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Some computations in Grothendieck rings

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0.1 Introduction

Let p be a prime number and let $\mathbb{Z}_p$ be the ring of p-adic integers. The product $\mathbb{Z}_p^n$ is a compact topological group and hence has a Haar measure μ which we normalize so that $\mu(\mathbb{Z}_p^n) = 1$. Identify the monic polynomials of degree n in $\mathbb{Z}_p[X]$ with $\mathbb{Z}_p^n$ via $X^n + a_1X^{n-1} + \cdots + a_n \mapsto (a_1, \ldots, a_n)$. In [Sko] the measure of the nth degree polynomials that satisfy certain factorization patterns are computed. Firstly the measure of the polynomials of degree n that split completely is a rational function of p. For example, the measure of the 2nd degree polynomials that split completely is $\frac{1}{2(1+p)}$. Secondly, the measure of the polynomials of degree n that are irreducible and unramified is also a rational function of p.

Associated to p is a ring scheme called the Witt vectors and which we denote by $\mathbf{W}$, with the property that the $\mathbb{F}_p$-rational points on $\mathbf{W}$, $\mathbf{W}(\mathbb{F}_p)$, are isomorphic to $\mathbb{Z}_p$. Moreover, if $q = p^r$ then $\mathbf{W}(\mathbb{F}_q)$ is isomorphic to the integral closure of $\mathbb{Z}_p$ in the unramified field extension of degree r of $\mathbb{Q}_p$. (For every r there is exactly one such extension in a fixed algebraic closure of $\mathbb{Q}_p$.) If we now compute the measure of the n'th degree polynomials with coefficients in $\mathbf{W}(\mathbb{F}_q)$ that split completely we get the same rational function as for $\mathbb{Z}_p$, but with p replaced with q. For example, the measure of the degree 2 polynomials with coefficients in $\mathbf{W}(\mathbb{F}_q)$ that split completely is $\frac{1}{2(1+q)}$.

To explain this phenomenon we will define the measure of certain subschemes of $\mathbf{W}^n$. This measure will take its values in the completion of a localization of the Grothendieck ring of finite type schemes over $\mathbb{F}_p$, so that the measure of a subscheme of $\mathbf{W}^n$ can be represented by a fraction of linear combinations of $\mathbb{F}_p$-schemes. It will have the following property: If the measure of the scheme $X \subset \mathbf{W}^n$ is represented by $[X_1]/[X_2]$ where X_1 and X_2 are $\mathbb{F}_p$-schemes then the Haar measure of the $\mathbb{F}_q$-rational points of X equals the number of $\mathbb{F}_q$-rational points of X_1 divided by the number of $\mathbb{F}_q$-rational points of X_2. For example, what was said above indicates that the measure of the scheme of degree 2 polynomials with coefficients in $\mathbf{W}(\mathbb{F}_p)$ that splits completely should be $\frac{1}{2(1+[\mathbb{A}_{\mathbb{F}_p}])}$.

We will define this measure and compute it in the case of polynomials that split completely. This will be done in chapter 3. We have also tried to compute the measure of the scheme of irreducible unramified polynomials, but so far without success. As a warm-up for that problem we do the computations in chapter 2 which turns out to be interesting in their own right.

The type of measure discussed above is called a motivic measure, referring to the fact that the Haar measures for different $\mathbf{W}(\mathbb{F}_q)$ could be perceived as different paintings of the same motive, the measure in the Grothendieck ring. Hence the name has the same explanation as the name of the category of motives, which is a category through which every Weil cohomology factors. Here the cohomology theories are the paintings. (This is the explanation given in [Man68].) Also, the fact that there already is a category of motives prevents us from calling the elements of our Grothendieck ring motives. We say that we compute motivic measures and motivic integrals but never that the integral is a motive. This is further complicated by the fact that there are several different theories of motivic integration, and in some of them the measure takes values in the Grothendieck ring over the category of motives.

For an overview of motivic integration together with further references, see [Loo00]. The variant which we use in chapter 3 is developed to suit our particular problem.

Outline of the thesis

In chapter 1 we collect some of the background material that is needed in order to understand this thesis, and which we do not consider to be well known. We introduce the notions of Grothendieck rings and λ-rings and define the particular rings that we are interested in. We also give an introduction to a ring scheme that is called the Witt vectors. Even though all the material in this chapter is already known we still give proofs of some of the results. Occasionally we just give a reference to a proof and sometimes we do neither.

Chapter 2 contains the computation of the class of an algebraic torus in the Grothendieck ring of varieties over a field. We arrive at a closed formula expressed in terms of elements of the Burnside ring of the symmetric group Σ_n. We then express these elements in terms of the λ-ring structure on $\mathcal{B}(\Sigma_n)$, so this chapter also contains an investigation of this particular λ-ring.

Chapter 3 is our attempt to generalize the above-mentioned Haar measure computations to a computation of a motivic measure.

Notation and prerequisites

We use the following standard notations: $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{Z}$ = ring of integers, $\mathbb{Q}$ = field of rational numbers, $\mathbb{R}$ = field of real numbers, $\mathbb{C}$ = field of complex numbers and $\mathbb{F}_q$ = field with q elements, q a power of a prime. Also, $\mathbb{Z}_p$ = ring of p-adic integers and $\mathbb{Q}_p$ = field of p-adic numbers. To denote a general field we use the letter k.

By a ring we will mean a commutative ring with unit.

When X is isomorphic to Y we write $X \simeq Y$. If X is defined to be Y we write $X := Y$. Finally $X \subset Y$ means that X is a, not necessarily proper, subset of Y.

We use **Sets**, **Rings**, $\mathbf{Alg}_A$ and **Sch** to denote the categories of sets, rings, A-algebras and schemes respectively.

We assume knowledge of the language of schemes as presented in chapter II of [Har77]. In particular if A is a ring, B an A-algebra and X is a scheme over A, then $X(B)$ is the set of points of X with coordinates in B, $\mathrm{Hom}_{A-\mathrm{schemes}}(\operatorname{Spec} B, X)$, whereas X_B is the scheme over B obtained from X by base extension, $X \times_A \operatorname{Spec} B$. We also frequently use the following two facts about schemes: An A-scheme X is determined by its functor of points $X(-)\colon \mathbf{Alg}_A \to \mathbf{Sets}$, and a functor $F\colon \mathbf{Alg}_A \to \mathbf{Sets}$ is an affine scheme if and only if it is representable by an A-algebra C, so that $F(B) = \mathrm{Hom}_{A-\mathrm{alg}}(C, B)$ for every A-algebra B. This also makes it easy to define an affine ring scheme over A, which is just a functor $\mathbf{Alg}_A \to \mathbf{Rings}$ whose composition with the forgetful functor to **Sets** is representable.

We use $\mathbb{G}_m$ and $\mathbb{G}_a$ to denote the multiplicative and additive group schemes respectively. By a torus we mean a group scheme that becomes isomorphic to $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$ after an extension of the base.

If A is a ring we use $\mathrm{D}(f)$ and $\mathrm{V}(I)$ to denote the open respectively closed subschemes of $\operatorname{Spec} A$ determined by the element $f \in A$ respectively the ideal $I \subset A$. If A is graded we use D_+ and V_+ to denote the corresponding subschemes of $\operatorname{Proj} A$.

If R is a ring then $R^\times$ is the group of invertible elements in R.

By Σ_n we mean the symmetric groups, i.e., Σ_n is the permutation group of $\{1, \dots, n\}$.

By a partition of n we mean a weakly decreasing sequence of positive integers which sum to n. We write $\lambda \vdash n$ to indicate that λ is a partition of n.

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Chapter 1

Background material

In this chapter we will bring up definitions and theorems that are of repeated use in this thesis.

1.1 Grothendieck rings

Let $\mathcal{A}$ be an abelian category. The Grothendieck group of $\mathcal{A}$ is then defined to be the free abelian group generated by $\{[A] : A \in \mathrm{ob}(\mathcal{A})\}$, subject to the relations that if $0 \to A \to B \to C \to 0$ is exact then $[B] = [A] + [C]$. (It follows in particular that $[A] = [B]$ if $A \simeq B$.) We denote this group by $\mathrm{K}_0(\mathcal{A})$.

Let $\mathcal{C}$ be a non-abelian category. As above we can form the free abelian group generated by $\{[A] : A \in \mathrm{ob}(\mathcal{C})\}$ and then form a quotient of this group. We will sometimes call this a Grothendieck group of $\mathcal{C}$, in analogy with the case when $\mathcal{C}$ is abelian.

A much studied example of a Grothendieck group is $\mathrm{K}_0(\mathbf{Mod}_R)$ where $\mathbf{Mod}_R$ is the category of finitely generated projective modules over the commutative ring R. This category is in general not abelian but we use the same defining relations as if it where. Since every short exact sequence of projective modules is split this means equivalently that the defining relations of $\mathrm{K}_0(\mathbf{Mod}_R)$ is $[P_1] = [P_2]$ if $P_1 \simeq P_2$ and $[P_1 \oplus P_2] = [P_1] + [P_2]$. This group can also be given the structure of a ring by defining $[P_1]\cdot[P_2] := [P_1 \otimes_R P_2]$. This product is well defined: If $P_2 \simeq P_1 \oplus P_3$ then $P \otimes_R P_2 \simeq (P \otimes_R P_1) \oplus (P \otimes_R P_3)$. It hence follows that $[P] \cdot [P_2] = [P] \cdot [P_1] + [P] \cdot [P_3]$ if $[P_2] = [P_1] + [P_3]$.

By a Grothendieck ring we mean a ring constructed in analogy with the above example, i.e., an abelian group where the generators are isomorphism classes of objects in some category. These generators are subject to relations that were given above if the category is abelian and that we have to define from case to case if the category is not abelian. The ring also has a multiplication that we have to define in each particular case.

If R is an abelian group and we want to define a group homomorphism $\mathrm{K}_0(\mathcal{A}) \to R$ then we often do this by giving a map $\phi\colon \mathrm{ob}(\mathcal{A}) \to R$. We then have to check that ϕ respects the relations. In the abelian case this means that $\phi(B) = \phi(A) + \phi(C)$ if $0 \to A \to B \to C \to 0$ is exact. Moreover, if we have given $\mathrm{K}_0(\mathcal{A})$ the structure of a ring and R is a ring, then we get a ring homomorphism if ϕ respects the multiplication on the generators.

1.1.1 The Grothendieck ring of schemes of finite type over a field

Fix a field k, and let $\mathbf{Sch}_k$ be the category of schemes of finite type over k. We now construct a Grothendieck ring over this category. Since $\mathbf{Sch}_k$ is not abelian we have to define the relations that we use.

Definition 1.1.1. *Let k be a field. The Grothendieck ring of schemes of finite type over k, $K_0(\mathbf{Sch}_k)$, is the free abelian group generated by symbols $[Y]$, for $Y \in \mathbf{Sch}_k$, with a multiplication given by*

$$[Y] \cdot [Z] := [Y \times_{\operatorname{Spec} k} Z]$$

and subject to the following relations:

$$\begin{aligned} &[Y] = [Z] && \text{if } Y \simeq Z \\ &[Y] = [Y \setminus Z] + [Z] && \text{if } Z \text{ is a closed subscheme of } [Y]. \end{aligned}$$

This is well defined for if Z is a closed subscheme of Y then $Z \times_k X$ is a closed subscheme of $Y \times_k X$ with complement $(Y \setminus Z) \times_k X$.

In $\mathrm{K}_0(\mathbf{Sch}_k)$ we have that $0 = [\emptyset]$ and $1 = [\operatorname{Spec} k]$. As a special case of the second relation we get $[\operatorname{Spec} A \oplus B] = [\operatorname{Spec} A] + [\operatorname{Spec} B]$. Also, every scheme Y has a unique reduced closed subscheme $Y_{\mathrm{red}} \to Y$ having the same underlying topological space as Y. From the second relation we see that $[Y] = [\emptyset] + [Y_{\mathrm{red}}]$ and hence $[Y_{\mathrm{red}}] = [Y]$.

Define $\mathbb{L} := [\mathbb{A}^1_k]$. ($\mathbb{L}$ is for Lefschetz). Since $\mathbb{A}^n_k = \mathbb{A}^1_k \times \cdots \times \mathbb{A}^1_k$ we get $[\mathbb{A}^n_k] = \mathbb{L}^n$. To find $[\mathbb{P}^n_k]$ recall that $\mathbb{P}^{n-1}_k \simeq \operatorname{Proj} k[X_0 \dots, X_n]/(X_0)$ is isomorphic to a closed subscheme of $\mathbb{P}^n_k = \operatorname{Proj} k[X_0 \dots, X_n]$ with support $V_+(X_0)$. Since $D_+(X_0) \simeq \operatorname{Spec} k[X_1/X_0, \dots, X_n/X_0] \simeq \mathbb{A}^n_k$ we get $[\mathbb{P}^n_k] = \mathbb{L}^n + [\mathbb{P}^{n-1}_k]$ when $n \geq 1$. When $n = 0$ we get $[\mathbb{P}^0_k] = [\operatorname{Spec} k] + [\operatorname{Proj} k] = 1 + 0$ so the formula $[\mathbb{P}^n_k] = \mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + 1$ follows by induction.

Remark. Perhaps a more common construction is to use instead the category of varieties over k, where variety here means a reduced, separated scheme of finite type over k. However, this construction gives an isomorphic ring, so we could refer to $\mathrm{K}_0(\mathbf{Sch}_k)$ as the Grothendieck ring of varieties over k. (Some authors even use the term variety to mean a scheme of finite type over a field, see for example [Liu02].) We also remark that if we instead do this construction over the category of all schemes over k then we end up with the zero ring, because if Y is any k-scheme and Z is an infinite disjoint union of copies of Y then $[Z] = [Z] + [Y]$ and hence $[Y] = 0$. On the other hand, there are many open questions about the structure of $\mathrm{K}_0(\mathbf{Sch}_k)$. For example, if $\operatorname{char}(k) \neq 0$ then it is not known whether $\mathrm{K}_0(\mathbf{Sch}_k)$ is a domain. If instead $\operatorname{char}(k) = 0$ then, by [Poo02], $\mathrm{K}_0(\mathbf{Sch}_k)$ is not a domain.

We next consider some examples of how the relations work.

Exampel 1.1.2. *Consider the open subscheme $D(X) \subset \operatorname{Spec} k[X] = \mathbb{A}^1_k$. Since $V(X)$ can be given the structure of a closed subscheme isomorphic to $\operatorname{Spec} k[X]/(X) \simeq \operatorname{Spec} k$ we get*

$$[D(X)] = [\mathbb{A}^1_k] - [V(X)] = \mathbb{L} - [\operatorname{Spec} k] = \mathbb{L} - 1.$$

On the other hand, $D(X) \simeq \operatorname{Spec} k[X]_X \simeq \operatorname{Spec} k[X, 1/X] \simeq \mathbb{G}_m$, so $[\mathbb{G}_m] = \mathbb{L} - 1$.

Exampel 1.1.3. *Let k be a field of characteristic different from 2 and let C_P be an irreducible projective conic with a k-rational point. (Every conic has a k-rational point if k is algebraically closed or finite.) We then know that there exists an isomorphism $C_P \simeq \mathbb{P}^1_k$ so $[C_P] = \mathbb{L} + 1$. The situation is more complicated in the case of an affine conic. Consider for example the unit circle $C_A := \operatorname{Spec} k[X, Y]/(X^2 + Y^2 - 1)$. We have*

$$\begin{aligned} \mathbb{L} + 1 =& \big[\operatorname{Proj} k[X, Y, Z]/(X^2 + Y^2 - Z^2)\big] \\ =& [V_+(Z)] + [D_+(Z)] \\ =& \big[\operatorname{Proj} k[X, Y]/(X^2 + Y^2)\big] + [C_A], \end{aligned}$$

where

$$\big[\operatorname{Proj} k[X, Y]/(X^2 + Y^2)\big] = [V_+(Z)] + [D_+(Z)] = 0 + \big[\operatorname{Spec} k[X]/(X^2 + 1)\big].$$

So if -1 is a square in k then $[C_A] = \mathbb{L} + 1 - [\operatorname{Spec} k^2] = \mathbb{L} - 1$ whereas if -1 is a non-square then $[C_A] = \mathbb{L} + 1 - [\operatorname{Spec} K]$ where $K = k(\sqrt{-1})$. An element of the form $[\operatorname{Spec} K]$ where K/k is a finite field extension is an example of what later will be called an artin class.

We expand the definition of the class of a scheme in $\mathrm{K}_0(\mathbf{Sch}_k)$ so that we also can talk about the class of a constructible subset of a scheme (i.e., a finite disjoint union of locally closed sets). For the following proposition, see the introduction to [DL99].

Proposition 1.1.4. *If Y is a scheme of finite type over k then the map $Y' \to [Y']$ from the set of closed subschemes of Y extends uniquely to a map $Z \to [Z]$ from the set of constructible subsets of Y to $\mathrm{K}_0(\mathbf{Sch}_k)$, satisfying $[Z \cup Z'] = [Z] + [Z'] - [Z \cap Z']$.*

So if we are given a constructible subset Z of a finite type scheme over k then it has a well defined image $[Z] \in \mathrm{K}_0(\mathbf{Sch}_k)$.

If $k \subset K$ is a finite field extension, then extension and restriction of scalars give rise to maps between $\mathrm{K}_0(\mathbf{Sch}_k)$ and $\mathrm{K}_0(\mathbf{Sch}_K)$.

Definition-Lemma 1.1.5. *Let K be a field extension of k of finite degree. Define $\operatorname{Res}_k^K \colon \mathrm{K}_0(\mathbf{Sch}_K) \to \mathrm{K}_0(\mathbf{Sch}_k)$ by the map $\mathbf{Sch}_K \to \mathrm{K}_0(\mathbf{Sch}_k)$ that take the K-scheme X to the class of X, viewed as a k-scheme. We have that Res_k^K is additive but not multiplicative.*

*Also, define $\operatorname{Sce}_k^K \colon \mathrm{K}_0(\mathbf{Sch}_k) \to \mathrm{K}_0(\mathbf{Sch}_K)$ by the map $\mathbf{Sch}_k \to \mathrm{K}_0(\mathbf{Sch}_K)$ that take the k-scheme X to the class of $X \times_k \operatorname{Spec} K$, viewed as a K-scheme. This is a ring homomorphism. (*Sce *is short for scalar extension.)*

The fact that Res_k^K fails to be multiplicative can be seen for example if we let $k = \mathbb{F}_p$ and $K = \mathbb{F}_{p^2}$, for then we have $\operatorname{Res}_k^K\big(1 \cdot [\operatorname{Spec} \mathbb{F}_{p^6}]\big) = [\operatorname{Spec} \mathbb{F}_{p^6}]$ whereas $\mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^6} = \mathbb{F}_{p^6}^2$ so

$$\operatorname{Res}_k^K(1) \cdot \operatorname{Res}_k^K\big([\operatorname{Spec} \mathbb{F}_{p^6}]\big) = [\operatorname{Spec} \mathbb{F}_{p^2}] \cdot [\operatorname{Spec} \mathbb{F}_{p^6}] = [\operatorname{Spec} \mathbb{F}_{p^6}^2] = 2 \cdot [\operatorname{Spec} \mathbb{F}_{p^6}].$$

Rather then being multiplicative, Res_k^K has a similar property: If X is k-scheme and Z is a K-scheme then from the universal property defining fibre products we get that $Z \times_K (\operatorname{Spec} K \times_k X) \simeq Z \times_k X$ as k-schemes. It follows that if we apply the restriction map to $[Z] \cdot [X_K] \in \mathrm{K}_0(\mathbf{Sch}_K)$ we get $[Z] \cdot [X] \in \mathrm{K}_0(\mathbf{Sch}_k)$. We will use this in the special case when $X = \mathbb{A}_k^n$ and $Z = \operatorname{Spec} L$ where L is a finite dimensional K-algebra, so we state this as the following proposition.

Proposition 1.1.6. *If K is a finite field extension of k and L is a finite dimensional K-algebra, then for every $n \in \mathbb{N}$ we have*

$$\operatorname{Res}_k^K([\operatorname{Spec} L] \cdot \mathbb{L}^n) = [\operatorname{Spec} L] \cdot \mathbb{L}^n.$$

In particular, $\operatorname{Res}_k^K(1) = [\operatorname{Spec} K]$ and $\operatorname{Res}_k^K(\mathbb{L}^n) = [\operatorname{Spec} K] \cdot \mathbb{L}^n$.

If we work over a finite field then the construction of $\mathrm{K}_0(\mathbf{Sch}_k)$ is compatible with point counting, as the following shows.

Definition-Lemma 1.1.7. *For q a prime power, let $C_q \colon \mathrm{K}_0(\mathbf{Sch}_{\mathbb{F}_q}) \to \mathbb{Z}$ be the map defined by $X \mapsto |X(\mathbb{F}_q)| \colon \operatorname{ob}(\mathbf{Sch}_{\mathbb{F}_q}) \to \mathbb{Z}$. Then C_q is a ring homomorphism.*

Proof. Let $T \in \mathbf{Sch}_{\mathbb{F}_q}$ be arbitrary. If $f \colon Y \to Z$ is an isomorphism then $g \mapsto f \circ g \colon Y(T) \to Z(T)$ is a bijection. Moreover, since $\operatorname{Spec} \mathbb{F}_q$ is a point, $|\operatorname{Hom}(\operatorname{Spec} \mathbb{F}_q, Y)| = |\operatorname{Hom}(\operatorname{Spec} \mathbb{F}_q, Z)| + |\operatorname{Hom}(\operatorname{Spec} \mathbb{F}_q, Y \setminus Z)|$ if Z is a closed subscheme of Y. Hence C_q is well defined. It is multiplicative since

$$\begin{aligned} (Y \times_{\mathbb{F}_q} Z)(T) &\to Y(T) \times Z(T) \\ f &\mapsto (\pi_Y \circ f, \pi_Z \circ f) \end{aligned}$$

is a bijection by the universal property that defines the fibre product. $\square$

Exampel 1.1.8. *In example 1.1.3 we saw that if p is an odd prime and if -1 is a square in $\mathbb{F}_p$, that is if $p \equiv 1 \pmod 4$, then the class of the circle $X^2 + Y^2 - 1$ equals $\mathbb{L} - 1$ in $\mathrm{K}_0(\mathbf{Sch}_{\mathbb{F}_p})$. Hence for every q that is a power of p, the number of $\mathbb{F}_q$-points on the circle is $q - 1$.*

If instead $p \equiv 3 \pmod 4$ then the class of the circle in $\mathrm{K}_0(\mathbf{Sch}_{\mathbb{F}_p})$ equals $\mathbb{L} + 1 - [\operatorname{Spec} \mathbb{F}_{p^2}]$. Hence if q is an odd power of p then the number of $\mathbb{F}_q$-points on the circle is $q + 1$ whereas if q is an even power of p then the number of $\mathbb{F}_q$-points equals $q - 1$.

1.1.2 The Burnside ring

Let G be a finite group. In this section we consider the Grothendieck ring of finite G-sets. However, this ring already has a name, namely the Burnside ring of G. For more on the Burnside ring, as well as proofs of the statements below, see [Knu73] chapter II, 4. There are several ways to generalize the construction of the Burnside ring to certain classes of infinite groups. We will need to do this when G is profinite, i.e., G is the inverse limit of a directed system of finite groups, with the inverse limit topology.

Let G be a group. Recall that a G-set is a set with a G action, and if S and T are G-sets then $f \colon S \to T$ is G-equivariant if $g \cdot f(s) = f(g \cdot s)$ for every $s \in S$ and $g \in G$.

Definition 1.1.9. *Let G be a finite group. Define $G - \mathbf{Sets}$ to be the category where an object is a finite set with a G-action and where a morphisms is a G-equivariant map of such sets. We will denote the morphisms between the G-sets S and T by $\operatorname{Hom}_G(S, T)$.*

More generally, if G is profinite we let $G - \mathbf{Sets}$ be the category of finite sets with a continuous G-action.

Definition 1.1.10. *Let G be a finite group. The Burnside ring of G, which we denote by $\mathcal{B}(G)$, is the free abelian group generated by the symbols $[S]$, for every G-set S, subject to the relations $[S \dot\cup T] = [S] + [T]$ and with a multiplication given by $[S] \cdot [T] := [S \times T]$, where G acts diagonally on $S \times T$.*

If G is profinite, $\mathcal{B}(G)$ is constructed in the same way but using the finite continuous G-sets.

Since every G-set can be written as a disjoint union of transitive G-sets we see that the transitive sets generates $\mathcal{B}(G)$, and in fact it is free on the isomorphism classes of transitive G-sets. Moreover, every finite transitive G-set is isomorphic to G/H where H is a (not necessarily normal) subgroup, and $G/H \simeq G/H'$ if and only if H and H' are conjugate subgroups. So every element of $\mathcal{B}(G)$ can be written uniquely as

$$\sum_{H \in R} a_H [G/H]$$

where R is a system of representatives of the set of conjugacy classes of subgroups of G and where $a_H \in \mathbb{Z}$ for every H.

Next, let $\phi \colon H \to G$ be a group homomorphism. If S is a G-set we can consider it as a H-set by defining $h \cdot s := \phi(h)s$. Also, if instead S is a H-set then we can construct a G-set in the following way.

Definition 1.1.11. *Let $\phi \colon H \to G$ be a group homomorphism and let S be an H-set. Define an equivalence relation on $G \times S$ by $(g \cdot \phi(h), s) \sim (g, hs)$ for $(g, s) \in G \times S$ and $h \in H$. Let $G \times_H S := G \times S/\sim$ with a G-action given by*

$$g' \cdot \overline{(g, s)} := \overline{(g'g, s)}.$$

We will only use this definition in the case case when H is a subgroup of G. In this case, note that if we choose a set of coset representatives of G/H, $R = \{g_1, \ldots, g_r\}$, then we can represent $G \times_H S$ as $R \times S$ with G-action given by $g \cdot (g_i, s) = (g_j, hs)$, where $gg_i = g_j h$ for $h \in H$.

This gives rise to two maps betwen Burnside rings.

Definition-Lemma 1.1.12. *Let $\phi\colon H \to G$ be a homomorphism of profinite groups. Then $\mathrm{Res}_H^G\colon \mathcal{B}(G) \to \mathcal{B}(H)$ is the map induced by restricting the G-action on a G-set S to a H-action, i.e., S is considered as a H-set via $h \cdot s := \phi(h)s$ for $h \in H$ and $s \in S$. This map is a ring homomorphism.*

Also, we define the induction map $\mathrm{Ind}_H^G\colon \mathcal{B}(H) \to \mathcal{B}(G)$ by associating to the H-set S the class of the G-set $G \times_H S$ in $\mathcal{B}(G)$. This map is additive but not multiplicative.

1.1.3 The subring of artin classes in $\mathrm{K}_0(\mathbf{Sch}_k)$

Given a field k with absolute Galois group $\mathcal{G}$. In this subsection we define a map from the Burnside ring of $\mathcal{G}$ (where $\mathcal{G}$ is given the profinite topology) to $\mathrm{K}_0(\mathbf{Sch}_k)$. The image of this consists of linear combinations of classes of zero dimensional schemes. An element in the image will be called an artin class.

We shall use the notion of a finite separable algebra:

Definition 1.1.13. *A finite separable algebra over the field k is a k-algebra L with the property that if k^s is a separable closure of k then $L \otimes_k k^s \simeq k^s \times \cdots \times k^s$.*

For a list of equivalent conditions, see [Wat79], page 46. The notion of a separable k-algebra is used in the following formulation of Galois theory. For a proof see for example *loc.cit.*, page 48.

Theorem 1.1.14. *Fix a field k together with a separable closure k^s. Set $\mathcal{G} := \mathcal{G}\mathrm{al}(k^s/k)$. Then we have a contravariant equivalence between the category of finite separable k-algebras and the category of finite continuous $\mathcal{G}$-sets (where the morphisms in the latter category are $\mathcal{G}$-equivariant maps of sets).*

This equivalence takes the k-algebra L to $\mathrm{Hom}_k(L, k^s)$ with $\mathcal{G}$-action given by $f^\sigma(l) := \sigma \circ f(l)$. Its pseudo-inverse takes the $\mathcal{G}$-set S to $\mathrm{Hom}_{\mathcal{G}}(S, k^s)$, i.e., the $\mathcal{G}$-equivariant maps of sets from S to k^s, considered as a ring by pointwise addition and multiplication and with a k-algebra structure given by $(\alpha \cdot f)(s) := \alpha \cdot f(s)$.

Proposition 1.1.15. *Under the correspondence in theorem 1.1.14, if L corresponds to S then the dimension of L equals the number of elements in S. Moreover, if also L' corresponds to S', then $L \otimes_k L'$ corresponds to $S \times S'$ with diagonal $\mathcal{G}$-action and the algebra $L \times L'$ corresponds to $S \dot{\cup} S'$. In particular, separable field extensions of k correspond to transitive $\mathcal{G}$-sets.*

Proof. The first statement is true because the equality $\dim_k L = |S|$ is equivalent to L being separable. (See [Wat79]). The second statement is true since $S \times S' \simeq \mathrm{Hom}_k(L \otimes_k L', k^s)$ follows from the universal property defining the tensor product in the category of k-algebras. The third statement follows since k^s contains no non-trivial idempotents so a homomorphism $L \times L' \to k^s$ is zero on one of the coordinates. □

Definition-Lemma 1.1.16. *Let $\mathrm{Art}_k\colon \mathcal{B}(\mathcal{G}) \to \mathrm{K}_0(\mathbf{Sch}_k)$ be induced by the map $\mathcal{G}-\mathbf{Sets} \to \mathrm{K}_0(\mathbf{Sch}_k)$ that takes the $\mathcal{G}$-set S to the class of $\mathrm{Spec}\,\mathrm{Hom}_{\mathcal{G}}(S, k^s)$. (If the field k is clear from the context then we just write Art.) Then Art_k is a ring homomorphism.*

Remark. It is also true that Art_k is injective, so we can think of $\mathcal{B}(\mathcal{G})$ as a subring of $\mathrm{K}_0(\mathbf{Sch}_k)$.

Definition 1.1.17. *Define an artin class to be an element in the image of Art_k. Let $\mathrm{ArtCl}_k \subset \mathrm{K}_0(\mathbf{Sch}_k)$, the subring of artin classes, be the image of $\mathcal{B}(\mathcal{G})$ under Art_k.*

We next study how Art behaves with respect to restriction of scalars. The following proposition is due to Grothendieck but we have not been able to find a reference so we include a proof for completeness.

Proposition 1.1.18. *Fix a field k together with a separable closure k^s and let $\mathcal{G} := \mathcal{G}\mathrm{al}(k^s/k)$. Let K be a finite field extension of k such that $K \subset k^s$. Let L be a finite separable K-algebra and let S be the corresponding $\mathcal{G}\mathrm{al}(k^s/K)$-set. View L as a k-algebra and let S' be the corresponding $\mathcal{G}$-set. Then $S' \simeq \mathcal{G} \times_{\mathcal{G}\mathrm{al}(k^s/K)} S$.*

Proof. The map

$$\begin{aligned}\phi\colon\ \mathcal{G}\times S &\to S'\\ (\sigma, f) &\mapsto \sigma f\end{aligned}$$

has the property that if $\tau \in \mathcal{G}\mathrm{al}(k^s/K)$ then $\phi(\sigma\tau, f) = \sigma\tau f = \phi(\sigma, \tau f)$. Hence it gives rise to a map of $\mathcal{G}$-sets $\varphi\colon\ \mathcal{G}\times_{\mathcal{G}\mathrm{al}(k^s/K)}S \to S'$. If $\phi(\sigma, f) = \phi(\tau, g)$ then $\tau^{-1}\sigma f = g$ so since f and g fixes K pointwise we must have that $\tau^{-1}\sigma \in \mathcal{G}\mathrm{al}(k^s/K)$. It follows that $(\tau, g) = (\tau, \tau^{-1}\sigma f) \sim (\tau\tau^{-1}\sigma, f) = (\sigma, f)$ so φ is injective.

Let $d := [K : k]$. Suppose that L has dimension n as a K-algebra, i.e., S has n elements. Then L has dimension nd as a k-algebra so S' has nd elements. On the other hand, by Galois theory, $|\mathcal{G}\,/\,\mathcal{G}\mathrm{al}(k^s/K)| = [K : k] = d$. So by the remark after definition 1.1.11, $\mathcal{G}\times_{\mathcal{G}\mathrm{al}(k^s/K)}S$ also has nd elements. Since φ is injective it follows that it also is surjective, hence an isomorphism of $\mathcal{G}$-sets. $\square$

Proposition 1.1.18 has the following consequence.

Proposition 1.1.19. *Let k be a field and k^s a separable closure. Define $\mathcal{G} := \mathcal{G}\mathrm{al}(k^s/k)$ and let K be a finite field extension of k such that $K \subset k^s$. Then the following diagram commutes:*

$$\begin{tikzcd} \mathcal{B}\big(\mathcal{G}\mathrm{al}(k^s/K)\big) \arrow[r, "\mathrm{Art}_K"] \arrow[d, "\mathrm{Ind}^{\mathcal{G}}_{\mathcal{G}\mathrm{al}(k^s/K)}"'] & \mathrm{K}_0(\mathbf{Sch}_K) \arrow[d, "\mathrm{Res}^K_k"] \\ \mathcal{B}(\mathcal{G}) \arrow[r, "\mathrm{Art}_k"] & \mathrm{K}_0(\mathbf{Sch}_k) \end{tikzcd}$$

1.1.4 The representation ring

The final Grothendieck ring that we introduce is the representation ring of a profinite group. We will not work so much in this ring; we use it only to prove facts about the Burnside ring, for example proposition 2.4.10. For this reason we just define the ring of $\mathbb{Q}$-representation, even though the same construction works over any field. The representation ring is a standard tool in representation theory, see for example [Ser77].

