. As we have mentioned, one can use any one of the norms (2.1) or (2.2) in (3.1).

Remark 3.1. By applying the Lagrange multiplier method, we see that any properly normalized minimizer of (3.1), when $k = 1$, solves

$$-\sum_{|\alpha|=1} D^\alpha(|D^\alpha u|^{p-2} D^\alpha u) = Q|u|^{p^*-2}u,$$

if we use the norm in (2.1) and

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = Q(x)|u|^{p^*-2}u,$$

if we use the norm in (2.2). Of course, the value of the constant $\bar{S}$ depends on the norm as well.

Remark 3.2. For general Q we will show that minimizers of (3.1) do not always exist. This is a well-known fact which can be deduced from the work of Lions [20, 21]. Our motivation for presenting it here is to show the contrast between the results when Q does and does not have any symmetry.

When Q is invariant under the action of G we have the following minimization problem

$$\bar{S}_G = \inf\{\|u\|_{k,p}^p : u \in D_G^{k,p}(\mathbb{R}^N), \int Q|u|^{p^*} = 1\}. \tag{3.2}$$

Here $D_G^{k,p}(\mathbb{R}^N)$ is the subspace of $D^{k,p}(\mathbb{R}^N)$ consisting of functions which are G-symmetric (or G-invariant). We say that $u : \mathbb{R}^N \to \mathbb{R}$ is G-symmetric if $u(gx) = u(x)$ for all $g \in G$ and a.e. $x \in \mathbb{R}^N$. In the sequel the minimizers of (3.2) will be called symmetric minimizers.

The partial differential equation associated with (3.2) was studied in [6] when $p = 2$, $k = 1$ and the second norm was used. There the authors used the mountain-pass theorem and the principle of symmetric criticality [31, Theorem 1.28] to show the existence of G-symmetric solutions. Here we will not appeal to the mountain-pass theorem but will use more direct methods. In fact, the results we obtain improve the results given there.

The case when $Q = 1$ was studied by Lions in [20], where it was shown that there exists a $u \neq 0$ which achieves

$$S = \inf_{\substack{u \in D^{k,p}(\mathbb{R}^N)\\ u \neq 0}} \frac{\|u\|_{k,p}^p}{\left(\int |u|^{p^*}\right)^{p/p^*}}.$$

Equivalently we have

$$S = \inf\{\|u\|_{k,p}^p : u \in D^{k,p}(\mathbb{R}^N), \int |u|^{p^*} = 1\}. \tag{3.3}$$

The crucial tool here is the concentration-compactness lemma, originally due to Lions, with extensions made by Bianchi, Chabrowski, Szulkin, Ben-Naoum, Troestler, Willem [6, 20, 21, 31].

4. The concentration-compactness lemma

Before we go on to state and prove the concentration-compactness lemma, we prove a few preliminary results. We note that for every $\epsilon > 0$ there exists a constant $C(\epsilon, p) > 0$ such that

$$||x+y|^p - |x|^p| \leq \epsilon |x|^p + C(\epsilon, p)|y|^p \qquad \forall x, y \in \mathbb{R}. \tag{4.1}$$

Proposition 4.1. *Suppose $kp < N$, $|\alpha| = k$, $\xi \in C_0^\infty(\mathbb{R}^N)$ and $u_n \rightharpoonup 0$ in $D^{k,p}(\mathbb{R}^N)$, then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |D^\alpha(\xi u_n)|^p dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |\xi D^\alpha u_n|^p dx.$$

Proof. The Leibniz formula gives

$$D^\alpha(\xi u_n) = \xi D^\alpha u_n + \sum_{0<\beta\leq\alpha} C_{\alpha,\beta} D^\beta \xi D^{\alpha-\beta} u_n.$$

For $\epsilon > 0$ put $x = \xi D^\alpha u_n$ and $y = \sum_{0<\beta\leq\alpha} C_{\alpha,\beta} D^\beta \xi D^{\alpha-\beta} u_n$ in (4.1) to get

$$\begin{aligned} &||D^\alpha(\xi u_n)|^p - |\xi D^\alpha u_n|^p| \\ &\leq \epsilon |\xi D^\alpha u_n|^p + C(\epsilon, p)|\sum_{0<\beta\leq\alpha} C_{\alpha,\beta} D^\beta \xi D^{\alpha-\beta} u_n|^p. \end{aligned}$$

Now an application of Höder's inequality (for sums) gives

$$\begin{aligned} &||D^\alpha(\xi u_n)|^p - |\xi D^\alpha u_n|^p| \\ &\leq \epsilon |\xi D^\alpha u_n|^p + C_1(\epsilon, p) \sum_{0<\beta\leq\alpha} |C_{\alpha,\beta} D^\beta \xi D^{\alpha-\beta} u_n|^p. \end{aligned}$$

Since $D^\beta \xi \in C_0^\infty(\mathbb{R}^N)$, we have $D^\beta \xi D^{\alpha-\beta} u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $0 < \beta \leq \alpha$, by the Rellich-Kondrachov theorem. So

$$\begin{aligned} &\lim_{n\to\infty} |\int_{\mathbb{R}^N} |D^\alpha(\xi u_n)|^p dx - \int_{\mathbb{R}^N} |\xi D^\alpha u_n|^p dx| \\ &\leq \lim_{n\to\infty} \int_{\mathbb{R}^N} ||D^\alpha(\xi u_n)|^p - |\xi D^\alpha u_n|^p| dx \\ &\leq \epsilon \lim_{n\to\infty} \int_{\mathbb{R}^N} |\xi D^\alpha u_n|^p dx. \end{aligned}$$

Since ϵ is arbitrary, we reach the desired conclusion. □

We next give a proposition which is an essential part in the proof of the concentration-compactness lemma. We provide a slightly different argument than that in Lions [20].

Proposition 4.2. *Let μ,ν be two bounded nonnegative measures on $\mathbb{R}^N$ satisfying for some constant $C \geq 0$*

$$\Big(\int_{\mathbb{R}^N} |\phi|^q d\nu\Big)^{1/q} \leq C\Big(\int_{\mathbb{R}^N} |\phi|^p d\mu\Big)^{1/p}, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N) \tag{4.2}$$

where $1 \le p < q < \infty$, and let μ_s be the atomic part of μ. Then there exists an at most countable set $(x_j)_{j\in J}$ of distinct points in $\mathbb{R}^N$ and a set of numbers $(\nu_j)_{j\in J}$ in $]0,\infty[$ such that

$$\nu = \sum_{j\in J} \nu_j \delta_{x_j}, \qquad \mu_s \ge C^{-p} \sum_{j\in J} \nu_j^{p/q} \delta_{x_j}.$$

Proof. From inequality (4.2) we obtain

$$(\nu(A))^{p/q} \le C^p \mu(A) \qquad \text{for all Borel sets } A.$$

We decompose ν into the atomic and non-atomic parts, i.e. we write

$$\nu = \tilde{\nu} + \sum_{j\in J} \nu_j \delta_{x_j}.$$

The set J is at most countable since ν is a bounded measure. Since $\nu(\{x\}) = \lim_{\epsilon\to 0} \nu(B(x,\epsilon))$, we have

$$(\nu_j)^{p/q} = \nu(\{x_j\})^{p/q} \le C^p \mu(\{x_j\}).$$

We further conclude that $\tilde{\nu}$ is absolutely continuous with respect to μ, and by the Radon-Nikodym theorem $\tilde{\nu} = f\mu$ where $f \in L^1_+(\mu)$. For μ- a.e. x which is not an atom of μ we have

$$C^{-p} f(x)^{p/q} = \lim_{\rho\to 0} \frac{C^{-p}\big(\int_{B_\rho(x)} d\tilde{\nu}\big)^{p/q}}{\big(\int_{B_\rho(x)} d\mu\big)^{p/q}} \le \lim_{\rho\to 0} \big(\int_{B_\rho(x)} d\mu\big)^{(q-p)/q} = 0,$$

Since $\tilde{\nu}$ is atom free and μ has at most countably many atoms, the result follows.

□

We point out here that if the reverse inequality in (4.2) also holds then μ and ν concentrate at a single point (see [20]). Recall the definition (3.1) of $\bar{S}$ and let $M(\mathbb{R}^N)$ denote the space of finite measures in $\mathbb{R}^N$.

Lemma 4.3. *(Concentration-compactness lemma). Let Q be a non-negative continuous bounded function on $\mathbb{R}^N$ and $\{u_n\}_{n=1}^\infty \subset D^{k,p}(\mathbb{R}^N)$ be a sequence such that*

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } D^{k,p}(\mathbb{R}^N) \\ |\nabla^k(u_n - u)|^p &\overset{*}{\rightharpoonup} \mu && \text{in } M(\mathbb{R}^N) \\ Q|(u_n-u)|^{p^*} &\overset{*}{\rightharpoonup} \nu && \text{in } M(\mathbb{R}^N) \\ u_n &\to u && \text{a.e. on } \mathbb{R}^N \end{aligned}$$

and define

$$\begin{aligned} \mu_\infty &:= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|>R} |\nabla^k u_n|^p \ , \\ \nu_\infty &:= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|>R} Q|u_n|^{p^*} \ . \end{aligned} \tag{4.3}$$

If μ_s is the atomic part of μ, then it follows that

$$\nu = \sum_{j\in J} \nu_j \delta_{x_j}, \tag{4.4}$$

$$\|\nu\|^{p/p^*} \le \bar{S}^{-1}\|\mu_s\|, \tag{4.5}$$

$$\nu_\infty^{p/p^*} \le \bar{S}^{-1}\mu_\infty, \tag{4.6}$$

$$\overline{\lim_{n\to\infty}}\|u_n\|_{k,p}^p \ge \|u\|_{k,p}^p + \|\mu_s\| + \mu_\infty, \tag{4.7}$$

$$\overline{\lim_{n\to\infty}}\|u_n\|_{p^*,Q}^{p^*} = \|u\|_{p^*,Q}^{p^*} + \|\nu\| + \nu_\infty. \tag{4.8}$$

Moreover, if $u = 0$ *and* $\|\nu\|^{p/p^*} = \bar{S}^{-1}\|\mu\|$*, then* ν *and* μ *are concentrated at a single point.*

Proof. Our argument is patterned on the proof of Lemma 1.40 in [31].
i) Assume first $u = 0$. Let $\xi \in C_0^\infty(\mathbb{R}^N)$, then we have

$$\Big(\int Q|\xi u_n|^{p^*}dx\Big)^{p/p^*} \le \bar{S}^{-1}\int |\nabla^k(\xi u_n)|^p dx.$$

Taking limits on both sides and using Proposition 4.1 gives

$$\Big(\int |\xi|^{p^*}d\nu\Big)^{p/p^*} \le \bar{S}^{-1}\int |\xi|^p d\mu. \tag{4.9}$$

Inequality (4.5) and equation (4.4) now follow from Proposition 4.2 and the strict concavity of the map $\lambda \to \lambda^{p/p^*}$.
ii) For $R > 1$, let $\psi_R \in C^\infty(\mathbb{R}^N)$ be such that $\psi_R(x) = 1$ for $|x| > R+1$, $\psi_R(x) = 0$ for $|x| < R$ and $0 \le \psi_R(x) \le 1$ on $\mathbb{R}^N$. We then obtain

$$\Big(\int Q|\psi_R u_n|^{p^*}dx\Big)^{p/p^*} \le \bar{S}^{-1}\int |\nabla^k(\psi_R u_n)|^p dx.$$