Let G be a finite group. A $\mathbb{Q}$-representation of G is a finitely generated $\mathbb{Q}[G]$-module, or equivalently a finite dimensional $\mathbb{Q}$-vector space with a G-action. A morphism of such representations is a homomorphism of $\mathbb{Q}[G]$-modules, or equivalently a G-equivariant linear map of $\mathbb{Q}$-vector spaces. The $\mathbb{Q}$-representations of G form an abelian category. More generally, if G is profinite then we define a $\mathbb{Q}$-representation of G in the same way as above but we also require the G-action to factor through a finite continuous quotient of G.

Definition 1.1.20. *The representation ring of G (over $\mathbb{Q}$) is the Grothendieck group of the category of $\mathbb{Q}$-representations of G with a multiplication given by $[V_1] \cdot [V_2] := [V_1 \otimes_{\mathbb{Q}} V_2]$, the G-action on the tensor product being given by $g \cdot (v_1 \otimes v_2) = gv_1 \otimes gv_2$. We denote this ring with $\mathrm{R}_{\mathbb{Q}}(G)$.*

By Maschke's theorem, every short exact sequence of $\mathbb{Q}$-representations splits, hence we can think of the relations just as $[V_1 \oplus V_2] = [V_1] + [V_2]$.

As an abelian group, the rational representation ring is free on the isomorphism classes of irreducible $\mathbb{Q}$-representations. Since we have a bijection between the set of such classes and the conjugacy classes of cyclic subgroups of G, the rank of $\mathrm{R}_{\mathbb{Q}}(G)$ equals the number of conjugacy classes of cyclic subgroups in G. (See [Ser77], 12.4 for this.)

In this ring we also get a restriction map and an induction map.

Definition-Lemma 1.1.21. *Let $H \to G$ be a group homomorphism. Then $\operatorname{Res}_H^G\colon \mathrm{R}_{\mathbb{Q}}(G) \to \mathrm{R}_{\mathbb{Q}}(H)$ is the map induced by restricting the G-action on the $\mathbb{Q}$-vector space V to a H-action on V. This map is a ring homomorphism.*

Moreover, we define a map $\operatorname{Ind}_H^G\colon \mathrm{R}_k(H) \to \mathrm{R}_k(G)$ by associating to the $\mathbb{Q}[H]$-module V the class of the $\mathbb{Q}[G]$-module $\mathbb{Q}[G] \otimes_{\mathbb{Q}[H]} V$ in $\mathrm{R}_{\mathbb{Q}}(G)$. This map is additive but not multiplicative.

We need to know a little about how the induction map works.

Proposition 1.1.22. *Let H be a subgroup of G and let $R = \{g_1, \dots, g_r\}$ be a system of coset representatives for G/H. Then R is a basis for $\mathbb{Q}[G]$ considered as a right $\mathbb{Q}[H]$-module. Hence $\mathbb{Q}[G]$ is free of rank $|G/H|$.*

Let V be an H-representation of dimension n. Let B be a $\mathbb{Q}$-basis for V. Then a basis for $\mathbb{Q}[G] \otimes_{\mathbb{Q}[H]} V$ as a $\mathbb{Q}$-vector space is $\{g_i \otimes v\}_{g_i \in R, v \in B}$ and the G-action is given by $g \cdot g_i \otimes v = g_j \otimes hv$ where $gg_i = g_j h$ for $h \in H$.

Proof. Let $r := |G/H|$ and let $g_1, \dots, g_r$ be coset representatives. Define

$$\varphi\colon \mathbb{Q}[G] \to \bigoplus_{i=1}^{r} \mathbb{Q}[H]$$

on the canonical basis for $\mathbb{Q}[G]$ by mapping $g = g_i h$, where $h \in H$, to the tuple with ith component h and zeros elsewhere. Since G is the disjoint union of its cosets this is a bijection. It is additive and $\mathbb{Q}$-linear by definition. Finally, if $g = g_i h$ then

$$\varphi(gh') = \varphi\big(g_i(hh')\big) = (0, \dots, hh', \dots, 0) = (0, \dots, h, \dots, 0) \cdot h' = \varphi(g)h'$$

for $h' \in H$, hence φ is H-equivariant. □

To be able to use the representation ring to prove facts about the Burnside ring we will need a map between them.

Definition-Lemma 1.1.23. *Let G be a profinite group. Let S be a finite continuous G-set. We can associate to S the permutation representation $\mathbb{Q}[S]$, i.e., the $\mathbb{Q}$-vector space with basis S and G-action on the basis elements. This gives rise to a ring homomorphism $\mathcal{B}(G) \to \mathrm{R}_{\mathbb{Q}}(G)$ which we denote by* h.

Proof. If S and T are G-sets then

$$\mathbb{Q}[S \,\dot{\cup}\, T] \simeq \mathbb{Q}[S] \oplus \mathbb{Q}[T] \qquad \text{and}$$
$$\mathbb{Q}[S \times T] \simeq \mathbb{Q}[S] \otimes_{\mathbb{Q}} \mathbb{Q}[T]$$

as $\mathbb{Q}[G]$-modules, hence this construction really defines a ring homomorphism from $\mathcal{B}(G)$ to $\mathrm{R}_{\mathbb{Q}}(G)$. □

This map is studied in [Seg71], where it is proved that if every element in G has prime power order, then h is surjective. It is an isomorphism if and only if G is cyclic. Since $\mathcal{B}(G)$ has rank equal to the number of conjugacy classes of subgroups of G whereas $\mathrm{R}_{\mathbb{Q}}(G)$ has rank equal to the number of conjugacy classes of cyclic subgroups of G, it is in general not injective. We will later prove that the restriction of h to a certain subring of $\mathcal{B}(\Sigma_n)$ is injective.

The map h commutes with the induction and restriction maps.

Proposition 1.1.24. *Let H be a subgroup of G. Then the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{B}(G) \arrow[r, "\mathrm{h}"] & \mathrm{R}_{\mathbb{Q}}(G) \\
\mathcal{B}(H) \arrow[r, "\mathrm{h}"] \arrow[u, "\operatorname{Ind}_H^G"] & \mathrm{R}_{\mathbb{Q}}(H) \arrow[u, "\operatorname{Ind}_H^G"']
\end{tikzcd}$$

Also, if $H \to G$ is a group homomorphism then h *commutes with the restriction maps* Res^G_H.

Proof. We prove the first part of the proposition. For every H-set S we have to find a G-equivariant isomorphism of $\mathbb{Q}$-vector spaces

$$\varphi\colon \mathbb{Q}[G \times_H S] \to \mathbb{Q}[G] \otimes_{\mathbb{Q}[H]} \mathbb{Q}[S].$$

Define a map $\phi\colon G \times S \to \mathbb{Q}[G] \otimes_{\mathbb{Q}[H]} \mathbb{Q}[S]$ as $(g, s) \mapsto g \otimes s$. Since $\phi(gh, s) = \phi(g, hs)$ this factors through $G \times_H S$ and by linear extension we get our map φ.

Choose a system of coset representatives $R = \{g_1, \dots, g_r\}$ for G/H. We have seen in 1.1.22 that $\mathbb{Q}[G] \otimes_{\mathbb{Q}[H]} \mathbb{Q}[S]$ has a basis given by $\{g_i \otimes s\}_{g_i \in R, s \in S}$ and G-action $g \cdot (g_i \otimes s) = (g_j \otimes hs)$, where $gg_i = g_j h$, hence φ is G-equivariant and surjective. Also, we have seen that $G \times_H S$ can be represented as $\{(g_i, s)\}_{g_i \in R, s \in S}$ so the two vector spaces have the same dimension. Hence φ is an isomorphism. □

We next define a map from the representation ring.

Definition-Lemma 1.1.25. *Let G be a profinite group. If g is an element of G then we have a map from the category of G-representations to $\mathbb{Q}$ that sends the G-representation V to $\chi_V(g)$. This induces a ring homomorphism $C_g\colon \mathrm{R}_{\mathbb{Q}}(G) \to \mathbb{Q}$.*

We have that if g and g' are conjugate then $C_g = C_{g'}$. Let R be a system of representatives of the set of conjugacy classes of G. Since a representation is determined by its character the following proposition is expected.

Proposition 1.1.26. *With the above notation, the map $\prod_{g \in R} C_g\colon \mathrm{R}_{\mathbb{Q}}(G) \to \prod_{g \in R} \mathbb{Q}$ is injective.*

We have the following commutation property.

Proposition 1.1.27. *Let $\phi\colon G \to H$ be a group homomorphism and let $g \in G$. The following diagram commutes.*

$$\begin{tikzcd} \mathrm{R}_{\mathbb{Q}}(H) \arrow[r, "\mathrm{Res}^H_G"] \arrow[dr, "C_{\phi(g)}"'] & \mathrm{R}_{\mathbb{Q}}(G) \arrow[d, "C_g"] \\ & \mathbb{Z} \end{tikzcd}$$

Proof. Let V be a $\mathbb{Q}$-representation of H and denote it by V' when we consider it as a representation of G via ϕ. Then g acts on V' by $\phi(g)$ so $\chi_{V'}(g) = \chi_V(\phi(g))$, hence $C_g(\mathrm{Res}^H_G[V]) = C_{\phi(g)}([V])$. Since every element of $\mathrm{R}_{\mathbb{Q}}(H)$ is a difference of classes of H-representations the result follows. □

1.2 The motivic ring

In this section we define the ring in which our motivic measure will take its values. It is obtained from the Grothendieck ring of varieties by a process of localization and completion. This material together with references can be found in [Bli05]. For basic facts about filtrations and completions, see [Ser00], chapter II.

Definition 1.2.1. *Given a field k, let $\mathcal{M}_k$ be the localization of $\mathrm{K}_0(\mathbf{Sch}_k)$ with respect to $\{\mathbb{L}^n\}_{n \in \mathbb{N}}$.*

Definition 1.2.2. *If $x \in \mathrm{K}_0(\mathbf{Sch}_k)$ we say that $\dim x \leq n$ if x can be expressed as a linear combination of classes of schemes, each of dimension $\leq n$. (By convention, the empty scheme has dimension $-\infty$.) We define a filtration of $\mathcal{M}_k$, $\{\mathcal{F}^n(\mathcal{M}_k)\}_{n \in \mathbb{Z}}$ by letting $\mathcal{F}^n(\mathcal{M}_k)$ be the subgroup of $\mathcal{M}_k$ generated by elements of the form $x \cdot \mathbb{L}^{-i}$ with $\dim x - i \leq n$. Let $\widehat{\mathcal{M}}_k$ be the completion of $\mathcal{M}_k$ with respect to this filtration.*

So we have homomorphisms

$$\mathrm{K}_0(\mathbf{Sch}_k) \to \mathcal{M}_k \to \widehat{\mathcal{M}}_k.$$

It is not known whether any of these maps are injective but we still denote by $[Y]$ the images in $\mathcal{M}_k$ and $\widehat{\mathcal{M}}_k$ of $[Y] \in \mathrm{K}_0(\mathbf{Sch}_k)$.

The following holds since $\widehat{\mathcal{M}}_k$ is a completion with respect to a filtration.

Proposition 1.2.3. *A sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\widehat{\mathcal{M}}_k$ is Cauchy, hence convergent, if and only if $a_{n+1} - a_n \to 0$. In particular, the sum $\sum_{n\in\mathbb{N}} a_n$ is convergent if and only if $a_n \to 0$.*

The following is a consequence.

Proposition 1.2.4. *If $\{a_n\}_{n\in\mathbb{N}} \subset \widehat{\mathcal{M}}_k$ and if $\sum_{n\in\mathbb{N}} a_n$ is convergent then every rearrangement of $\sum_{n\in\mathbb{N}} a_n$ is convergent, and they all converges to the same limit.*

We will use this result a great deal so we will not refer to it every time. The same holds for its consequence that if $\{a_{nm}\}_{(n,m)\in\mathbb{N}^2} \subset \widehat{\mathcal{M}}_k$ then $\sum_{(n,m)} a_{nm}$ is well defined and if it converges then it equalls $\sum_n \sum_m a_{nm}$.

Finally we have the following formula:

Proposition 1.2.5. *In $\widehat{\mathcal{M}}_k$ we have the equality $\sum_{i\in\mathbb{N}} \mathbb{L}^{ni} = (1 - \mathbb{L}^{-n})^{-1}$ for every positive integer n.*

1.3 λ-rings

The definition of a λ-ring is due to Grothendieck. An introduction to this subject is given for example in the first part of [AT69] or in [Knu73]. We define only the part of theory that we need.

Definition 1.3.1. *A λ-ring is a commutative ring R with identity together with a set of maps $\lambda^n \colon R \to R$, for each $n \in \mathbb{N}$, such that for all $x, y \in R$*

$$\begin{aligned} \lambda^0(x) &= 1 \\ \lambda^1(x) &= x \\ \lambda^n(x+y) &= \sum_{i=0}^{n} \lambda^i(x)\lambda^{n-i}(y). \end{aligned} \tag{1.1}$$

A morphism of λ-rings is a homomorphism of commutative rings, commuting with the λ-operations.

For an indeterminate t, define $\lambda_t(x) := \sum_{n\geq 0} \lambda^n(x)t^n \in R[[t]]$. The last axiom can then be expressed as the equality

$$\lambda_t(x+y) = \lambda_t(x)\lambda_t(y) \tag{1.2}$$

in $R[[t]]$, so a λ-ring structure on R is the same thing as homomorphism λ_t from the additive group of R to the multiplicative group of $R[[t]]$ fullfilling the two first axioms of (1.1).

Sometimes when defining λ-structures it is more convenient to define the λ^n:s implicitly. One way is to first define functions σ^n, fullfilling the same axioms as the λ^n:s (so R is a λ-ring also with respect to the σ^n:s), define $\sigma_t(x) := \sum_{n\geq 0} \sigma^n(x)t^n \in R[[t]]$ and then define the λ-operations by

$$\sigma_t(x)\lambda_{-t}(x) = 1. \tag{1.3}$$

Proposition 1.3.2. *Given a collection of maps σ^n on R fullfilling the axioms* (1.1)*. Then* (1.3) *define a unique λ-ring structure on R.*

Moreover, if $f \colon R \to R'$ is a ring homomorphism commuting with the σ^n:s, then it is a morphism of λ-rings.

Proof. First, $\lambda_{-t}(x)$ exists uniquely since $\sigma_t(x)$ has constant coefficient equal to 1, hence is invertible, and inverses are unique when they exist. Moreover, we have that $\lambda^0(x) = \sigma^0(x) = 1$ and $-\sigma^0(x)\lambda^1(x) + \sigma^1(x)\lambda^0(x) = 0$ so that $\lambda^1(x) = \sigma^1(x) = x$. The third axiom follows from its formulation as (1.2) for we have

$$\begin{aligned}\lambda_{-t}(x+y) =& \sigma_t(x+y)^{-1}\\ =& \sigma_t(x)^{-1}\sigma_t(y)^{-1}\\ =& \lambda_{-t}(x)\lambda_{-t}(y),\end{aligned}$$

so it follows that $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$.

For the last part of the proposition, Since f is a homomorphism it induces a homomorphism on the power series rings $R[[t]] \to R'[[t]]$, $\sum_{i\geq 0} a_i t^i \mapsto \sum_{i\geq 0} f(a_i)t^i$ which we also denote by f. Since f commutes with σ_t we then have

$$\begin{aligned}\sigma_t\big(f(x)\big)\cdot f\big(\lambda_{-t}(x)\big) =& f\big(\sigma_t(x)\big)\cdot f\big(\lambda_{-t}(x)\big)\\ =& f\big(\sigma_t(x)\cdot\lambda_{-t}(x)\big)\\ =& f(1)\\ =& 1 \in R'[[t]].\end{aligned}$$

Since $\lambda_{-t}\big(f(x)\big)$ is unique with this property it follows that $f\big(\lambda_{-t}(x)\big) = \lambda_{-t}\big(f(x)\big)$. $\square$

The aim of this section is to define a λ-ring structure on $\mathcal{B}(\Sigma_n)$. In section 2.4 we will then prove an explicit formula for $\lambda^i(\{1,\dots,n\})$. However, we are not able to prove this formula directly so we have to move it to the representation ring $\mathrm{R}_{\mathbb{Q}}(\Sigma_n)$ and prove it there instead. So we begin by describing a λ-ring structure on the representation ring. This structure is also one of the best-known and most studied of all λ-rings.

Remark. In the theory of λ-rings a great part centers around the concept of a special λ-ring, which is a λ-ring where $\lambda^n(xy)$ is a universal polynomial in $\lambda^i(x)$ and $\lambda^i(y)$ for $i \leq n$, and $\lambda^n(\lambda^m(x))$ is a universal polynomial in $\lambda^i(x)$ for $i \leq mn$. In that theory there is not such a symmetry between λ and σ, for R can be special with respect to λ bur not with respect to σ. Of the rings we shall encounter, the representation ring is special but the Burnside ring is not.

1.3.1 The λ-ring structure on the representation ring

Define $\lambda_t\colon \mathrm{R}_{\mathbb{Q}}(G) \to \mathrm{R}_{\mathbb{Q}}(G)[[t]]$ by associating to the G-representation V the power series

$$\sum_{n\geq 0}[\wedge^n V]\cdot t^n$$

where $\wedge^n V$ has the G-action $g\cdot v_1\wedge\cdots\wedge v_n := gv_1\wedge\cdots\wedge gv_n$. Then λ_t is a well defined homomorphism from the additive group of $\mathrm{R}_{\mathbb{Q}}(G)$ to the multiplicative group of $\mathrm{R}_{\mathbb{Q}}(G)[[t]]$ because for every $n\in\mathbb{N}$ we have an isomorphism

$$\wedge^n(U\oplus V) \xrightarrow{\sim} \bigoplus_{i=0}^{n} \wedge^i U \otimes_{\mathbb{Q}} \wedge^{n-i} V,$$

which is G-equivariant. When referring to $\mathrm{R}_{\mathbb{Q}}(G)$ as a λ-ring we will always use this λ-ring structure.

When G is the trivial group we see that $\mathrm{R}_{\mathbb{Q}}(G)$ is isomorphic to $\mathbb{Z}$ via $V \mapsto \dim_{\mathbb{Q}} V$. Under this isomorphism, the corresponding λ-ring structure on $\mathbb{Z}$ is $\lambda^n(m) = \binom{m}{n}$.

Next we define σ_t by associating to the G-representation V the power series

$$\sum_{n\geq 0}[\mathrm{S}^n(V)]\cdot t^n \in \mathrm{R}_{\mathbb{Q}}(G)[[t]],$$

where $\mathrm{S}^n(V)$ is the symmetric n:th-power of V as a $\mathbb{Q}$-vector space and with a G-action given by $g\cdot v_1^{e_1}\cdots v_j^{e_j} := (gv_1)^{e_1}\cdots(gv_j)^{e_j}$, where $e_1+\cdots+e_j=n$. This is really the σ corresponding to λ that we defined previously, for $\sigma_t(x)\cdot\lambda_{-t}(x)=1$ follows for example from an investigation of the Koszul complex given in [McD84], chapter V.G.

1.3.2 The λ-ring structure on the Burnside ring

We are now going to define a λ-ring structure on $\mathcal{B}(G)$. This will be used to define elements in $\mathcal{B}(\Sigma_n)$ that will give us a very compact way of writing the formula for $[L^*]\in \mathrm{K}_0(\mathbf{Sch}_k)$ that we will find in chapter 2. It turns out that our λ-structure on $\mathcal{B}(G)$ will be rather hard to work with. We will therefore use the homomorphism $\mathrm{h}\colon \mathcal{B}(\Sigma_n)\to \mathrm{R}_{\mathbb{Q}}(\Sigma_n)$ (definition 1.1.23) which will allow us to move a crucial part of the computations in $\mathcal{B}(\Sigma_n)$ to the corresponding computations in $\mathrm{R}_{\mathbb{Q}}(\Sigma_n)$ which will be easier to handle. For this we will have to prove that h respects the λ-structures.

We begin by defining the λ-structure on $\mathcal{B}(G)$. We do this implicitly by first defining σ_t. Define a map that takes the G-set S to the power series

$$\sum_{n\geq 0}[S^n/\,\Sigma_n]\in \mathcal{B}(G)[[t]],$$

where Σ_n acts on S^n by permuting the entries. There is an isomorphism of G-sets

$$(S\,\dot{\cup}\,T)^n/\Sigma_n \to \dot{\bigcup}_{i+j=n} S^i/\Sigma_i\times T^j/\,\Sigma_j,$$

so this defines a homomorphism from the additive group of $\mathcal{B}(G)$ to the multiplicative group of $\mathcal{B}(G)[[t]]$ which is our σ_t.

We then define λ_t by the formula

$$\sigma_t(x)\lambda_{-t}(x)=1$$

for every $x\in\mathcal{B}(G)$. By proposition 1.3.2 this defines a λ-ring structure on $\mathcal{B}(G)$.

Next we describe a connection between $\mathcal{B}(G)$ and $\mathrm{R}_{\mathbb{Q}}(G)$ with the λ-structures we have given them.

Lemma 1.3.3. *Let G be a finite group and let* $\mathrm{h}\colon \mathcal{B}(G)\to\mathrm{R}_{\mathbb{Q}}(G)$ *be the map defined in 1.1.23. Then* h *is a homomorphism of λ-rings.*

Proof. To show that h commutes with the λ-operations we begin by showing that it commutes with σ^i for every i. For this we have to show that if T is a G-set then

$$\mathbb{Q}[T^i/\,\Sigma_i]\simeq \mathrm{S}^i\big(\mathbb{Q}[T]\big)$$

as $\mathbb{Q}[G]$-modules. Let $T=\{t_1,\dots,t_j\}$ and identify $T^i/\,\Sigma_i$ with the set of monomials of degree i,

$$\{t_1^{e_1}\cdots t_j^{e_j}\,:\,e_1+\cdots+e_j=i\}.$$

Then $\mathbb{Q}[T^i/\,\Sigma_i]$ is the $\mathbb{Q}$-vector space with this basis and G-action given by

$$g\cdot t_1^{e_1}\cdots t_j^{e_j}=(gt_1)^{e_1}\cdots(gt_j)^{e_j}.$$

The same holds for $\mathrm{S}^i\big(\mathbb{Q}[T]\big)$. Hence h commutes with the σ^n:s, so it follows from the second part of proposition 1.3.2 that h is a morphism of λ-rings. $\square$

Remark. This is not the only possible λ-structure on $\mathcal{B}(G)$, for example one could have defined $\lambda^n([S]) = [\mathcal{P}_n(S)]$, the subsets of S of cardinality i. This would almost make h a morphism of λ-rings for then $\mathrm{h}\big(\lambda^n([S])\big)$ is naturally isomorphic to $\lambda^n\big(\mathrm{h}([S])\big)$ as $\mathbb{Q}$-vectorspaces. However, this isomorphism is not in general G-equivariant.

1.4 The Witt vectors

In this section we define a ring scheme called the Witt vectors and denoted by $\mathbf{W}$. This material is essentially in [Ser79] pp. 40-44 and in [Dem72].

1.4.1 Definitions

Fix a prime p. Consider the following sequence of polynomials in $\mathbb{Z}[X_0, \ldots, X_n, \ldots]$:

$$\begin{aligned}
W_0 =& X_0 \\
W_1 =& X_0^p + pX_1 \\
&\vdots \\
W_n =& \sum_{i=0}^{n} p^i X_i^{p^{n-i}} = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n \\
&\vdots
\end{aligned}$$

It is a fact (see [Ser79] for a proof) that for every $\Phi \in \mathbb{Z}[X, Y]$ there exists a unique sequence $(\varphi_0, \ldots, \varphi_n, \ldots)$ of polynomials in $\mathbb{Z}[X_0, \ldots, X_n, \ldots; Y_0, \ldots, Y_n, \ldots]$ such that

$$W_n(\varphi_0, \ldots, \varphi_n) = \Phi\big(W_n(X_0, \ldots, X_n), W_n(Y_0, \ldots, Y_n)\big) \qquad n \in \mathbb{N}.$$

Note that φ_n only involves the variables $X_0, \ldots, X_n$ and $Y_0, \ldots, Y_n$. If $\Phi = X + Y$ we denote the associated φ_n with S_n and we get

$$\begin{aligned}
S_0 =& X_0 + Y_0 \\
S_1 =& X_1 + Y_1 + \frac{X_0^p + Y_0^p - (X_0 + Y_0)^p}{p} \\
S_2 =& X_2 + Y_2 + \frac{1}{p}(X_1^p + Y_1^p) - \frac{1}{p}\left(X_1 + Y_1 + \frac{X_0^p + Y_0^p - (X_0 + Y_0)^p}{p}\right)^p \\
&+ \frac{1}{p^2}\big(X_0^{p^2} + Y_0^{p^2} - (X_0 + Y_0)^{p^2}\big) \\
&\vdots
\end{aligned}$$

If instead $\Phi = XY$ we set $P_n := \varphi_n$ and we get

$$\begin{aligned}
P_0 =& X_0 Y_0 \\
P_1 =& X_1 Y_0^p + X_0^p Y_1 + pX_1 Y_1 \\
&\vdots
\end{aligned}$$

We are now ready to define the Witt vectors as the functor $\mathbf{W}\colon \mathbf{Rings} \to \mathbf{Rings}$ that takes the ring A to $A^{\mathbb{N}}$ with the ring operations defined as follows: Let $\mathbf{a} = (a_0, \dots, a_n, \dots)$ and $\mathbf{b} = (b_0, \dots, b_n, \dots)$ be two elements of $A^{\mathbb{N}}$ and set

$$\begin{aligned} \mathbf{a} + \mathbf{b} :=& \big(S_0(\mathbf{a}, \mathbf{b}), \dots, S_n(\mathbf{a}, \mathbf{b}), \dots\big) \\ \mathbf{a} \cdot \mathbf{b} :=& \big(P_0(\mathbf{a}, \mathbf{b}), \dots, P_n(\mathbf{a}, \mathbf{b}), \dots\big). \end{aligned}$$

(Where we view polynomials $Q \in \mathbb{Z}[X_0, \dots, X_n, \dots; Y_0, \dots, Y_n, \dots]$, i.e., S_n and P_n, as functions $A^{\mathbb{N}} \times A^{\mathbb{N}} \to A$ by defining $Q(\mathbf{a}, \mathbf{b})$ to be the value of Q when we replace X_i by a_i and Y_i by b_i.) To prove that $\mathbf{W}(A)$ is a ring one observes that the map

$$\begin{aligned} W_*(A) : \mathbf{W}(A) &\to A^{\mathbb{N}} \\ \mathbf{a} &\mapsto \big(W_0(\mathbf{a}), \dots, W_n(\mathbf{a}), \dots\big) \end{aligned}$$

is a homomorphism. (It actually defines a morphism of ring schemes from $\mathbf{W}$ to $\mathbb{A}_{\mathbb{Z}}^{\mathbb{N}}$, where the latter is viewed as a ring scheme using the product ring structure.) If p is invertible in A, $W_*(A)$ is an isomorphism. (That is $\mathbf{W}_{\mathbb{Z}[1/p]} \simeq \mathbb{A}_{\mathbb{Z}[1/p]}^{\mathbb{N}}$ as ring schemes.) So if p is invertible in A then $\mathbf{W}(A)$ is a ring with identity element $(1, 0, 0, \dots)$. But if $\mathbf{W}(A)$ is a ring and B is any sub- or quotient ring of A then $\mathbf{W}(B)$ is a ring. Since $\mathbf{W}(\mathbb{Z}[1/p, X_\alpha])$ is a ring for any family $\{X_\alpha\}$ of indeterminates, it follows that $\mathbf{W}(\mathbb{Z}[X_\alpha])$ is a ring. But if A is an arbitrary ring it is a quotient of some polynomial ring, hence $\mathbf{W}(A)$ is a ring. (One can verify that $\mathbf{W}(A)$ is a ring directly from the definitions but the proof of the associative and the distributive laws becomes very complicated.)

It can be of interest to see the underlying double Hopf-algebra of $\mathbf{W}$. As a functor to $\mathbf{Sets}$ it is clear that $\mathbf{W}$ is represented by $\mathbb{Z}[X_i]_{i=0}^{\infty}$. We also need two comultiplications,

$$\Delta_a, \Delta_m \colon \mathbb{Z}[X_i]_{i=0}^{\infty} \to \mathbb{Z}[X_i]_{i=0}^{\infty} \otimes_{\mathbb{Z}} \mathbb{Z}[X_i]_{i=0}^{\infty},$$

one for addition and one for multiplication. If now $(a_0, a_1, \dots)$ and $(b_0, b_1, \dots)$ in $\mathbf{W}(A)$ correspond to f and g in $\operatorname{Hom}\big(\mathbb{Z}[X_i]_{i=0}^{\infty}, A\big)$, that is $f(X_i) = a_i$ and $g(X_i) = b_i$, then we shall have $(f, g)\Delta_a(X_i) = S_i(a_0 \dots a_i, b_0 \dots b_i)$. It is now clear hove to construct Δ_a, given that we know S_i for all $i \in \mathbb{N}$. We get

$$\begin{aligned} \Delta_a(X_0) =& X_0 \otimes 1 + 1 \otimes X_0 \\ \Delta_a(X_1) =& X_1 \otimes 1 + 1 \otimes X_1 + \frac{(X_0 \otimes 1)^p + (1 \otimes X_0)^p - (X_0 \otimes 1 + 1 \otimes X_0)^p}{p} \\ &\vdots \end{aligned}$$

In the same way one constructs Δ_m from P_i, $i \in \mathbb{N}$.

We have seen that $\mathbf{W}$ is an affine ring scheme, but it is not of finite type over $\operatorname{Spec} \mathbb{Z}$. However we are going to work in a Grothendieck ring generated by schemes of finite type over $\operatorname{Spec} \mathbb{F}_p$. Now the S_n and P_n that define the ring operations in $\mathbf{W}(A)$ only involve variables of index $\leq n$. Hence we can define the Witt vectors of length n, $\mathbf{W}_n$, to be the functor that takes the ring A to A^n, with addition and multiplication defined in the same way as for $\mathbf{W}$, that is if $\mathbf{a} = (a_0, \dots, a_{n-1})$ and $\mathbf{b} = (b_0, \dots, b_{n-1}) \in A^n$ then

$$\begin{aligned} \mathbf{a} + \mathbf{b} :=& \big(S_0(\mathbf{a}, \mathbf{b}), \dots, S_{n-1}(\mathbf{a}, \mathbf{b})\big) \\ \mathbf{a} \cdot \mathbf{b} :=& \big(P_0(\mathbf{a}, \mathbf{b}), \dots, P_{n-1}(\mathbf{a}, \mathbf{b})\big). \end{aligned}$$

This scheme is of finite type over $\operatorname{Spec} \mathbb{Z}$. One has that $\mathbf{W}_1$ is the identity functor, that is $\mathbf{W}_1(A) = A$. We also have that the ring $\mathbf{W}(A)$ is the inverse limit of the rings $\mathbf{W}_n(A)$ as $n \to \infty$. We define the projection map $\pi_n \colon \mathbf{W} \to \mathbf{W}_n$ by

$$(a_0, a_1, \dots) \mapsto (a_0, \dots, a_{n-1}) \colon \mathbf{W}(A) \to \mathbf{W}_n(A)$$

for every ring A.

We will be interested in the $\mathbb{F}_q$-rational points on $\mathbf{W}$. This is because $\mathbf{W}(\mathbb{F}_p) = \mathbb{Z}_p$ and if $q = p^n$ then $\mathbf{W}(\mathbb{F}_q)$ is the integral closure of $\mathbb{Z}_p$ in the unique unramified degree n extension of $\mathbb{Q}_p$. (In a fixed algebraic closure of $\mathbb{Q}_p$.) See [Ser79] for a proof.

1.4.2 Operations on W

Define $\mathrm{V}\colon \mathbf{W} \to \mathbf{W}$ by $\mathrm{V}\,\mathbf{a} = (0, a_0, \ldots, a_{n-1}, \ldots)$. V is short for "Verschiebung". It is not a morphism of ring schemes but it is additive. To see this we use the same observation as above; it suffices to prove additivity for $\mathbf{W}(A)$ when p is invertible in A, and in this case $W_*(A)$ is an isomorphism so it suffices to show that $W_*(A)$ transforms V to an additive map. But this is true since

$$W_n(\mathrm{V}\,\mathbf{a}) = \sum_{i=1}^{n} p^i a_{i-1}^{p^{n-i}} = p\sum_{i=1}^{n} p^{i-1} a_{i-1}^{p^{(n-1)-(i-1)}} = pW_{n-1}(\mathbf{a})$$

so W_* transforms $\mathrm{V}(A)$ to the map $A^{\mathbb{N}} \to A^{\mathbb{N}}$ that sends $(w_0, w_1, \ldots)$ to $(0, pw_0, pw_1, \ldots)$ and this is clearly additive. Note that $\mathbf{W}/\mathrm{V}^n\,\mathbf{W} \simeq \mathbf{W}_n$. This identification will be used a lot.

Next we define the map $\mathrm{r}\colon \mathbf{W}_1 \to \mathbf{W}$ by $a \mapsto (a, 0, \ldots, 0, \ldots)$. Since

$$W_n\big(\mathrm{r}(a)\big) = (a, a^p, \ldots, a^{p^n}, \ldots)$$

we see that W_* transforms $\mathrm{r}(A)$ to the map $A \to A^{\mathbb{N}}$ that sends w to $(w, w^p, w^{p^2}, \ldots)$. This map is multiplicative so when p is invertible in A it follows that $\mathrm{r}(A)$ is multiplicative. As above this implies that r is multiplicative.

Finally over $\mathbb{F}_p$ (where p is the prime that was fixed in the beginning of this section) we define the Frobenius morphism $\mathrm{F}\colon \mathbf{W}_{\mathbb{F}_p} \to \mathbf{W}_{\mathbb{F}_p}$ by $\mathrm{F}\,\mathbf{a} = (a_0^p, \ldots, a_n^p, \ldots)$. It is a morphism of ring schemes. The next proposition will be very useful to us.

Proposition 1.4.1. *If A is an $\mathbb{F}_p$-algebra and $\mathbf{a}, \mathbf{b} \in \mathbf{W}(A)$ the following formulas hold:*

$$\begin{aligned} \mathrm{V}\,\mathrm{F}\,\mathbf{a} &= \mathrm{F}\,\mathrm{V}\,\mathbf{a} = p\mathbf{a} \\ \mathbf{a}\cdot\mathrm{V}\,\mathbf{b} &= \mathrm{V}(\mathrm{F}\,\mathbf{a}\cdot\mathbf{b}). \end{aligned}$$

Proof. For the first formula see [Ser79]. For the second formula it suffices to prove this when A is perfect so we may assume that $\mathbf{b} = \mathrm{F}\,\mathbf{c}$. The first formula, the distributive law and the fact that F is a ring homomorphism then give

$$\mathrm{V}(\mathrm{F}\,\mathbf{a}\cdot\mathbf{b}) = \mathrm{V}(\mathrm{F}\,\mathbf{a}\cdot\mathrm{F}\,\mathbf{c}) = \mathrm{V}\,\mathrm{F}(\mathbf{a}\cdot\mathbf{c}) = p(\mathbf{a}\cdot\mathbf{c}) = \mathbf{a}\cdot(p\mathbf{c}) = \mathbf{a}\cdot\mathrm{V}\,\mathrm{F}\,\mathbf{c} = \mathbf{a}\cdot\mathrm{V}\,\mathbf{b}.$$

□

Corollary 1.4.2. *If A is an $\mathbb{F}_p$-algebra, $\mathbf{a}, \mathbf{b} \in \mathbf{W}(A)$ and $i, j \in \mathbb{N}$ then*

$$\mathrm{V}^i\,\mathbf{a}\cdot\mathrm{V}^j\,\mathbf{b} = \mathrm{V}^{i+j}\big(\mathrm{F}^j\,\mathbf{a}\cdot\mathrm{F}^i\,\mathbf{b}\big).$$

Exampel 1.4.3. *Let $\mathbf{b} := (b_0, \ldots, b_n) \in \mathbf{W}_{n+1}(A)$. We then have*

$$(0, \ldots, 0, a)\cdot\mathbf{b} = \mathrm{V}^n\,\mathrm{r}\,a\cdot\mathbf{b} = \mathrm{V}^n(\mathrm{r}\,a\cdot\mathrm{F}^n\,\mathbf{b}) = (0, \ldots, 0, a\cdot b_0^{p^n})$$

Corollary 1.4.4. *Let $\Delta \in \mathbf{W}(A)[X_1, \ldots, X_n]$ be a form of degree d. If $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbf{W}(A)$ then*

$$\Delta(\mathrm{V}\,\mathbf{a}_1, \ldots, \mathrm{V}\,\mathbf{a}_n) = \mathrm{F}^{d-1}\,\mathrm{V}^d\,\Delta(\mathbf{a}_1, \ldots, \mathbf{a}_n)$$

Proof. Let $\Delta = X_1^d$. The formula is true for $d=1$. Suppose that it is true for $d-1$. Then with the help of corollary 1.4.2,

$$\begin{aligned}\Delta(\mathrm{V}\,\mathbf{a}) =&(\mathrm{V}\,\mathbf{a})(\mathrm{V}\,\mathbf{a})^{d-1}\\ =&(\mathrm{V}\,\mathbf{a})(\mathrm{F}^{d-2}\,\mathrm{V}^{d-1}\,\mathbf{a}^{d-1})\\ =&\mathrm{V}^d(\mathrm{F}^{d-1}\,\mathbf{a}\cdot\mathrm{F}^{d-1}\,\mathbf{a}^{d-1})\\ =&\mathrm{F}^{d-1}\,\mathrm{V}^d\,\Delta(\mathbf{a}).\end{aligned}$$

Next, let d and n be arbitrary and suppose the formula is proved for every $X_1^{d_1}\cdots X_{n-1}^{d_{n-1}}$ with $d_1+\cdots+d_{n-1}\le d$. Let $\Delta = X_1^{d_1}\cdots X_n^{d_n}$ with $d_1+\cdots+d_n=d$. Then

$$\begin{aligned}\Delta(\mathrm{V}\,\mathbf{a}_1,\dots,\mathrm{V}\,\mathbf{a}_n) =&(\mathrm{V}\,\mathbf{a}_1)^{d_1}\prod_{i=2}^n(\mathrm{V}\,\mathbf{a}_i)^{d_i}\\ =&\mathrm{F}^{d_1-1}\,\mathrm{V}^{d_1}\,\mathbf{a}_1^{d_1}\cdot\mathrm{F}^{d-d_1-1}\,\mathrm{V}^{d-d_1}\prod_{i=2}^n\mathbf{a}_i^{d_i}.\end{aligned}$$

Since F and V commutes we can use corollary 1.4.2 on this expression to get

$$\mathrm{V}^d\left(\mathrm{F}^{d-1}\,\mathbf{a}_1^{d_1}\cdot\mathrm{F}^{d-1}\prod_{i=2}^n\mathbf{a}_i^{d_i}\right)$$

and because F is a homomorphism this equals $\mathrm{F}^{d-1}\,\mathrm{V}^d\,\Delta(\mathbf{a}_1,\dots,\mathbf{a}_n)$.

Now, for an arbitrary degree d form, the result follows since V is additive. □

1.5 Miscellaneous results

1.5.1 The norm map

Definition 1.5.1. *Let $A\to B$ be an algebra such that B is free of rank n as an A-module. If $f\colon B\to B$ is a morphism of A-modules, define* $\det f$ *to be the determinant of the matrix of f in some basis. Since the determinant is multiplicative this definition is independent of the choice of basis. If $x\in B$, let $f_x\colon B\to B$ be the map $y\mapsto xy$. Define* $\mathrm{N}_{B/A}\colon B\to A$ *as* $x\mapsto\det f_x$.

It follows from the definition that $\mathrm{N}_{B/A}$ is multiplicative and $\mathrm{N}_{B/A}(1)=1$. Hence if $x\in B^\times$ then $\mathrm{N}_{B/A}(x)\in A^\times$. On the other hand if $\mathrm{N}_{B/A}(x)\in A^\times$, i.e., if $\det f_x\in A^\times$, then by Cramer's rule (which holds over every commutative ring) we have that f_x is invertible so there exists $y\in B$ such that $1=f_x(y)=xy$, hence $x\in B^\times$. We therefore have $B^\times=\mathrm{N}_{B/A}^{-1}(A^\times)$.

1.5.2 Equalizers in the category of schemes

Definition 1.5.2. *If $f,g\colon X\to Y$ are morphisms of schemes, define the equalizer* $\mathrm{Equal}(f,g)\to X$ *of f and g as the scheme that represents the functor* $\mathrm{Equal}(f,g)(S)=\{x\in X(S):\ f(x)=g(x)\}$.

To see that this scheme exists let Z be the fibre product

$$\begin{array}{ccc} Z & \xrightarrow{p} & X\\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle g}\\ X & \xrightarrow{f} & Y\end{array}$$

Define $\mathrm{Equal}(f,g)$ as the fibre product

$$\begin{tikzcd} \mathrm{Equal}(f,g) \arrow[r, "s"] \arrow[d] & X \arrow[d, "\Delta"] \\ Z \arrow[r, "(p,q)"] & X \times X \end{tikzcd}$$

By the universal property of the fibre product, $s\colon \mathrm{Equal}(f,g) \to X$ has the properties that $fs = gs$ and if $x\colon S \to X$ is a map of schemes such that $fx = gx$ then there exists a unique map $x'\colon S \to \mathrm{Equal}(f,g)$ such that $x = sx'$. This implies that $\mathrm{Equal}(f,g)(S) = \{x \in X(S) : f(x) = g(x)\}$ for every S.

1.5.3 Descent

Sometimes one can prove that a morphism of schemes has some property by extending the scalars and see that the property holds for the extension. We collect here some results of this kind that will be of use to us.

Lemma 1.5.3. *Let $f\colon X \to Y$ be a morphism of A-schemes, A a ring. If $A \to B$ is faithfully flat and $f_B\colon X_B \to Y_B$ is an isomorphism, then f is an isomorphism.*

See [Gro71], page 213 for this.

Lemma 1.5.4. *Let $A \to B$ be a flat ring homomorphism and let X be a noetherian A-scheme. Then the canonical homomorphism $\mathcal{O}_X(X) \otimes_A B \to \mathcal{O}_{X_B}(X_B)$ is an isomorphism.*

For a proof of this see [Liu02], page 85.

Lemma 1.5.5. *Let $A \to B$ be a faithfully flat ring homomorphism and let X be a noetherian A-scheme. If X_B is affine then X is affine.*

Proof. The identity map $\mathcal{O}_X(X) \to \mathcal{O}_X(X)$ gives a morphism of schemes $X \to \operatorname{Spec} \mathcal{O}_X(X)$ which is an isomorphism if and only if X is affine. By extension of scalars we get the morphism $X_B \to \operatorname{Spec} \mathcal{O}_X(X) \times_A \operatorname{Spec} B = \operatorname{Spec}(\mathcal{O}_X(X) \otimes_A B) = \operatorname{Spec} \mathcal{O}_{X_B}(X_B)$ and since X_B is affine this is an isomorphism. Hence by lemma 1.5.3 the original morphism is an isomorphism. $\square$

Chapter 2

The class of a torus in $\mathrm{K}_0(\mathbf{Sch}_k)$

Given a field k and a separable k-algebra L of dimension n we define an affine group scheme L^* by letting $L^*(M) = (L \otimes_k M)^\times$ for every k-algebra M. If we extend the base of L^* to a separable closure of k then L^* becomes isomorphic to $\mathbb{G}_m^n$ as an algebraic group, hence L^* is a torus. The objective of this chapter is to compute, for an arbitrary separable k-algebra L, the class of L^* in $\mathrm{K}_0(\mathbf{Sch}_k)$ in terms of the Lefschetz class $\mathbb{L}$ and artin classes.

2.1 Definitions

Definition-Lemma 2.1.1. *Let K be a ring and let L be a free K-algebra of finite rank. We define the affine ring scheme $\widetilde{L}$ over K as the functor $\widetilde{L}\colon \mathbf{Alg}_K \to \mathbf{Rings}$ given by*

$$\widetilde{L}(M) = L \otimes_K M \qquad \text{for every } K\text{-algebra } M$$

and if $f\colon M \to N$ is morphism of K-algebras then $\widetilde{L}(f)\colon \widetilde{L}(M) \to \widetilde{L}(N)$ maps $l \otimes m \in L \otimes_K M$ to $l \otimes f(m) \in L \otimes_k N$.

Proof. We have to show that this functor really defines an affine ring scheme, i.e., that its composition with the forgetful functor to **Sets** is representable. This is true because if M is a K-algebra then we have canonical isomorphims of K-modules

$$L \otimes_K M \simeq L^{\vee\vee} \otimes_K M \simeq \mathrm{Hom}_K(L^\vee, M) \simeq \mathrm{Hom}_{K-\mathrm{alg}}\big(\mathrm{S}(L^\vee), M\big),$$

hence the composition is represented by $\mathrm{S}(L^\vee)$. Therefore $\widetilde{L}$ is an affine ring scheme. (This ring scheme structure can also be given by coalgebra structures $\Delta_a, \Delta_m\colon \mathrm{S}(L^\vee) \to \mathrm{S}(L^\vee) \otimes \mathrm{S}(L^\vee)$ such that $\mathrm{Hom}_{K-\mathrm{alg}}\big(\mathrm{S}(L^\vee), M\big) \simeq L \otimes_k M$ as K-algebras. Then Δ_a is defined by $f \mapsto 1 \otimes f + f \otimes 1$ when $f \in L^\vee$ and Δ_m is defined by the map $L^\vee \to L^\vee \otimes L^\vee$ that is the composition of the dual of the multiplication $L \otimes L \to L$ with the inverse of the canonical isomorphism $L^\vee \otimes L^\vee \to (L \otimes L)^\vee$. However, we will not use this.) $\square$

The proof shows that as a scheme, $\widetilde{L} = \operatorname{Spec} \mathrm{S}(L^\vee)$. Hence $\widetilde{L}$ is the vector bundle associated to the free $\mathcal{O}_{\operatorname{Spec} K}$-module L.

Note in particular that $\widetilde{K}$ is the ring scheme with additive group $(\mathbb{G}_a)_K$ and multiplicative group $(\mathbb{G}_m)_K$. Also, if we choose a K-basis of L we get an isomorphism $\mathrm{S}(L^\vee) \simeq K[X_1, \ldots, X_n]$, where n is the rank of L. Hence $\widetilde{L} \simeq \mathbb{A}_K^n$ as schemes.

We next define the object that we are interested in.

Definition 2.1.2. *Let K be a ring and let L be a free K-algebra of finite rank. With $\widetilde{L}$ as above, define L^* as the subfunctor given by $L^*(M) = (L \otimes_K M)^\times$. We will see that this is an affine group scheme.*

We now give another construction of L^*, which will be useful to us. It also shows that L^* really is a scheme. For this we use the general definition of the norm map. (A discussion of the norm map can be found in section 1.5.1.)

Since L is assumed to be free of rank n over K the norm map $\mathrm{N}_{L/K}\colon L \to K$ is defined. Also, if M is a K-algebra then $L \otimes_K M$ is free of rank n as an M-module so $\mathrm{N}_{L\otimes_K M/M}$ is defined. Hence we can define a map of K-schemes $\widetilde{\mathrm{N}}_{L/K}\colon \widetilde{L} \to \widetilde{K}$ by

$$\widetilde{\mathrm{N}}_{L/K}(M) := \mathrm{N}_{L\otimes_K M/M} \qquad \text{for every } k\text{-algebra } M.$$

This is functorial, if $f\colon M \to N$ is a morphism of K-algebras then

$$\begin{CD} \widetilde{L}(M) @>{\widetilde{\mathrm{N}}_{L/K}}>> \widetilde{K}(M) \\ @V{\widetilde{L}(f)}VV @VV{\widetilde{K}(f)}V \\ \widetilde{L}(N) @>>> \widetilde{K}(N) \end{CD}$$

commutes. In fact, $\mathrm{N}_{L\otimes M/M}(l \otimes m) = \mathrm{N}_{L/K}(l)\cdot m^n$ so the upper half of the diagram maps $l \otimes m$ to $\mathrm{N}_{L/K}(l)\cdot f(m^n)$ whereas the lower half maps it to $\mathrm{N}_{L/K}(l)\cdot f(m)^n$. Therefore $\widetilde{\mathrm{N}}_{L/K}$ really is a morphism of schemes. Note however that it is not a morphism of ring schemes.

We now claim that $L^* = \widetilde{\mathrm{N}}^{-1}_{L/K}(\mathbb{G}_m)$ as subfunctors of $\widetilde{L}$. We need a fact from section 1.5.1: If S is a free R-algebra of finite rank then $\mathrm{N}^{-1}_{S/R}(R^\times) = S^\times$. Using this we see that for every K-algebra M,

$$\begin{aligned} \widetilde{\mathrm{N}}^{-1}_{L/K}(\mathbb{G}_m)(M) =& \{x \in \widetilde{L}(M) : \widetilde{\mathrm{N}}_{L/K}(M)(x) \in \mathbb{G}_m(M)\} \\ =& \{x \in L \otimes_K M : \mathrm{N}_{L\otimes_K M/M}(x) \in M^\times\} \\ =& (L \otimes_K M)^\times \\ =& L^*(M). \end{aligned}$$

Therefore $L^* = \widetilde{\mathrm{N}}^{-1}_{L/K}(\mathbb{G}_m)$ as functors. In particular, L^* is an open affine subscheme of $\widetilde{L}$.

We will be interested in the case when K is a field. However we will also be forced to consider the case when K is a finite product of fields. The following proposition shows that the latter case can always be reduced to the former.

Proposition 2.1.3. *Let $K = \prod_{v\in I} K_v$ where the K_v:s are fields and I is finite. Let L be a free K-algebra of rank n. Then L must be of the form $\prod_{v\in I} L_v$ where, for each v, L_v is a K_v-algebra of dimension n. For each v, construct the K_v-ring scheme $\widetilde{L_v}$ and view this as a K-scheme. Then as K-schemes, $\widetilde{L} \simeq \dot{\cup}_v \widetilde{L_v}$. Moreover, let L_v^* be constructed with respect to the K_v-algebra structure on L_v. Then L^* is isomorphic to $\dot{\cup}_v L_v^*$ as schemes over K.*

Proof. Since $\mathrm{S}(L^\vee) \simeq \mathrm{S}\big(\prod_{v\in I} L_v^\vee\big) \simeq \prod_{v\in I} \mathrm{S}(L_v^\vee)$ as K-algebras we have

$$\widetilde{L} = \operatorname{Spec} \mathrm{S}(L^\vee) \simeq \operatorname{Spec} \prod_{v\in I} \mathrm{S}(L_v^\vee) \simeq \dot{\cup}_{v\in I} \operatorname{Spec} \mathrm{S}(L_v^\vee) = \dot{\cup}_{v\in I} \widetilde{L_v}$$

as K-schemes.

To prove that $L^* \simeq \dot{\bigcup}(L_v/K_v)^*$ as K-schemes we prove that their functors of points are equal. Let M be a K-algebra. Then $M = \prod_v M_v$ where M_v is a K_v-algebra (possibly equal to zero). An M-point on $\dot{\bigcup}(L_v/K_v)^*$ is a morphism $f\colon \dot{\bigcup}_v \operatorname{Spec} M_v \to \dot{\bigcup}(L_v/K_v)^*$ that commutes with the structural morphisms to $\dot{\bigcup}_v \operatorname{Spec} K_v$. Since the image of $\operatorname{Spec} M_v$ under the structural morphism is contained in $\operatorname{Spec} K_v$ we must have $f(\operatorname{Spec} M_v) \subset L_v^*$. Therefore f is determined by a set of morphisms $\{f_v\colon \operatorname{Spec} M_v \to L_v^*\}_{v \in I}$ where f_i is a morphism of K_v-schemes. Hence we can identify f with an element in $\prod L_v^*(M_v)$. The same is true for an M-point on L^* for

$$L^*(M) = \left(\left(\prod L_v\right) \otimes_{\prod K_v} \left(\prod M_v\right)\right)^{\times} \simeq \prod (L_v \otimes_{K_v} M_v)^{\times} = \prod L_v^*(M_v).$$

So by Yoneda's lemma, $L^* \simeq \dot{\bigcup} L_v^*$. (This method could also have been used to prove the first part of the proposition, but there we knew the algebra representing $\widetilde{L}$ and that gave a shorter proof.) $\square$

2.2 A recursive computation of $[L^*]$

Now that we have defined the scheme that we are interested in we can start the computations. Let k be a field and let L be a separable k-algebra of dimension n. We are going to show that $[L^*] \in \mathrm{K}_0(\mathbf{Sch}_k)$ is a polynomial in $\mathbb{L}$ with coefficients that are artin classes. We begin with the simplest case.

Theorem 2.2.1. *If $L = k^n$ then $[L^*] = (\mathbb{L} - 1)^n \in \mathrm{K}_0(\mathbf{Sch}_k)$.*

Proof. We have $\widetilde{L}(M) = M^n$ and hence $L^*(M) = (M^n)^\times = (M^\times)^n$. We therefore have an isomorphism $L^*(M) \to \mathbb{G}_m^n(M)$ for every M and this isomorphism is functorial in M, hence L^* is isomorphic to $\mathbb{G}_m^n$ as schemes so $[L^*] = (\mathbb{L} - 1)^n \in \mathrm{K}_0(\mathbf{Sch}_k)$. $\square$

We next consider a simple example which still will take up some space since we work it out in detail.

Exampel 2.2.2. *Let K be a separable extension field of k of degree 2. We can think of K as $k[T]/\big(f(T)\big)$ where $f(T) = T^2 + \alpha T + \beta$ is irreducible, in particular $\beta \neq 0$. If $\operatorname{char} k \neq 2$ we may and will assume that $\alpha = 0$.*

We can now describe $\widetilde{K}$. We have

$$\widetilde{K}(M) = K \otimes_k M \simeq M[T]/\big(f(T)\big)$$

for every k-algebra M. A basis for the M-algebra $\widetilde{K}(M)$ is $\{1, t\}$ where t is the class of T modulo $f(T)$. If $m_1, m_2 \in M$ then $(m_1 + m_2 t) \cdot t = -m_2\beta + (m_1 - m_2\alpha)t$, hence

$$\mathrm{N}_{\widetilde{K}(M)/M}(m_1 + m_2 t) = m_1^2 - m_1 m_2 \alpha + m_2^2 \beta.$$

So if we identify $\widetilde{K}$ with $\operatorname{Spec} k[X_1, X_2]$ then

$$K^* = \mathrm{D}\big(X_1^2 - \alpha X_1 X_2 + \beta X_2^2\big) \subset \widetilde{K},$$

for we have

$$\begin{aligned} K^*(M) =& \Big(M[T]/\big(f(T)\big)\Big)^{\times} \\ =& \{m_1 + m_2 t : \mathrm{N}_{\widetilde{K}(M)/M}(m_1 + m_2 t) \in M^\times\} \\ =& \{(m_1, m_2) \in M^2 : m_1^2 - \alpha m_1 m_2 + \beta m_2^2 \in M^\times\} \\ =& \mathrm{D}\big(X_1^2 - \alpha X_1 X_2 + \beta X_2^2\big)(M). \end{aligned}$$

for every k-algebra M. We now have an explicit equation describing K^*. To compute $[K^*]$ we first compute its complement in $\widetilde{K}$, $X := \operatorname{Spec} k[X_1, X_2]/(X_1^2 - \alpha X_1 X_2 + \beta X_2^2) \subset \widetilde{K}$. With respect to X we have

$$\mathrm{V}(\overline{X}_2) \simeq \operatorname{Spec} k[X_1]/(X_1^2) \subset X,$$

hence $\left[\mathrm{V}(\overline{X}_2)\right] = 1$. And

$$\begin{aligned}
\mathrm{D}(\overline{X}_2) &\simeq \operatorname{Spec} \frac{k[X_1, X_2, 1/X_2]}{\left(X_1^2 - \alpha X_1 X_2 + \beta X_2^2\right)} \\
&= \operatorname{Spec} \frac{k[X_1, X_2, 1/X_2]}{\left((X_1/X_2)^2 - \alpha X_1/X_2 + \beta\right)} \\
&\simeq \operatorname{Spec} \frac{k[Y_1, Y_2, 1/Y_2]}{\left(Y_1^2 - \alpha Y_1 + \beta\right)}.
\end{aligned}$$

Now if $\operatorname{char} k \neq 2$ then $\alpha = 0$ so $Y_1^2 - \alpha Y_1 + \beta = f(Y_1)$ and this is also true if $\operatorname{char} k = 2$ for then $-\alpha = \alpha$. Hence the above expression equals

$$\operatorname{Spec} k[Y_2, 1/Y_2] \times_k \operatorname{Spec} K$$

so $\left[\mathrm{D}(\overline{X}_2)\right] = (\mathbb{L} - 1) \cdot [\operatorname{Spec} K]$.

We therefore have $[X] = 1 + (\mathbb{L} - 1) \cdot [\operatorname{Spec} K]$, hence

$$[K^*] = \left[\widetilde{K}\right] - [X] = \mathbb{L}^2 - [\operatorname{Spec} K] \cdot \mathbb{L} + [\operatorname{Spec} K] - 1.$$

Next we look at an example which suggests what the answer should be in a more complicated case.

Exampel 2.2.3. *Suppose that $k = \mathbb{F}_q$ and $L = \mathbb{F}_{q^3}$. We know that*

$$L \otimes_k \mathbb{F}_{q^m} = \begin{cases} \mathbb{F}_{q^m}^3 & \text{if } 3 \mid m \\ \mathbb{F}_{q^{3m}} & \text{if } 3 \nmid m. \end{cases}$$

It follows that

$$L^*(\mathbb{F}_{q^m}) = \begin{cases} (\mathbb{F}_{q^m}^\times)^3 & \text{if } 3 \mid m \\ (\mathbb{F}_{q^{3m}})^\times & \text{if } 3 \nmid m, \end{cases}$$

and therefore

$$\left|L^*(\mathbb{F}_{q^m})\right| = \begin{cases} (q^m - 1)^3 & \text{if } 3 \mid m \\ q^{3m} - 1 & \text{if } 3 \nmid m. \end{cases}$$

Since $\left|\operatorname{Spec} \mathbb{F}_{q^3}(\mathbb{F}_{q^m})\right| = 3$ if $3 \mid m$ and 0 otherwise, we have reason to believe that

$$[L^*] = \mathbb{L}^3 - [\operatorname{Spec} \mathbb{F}_{q^3}] \cdot \mathbb{L}^2 + [\operatorname{Spec} \mathbb{F}_{q^3}] \cdot \mathbb{L} - 1.$$

In example 2.3.6 we will see that this formula is true.

Our first result concerning the general problem will be the following.

Theorem 2.2.4. *Let L be a separable k-algebra of dimension n. Then there exist artin classes $a_1, \ldots, a_n \in \operatorname{ArtCl}_k \subset \mathrm{K}_0(\mathbf{Sch}_k)$ such that*

$$[L^*] = \mathbb{L}^n + a_1 \mathbb{L}^{n-1} + a_2 \mathbb{L}^{n-2} + \cdots + a_n \in \mathrm{K}_0(\mathbf{Sch}_k).$$

Moreover, there exists an algorithm for computing the a_i:s.

The rest of this section will be devoted to proving this theorem by describing the algorithm. This will be done in the following way. We first describe subschemes of $\widetilde{L}$, denoted $L_1, \dots, L_n$ such that $[L^*] = \mathbb{L}^n - \sum_{i=1}^n [L_i]$. We are then reduced to compute $[L_i]$ for every i. For every i we find a subscheme T_i of L_i and an $\mathcal{O}_{T_i}$-algebra of dimension less that n such that $L_i \simeq (L'_i/T_i)^*$ as k-schemes. We show that T_i is the spectrum of a product of fields, $\prod K_v$, and that $(L'_i/T_i)^* \simeq \dot{\bigcup}(L_v/K_v)^*$ where L_v is a K_v-algebra of dimension less than n. We are then in the situation we started with, only that the algebras have dimension less then n, for having computed $[(L_v/K_v)^*] \in \mathrm{K}_0(\mathbf{Sch}_{K_v})$ we can find $[(L_v/K_v)^*] \in \mathrm{K}_0(\mathbf{Sch}_k)$ with the help of proposition 1.1.6.

We will now give the definitions of L_i, T_i and L'_i. To prove that $L_i \simeq (L'_i/T_i)^*$ we will construct a map between them. It will then suffice to show that this map is an isomorphism when $L = k^n$. For this reason we give an explicit description of L_i, T_i and L'_i in this case.

Description of L_i

The norm map $\mathrm{N}_{L/k}$ factors as

$$\begin{aligned} L &\to \mathrm{End}(L) \to k \\ x &\mapsto f_x \mapsto \det f_x \end{aligned}$$

where f_x is the map that takes y to xy and $\det f_x$ is the determinant of the matrix of f_x in some basis for L. Consider the subscheme of endomorphisms of corank i in $\widetilde{\mathrm{End}(L)}$. To be more precise we want the M-rational points of this scheme to be the elements of $\widetilde{\mathrm{End}(L)}(M)$ of corank i, i.e., the locally closed subscheme

$$\mathrm{V}(n-i+1\text{-minors}) \setminus \mathrm{V}(n-i\text{-minors}) \subset \operatorname{Spec} k[X_{ij}]_{1\le i,j\le n} \simeq \widetilde{\mathrm{End}(L)}.$$

Here a j-minor is the determinant of a $j \times j$ submatrix of $(X_{ij})_{1\le i,j\le n}$.

Let L_i be the inverse image in $\widetilde{L}$ of the subscheme of endomorphism of corank i in $\widetilde{\mathrm{End}(L)}$. Then $L^* = L_0$ and $\widetilde{L} = \dot{\bigcup}_{0\le i\le n} L_i$, hence $[L^*] = \mathbb{L}^n - \sum_{i=1}^n [L_i]$.

We next describe L_i when $L = k^n$. First we choose the standard basis for $L = k^n$. When we then let $k[X_1, \dots, X_n]$ represent $\widetilde{L}$ we see that, under the isomorphism $\widetilde{L}(M) = M^n \simeq \mathrm{Hom}_{k\text{-alg}}(k[X_1, \dots, X_n], M)$, the element $(m_j)_{j=1}^n$ corresponds to $X_j \mapsto m_j\colon k[X_1, \dots, X_n] \to M$. We use this to identify the M-rational points on L_i:

We have that $X_j \mapsto m_j \in \widetilde{L}(M)$ is in $L_i(M)$ if and only if $X_{ij} \mapsto \delta_{ij} X_i \mapsto \delta_{ij} m_i$ is in

$$\big(\mathrm{V}(n-i+1\text{-minors}) \setminus \mathrm{V}(n-i\text{-minors})\big)(M),$$

i.e., if it maps all $n-i+1$-minors to 0 but maps some $n-i$ minor to an invertible number. Now the map $X_{ij} \mapsto \delta_{ij} X_i$ maps every minor to zero, except those coming from sub-matrices on the diagonal. They map to $\prod_{j\in S} X_j$ where

$$S \in \mathcal{P}_l := \text{ the } l\text{-subsets of } \{1, \dots, n\}$$

for some l. Hence the condition for $(X_j \mapsto m_j) \in \widetilde{L}(M)$ to lie in $L_i(M)$ is that $\prod_{j\in S} m_j = 0$ for every $S \in \mathcal{P}_{n-i+1}$ and that there exists an $S \in \mathcal{P}_{n-i}$ such that $\prod_{j\in S} m_j \in M^\times$. This means that there is an $S \in \mathcal{P}_{n-i}$ such that $m_j \in M^\times$ if $j \in S$. Moreover if $j' \notin S$ then $m_{j'} \prod_{j\in S} m_j = 0$ so $m_{j'} = 0$.

For $S \in \mathcal{P}_i$, let e_S be the n-tuple of zeros and ones such that

$$(e_S)_j = \begin{cases} 0 & j \in S \\ 1 & j \notin S \end{cases}. \tag{2.1}$$

Then the M-points on L_i can be given as

$$L_i(M) = \dot{\bigcup}_{S \in \mathcal{P}_i} (e_S M^n)^\times. \tag{2.2}$$

for every k-algebra M.

We also give a description of L_i as a locally ringed space: Let $I := (\prod_{j \in S} X_j)_{S \in \mathcal{P}_{n-i+1}}$ and consider the closed subscheme $\mathrm{V}(I)$ of $\widetilde{L}$, i.e., $\mathrm{V}(I) = \operatorname{Spec} k[X_1, \ldots, X_n]/I$. Let $P_S := \prod_{j \in S} X_j$. Then

$$L_i = \bigcup_{S \in \mathcal{P}_{n-i}} \mathrm{D}(P_S) \subset \mathrm{V}(I),$$

where we have identified P_S with its image in $k[X_1, \ldots, X_n]/I$. So L_i is an open subscheme of $\mathrm{V}(I)$, hence a locally closed subscheme of $\widetilde{L}$.

Description of T_i

First we construct the subscheme $\operatorname{Idem} \widetilde{L} = \{e \in \widetilde{L} : e^2 = e\} \subset \widetilde{L}$, by which we mean the scheme such that for every k-algebra M, $(\operatorname{Idem} \widetilde{L})(M) = \{e \in \widetilde{L}(M) : e^2 = e\}$. It is not obvious that this scheme exists but we can show that it does by using the more general construction of an equalizer. (It is a standard fact that equalizers exist in the category of k-schemes, see section 1.5.2 for the definition of an equalizer and a construction.) Let x^2 be the composition $\widetilde{L} \xrightarrow{\Delta} \widetilde{L} \times_k \widetilde{L} \to \widetilde{L}$ and let $x\colon \widetilde{L} \to \widetilde{L}$ be the identity. Then $\operatorname{Idem} \widetilde{L} = \operatorname{Equal}(x, x^2)$.

Now we fix an i and define the scheme of connected components of L_i, denoted T_i, as $L_i \cap \operatorname{Idem} \widetilde{L}$, i.e., the fibre product

$$\begin{tikzcd} T_i \arrow[r] \arrow[d] & L_i \arrow[d] \\ \operatorname{Idem} \widetilde{L} \arrow[r] & \widetilde{L} \end{tikzcd} \tag{2.3}$$

It follows that if M is a k-algebra then $T_i(M) = \{m \in L_i(M) : m^2 = m\}$.