Since $D^{\alpha-\beta}u_n \to 0$ in $L^p_{loc}(\mathbb{R}^N)$ and $D^\beta\psi_R \in C_0^\infty(\mathbb{R}^N)$ for $0 < \beta \le \alpha$, we obtain the following inequality by applying Proposition 4.1:

$$\overline{\lim_{n\to\infty}}\Big(\int Q|\psi_R u_n|^{p^*}dx\Big)^{p/p^*} \le \bar{S}^{-1}\,\overline{\lim_{n\to\infty}}\int |\nabla^k u_n|^p\psi_R^p dx. \tag{4.10}$$

We also have that

$$\int_{|x|>R+1}|\nabla^k u_n|^p dx \le \int |\nabla^k u_n|^p\psi_R^p dx \le \int_{|x|>R}|\nabla^k u_n|^p dx$$

and

$$\int_{|x|>R+1} Q|u_n|^{p^*}dx \le \int Q|u_n|^{p^*}\psi_R^{p^*}dx \le \int_{|x|>R} Q|u_n|^{p^*}dx.$$

Hence

$$\mu_\infty = \lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int |\nabla^k u_n|^p\psi_R^p dx, \quad \nu_\infty = \lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int Q|u_n|^{p^*}\psi_R^{p^*}dx.$$

Inequality (4.6) now follows from (4.10).
iii) Further assume that $\|\nu\|^{p/p^*} = \bar{S}^{-1}\|\mu\|$. From Hölder's inequality we have, for $\xi \in C_0^\infty(\mathbb{R}^N)$

$$\Big(\int |\xi|^p d\mu\Big)^{1/p} \le \|\mu\|^{k/N}\Big(\int |\xi|^{p^*}d\mu\Big)^{1/p^*}.$$

Combining this with (4.9) gives

$$\Big(\int |\xi|^{p^*}d\nu\Big)^{1/p^*} \le \bar{S}^{-1/p}\|\mu\|^{k/N}\Big(\int |\xi|^{p^*}d\mu\Big)^{1/p^*}.$$

The above inequality gives $\nu \le \bar{S}^{-p^*/p}\|\mu\|^{kp^*/N}\mu$, which combined with the equality $\|\nu\|^{p/p^*} = \bar{S}^{-1}\|\mu\|$ implies

$$\nu = \bar{S}^{-p^*/p}\|\mu\|^{kp^*/N}\mu \quad \text{and} \quad \mu = \bar{S}\|\nu\|^{-pk/N}\nu.$$

So for $\xi \in C_0^\infty(\mathbb{R}^N)$ we have from (4.9)

$$\begin{aligned} (\int |\xi|^{p^*} d\nu)^{p/p^*} &\le \int |\xi|^p \|\nu\|^{-pk/N} d\nu \\ \text{and} \quad \|\nu\|^{k/N} (\int |\xi|^{p^*} d\nu)^{1/p^*} &\le (\int |\xi|^p d\nu)^{1/p}. \end{aligned} \tag{4.11}$$

Hence for each open set $\Omega \subset \mathbb{R}^N$

$$\nu(\Omega)^{1/p^*} \nu(\mathbb{R}^N)^{k/N} \le \nu(\Omega)^{1/p}.$$

It follows that either $\nu(\Omega) = 0$ or $\nu(\mathbb{R}^N) \le \nu(\Omega)$. Therefore ν is concentrated at a single point, and so is μ.
iv) Consider now the general case. Set $v_n = u_n - u$, then $v_n \rightharpoonup 0$ in $D^{k,p}(\mathbb{R}^N)$ and inequality (4.5) follows from part (i) of the proof.
v) For any $\epsilon > 0$, set $x = D^\alpha u_n$ and $y = -D^\alpha u$ in inequality (4.1) to obtain,

$$||D^\alpha v_n|^p - |D^\alpha u_n|^p| \le \epsilon |D^\alpha u_n|^p + C(\epsilon, p)|D^\alpha u|^p.$$

It follows that

$$\begin{aligned} &|\int_{|x|>R} (|\nabla^k v_n|^p - |\nabla^k u_n|^p) dx| = |\int_{|x|>R} \sum_{|\alpha|=k} (|D^\alpha v_n|^p - |D^\alpha u_n|^p) dx| \\ &\le \int_{|x|>R} \sum_{|\alpha|=k} (||D^\alpha v_n|^p - |D^\alpha u_n|^p|) dx \\ &\le \epsilon \int_{|x|>R} \sum_{|\alpha|=k} |D^\alpha u_n|^p dx + C(\epsilon, p) \int_{|x|>R} \sum_{|\alpha|=k} |D^\alpha u|^p dx \\ &= \epsilon \int_{|x|>R} |\nabla^k u_n|^p dx + C(\epsilon, p) \int_{|x|>R} |\nabla^k u|^p dx. \end{aligned}$$

Since ϵ is arbitrary, by letting $n \to \infty$ and $R \to \infty$, we conclude that

$$\lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|>R} |\nabla^k v_n|^p = \mu_\infty.$$

From the Brézis-Lieb lemma (see [31, Lemma 1.32]) we have

$$\overline{\lim_{n\to\infty}} (\int_{|x|>R} Q|u_n|^{p^*} dx - \int_{|x|>R} Q|v_n|^{p^*} dx) = \int_{|x|>R} Q|u|^{p^*} dx.$$

So

$$\lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|>R} Q|v_n|^{p^*} = \nu_\infty.$$

Inequality (4.6) now follows from part (ii) of the proof.
vi) There exists a finite measure $\tilde{\mu}$ such that $|\nabla^k u_n|^p \overset{*}{\rightharpoonup} \tilde{\mu}$ in $M(\mathbb{R}^N)$.

Let $\phi_\eta \in C_0^\infty(B(x_j, \eta))$, $0 \le \phi \le 1$ and $\phi(x_j) = 1$ where x_j is an atom of μ. Set $x = D^\alpha v_n$ and $y = D^\alpha u$ in inequality (4.1) to get

$$\begin{aligned} |\tilde{\mu}(\phi_\eta) - \mu(\phi_\eta)| &\le \lim_{n\to\infty} \int_{\mathbb{R}^N} \sum_{|\alpha|=k} \phi_\eta ||D^\alpha u_n|^p - |D^\alpha v_n|^p| \\ &\le \lim_{n\to\infty} \int_{\mathbb{R}^N} \sum_{|\alpha|=k} (\epsilon \phi_\eta |D^\alpha v_n|^p + C(\epsilon, p)\phi_\eta |D^\alpha u|^p) \\ &= \epsilon \mu(\phi_\eta) + C(\epsilon, p) \int_{\mathbb{R}^N} |\nabla^k u|^p \phi_\eta. \end{aligned} \tag{4.12}$$

Letting $\eta \to 0$ we have

$$|\tilde{\mu}_s(\{x_j\}) - \mu_s(\{x_j\})| \leq \epsilon \mu_s(\{x_j\}).$$

From the fact that ϵ is arbitrary, we see that the atomic part of $\tilde{\mu}$ is equal to μ_s. Since $\xi D^\alpha u_n \rightharpoonup \xi D^\alpha u$ in $L^p(\mathbb{R}^N)$ for all positive $\xi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\lim_{n\to\infty} \int |\xi \nabla^k u_n|^p \geq \int |\xi \nabla^k u|^p.$$

Now, $|\nabla^k u|^p$ seen as a measure is relatively singular to the Dirac measures δ_{x_j}, and it follows that

$$\|\tilde{\mu}\| \geq \|u\|_{k,p}^p + \|\mu_s\|. \tag{4.13}$$

For $R > 1$ we have

$$\overline{\lim_{n\to\infty}} \int |\nabla^k u_n|^p = \overline{\lim_{n\to\infty}} \Big(\int \psi_R |\nabla^k u_n|^p + \int (1-\psi_R)|\nabla^k u_n|^p \Big).$$

As $R \to \infty$, by Lebesgue's dominated convergence theorem we have

$$\overline{\lim_{n\to\infty}} \|u_n\|_{k,p}^p = \mu_\infty + \|\tilde{\mu}\| \geq \mu_\infty + \|u\|_{k,p}^p + \|\mu_s\|. \tag{4.14}$$

An application of the Brézis-Lieb lemma gives, for $R > 1$

$$\begin{aligned} \overline{\lim_{n\to\infty}} \int Q|u_n|^{p^*} &= \overline{\lim_{n\to\infty}} \Big(\int \psi_R Q|u_n|^{p^*} + \int (1-\psi_R) Q|u_n|^{p^*} \Big) \\ &= \overline{\lim_{n\to\infty}} \int \psi_R Q|u_n|^{p^*} + \int (1-\psi_R) d\nu + \int (1-\psi_R) Q|u|^{p^*}. \end{aligned}$$

As $R \to \infty$,

$$\overline{\lim_{n\to\infty}} \|u_n\|_{p^*,Q}^{p^*} = \|u\|_{p^*,Q}^{p^*} + \|\nu\| + \nu_\infty$$

follows from Lebesgue's dominated convergence theorem. Hence we have proved (4.7) and (4.8). □

It is important to make the following remarks.

Remark 4.4. There are many variants of the above Lemma as we will see later on. We mention here two of them. It is clear that we could have used the norm in (2.2). The only difference in this case would be that the conclusion of Proposition (4.1) needs to be replaced by $\lim_{n\to\infty} \|(-\Delta)^{k/2}(\psi_n u)\|_p = \lim_{n\to\infty} \|\psi(-\Delta)^{k/2} u\|_p$ (even k) and $\lim_{n\to\infty} \|\nabla(-\Delta)^{(k-1)/2}(\psi_n u)\|_p = \lim_{n\to\infty} \|\psi \nabla(-\Delta)^{(k-1)/2} u\|_p$ (odd k). The argument is similar. Secondly, we may change the space $D^{k,p}(\mathbb{R}^N)$ to $D_G^{k,p}(\mathbb{R}^N)$, in which case we would also have to replace $\bar{S}$ with $\bar{S}_G$ as defined in (3.2).

Remark 4.5. Looking back at the proof of the above lemma, we see that part (vi) is rather cumbersome and forces (4.7) to be an inequality rather than an equality. However, in the case when $p = 2$, we can avoid the argument in part (vi) of the proof above by using the following argument which exploits the Hilbert structure of $D^{k,2}(\mathbb{R}^N)$.

$$\lim_{n\to\infty} \int |\nabla^k v_n|^2 \psi_R^2 dx = \lim_{n\to\infty} \int |\nabla^k u_n|^2 \psi_R^2 dx - \int |\nabla^k u|^2 \psi_R^2 dx,$$

since the Brézis-Lieb lemma holds when a.e. convergence is replaced by weak convergence (see [31, Remarks 1.33]). Hence,

$$\lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int|\nabla^k v_n|^2\psi_R^2dx=\lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int|\nabla^k u_n|^2\psi_R^2dx=\mu_\infty.$$

For $R>1$ we have, once again by the Brézis-Lieb lemma

$$\begin{aligned}\overline{\lim_{n\to\infty}}\int|\nabla^k u_n|^2&=\overline{\lim_{n\to\infty}}\Big(\int\psi_R|\nabla^k u_n|^2+\int(1-\psi_R)|\nabla^k u_n|^2\Big)\\&=\overline{\lim_{n\to\infty}}\int\psi_R|\nabla^k u_n|^2+\int(1-\psi_R)d\mu+\int(1-\psi_R)|\nabla^k u|^2.\end{aligned}$$

As $R\to\infty$, by Lebesgue's dominated convergence theorem we have

$$\overline{\lim_{n\to\infty}}\|u_n\|_{k,p}^2=\mu_\infty+\|\mu\|+\int|\nabla^k u|^2. \tag{4.15}$$

So we arrive at the stronger conclusion

$$\overline{\lim_{n\to\infty}}\|u_n\|_{k,2}^2=\|u\|_{k,2}^2+\|\mu\|+\mu_\infty.$$

Further, one can replace μ_s with μ in inequality (4.5). We point out that in this case Proposition 4.2 is redundant, since the fact that $\nu=\sum_{j\in J}\nu_j\delta_{x_j}$ will not be used in our applications as we will see.