We next describe T_i when $L = k^n$. We have $T_i(M) = \{\mathbf{m} = (m_1, \ldots, m_n) \in L_i(M) : \mathbf{m}^2 = \mathbf{m}\}$ so if M has no non-trivial idempotents then $m_j = 0$ or 1 for each j, hence the above description of $L_i(M)$ gives that $T_i(M) = \{e_S\}_{S \in \mathcal{P}_i}$ where e_S was defined in (2.1).

Let

$$R_S := \frac{k[X_1, \ldots, X_n]}{(X_j)_{j \in S} \cdot (X_j - 1)_{j \notin S}}.$$

We claim that T_i is represented by $\prod_{S \in \mathcal{P}_i} R_S$. For this, define

$$T_i' := \operatorname{Spec} \prod_{S \in \mathcal{P}_i} R_S.$$

We have to show that $T_i = T_i'$ as subschemes of $\widetilde{L}$, i.e., that $T_i(M) = T_i'(M)$ for every k-algebra M. But by the construction of T_i as a fibre product it is a closed subscheme of L_i, which in turn is a locally closed subscheme of $\widetilde{L}$. Since $\widetilde{L}$ is noetherian it follows that T_i is noetherian. Also T_i' is noetherian, hence to show that $T_i = T_i'$ it suffices to show that $T_i(M) = T_i'(M)$ for every noetherian k-algebra M.

We first show that the equality is true if M has no non-trivial idempotents and for this we just have to show that $T_i'(M) = \{e_S\}_{S \in \mathcal{P}_i}$ since we just noticed that this holds for $T_i(M)$. Let $f_T \in \prod_{S \in \mathcal{P}_i} R_S$ have the entry with index T equal to 1 and zeros in the other entries. An element of $\operatorname{Hom}_{k\text{-alg}}(\prod_{S \in \mathcal{P}_i} R_S, M)$

has to send idempotents to idempotents and $(1,\ldots,1)$ to 1, hence every f_T maps to 0 or 1, and $\sum_{T\in\mathcal{P}_i} f_T$ maps to 1. Moreover if $T \neq T'$ then $f_T f_{T'}$ maps to 0 so at least one of f_T and $f_{T'}$ maps to 0. Hence all f_T are mapped to 0 except one which are mapped to 1. So

$$T_i'(M) = \mathrm{Hom}_{k\text{-alg}}\left(\prod_{S\in\mathcal{P}_i} R_S, M\right) = \{\phi_S : \phi_S(f_T) = 1 \text{ if } T = S, \text{ and } 0 \text{ otherwise}\}_{S\in\mathcal{P}_i}.$$

To see which element in $\widetilde{L}(M)$ that corresponds to ϕ_S we have to compose the map $k[X_1,\ldots,X_n] \to \prod_{S\in\mathcal{P}_i} R_S$ with ϕ_S. The first map sends X_j to

$$\sum_{T\in\mathcal{P}_i : j\notin T} f_T$$

and this in turn is mapped by ϕ_S to 0 if $j \in S$ and to 1 if $j \notin S$. Hence $\phi_S = e_S$ as elements of $\widetilde{L}(M)$ so $T_i'(M) = \{e_S\}_{S\in\mathcal{P}_i} = T_i(M)$ in this case.

For the general case we may assume that M has only a finite number of orthogonal idempotents, for if $x_1,\ldots,x_{l+1} \in M$ are orthogonal idempotents and $x_{l+1} = \sum_{j=1}^{l} h_j x_j$ then if we multiply with x_{l+1} we get $x_{l+1}^2 = 0$, i.e., $x_{l+1} = 0$, hence if there are an infinite number of orthogonal idempotents then M is not noetherian. Therefore we can write $M = \prod_{j=1}^{l} M_j$ where each M_j contains no non-trivial idempotents. Since the product is finite we have $\operatorname{Spec} M = \dot\cup_{1\le j\le l} \operatorname{Spec} M_j$ (by this we mean open disjoint union, i.e., the coproduct in the category of schemes). So by the defining universal property of coproducts,

$$T_i(M) = \mathrm{Hom}(\dot\cup \operatorname{Spec} M_j, T_i) = \prod_{j=1}^{l} \mathrm{Hom}(\operatorname{Spec} M_j, T_i) = \prod_{j=1}^{l} T_i(M_j).$$

By the same reasoning, $T_i'(M) = \prod_{j=1}^{l} T_i'(M_j)$ so $T_i(M) = T_i'(M)$. (Note that since we know that T_i' is affine we don't need to know that the product is finite in this case, for $T_i' = \operatorname{Spec} R$ so $T_i'(\prod M_j) = \mathrm{Hom}_{\text{k-alg}}(R, \prod M_j) = \prod \mathrm{Hom}_{\text{k-alg}}(R, M_j) = \prod T_i'(M_j)$ for any product. So if we knew a priori that T_i where affine then the above proof would be shorter.)

Hence we have identified T_i as a closed subscheme of $\widetilde{L} = \operatorname{Spec} k[X_1,\ldots,X_n]$, namely we have

$$T_i = \operatorname{Spec} \prod_{S\in\mathcal{P}_i} R_S = \dot{\bigcup}_{S\in\mathcal{P}_i} \operatorname{Spec} R_S. \tag{2.4}$$

Let $T_S := \operatorname{Spec} R_S$. We see that $R_S \simeq k$ for every S so $T_S \simeq \operatorname{Spec} k$, hence T_i consists of $\binom{n}{i}$ points.

We have now seen that T_i is affine when $L = k^n$. It follows that this is true also in the general case.

Proposition 2.2.5. *Let L be a separable k-algebra and construct T_i with respect to L. Then T_i is affine. In fact it is the spectrum of a product of fields.*

Proof. Let k^s be a separable closure of k. Since L is separable, $L \otimes_k k^s \simeq (k^s)^n$. Hence, by the above, $(T_i)_{k^s}$ is the spectrum of $(k^s)^{\binom{n}{i}}$. In particular it is affine. From lemma 1.5.5 it follows that T_i is affine and then that it is the spectrum of a separable algebra. Since we also have that $(T_i)_{k^s}$ is zero dimensional it follows that $\dim T_i = 0$ (dimension is invariant under base extension from a field to an algebraic extension.). Hence T_i is the spectrum of a product of fields. (Alternatively, if we use that any scheme whose underlying topological space has finite cardinality and dimension 0 is affine then we don't need lemma 1.5.5.) $\square$

Description of L_i'

Next let $\pi\colon T_i \to \operatorname{Spec} k$ be the structural morphism. From proposition 2.2.5 we know that T_i is affine, say $T_i = \operatorname{Spec} R$. The $\mathcal{O}_{T_i}$-algebra π^*L is then isomorphic to $L \otimes_k R$, hence it is free and we can define $\widetilde{\pi^*L}$. The dual of the R-module π^*L is $L^\vee \otimes_k R$. Since the symmetric algebra commutes with base change we then have $\mathrm{S}\big((\pi^*L)^\vee\big) \simeq \mathrm{S}(L^\vee) \otimes_k R$. It follows that $\widetilde{\pi^*L}$ isomorphic to $\widetilde{L} \times_k T_i$ as a T_i-scheme.

We have a map $e\colon T_i \to \widetilde{L} \times_k T_i$, given by the identity map $T_i \to T_i$ together with the map $T_i \to \widetilde{L}$ from the definition of T_i (see (2.3)). The map e is a global section of $\widetilde{\pi^*L} \to T_i$. It hence corresponds to a global section $e \in (\pi^*L)(T_i)$.

Lemma 2.2.6. *The global section $e \in (\pi^*L)(T_i)$ is an idempotent.*

Proof. e was defined via the isomorphism $\big(\widetilde{L} \times_k T_i\big)(T_i) \simeq \widetilde{L}(T_i) \times T_i(T_i)$ and under this identification, the second coordinate of e is an idempotent by the definition of T_i and the first coordinate is an idempotent if it lies in $(\operatorname{Idem} \widetilde{L})(T_i) \subset \widetilde{L}(T_i)$. But this follows since it factors through $\operatorname{Idem} \widetilde{L}$ by its definition (2.3). □

Define $L_i' := e(\pi^*L)$. Then since $e^2 = e$, we have that L_i' is a free $\mathcal{O}_{T_i}$-algebra so the norm map $L_i' \to \mathcal{O}_{T_i}$ is defined. Hence we can form $(L_i')^*$ and we will see that $(L_i')^*$ and L_i are isomorphic as schemes over $\operatorname{Spec} k$. For this we define a map betwen them: First note that since $L_i' \subset \pi^*L$ we have a map $\widetilde{L_i'} \to \widetilde{\pi^*L}$. Since $(L_i')^* \subset \widetilde{L_i'}$ this gives a map $(L_i')^* \to \widetilde{\pi^*L} = \widetilde{L} \times_k T_i$. Composing this with the map from the fibre product to $\widetilde{L}$ gives the map $g\colon (L_i')^* \to \widetilde{L}$. We will see that g is an isomorphism onto $L_i \subset \widetilde{L}$.

We now describe L_i' when $L = k^n$. First we identify π^*L. Let π_S be the restriction of π to T_S. Then π_S is an isomorphism (corresponding to the isomorphism of k-algebras $k \to R_S$) so $(\pi_S^*L)(T_S) \simeq L(\operatorname{Spec} k) = L$. Therefore $(\pi^*L)(T_S) = (\pi_S^*L)(T_S) = L$ so if $I \subset \mathcal{P}_i$ then

$$(\pi^*L)\Big(\dot{\bigcup}_{S\in I} T_S\Big) = \prod_{S\in I} L.$$

Then to find e it suffices to find its component over T_S, $e_S \in (\pi^*L)(T_S)$. The canonical map $T_S \to T_S \times \widetilde{L}$ corresponds to the map $R_S \otimes_k k[X_1, \dots, X_n] \to R_S$ that maps X_i to its image in R_S, namely 0 if $i \in S$ and 1 otherwise. Next $R_S \otimes_k k[X_1, \dots, X_n]$ is canonically isomorphic to $\mathrm{S}\big((\pi_S^*L)(T_S)^\vee\big) = \mathrm{S}(L^\vee)$ under $X_i \mapsto f_i$, where f_i maps the i:th basis element of L to 1 and the rest to zero. Hence $T_i \to T_i \times \widetilde{L}$ corresponds to the element in $L^{\vee\vee}$ that maps f_i to 0 if $i \in S$ and to 1 otherwise. This in turn corresponds to $e_S \in L = (\pi_S^*L)(T_S)$ with j:th coordinate 0 if $j \in S$ and 1 otherwise. Therefore $e = (e_S) \in \prod_{S\in\mathcal{P}_i} L$.

Now by definition $L_i' = e(\pi^*L)$, hence

$$L_i'\Big(\dot{\bigcup}_{S\in I} T_S\Big) = \prod_{S\in I} (e_S \cdot L).$$

with T_i-algebra structure given by the map $\prod_{S\in\mathcal{P}_i} R_S \to \prod_{S\in\mathcal{P}_i} (e_S \cdot L)$. To find $(L_i')^*$ we first have to understand $\mathrm{N}_{L_i'/\mathcal{O}_{T_i}}$. This can be done on each connected component, $L_i'|_{T_S}$ is just the k-algebra $e_S \cdot L$. Then by the same reasoning as when we determined L^*, the M-points on $(L_i'|_{T_S})^*$ is $(L_i'|_{T_S})^*(M) = \{e_S \cdot \mathbf{m} : \mathbf{m} = (m_1, \dots, m_n) \in M^n, \prod_{j\notin S} m_j \in M^\times\} = (e_S M^n)^\times$.

L_i is isomorphic to $(L_i'/T_i)^*$

To prove that the map g defined previously really is an isomorphism we use lemma 1.5.3 which says that to check that a morphism of schemes is an isomorphism it suffices to check this after an extension of the base.

Now since L is separable there exists an extension field $K \supset k$ such that $L \otimes_k K \simeq K^n$. Because of lemma 1.5.3 we only have to prove that g is an isomorphism over $\operatorname{Spec} K$. We may therefore assume that $L = k^n$. In this case we have identified explicitly the rational points of $(L_i')^*$ and L_i and we now show that they are isomorphic via g:

Lemma 2.2.7. *If $L = k^n$ then $g\colon (L_i')^* \to L_i$ is an isomorphism.*

Proof. From (2.2) we know the M-points on L_i for every k-algebra M. Define a map $\rho\colon L_i \to T_i$ by $e_S \cdot (m_1, \dots, m_n) \in L_i(M) \mapsto e_S \in T_i(M)$. Since $L_i = \dot\bigcup \rho^{-1} T_S$ and $(L_i')^* = \dot\bigcup (L_i'|_{T_S})^* = \dot\bigcup (e_S L)^*$ it suffices to show that $g|_{(e_S L)^*}\colon (e_S L)^* \to \rho^{-1} T_S$ is an isomorphism for every S. We have already seen what the M-points on these schemes are, they have both been identified with $(e_S M^n)^\times$. It remains to see that $g|_{(e_S L)^*}(M)$ gives this identification.

Now $g|_{(e_S L)^*}(M)$ first maps $e_S \cdot \mathbf{m}$ to $(e_S, e_S \cdot \mathbf{m}) \in (T_i \times \widetilde{L})(M)$, then this is mapped to $e_S \cdot \mathbf{m} \in \rho^{-1}(T_S)(M) \subset L_i(M)$. Hence $g(M)$ is a bijection and it follows from Yoneda's lemma that g is an isomorphism. □

From this it now follows:

Proposition 2.2.8. *For any finite dimensional k-algebra L, we have that L_i is isomorphic to $(L_i'/T_i)^*$ as k-schemes via the map g defined above.*

Proposition 2.2.9. *Let L be a separable k-algebra of dimension n. Then*

$$[L^*] = \mathbb{L}^n - \sum_{j=1}^{n-1} \left[(L_i'/T_i)^*\right] - 1 \in \mathrm{K}_0(\mathbf{Sch}_k).$$

Proof of theorem 2.2.4

Above we were given k and L and we then constructed the k-schemes $\widetilde{L}$ and L^*. To be able to compute the class of L^* we constructed L_i for $i = 1, \dots, n$. Moreover we constructed a k-scheme T_i and a T_i-algebra L_i'. We then constructed $(L_i')^*$, which we also write as $(L_i'/T_i)^*$ to indicate that we construct it with respect to the T_i-algebra structure of L_i'. We showed that it is isomorphic to L_i as a k-scheme. When performing the induction we will have to repeat the above a number of times. We therefore use the notation $T_i(L/k)$, $L_i'(L/k)$ and $L_i(L/k)$ and we have $L_i(L/k) \simeq (L_i'(L/k)/T_i(L/k))^*$. To go further we will need a lemma.

Lemma 2.2.10. *Let L be a separable k-algebra. Then $T_i(L/k) = \dot\bigcup \operatorname{Spec} K_v$ where K_v are fields. And $L_i'(L/k) = \prod L_v$ where L_v is a K_v-algebra. Moreover, $(L_i'(L/k)/T_i(L/k))^*$ is isomorphic to $\dot\bigcup (L_v/K_v)^*$ as k-schemes.*

Proof. By proposition 2.2.5, T_i is a product of fields. It follows that L_i' is a product of algebras over the points of T_i. The last part was dealt with in proposition 2.1.3. □

This enables us to prove what we want.

Proof, theorem 2.2.4. We use induction over n, the dimension of L. For every field k the theorem is trivially true for $n = 1$ for then $[L^*] = [k^*] = \mathbb{L} - 1$. Suppose that for every field k and every separable k-algebra L of dimension $n' < n$ we have

$$[L^*] = \mathbb{L}^{n'} + a_1 \mathbb{L}^{n'-1} + \cdots + a_{n'} \in \mathrm{K}_0(\mathbf{Sch}_k).$$

where $a_s \in \mathrm{ArtCl}_k$.

Fix a separable k-algebra L of dimension n. By proposition 2.2.9 we have

$$[L^*] = \mathbb{L}^n - \sum_{j=1}^{n-1} \big[(L_i'/T_i)^*\big] - 1 \in \mathrm{K}_0(\mathbf{Sch}_k).$$

And by lemma 2.2.10 $(L_i'/T_i)^* \simeq \dot{\cup}_v (L_{i,v}/K_{i,v})^*$. Here the dimension of $L_{i,v}/K_{i,v}$ is $n-i$. We postpone the proof of this to corollary 2.3.6 because we will then be able to see it very easily. But assuming this result for the moment, the induction hypothesis gives

$$[(L_{i,v}/K_{i,v})^*] = \mathbb{L}^{n-i} + a_1\mathbb{L}^{n-i-1} + \cdots + a_{n-i} \in \mathrm{K}_0(\mathbf{Sch}_{K_v}), \tag{2.5}$$

with $a_j \in \mathrm{ArtCl}_{K_v}$, hence by proposition 1.1.6

$$[(L_{i,v}/K_{i,v})^*] = [\operatorname{Spec} K_{i-v}] \cdot \mathbb{L}^{n-i} + a_1'\mathbb{L}^{n-i-1} + \cdots + a_{n-i}' \in \mathrm{K}_0(\mathbf{Sch}_k),$$

with $a_j' \in \mathrm{ArtCl}_k$. Summation over every (i, v) gives that the formula holds for the k-algebra L. $\square$

A formula for the a_i:s

To get more compact formulas we use the following notation.

Definition 2.2.11. *Let K be a finite separable k-algebra and L a finite separable K-algebra, so $K = \prod_v K_v$ where K_v are separable extension fields of k and $L = \prod_v L_v$ where L_v is a separable K_v-algebra. Let*

$$L_i(L/K) := \dot{\cup}_v\, L_i(L_v/K_v).$$

Furthermore, define

$$T_i(L/K) := \dot{\cup}_v\, T_i(L_v/K_v)$$

and define $L_i'(L/K)$ to be the $T_i(L/K)$-algebra which is $L_i'(L_v/K_v)$ on $T_i(L_v/K_v)$.

With this notation proposition 2.2.8 and lemma 2.2.10 generalizes to:

Lemma 2.2.12. *Let K be a finite separable k-algebra and L a finite separable K-algebra, so $K = \prod_v K_v$ where K_v are separable extension fields of k and $L = \prod_v L_v$ where L_v is a separable K_v-algebra. Then $T_i(L/K) = \dot{\cup} \operatorname{Spec} K_v$ where K_v are fields. And $L_i'(L/K) = \prod L_v$ where L_v is a K_v-algebra. Moreover,*

$$L_i(L/K) \simeq \big(L_i'(L/K)/T_i(L/K)\big)^*$$

as k-schemes.

Proposition 2.2.8 may now be expressed as $L_i(L/k) \simeq \big(L_i'(L/k)/T_i(L/k)\big)^*$. In the next step we therefore want to compute $L_{i_2}\big(L_{i_1}'(L/k)/T_{i_1}(L/k)\big)$ for $1 \le i_2 \le n-i_1$. We then construct $T_{i_2}\big(L_{i_1}'(L/k)/T_{i_1}(L/k)\big)$ and its algebra $L_{i_2}'\big(L_{i_1}'(L/k)/T_{i_1}(L/k)\big)$ and we use that $L_{i_2}\big(L_{i_1}'(L/k)/T_{i_1}(L/k)\big)$ is isomorphic to

$$\Big(L_{i_2}'\big(L_{i_1}'(L/k)/T_{i_1}(L/k)\big)/T_{i_2}\big(L_{i_1}'(L/k)/T_{i_1}(L/k)\big)\Big)^*.$$

For the rest of this section, we fix a field k and a separable k-algebra L of dimension n. We now introduce some notation which allows us to write up a rather compact formula for $[L^*]$: Given a sequence of positive integers $i_1, \ldots, i_q$, construct the algebra $L_{i_1}'/T_{i_1} = L_{i_1}'(L/k)/T_{i_1}(L/k)$. Define the algebra $L_{i_2,i_1}'/T_{i_2,i_1}$ as $L_{i_2}'(L_{i_1}'/T_{i_1})/T_{i_2}(L_{i_1}'/T_{i_1})$ and define inductively $L_{i_{r+1},\ldots,i_1}'/T_{i_{r+1},\ldots,i_1}$ as

$$L_{i_{r+1}}'(L_{i_r,\ldots,i_1}'/T_{i_r,\ldots,i_1})/T_{i_{r+1}}(L_{i_r,\ldots,i_1}'/T_{i_r,\ldots,i_1}).$$

With this notation we have the following generalization of proposition 2.2.9.

Lemma 2.2.13. *Let $\alpha = (i_r, \dots, i_1)$ where $\sum_{s=1}^r i_s = i$. Then*

$$\big[(L'_\alpha/T_\alpha)^*\big] = [T_\alpha] \cdot \mathbb{L}^{n-i} - \sum_{j=1}^{n-i-1} \big[(L'_{j,\alpha}/T_{j,\alpha})^*\big] - [T_\alpha] \in \mathrm{K}_0(\mathbf{Sch}_k).$$

Proof. From the definitions of T_α and L'_α, and from lemma 2.2.10 it follows that $T_\alpha = \dot\bigcup_v \operatorname{Spec} K_v$ where K_v are fields and $L'_\alpha = \prod_v L_v$ where L_v is a K_v-algebra, where v is in some finite index set I. It then follows from lemma 2.2.10 that $(L'_\alpha/T_\alpha)^*$ is equal to the disjoint union of the $(L_v/K_v)^*$. Now by corollary 2.3.11, which we will prove later, L'_α has rank $n-i$ as a T_α-module, hence L_v has dimension $n-i$ as a K_v-vector space for every v. It follows that

$$(L_v/K_v)^* = \mathbb{L}^{n-i} - \sum_{j=1}^{n-i-1} [(L'_j(L_v/K_v)/T_j(L_v/K_v))^*] - 1 \in \mathrm{K}_0(\mathbf{Sch}_{K_v})$$

and hence by proposition 1.1.6

$$(L_v/K_v)^* = [\operatorname{Spec} K_v] \cdot \mathbb{L}^{n-i} - \sum_{j=1}^{n-i-1} [(L'_j(L_v/K_v)/T_j(L_v/K_v))^*] - [\operatorname{Spec} K_v] \in \mathrm{K}_0(\mathbf{Sch}_k). \tag{2.6}$$

Since $[T_\alpha] = \sum_v [\operatorname{Spec} K_v] \in \mathrm{K}_0(\mathbf{Sch}_k)$ and

$$\begin{aligned}(L'_{j,\alpha}/T_{j,\alpha})^* &= \big(L'_j(L'_\alpha/T_\alpha)/T_j(L'_\alpha/T_\alpha)\big)^* \\ &= \Big(\dot\bigcup\nolimits_v L'_j(L_v/K_v)/T_j(L_v/K_v)\Big)^* \\ &= \dot\bigcup\nolimits_v \big(L'_j(L_v/K_v)/T_j(L_v/K_v)\big)^*,\end{aligned}$$

so $\big[(L'_{j,\alpha}/T_{j,\alpha})^*\big] = \sum_v \big[\big(L'_j(L_v/K_v)/T_j(L_v/K_v)\big)^*\big]$, the result follows when we add together the equations (2.6) for every v. ☐

We are now ready to prove the main theorem of this section.

Theorem 2.2.14. *With the same notation as above we have*

$$[L^*] = \mathbb{L}^n + a_1\mathbb{L}^{n-1} + \dots + a_{n-1}\mathbb{L} + a_n$$

where

$$a_j = \sum_{r=1}^{j} (-1)^r \sum_{\substack{(i_1,\dots,i_r):\\ i_1+\dots+i_r=j\\ i_s \ge 1}} [T_{i_r,\dots,i_1}]$$

for $j = 1, \dots, n$.

Proof. We evaluate $[L^*]$ in n steps, using lemma 2.2.13. In the first step we write

$$[(L/k)^*] = \mathbb{L}^n - [(L'_1/T_1)^*] - \dots - [(L'_{n-1}/T_{n-1})^*] - 1$$

so we get the contribution $\mathbb{L}^n - 1$. We then evaluate the remaining terms, using lemma 2.2.13, so in step two we get a sum consisting of two parts. First, $\big[(L'_{i_2,i_1}/T_{i_2,i_1})^*\big]$ shows up with sign $(-1)^2$, for

$2 \leq i_2 + i_1 < n$ (we always have $i_s \geq 1$). This is the terms that we will take care of in step three. The second part of the sum contributes to our formula. It consists of the terms

$$(-1)^2\big(-[T_j]\cdot\mathbb{L}^{n-j} + [T_j]\big) \qquad\qquad 1 \leq j < n.$$

Continuing in this way we find that in step r we get a sum consisting of two parts. Firstly, every term of the form $\big[(L'_{i_r,\dots,i_1}/T_{i_r,\dots,i_1})^*\big]$ with coefficient $(-1)^r$, for $\sum_{s=1}^{r} i_s < n$. This part is taken care of in step $r+1$. And secondly we get a contribution to our formula consisting of

$$(-1)^r\big(-[T_{i_{r-1},\dots,i_1}]\cdot\mathbb{L}^{n-j} + [T_{i_{r-1},\dots,i_1}]\big) \qquad\qquad r-1 \leq j < n$$

for every $r-1$-tuple $(i_{r-1},\dots,i_1)$ such that $\sum_{s=1}^{r-1} i_s = j$. This process ends in step n.

Collecting terms we now see that if $1 \leq j \leq n-1$ then the coefficient in front of $\mathbb{L}^{n-j}$ becomes

$$\sum_{r=2}^{j+1}(-1)^{r+1}\sum_{\substack{(i_1,\dots,i_{r-1}):\\ i_1+\dots+i_{r-1}=j\\ i_s\geq 1}}[T_{i_{r-1},\dots,i_1}].$$

This equals

$$\sum_{r=1}^{j}(-1)^{r}\sum_{\substack{(i_1,\dots,i_r):\\ i_1+\dots+i_r=j\\ i_s\geq 1}}[T_{i_r,\dots,i_1}]. \tag{2.7}$$

The constant coefficient becomes

$$-1+\sum_{r=2}^{n}(-1)^{r}\sum_{j=r-1}^{n-1}\sum_{\substack{(i_1,\dots,i_{r-1}):\\ i_1+\dots+i_{r-1}=j\\ i_s\geq 1}}[T_{i_{r-1},\dots,i_1}].$$

Since $[T_n]=1$ it follows that if $1 \leq \sum_{s=1}^{r-1} i_s = j < n$ then $T_{n-j,i_{r-1},\dots,i_1} = T_{i_{r-1},\dots,i_1}$ so this becomes

$$-1+\sum_{r=2}^{n}(-1)^{r}\sum_{\substack{(i_1,\dots,i_r):\\ i_1+\dots+i_r=n\\ i_s\geq 1}}[T_{i_r,\dots,i_1}].$$

Hence formula (2.7) holds also when $j=n$. □

2.3 The formula for $[L^*]$ expressed using the Burnside ring

In the preceding section we only gave explicit descriptions of L_i and T_i when L is a product of copies of k. In this section we want to describe them when L is an arbitrary separable k-algebra. The strategy for this will be to lift them to k^s where we know what they look like. Then we have to be able to go back again and this will be achieved with the help of some Galois theory.

Galois theory

To be able to make explicit computations using the results in the previous section we use the following formulation of Galois theory.

Definition 2.3.1. *Let $k \subset K$ be Galois and $G := \mathcal{G}\mathrm{al}(K/k)$. Then the category of separable $K - G$-algebras is defined to be the category whose objects is separable K-algebras L together with a G-action on the underlying ring such that $K \to L$ is G-equivariant, and whose morphisms are G-equivariant maps of K-algebras.*

Theorem 2.3.2. *Fix a field k together with a separable closure k^s. Set $\mathcal{G} := \mathcal{G}\mathrm{al}(k^s/k)$. Then we have an equivalence between the category of finite separable k-algebras and the category of finite separable $k^s - \mathcal{G}$-algebras.*

This equivalence takes the k-algebra L to $L \otimes_k k^s$ with $\mathcal{G}$-action $\sigma(l \otimes \alpha) := l \otimes \sigma(\alpha)$. Its pseudo-inverse takes the $k^s - \mathcal{G}$-algebra U to $U^{\mathcal{G}}$.

If we have a $\mathcal{G}$-set T and a k-algebra A then the following lemma gives a criterion for whether T corresponds to A under the Galois correspondence or theorem 1.1.14.

Lemma 2.3.3. *Fix a field k with absolute Galois group $\mathcal{G}$.*

Let A be a separable finite dimensional k-algebra and give $A \otimes_k k^s$ the structure of a $k - \mathcal{G}$-algebra by $\sigma(x \otimes \alpha) := x \otimes \sigma(\alpha)$.

Let T be a $\mathcal{G}$-set and define a $k - \mathcal{G}$-algebra as $\prod_{t\in T} k^s e_t$ with $\mathcal{G}$-action

$$\sigma\Big(\sum_{t\in T} \alpha_t e_t\Big) := \sum_{t\in T} \sigma(\alpha_t) e_{\sigma(t)}. \tag{2.8}$$

Then T corresponds to A under the Galois correspondence of theorem 1.1.14, (i.e., $\mathrm{Art}_k[T] = [\mathrm{Spec}\, A]$) if and only if $A \otimes_k k^s$ and $\prod_{t\in T} k^s e_t$ are isomorphic as $k - \mathcal{G}$-algebras.

Proof. We have that $\prod_{t\in T} k^s e_t$ is isomorphic to $\mathrm{Hom}_{\mathbf{Sets}}(T, k^s)$ as $k - \mathcal{G}$ algebras, the $\mathcal{G}$-action on the latter being given by $(\sigma f)(t) = \sigma \circ f \circ \sigma^{-1}(t)$. And with this $\mathcal{G}$-action we get $\mathrm{Hom}_{\mathbf{Sets}}(T, k^s)^{\mathcal{G}} = \mathrm{Hom}_{\mathcal{G}}(T, k^s)$. It follows from theorem 2.3.2 that if $A \otimes_k k^s \simeq \prod_{t\in T} k^s e_t$ then

$$A \simeq (A \otimes_k k^s)^{\mathcal{G}} \simeq \left(\prod_{t\in T} k^s e_t\right)^{\mathcal{G}} \simeq \mathrm{Hom}_{\mathcal{G}}(T, k^s),$$

which means that A corresponds to T.

On the other hand, suppose that $A \simeq \mathrm{Hom}_{\mathcal{G}}(T, k^s)$. Then $A \otimes_k k^s \simeq \mathrm{Hom}_{\mathbf{Sets}}(T, k^s)^{\mathcal{G}} \otimes_k k^s$ as $k - \mathcal{G}$-algebras, and the latter is isomorphic to $\mathrm{Hom}_{\mathbf{Sets}}(T, k^s)$ by theorem 2.3.2. ☐

Computations

We now go back to our problem, we have a separable n-dimensional k-algebra L and we want to describe T_i and L'_i.

Definition 2.3.4. *If S is a set then we define $\mathcal{P}_i(S)$ to be the set of subsets of S of cardinality i. If $S = \{1, \ldots, n\}$ then we sometimes (as in the preceding section) write $\mathcal{P}_i$ or $\mathcal{P}_i^{(n)}$.*

If S is a G-set for a group G then $\mathcal{P}_i(S)$ is a G-set because if $T \subset S$ then $gT \subset S$ has the same cardinality as T for every $g \in G$.

Lemma 2.3.5. *Let S be the $\mathcal{G}$-set corresponding to L under the equivalence of theorem 1.1.14, i.e., $S = \mathrm{Hom}_k(L, k^s)$ so $\mathrm{Art}_k[S] = [\mathrm{Spec}\, L]$. Consider $\mathcal{P}_i(S)$ as a $\mathcal{G}$-set with the action induced from that on S. Then T_i corresponds to $\mathcal{P}_i(S)$, so*

$$T_i \simeq \mathrm{Spec}\, \mathrm{Hom}_{\mathcal{G}}(\mathcal{P}_i(S), k^s).$$

Moreover, L'_i corresponds to the set $\{(s, T) \in S \times \mathcal{P}_i(S) : s \notin T\}$ (with componentwise $\mathcal{G}$-action) and the T_i-algebra structure on L'_i corresponds to the projection $(s, T) \mapsto T$.

Proof. Let $T_i = \operatorname{Spec} R_i$. From (2.4) we know that $R_i \otimes_k k^s \simeq (k^s)^{\binom{n}{i}}$. For every $T \in \mathcal{P}_i(S)$, let e'_T be the tuple indexed by $\mathcal{P}_i(S)$ with 1 in position T and zeros elsewhere. Since $\mathcal{P}_i(S)$ has $\binom{n}{i}$ elements we have $R_i \otimes_k k^s \simeq \prod_{T \in \mathcal{P}_i(S)} k^s e'_T$. If we now can prove that $\mathcal{G}$ acts on this as $\sigma(\sum_{T\in\mathcal{P}_i(S)} \alpha_T e'_T) = \sum_{T\in\mathcal{P}_i(S)} \sigma(\alpha_T) e'_{\sigma(T)}$, i.e., with action (2.8), then by lemma 2.3.3, R_i corresponds to $\mathcal{P}_i(S)$.