Remark 4.6. If $u=0$, then by definition, $\tilde{\mu}=\mu$. Hence it follows from (4.14) that $\overline{\lim}_{n\to\infty}\|u_n\|_{k,p}^p=\mu_\infty+\|\mu\|$.

If $\{u_n\}_{n=1}^\infty\subset D^{k,p}(\mathbb{R}^N)$ is a bounded sequence such that $Q|(u_n-u)|^{p^*}\overset{*}{\rightharpoonup}\nu$, then we may assume that $|u_n-u|^p\overset{*}{\rightharpoonup}\gamma$. Hence, by defining γ_∞ in the same way as ν_∞, we see that $\nu(\{x\})=Q(x)\gamma(\{x\})$ and $\nu_\infty\le Q_\infty\gamma_\infty$. So γ and ν concentrate at exactly the same points, if $Q>0$. Further, $\nu_\infty=Q_\infty\gamma_\infty$ if $Q_\infty=\lim_{|x|\to\infty}Q(x)$.

5. Non-existence result

The proposition given below is the essential part in showing that for general Q a minimizer of (3.1) does not exist.

Proposition 5.1. *If Q is a bounded nonnegative continuous function in $\mathbb{R}^N$, then $S=\bar{S}\|Q\|_\infty^{p/p^*}$.*

Proof. We have,

$$\bar{S}=\inf_{\substack{u\in D^{k,p}(\mathbb{R}^N)\\u\neq0}}\frac{\int|\nabla^k u|^p}{\left(\int Q|u|^{p^*}\right)^{p/p^*}}\ge\inf_{\substack{u\in D^{k,p}(\mathbb{R}^N)\\u\neq0}}\frac{\int|\nabla^k u|^p}{\|Q\|_\infty^{p/p^*}\left(\int|u|^{p^*}\right)^{p/p^*}}=\frac{S}{\|Q\|_\infty^{p/p^*}}.$$

So, $S\le\bar{S}\|Q\|_\infty^{p/p^*}$ follows. Let u be a function which achieves S in (3.3) and for $x_0\in\mathbb{R}^N$ set

$$u_\epsilon(x)=\epsilon^{\frac{-N}{p^*}}u\Big(\frac{x-x_0}{\epsilon}\Big).$$

Through a variable substitution we have

$$\bar{S}\le\frac{\int|\nabla^k u_\epsilon|^pdx}{\left(\int Q(x)|u_\epsilon|^{p^*}dx\right)^{p/p^*}}=\frac{\int|\nabla^k u|^pdy}{\left(\int Q(\epsilon y+x_0)|u|^{p^*}dy\right)^{p/p^*}}.$$

As $\epsilon \to 0$, by Lebesgue's dominated convergence theorem we obtain

$$\bar{S} \le \frac{S}{Q(x_0)^{p/p^*}}.$$

The assertion follows, since we have $(Q(x_0))^{p/p^*}\bar{S} \le S \le \bar{S}\|Q\|_\infty^{p/p^*}$, $\forall x_0 \in \mathbb{R}^N$.

□

To see that minimizers of (3.1) usually do not exist, assume that u is such a minimizer. Then in view of Proposition 5.1 we have

$$\begin{aligned}\Big(\int Q|u|^{p^*}\Big)^{p/p^*} &\le \|Q\|_\infty^{p/p^*}\Big(\int |u|^{p^*}\Big)^{p/p^*} \\ &\le \|Q\|_\infty^{p/p^*} S^{-1}\int |\nabla^k u|^p = \Big(\int Q|u|^{p^*}\Big)^{p/p^*}.\end{aligned}$$

So it follows that

$$\Big(\int_{\mathbb{R}^N}(\|Q\|_\infty - Q)|u|^{p^*}\Big) = 0.$$

We now deduce that if the set $E = \{x \in \mathbb{R}^N : \|Q\|_\infty = Q(x)\}$ has measure zero, then a minimizer of (3.1) does not exist. We can further conclude, since the minimizers for S are positive everywhere when $p > 1$, $k = 1$ or $p = 2$ and $k > 2$ (see Section 7), that the minimizers of (3.1) exist if and only if Q is constant. We state these observations in the following proposition.

Proposition 5.2. *If the set $E = \{x \in \mathbb{R}^N : \|Q\|_\infty = Q(x)\}$ has measure zero, then problem (3.1) has no minimizer. Further, when $p > 1$, $k = 1$ or when $p = 2$, $k \ge 2$, minimizers of (3.1) exist if and only if Q is constant.*

6. Sufficient condition for existence of minimizers

We now give a sufficient condition for the existence of symmetric minimizers for problem (3.2). We will then give an example which shows that there are functions Q so that the condition holds.

Theorem 6.1. *If $\bar{S}_G \max\{Q_0^{p/p^*}, Q_\infty^{p/p^*}\} < S$ then problem (3.2) has a minimizer.*

Proof. Let $\{u_n\}$ be a minimizing sequence for $\bar{S}_G$ such that $\|u_n\|_{p^*,Q} = 1$. For some subsequence, still denoted $\{u_n\}$, we may assume that the conditions of the modified version of Lemma 4.3, as mentioned in Remark 4.4, are fulfilled and so the conclusion holds with $\bar{S}$ replaced by $\bar{S}_G$. We need to show that $\|\nu\| = \nu_\infty = 0$. We have

$$\bar{S}_G = \lim_{n\to\infty}\|u_n\|_{k,p}^p \ge \|u\|_{k,p}^p + \|\mu_s\| + \mu_\infty$$

$$\text{and} \qquad 1 = \lim_{n\to\infty}\|u_n\|_{p^*,Q}^{p^*} = \|u\|_{p^*,Q}^{p^*} + \|\nu\| + \nu_\infty.$$

Combining these with inequalities (4.5) and (4.6) gives

$$\begin{aligned}&\bar{S}_G(\|u\|_{p^*,Q}^{p^*} + \|\nu\| + \nu_\infty)^{p/p^*} \ge \|u\|_{k,p}^p + \|\mu_s\| + \mu_\infty \\ &\ge \bar{S}_G((\|u\|_{p^*,Q}^{p^*})^{p/p^*} + \|\nu\|^{p/p^*} + \nu_\infty^{p/p^*}).\end{aligned} \tag{6.1}$$

So, only one of the three quantities, $\|u\|_{p^*,Q}^{p^*}$, $\|\nu\|$ and ν_∞, is equal to 1 and the other two are zero. If $Q_\infty \neq 0$ and $\nu_\infty = 1$, then using the hypothesis, Remark 4.6, (6.1) and (4.6) with $Q = 1$ (i.e. $\bar{S} = S$) we have

$$\begin{aligned} S(\gamma_\infty)^{p/p^*} &> \bar{S}_G(Q_\infty\gamma_\infty)^{p/p^*} \geq \bar{S}_G(\nu_\infty)^{p/p^*} \\ &\geq \mu_\infty \geq S(\gamma_\infty)^{p/p^*}, \end{aligned}$$

a contradiction. So $\nu_\infty = 0$. If $Q_0 \neq 0$ and $\|\nu\| = 1$ then $u = 0$ and $\|\nu\|^{p/p^*} = \bar{S}_G^{-1}\|\mu\|$, and so ν is concentrated at a single point. Since the set of concentration points of ν are G-invariant and $\mathrm{Fix}(G) = \{0\}$, we conclude that ν and μ are concentrated at the origin. Once again we get a contradiction, since

$$\begin{aligned} S(\gamma(\{0\}))^{p/p^*} &> \bar{S}_G(Q_0\gamma(\{0\}))^{p/p^*} = \bar{S}_G(\nu(\{0\}))^{p/p^*} \\ &\geq \|\mu_s\| \geq S(\gamma(\{0\}))^{p/p^*}. \end{aligned}$$

It follows that $\|u\|_{p^*,Q} = 1$, and u is a minimizer of (3.2). If $Q_0^{p/p^*} = Q_\infty^{p/p^*} = 0$ then it is easy to see that $\|\nu\| = \nu_\infty = 0$ and we are done. □

Remark 6.2. The results presented above are independent of the norm chosen on $D^{k,p}(\mathbb{R}^N)$. But we still have to show that there are functions Q for which the above theorem holds. To this end, we will assume that the norm used on $D^{k,p}(\mathbb{R}^N)$ is the norm given in (2.2). This will guarantee that there are radially symmetric, nonnegative and decreasing minimizers of problem (3.3), (see [20, Corollary I.2]). Now, if

$$S_G = \inf\{\|u\|_{k,p}^p : u \in D_G^{k,p}(\mathbb{R}^N), \int |u|^{p^*} = 1\}, \tag{6.2}$$

then $S_G = S$. This is because there exists a radially symmetric and hence G-symmetric function which minimizes S and $S \leq S_G$.

We observe that if u is a function which achieves $S = S_G$ in (6.2) then

$$u_\epsilon(x) = \epsilon^{\frac{-N}{p^*}} u\Big(\frac{x}{\epsilon}\Big),$$

also achieves S_G. Through a variable substitution we have

$$\bar{S}_G \leq \frac{\|u_\epsilon\|_{k,p}^p}{\left(\int Q(x)|u_\epsilon|^{p^*}dx\right)^{p/p^*}} = \frac{\|u\|_{k,p}^p}{\left(\int Q(\epsilon y)|u|^{p^*}dy\right)^{p/p^*}}$$

By Lebesgue's dominated convergence theorem we may take the limit under the integral sign. Hence by letting ϵ go to 0 or ∞, we obtain, $\bar{S}_G \max\{Q_0^{p/p^*}, Q_\infty^{p/p^*}\} \leq S$, provided $Q_\infty = \lim_{|x|\to\infty} Q(x)$. Since $\frac{S}{\|Q\|_\infty^{p/p^*}} \leq \bar{S}_G$ (see the argument of Proposition 5.2), we may conclude that

$$\frac{S}{\|Q\|_\infty^{p/p^*}} \leq \bar{S}_G \leq S\min\{Q_0^{-p/p^*}, Q_\infty^{-p/p^*}\}.$$

We further observe that if $\|Q\|_\infty^{-p/p^*} = \min\{Q_0^{-p/p^*}, Q_\infty^{-p/p^*}\}$ then the assumption of Theorem 6.1 cannot be satisfied. In this case we can state a result similar to Proposition 5.2.

We now give two simple examples. In the next section we will show how to find conditions on the behavior of Q at zero and infinity so that the assumption of Theorem 6.1 is satisfied.

Example 6.3. The most trivial example is when $Q_0 = Q_\infty = 0$. This condition immediately guarantees that concentration cannot occur at zero or infinity, and the assumption of Theorem 6.1 is satisfied.

The following example shows that the condition $\bar{S}_G \max\{Q_0^{p/p^*}, Q_\infty^{p/p^*}\} < S = S_G$, is not always necessary to conclude that minimizers of $\bar{S}_G$ exist.