To prove that the $\mathcal{G}$-action on $\prod_{T\in\mathcal{P}_i(S)} k^s e'_T$ is action (2.8) we use that we know the action on $L \otimes_k k^s$, for S corresponds to L by definition, hence by lemma 2.3.3 we have that $L \otimes k^s \simeq \prod_{s\in S} k^s e_s$ with action (2.8). So if $e_T := \sum_{s \notin T} e_s$ then $\sigma(e_T) = e_{\sigma T}$. We now look at the element e in the R_i-algebra $\pi^* L$ that we defined previously. Its image in the $R_i \otimes k^s$-algebra $\pi^* L \otimes k^s$ is $e \otimes 1$, hence $\sigma(e\otimes 1) = e \otimes \sigma(1) = e \otimes 1$. But at the same time, $\pi^* L \otimes_k k^s = (\pi \otimes 1)^* (L \otimes_k k^s)$ as an $R_i \otimes k^s$-algebra and the latter we have already computed, it becomes $\prod_{T\in\mathcal{P}_i(S)} (L \otimes k^s) e'_T$ when we identify $R_i \otimes k^s$ with $\prod_{T\in\mathcal{P}_i(S)} k^s e'_T$. We also know what $e \otimes 1$ is in this algebra,

$$e \otimes 1 = \sum_{T\in\mathcal{P}_i(S)} e_T e'_T. \tag{2.9}$$

Hence $\sigma(e\otimes 1) = \sum_{T\in\mathcal{P}_i(S)} e_{\sigma T} \sigma(e'_T)$. Since $\sigma(e \otimes 1) = e \otimes 1$ we must have $\sum_{T\in\mathcal{P}_i(S)} e_T e'_T = \sum_{T\in\mathcal{P}_i(S)} e_{\sigma T} \sigma(e'_T)$, hence $\sigma(e'_T) = e'_{\sigma(T)}$.

Now when we know the $\mathcal{G}$-action on e'_T we can also determine which $\mathcal{G}$-set corresponds to $\pi^* L$. For we have

$$\pi^* L \otimes_k k^s = \prod_{T\in\mathcal{P}_i(S)} \left(\prod_{s \in S} k^s e_s \right) e'_T \simeq \prod_{(s,T)\in S\times\mathcal{P}_i(S)} k^s e'_{s,T} \tag{2.10}$$

as $R_i = \prod_{T\in\mathcal{P}_i(S)} k^s e'_T$-algebras, where $e_{s,T}$ has 1 in position (s,T) and zeros elsewhere. Here $\sum_T (\sum_s \alpha_{s,T} e_s) e'_T$ corresponds to $\sum_{(s,T)} \alpha_{s,T} e'_{s,T}$ and $\sigma\big(\sum_T (\sum_s \alpha_{s,T} e_s) e'_T\big) = \sum_T (\sum_s \sigma(\alpha_{s,T}) e_{\sigma s}) e'_{\sigma T}$ so $\sigma(\sum_{(s,T)} \alpha_{s,T} e'_{s,T}) = \sum_{(s,T)} \sigma(\alpha_{s,T}) e'_{\sigma s, \sigma T}$. Therefore $\pi^* L$ corresponds to $S \times \mathcal{P}_i(S)$ with componentwise $\mathcal{G}$-action. (This can also be seen more directly, $\pi^* L \simeq L \otimes_k R_i$, hence corresponds to $S \times \mathcal{P}_i(S)$.)

Using this together with (2.9) we get

$$\begin{aligned}
L'_i \otimes_k k^s &= (e \otimes 1)(\pi^* L \otimes_k k^s) \\
&= \left(\sum_{T\in\mathcal{P}_i(S)} e_T e'_T \right) \cdot \prod_{T\in\mathcal{P}_i(S)} \left(\prod_{s\in S} k^s e_s \right) e'_T \\
&= \prod_{T\in\mathcal{P}_i(S)} \left(e_T \prod_{s\in S} k^s e_s \right) e'_T \\
&= \prod_{T\in\mathcal{P}_i(S)} \left(\prod_{s\in S\setminus T} k^s e_s \right) e'_T.
\end{aligned}$$

Under the correspondence in (2.10) this becomes

$$\prod_{\substack{(s,T)\in S\times\mathcal{P}_i(S) \\ s\notin T}} k^s e'_{s,T}$$

with the same $\mathcal{G}$-action as that in (2.10). Hence L'_i corresponds to $\{(s,T) \in S \times \mathcal{P}_i(S) : s \notin T\}$.

Finally, the $T_i \otimes_k k^s$-algebra structure on $L'_i \otimes_k k^s$ is given by $\sum_{T\in\mathcal{P}_i(S)} e'_T \mapsto \sum_{T\in\mathcal{P}_i(S)} (\sum_{s\in S\setminus T} e_s) e'_T$ and this comes from the projection map $(s,T) \mapsto T$. $\square$

Corollary 2.3.6. *L'_i has rank $n - i$ as a T_i-module.*

Proof. The $\mathcal{G}$-set $\{(s,T) \in S \times \mathcal{P}_i(S) : s \notin T\}$ has cardinality $(n-i)\binom{n}{i}$, hence by proposition 1.1.15 L_i' has dimension $(n-i)\binom{n}{i}$ as a k-algebra. Since the dimension of the coordinate ring of T_i is $\binom{n}{i}$ the result follows. □

Exampel 2.3.7. *Let $k = \mathbb{F}_q$ and $L = \mathbb{F}_{q^3}$. We then have*

$$[L^*] = \mathbb{L}^3 - [L_1] - [L_2] - 1 \in \mathrm{K}_0(\mathrm{Sch}_k). \tag{2.11}$$

Let $\mathcal{G} := \mathcal{G}\mathrm{al}(\overline{k}/k)$ and let σ be $\mathcal{G}$:s topological generator, the Frobenius automorphism $\alpha \mapsto \alpha^q$. Then L corresponds to the $\mathcal{G}$-set $S := \mathrm{Hom}_k(L, \overline{k}) = \{1, \sigma, \sigma^2\}$, where we have identified σ with its restriction to L.

We have $\mathcal{P}_1(S) = \{\{1\}, \{\sigma\}, \{\sigma^2\}\} \simeq S$. Therefore $T_1 \simeq \operatorname{Spec} L$. Moreover, L_1' corresponds to

$$\big\{(1, \{\sigma\}), (1, \{\sigma^2\}), (\sigma, \{1\}), (\sigma, \{\sigma^2\}), (\sigma^2, \{1\}), (\sigma^2, \{\sigma\})\big\}$$

and this is the union of two sets on which $\mathcal{G}$ acts transitive, hence it is isomorphic to $S \dot\cup S$ as a $\mathcal{G}$-set. So $L_1' \simeq L^2$. Therefore $[(L_1'/T_1)^*] = (\mathbb{L}-1)^2 \in \mathrm{K}_0(\mathrm{Sch}_L)$ and hence by proposition 1.1.6

$$[L_1] = \mathrm{Res}_k^L\big((\mathbb{L}-1)^2\big) = [\operatorname{Spec} L] \cdot (\mathbb{L}-1)^2 \in \mathrm{K}_0(\mathrm{Sch}_k)$$

Next $\mathcal{P}_2(S) = \{\{1,\sigma\}, \{\sigma, \sigma^2\}, \{1, \sigma^2\}\}$. Since $\mathcal{G}$ acts transitively on this we have $\mathcal{P}_2(S) \simeq S$ so $T_2 \simeq \operatorname{Spec} L$. Moreover, L_2' corresponds to

$$\big\{(\sigma^2, \{1, \sigma\}), (1, \{\sigma, \sigma^2\}), (\sigma, \{1, \sigma^2\})\big\}$$

and this is also isomorphic to S so $L_2' \simeq L$. Therefore $[(L_2'/T_2)^*] = \mathbb{L} - 1 \in \mathrm{K}_0(\mathrm{Sch}_L)$ and hence

$$[L_2] = \mathrm{Res}_k^L(\mathbb{L}-1) = [\operatorname{Spec} L] \cdot (\mathbb{L}-1) \in \mathrm{K}_0(\mathrm{Sch}_k)$$

Putting this into (2.11) now give that

$$[L^*] = \mathbb{L}^3 - [\operatorname{Spec} L] \cdot \mathbb{L}^2 + [\operatorname{Spec} L] \cdot \mathbb{L} - 1 \in \mathrm{K}_0(\mathrm{Sch}_k),$$

in agreement with example 2.2.3.

We now want to prove a more general version of lemma 2.3.5.

Lemma 2.3.8. *Let k be a field and K a separable k-algebra of dimension t. Let L be a separable K-algebra of rank n. Let $\mathcal{G} := \mathcal{G}\mathrm{al}(k^s/k)$ and let K and L correspond to T respectively S as $\mathcal{G}$-sets. Write $T = \mathrm{Hom}_k(K, k^s) = \{\tau_1, \ldots, \tau_t\}$. The map $S \to T$ corresponding to $K \to L$ is $n : 1$. Let S_j be the inverse image of τ_j. We use the notation $T_i(L/K)$ and $L_i'(L/K)$ from definition 2.2.11. Then $T_i(L/K)$ corresponds to the $\mathcal{G}$-set*

$$\bigcup_{j=1}^{t} \mathcal{P}_i(S_j)$$

and $L_i'(L/K)$ corresponds to

$$\left\{(f, U) \in \bigcup_{j=1}^{t} S_j \times \mathcal{P}_i(S_j) : f \notin U\right\}$$

Proof. Suppose first that K is a field. According to lemma 2.3.5, $T_i(L/K)$ corresponds to $\mathcal{P}_i\big(\mathrm{Hom}_K(L, k^s)\big)$ as a $\mathcal{G}\mathrm{al}(k^s/K)$-set. Hence by proposition 1.1.18 it corresponds to

$$\mathcal{G} \times_{\mathcal{G}\mathrm{al}(k^s/K)} \mathcal{P}_i\big(\mathrm{Hom}_K(L, k^s)\big)$$

as a $\mathcal{G}$-set, with the $\mathcal{G}$-action given in that proposition. Since we assumed that K is a field we may write T as $\{\tau_1|_K, \dots, \tau_t|_K\}$, where $\tau_j \in \mathcal{G}$, and this in turn can be identified with a system of coset representatives of $\mathcal{G} \,/\, \mathcal{G}\mathrm{al}(k^s/K)$. We hence want to show that we have an isomorphism of $\mathcal{G}$-sets,

$$\phi\colon T \times \mathcal{P}_i\big(\mathrm{Hom}_K(L, k^s)\big) \to \bigcup_{j=1}^{t} \mathcal{P}_i(S_j)$$

To construct this, define ϕ as $(\tau_j|_K, U) \mapsto \tau_j U$. (Note that τ_j have to be fixed for every j, if we replace it with τ_j' such that $\tau_j|_K = \tau_j'|_K$ we may get another ϕ.) First ϕ is well defined because every element in U fixes K, so every element of $\tau_j U$ is in S_j, the inverse image of $\tau_j|_K$ in S. Hence $\phi(\tau_j|_K, U) \in \mathcal{P}_i(S_j)$. It is also $\mathcal{G}$-equivariant, because if $\sigma \in \mathcal{G}$ is such that $\sigma\tau_j = \tau_l\tau'$, where $\tau' \in \mathcal{G}\mathrm{al}(k^s/K)$, then

$$\phi\big(\sigma(\tau_j|_K, U)\big) = \phi(\tau_l, \tau' U) = \tau_l \tau' U$$

and

$$\sigma\phi(\tau_j|_K, U) = \sigma(\tau_j U) = \sigma\tau_j U = \tau_l\tau' U.$$

Next ϕ is injective: If $\phi(\tau_j|_K, U) = \phi(\tau_l|_K, U')$ then they both must be in $\mathcal{P}_i(S_j)$, so $l = j$. Hence $\tau_j U = \tau_j U'$ and since τ_j is an isomorphism, $U = U'$. So ϕ is an injective morphism between two $\mathcal{G}$-sets of cardinality $t \cdot \binom{n}{i}$, hence an isomorphism.

For the general case when K is a separable k-algebra of dimension t, note that we can identify T with

$$\dot{\bigcup_v} \mathrm{Hom}_k(K_v, k^s)$$

where $K = \prod_v K_v$, by sending $f \in \mathrm{Hom}_k(K_{v_0}, k^s)$ to $(\alpha_v) \mapsto f(\alpha_{v_0}) \in T$. Denote the map $S \to T$ by π. We have that $T_i(L/K) = \dot{\bigcup}_v T_i(L_v/K_v)$. This corresponds to the $\mathcal{G}$-set

$$\dot{\bigcup_v} \bigcup_{\tau \in \mathrm{Hom}_k(K_v, k^s)} \mathcal{P}_i(\pi^{-1}\tau) = \bigcup_{\tau \in T} \mathcal{P}_i(\pi^{-1}\tau) = \bigcup_{j=1}^{t} \mathcal{P}_i(S_j)$$

As for $L_i'(L/K)$, assume first that K is a field. As a $\mathcal{G}\mathrm{al}(k^s/K)$-set, $L_i'(L/K)$ corresponds to

$$M := \{(f, U) \in \mathrm{Hom}_K(L, k^s) \times \mathcal{P}_i\big(\mathrm{Hom}_K(L, k^s)\big) : \, f \notin U\},$$

hence it corresponds to $T \times M$ as a $\mathcal{G}$-set. Define a map

$$T \times M \to \left\{ (f, U) \in \bigcup_{j=1}^{t} S_j \times \mathcal{P}_i(S_j) : \, f \notin U \right\}$$

by

$$\big(\tau_j|_K, (f, U)\big) \mapsto (\tau_j \circ f, \tau_j U).$$

As above one shows that this is an isomorphism of $\mathcal{G}$-sets. The case when K is an arbitrary separable k-algebra is handled in the same way as T_i. □

A formula for $[L^*]$ in terms of the $\mathcal{G}$-set corresponding to L

We are now ready to give a closed formula for $[L^*]$. To express this we first generalize definition 2.3.4.

Definition 2.3.9. *Given a G-set S of cardinality n and a positive integer r. Moreover, let $(i_1, \dots, i_r)$ be an r-tuple of positive integers such that $i_1 + \cdots + i_r \le n$. Then $\mathcal{P}_{i_r, \dots, i_1}(S)$ is the G-set of r-tuples $(S_r, \dots, S_1)$ where S_j is a subset of S of cardinality i_j and the S_j:s are pairwise disjoint. In particular $\mathcal{P}_i(S)$ has the same meaning as before (up to isomorphism).*

Note that if $i_1+\cdots+i_r=j$ then $\mathcal{P}_{i_r,\ldots,i_1}(S)\simeq\mathcal{P}_{i_r,\ldots,i_1,n-j}(S)$ and also that if $(i'_1,\ldots,i'_r)$ is a permutation of $(i_1,\ldots,i_r)$ then $\mathcal{P}_{i'_r,\ldots,i'_1}(S)\simeq\mathcal{P}_{i_r,\ldots,i_1}(S)$. In particular, if λ is the partition of n corresponding to $(i_r,\ldots,i_1,n-j)$ then $\mathcal{P}_{i_r,\ldots,i_1,n-j}(S)\simeq\mathcal{P}_\lambda(S)$.

For the rest of this section, we fix a field k and a separable k-algebra L of dimension n such that L corresponds to the $\mathcal{G}$-set S. Recall the notation used at the end of the preceding section: Given a sequence of positive integers $i_1,\ldots,i_q$. Construct the algebra $L'_{i_1}/T_{i_1}=L'_{i_1}(L/k)/T_{i_1}(L/k)$. Define the algebra $L'_{i_2,i_1}/T_{i_2,i_1}$ as $L'_{i_2}(L'_{i_1}/T_{i_1})/T_{i_2}(L'_{i_1}/T_{i_1})$ and define inductively $L'_{i_{r+1},\ldots,i_1}/T_{i_{r+1},\ldots,i_1}$ as

$$L'_{i_{r+1}}(L'_{i_r,\ldots,i_1}/T_{i_r,\ldots,i_1})/T_{i_{r+1}}(L'_{i_r,\ldots,i_1}/T_{i_r,\ldots,i_1}).$$

Proposition 2.3.10. *Let $\alpha=(i_r,\ldots,i_1)$ be an r-tuple of positive integers such that $i_1+\cdots+i_r=i$ where $1\le i\le n$. The algebra L'_α/T_α in the category of k-algebras corresponds to the $\mathcal{G}$-sets*

$$\Big\{\big(s,(S_r,\ldots,S_1)\big)\in S\times\mathcal{P}_\alpha(S):\, s\notin\cup_{t=1}^r S_t\Big\}$$

and $\mathcal{P}_\alpha(S)$ together with the projection morphism.

Proof. By lemma 2.3.5 the proposition holds for $r=1$. Suppose the formula has been proved for r. We have $T_{i_{r+1},i_r,\ldots,i_1}=T_{i_{r+1}}(L'_{i_r,\ldots,i_1}/T_{i_r,\ldots,i_1})$. By the induction hypothesis and lemma 2.3.8 this corresponds to

$$\bigcup_{(S_r,\ldots,S_1)\in\mathcal{P}_{i_r,\ldots,i_1}(S)}\mathcal{P}_{i_{r+1}}\Big(\Big\{\big(s,(S_r,\ldots,S_1)\big):\, s\notin\cup_{t=1}^r S_t\Big\}\Big)$$

which is isomorphic to

$$\bigcup_{(S_r,\ldots,S_1)\in\mathcal{P}_{i_r,\ldots,i_1}(S)}\Big\{\big(\{s_1,\ldots,s_{i_{r+1}}\},S_r,\ldots,S_1\big):\, s_{i_t}\notin\cup_{t=1}^r S_t\Big\}$$

and this in turn is equal to $\mathcal{P}_{i_{r+1},i_r,\ldots,i_1}(S)$.

And $L'_{i_{r+1},i_r,\ldots,i_1}$ corresponds to the pairs (f,U) in

$$\bigcup_{(S_r,\ldots,S_1)\in\mathcal{P}_{i_r,\ldots,i_1}(S)}\Big\{\big(s,(S_r,\ldots,S_1)\big):\, s\notin\cup_{t=1}^r S_t\Big\}\times\mathcal{P}_{i_{r+1}}\Big(\Big\{\big(s,(S_r,\ldots,S_1)\big):\, s\notin\cup_{t=1}^r S_t\Big\}\Big)$$

such that $f\notin U$. This is isomorphic to

$$\bigcup_{(S_r,\ldots,S_1)\in\mathcal{P}_{i_r,\ldots,i_1}(S)}\Big\{\big(s,(S_{r+1},S_r,\ldots,S_1)\big)\in S\times\mathcal{P}_{i_{r+1},i_r,\ldots,i_1}(S):\, s\notin\cup_{t=1}^{r+1}S_t\Big\}$$

which equals

$$\Big\{\big(s,(S_{r+1},S_r,\ldots,S_1)\big)\in S\times\mathcal{P}_{i_{r+1},\ldots,i_1}(S):\, s\notin\cup_{t=1}^{r+1}S_t\Big\}.$$

□

Since the projection is $n-i:1$ we have the following.

Corollary 2.3.11. *L'_α has rank $n-i$ as a T_α-module.*

We are now ready to give our first closed formula for $[L^*]$. It follows from theorem 2.2.14 and proposition 2.3.10.

Theorem 2.3.12. *Let L be a k-algebra of dimension n and S a $\mathcal{G}$-set such that* $\mathrm{Art}\big([S]\big) = [\mathrm{Spec}\, L]$. *Then we have*

$$[L^*] = \mathbb{L}^n + a_1 \cdot \mathbb{L}^{n-1} + \cdots + a_{n-1} \cdot \mathbb{L} + a_n \in \mathrm{K}_0(\mathbf{Sch}_k)$$

where $a_i = \mathrm{Art}_k(\rho_i(S))$ *and*

$$\rho_i(S) = \sum_{t=1}^{i} \sum_{\substack{(i_1,\ldots,i_t): \\ i_1+\cdots+i_t=i \\ i_s \geq 1}} (-1)^t [\mathcal{P}_{i_t,\ldots,i_1}(S)] \in \mathcal{B}(\mathcal{G}).$$

The universal nature of the formula

Fix a field k with absolute Galois group $\mathcal{G}$. Also, fix a separable k-algebra L of dimension n corresponding to the $\mathcal{G}$-set S. Define a homomorphism $\phi\colon \mathcal{G} \to \Sigma_n$ as the composition of $\mathcal{G} \to \mathrm{Aut}(S)$ with an isomorphism $\mathrm{Aut}(S) \to \Sigma_n$. Let $\mathrm{Res}_{\mathcal{G}}^{\Sigma_n}$ denote the restriction maps with respect to ϕ. Then $\mathrm{Res}_{\mathcal{G}}^{\Sigma_n}$ is independent of the chosen isomorphism $\mathrm{Aut}(S) \to \Sigma_n$.

We have that

$$\mathrm{Res}_{\mathcal{G}}^{\Sigma_n} \big[\{1, \ldots, n\}\big] = [S] \in \mathcal{B}(\mathcal{G}).$$

Also, $\mathrm{Res}_{\mathcal{G}}^{\Sigma_n} \big([\mathcal{P}_\alpha]\big) = [\mathcal{P}_\alpha(S)]$. We therefore use the notation that if $\rho \in \mathcal{B}(\Sigma_n)$ then $\rho(S) := \mathrm{Res}_{\mathcal{G}}^{\Sigma_n}(\rho) \in \mathcal{B}(\mathcal{G})$.

This discussion gives the following formulation of theorem 2.3.12.

Theorem 2.3.13. *Fix a positive integer n. There exist elements $\rho_i^{(n)} \in \mathcal{B}(\Sigma_n)$, $i = 1, \ldots, n$, with the property that for every field k with absolute Galois group $\mathcal{G}$ and every separable k-algebra of dimension n corresponding to the $\mathcal{G}$-set S,*

$$[L^*] = \mathbb{L}^n + a_1 \cdot \mathbb{L}^{n-1} + \cdots + a_{n-1} \cdot \mathbb{L} + a_n \in \mathrm{K}_0(\mathbf{Sch}_k)$$

where $a_i = \mathrm{Art}_k(\rho_i^{(n)}(S))$.

The $\rho_i^{(n)}$:s can be given explicit as

$$\rho_i^{(n)} = \sum_{t=1}^{i} \sum_{\substack{(i_1,\ldots,i_t): \\ i_1+\cdots+i_t=i \\ i_s \geq 1}} (-1)^t [\mathcal{P}_{i_t,\ldots,i_1}] \in \mathcal{B}(\Sigma_n).$$

We illustrate with two examples.

Exampel 2.3.14. *We have*

$$\begin{aligned}
\rho_1^{(3)} &= -[\mathcal{P}_1^{(3)}] \\
\rho_2^{(3)} &= -[\mathcal{P}_2^{(3)}] + [\mathcal{P}_{1,1}^{(3)}] \\
\rho_3^{(3)} &= -[\mathcal{P}_3^{(3)}] + 2 \cdot [\mathcal{P}_{2,1}^{(3)}] - [\mathcal{P}_{1,1,1}^{(3)}].
\end{aligned}$$

We apply this to example 2.3.7, where $L/k = \mathbb{F}_{q^3}/\mathbb{F}_q$. Then $\mathcal{G}$ is generated by the Frobenius map F *and we can identify S, the $\mathcal{G}$-set corresponding to L, with $\{1, \mathrm{F}, \mathrm{F}^2\}$. As in that example we get* $[\mathcal{P}_1(S)] = [\mathcal{P}_2(S)] = [S]$. *We also have that* $[\mathcal{P}_{2,1}^{(3)}] = [\mathcal{P}_2^{(3)}]$, *hence* $[\mathcal{P}_{2,1}^{(3)}(S)] = [S]$. *Moreover,*

$$\begin{aligned}
\mathcal{P}_{1,1}(S) =& \big\{(\{1\}, \{\mathrm{F}\}), (\{\mathrm{F}\}, \{\mathrm{F}^2\}), (\{\mathrm{F}^2\}, \{1\})\big\} \\
& \dot\cup \big\{(\{1\}, \{\mathrm{F}^2\}), (\{\mathrm{F}\}, \{1\}), (\{\mathrm{F}^2\}, \{\mathrm{F}\})\big\} \\
\simeq& S \,\dot\cup\, S.
\end{aligned}$$

and hence $[\mathcal{P}_{1,1,1}(S)] = [\mathcal{P}_{1,1}(S)] = 2\cdot[S]$*. Finally* $\mathcal{P}_3(S) = \big\{(\{1,\mathrm{F},\mathrm{F}^2\})\big\}$ *so* $[\mathcal{P}_3(S)] = 1$*. We therefore have*

$$\begin{aligned}\rho_1^{(3)}(S) &= -[S]\\ \rho_2^{(3)}(S) &= -[S]+2\cdot[S] = [S]\\ \rho_3^{(3)}(S) &= -1+2\cdot[S]-2\cdot[S] = -1\end{aligned}$$

which gives the same formula for $[L^*]$ *as in example 2.3.7.*

Exampel 2.3.15. *It follows from theorem 2.3.13 that*

$$\begin{aligned}\rho_1^{(4)} &= -[\mathcal{P}_1^{(4)}]\\ \rho_2^{(4)} &= -[\mathcal{P}_2^{(4)}]+[\mathcal{P}_{1,1}^{(4)}]\\ \rho_3^{(4)} &= -[\mathcal{P}_3^{(4)}]+2\cdot[\mathcal{P}_{2,1}^{(4)}]-[\mathcal{P}_{1,1,1}^{(4)}]\\ \rho_4^{(4)} &= -[\mathcal{P}_4^{(4)}]+2\cdot[\mathcal{P}_{3,1}^{(4)}]+[\mathcal{P}_{2,2}^{(4)}]-3\cdot[\mathcal{P}_{2,1,1}^{(4)}]+[\mathcal{P}_{1,1,1,1}^{(4)}].\end{aligned}$$

Let $L/k = \mathbb{F}_{q^4}/\mathbb{F}_q$*. Since* $\mathcal{G}$ *is generated by the Frobenius map* F *we can identify* S*, the* $\mathcal{G}$*-set corresponding to* L*, with* $\{1,\mathrm{F},\mathrm{F}^2,\mathrm{F}^3\}$*. We compute the* $[\mathcal{P}_\mu(S)]$*:s in the same way as in the preceding example. For example,*

$$\mathcal{P}_2^{(4)}(S) = \big\{\{1,\mathrm{F}\},\{\mathrm{F},\mathrm{F}^2\},\{\mathrm{F}^2,\mathrm{F}^3\},\{1,\mathrm{F}^3\}\big\}\,\dot\cup\big\{\{1,\mathrm{F}^2\},\{\mathrm{F},\mathrm{F}^3\}\big\}.$$

The first of these sets is isomorphic to S*. The second is transitive of cardinality* 2 *so it corresponds to a field extension of* k *of degree* 2*, i.e.,* $\mathbb{F}_{q^2}$*. Reasoning in this way we find that*

$$[L^*] = \mathbb{L}^4 - [\operatorname{Spec}\mathbb{F}_{q^4}]\cdot\mathbb{L}^3 + \big(2[\operatorname{Spec}\mathbb{F}_{q^4}] - [\operatorname{Spec}\mathbb{F}_{q^2}]\big)\cdot\mathbb{L}^2 - [\operatorname{Spec}\mathbb{F}_{q^4}]\cdot\mathbb{L} + [\operatorname{Spec}\mathbb{F}_{q^2}] - 1.$$

If instead $L/k = \mathbb{F}_{q^2}\times\mathbb{F}_{q^2}/\mathbb{F}_q$ *then* $S = \{e_1,\mathrm{F}\,e_1\}\,\dot\cup\{e_2,\mathrm{F}\,e_2\}$ *where* e_1 *and* e_2 *are the projection maps. We then get, for example,*

$$\mathcal{P}_2^{(4)}(S) = \big\{\{e_1,\mathrm{F}\,e_1\}\big\}\,\dot\cup\big\{\{e_2,\mathrm{F}\,e_2\}\big\}\,\dot\cup\big\{\{e_1,e_2\},\{\mathrm{F}\,e_1,\mathrm{F}\,e_2\}\big\}\,\dot\cup\big\{\{e_1,\mathrm{F}\,e_2\},\{\mathrm{F}\,e_1,e_2\}\big\}.$$

This kind of computations show that

$$[L^*] = \mathbb{L}^4 - 2[\operatorname{Spec}\mathbb{F}_{q^2}]\cdot\mathbb{L}^3 + \big(4[\operatorname{Spec}\mathbb{F}_{q^2}] - 2\big)\cdot\mathbb{L}^2 - 2[\operatorname{Spec}\mathbb{F}_{q^2}]\cdot\mathbb{L} + 1.$$

2.4 $[L^*]$ expressed in terms of the λ-ring structure on $\mathcal{B}(\Sigma_n)$

In section 1.3.2 we defined a λ-ring structure on Burnside rings. Define $\ell_i := \lambda^i\big([\{1,\dots,n\}]\big) \in \mathcal{B}(\Sigma_n)$. In this section we will see that the ρ_i:s that where introduced in theorem 2.3.13 can be described in terms of this λ-structure. Namely we will prove that $\rho_i = (-1)^i\ell_i$. This formula is suggested in the following way. We can give $\mathrm{K}_0(\mathbf{Sch}_k)$ the structure of a λ-ring that extend the structure already defined on the subring $\mathrm{h}\big(\mathcal{B}(\mathcal{G})\big)$. (Recall that h is injective.) See [LL02] for this construction. Moreover, let $\mathrm{K}_0(\mathbb{Q}_l-\mathcal{G})$ be the Grothendieck ring of continuous $\mathbb{Q}_l$-representations of $\mathcal{G}$. We then have a commutative square of λ-rings

$$\begin{tikzcd}\mathcal{B}(\mathcal{G}) \arrow[r] \arrow[d] & \mathrm{K}_0(\mathbf{Sch}_k) \arrow[d]\\ \mathrm{R}_{\mathbb{Q}}(\mathcal{G}) \arrow[r] & \mathrm{K}_0(\mathbb{Q}_l-\mathcal{G})\end{tikzcd}$$

where the map $\mathrm{K}_0(\mathbf{Sch}_k) \to \mathrm{K}_0(\mathbb{Q}_l - \mathcal{G})$ sends the class of X to the class of its l-adic cohomology. By the classical computation of the cohomology of a torus then, the image of $[L^*]$ can be expressed in terms of the λ-structure on $\mathrm{K}_0(\mathbb{Q}_l - \mathcal{G})$. This suggest that a similar formula should hold in $\mathrm{K}_0(\mathbf{Sch}_k)$. And even though that is not the case for an arbitrary torus, it is true for L^*.

The result is the following theorem.

Theorem 2.4.1. *Let $\rho_i^{(n)}$ be the elements defined in theorem 2.3.13, i.e., the elements in $\mathcal{B}(\Sigma_n)$ describing $[L^*] \in \mathrm{K}_0(\mathbf{Sch}_k)$ for every separable, n-dimensional algebra $k \to L$. Then $\rho_i^{(n)} = (-1)^i \ell_i^{(n)}$ where $\ell_i^{(n)} = \lambda^i([\{1, \ldots, n\}])$.*

Proof. We will prove an explicit formula for $\ell_i \in \mathcal{B}(\Sigma_n)$, namely the one in theorem 2.4.13. The theorem then follows when we compare it with the formula for ρ_i obtained in theorem 2.3.12. ☐

So from now on this section contains no reference to the algebra L that we started with, it is an independent investigation of $\mathcal{B}(\Sigma_n)$. We begin by proving a proposition in the representation ring that will help us prove a theorem in the Burnside ring that we are not able to prove directly.

The representation ring $\mathrm{R}_{\mathbb{Q}}(\Sigma_n)$

The theorem that we are not able to prove directly in the Burnside ring corresponds to the following in the representation ring.