Example 6.4. Suppose that $Q(x) \geq Q_0 = Q_\infty = \lim_{|x|\to\infty} Q(x) > 0$ for all $x \in \mathbb{R}^N$, then it is clear that $\bar{S}_G \max\{Q_0^{p/p^*}, Q_\infty^{p/p^*}\} \leq S_G = S$. If strict inequality holds then a minimizer of $\bar{S}_G$ exists by Theorem 6.1. On the other hand if $\bar{S}_G \max\{Q_0^{p/p^*}, Q_\infty^{p/p^*}\} = S_G$ then a minimizer of $\bar{S}_G$ also exists. To see this, let u be a minimizer for S_G. We then have

$$\bar{S}_G \leq \frac{\|u\|_{k,p}^p}{\|u\|_{p^*,Q}^p} \leq \frac{\|u\|_{k,p}^p}{Q_0^{p/p^*}\|u\|_{p^*}^p} = \frac{S_G}{Q_0^{p/p^*}} = \bar{S}_G. \tag{6.3}$$

So u is a minimizer for $\bar{S}_G$ as well.

The two examples above show that we require very little knowledge of Q to guarantee the existence of symmetric solutions to a large class of partial differential equations of arbitrary order. We state the above observation in the next corollary.

Corollary 6.5. *Problem (3.2) has a solution if* $Q(x) \geq Q_0 = Q_\infty = \lim_{|x|\to\infty} Q(x) \geq 0$ *for all* $x \in \mathbb{R}^N$.

The above corollary together with Proposition 5.2 shows that $\bar{S} < \bar{S}_G$ if $E = \{x \in \mathbb{R}^N : \|Q\|_\infty = Q(x)\}$ has measure zero and $Q(x) \geq Q_0 = Q_\infty = \lim_{|x|\to\infty} Q(x) > 0$ for all $x \in \mathbb{R}^N$.

7. Application to partial differential equations

7.1. **The case $p = 2$, $k = 1$.** In [6] the authors studied the solutions to the following problem

$$-\Delta u = Q(x)|u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, u \in D_G^{1,2}(\mathbb{R}^N), \tag{7.1}$$

where $N > 2$, $2^* = \frac{2N}{N-2}$ and Q is G-symmetric. We know that any minimizer of problem (3.2) with $p = 2$ and $k = 1$ will then give a solution of the above problem. In Proposition 2 in [6] the authors show that a solution to problem (3.2) exists if $\bar{S}_G \max\{Q_0^{2/2^*}, Q_\infty^{2/2^*}, |G|^{-2/N}\|G\|_\infty^{2/2^*}\} < S$ where $|G| = \inf_{x\in\mathbb{R}^N, x\neq 0}|G_x|$ and $|G_x|$ is the cardinality of the set $G_x = \{gx : g \in G\}$. Comparing this to Theorem 6.1 shows that our result is an improvement upon the result given there.

We can now state some conditions on Q which will guarantee that the assumption of Theorem 6.1 is satisfied. The proofs are similar to those of Corollary 1 and 2 in [6].

Corollary 7.1. *Suppose that Q is G-symmetric, $Q_0 \geq Q_\infty > 0$ and either*
(i) $Q(x) \geq Q_0 + \epsilon|x|^N$ for some $\epsilon > 0$ and $|x|$ small or
(ii) $|Q(x) - Q_0| \leq C|x|^\alpha$ for some constant $C > 0$,$\alpha > N$, $|x|$ small and

$$\int_{\mathbb{R}^N} (Q(x) - Q_0)|x|^{-2N} dx > 0.$$

Then there exists a nontrivial solution to problem (7.1).

Proof. We know that the instanton $v(x) = (1 + |x|^2)^{-N/2^*}$ is the unique minimizer for (3.3) with $k = 1$ and $p = 2$, up to translation and dilation. In view of Theorem 6.1 it suffices to show that for some $\eta > 0$

$$\bar{S}_G^{-2^*/2} \geq \int_{\mathbb{R}^N} Q(x)|Av(x/\eta)|^{2^*} > \int_{\mathbb{R}^N} Q_0|Av(x/\eta)|^{2^*} = Q_0 S^{-2^*/2},$$

where $A > 0$ is a constant chosen so that $\|Av(x/\eta)\|_{1,2} = 1$. Of course this is equivalent to showing that for some $\eta > 0$

$$\int_{\mathbb{R}^N} Q(x)\Big(\frac{1}{\eta^2 + |x|^2}\Big)^N - \int_{\mathbb{R}^N} Q_0\Big(\frac{1}{\eta^2 + |x|^2}\Big)^N > 0.$$

(i) By the hypothesis, for some $\delta > 0$,

$$\int_{|x|\leq\delta} (Q(x) - Q_0)\Big(\frac{1}{\eta^2 + |x|^2}\Big)^N \geq \epsilon \int_{|x|\leq\delta} \Big(\frac{|x|}{\eta^2 + |x|^2}\Big)^N \to \infty$$

as $\eta \to 0$. On the other hand, for all $\eta > 0$ we have

$$\Big|\int_{|x|>\delta} (Q(x) - Q_0)\Big(\frac{1}{\eta^2 + |x|^2}\Big)^N\Big| \leq C_1 \int_{|x|>\delta} \frac{1}{|x|^{2N}} = C_2$$

for some constants C_1, C_2 greater than zero and independent of η. We now obtain the required conclusion.

(ii) By the hypothesis, $|Q(x) - Q_0||x|^{-2N} \in L^1(\mathbb{R}^N)$, and by Lebesgue's dominated convergence theorem we have

$$\int_{\mathbb{R}^N} (Q(x) - Q_0)\Big(\frac{1}{\eta^2 + |x|^2}\Big)^N \to \int_{\mathbb{R}^N} (Q(x) - Q_0)|x|^{-2N}$$

as $\eta \to 0$. Hence, we deduce the required conclusion. □

Corollary 7.2. *Suppose that Q is G-symmetric, $Q_\infty = \lim_{|x|\to\infty} Q(x)$, $Q_\infty \geq Q_0 > 0$ and either*
(i) $Q(x) \geq Q_\infty + \epsilon|x|^{-N}$ for some $\epsilon > 0$ and $|x|$ large or
(ii) $|Q(x) - Q_\infty| \leq C|x|^{-\alpha}$ for some constant $C > 0$,$\alpha > N$, $|x|$ large and

$$\int_{\mathbb{R}^N} (Q(x) - Q_\infty) dx > 0.$$

Then there exists a nontrivial solution to problem (7.1).

Proof. As mentioned in the proof of the previous corollary, in view of Theorem 6.1 it suffices to show that for some $\eta > 0$

$$\bar{S}_G^{-2^*/2} \geq \int_{\mathbb{R}^N} Q(x)|Av(x/\eta)|^{2^*} > \int_{\mathbb{R}^N} Q_\infty|Av(x/\eta)|^{2^*} = Q_\infty S^{-2^*/2}.$$

(i) Hence, we need to show that

$$\int_{\mathbb{R}^N} (Q(x) - Q_\infty)\Big(\frac{1}{1 + |x/\eta|^2}\Big)^N > 0$$

for some $\eta > 0$. By the hypothesis, we can find $R > 0$ such that $Q(x) \geq Q_\infty + \epsilon|x|^{-N}$ for all $|x| \geq R$. It follows that

$$\int_{|x|>R} (Q(x) - Q_\infty)\Big(\frac{1}{1+|x/\eta|^2}\Big)^N \to \infty$$

as $\eta \to \infty$. We also have

$$\Big|\int_{|x|\leq R} (Q(x) - Q_\infty)\Big(\frac{1}{1+|x/\eta|^2}\Big)^N\Big| \leq C_1$$

where $C_1 > 0$ is independent of η. By putting these two observations together, we obtain the desired result.

(ii) By the hypothesis, $|Q(x) - Q_\infty| \in L^1(\mathbb{R}^N)$ and so

$$\lim_{\eta\to\infty} \int_{\mathbb{R}^N} (Q(x) - Q_\infty)\Big(\frac{1}{1+|x/\eta|^2}\Big)^N = \int_{\mathbb{R}^N} (Q(x) - Q_\infty)dx > 0,$$

we immediately conclude the desired result. □

Remark 7.3. We observe that $Q_\infty = \lim_{|x|\to\infty} Q(x)$ is now a part of the assumption.

7.2. **The case $p = 2$ and $k > 1$.** We continue with a higher order variant of the above example. We wish to find non-trivial solutions to the following non-linear partial differential equation

$$(-\Delta)^k u = Q(x)|u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, u \in D_G^{1,2}(\mathbb{R}^N), \tag{7.2}$$

where $N > 2k$, $2^* = \frac{2N}{N-2k}$ and Q is G-symmetric. Keeping in mind the norm (2.2), a minimizer for (3.2) with $p = 2$, will then give a solution of the above problem. In the previous example, by knowing explicitly the instanton which minimizes (3.3), we could state explicit conditions on Q under which problem (3.2) has a minimizer. We do the same thing here, since we know that up to translation and dilation the instanton $v(x) = (1 + |x|^2)^{-N/2^*}$ is a minimizer for (3.2) (see [26]). By the same arguments as in Corollary 7.1 and 7.2 we see that the following results hold.

Corollary 7.4. *Suppose that Q is G-symmetric, $Q_0 \geq Q_\infty > 0$ and either*
(i) $Q(x) \geq Q_0 + \epsilon|x|^N$ for some $\epsilon > 0$ and $|x|$ small or
(ii) $|Q(x) - Q_0| \leq C|x|^\alpha$ for some constant $C > 0$,$\alpha > N$, $|x|$ small and

$$\int_{\mathbb{R}^N} (Q(x) - Q_0)|x|^{-2N} dx > 0.$$

Then there exists a nontrivial solution to problem (7.2).

Corollary 7.5. *Suppose that Q is G-symmetric, $Q_\infty = \lim_{|x|\to\infty} Q(x)$, $Q_\infty \geq Q_0 > 0$ and either*
(i) $Q(x) \geq Q_\infty + \epsilon|x|^{-N}$ for some $\epsilon > 0$ and $|x|$ large or
(ii) $|Q(x) - Q_\infty| \leq C|x|^{-\alpha}$ for some constant $C > 0$,$\alpha > N$, $|x|$ large and

$$\int_{\mathbb{R}^N} (Q(x) - Q_\infty)dx > 0.$$

Then there exists a nontrivial solution to problem (7.2).

For some results in the non-critical case we refer to [5] and references therein.

7.3. **The case $p > 1$ and $k = 1$.** Here we obtain an equation involving the p-Laplace operator. We have

$$-\Delta_p u = Q(x)|u|^{p^*-2}u \quad \text{in } \mathbb{R}^N, u \in D_G^{1,2}(\mathbb{R}^N), \tag{7.3}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $N > p$, $p^* = \frac{Np}{N-p}$ and Q is G-symmetric. It is known from the work of Aubin [2] and Talenti [29] that $v(x) = (1 + |x|^{p/(p-1)})^{-N/p^*}$ is the unique minimizer up to translation and dilation, for problem (3.3) with $k = 1$. In this case also Corollaries 7.1 and 7.2 hold with minor changes. Since the proofs are similar we skip them.

Corollary 7.6. *Suppose that Q is G-symmetric, $Q_0 \geq Q_\infty > 0$ and either*
(i) $Q(x) \geq Q_0 + \epsilon|x|^{N/(p-1)}$ for some $\epsilon > 0$ and $|x|$ small or
(ii) $|Q(x) - Q_0| \leq C|x|^\alpha$ for some constant $C > 0$,$\alpha > N/(p-1)$, $|x|$ small and

$$\int_{\mathbb{R}^N} (Q(x) - Q_0)|x|^{-pN/(p-1)}dx > 0.$$

Then there exists a nontrivial solution to problem (7.3).

Corollary 7.7. *Suppose that Q is G-symmetric, $Q_\infty = \lim_{|x|\to\infty} Q(x)$, $Q_\infty \geq Q_0 > 0$ and either*
(i) $Q(x) \geq Q_\infty + \epsilon|x|^{-N}$ for some $\epsilon > 0$ and $|x|$ large or
(ii) $|Q(x) - Q_\infty| \leq C|x|^{-\alpha}$ for some constant $C > 0$,$\alpha > N$, $|x|$ large and

$$\int_{\mathbb{R}^N} (Q(x) - Q_\infty)dx > 0.$$

Then there exists a nontrivial solution to problem (7.3).

The p-Laplace operator in equation (7.3) has been the object of many studies, where both critical and non-critical exponents have been considered. We refer the reader e.g. to [1, 12, 23, 25, 27] and the references therein.

7.4. **The p-biharmonic operator.** Let

$$F(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\Delta u|^p,$$

then

$$F'(u)\phi = \frac{1}{p}\int_{\mathbb{R}^N} |\Delta u|^{p-2}\Delta u \Delta\phi \quad \forall \phi \in C_0^\infty(\mathbb{R}^N),$$

i.e. any minimizer of problem (3.2) with $k = 2$ will satisfy

$$\Delta(|\Delta u|^{p-2}\Delta u) = Q|u|^{p^*-2}u \quad \text{in } \mathbb{R}^N, u \in D_G^{2,p}(\mathbb{R}^N). \tag{7.4}$$

In this case the explicit form of the minimizers of S_G is not known, therefore we are not able to give explicit conditions on Q so that a solution to (7.4) exists. However, by using Corollary 6.5 we may conclude that if $Q(x) \geq Q_0 = Q_\infty = \lim_{|x|\to\infty} Q(x) \geq 0$ then equation (7.4) has a G-invariant solution.

The operator $\Delta(|\Delta u|^{p-2}\Delta u)$ is called the p-biharmonic operator. In comparison to the p-Laplace operator, very little is known about it. However see [13, 14, 28].

8. Double Weights

In this section, we will apply the methods developed in the previous sections to a more general problem. Let H be a bounded continuous function in $\mathbb{R}^N$. Assume that $\bar{H} = \inf_{x\in\mathbb{R}^N} H(x) > 0$ and $H_\infty := \lim_{|x|\to\infty} H(x)$ exists. We will look at the following problem:

$$I = \inf\{\|u\|_{k,p,H}^p : u \in D^{k,p}(\mathbb{R}^N), \|u\|_{p^*,Q} = 1\}. \tag{8.1}$$

Here $\|u\|_{k,p,H}$ can either be $\|\nabla^k u\|_{p,H}$ or $\|(-\Delta)^{k/2}u\|_{p,H}$ when k is even and $\|\nabla(-\Delta)^{(k-1)/2}u\|_{p,H}$ when k is odd. There is no problem in doing so since our hypothesis on H shows that for even k

$$\int_{\mathbb{R}^N} H|\Delta^{k/2}u|^p \sim \int_{\mathbb{R}^N} |\Delta^{k/2}u|^p \sim \int_{\mathbb{R}^N} |\nabla^k u|^p \sim \int_{\mathbb{R}^N} H|\nabla^k u|^p,$$

where $\sim$ indicates the equivalence of norms. The same is true for odd k. Similarly as in Section 4, we first assume that $\|u\|_{k,p,H} = \|\nabla^k u\|_{p,H}$.

We note that the condition $\bar{H} > 0$ guaranties the positivity of I and also that $\|\cdot\|_{k,p,H}$ is an equivalent norm to $\|\cdot\|_{k,p}$ in $D^{k,p}(\mathbb{R}^N)$. To keep things simple we will also assume that $Q_\infty := \lim_{|x|\to\infty} Q(x)$. It is easy to see that the methods applied in the previous sections can be adapted to handle the case of double weights.

This type of problems with double weights have been studied by some authors. We refer the reader to [3, 4, 9, 16] and references therein.

We start by studying the effect of dilation and translation in order to obtain a relationship between the values I and S. Let u be a function which achieves S in (3.3) and for $x_0 \in \mathbb{R}^N$ set

$$u_\epsilon(x) = \epsilon^{\frac{-N}{p^*}} u\left(\frac{x - x_0}{\epsilon}\right).$$

Through a variable substitution we have

$$I \le \frac{\int H(x)|\nabla^k u_\epsilon|^p dx}{\left(\int Q(x)|u_\epsilon|^{p^*} dx\right)^{p/p^*}} = \frac{\int H(\epsilon y + x_0)|\nabla^k u|^p dy}{\left(\int Q(\epsilon y + x_0)|u|^{p^*} dy\right)^{p/p^*}}.$$

As $\epsilon \to 0$, by Lebesgue's dominated convergence theorem we obtain,

$$I \le \frac{SH(x_0)}{Q(x_0)^{p/p^*}}.$$

Since the above inequality holds for all $x_0 \in \mathbb{R}^N$, we conclude $I \le S \inf_{x\in\mathbb{R}^N} \frac{H(x)}{(Q(x))^{p/p^*}}$. On the other hand, we have

$$\left(\int Q|u|^{p^*}\right)^{p/p^*} \le \|Q\|_\infty^{p/p^*} \left(\int |u|^{p^*}\right)^{p/p^*} \le S^{-1} \frac{\|Q\|_\infty^{p/p^*}}{\bar{H}} \int H|\nabla^k u|^p$$

for all $u \in D^{k,p}(\mathbb{R}^N)$. Hence we deduce that

$$S\frac{\bar{H}}{\|Q\|_\infty^{p/p^*}} \le I \le S \inf_{x\in\mathbb{R}^N} \frac{H(x)}{(Q(x))^{p/p^*}}. \tag{8.2}$$

Next, we require the concentration-compactness lemma, which gives us information regarding weakly converging sequences and in particular minimizing sequences. Since Proposition 4.1 holds even when we use Hdx as weights, we can state another version of the concentration-compactness lemma. Since the proof is similar to that of Lemma 4.3 we omit it.

Lemma 8.1. *(Concentration-compactness lemma). Assume that our hypothesis on H and Q hold, and $\{u_n\}_{n=1}^{\infty} \subset D^{k,p}(\mathbb{R}^N)$ is a sequence such that*

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } D^{k,p}(\mathbb{R}^N)\\ H|\nabla^k(u_n-u)|^p &\overset{*}{\rightharpoonup} \mu && \text{in } M(\mathbb{R}^N)\\ Q|(u_n-u)|^{p^*} &\overset{*}{\rightharpoonup} \nu && \text{in } M(\mathbb{R}^N)\\ u_n &\to u && \text{a.e. on } \mathbb{R}^N \end{aligned}$$

and define

$$\begin{aligned} \mu_\infty &:= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|>R} H|\nabla^k u_n|^p \ , \\ \nu_\infty &:= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|>R} Q|u_n|^{p^*} \ . \end{aligned} \tag{8.3}$$

If μ_s is the atomic part of μ, then it follows that

$$\nu = \sum_{j\in J} \nu_j \delta_{x_j}, \tag{8.4}$$

$$\|\nu\|^{p/p^*} \le I^{-1}\|\mu_s\|, \tag{8.5}$$

$$\nu_\infty^{p/p^*} \le I^{-1}\mu_\infty, \tag{8.6}$$

$$\overline{\lim_{n\to\infty}} \|u_n\|_{k,p,H}^p \ge \|u\|_{k,p}^p + \|\mu_s\| + \mu_\infty, \tag{8.7}$$

$$\overline{\lim_{n\to\infty}} \|u_n\|_{p^*,Q}^{p^*} = \|u\|_{p^*,Q}^{p^*} + \|\nu\| + \nu_\infty. \tag{8.8}$$

Moreover, if $u = 0$ and $\|\nu\|^{p/p^*} = I^{-1}\|\mu\|$, then ν and μ are concentrated at a single point.

Remark 8.2. Those changes mentioned in remark 4.4 can also be made here.

Remark 8.3. If $\{u_n\}_{n=1}^{\infty} \subset D^{k,p}(\mathbb{R}^N)$ is a bounded sequence such that $H|\nabla^k(u_n-u)|^p \overset{*}{\rightharpoonup} \mu$, $Q|(u_n-u)|^{p^*} \overset{*}{\rightharpoonup} \nu$, then we may assume that $|\nabla^k(u_n-u)|^p \overset{*}{\rightharpoonup} \alpha$, and $|(u_n-u)|^{p^*} \overset{*}{\rightharpoonup} \beta$. Hence, by defining α_∞ and β_∞ in the way μ_∞ is defined, we see that $\mu(\{x\}) = H(x)\alpha(\{x\})$, $\nu(\{x\}) = Q(x)\beta(\{x\})$, $\mu_\infty = H_\infty\alpha_\infty$ and $\nu_\infty = Q_\infty\beta_\infty$.

We can now state a result which basically, is a necessary and sufficient condition for all minimizing sequences to be relatively compact. That it is sufficient follows from the following theorem. To see that this is also necessary we refer the reader to the work of Lions [18, 19]. We would like to mention that the hypothesis of the next theorem is hard to check, but we give an example which will show that the theorem is not empty, i.e. there exist H and Q such that the assumption is satisfied.

Theorem 8.4. *If $I < S \inf_{x\in\mathbb{R}^N} \frac{H(x)}{(Q(x))^{p/p^*}}$ then all minimizing sequences are relatively compact. In particular, a minimizer for I exists.*

Proof. Let $\{u_n\}_{n=1}^{\infty} \subset D^{k,p}(\mathbb{R}^N)$ be a minimizing sequence for I. Arguing exactly as in Theorem 6.1 we see that only one of the three quantities, $\|u\|_{p^*,Q}^{p^*}$, $\|\nu\|$ and ν_∞, is equal to 1 and the other two are zero.

i) If $\nu_\infty = 1$, then

$$I = I(\nu_\infty)^{p/p^*} = I(Q_\infty \beta_\infty)^{p/p^*} \geq \mu_\infty = H_\infty \alpha_\infty \geq S H_\infty (\beta_\infty)^{p/p^*} = S \frac{H_\infty}{Q_\infty^{p/p^*}}.$$

Hence, $I \geq S \frac{H_\infty}{Q_\infty^{p/p^*}} \geq S \inf_{x\in\mathbb{R}^N} \frac{H(x)}{(Q(x))^{p/p^*}}$ contradicts our assumption.

ii) If $\|\nu\| = 1$, then $u = 0$, $I_1\|\nu\|^{p/p^*} \geq \|\mu\|$ and so by the previous lemma ν concentrates at a point $x \in \mathbb{R}^N$. We now have

$$\begin{aligned} I = I(\nu(\{x\}))^{p/p^*} &= I(Q(x)\beta(\{x\}))^{p/p^*} \geq \mu(\{x\} \\ &= H(x)\alpha(\{x\}) \geq S H(x)(\beta(\{x\}))^{p/p^*}. \end{aligned}$$

Once again $I \geq S \frac{H(x)}{(Q(x))^{p/p^*}}$ will contradict our assumption. It follows that $\|u\|_{p^*,Q}^{p^*} = 1$ and so the proof is complete. □

We now give the example mentioned above.