Proposition 2.4.2. *Let $S_n := \{1, \ldots, n\}$ and let $\mathbb{Q}[S_n]$ be the associated permutation representation of Σ_n. Given n and i, view Σ_n and Σ_i as the permutation groups of S_n and S_i respectively. View Σ_{n-i} as the permutation group of $\{i+1, \ldots, n\}$. We get a restriction map $\mathrm{R}_{\mathbb{Q}}(\Sigma_i) \to \mathrm{R}_{\mathbb{Q}}(\Sigma_i \times \Sigma_{n-i})$ with respect to the map $\Sigma_i \times \Sigma_{n-i} \to \Sigma_i$ which is projection on the first coordinate. We also get an induction map $\mathrm{R}_{\mathbb{Q}}(\Sigma_i \times \Sigma_{n-i}) \to \mathrm{R}_{\mathbb{Q}}(\Sigma_n)$ given by the inclusion $(\tau, \rho) \mapsto \tau\rho = \rho\tau \colon \Sigma_i \times \Sigma_{n-i} \to \Sigma_n$. Putting these together we get a map $\mathrm{R}_{\mathbb{Q}}(\Sigma_i) \to \mathrm{R}_{\mathbb{Q}}(\Sigma_n)$. We have*

$$\mathrm{Ind}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n} \circ \mathrm{Res}_{\Sigma_i}^{\Sigma_i \times \Sigma_{n-i}} \Big(\lambda^i\big(\big[\mathbb{Q}[S_i]\big]\big)\Big) \simeq \lambda^i\big(\big[\mathbb{Q}[S_n]\big]\big) \in \mathrm{R}_{\mathbb{Q}}(\Sigma_n).$$

Proof. To see what we are doing, identify S_n with $\{e_1, \ldots, e_n\}$ with Σ_n-action given $\sigma(e_i) = e_{\sigma(i)}$. Then $\lambda^i\big(\big[\mathbb{Q}[S_n]\big]\big)$ is the class of the $\mathbb{Q}$-vectorspace with basis $\{e_{j_1} \wedge \cdots \wedge e_{j_i}\}_{1 \le j_1 < \cdots < j_i \le n}$ and Σ_n-action given by

$$\sigma(e_{j_1} \wedge \cdots \wedge e_{j_i}) = e_{\sigma j_1} \wedge \cdots \wedge e_{\sigma j_i}.$$

In particular, in $\mathrm{R}_{\mathbb{Q}}(\Sigma_i)$ we have that $\lambda^i\big(\big[\mathbb{Q}[S_i]\big]\big)$ is the class of the $\mathbb{Q}$-vectorspace with basis $e_1 \wedge \cdots \wedge e_i$ and Σ_i-action given by

$$\tau(e_1 \wedge \cdots \wedge e_i) = \mathrm{sgn}(\tau) \cdot e_1 \wedge \cdots \wedge e_i.$$

We have

$$\mathrm{Ind}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n} \circ \mathrm{Res}_{\Sigma_i}^{\Sigma_i \times \Sigma_{n-i}} \big(\big[\wedge^i \mathbb{Q}[S_i]\big]\big) = \big[\mathbb{Q}[\Sigma_n] \otimes_{\mathbb{Q}[\Sigma_i \times \Sigma_{n-i}]} \wedge^i \mathbb{Q}[S_i]\big]$$

and we want to define a Σ_n-equivariant isomorphism of $\mathbb{Q}$-vectorspaces

$$\varphi \colon \mathbb{Q}[\Sigma_n] \otimes_{\mathbb{Q}[\Sigma_i \times \Sigma_{n-i}]} \wedge^i \mathbb{Q}[S_i] \to \wedge^i \mathbb{Q}[S_n].$$

Let $r := \binom{n}{i}$ and let $\sigma_1, \ldots, \sigma_r$ be coset representatives for $\Sigma_n \,/\, \Sigma_i \times \Sigma_{n-i}$. By proposition 1.1.22, we may identify the left hand side with a $\mathbb{Q}$-vector space with basis $\{\sigma_j \otimes e_1 \wedge \cdots \wedge e_i\}_{j=1}^r$. For $\sigma \in \Sigma_n$, let $\sigma\sigma_j = \sigma_k \tau\rho$ where $(\tau, \rho) \in \Sigma_i \times \Sigma_{n-i}$. The Σ_n-action is then given by

$$\sigma(\sigma_j \otimes e_1 \wedge \cdots \wedge e_i) = (\sigma_k \tau\rho) \otimes e_1 \wedge \cdots \wedge e_i = \mathrm{sgn}\,\tau \cdot (\sigma_k \otimes e_1 \wedge \cdots \wedge e_i).$$

Now define φ on this basis by $\varphi(\sigma_j \otimes e_1 \wedge \cdots \wedge e_i) := e_{\sigma_j 1} \wedge \cdots \wedge e_{\sigma_j i}$. This is surjective for given $1 \leq k_1 < \cdots < k_i \leq n$, choose $\sigma \in \Sigma_n$ such that $\sigma(j) = k_j$ for $k = 1, \ldots, i$. Let $\sigma = \sigma_j \tau \rho$. Then $\varphi\big(\sigma_j \otimes (\operatorname{sgn}\tau \cdot e_1 \wedge \cdots \wedge e_i)\big) = e_{k_1} \wedge \cdots \wedge e_{k_i}$. Since φ is a surjective map of $\mathbb{Q}$-vector spaces of dimension $\binom{n}{i}$ it is an isomorphism of vector spaces. Finally, φ is Σ_n-equivariant for if $\sigma\sigma_j = \sigma_k \tau \rho$ then

$$\begin{aligned}\sigma\varphi(\sigma_j \otimes e_1 \wedge \cdots \wedge e_i) =& \operatorname{sgn}\tau \cdot (\sigma_k(e_1 \wedge \cdots \wedge e_i)) \\ =& \varphi\big(\sigma(\sigma_j \otimes e_1 \wedge \cdots \wedge e_i)\big).\end{aligned}$$

□

The Burnside ring $\mathcal{B}(\Sigma_n)$

In what follows we will prove some facts about the λ-operations on $\mathcal{B}(\Sigma_n)$. For this we use the map $\mathrm{h}\colon \mathcal{B}(\Sigma_n) \to \mathrm{R}_{\mathbb{Q}}(\Sigma_n)$ defined in 1.1.23. Much of the below could have been done in greater generality, i.e., for any finite group. However, the general case often follows by restriction from the special case, since every finite group can be embedded in some Σ_n. In any case we are only interested in $\mathcal{B}(\Sigma_n)$.

We will use S_n to denote the set $\{1, \ldots, n\}$. Recall that if $i_1 + \cdots + i_j = n$ then $\mathcal{P}^{(n)}_{i_1,\ldots,i_j}$ is the Σ_n-set consisting of j-tuples of pairwise disjoint subsets of S_n, where the first subset has cardinality i_1 and so on. When the integer n is clear from the context we just write $\mathcal{P}_{i_1,\ldots,i_j}$. Recall also that if μ is the partition of n corresponding to $(i_1, \ldots, i_j)$, then $\mathcal{P}_{i_1,\ldots,i_j}$ are isomorphic to $\mathcal{P}_\mu$ as a Σ_n-set, hence they define the same element in $\mathcal{B}(\Sigma_n)$. Also if $i_1 + \cdots + i_j = n' < n$ then we sometimes write $\mathcal{P}_{i_1,\ldots,i_j}$ for $\mathcal{P}_{i_1,\ldots,i_j,n-n'}$.

One sees that $\mathcal{P}_\mu$ is a transitive Σ_n-set. Moreover, if μ and μ' are two partitions such that $\mu \neq \mu'$ then $\mathcal{P}_\mu \not\cong \mathcal{P}_{\mu'}$ as Σ_n-sets.

In what follows, recall that we write $\mu \vdash n$ when μ is a partition of n.

Definition 2.4.3. *Let* $\mathrm{Sch}_n \subset \mathcal{B}(\Sigma_n)$ *be the additive subgroup generated by* $\{[\mathcal{P}_\mu]\}_{\mu\vdash n}$*. Here,* Sch *is short for Schur.*

The following proposition will show that Sch_n is a ring.

Proposition 2.4.4. *Let* $\alpha = (\alpha_1, \ldots, \alpha_s)$ *and* $\beta = (\beta_1, \ldots, \beta_t)$*. Then* $[\mathcal{P}^{(n)}_\alpha] \cdot [\mathcal{P}^{(n)}_\beta]$ *belongs to* Sch_n*.*

Proof. For every $s \times t$ integer matrix $M = (m_{ij})$, define the Σ_n-set

$$\mathcal{P}_M := \{(S_1, \ldots, S_s, T_1, \ldots, T_t) : |S_i \cap T_j| = m_{ij}\}$$

where S_i has cardinality α_i, T_i has cardinality β_i, $S_i \cap S_j = \emptyset$ and $T_i \cap T_j = \emptyset$. (S_i and T_i are subsets of $\{1, \ldots, n\}$.) We have that $\mathcal{P}_\alpha \times \mathcal{P}_\beta = \dot{\cup}_M \mathcal{P}_M$.

Let $m_{i\bullet} := \sum_{j=1}^t m_{ij}$ and $m_{\bullet j} := \sum_{i=1}^t m_{ij}$. Mapping the element $(S_1, \ldots, S_s, T_1, \ldots, T_t) \in \mathcal{P}_M$ to

$$\begin{aligned}\big(S_1 \setminus \cup_{j=1}^t S_1 \cap T_j, \ldots, S_s \setminus \cup_{j=1}^t S_s \cap T_j,\\ T_1 \setminus \cup_{i=1}^1 S_1 \cap T_i, \ldots, T_t \setminus \cup_{i=1}^t S_i \cap T_t,\\ S_1 \cap T_1, S_1 \cap T_2, \ldots, S_s \cap T_t\big)\end{aligned}$$

in

$$\mathcal{P}_{i_1 - m_{i\bullet}, \ldots, i_s - m_{s\bullet}, j_1 - m_{\bullet 1}, \ldots, j_t - m_{\bullet t}, m_{11}, m_{12}, \ldots, m_{st}} \tag{2.12}$$

gives an isomorphism, hence $[\mathcal{P}_\alpha] \cdot [\mathcal{P}_\beta] = \sum_M [\mathcal{P}_M]$ belongs to Sch_n. □

Corollary 2.4.5. Sch_n *is a subring of* $\mathcal{B}(\Sigma_n)$*.*

Remark. Sch_n is not a λ-ring since it is not closed under the λ-operations.

We next describe the action of σ^i and λ^i on $[\{1,\ldots,n\}]$.

Notation 2.4.6. *Define* $s_i^{(n)} := \sigma^i\big([\{1,\ldots,n\}]\big) \in \mathcal{B}(\Sigma_n)$ *and* $\ell_i^{(n)} := \lambda^i\big([\{1,\ldots,n\}]\big) \in \mathcal{B}(\Sigma_n)$. *Here, if the superscript* n *is clear from the context we leave it out.*

We will give a formula for $s_i^{(n)}$ which shows that it lies in Sch_n and then deduce from this that also $\ell_i^{(n)}$ is in Sch_n.

Lemma 2.4.7. *To any partition of* i, $\mu = (\mu_1,,\ldots,\mu_j)$, *where* $\mu_1 = \cdots = \mu_{\alpha_1} > \mu_{\alpha_1+1} = \cdots = \mu_{\alpha_1+\alpha_2} > \cdots > \mu_{j-\alpha_l+1} = \cdots = \mu_j$, *associate the tuple* $\alpha(\mu) := (\alpha_1,\ldots,\alpha_l)$. *Then*

$$s_i^{(n)} = \sum_{\substack{\mu\vdash i:\\ \ell(\mu)\leq n}} [\mathcal{P}^{(n)}_{\alpha(\mu)}].$$

Proof. Identify $\{1,\ldots,n\}$ with $\{s_1,\ldots,s_n\}$. Then $\{1,\ldots,n\}^i/\,\Sigma_i$ is identified with the set of monomials

$$\{s_1^{e_1}\cdots s_n^{e_n} \,:\, e_1+\cdots+e_n = 1\} = \dot{\bigcup}_{\substack{e_1+\cdots+e_n=i\\ e_1\geq e_2\cdots\geq e_n\geq 0}} \Sigma_n \cdot s_1^{e_1}\cdots s_n^{e_n},$$

where the index set on the disjoint union can be identified with the set of $\mu\vdash i$ such that $\ell(\mu)\leq n$. Now let $e_1 = \cdots = e_{\alpha_1} > e_{\alpha_1+1} = \cdots = e_{\alpha_1+\alpha_2} > \cdots > e_{n-\alpha_l+1} = \cdots = e_n$ Then

$$\begin{aligned}\Sigma_n \cdot s_1^{e_1}\cdots s_n^{e_n} &= \Sigma_n \cdot (s_1\cdots s_{\alpha_1})^{e_1}(s_{\alpha_1+1}\cdots s_{\alpha_1+\alpha_2})^{e_{\alpha_1+1}}\cdots(s_{n-\alpha_l+1}\cdots s_n)^{e_{n-\alpha_l+1}}\\ &\simeq \Sigma_n\big(\{s_1,\ldots,s_{\alpha_1}\},\{s_{\alpha_1+1},\ldots,s_{\alpha_1+\alpha_2}\},\ldots,\{s_{n-\alpha_l+1},\ldots,s_n\}\big)\\ &\simeq \mathcal{P}^{(n)}_{\alpha_1,\ldots,\alpha_l},\end{aligned}$$

hence the lemma follows. ☐

Proposition 2.4.8. $\ell_i^{(n)} \in \mathrm{Sch}_n$ *for every* i.

Proof. From the definition of ℓ_i we have that

$$-(-1)^i\ell_i^{(n)} = \sum_{j=0}^{i-1}(-1)^j\ell_j^{(n)}s_{i-j}^{(n)}. \tag{2.13}$$

Since we know that Sch_n is a ring, and that the $s_j^{(n)}$:s and $\ell_1^{(n)} = [\mathcal{P}_n^{(n)}]$ are in Sch_n, the formula follows by induction. ☐

We need some facts about the behavior of the the induction operation. In what follows we view Σ_i as the permutation group of S_i and embed it in Σ_n, the permutation group of S_n. Moreover we view Σ_{n-i} as the permutation group of $\{i+1,\ldots,n\}$.

Proposition 2.4.9. *Let* $\mu\vdash i$. *Then* $\mathrm{Ind}_{\Sigma_i\times\Sigma_{n-i}}^{\Sigma_n}\circ\mathrm{Res}_{\Sigma_i}^{\Sigma_i\times\Sigma_{n-i}}\big([\mathcal{P}_\mu^{(i)}]\big) = [\mathcal{P}_\mu^{(n)}] \in \mathcal{B}(\Sigma_n)$.

Proof. Let $R = \{\sigma_1,\ldots,\sigma_r\}$, where $r = \binom{n}{i}$, be a system of coset representatives for $\Sigma_n\,/\,\Sigma_i\times\Sigma_{n-i}$. We know that $\Sigma_n\times_{\Sigma_i\times\Sigma_{n-i}}\mathcal{P}_\mu^{(i)}$ can be identified with the set of pairs (σ_j,t), where $\sigma_j\in R$ and $t = (T_1,\ldots,T_l)\in\mathcal{P}_\mu^{(i)}$. From this set we define a map to $\mathcal{P}_\mu^{(n)}$ by

$$(\sigma_j,t)\mapsto\big(\sigma_jT_1,\ldots,\sigma_jT_l,\sigma_j\{i+1,\ldots,n\}\big).$$

This map is surjective for given $t' = (T'_1, \dots, T'_l, T'_{l+1}) \in \mathcal{P}^{(n)}_\mu$, there is a $\sigma \in \Sigma_n$ such that $\sigma\{1, \dots, \mu_1\} = T'_1, \dots, \sigma\{i - \mu_l + 1, \dots, i\} = T'_l$. Let $\sigma_j \in R$ be such that $\sigma = \sigma_j \tau \rho$. Then

$$\big(\sigma_j, \tau(\{1, \dots, \mu_1\}, \dots, \{i - \mu_l + 1, \dots, i\})\big) \mapsto t'.$$

Since both sets have $n!/(\mu_1! \cdots \mu_l!(n-i)!)$ elements this is a bijection. Finally, the map is G-equivariant, hence it is an isomorphism. $\square$

Proposition 2.4.10. *We have* $\operatorname{Ind}^{\Sigma_n}_{\Sigma_i} \circ \operatorname{Res}^{\Sigma_i \times \Sigma_{n-i}}_{\Sigma_i} \big(\ell^{(i)}_i\big) = \ell^{(n)}_i \in \mathcal{B}(\Sigma_n)$.

It is this proposition that forces us to go over to the representation ring, for we haven't been able to prove it directly in the Burnside ring. To prove it we need the following theorem.

Theorem 2.4.11. *Let* $\mathrm{h}\colon \mathcal{B}(\Sigma_n) \to \mathrm{R}_{\mathbb{Q}}(\Sigma_n)$ *be the λ-ring homomorphism defined in 1.1.23. The restriction of* h *to* Sch_n *is injective.*

Proof. For every $\lambda \vdash n$ let $\sigma_\lambda \in \Sigma_n$ be an element in the conjugacy class determined by λ and let $C_{\sigma_\lambda}\colon \mathrm{R}_{\mathbb{Q}}(\Sigma_n) \to \mathbb{Z}$ be the homomorphism from definition 1.1.25, i.e., the map defined by $V \mapsto \chi_V(\sigma_\lambda)$. (This is independent of the choice of σ_λ.) This gives a homomorphism

$$\mathrm{R}_{\mathbb{Q}}(\Sigma_n) \to \prod_{\lambda \vdash n} \mathbb{Z}$$

and it suffices to show that the composition of this with the restriction of h to Sch_n is injective, i.e., that

$$\begin{aligned} \varphi\colon \mathrm{Sch}_n &\to \prod_{\lambda \vdash n} \mathbb{Z} \\ [T] &\mapsto \big(|T^{\sigma_\lambda}|\big)_{\lambda \vdash n} \end{aligned}$$

is injective. To do this, define a total ordering on the set of partitions of n by $\lambda > \lambda'$ if $\lambda_1 = \lambda'_1, \dots, \lambda_{j-1} = \lambda'_{j-1}$ and $\lambda_j > \lambda'_j$ for some j (i.e., lexicographic order). We claim that $\mathcal{P}^{\sigma_\lambda}_{\lambda'} = \emptyset$ if $\lambda > \lambda'$ whereas $|\mathcal{P}^{\sigma_\lambda}_\lambda| \neq 0$. For the second assertion, choose for example

$$\sigma_\lambda = (1, \dots, \lambda_1)(\lambda_1 + 1, \dots, \lambda_1 + \lambda_2) \cdots (n - \lambda_{\ell(\lambda)} + 1, \dots, n).$$

Then

$$\big(\{1, \dots, \lambda_1\}, \{\lambda_1 + 1, \dots, \lambda_1 + \lambda_2\}, \dots, \{n - \lambda_{\ell(\lambda)} + 1, \dots, n\}\big) \in \mathcal{P}_\lambda$$

is fixed by σ_λ.

For the first assertion, suppose $\lambda' < \lambda$ and $t = (T_1, \dots, T_l) \in \mathcal{P}_{\lambda'}$, where $l = \ell(\lambda')$. Suppose moreover that t is fixed by σ_λ. If now $\lambda_1 > \lambda_2 > \cdots > \lambda_{\ell(\lambda)}$ then, with the same σ_λ as above, we must have $T_1 = \{1, \dots, \lambda_1\}, \dots, T_l = \{n - \lambda_l + 1, \dots, n\}$ (because $\lambda_j \geq \lambda'_j$ for every j and if 1 lies in T_j then so does $\sigma_\lambda(1) = 2$, hence also $3, , \dots, , \lambda_1$. So T_j has cardinality at least λ_1 and the only λ'_j that can be that big is λ'_1.). But if λ and λ' differs in position j it is impossible for T_j to fullfill this since it has cardinality $\lambda'_j < \lambda_j$. In the general case, when we may have $\lambda_j = \lambda_{j+1}$, the above argument works the same only that we for example can have $T_1 = \{\lambda_1 + 1, \dots, \lambda_1 + \lambda_2\}$ and $T_2 = \{1, \dots, \lambda_1\}$ if $\lambda_1 = \lambda_2$.

We are now ready to prove that φ is injective. Let $x = \sum_{\lambda \vdash n} a_\lambda [\mathcal{P}_\lambda]$, where $a_\lambda \in \mathbb{Z}$, and suppose that $x \neq 0$. Choose the maximal λ_0 such that $a_{\lambda_0} \neq 0$. Let φ_{λ_0} be the λ_0:t component of φ. Then

$$\varphi_{\lambda_0}(x) = \sum_{\lambda \vdash n} a_\lambda |\mathcal{P}^{\sigma_{\lambda_0}}_\lambda| = a_{\lambda_0} |\mathcal{P}^{\sigma_{\lambda_0}}_{\lambda_0}| \neq 0,$$

hence $\varphi(x) \neq 0$. $\square$

We are now ready to prove proposition 2.4.10.

Proof, proposition 2.4.10. Since h is a morphism of λ-rings that commutes with the induction and restriction maps we have that if we write f for $\mathrm{Ind}_{\Sigma_i}^{\Sigma_n} \circ \mathrm{Res}_{\Sigma_i}^{\Sigma_i \times \Sigma_{n-i}}$ then

$$\begin{aligned} \mathrm{h} \circ f\big(\ell_i^{(i)}\big) =& f \circ \mathrm{h}\big(\lambda^i([S_i])\big) \\ =& f\Big(\lambda^i\big([\mathbb{Q}[S_i]]\big)\Big) \\ =& \lambda^i\big([\mathbb{Q}[S_n]]\big) & \in \mathrm{R}_{\mathbb{Q}}(\Sigma_n). \end{aligned}$$

where the first equality is proposition 1.1.24, the second is lemma 1.3.3 and the last equality is proposition 2.4.2. Since h is injective on Sch_n and $\mathrm{h}\big(\ell_i^{(n)}\big) = \lambda^i\big([\mathbb{Q}[S_n]]\big)$ in $\mathrm{R}_{\mathbb{Q}}(\Sigma_n)$ we have that $f\big(\ell_i^{(i)}\big) = \ell_i^{(n)}$ in $\mathcal{B}(\Sigma_n)$. □

Now when this proposition is proved we may forget everything about the representation ring; from now on we work exclusively in the Burnside ring.

As before, let $S_n = \{1, \dots, n\}$ and $\ell_i^{(n)} := \lambda^i([S_n]) \in \mathcal{B}(\Sigma_n)$.

Proposition 2.4.12. *There exists integers a_μ, where $\mu \vdash i$, such that*

$$\ell_i^{(n)} = \sum_{\mu \vdash i} a_\mu \big[\mathcal{P}_{\lambda, n-i}^{(n)}\big].$$

Proof. Since $\ell_i^{(i)} \in \mathrm{Sch}_i$ we have $\ell_i^{(i)} = \sum_{\mu \vdash i} a_\mu \big[\mathcal{P}_\mu^{(i)}\big]$ so since the induction map is additive it follows from proposition 2.4.9 and proposition 2.4.10 that $\ell_i^{(n)} = \sum_{\mu \vdash i} a_\mu \big[\mathcal{P}_{\mu, n-i}^{(n)}\big]$ in Sch_n. □

Theorem 2.4.13. *Given n, for $i = 1, \dots, n$ we have that*

$$\ell_i^{(n)} = (-1)^i \sum_{t=1}^{i} \sum_{\substack{(i_1, \dots, i_t): \\ i_1 + \cdots + i_t = i \\ i_s \geq 1}} (-1)^t \big[\mathcal{P}_{i_t, \dots, i_1}^{(n)}\big] \in \mathcal{B}(\Sigma_n).$$

Proof. For $i = 1$ the formula becomes $\ell_1^{(n)} = [\mathcal{P}_{1, n-1}^{(n)}]$ which is true for every n.

Given i, suppose the formula is true for every pair (i', n) where $i' < i$ and n is an arbitrary integer greater than or equal to i'. We want to show that it holds for (i, n) where n is an arbitrary integer greater than or equal to i.

Assume first that n is much greater than i. From proposition 2.4.12 we see that there are integers a_μ such that

$$\ell_i^{(n)} = \sum_{\mu \vdash i} a_\mu \big[\mathcal{P}_{\mu, n-i}^{(n)}\big]. \tag{2.14}$$

Because of our assumption, $n - i$ is much greater than all the entries in μ so we may define the degree of $\big[\mathcal{P}_\mu^{(n)}\big]$, where $\mu \vdash j$ and $j \leq i$, to be j. Proposition 2.4.12 then tells us that $\ell_i^{(n)}$ is a linear combination of elements of degree i.

On the other hand, by the definition of $\ell_i^{(n)}$ we have

$$-(-1)^i \ell_i^{(n)} = \sum_{j=0}^{i-1} (-1)^j \ell_j^{(n)} s_{i-j}^{(n)}. \tag{2.15}$$

By induction and the formula for $s_j^{(n)}$, the right hand side equals

$$\sum_{\mu\vdash i}[\mathcal{P}^{(n)}_{\alpha(\mu)}]+\sum_{j=1}^{i-1}(-1)^j\left((-1)^j\sum_{t=1}^{j}\sum_{\substack{(i_1,\dots,i_t):\\ i_1+\dots+i_t=j\\ i_s\geq 1}}(-1)^t[\mathcal{P}^{(n)}_{i_t,\dots,i_1}]\right)\cdot\left(\sum_{\mu\vdash i-j}[\mathcal{P}^{(n)}_{\alpha(\mu)}]\right) \tag{2.16}$$

To evaluate this expression seems to be very complicated, and we haven't managed to to so. However, we only have to evaluate it in degree i, for we have already seen that $\ell_i^{(n)}$ is zero in every other degree.

So we next compute the degree i part of (2.16). We see that for every j such that $0\leq j<i$ we have a product of two sums, one consisting of elements of degree j and one consisting of elements of degree less that of equal to $i-j$, for if $\mu\vdash i-j$ then $[\mathcal{P}_{\alpha(\mu)}]$ has degree $\leq i-j$ with equality if and only if $\mu=(1,1,\dots,1)$, in which case $\alpha(\mu)=(i-j)$.

Also, if $[\mathcal{P}^{(n)}_{i_t,\dots,i_1}]$ has degree j, i.e., $i_1+,\dots,+i_t=j$, and $[\mathcal{P}^{(n)}_{\alpha_s,\dots,\alpha_1}]$ has degree $m\leq i-j$ then by equation (2.12) in the proof of proposition 2.4.4,

$$[\mathcal{P}^{(n)}_{i_t,\dots,i_1}]\cdot[\mathcal{P}^{(n)}_{\alpha_s,\dots,\alpha_1}]=[\mathcal{P}^{(n)}_{i_t,\dots,i_1,\alpha_s,\dots,\alpha_1}]+\text{ terms of degree }<j+m.$$

Hence only the degree $i-j$ part of $\sum_{\mu\vdash i-j}[\mathcal{P}^{(n)}_{\alpha(\mu)}]$ contributes to the degree i part of (2.16). Therefore the only part of (2.16) that contains elements of degree i is

$$[\mathcal{P}^{(n)}_i]+\sum_{j=1}^{i-1}(-1)^j\left((-1)^j\sum_{t=1}^{j}\sum_{\substack{(i_1,\dots,i_t):\\ i_1+\dots+i_t=j\\ i_s\geq 1}}(-1)^t[\mathcal{P}^{(n)}_{i_t,\dots,i_1}]\right)\cdot[\mathcal{P}^{(n)}_{i-j}]$$

and the degree i part of this is

$$[\mathcal{P}^{(n)}_i]+\sum_{j=1}^{i-1}\sum_{t=1}^{j}\sum_{\substack{(i_1,\dots,i_t):\\ i_1+\dots+i_t=j\\ i_s>0}}(-1)^t[\mathcal{P}^{(n)}_{i_1,\dots,i_t,i-j}]. \tag{2.17}$$

Fix $(i'_1,\dots,i'_{t'})$ such that $i'_1+\dots+i'_{t'}=i$ and $i_s>0$. If $t'>1$ then $[\mathcal{P}^{(n)}_{i'_1,\dots,i'_{t'}}]$ occurs in (2.17) when $i_1=i'_1,\dots,i_t=i'_{t'-1}$ and $i-j=i'_{t'}$. So it occurs exactly one time and the coefficient is then $(-1)^t=-(-1)^{t'}$. If $t'=1$ then $i'_1=i$ and $[\mathcal{P}^{(n)}_i]$ occurs one time in (2.17), namely as the first term, the coefficient being $1=-(-1)^{t'}$. Hence (2.17) equals

$$-\sum_{t=1}^{i}\sum_{\substack{(i_1,\dots,i_t):\\ i_1+\dots+i_t=i\\ i_s>0}}(-1)^t[\mathcal{P}^{(n)}_{i_1,\dots,i_t}].$$

So (2.15) together with the knowledge that $\ell_i^{(n)}$ is zero in degree different from i give that

$$\ell_i^{(n)}=(-1)^i\sum_{t=1}^{i}\sum_{\substack{(i_1,\dots,i_t):\\ i_1+\dots+i_t=i\\ i_s>0}}(-1)^t[\mathcal{P}^{(n)}_{i_1,\dots,i_t}]$$

when n is much greater then i. Now by the proof of proposition 2.4.12 the coefficients in (2.14) are the same for every n. Since we have determined them for every n big enough it follows that they are determined for every n, we are through. $\square$

2.5 An alternative proof of theorem 2.4.1

In this section we give a proof of the equality $\rho_i^{(n)} = (-1)^i \lambda_i^{(n)}$ that does not depend on the explicit formula for $\ell_i^{(n)}$ from theorem 2.4.13. We begin with some lemmas.

Lemma 2.5.1. *Let L be a separable $\mathbb{F}_q$-algebra of dimension n, corresponding to the $\mathcal{G} := \mathcal{G}\mathrm{al}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-set S. Choose an isomorphism $\mathrm{Aut}(S) \to \Sigma_n$ and compose it with the homomorphism $\mathcal{G} \to \mathrm{Aut}(S)$ to get a homomorphism $\phi\colon \mathcal{G} \to \Sigma_n$. Let F be the topological generator of $\mathcal{G}$ and define $\sigma := \phi(\mathrm{F}) \in \Sigma_n$. Let $\mathrm{Res}_{\mathcal{G}}^{\Sigma_n}$ denote the restriction maps with respect to ϕ for Burnside as well as representation rings. Let C_σ and C_F be the maps from the representation rings to $\mathbb{Z}$ defined in* 1.1.25. *Let the map from $\mathrm{K}_0(\mathbf{Sch}_{\mathbb{F}_q})$ to $\mathbb{Z}$ be the counting function C_q. Then the following diagram commutes*

$$\begin{tikzcd}
\mathcal{B}(\Sigma_n) \arrow[r, "\mathrm{Res}_{\mathcal{G}}^{\Sigma_n}"] \arrow[d, "\mathrm{h}"] & \mathcal{B}(\mathcal{G}) \arrow[d, "\mathrm{h}"] \arrow[dr, "\mathrm{Art}"] & \\
\mathrm{R}_\mathbb{Q}(\Sigma_n) \arrow[r, "\mathrm{Res}_{\mathcal{G}}^{\Sigma_n}"] \arrow[dr, "C_\sigma"'] & \mathrm{R}_\mathbb{Q}(\mathcal{G}) \arrow[d, "C_\mathrm{F}"] & \mathrm{K}_0(\mathrm{Sch}_{\mathbb{F}_q}) \arrow[dl, "C_q"] \\
& \mathbb{Z} &
\end{tikzcd}$$

Proof. From Proposition 1.1.24 we know that the square in the upper left corner of the diagram commutes. The triangle in the lower left corner commutes by proposition 1.1.27. Finally, for the right triangle, if T is $\mathcal{G}$-set then $\chi_{\mathbb{Q}[T]}(\mathrm{F}) = |T^\mathrm{F}|$. At the same time, if T maps to X in $\mathrm{K}_0(\mathbf{Sch}_{\mathbb{F}_q})$ then

$$|X(\mathbb{F}_q)| = |\mathrm{Hom}_{\mathbb{F}_q}(\mathrm{Spec}\,\mathbb{F}_q, X)| = |\mathrm{Hom}_{\mathcal{G}}(\{\bullet\}, T)| = |T^\mathrm{F}|.$$

$\square$

Lemma 2.5.2. *Let M be a transitive $n \times n$ permutation matrix. Then the characteristic polynomial of M equals $T^n - 1$.*

Proof. Since the transitive permutation matrices form a conjugacy class (they correspond to the permutations of cycle type (n)) it suffices to compute the characteristic polynomial for one particular such matrix, for example

$$\begin{pmatrix} 0 & 1 & 0 & & 0 \\ & 0 & 1 & & \\ & & & \ddots & \\ 0 & 0 & & 0 & 1 \\ 1 & 0 & & & 0 \end{pmatrix}$$

Using induction one shows that the characteristic polynomial of this matrix is $T^n - 1$. $\square$

We are now ready to give the alternative proof of theorem 2.4.1, which is based on counting points over finite fields. What we need to know is the following: We need to know the existence of the universal elements $\rho_i^{(n)}$ proved in theorem 2.3.13. We do not need the explicit description of them given in that theorem, however when one has proved the existence it is not such a long step to describe the elements. For the $\ell_i^{(n)}$ we only need to know that they lie in Sch_n which was one of the first things we proved about them. We also need to know that h is injective on Sch_n.

Proof of theorem 2.4.1. Fix a positive integer n. We want to prove that $\rho_i = (-1)^i \ell_i \in \mathcal{B}(\Sigma_n)$. Since they both lie in Sch_n it suffices to show that $\mathrm{h}(\rho_i) = (-1)^i \mathrm{h}(\ell_i) \in \mathrm{R}_{\mathbb{Q}}(\Sigma_n)$ and by proposition 1.1.26 we can prove this by proving that if R is a set of representatives of the conjugacy classes of Σ_n then for every $\sigma \in R$,

$$C_\sigma \mathrm{h}(\rho_i) = (-1)^i C_\sigma \mathrm{h}(\ell_i).$$

We do this simultaneously for $i = 0, \dots, n$ by showing that

$$\sum_{i=0}^{n} C_\sigma \mathrm{h}(\rho_i) X^{n-i} = \sum_{i=0}^{n} (-1)^i C_\sigma \mathrm{h}(\ell_i) X^{n-i} \in \mathbb{Z}[X] \tag{2.18}$$

for every $\sigma \in R$.

From now on, fix a $\sigma \in R$. Let q be an arbitrary prime power, let $k = \mathbb{F}_q$ and let $\mathcal{G} := \mathcal{G}\mathrm{al}(\overline{k}/k)$. As before, if S is a $\mathcal{G}$-set of cardinality n, then choosing any enumeration of S, the action of $\mathcal{G}$ on S gives a map $\phi\colon \mathcal{G} \to \mathrm{Aut}(S) \simeq \Sigma_n$ which in turn gives our $\mathrm{Res}_{\mathcal{G}}^{\Sigma_n}$. (Independent of the chosen ϕ.) Choose S such that the topological generator for $\mathcal{G}$, the Frobenius automorphism F, maps to (a permutation in the same conjugacy class as) σ under ϕ. Equivalently, let $S = \dot\bigcup_{1 \le j \le m} T_j$ such that T_j is a transitive $\mathcal{G}$-set of cardinality n_j, where σ has cycle-type $(n_1, \dots, n_m)$. Such an S always exists for by theorem 1.1.14 it comes from $L = \prod_{j=1}^{m} K_j$ where K_j is a degree n_j field extension of k, i.e., $K_j = \mathbb{F}_{q^{n_j}}$.