Example 8.5. Let $k = 1$ and $H = Q^{p/p^*}$. We shall construct a Q such that

$$I < S \inf_{x\in\mathbb{R}^N} \frac{H(x)}{(Q(x))^{p/p^*}} = S.$$

Set $u(x) = (1 + |x|^{p/(p-1)})^{-N/p^*}$, so that $|\nabla u|^p = C|x|^{p/(p-1)}|u|^{p^*}$ and

$$S = \frac{\int |\nabla u|^p}{\left(\int |u|^{p^*}\right)^{p/p^*}}.$$

For some small $\eta > 0$, let $1 \leq Q(x) \leq 1 + \eta$ and set $Q(x) = 1$ if $|x| > 2\delta$, $Q(x) = 1 + \eta$ if $|x| < \delta$. We shall show that $\delta > 0$ can be chosen such that

$$I \leq \frac{\int Q^{p/p^*}|\nabla u|^p}{\left(\int Q|u|^{p^*}\right)^{p/p^*}} < S. \tag{8.9}$$

We have

$$\begin{aligned} &\int Q^{p/p^*}|\nabla u|^p \\ &= \int_{|x|<\delta} (1+\eta)^{p/p^*}|\nabla u|^p + \int_{\delta<|x|<2\delta} Q^{p/p^*}|\nabla u|^p + \int_{2\delta<|x|} |\nabla u|^p \\ &= \int_{|x|<\delta} (1+\eta)^{p/p^*}|\nabla u|^p + \int_{\delta<|x|<2\delta} Q^{p/p^*}|\nabla u|^p + \int |\nabla u|^p - \int_{|x|<2\delta} |\nabla u|^p \\ &= \int_{|x|<\delta} ((1+\eta)^{p/p^*} - 1) C|x|^{p/(p-1)}|u|^{p^*} \\ &+ \int_{\delta<|x|<2\delta} (Q^{p/p^*} - 1) C|x|^{p/(p-1)}|u|^{p^*} + S\left(\int |u|^{p^*}\right)^{p/p^*} \end{aligned}$$

$$\leq \frac{p}{p^*}\eta C\delta^{p/(p-1)}\int_{|x|<\delta}|u|^{p^*} + C\frac{p}{p^*}\eta(2\delta)^{p/(p-1)}\int_{\delta<|x|<2\delta}|u|^{p^*}$$
$$+ S(\int|u|^{p^*})^{p/p^*}$$
$$\leq \frac{p}{p^*}\eta C\delta^{p/(p-1)}\int_{|x|<\delta}|u|^{p^*} + C_1\frac{p}{p^*}\eta(2\delta)^{p/(p-1)}\int_{|x|<\delta}|u|^{p^*} + S(\int|u|^{p^*})^{p/p^*}.$$

We have used the inequalities $(1+\eta)^{p/p^*} \leq 1 + \frac{p}{p^*}\eta$ and $\int_{\delta<|x|<2\delta}|u|^{p^*} \leq C_2\int_{|x|<\delta}|u|^{p^*}$. The second one follows easily from the fact that u is decreasing in $|x|$. Also,

$$(\int Q|u|^{p^*})^{p/p^*} = (\eta\int_{|x|<\delta}|u|^{p^*} + \int_{\delta<|x|<2\delta}(Q-1)|u|^{p^*} + \int|u|^{p^*})^{p/p^*}$$
$$\geq (\eta\int_{|x|<\delta}|u|^{p^*} + \int|u|^{p^*})^{p/p^*}.$$

Taylor expansion of $f(x) = x^{p/p^*}$ about $\int|u|^{p^*}$ gives

$$(\int Q|u|^{p^*})^{p/p^*} \geq (\int|u|^{p^*})^{p/p^*} + \frac{p}{p^*}(\int|u|^{p^*})^{p/p^*-1}\eta\int_{|x|<\delta}|u|^{p^*}$$
$$+ o((\eta\int_{|x|<\delta}|u|^{p^*})^2).$$

So we see that (8.9) holds if we can show that

$$\frac{p}{p^*}\eta C\delta^{p/(p-1)}\int_{|x|<\delta}|u|^{p^*} + C_1\frac{p}{p^*}\eta(2\delta)^{p/(p-1)}\int_{|x|<\delta}|u|^{p^*}$$
$$< S\frac{p}{p^*}(\int|u|^{p^*})^{p/p^*-1}\eta\int_{|x|<\delta}|u|^{p^*} + o((\eta\int_{|x|<\delta}|u|^{p^*})^2)$$

Since $\int_{|x|<\delta}|u|^{p^*} = o(\delta)$, the above inequality can be re-written in the form $A_1\delta^{p/(p-1)} < A_2 + o(\delta)$. Hence it suffices to choose $\delta > 0$ small enough.

Remark 8.6. The above theorem together with the example reveals a rather surprising fact regarding the embedding $D^{k,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N, Q)$. In Section 5 we saw that in general not all minimizing sequences are relatively compact if the norms (2.1) or (2.2) are used in $D^{k,p}(\mathbb{R}^N)$. But, there may exist an equivalent norm in $D^{k,p}(\mathbb{R}^N)$ so that all minimizing sequences are relatively compact.

Returning to inequality (8.2) we see that if

$$\inf_{x\in\mathbb{R}^N}\frac{H(x)}{(Q(x))^{p/p^*}} = \frac{\bar{H}}{\|Q\|_\infty^{p/p^*}}$$

then the hypothesis of Theorem 8.4 cannot be satisfied. In this case minimizing sequences are not relatively compact and minimizers do not exist. More precisely, we have the following proposition which of course is a straight forward generalization of the observations made in Section 5.

Proposition 8.7. *Suppose that* $I = \frac{S\bar{H}}{\|Q\|_\infty^{p/p^*}}$. *If* $E_Q = \{x \in \mathbb{R}^N : \|Q\|_\infty = Q(x)\}$ *or* $E_H = \{x \in \mathbb{R}^N : \bar{H} = H(x)\}$ *has measure zero, then there are no minimizers to problem* (8.1).

Proof. We argue as we did in Proposition 5.2. Suppose that $I = S\frac{\bar{H}}{\|Q\|_\infty^{p/p^*}}$ and $u \in D^{k,p}(\mathbb{R}^N)$ is a minimizer for I. Then

$$\Big(\int Q|u|^{p^*}\Big)^{p/p^*} \leq \|Q\|_\infty^{p/p^*}\Big(\int |u|^{p^*}\Big)^{p/p^*} \leq \frac{S^{-1}\|Q\|_\infty^{p/p^*}}{\bar{H}}\int \bar{H}|\nabla^k u|^p$$
$$= I^{-1}\int \bar{H}|\nabla^k u|^p \leq I^{-1}\int H|\nabla^k u|^p = \Big(\int Q|u|^{p^*}\Big)^{p/p^*}$$

So it follows that $\int(\|Q\|_\infty - Q)|u|^{p^*}dx = 0$ and $\int(H-\bar{H})|\nabla^k u|^p = 0$. Hence there are no minimizers.

□

Remark 8.8. Combining the above proposition and Theorem 8.4 gives another interesting result. Suppose that $E_Q = \{x \in \mathbb{R}^N : \|Q\|_\infty = Q(x)\}$ or $E_H = \{x \in \mathbb{R}^N : \bar{H} = H(x)\}$ has measure zero, then $I < \inf_{x\in\mathbb{R}^N}\frac{H(x)}{(Q(x))^{p/p^*}}$ implies $I > \frac{S\bar{H}}{\|Q\|_\infty^{p/p^*}}$.

Now we turn to the problem of finding symmetric minimizers. Assuming that H and Q are G-invariant, we consider the following problem

$$I_G = \inf\{\|u\|_{k,p,H}^p : u \in D_G^{k,p}(\mathbb{R}^N), \|u\|_{p^*,Q} = 1\}, \tag{8.10}$$

where $\|u\|_{k,p,H} = \|(-\Delta)^{k/2}u\|_{p,H}$ when k is even and $\|\nabla(-\Delta)^{(k-1)/2}u\|_{p,H}$ when k is odd. We can now state the conditions under which a minimizer to the above problem exists. We use the same notation for H as we do for Q.

Theorem 8.9. *If $I_G < \min\{\frac{H_0}{Q_0^{p/p^*}}, \frac{H_\infty}{Q_\infty^{p/p^*}}\}S$ then the infimum in (8.10) is attained.*

The above theorem is a straight forward generalization of Theorem 6.1 and the proof is an obvious adaptation of that of Theorem 6.1. From the above theorem we can immediately conclude that if $Q_0 = Q_\infty = 0$ then a minimizer to problem (8.10) exists. By using explicitly the properties of the minimizers of problem (3.3) we can state explicit conditions on H and Q so that the minimizer of problem (8.10) exists.

The following corollaries are generalizations of Corollaries 7.6 and 7.7.

Corollary 8.10. *Assume that H and Q are G-symmetric functions, $\frac{H_0}{Q_0^{p/p^*}} < \frac{H_\infty}{Q_\infty^{p/p^*}}$ and $H_0 = \sup H$. If either*
(i) $Q(x) \geq Q_0 + \epsilon|x|^{N/(p-1)}$ for some $\epsilon > 0$ and $|x|$ small or
(ii) $|Q(x) - Q_0| \leq C|x|^\alpha$ for some constant $C > 0$,$\alpha > N/(p-1)$, $|x|$ small and

$$\int_{\mathbb{R}^N}(Q(x) - Q_0)|x|^{-pN/(p-1)}dx > 0$$

then there exists a minimizer for problem (8.10) with $p > 1$ and $k = 1$.

Proof. We know that the instanton $v(x) = (1+|x|^{p/(p-1)})^{-N/p^*}$ is the unique minimizer for problem (3.3) with $k = 1$ and $p > 1$, up to translation and dilation. In view of Theorem 8.9,

$$I_G \leq \frac{\int H|\nabla A(v(x/\eta))|^p}{(\int Q|Av(x/\eta)|^{p^*})^{p/p^*}} < \frac{H_0 \int |\nabla(Av(x/\eta))|^p}{(Q_0 \int |Av(x/\eta)|^{p^*})^{p/p^*}} = S\frac{H_0}{Q_0^{p/p^*}},$$

where $A > 0$ is a constant chosen so that $\|Av(x/\eta)\|_{1,p} = 1$. Since $\int H|\nabla(Av(x/\eta))|^p \leq H_0 \int |\nabla(Av(x/\eta))|^p$, it suffices to show that for some $\eta > 0$

$$\int_{\mathbb{R}^N} (Q(x) - Q_0)\Big(\frac{1}{\eta^{p/(p-1)} + |x|^{p/(p-1)}}\Big)^N > 0.$$

The proof is as in Corollary 7.1 (cf. Corollary 7.6). □

Corollary 8.11. *Assume that H and Q are G-symmetric functions, $\frac{H_\infty}{Q_\infty^{p/p^*}} < \frac{H_0}{Q_0^{p/p^*}}$ and $H_\infty = \sup H$. If either*
(i) $Q(x) \geq Q_\infty + \epsilon|x|^{-N}$ for some $\epsilon > 0$ and $|x|$ large or
(ii) $|Q(x) - Q_\infty| \leq C|x|^{-\alpha}$ for some constant $C > 0$,$\alpha > N$, $|x|$ large and

$$\int_{\mathbb{R}^N} (Q(x) - Q_\infty)dx > 0,$$

then there exists a minimizer for problem (8.10) with $p > 1$ and $k = 1$.