We begin by computing the right hand side of (2.18) in terms of $(n_1, \dots, n_m)$. Let f be an endomorphism of the vector space V of dimension n. From linear algebra ([McD84] or [Knu73], page 83) we know the following expression for the characteristic polynomial of f:

$$\det(X \cdot E_n - f) = \sum_{i=0}^{n} (-1)^i \mathrm{Tr}(\wedge^i f) X^{n-i}.$$

Putting $f = \mathrm{F}$ gives

$$\det(X \cdot E_n - \mathrm{F}) = \sum_{i=0}^{n} (-1)^i \chi_{\wedge^i \mathbb{Q}[S]}(\mathrm{F}) X^{n-i}. \tag{2.19}$$

Since $\mathrm{h}(\ell_i(S)) = \left[\wedge^i \mathbb{Q}[S]\right] \in \mathrm{R}_{\mathbb{Q}}(\mathcal{G})$ we have that $C_{\mathrm{F}} \mathrm{h}(\ell_i(S)) = \chi_{\wedge^i \mathbb{Q}[S]}(\mathrm{F})$, hence lemma 2.5.1 gives that the right hand side of (2.19) equals

$$\sum_{i=0}^{n} (-1)^i C_\sigma \mathrm{h}(\ell_i) X^{n-i}.$$

As for the left hand side of (2.19), since S is a union of transitive $\mathcal{G}$-sets T_j we have $\mathbb{Q}[S] = \oplus_{j=1}^{m} \mathbb{Q}[T_j]$ where $\mathbb{Q}[T_j]$ is irreducible, hence the matrix for F is of the form

$$\begin{pmatrix} M_1 & & & 0 \\ & M_2 & & \\ & & \ddots & \\ 0 & & & M_m \end{pmatrix}$$

where M_j is a transitive $n_j \times n_j$ permutation matrix. Therefore by lemma 2.5.2 $\det(XE_n - F) = \prod_{j=1}^{m} \det(XE_{n_j} - M_j) = \prod_{j=1}^{m} (X^{n_j} - 1)$. From (2.19) we therefore get

$$\prod_{j=1}^{m} (X^{n_j} - 1) = \sum_{i=0}^{n} (-1)^i C_\sigma \mathrm{h}(\ell_i) X^{n-i}. \tag{2.20}$$

We next compute the left hand side of (2.18). By the definition of the ρ_i:s we have

$$[L^*] = \sum_{i=0}^{n} \operatorname{Art}\big(\rho_i(S)\big)\mathbb{L}^{n-i} \in \mathrm{K}_0(\mathbf{Sch}_k).$$

Applying C_q to this gives

$$|L^*(k)| = \sum_{i=0}^{n} (-1)^i C_q \operatorname{Art}(\rho_i(S)) \cdot q^{n-i}. \tag{2.21}$$

By lemma 2.5.1, $C_q \operatorname{Art}(\rho_i(S)) = C_\sigma \operatorname{h}(\rho_i)$, so the right hand side of (2.21) equals

$$\sum_{i=0}^{n} C_\sigma \operatorname{h}(\rho_i) q^{n-i}.$$

On the other hand, since we saw that $L = \prod_{j=1}^{m} \mathbb{F}_{q^{n_j}}$ we have $L^*(k) = L^\times = \prod_{j=1}^{m} \mathbb{F}_{q^{n_j}}^\times$ so $|L^*(k)| = \prod_{j=1}^{m}(q^{n_j} - 1)$. Hence (2.21) says that

$$\prod_{j=1}^{m}(q^{n_j} - 1) = \sum_{i=0}^{n} C_\sigma \operatorname{h}(\rho_i) q^{n-i}.$$

Since q is an arbitrary prime power it follows that

$$\prod_{j=1}^{m}(X^{n_j} - 1) = \sum_{i=0}^{n} C_\sigma \operatorname{h}(\rho_i) X^{n-i}. \tag{2.22}$$

Comparing (2.20) to (2.22) now gives (2.18). □

Chapter 3

Calculation of an integral

The aim of this chapter is to generalize the computations of some p-adic integrals to computations in the Grothendieck ring of varieties. For this we first have to define a suitable version of motivic integration. There are already existing theories for this but we content ourselves with a definition that only includes as much as we need for the integrals that we are interested in computing.

We must emphasize that many of the definitions in this chapter are not suitable in general.

3.1 Definition of the motivic integral

Fix a field k of characteristic p. Let $\mathcal{M}_k$ and $\widehat{\mathcal{M}}_k$ be the rings defined in section 1.2. We now define the measures of certain subsets of $\mathbf{W}_k^n$. (Here $\mathbf{W}$ is the Witt vectors constructed with respect to p and $\mathbf{W}_k$ is the scalar extension to k of $\mathbf{W}$.) This measure will take values in $\widehat{\mathcal{M}}_k$.

Fix an $n > 0$ and let $Z \subset \mathbf{W}_k^n$ be a disjoint union of locally closed subschemes. Let $\pi_m : \mathbf{W}^n \to \mathbf{W}_m^n$ be the projection map, and let $Z_m = \pi_m Z$. So

$$Z_m(A) = \{[a] \in (\mathbf{W}(A)/\operatorname{V}^m \mathbf{W}(A))^n\}_{a \in Z(A)} \qquad \forall A \in \mathbf{Alg}_k.$$

If Z_m is a constructible subset of $\mathbf{W}_N^n$ for every m and if $\lim_{m\to\infty}[Z_m]/\mathbb{L}^{nm}$ exists in $\widehat{\mathcal{M}}_k$ then we define $\operatorname{vol} Z \in \widehat{\mathcal{M}}_k$ to be this limit. We then also say that Z is measurable.

We will be interested in the measure of the following type of subschemes. Let $f_1, \ldots, f_s$ be polynomials in $\mathbf{W}(k)[X_1, \ldots, X_n]$. Let $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s$. Let $\{\operatorname{ord} f_i \geq \alpha_i\}_{i=1}^s \subset \mathbf{W}_k^n$ be the functor whose A-points are

$$\{a \in \mathbf{W}^n(A) : \ f_i(a) \equiv 0 \pmod{\operatorname{V}^{\alpha_i}} \text{ for } i = 1, \ldots, s\} \qquad \forall A \in \mathbf{Alg}_k.$$

This is well defined since $\mathbf{W}(A)$ is a $\mathbf{W}(k)$-algebra when A is a k-algebra. We will also write, for example, $\{\operatorname{ord} f_1 \geq \alpha_1 \wedge \operatorname{ord} f_2 \geq \alpha_2\}$ for $\{\operatorname{ord} f_i \geq \alpha_i\}_{i=1}^2$.

Proposition 3.1.1. *The functor $\{\operatorname{ord} f_i \geq \alpha_i\}_{i=1}^s$ is a closed subscheme of $\mathbf{W}_k^n$. Moreover, it is measurable.*

Proof. Write $P := k[X_{i0}, \ldots, X_{iN}, \ldots]_{i=1}^n$ and let $\mathbf{W}^n$ be represented by P, i.e., if $a = (a_1, \ldots, a_n) \in \mathbf{W}^n(A)$ where $a_i = (a_{i0}, \ldots, a_{iN}, \ldots) \in \mathbf{W}(A)$ for $i = 1, \ldots, n$, then a is identified with

$$(X_{iN} \mapsto a_{iN})_{1 \leq i \leq n, N \in \mathbb{N}} \in \operatorname{Hom}_k(P, A).$$

We now want to show that $\{\operatorname{ord} f_i \geq \alpha_i\}_{i=1}^s$ is represented by a quotient of P. For $i = 1, \ldots, n$ let $x_i := (X_{i0}, \ldots, X_{iN}, \ldots) = (X_{iN})_{N \in \mathbb{N}} \in \mathbf{W}(P)$. In $\mathbf{W}(P)$ we then have, for $j = 1, \ldots, s$,

$$f_j(x_1, \ldots, x_n) = (f_{j0}(X_{\bullet 0}), f_{j1}(X_{\bullet 0}, X_{\bullet 2}), \ldots, f_{jN}(X_{\bullet 0}, \ldots, X_{\bullet N}), \ldots)$$

where $f_{jN}(X_{\bullet 0}, \ldots, X_{\bullet N}) := f_{jN}(X_{10}, \ldots, X_{1N}, \ldots, X_{n0}, \ldots, X_{nN})$ is an element of P that lies in the subring $P_N := k[X_{i0}, \ldots, X_{iN}]_{i=1}^n$. Now $a = (a_1, \ldots, a_n) \in \mathbf{W}^n(A)$ is such that $f_j(a) \equiv 0 \pmod{\mathrm{V}^{\alpha_j + 1}}$ if and only if $f_{j0}(a_{\bullet 0}) = \cdots = f_{j\alpha_j}(a_{\bullet 0}, \ldots, a_{\bullet \alpha_j}) = 0$, i.e., if $(X_{iN} \mapsto a_{iN})_{1 \leq i \leq n, N \in \mathbb{N}}$ maps $f_{j0}, \ldots, f_{j\alpha_j}$ to zero. It hence follows that $\{\operatorname{ord} f_i \geq \alpha_i + 1\}_{i=1}^s$ is represented by

$$\frac{P}{(f_{j0}, \ldots, f_{j\alpha_j})_{j=0}^s}.$$

We next want to show that $\{\operatorname{ord} f_i \geq \alpha_i + 1\}_{i=1}^s$ is measurable. We have that $\mathbf{W}_{N+1}^n$ is represented by P_N. Let $Z := \{\operatorname{ord} f_i \geq \alpha_i + 1\}_{i=1}^s$. If $N > m := \max\{\alpha_1 + 1, \ldots, \alpha_s + 1\}$ then Z_{N+1} is represented by

$$\frac{P_N}{(f_{j0}, \ldots, f_{j\alpha_j})_{j=0}^s} \simeq \frac{P_{m-1}}{(f_{j0}, \ldots, f_{j\alpha_j})_{j=0}^s} \otimes_k k[X_{im}, \ldots, X_{iN}]_{i=1}^n,$$

hence $[Z_{N+1}] = [Z_m] \cdot \mathbb{L}^{(N-m+1)n} \in \mathrm{K}_0(\mathbf{Sch}_k)$. It follows that

$$\operatorname{vol} Z = \lim_{N \to \infty} \frac{[Z_{N+1}]}{\mathbb{L}^{(N+1)n}} = \frac{[Z_m]}{\mathbb{L}^{mn}} \in \widehat{\mathcal{M}}_k.$$

□

If $\beta_j \geq \alpha_j$ for $j = 1, \ldots, s$ then the above proof shows that $\{\operatorname{ord} f_i \geq \beta_i\}_{i=1}^s$ is a closed subscheme of $\{\operatorname{ord} f_i \geq \alpha_i\}_{i=1}^s$. Let $f \in \mathbf{W}(k)[X_1, \ldots, X_n]$. For $\alpha \in \mathbb{N}$ we define

$$\{\operatorname{ord} f = \alpha\} := \{\operatorname{ord} f \geq \alpha\} \setminus \{\operatorname{ord} f \geq \alpha + 1\}.$$

This is an open subscheme of $\{\operatorname{ord} f \geq \alpha\}$, hence a locally closed subscheme of $\mathbf{W}^n$.

Proposition 3.1.2. *If $\alpha \in \mathbb{N}$ and $f \in \mathbf{W}(k)[X_1, \ldots, X_n]$ then $\{\operatorname{ord} f = \alpha\}$ is measurable. We have* $\operatorname{vol}\{\operatorname{ord} f = \alpha\} = \operatorname{vol}\{\operatorname{ord} f \geq \alpha\} - \operatorname{vol}\{\operatorname{ord} f \geq \alpha + 1\}$.

Proof. Looking back at the preceding proof we see that $\{\operatorname{ord} f \geq \alpha + 1\}_N$ is a closed subscheme of $\{\operatorname{ord} f \geq \alpha\}_N$ and that $\{\operatorname{ord} f = \alpha\}_N = \{\operatorname{ord} f \geq \alpha\}_N \setminus \{\operatorname{ord} f \geq \alpha + 1\}_N$ when $N > \alpha$. It then follows that $[\{\operatorname{ord} f = \alpha\}_N] = [\{\operatorname{ord} f \geq \alpha\}_N] - [\{\operatorname{ord} f \geq \alpha + 1\}_N] \in \mathrm{K}_0(\mathbf{Sch}_k)$ and hence that $\operatorname{vol}\{\operatorname{ord} f = \alpha\} = \operatorname{vol}\{\operatorname{ord} f \geq \alpha\} - \operatorname{vol}\{\operatorname{ord} f \geq \alpha + 1\} \in \widehat{\mathcal{M}}_k$. □

We are now ready to define the type of integrals we are interested in. For $f \in \mathbf{W}(k)[X_1, \ldots, X_n]$, define

$$\int_{\mathbf{W}^n} \left| f(X_1, \ldots, X_n) \right| dX_1 \cdots dX_n := \sum_{i \geq 0} \operatorname{vol}\{\operatorname{ord} f(X_1, \ldots, X_n) = i\} \cdot \mathbb{L}^{-i} \in \widehat{\mathcal{M}}_k. \tag{3.1}$$

This sum always converges as the following proposition shows.

Proposition 3.1.3. *Let $f \in \mathbf{W}(k)[X_1, \ldots, X_n]$. Then $\int_{\mathbf{W}^n} \left| f(X_1, \ldots, X_n) \right| dX_1 \cdots dX_n$ exists in $\widehat{\mathcal{M}}_k$.*

Proof. By proposition 1.2.3 it suffices to show that $\operatorname{vol}(\operatorname{ord} f = m)/\mathbb{L}^m \to 0$ as $m \to \infty$. And for this it suffices to show that $\operatorname{vol}(\operatorname{ord} f \geq m)/\mathbb{L}^m \to 0$ as $m \to \infty$ for then we also have $\operatorname{vol}(\operatorname{ord} f \geq m+1)/\mathbb{L}^m \to 0$ so the result follows by proposition 3.1.2. We proceed to prove that $\operatorname{vol}(\operatorname{ord} f \geq m)/\mathbb{L}^m \to 0$.

By the proof of proposition 3.1.1 we have that

$$\operatorname{vol}(\operatorname{ord} f \geq m)/\mathbb{L}^m = \frac{[\{\operatorname{ord} f \geq m\}_{m+1}]}{\mathbb{L}^{(m+1)n}} \cdot \frac{1}{\mathbb{L}^m}.$$

Since $\{\operatorname{ord} f \geq m\}_{m+1} \subset \mathbf{W}^n_{m+1} \simeq \mathbb{A}_k^{n(m+1)}$ it has dimension $\leq n(m+1)$. It follows that $[\{\operatorname{ord} f \geq m\}_{m+1}]/\mathbb{L}^{(m+1)n+m} \in \mathcal{F}^{-m}(\mathcal{M}_k)$, hence it tends to zero as $m \to \infty$. $\square$

We conclude this section by defining some notions that we will need when computing integrals of this kind. To begin with, if $f_1, \ldots, f_s \in \mathbf{W}(k)[X_1, \ldots, X_n]$ then

$$\cup_{j=1}^s \{\operatorname{ord} f_i \geq \alpha_i \text{ if } i \neq j, \operatorname{ord} f_j \geq \alpha_j + 1\}$$

is a closed subscheme of $\{\operatorname{ord} f_j \geq \alpha_j\}_{j=1}^s$. Define a locally closed subscheme of $\mathbf{W}^n$

$$\{\operatorname{ord} f_j = \alpha_j\}_{j=1}^s := \{\operatorname{ord} f_j \geq \alpha_j\}_{j=1}^s \setminus \cup_{j=1}^s \{\operatorname{ord} f_i \geq \alpha_i \text{ if } i \neq j, \operatorname{ord} f_j \geq \alpha_j + 1\}$$

Proposition 3.1.4. *The scheme* $\{\operatorname{ord} f_j = \alpha_j\}_{j=1}^s$ *is measurable, its volume is*

$$\begin{aligned}
\operatorname{vol}\{\operatorname{ord} f_j = \alpha_j\}_{j=1}^s = &\operatorname{vol}\{\operatorname{ord} f_j \geq \alpha_j\}_{j=1}^s \\
&- \sum_{j=1}^s \operatorname{vol}\{\operatorname{ord} f_i \geq \alpha_i \textit{ if } i \neq j, \operatorname{ord} f_j \geq \alpha_j + 1\} \\
&+ \sum_{j,l} \operatorname{vol}\{\operatorname{ord} f_i \geq \alpha_i \textit{ if } i \notin \{j,l\}, \operatorname{ord} f_j \geq \alpha_j + 1, \operatorname{ord} f_l \geq \alpha_l + 1\} \\
&\vdots \\
&(-1)^s \operatorname{vol}\{\operatorname{ord} f_j \geq \alpha_j + 1\}_{j=1}^s.
\end{aligned}$$

Proof. By the same argument as in the proof of proposition 3.1.1 we have that the reduction of $\{\operatorname{ord} f_j = \alpha_j\}_{j=1}^s$ modulo V^N is a closed subscheme of the reduction of $\{\operatorname{ord} f_j \geq \alpha_j\}_{j=1}^s$ modulo V^N, it follows that $\operatorname{vol}\{\operatorname{ord} f_j = \alpha_j\}_{j=1}^s = \operatorname{vol}\{\operatorname{ord} f_j \geq \alpha_j\}_{j=1}^s - \operatorname{vol}\big(\cup_{j=1}^s \{\operatorname{ord} f_i \geq \alpha_i \text{ if } i \neq j, \operatorname{ord} f_j \geq \alpha_j + 1\}\big)$.

Again counting modulo V^N we see that the intersection of $\{\operatorname{ord} f_i \geq \alpha_i \text{ if } i \neq j, \operatorname{ord} f_j \geq \alpha_j + 1\}$ and $\{\operatorname{ord} f_i \geq \alpha_i \text{ if } i \neq l, \operatorname{ord} f_j \geq \alpha_l + 1\}$ is $\{\operatorname{ord} f_i \geq \alpha_i \text{ if } i \notin \{j,l\}, \operatorname{ord} f_j \geq \alpha_j + 1, \operatorname{ord} f_l \geq \alpha_l + 1\}$, hence $\operatorname{vol}\big(\cup_{j=1}^s \{\operatorname{ord} f_i \geq \alpha_i \text{ if } i \neq j, \operatorname{ord} f_j \geq \alpha_j + 1\}\big) = \sum_{j=1}^s \operatorname{vol}\{\operatorname{ord} f_i \geq \alpha_i \text{ if } i \neq j, \operatorname{ord} f_j \geq \alpha_j + 1\} - \operatorname{vol}\big(\cup_{j,l} \{\operatorname{ord} f_i \geq \alpha_i \text{ if } i \notin \{j,l\}, \operatorname{ord} f_j \geq \alpha_j + 1, \operatorname{ord} f_l \geq \alpha_l + 1\}\big)$. Continuing in this way the result follows. $\square$

We are now ready to define the most general sets that we will work with. For a (finite of infinite) subset $I \subset \mathbb{N}^n$, $m \in \mathbb{N}$ and $f \in \mathbf{W}(k)[X_1, \ldots, X_n]$, let

$$U_{I,m}(f) := \cup_{(\alpha_1, \ldots, \alpha_n) \in I} \{\operatorname{ord} X_i = \alpha_i, \operatorname{ord} f = m\}_{i=1}^n$$

(This is a subset of $\mathbf{W}^n$ but not in general a subscheme). When f is clear from the context we write this as just $U_{I,m}$.

Proposition 3.1.5. $U_{I,m}(f)$ *is measurable. We have* $\operatorname{vol} U_{I,m}(f) = \sum_{(\alpha_1, \ldots, \alpha_n) \in I} \operatorname{vol}\{\operatorname{ord} X_i = \alpha_i, \operatorname{ord} f = m\}_{i=1}^n$, *where the sum to the right is convergent.*

Proof. If $\alpha = (\alpha_1, \ldots, \alpha_n)$, write $U_{\alpha,m} := \{\operatorname{ord} X_i = \alpha_i, \operatorname{ord} f = m\}_{i=1}^n$.

First assume that I is finite. Then $(U_{I,m})_N = \dot\cup_{\alpha \in I} (U_{\alpha,m})_N$, hence $[(U_{I,m})_N] = \sum_{\alpha \in I} [(U_{\alpha,m})_N]$. Dividing by $\mathbb{L}^{nN}$ and letting N tend to infinity proves the proposition since $U_{\alpha,m}$ is measurable by proposition 3.1.4.

If I is infinite, then the N-projection still is a finite disjoint union since it can't see $\alpha > N$. More precisely,

$$[(U_{I,m})_N] = \sum_{\substack{\alpha:\\ \alpha_i \leq N}} [(U_{\alpha,m})_N] = \sum_{\alpha \in I} [(U_{\alpha,m})_N].$$

Therefore

$$\operatorname{vol} U_{I,m} = \lim_{N\to\infty} \frac{[(U_{I,m})_N]}{\mathbb{L}^{nN}} = \lim_{N\to\infty} \sum_{\alpha\in I} \frac{[(U_{\alpha,m})_N]}{\mathbb{L}^{nN}} = \sum_{\alpha\in I} \operatorname{vol}\{\operatorname{ord} X_i = \alpha_i, \operatorname{ord} f = m\}_{i=1}^n$$

if this sum converges, or equivalently if $\operatorname{vol}\{\operatorname{ord} X_i = \alpha_i, \operatorname{ord} f = m\}_{i=1}^n \to 0$ as $M := \min(\alpha_1, \ldots, \alpha_n) \to \infty$. Because of proposition 3.1.4 it suffices to show that $\operatorname{vol}\{\operatorname{ord} X_i \geq \alpha_i, \operatorname{ord} f \geq m\}_{i=1}^n \to 0$ and this is true if we just assume that $\alpha_1 \to \infty$. For by the proof of proposition 3.1.1, if $\alpha_1 \geq \alpha_i$ and $\alpha_1 \geq m$ then

$$\operatorname{vol}\{\operatorname{ord} X_i \geq \alpha_i, \operatorname{ord} f \geq m\}_{i=1}^n = [(\{\operatorname{ord} X_i \geq \alpha_i, \operatorname{ord} f \geq m\}_{i=1}^n)_{\alpha_1}]/\mathbb{L}^{n\alpha_1}$$

and the dimension of $(\{\operatorname{ord} X_i \geq \alpha_i, \operatorname{ord} f \geq m\}_{i=1}^n)_{\alpha_1}$ is less than or equal to $\alpha_1(n-1)$ so $[(\{\operatorname{ord} X_i \geq \alpha_i, \operatorname{ord} f \geq m\}_{i=1}^n)_{\alpha_1}]/\mathbb{L}^{n\alpha_1} \in \mathcal{F}^{-\alpha_1}(\mathcal{M}_k)$. ☐

We use this to define a more general integral. Let $I \subset \mathbb{N}^n$ and let $U_{I,m}(f)$ have the same meaning as above. Also, let $U_I := \cup_{(\alpha_1,\ldots,\alpha_n)\in I}\{\operatorname{ord} X_i = \alpha_i\}_{i=1}^n$. (This is measurable by the same argument as for $U_{I,m}$.) Define

$$\int_{U_I} \big|f(X_1, \ldots, X_n)\big| dX_1 \cdots dX_n := \sum_{m\geq 0} \operatorname{vol} U_{I,m}(f) \cdot \mathbb{L}^{-m} \in \widehat{\mathcal{M}}_k.$$

Proposition 3.1.6. *Let $I \subset \mathbb{N}^n$ and $f \in \mathbf{W}(k)[X_1, \ldots, X_n]$. Then $\int_{U_I} \big|f(X_1, \ldots, X_n)\big| dX_1 \ldots dX_n$ exists and we have*

$$\int_{U_I} \big|f(X_1, \ldots, X_n)\big| dX_1 \cdots dX_n = \sum_{(\alpha_1,\ldots,\alpha_n)\in I} \int_{\{\operatorname{ord} X_i = \alpha_i\}_{i=1}^n} \big|f(X_1, \ldots, X_n)\big| dX_1 \cdots dX_n$$

Proof. To prove existence we have to prove that $\operatorname{vol} U_{I,m}/\mathbb{L}^m \to 0$ as $m \to \infty$. This is done in the same way as in the proof of proposition 3.1.3. The equality follows from proposition 3.1.5. ☐

Corollary 3.1.7.

$$\int_{\mathbf{W}^n} \big|f(X_1, \ldots, X_n)\big| dX_1 \cdots dX_n = \sum_{(\alpha_1,\ldots,\alpha_n)\in \mathbb{N}^n} \int_{\{\operatorname{ord} X_i = \alpha_i\}_{i=1}^n} \big|f(X_1, \ldots, X_n)\big| dX_1 \cdots dX_n$$

3.2 The motivic integral of a polynomial in one variable

In this section we compute the integral over $\mathbf{W}$ of $Q(X)$ in $\mathbf{W}(k)[X]$ in the case when Q is separable modulo V. We begin by repeating some arguments from the preceding section in this special case.

Let k be a field of characteristic p and let $Q = a_d X^d + \cdots + a_1 X + a_0 \in \mathbf{W}(k)[X]$. Let $T^\alpha := \{\operatorname{ord} Q \geq \alpha\} \subset \mathbf{W}$ be the scheme defined above, i.e., $T^\alpha(A) = \{a \in \mathbf{W}(A) : Q(a) \equiv 0 \pmod{V^\alpha}\}$ when A is a k-algebra. We want to compute $\operatorname{vol} T^\alpha \in \widehat{\mathcal{M}}_k$.

Let $P := k[X_0, X_1, \ldots, X_N, \ldots]$ and set $x := (X_0, X_1, \ldots, X_N, \ldots) \in \mathbf{W}(P)$. Since $k \subset P$ we have $\mathbf{W}(k) \subset \mathbf{W}(P)$ and hence we can compute $Q(x)$ in $\mathbf{W}(P)$. We then get

$$Q(x) = \big(Q_0(X_0), Q_1(X_0, X_1), \ldots, Q_N(X_0, \ldots, X_N), \ldots\big) \in \mathbf{W}(P) \tag{3.2}$$

where $Q_N \in k[X_0, \dots, X_N]$. Therefore T^α is defined by the set of formulas

$$\begin{aligned}
0 &= Q_0(X_0) \\
0 &= Q_1(X_0, X_1) \\
&\vdots \\
0 &= Q_{\alpha-1}(X_0, \dots, X_{\alpha-1}).
\end{aligned}$$

Hence

$$T_N^\alpha = \operatorname{Spec} \frac{k[X_0, \dots, X_{N-1}]}{(Q_0, \dots, Q_{\alpha-1})} \qquad \text{for } N \geq \alpha$$

and

$$T_N^\alpha = \operatorname{Spec} \frac{k[X_0, \dots, X_{N-1}]}{(Q_0, \dots, Q_{N-1})} \qquad \text{for } N = 1, \dots, \alpha.$$

In the same way as before we now prove that $\operatorname{vol} T^\alpha \in \widehat{\mathcal{M}}_k$. If $N > \alpha$ then

$$\frac{k[X_0, \dots, X_{N-1}]}{(Q_0, \dots, Q_{\alpha-1})} \simeq \frac{k[X_0, \dots, X_{\alpha-1}]}{(Q_0, \dots, Q_{\alpha-1})} \otimes_k k[X_\alpha, \dots X_{N-1}]$$

and hence $[T_N^\alpha] = [T_\alpha^\alpha \times_k \mathbb{A}_k^{N-\alpha}] = [T_\alpha^\alpha] \cdot \mathbb{L}^{N-\alpha} \in \mathrm{K}_0(\mathbf{Sch}_k)$. Therefore $\operatorname{vol} T^\alpha = \lim_{N\to\infty} [T_N^\alpha]/\mathbb{L}^N = [T_\alpha^\alpha]/\mathbb{L}^\alpha \in \widehat{\mathcal{M}}_k$.

We are now ready to prove an analogue of the p-adic Newton's lemma.

Proposition 3.2.1 (Motivic Newton's lemma)**.** *Assume that we are in the above situation and assume also that Q is separable modulo* V*, that is Q_0 is separable. Let $T^\alpha := \{\operatorname{ord} Q \geq \alpha\}$. If α is a positive integer then there is an isomorphism of k-schemes $T_\alpha^\alpha \to T_1^\alpha = \operatorname{Spec} k[X_0]/Q_0(X_0)$. It follows that $[T_\alpha^\alpha] = [T_1^\alpha] \in \mathrm{K}_0(\mathbf{Sch}_k)$ and hence $\operatorname{vol} T^\alpha = [T_1^\alpha]/\mathbb{L}^\alpha \in \widehat{\mathcal{M}}_k$.*

Proof. Let

$$R_i := \frac{k[X_0, \dots, X_{i-1}]}{(Q_0, \dots, Q_{i-1})} \qquad i \geq 1.$$

Then $T_i^\alpha = \operatorname{Spec} R_i$ for $i = 1, \dots, \alpha$ and we want to prove that $R_\alpha \simeq R_1$. We do this by proving that the canonical homomorphism $R_i \to R_{i+1}$ is an isomorphism for $i \geq 1$.

Let $x := (X_0, \dots, X_i) \in \mathbf{W}_{i+1}\big(k[X_0, \dots, X_i]\big)$. We let $\tilde{x} := (X_0, \dots, X_{i-1}, 0)$, so $x = \tilde{x} + \mathrm{V}^i \mathrm{r}(X_i)$, and then Taylor expand:

$$\begin{aligned}
Q(x) =& Q\big(\tilde{x} + \mathrm{V}^i \mathrm{r}(X_i)\big) \\
=& Q(\tilde{x}) + Q'(\tilde{x}) \cdot \mathrm{V}^i \mathrm{r}(X_i) + \mathcal{O}\big(\mathrm{V}^i \mathrm{r}(X_i)\big)^2 \in \mathbf{W}_{i+1}(A_{i+1}).
\end{aligned} \tag{3.3}$$

Here $Q(\tilde{x}) = (Q_0, \dots, Q_{i-1}, P)$, where P is a polynomial in $k[X_0, \dots, X_{i-1}]$. Moreover, since $\pi_1 Q = Q_0$, it follows that if $Q'(x) = (Q_0^*, \dots, Q_i^*)$ then $Q^* = Q_0'$. Hence $Q'(\tilde{x}) = (Q_0', \dots)$. Finally by proposition 1.4.1 $\big(\mathrm{V}^i \mathrm{r}(X_i)\big)^2 = F^i \mathrm{V}^{2i}\big(\mathrm{r}(X_i)\big) = 0 \in \mathbf{W}_i\big(k[X_1, \dots, X_i]\big)$. Hence if we write explicitly we see that the right hand side of (3.3) is

$$\begin{aligned}
(Q_0, \dots, Q_{i-1}, P) + (Q_0', \dots) \cdot (0, \dots, 0, X_i) =& (Q_0, \dots, Q_{i-1}, P) + (0, \dots, 0, Q_0'^{p^i} X_i) \\
=& (Q_0, \dots, Q_{i-1}, P + Q_0'^{p^i} X_i).
\end{aligned}$$

Since the left hand side of (3.3) equals $(Q_0, \dots, Q_i)$ we get the identity

$$Q_i(X_0, \dots, X_i) = P(X_0, \dots, X_{i-1}) + Q_0'(X_0)^{p^i} \cdot X_i \tag{3.4}$$

in $k[X_0, \dots, X_i]$.

We shall also use the hypothesis that Q is separable modulo V. This means that Q_0' is invertible in R_1. Let $Q_0'^{-1}$ be such that $Q_0'(X_0)Q_0'^{-1}(X_0) = 1 + h(X_0)Q_0(X_0)$ in $k[X_0]$.

We now prove that $R_i \to R_{i+1}$ is injective. Let $f(X_0, \dots, X_{i-1}) \in k[X_1, \dots, X_{i-1}]$. We have to prove that if $\overline{f} = 0 \in R_{i+1}$ then $\overline{f} = 0 \in R_i$. So suppose that $f = h_0Q_0 + \cdots + h_iQ_i$ where $h_j \in k[X_1, \dots, X_i]$. By (3.4) this gives

$$\begin{aligned} f(X_0 \dots, X_{i-1}) = h_0 \cdot Q_0(X_0) + \cdots + h_{i-1} \cdot Q_{i-1}(X_0, \dots, X_{i-1}) \\ + h_i \cdot \big(P(X_0, \dots, X_{i-1}) + Q_0'(X_0)^{p^i} \cdot X_i\big). \end{aligned}$$

Substituting $-P(X_0, \dots, X_{i-1}) \cdot Q_0'(X_0)^{-p^i}$ for X_i then gives

$$\begin{aligned} f(X_0 \dots, X_{i-1}) = h_0^* \cdot Q_0(X_0) + \cdots + h_{i-1}^* \cdot Q_{i-1}(X_0, \dots, X_{i-1}) \\ + h_i^* \cdot \big(P(X_0, \dots, X_{i-1}) - P(X_0, \dots, X_{i-1}) + h^* \cdot Q_0(X_0)\big) \end{aligned}$$

where the h_j^* and h^* are polynomials in $k[X_0, \dots, X_{i-1}]$. Hence $\overline{f} = 0 \in R_i$ and consequently $R_i \to R_{i+1}$ is injective.

Finally we prove that $R_i \to R_{i+1}$ is surjective. Identifying R_i with its image in R_{i+1} it suffices to show that $\overline{X}_i \in R_i$. Working in R_{i+1}, (3.4) becomes

$$0 = \overline{P(X_0, \dots, X_{i-1})} + \overline{Q_0'(X_0)}^{p^i} \cdot \overline{X_i}.$$

Since Q_0 is separable we can write this as

$$\overline{X_i} = -\overline{P(X_0, \dots, X_{i-1})} \cdot \overline{Q_0'(X_0)}^{-p^i}.$$

and the right hand side involves only the variables $X_0, \dots, X_{i-1}$ and hence is in R_i. □

We are now going to compute the motivic integral of a polynomial in one variable. Let $Q \in \mathbb{Z}_p[X]$ be separable modulo p. Using Newton's lemma one shows that

$$\int_{\mathbf{W}(\mathbb{F}_q)} |Q(X)|_p dX = 1 + \big|\{x \in \mathbb{F}_q : \overline{Q}(x) \equiv 0 \quad (p)\}\big| \cdot \left(\frac{q}{q+1} - 1\right) \tag{3.5}$$

where $\overline{Q}$ is the reduction of Q modulo p and $q = p^k$. For fixed p we are going to prove that this is true motivically.