Proof. The instanton $v(x) = (1 + |x|^{p/(p-1)})^{-N/p^*}$ is the unique minimizer for problem (3.3) with $k = 1$ and $p > 1$, up to translation and dilation. In view of theorem 8.9, we have to show that for some $\eta > 0$

$$I_G \leq \frac{\int H|\nabla A(v(x/\eta))|^p}{(\int Q|Av(x/\eta)|^{p^*})^{p/p^*}} < \frac{H_\infty \int |\nabla(Av(x/\eta))|^p}{(Q_\infty \int |Av(x/\eta)|^{p^*})^{p/p^*}} = S\frac{H_\infty}{Q_\infty^{p/p^*}},$$

where the $A > 0$ is a constant chosen such that $\|Av(x/\eta)\|_{1,2} = 1$. Since $\int H|\nabla A(v(x/\eta))|^p \leq H_\infty \int |\nabla(Av(x/\eta))|^p$, it suffices to show that for some $\eta > 0$

$$\int_{\mathbb{R}^N} (Q(x) - Q_\infty)\Big(\frac{1}{1 + |x/\eta|^{p/(p-1)}}\Big)^N > 0.$$

The proof is as in Corollary 7.2 (cf. Corollary 7.7). □

We see that similar proofs to the ones given for the two preceding corollaries above is valid even when $p = 2$ and $k \geq 1$, and so we have

Corollary 8.12. *Assume that H and Q are G-symmetric functions, $\frac{H_0}{Q_0^{2/2^*}} < \frac{H_\infty}{Q_\infty^{2/2^*}}$ and $H_0 = \sup H$, $2^* = \frac{2N}{N-2k}$. If either*
(i) $Q(x) \geq Q_0 + \epsilon|x|^N$ for some $\epsilon > 0$ and $|x|$ small or
(ii) $|Q(x) - Q_0| \leq C|x|^\alpha$ for some constant $C > 0$,$\alpha > N$, $|x|$ small and

$$\int_{\mathbb{R}^N} (Q(x) - Q_0)|x|^{-2N}dx > 0.$$

Then there exists a minimizer for problem (8.10) with $p = 2$ and $k \geq 1$.

Corollary 8.13. *Assume that H and Q are G-symmetric functions, $\frac{H_\infty}{Q_\infty^{2/2^*}} < \frac{H_0}{Q_0^{2/2^*}}$ and $H_\infty = \sup H$, $2^* = \frac{2N}{N-2k}$. If either*

(i) $Q(x) \geq Q_\infty + \epsilon |x|^{-N}$ *for some* $\epsilon > 0$ *and* $|x|$ *large or*
(ii) $|Q(x) - Q_\infty| \leq C|x|^{-\alpha}$ *for some constant* $C > 0$,$\alpha > N$, $|x|$ *large and*

$$\int_{\mathbb{R}^N} (Q(x) - Q_\infty) dx > 0.$$

Then there exists a minimizer for problem (8.10) with $p = 2$ *and* $k \geq 1$.

9. Singular weights

Let $D_a^{1,2}(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm $(\int_{\mathbb{R}^N} ||x|^{-a}\nabla u|^2\, dx)^{1/2}$. We define,

$$S(a,b) := \inf_{\substack{u \in D_a^{1,2}(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} ||x|^{-a}\nabla u|^2\, dx}{(\int_{\mathbb{R}^N} ||x|^{-b}u|^p\, dx)^{2/p}}, \tag{9.1}$$

and

$$S(a,b,\lambda) := \inf_{\substack{u \in D_a^{1,2}(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} ||x|^{-a}\nabla u|^2 + \lambda ||x|^{-(a+1)}u|^2\, dx}{(\int_{\mathbb{R}^N} ||x|^{-b}u|^p\, dx)^{2/p}}, \tag{9.2}$$

where $N \geq 3$, $0 \leq a < (N-2)/2$, $a \leq b < a+1$,

$$p = p(a,b) := \frac{2N}{N - 2 + 2(b-a)}$$

and λ is a real parameter. Due to an inequality by Caffarelli, Kohn and Nirenberg [7] $S(a,b)$ and $S(a,b,\lambda)$ are positive for $a \leq b \leq a+1$ and suitable λ (see [30]).

The first problem was studied in [17] when $a = 0$, and for positive a it was studied in [10]. There one can also find an explicit form of the minimizer. Both problems were then studied in [30] by using a different method. There the authors proved the existence of minimizers provided $-S(a, a+1) < \lambda < 0$. Some results can also be found in [8]. Due to these results, the method we have developed in the previous sections allows us now to study

$$I(a,b) = \inf_{\substack{u \in D_a^{1,2}(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} ||x|^{-a}\nabla u|^2\, dx}{(\int_{\mathbb{R}^N} Q||x|^{-b}u|^p\, dx)^{2/p}} \tag{9.3}$$

and

$$I(a,b,\lambda) = \inf_{\substack{u \in D_a^{1,2}(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} ||x|^{-a}\nabla u|^2 + \lambda ||x|^{-(a+1)}u|^2\, dx}{(\int_{\mathbb{R}^N} Q||x|^{-b}u|^p\, dx)^{2/p}}. \tag{9.4}$$

In a recent paper by Deng and Jin [11] the authors studied the second problem when $a = 0$ and Q is G-symmetric. Our method will allow us to improve the results given in [11]. We mention here that the above problems are delicate when $a = b$ since then we are dealing with the critical Sobolev constant.

In our present work, we are mainly interested in the case when Q is G-symmetric, but as an illustration of the advantage of our method, we give the following simple result. Since problems (9.1) and (9.2) are dilation invariant, we have by the same argument as in the beginning of the previous section that

$$\frac{S(a,b)}{\|Q\|_\infty^{2/p}} \leq I(a,b) \leq \min\{(Q_0)^{-2/p}, (Q_\infty)^{-2/p}\} S(a,b)$$

and

$$\frac{S(a,b,\lambda)}{\|Q\|_\infty^{2/p}} \leq I(a,b,\lambda) \leq \min\{(Q_0)^{-2/p}, (Q_\infty)^{-2/p}\} S(a,b,\lambda)$$

provided $Q_\infty = \lim_{|x|\to\infty} Q(x)$ exists. This shows that the assumption of the following proposition is satisfied by some Q. With this in mind, we state conditions under which minimizers to problems (9.3) and (9.4) will not exist.

Proposition 9.1. *If $I(a,b) = \frac{S(a,b)}{\|Q\|_\infty^{2/p}}$, $I(a,b,\lambda) = \frac{S(a,b,\lambda)}{\|Q\|_\infty^{2/p}}$ and if $E = \{x \in \mathbb{R}^N : Q(x) = \|Q\|_\infty\}$ has measure zero then there are no minimizers for $I(a,b)$ and $I(a,b,\lambda)$.*

Proof. The argument is the same as in Proposition 8.7 but somewhat simpler. □

Assume now that Q is a G-symmetric function. Denote by $D^{1,2}_{a,G}(\mathbb{R}^N)$ the subspace of $D^{1,2}_a(\mathbb{R}^N)$ consisting of G-symmetric functions. $S_G(a,b)$, $S_G(a,b,\lambda)$, $I_G(a,b)$ and $I_G(a,b,\lambda)$ will denote the infima as in (9.1) - (9.4), but with $D^{1,2}_a(\mathbb{R}^N)$ replaced by $D^{1,2}_{a,G}(\mathbb{R}^N)$. Of course we have a similar result to Proposition 9.1 with identical proof, in this symmetric case.

Proposition 9.2. *If $I_G(a,b) = \frac{S_G(a,b)}{\|Q\|_\infty^{2/p}}$, $I_G(a,b,\lambda) = \frac{S_G(a,b,\lambda)}{\|Q\|_\infty^{2/p}}$ and if $E = \{x \in \mathbb{R}^N : Q(x) = \|Q\|_\infty\}$ has measure zero then there are no minimizers for $I_G(a,b)$ and $I_G(a,b,\lambda)$.*

We start by stating one more version of the concentration-compactness lemma.

Lemma 9.3. *(Concentration-compactness lemma). Assume that Q is a G-symmetric continuous, bounded function and let $N \geq 3$, $0 \leq a < (N-2)/2$, $a \leq b < a+1$, $p = p(a,b)$ and $-I(a,a+1) < \lambda$. Let $\{u_n\}_{n=1}^\infty \subset D^{1,2}_{a,G}(\mathbb{R}^N)$ be a sequence such that*

$$\begin{aligned}
u_n &\rightharpoonup u && \text{in } D^{1,2}_{a,G}(\mathbb{R}^N)\\
||x|^{-a}\nabla(u_n-u)|^2 &\overset{*}{\rightharpoonup} \mu && \text{in } M(\mathbb{R}^N)\\
Q||x|^{-b}(u_n-u)|^p &\overset{*}{\rightharpoonup} \nu && \text{in } M(\mathbb{R}^N)\\
||x|^{-a}\nabla(u_n-u)|^2 + \lambda||x|^{-(a+1)}u-u_n|^2 &\overset{*}{\rightharpoonup} \gamma && \text{in } M(\mathbb{R}^N)\\
u_n &\to u && \text{a.e. on } \mathbb{R}^N
\end{aligned}$$

and define

$$\begin{aligned}
\mu_\infty &:= \lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int_{|x|>R} ||x|^{-a}\nabla u|^2\ ,\\
\nu_\infty &:= \lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int_{|x|>R} Q||x|^{-b}u|^p\ ,\\
\gamma_\infty &:= \lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int_{|x|>R} ||x|^{-a}\nabla u|^2 + \lambda||x|^{-(a+1)}u|^2.
\end{aligned} \tag{9.5}$$

Then it follows that

$$\|\nu\|^{2/p} \leq I_G(a,b)^{-1}\|\mu\|, \tag{9.6}$$

$$\|\nu\|^{2/p} \leq I_G(a,b,\lambda)^{-1}\|\gamma\|, \tag{9.7}$$

$$\nu_\infty^{2/p} \leq I_G(a,b)^{-1}\mu_\infty, \tag{9.8}$$

$$\nu_\infty^{2/p} \leq I_G(a,b,\lambda)^{-1}\gamma_\infty, \tag{9.9}$$

$$\overline{\lim_{n\to\infty}}\||x|^{-a}\nabla u_n\|_2^2 = \||x|^{-a}\nabla u\|_2^2 + \|\mu\| + \mu_\infty, \tag{9.10}$$

$$\begin{aligned}&\overline{\lim_{n\to\infty}}\||x|^{-a}\nabla u_n\|_2^2 + \lambda\||x|^{-(a+1)}u_n\|_2^2 \\ &\qquad = \||x|^{-a}\nabla u_n\|_2^2 + \lambda\||x|^{-(a+1)}u_n\|_2^2 + \|\mu\| + \mu_\infty,\end{aligned} \tag{9.11}$$

$$\overline{\lim_{n\to\infty}}\|u_n\|_{p,Q}^p = \|u\|_{p,Q}^p + \|\nu\| + \nu_\infty. \tag{9.12}$$

Further, suppose $u = 0$, *then* $\|\nu\|^{p/p^*} = I_G(a,b)^{-1}\|\mu\|$ *implies that* ν, μ *are concentrated at a single point and* $\|\nu\|^{p/p^*} = I_G(a,b,\lambda)^{-1}\|\gamma\|$ *implies that* ν, γ *are concentrated at a single point. This point must be the origin.*

The last statement follows from the fact that, if concentration occurs at x, then it must occur at $g(x)$ for all $g \in G$. The proof is similar to that of Lemma 4.3, keeping in mind that $D_{a,G}^{1,2}(\mathbb{R}^N)$ is a Hilbert space, and so Remark 4.5 is applicable. The only technical point is the verification of a result similar to Proposition 4.1. This can be easily deduced by using the following lemma, which is actually similar to Lemma 2 in [30] and its proof is easily adapted.

Lemma 9.4. *Let* $N \geq 3$ *and* $0 \leq a < (N-2)/2$. *If* $u_n \rightharpoonup u$ *in* $D_{a,G}^{1,2}(\mathbb{R}^N)$ *then* $|x|^{-a}u_n \to |x|^{-a}u$ *in* $L_{loc}^2(\mathbb{R}^N)$.

Remark 9.5. If $\{u_n\}_{n=1}^\infty \subset D_{a,G}^{1,2}(\mathbb{R}^N)$ is a bounded sequence such that $Q||x|^{-b}(u_n-u)|^p \overset{*}{\rightharpoonup} \nu$ then we may assume that $||x|^{-b}(u_n-u)|^p \overset{*}{\rightharpoonup} \alpha$, for some α. Hence, by defining α_∞ in the way ν_∞ is defined, we see that $\nu(\{x\}) = Q(x)\alpha(\{x\})$ and $\nu_\infty \leq Q_\infty\alpha_\infty$ where $Q_\infty = \overline{\lim}_{|x|\to\infty} Q(x)$. Further, $\nu_\infty = Q_\infty\alpha_\infty$ if $Q_\infty = \overline{\lim}_{|x|\to\infty} Q(x) = \lim_{|x|\to\infty} Q(x)$.

With concentration-compactness lemma at our disposal, we may proceed to compare $I_G(a,b)$ and $S_G(a,b)$ as required by our method. We know from [10] that function

$$u(x) = (1+|x|^{2a-bp+2})^{\frac{N-2a-2}{2a-bp+2}} \tag{9.13}$$

is, up to dilation and multiplication by a constant, a minimizer for $S(a,b)$. Since $S(a,b) \leq S_G(a,b)$ and the above minimizer is radially symmetric, we have $S(a,b) = S_G(a,b)$.

The following theorem is the main result of this section.

Theorem 9.6. *If* $I_G(a,b) < \min\{Q_0^{-2/p}, Q_\infty^{-2/p}\}S_G(a,b)$ *then all minimizing sequences are relatively compact. In particular, there is a minimizer for* $I_G(a,b)$.

Proof. The argument is similar to the ones given in the previous sections. Therefore we omit some details. Let $\{u_n\}_{n=1}^{\infty} \subset D^{1,2}_{a,G}(\mathbb{R}^N)$ be a minimizing sequence for $I_G(a,b)$. Going if necessary to a subsequence, still denoted by u_n, we may assume that the conditions of Lemma 9.3 are fullfilled. Hence

$$I_G(a,b) = \overline{\lim_{n\to\infty}} \||x|^{-a}\nabla u_n\|_2^2 = \||x|^{-a}\nabla u\|_2^2 + \|\mu\| + \mu_\infty$$

and

$$1 = \overline{\lim_{n\to\infty}} \|u_n\|_{p,Q}^p = \|u\|_{p,Q}^p + \|\nu\| + \nu_\infty.$$

So we have using inequalities (9.6) and (9.8)

$$\begin{aligned} I_G(a,b)(\|u\|_{p,Q}^p + \|\nu\| + \nu_\infty)^{2/p} &= \||x|^{-a}\nabla u\|_2^2 + \|\mu\| + \mu_\infty \\ &\geq I_G(a,b)((\|u\|_{p,Q}^p)^{2/p} + \|\nu\|^{2/p} + \nu_\infty^{2/p}). \end{aligned}$$

Since $p > 2$, we deduce that only one of the quantities $\|u\|_{p,Q}^p$, $\|\nu\|$ and ν_∞ is 1 and the other are zero. Suppose $Q_\infty \neq 0$. If $\nu_\infty = 1$, we obtain a contradiction, since from Remark 9.5 and Lemma 9.3 we have

$$\begin{aligned} S_G(a,b)(\alpha_\infty)^{2/p} &> I_G(a,b)(Q_\infty \alpha_\infty)^{2/p} \geq I_G(a,b)(\nu_\infty)^{2/p} \\ &\geq \mu_\infty \geq S_G(a,b)(\alpha_\infty)^{2/p}. \end{aligned}$$

If $Q_0 \neq 0$ and $\|\nu\| = 1$ then $u = 0$ and $\|\nu\|^{p/p^*} = I_G(a,b)^{-1}\|\mu\|$ and so ν is concentrated at the origin. Once again we obtain a contradiction since

$$\begin{aligned} S_G(a,b)(\alpha(\{0\}))^{2/p} &> I_G(a,b)(Q_0\alpha(\{0\}))^{2/p} = I_G(a,b)(\nu(\{0\}))^{2/p} \\ &\geq \|\mu\| \geq S_G(a,b)(\alpha(\{0\}))^{2/p}. \end{aligned}$$

So it follows that $\|u\|_{p,Q}^p = 1$ and we reach the desired conclusion. When $Q_0 = 0$ then $\|\nu\| = 0$ and when $Q_\infty = 0$ then $\nu_\infty = 0$, so we will have $\|u\|_{p,Q}^p = 1$. □

Set $u_\eta(x) = u(x/\eta) = (1 + |x/\eta|^{2a-bp+2})^{\frac{N-2a-2}{2a-bp+2}}$, then

$$S_G(a,b) = \frac{\int_{\mathbb{R}^N} ||x|^{-a}\nabla u_\eta|^2 \, dx}{\int_{\mathbb{R}^N} ||x|^{-b} u_\eta|^p \, dx} \qquad \forall \eta > 0.$$

If we assume that $Q_\infty = \lim_{|x|\to\infty} Q(x)$ then by letting η tend to 0 and ∞, we obtain $I_G(a,b) \leq \min\{Q_0^{-2/p}, Q_\infty^{-2/p}\} S_G(a,b)$. At this point we can easily deduce that if $\min\{Q_0^{-2/p}, Q_\infty^{-2/p}\} = \|Q\|_\infty^{-2/p}$ then by Proposition 9.2, minimizers in general will not exist. However, we have the following corollary to Theorem 9.6, which is similar to Corollary 6.5.

Corollary 9.7. *If Q is G-symmetric, $Q(x) \geq Q_0 = Q_\infty = \lim_{|x|\to\infty} Q(x) \geq 0$ then there is a minimizer for $I_G(a,b)$.*

Proof. If $I_G(a,b) < \min\{Q_0^{-2/p}, Q_\infty^{-2/p}\} S_G(a,b)$ then we are done by theorem 9.6. If $I_G(a,b) = \min\{Q_0^{-2/p}, Q_\infty^{-2/p}\} S_G(a,b)$ let u be the function in (9.13). u is then a minimizer of $S_G(a,b)$, and

$$\begin{aligned} I_G(a,b) &\leq \frac{\int_{\mathbb{R}^N} ||x|^{-a}\nabla u|^2\ dx}{(\int_{\mathbb{R}^N} Q||x|^{-b}u|^p\ dx)^{2/p}} \\ &\leq \frac{\int_{\mathbb{R}^N} ||x|^{-a}\nabla u|^2\ dx}{Q_0^{2/p}(\int_{\mathbb{R}^N} ||x|^{-b}u|^p\ dx)^{2/p}} = \frac{S_G(a,b)}{Q_0^{2/p}} = I_G(a,b). \end{aligned}$$

It follows that u is a minimizer of $I_G(a,b)$. This concludes the proof. □

Of course knowing the explicit form of the minimizer for $S_G(a,b)$ allows us to give conditions on Q, similar to those given in the previous sections, so that minimizers exist.

Corollary 9.8. *Suppose that Q is G-symmetric, $Q_0 \geq Q_\infty > 0$ and either (i) $Q(x) \geq Q_0 + \epsilon|x|^{N-bp}$ for some $\epsilon > 0$ and $|x|$ small or (ii) $|Q(x) - Q_0| \leq C|x|^\alpha$ for some constant $C > 0$,$\alpha > N - bp$, $|x|$ small and*

$$\int_{\mathbb{R}^N} (Q(x) - Q_0)|x|^{-2N+bp}dx > 0.$$

Then there exists a minimizer for $I_G(a,b)$.

Corollary 9.9. *Suppose that Q is G-symmetric, $Q_\infty \geq Q_0 > 0$ and either (i) $Q(x) \geq Q_\infty + \epsilon|x|^{-N+bp}$ for some $\epsilon > 0$ and $|x|$ large or (ii) $|Q(x) - Q_\infty| \leq C|x|^{-\alpha}$ for some constant $C > 0$,$\alpha > N - bp$, $|x|$ large and*

$$\int_{\mathbb{R}^N} (Q(x) - Q_\infty)|x|^{-bp}dx > 0.$$

Then there exists a minimizer for $I_G(a,b)$.

The proofs are similar to the proofs of Corollaries 7.1 and 7.2.

Since we do not know whether there is a minimizer for $S_G(a,b,\lambda)$ when $a > 0$, we are not able to compare $I_G(a,b,\lambda)$ and $S_G(a,b,\lambda)$. However, in the case when $a = 0$ and $0 > \lambda > \bar{\lambda} = -(\frac{n-2}{2})^2$ we know from [11] that, up to multiplication by a constant and dilation, $S(0,b,\lambda)$ is achieved by

$$u(x) = \frac{1}{|x|^{\sqrt{-\bar{\lambda}}-\beta}(1+|x|^{\frac{(2-bp)\beta}{\sqrt{-\bar{\lambda}}}})^{\frac{n-2}{2-bp}}},$$

where $\beta = (\lambda - \bar{\lambda})^{1/2}$. Since the above function is radially symmetric, we deduce that $S_G(0,b,\lambda)$ has a minimizer. We may now continue by stating the following straightforward variants of Theorem 9.6 and its corollary.

Theorem 9.10. *If $I_G(0,b,\lambda) < \min\{Q_0^{-2/p}, Q_\infty^{-2/p}\}S_G(0,b,\lambda)$ then all minimizing sequences are relatively compact. In particular, there exists a minimizer for $I_G(0,b,\lambda)$.*

Corollary 9.11. *If Q is G-symmetric, $Q(x) \geq Q_0 = Q_\infty = \lim_{|x|\to\infty} Q(x) \geq 0$ then $I_G(0,b,\lambda)$ has a minimizer.*

In Deng's and Jin's article (see [11, Theorem 2.1]) the authors presented a result which in effect says that there exists a minimizer for $I_G(0,b,\lambda)$ provided that

$$I_G(0,b,\lambda) < \min\{Q_0^{-2/p}, Q_\infty^{-2/p}, T\}S_G(0,b,\lambda).$$

Here, the third term T depends on $|G| = \inf_{x\in\mathbb{R}^N, x\neq 0}|G_x|$, where $|G_x|$ is the cardinality of the set $G_x = \{gx : g \in G\}$. We see that Theorem 9.10 improves this result, since our condition does not require any knowledge of $|G|$.

Since we know the explicit form of the extremal function for $S(0, b, \lambda)$, we may proceed to formulate explicit conditions on Q so that a minimizer for $I_G(0, b, \lambda)$ exists.

Corollary 9.12. *Suppose that Q is G-symmetric $Q_0 \geq Q_\infty > 0$ and either*
(i) $Q(x) \geq Q_0 + \epsilon|x|^{\frac{2\beta(N-bp)}{N-2}}$ for some $\epsilon > 0$ and $|x|$ small or
(ii) $|Q(x) - Q_0| \leq C|x|^\alpha$ for some constant $C > 0$,$\alpha > \frac{2\beta(N-bp)}{N-2}$, $|x|$ small and

$$\int_{\mathbb{R}^N} (Q(x) - Q_0)|x|^{-N-\frac{2\beta(N-bp)}{N-2}} dx > 0.$$

Then there exists a minimizer for $I_G(0, b, \lambda)$.

Corollary 9.13. *Suppose that Q is G-symmetric $Q_\infty \geq Q_0 > 0$ and either*
(i) $Q(x) \geq Q_\infty + \epsilon|x|^{-\frac{2\beta(N-bp)}{N-2}}$ for some $\epsilon > 0$ and $|x|$ large or
(ii) $|Q(x) - Q_\infty| \leq C|x|^{-\alpha}$ for some constant $C > 0$,$\alpha > \frac{2\beta(N-bp)}{N-2}$, $|x|$ large and

$$\int_{\mathbb{R}^N} (Q(x) - Q_\infty)|x|^{n-\frac{2\beta(N-bp)}{N-2}} dx > 0.$$

Then there exists a minimizer for $I_G(0, b, \lambda)$.

The proofs are similar to those of Corollaries 7.1 and 7.2.

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