Proposition 3.2.2. *If $Q \in \mathbf{W}(k)[X]$ is separable modulo* V *we have that*

$$\int_{\mathbf{W}} |Q(X)| dX = 1 + [\operatorname{Spec} k[X_0]/(Q_0(X_0))] \cdot \left(\frac{\mathbb{L}}{\mathbb{L}+1} - 1\right) \in \widehat{\mathcal{M}}_k.$$

Proof. By definition we have

$$\int_{\mathbf{W}} |Q(X)| dX = \sum_{m \geq 0} \mathbb{L}^{-m} \operatorname{vol}\{\operatorname{ord} Q(X) = m\}.$$

Since $[(\operatorname{ord} Q(X) = m)_n] = [T_n^m \setminus T_n^{m+1}] = [T_n^m] - [T_n^{m+1}]$ for $n > m \geq 1$ we have

$$\begin{aligned}\operatorname{vol}\{\operatorname{ord} Q(X) = m\} &= \lim_{n\to\infty} \frac{[T_n^m] - [T_n^{m+1}]}{\mathbb{L}^n} \\ &= \frac{[T_1^1] \cdot (\mathbb{L}^{n-m} - \mathbb{L}^{n-m-1})}{\mathbb{L}^n} \\ &= [T_1^1] \cdot (\mathbb{L}^{-m} - \mathbb{L}^{-(m+1)})\end{aligned}$$

for $m \geq 1$. For $m = 0$ we have $\operatorname{vol}(\operatorname{ord} Q(X) = 0) = \lim[\mathbf{W}_n \setminus T_n^1]/\mathbb{L}^n = 1 - [T_1^1]/\mathbb{L}$. Therefore, with the help of proposition 1.2.5,

$$\begin{aligned}\int_{\mathbf{W}} |Q(X)| dX &= 1 + [T_1^1] \cdot \left(-\mathbb{L}^{-1} + \sum_{m\geq 1} \mathbb{L}^{-m}(\mathbb{L}^{-m} - \mathbb{L}^{-(m+1)}) \right) \\ &= 1 + [T_1^1] \cdot \left(\sum_{m\geq 1} \mathbb{L}^{-2m} - \sum_{m\geq 0} \mathbb{L}^{-2m-1} \right) \\ &= 1 + [T_1^1] \cdot \left(\frac{\mathbb{L}}{\mathbb{L}+1} - 1 \right).\end{aligned}$$

Note also that if Q is irreducible of degree k then $\mathbb{F}_p[X_0]/(Q_0) \simeq \mathbb{F}_{p^k}$ and so $[T_1^1] = [\operatorname{Spec} \mathbb{F}_{p^k}]$. $\square$

3.3 Many variables

The theorems in this section are all well known and rather trivial for ordinary integrals but for motivic integrals they need a great deal of space to prove. Throughout this section, let k be a field of characteristic p.

A primitive change of variables formula

Let $a_{ij} \in \mathbf{W}(k)$ for $1 \leq i, j \leq n$ be such that the determinant of the matrix $M := (a_{ij})$ is in $\mathbf{W}(k)^\times$.

Proposition 3.3.1. *Given $f \in \mathbf{W}(k)[X_1, \ldots, X_n]$, define $g(X_1, \ldots, X_n) := f\big((X_1, \ldots, X_n)M\big)$. Then for every $\alpha \in \mathbb{N}$,*

$$\operatorname{vol}(\operatorname{ord} g(X_1, \ldots, X_n) \geq \alpha) = \operatorname{vol}(\operatorname{ord} f(X_1, \ldots, X_n) \geq \alpha) \in \widehat{\mathcal{M}}_k$$

Proof. For every k-algebra A we have a map

$$\{\underline{a} \in \mathbf{W}^n(A) : f(a_1, \ldots, a_n) \equiv 0 \pmod{\mathrm{V}^\alpha}\} \to \{\underline{a} \in \mathbf{W}^n(A) : g(a_1, \ldots, a_n) \equiv 0 \pmod{\mathrm{V}^\alpha}\},$$

given by $(a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n)M^{-1}$. This is a bijection, for it is well defined since $g(\underline{a}M^{-1}) = f(\underline{a}M^{-1}M) = f(\underline{a}) = 0$, and it has a well defined inverse $\underline{a} \mapsto \underline{a}M$. Hence $\{\operatorname{ord} g(X_1, \ldots, X_n) \geq \alpha\}$ and $\{\operatorname{ord} f(X_1, \ldots, X_n) \geq \alpha\}$ are isomorphic as subschemes of $\mathbf{W}^n$ so their restrictions modulo V^N are isomorphic for every N, hence they have the same volume. $\square$

Proposition 3.3.2. *We have the following equality:*

$$\int_{\mathbf{W}^n} \big|f(X_1, \ldots, X_n)\big| dX_1 \cdots dX_n = \int_{\mathbf{W}^n} |g(X_1, \ldots, X_n)| dX_1 \cdots dX_n.$$

Proof. From proposition 3.3.1 it follows that $\operatorname{vol}\{\operatorname{ord} f = \alpha\} = \operatorname{vol}\{\operatorname{ord} g = \alpha\}$, hence the result. $\square$

Separation of variables

The main result of this section is theorem 3.3.6 and its consequence theorem 3.3.7. However, to prove theorem 3.3.6 requires very complicated notation. We therefore just prove it in a special case, namely proposition 3.3.5.

Lemma 3.3.3. *Let $P, Q \in \mathbf{W}(k)[X]$. For $\mu, \nu \in \mathbb{N}$ we have*

$$\operatorname{vol}\bigl\{\operatorname{ord} P(X) = \mu \wedge \operatorname{ord} Q(Y) = \nu\bigr\} = \operatorname{vol}\bigl\{\operatorname{ord} P(X) = \mu\bigr\} \cdot \operatorname{vol}\bigl\{\operatorname{ord} Q(Y) = \nu\bigr\} \in \widehat{\mathcal{M}}_k.$$

Proof. Let $x = (X_0, \dots, X_m) \in \mathbf{W}_{m+1}\bigl(k[X_0, \dots, X_m]\bigr)$. Then there are polynomials $P_i, Q_i \in k[X_0, \dots, X_m]$ with the property that

$$\begin{aligned} P(x) =& \bigl(P_0(X_0), \dots, P_m(X_0, \dots, X_m)\bigr) \\ Q(x) =& \bigl(Q_0(X_0), \dots, Q_m(X_0, \dots, X_m)\bigr) \in \mathbf{W}_{m+1}\bigl(k[X_0, \dots, X_m]\bigr) \end{aligned}$$

Let $T^{\mu,\nu} := \{\operatorname{ord} P(X) \geq \mu \wedge \operatorname{ord} Q(Y) \geq \nu\}$. Then $T^{\mu,\nu}$ is defined by the set of formulas

$$\begin{aligned} P_i(X_0, \dots, X_i) =& 0 \qquad\qquad i = 0, \dots, \mu - 1 \\ Q_j(Y_0, \dots, Y_j) =& 0 \qquad\qquad j = 0, \dots, \nu - 1. \end{aligned}$$

Also, let $U^\mu := \{\operatorname{ord} P(X) \geq \mu\}$ and $V^\nu := \{\operatorname{ord} Q(Y) \geq \nu\}$. Then for $n \geq \mu, \nu$ we have

$$\begin{aligned} T_n^{\mu,\nu} =& \operatorname{Spec} \frac{k[X_0, \dots, X_{n-1}, Y_0, \dots, Y_{n-1}]}{(P_0, \dots, P_{\mu-1}, Q_0, \dots, Q_{\nu-1})} \\ =& \operatorname{Spec} \frac{k[X_0, \dots, X_{n-1}]}{(P_0, \dots, P_{\mu-1})} \times_k \operatorname{Spec} \frac{k[Y_0, \dots, Y_{n-1}]}{(Q_0, \dots, Q_{\nu-1})} \\ =& U_n^\mu \times_k V_n^\nu \end{aligned}$$

and so $[T_n^{\mu,\nu}] = [U_n^\mu] \cdot [V_n^\nu] \in \mathrm{K}_0(\mathbf{Sch}_k)$. From this we get

$$\begin{aligned} \operatorname{vol}\bigl\{\operatorname{ord} P(X) =& \mu \wedge \operatorname{ord} Q(Y) = \nu\bigr\} \\ &= \lim_{n\to\infty} \frac{1}{\mathbb{L}^{2n}} \bigl([T_n^{\mu,\nu}] - [T_n^{\mu+1,\nu}] - [T_n^{\mu,\nu+1}] + [T_n^{\mu+1,\nu+1}]\bigr) \\ &= \lim_{n\to\infty} \frac{[U_n^\mu] - [U_n^{\mu+1}]}{\mathbb{L}^n} \cdot \frac{[V_n^\nu] - [V_n^{\nu+1}]}{\mathbb{L}^n} \\ &= \operatorname{vol}\bigl\{\operatorname{ord} P(X) = \mu\bigr\} \cdot \operatorname{vol}\bigl\{\operatorname{ord} Q(Y) = \nu\bigr\}. \end{aligned}$$

$\square$

Lemma 3.3.4. *Let $P, Q \in \mathbf{W}(k)[X]$. Then for $\xi \in \mathbb{N}$,*

$$\operatorname{vol}\bigl\{\operatorname{ord} P(X)Q(Y) = \xi\bigr\} = \sum_{\mu+\nu=\xi} \operatorname{vol}\bigl\{\operatorname{ord} P(X) = \mu \wedge \operatorname{ord} Q(Y) = \nu\bigr\}.$$

Proof. We are going to prove that

$$\{\operatorname{ord} P(X)Q(Y) \geq \xi\} = \bigcup_{\mu+\nu=\xi} \{\operatorname{ord} P(X) \geq \mu \wedge \operatorname{ord} Q(Y) \geq \nu\} \subset \mathbf{W}^2. \tag{3.6}$$

It then follows that

$$\begin{aligned}
&\{\mathrm{ord} P(X)Q(Y) = \xi\} \\
&\quad = \{\mathrm{ord}\, P(X)Q(Y) \geq \xi\} \setminus \{\mathrm{ord}\, P(X)Q(Y) \geq \xi + 1\} \\
&\quad = \bigcup_{\mu+\nu=\xi} \{\mathrm{ord}\, P(X) \geq \mu \wedge \mathrm{ord}\, Q(Y) \geq \nu\} \setminus \bigcup_{\mu+\nu=\xi+1} \{\mathrm{ord}\, P(X) \geq \mu \wedge \mathrm{ord}\, Q(Y) \geq \nu\} \\
&\quad = \dot{\bigcup}_{\mu+\nu=\xi} \{\mathrm{ord}\, P(X) = \mu \wedge \mathrm{ord}\, Q(Y) = \nu\}.
\end{aligned}$$

So the two sides define the same subscheme of $\mathbf{W}^2$. Moreover, the volume of this subscheme equals $\sum_{\mu+\nu=\xi} \mathrm{vol}\{\mathrm{ord}\, P(X) = \mu \wedge \mathrm{ord}\, Q(Y) = \nu\}$ because the union is disjoint.

To prove (3.6) we use the same method as in the proof of the preceding lemma: Let $x = (X_0, \dots, X_m, \dots)$ and $y = (Y_0, \dots, Y_m, \dots) \in \mathbf{W}\big(k[X_m, Y_m]_{m \in \mathbb{N}}\big)$. We then have

$$\begin{aligned}
P(x) =& \big(P_0(X_0), \dots, P_m(X_0, \dots, X_m), \dots\big) \\
Q(y) =& \big(Q_0(Y_0), \dots, Q_m(Y_0, \dots, Y_m), \dots\big) \in \mathbf{W}\big(k[X_m, Y_m]_{m \in \mathbb{N}}\big)
\end{aligned}$$

and

$$P(x)Q(y) = \big(S_0(X_0, Y_0), \dots, S_m(X_0, \dots, X_m, Y_0, \dots, Y_m), \dots\big) \in \mathbf{W}\big(k[X_m, Y_m]_{m \in \mathbb{N}}\big).$$

Therefore $\{\mathrm{ord}\, P(X)Q(Y) \geq \xi\}$ is defined by the set of formulas $S_i(X_0, \dots, X_i, Y_0, \dots, Y_i) = 0$ for $i = 0, \dots, \xi - 1$. Also $\cup_{\mu+\nu=\xi}\{\mathrm{ord}\, P(X) \geq \mu \wedge \mathrm{ord}\, Q(Y) \geq \nu\}$ is defined by $P_i(X_1, \dots, X_i) = 0$ and $Q_j(Y_0, \dots, Y_j) = 0$ for $i = 0, \dots, \mu - 1$ and $\nu = 0, \dots, \nu - 1$ for some μ, ν with $\mu + \nu = \xi$. Now if $\mathbf{a}$ and $\mathbf{b} \in \mathbf{W}(A)$ fulfills $P_i(\mathbf{a}) = 0$ for $i = 0, \dots, \mu - 1$ and $Q_j(\mathbf{b}) = 0$ for $j = 0, \dots, \nu - 1$, where $\mu + \nu = \xi$, then by corollary 1.4.2 we have

$$\begin{aligned}
P(\mathbf{a})Q(\mathbf{b}) =& \mathrm{V}^{\mu}(P_\mu(\mathbf{a}), \dots) \cdot \mathrm{V}^{\nu}(Q_\nu(\mathbf{b}), \dots) \\
=& \mathrm{V}^{\xi}(P_\mu(\mathbf{a})^{p^\nu} Q_\nu(\mathbf{b})^{p^\mu}, \dots) \qquad \in \mathbf{W}(A)
\end{aligned}$$

hence $S_0(\mathbf{a}, \mathbf{b}) = \cdots = S_{\xi-1}(\mathbf{a}, \mathbf{b}) = 0$. If instead $\mu + \nu < \xi$ and $P_\mu(\mathbf{a}) \neq 0$ and $Q_\nu(\mathbf{b}) \neq 0$ then $S_{\mu+\nu}(\mathbf{a}, \mathbf{b}) \neq 0$ and we have $\mu + \nu \leq \xi - 1$ so the converse also holds. □

The next proposition follows from these two lemmas.

Proposition 3.3.5. *Let $P, Q \in \mathbf{W}(k)[X]$. Then for $\xi \in \mathbb{N}$,*

$$\mathrm{vol}\{\mathrm{ord}\, P(X)Q(Y) = \xi\} = \sum_{\mu+\nu=\xi} \mathrm{vol}\big\{\mathrm{ord}\, P(X) = \mu\big\} \cdot \mathrm{vol}\big\{\mathrm{ord}\, Q(Y) = \nu\big\} \tag{3.7}$$

We proceed to give the more general versions of proposition 3.3.5. Let $P \in \mathbf{W}(k)[X_1, \dots, X_n]$ and $Q \in \mathbf{W}(k)[Y_1, \dots, Y_m]$. Let $I \subset \mathbb{N}^n$ and $J \subset \mathbb{N}^m$. Define, in the same way as before,

$$\begin{aligned}
U_{I,\mu}(P) :=& \cup_{\alpha \in I}\{\mathrm{ord}\, X_i = \alpha_i, \mathrm{ord}\, P(X_1, \dots, X_n) = \mu\}_{i=1}^n \subset \mathbf{W}^n \\
U_{J,\nu}(Q) :=& \cup_{\beta \in J}\{\mathrm{ord}\, Y_i = \beta_i, \mathrm{ord}\, Q(Y_1, \dots, Y_m) = \nu\}_{i=1}^m \subset \mathbf{W}^m \\
U_{I \times J, \xi}(PQ) :=& \cup_{(\alpha,\beta) \in I \times J}\{\mathrm{ord}\, X_i = \alpha_i, \mathrm{ord}\, Y_j = \beta_j, \mathrm{ord}\, P(X_1, \dots, X_n) \cdot Q(Y_1, \dots, Y_m) = \xi\} \subset \mathbf{W}^{n+m}.
\end{aligned}$$

Define U_I, U_J and $U_{I \times J}$ in the same way but with the restrictions of the orders of P and Q removed. With this notation we have the following theorem.

Theorem 3.3.6. *For every $\xi \in \mathbb{N}$ we have*

$$\operatorname{vol} U_{I\times J,\xi}(PQ) = \sum_{\mu+\nu=\xi} \operatorname{vol} U_{I,\mu}(P) \cdot \operatorname{vol} U_{J,\nu}(Q).$$

In particular, when $I = J$ we get

$$\operatorname{vol} U_{I,\xi}(PQ) = \sum_{\mu+\nu=\xi} \operatorname{vol} U_{I,\mu}(P) \cdot \operatorname{vol} U_{I,\nu}(Q).$$

The proof is identical to that of the special case given in proposition 3.3.6 but the notation is much more complicated. We omit it.

Theorem 3.3.7 (Separation of variables)**.** *For $P \in \mathbf{W}(k)[X_1, \dots, X_n]$ and $Q \in \mathbf{W}(k)[Y_1, \dots, Y_m]$ we have*

$$\int_{U_{I\times J}} \big|P(X_1, \dots, X_n)Q(Y_1, \dots, Y_m)\big| dX_1 \cdots dX_n dY_1 \cdots dY_m$$
$$= \int_{U_I} \big|P(X_1, \dots, X_n)\big| dX_1 \cdots dX_n \cdot \int_{U_J} \big|Q(Y_1, \dots, Y_m)\big| dY_1 \cdots Y_m.$$

Proof. Theorem 3.3.6 gives that

$$\begin{aligned}
\int_{U_{I\times J}} &|P(X)Q(Y)| dX dY = \\
&= \sum_{\xi \geq 0} \operatorname{vol}\big(U_{I\times J,\xi}(PQ)\big) \cdot \mathbb{L}^{-\xi} \\
&= \sum_{\xi \geq 0} \left(\sum_{\mu+\nu=\xi} \operatorname{vol} U_{I,\mu}(P) \cdot \operatorname{vol} U_{J,\nu}(Q) \right) \cdot \mathbb{L}^{-\xi} \\
&= \left(\sum_{\mu \geq 0} \operatorname{vol} U_{I,\mu}(P) \cdot \mathbb{L}^{-\mu} \right) \cdot \left(\sum_{\nu \geq 0} \operatorname{vol} U_{J,\nu}(Q)) \cdot \mathbb{L}^{-\nu} \right) \\
&= \int_{U_I} |P(X)| dX \cdot \int_{U_J} |Q(Y)| dY.
\end{aligned}$$

□

We will need the following.

Lemma 3.3.8 (Ultrametric inequality)**.** *If A is a ring, let $\mathbf{a}$ and $\mathbf{b} \in \mathbf{W}(A)$ be such that $\mathbf{a} = \mathrm{V}^i(a_0, a_1, \dots)$ where $a_0 \in A^\times$ and $\mathbf{b} = \mathrm{V}^i(0, b_1, \dots)$. Then $\mathbf{a} - \mathbf{b} = \mathrm{V}^i(a_0, a_1 - b_1, \dots)$.*

Proof. Because V is additive we have $\mathbf{a} - \mathbf{b} = \mathrm{V}^i\big((a_0, a_1, \dots) - (0, b_1, \dots)\big)$ so the lemma follows from the shape of the polynomials defining addition. □

Proposition 3.3.9. *Let $Q \in \mathbf{W}(k)[X_1, \dots, X_n]$. Fix $1 \leq i, j \leq n$. Let $I \subset \mathbb{N}^n$ be such that $\alpha_i < \alpha_j$ for every $\alpha = (\alpha_1, \dots, \alpha_n) \in I$. Then*

$$\int_{U_I} \big|Q(X_1, \dots, X_n) \cdot X_i\big| dX_1 \cdots dX_n = \int_{U_I} \big|Q(X_1, \dots, X_n) \cdot (X_i - X_j)\big| dX_1 \cdots dX_n.$$

Proof. This follows if we can prove that

$$\operatorname{vol} U_{I,\xi}\big(Q(X_1,\dots,X_n)\cdot X_i\big) = \operatorname{vol} U_{I,\xi}\big(Q(X_1,\dots,X_n)\cdot(X_i - X_j)\big) \tag{3.8}$$

for every $\xi \in \mathbb{N}$.

By theorem 3.3.6 we have

$$\operatorname{vol} U_{I,\xi}\big(Q(X_1,\dots,X_n)\cdot X_i\big) = \sum_{\mu+\nu=\xi} \operatorname{vol} U_{I,\mu}\big(Q(X_1,\dots,X_n)\big)\cdot \operatorname{vol} U_{I,\nu}(X_i).$$

Now $(a_1,\dots,a_n)$ is an A-point on $U_{I,\nu}(X_i)$ if and only if it fulfills the conditions for U_I and $a_i = \mathrm{V}^\nu(a_{i0},\dots)$ where $a_{i0} \in A^\times$. By lemma 3.3.8 this is equivalent to that $a_i - a_j = \mathrm{V}^\nu(a_{i0},\dots)$ and that $(a_1,\dots,a_n)$ fulfills the conditions for U_I, hence $U_{I,\nu}(X_i) = U_{I,\nu}(X_i - X_j)$. Therefore we have $\operatorname{vol} U_{I,\nu}(X_i) = \operatorname{vol} U_{I,\nu}(X_i - X_j)$ so (3.8) follows. □

3.4 Reducible polynomials

We are now ready to generalize the p-adic computations from [Sko] to the motivic case.

In [Sko] the author sets out to compute the measure of the set of points $(a_1,\dots,a_n) \in \mathbb{Z}_p^n$ such that $X^n + a_1X^{n-1} + \cdots + a_n$ splits completely over $\mathbb{Z}_p$. He starts by making the change of variables $a_i = (-1)^i\sigma_i(b_1,\dots,b_n)$, where the σ_i are the elementary symmetric polynomials, to get the integral

$$\frac{1}{n!}\int_{\mathbb{Z}_p^n} \prod_{1\le i<j\le n} |b_i - b_j|_p db_1 \dots db_n.$$

He then gives a recursive way to compute this integral. We are going to show that this recursion also works on the integral

$$\int_{\mathbf{W}^n} \Big|\prod_{1\le i<j\le n}(X_i - X_j)\Big| dX_1\dots dX_n \in \widehat{\mathcal{M}}_{\mathbb{F}_p}.$$

We have already proved that this integral exists (proposition 3.1.6). The recursive method will allow us to compute an explicit formula for it for any given n. In particular this will show that it actually is a rational function in $\mathbb{L}$.

Observe that we have not proved that the functor of polynomials that split completely is motivic. To do that would require a motivic change of variables formula.

Notation

Define

$$V^n := \frac{1}{n!}\int_{\mathbf{W}^n} |\Delta_n| dX_1 \dots dX_n \in \widehat{\mathcal{M}}_{\mathbb{F}_p} \otimes_{\mathbb{Z}} \mathbb{Q}$$

where $\Delta_n := \prod_{1\le i<j\le n}(X_i - X_j)$. The reason why we tensor $\widehat{\mathcal{M}}_{\mathbb{F}_p}$ with $\mathbb{Q}$ is to make it possible for us to divide by $n!$. We could avoid this but the notation would then be even more messy than it already is.

For an l-tuple α of positive integers with sum n, that is $\alpha = (n_1,\dots,n_l)$ where $n_1 + \cdots + n_l = n$, let $\alpha! := n_1!\cdots n_l!$. We will write

$$U_\alpha := \{\operatorname{ord}_p X_1 = \cdots = \operatorname{ord}_p X_{n_1} < \cdots < \operatorname{ord}_p X_{n-n_l+1} = \cdots = \operatorname{ord}_p X_n\} \subset \mathbf{W}^n.$$

By this we mean

$$\bigcup_{(\beta_1,\dots,\beta_n)\in I} \{\operatorname{ord} X_i = \beta_i\}_{i=1}^n$$

where $I := \{(\beta_1, \dots, \beta_n) \in \mathbb{N}^n : \beta_1 = \dots = \beta_{n_1} < \dots < \beta_{n-n_l+1} = \dots = \beta_n\}$.

Define

$$V_\alpha^n(s,t) := \frac{1}{\alpha!} \int_{U_\alpha} \Big| \Big(\prod_{i=1}^n X_i\Big)^s X_n^t \Delta_n \Big| dX_1 \dots dX_n.$$

Define $V^n(s,t)$ in the same way; the same integrand but integrating over $\mathbf{W}^n$. We then have $V^n = V^n(0,0)$ and we will also write $V_\alpha^n := V_\alpha^n(0,0)$.

Description of the recursion

Partitioning $\mathbf{W}^n$ and using that Δ_n is symmetric together with the change of variables formula, proposition 3.3.2, we see that for every $s \in \mathbb{N}$, $V^n(s,0) = \sum_\alpha V_\alpha^n(s,0)$ where the sum is taken over all tuples of positive integers which sum to n.

On the other hand, using the change of variables $Y_i = X_i - X_n$, $i = 1, \dots, n-1$ and $Y_n = X_n$, it follows from proposition 3.3.2 that

$$V^n = \frac{1}{n!} \int_{\mathbf{W}^n} \Big| \Big(\prod_{i=1}^{n-1} Y_i\Big) \Delta_{n-1}(Y_1, \dots, Y_{n-1}) \Big| dY_1 \dots dY_n.$$

By theorem 3.3.7 this equals $\frac{1}{n} V^{n-1}(1,0) \cdot \int_{\mathbf{W}} |1| dY_n$ Since the second integral is equal to 1 we find that

$$V^n = \tfrac{1}{n} V^{n-1}(1,0).$$

Together the above gives

$$\sum_{\text{tuples } \alpha \text{ with sum } n} V_\alpha^n = \frac{1}{n} \sum_{\text{tuples } \beta \text{ with sum } n-1} V_\beta^{n-1}(1,0). \tag{3.9}$$

If $\alpha = (n_1, \dots, n_l)$ with $n_1 + \dots + n_l = n$, let $(\alpha, m) = (n_1, \dots, n_l, m)$. The problem now comes down to proving the two formulas

$$V_{(\alpha,m)}^{n+m}(s,t) = \mathbb{L}^{-(ms+t+m(m+1)/2)} V_\alpha^n(s+m, sm+t+m(m+1)/2) V_{(m)}^m(s,t) \tag{3.10}$$

and

$$V_{(n)}^n(s,t) = \frac{1 - \mathbb{L}^{-n(n+1)/2}}{1 - \mathbb{L}^{-(sn+t+n(n+1)/2)}} V_{(n)}^n. \tag{3.11}$$

Using them, (3.9) takes the form $V_{(n)}^n =$ rational function in $\mathbb{L}, V_{(1)}^1, \dots, V_{(n-1)}^{n-1}$. Since $V_{(1)}^1 = 1$ this recursively gives us a formula for $V_{(n)}^n$, hence, again using (3.10) and (3.11), for V_α. Since $V_n = \sum_\alpha V_\alpha^n$ we are through.

Proofs of (3.10) and (3.11)

To prove (3.10) and (3.11) we need a lemma:

Lemma 3.4.1. *Let $\Delta \in \mathbb{Z}_p[X_1, \dots, X_n]$ be a form of degree d. For every pair of non-negative integers m and k we have*

$$\operatorname{vol}(\operatorname{ord}\Delta = m \wedge \operatorname{ord} X_i = k, i = 1, \dots, n) = \mathbb{L}^{-kn} \operatorname{vol}(\operatorname{ord}\Delta = m - dk \wedge \operatorname{ord} X_i = 0, i = 1, \dots, n).$$

Proof. By proposition 3.1.4 the left hand side equals

$$
\begin{aligned}
&\operatorname{vol}(\operatorname{ord}\Delta \geq m \wedge \operatorname{ord} X_i \geq k) \\
&\qquad - \sum_{i=1}^{n} \operatorname{vol}(\operatorname{ord}\Delta \geq m \wedge \operatorname{ord} X_j \geq k, j \neq i \wedge \operatorname{ord} X_i \geq k+1) + \\
&\qquad\qquad \cdots + (-1)^n \operatorname{vol}(\operatorname{ord}\Delta \geq m \wedge \operatorname{ord} X_i \geq k+1) \\
&- \Big(\operatorname{vol}(\operatorname{ord}\Delta \geq m+1 \wedge \operatorname{ord} X_i \geq k) \\
&\qquad - \sum_{i=1}^{n} \operatorname{vol}(\operatorname{ord}\Delta \geq m+1 \wedge \operatorname{ord} X_j \geq k, j \neq i \wedge \operatorname{ord} X_i \geq k+1) + \\
&\qquad\qquad \cdots + (-1)^n \operatorname{vol}(\operatorname{ord}\Delta \geq m+1 \wedge \operatorname{ord} X_i \geq k+1)\Big)
\end{aligned}
$$

so it suffices to show that, for $k_i \in \{k, k+1\}$,

$$\operatorname{vol}(\operatorname{ord}\Delta \geq m \wedge \operatorname{ord} X_i \geq k_i) = \mathbb{L}^{-kn} \operatorname{vol}(\operatorname{ord}\Delta \geq m - dk \wedge \operatorname{ord} X_i \geq k_i - k).$$

We do this for the special case $k_i = k$, $i = 1, \ldots, n$. The general case is similar but the indexing is even more complicated.

For $1 \leq i \leq n$, let $X_i = (X_{i0}, \ldots, X_{iN}) \in \mathbf{W}_{N+1}\big(\mathbb{F}_p[X_{i0}, \ldots, X_{iN}]_{i=1}^n\big)$. We then have

$$\Delta(X_1, \ldots, X_n) = \big(\Delta_0(X_{\bullet 0}), \ldots, \Delta_N(X_{\bullet 0}, \ldots, X_{\bullet N})\big)$$

where $\Delta_j(X_{\bullet 0}, \ldots, X_{\bullet j}) := \Delta_j(X_{10}, \ldots, X_{1j}, \ldots, X_{n0}, \ldots, X_{nj}) \in \mathbb{F}_p[X_{i0}, \ldots X_{iN}]_{i=1}^n$ for $j = 0, \ldots, N$. Let

$$T_{N+1} := \operatorname{Spec} \frac{\mathbb{F}_p[X_{i0}, \ldots, X_{iN}]_{i=1}^n}{\big(\Delta_0(X_{\bullet 0}), \ldots, \Delta_{m-dk}(X_{\bullet 0}, \ldots, X_{\bullet m-dk})\big)}.$$

Then

$$\operatorname{vol}(\operatorname{ord}\Delta \geq m - dk \wedge \operatorname{ord} X_i \geq 0) = \lim_{N \to \infty} \frac{[T_{N+1}]}{\mathbb{L}^{n(N+1)}}.$$

On the other hand, let $\mathbf{0}$ be an n-tuple of zeros and set

$$S_{N+1} := \operatorname{Spec} \frac{\mathbb{F}_p[X_{ik}, \ldots, X_{iN}]_{i=1}^n}{\big(\Delta_0(\mathbf{0}), \ldots, \Delta_m(\mathbf{0}, \ldots, \mathbf{0}, X_{\bullet k}, \ldots, X_{\bullet m})\big)}.$$

Then

$$\operatorname{vol}(\operatorname{ord}\Delta \geq m \wedge \operatorname{ord} X_i \geq k) = \lim_{N \to \infty} \frac{[S_{N+1}]}{\mathbb{L}^{n(N+1)}}.$$

Now, let A be an $\mathbb{F}_p$-algebra and let $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbf{W}(A)$. Corollary 1.4.4 says that

$$\Delta(V\mathbf{a}_1, \ldots, V\mathbf{a}_n) = F^{d-1}V^d\Delta(\mathbf{a}_1, \ldots, \mathbf{a}_n).$$

In $\mathbf{W}_{N+1}\big(\mathbb{F}_p[X_{i0}, \ldots, X_{iN}]_{i=1}^n\big)$ we therefore have

$$\Delta(V^k X_1, \ldots, V^k X_n) = F^{(d-1)k}V^{dk}\Delta(X_1, \ldots, X_n).$$

(We may assume that N is much bigger then m.) This gives the following equalities in $\mathbb{F}_p[X_{i0},\ldots,X_{iN}]_{i=1}^n$:

$$\begin{aligned}
\Delta_0(\mathbf{0}) &= 0\\
\Delta_1(\mathbf{0},\mathbf{0}) &= 0\\
&\vdots\\
\Delta_{kd-1}(\mathbf{0},\ldots,\mathbf{0},X_{\bullet 0},\ldots,X_{\bullet kd-1-k}) &= 0\\
\Delta_{kd}(\mathbf{0},\ldots,\mathbf{0},X_{\bullet 0},\ldots,X_{\bullet kd-k}) &= \big(\Delta_0(X_{\bullet 0})\big)^{p^{(d-1)k}}\\
&\vdots\\
\Delta_{m}(\mathbf{0},\ldots,\mathbf{0},X_{\bullet 0},\ldots,X_{\bullet m-k}) &= \big(\Delta_{m-dk}(X_{\bullet 0},\ldots,X_{\bullet m-dk})\big)^{p^{(d-1)k}}
\end{aligned}$$

So the change of variables $X_{ij}\mapsto X_{i,j-k}$ gives an isomorphism

$$\frac{\mathbb{F}_p[X_{ik},\ldots,X_{iN}]_{i=1}^n}{\big(\Delta_0(\mathbf{0}),\ldots,\Delta_m(\mathbf{0},\ldots,\mathbf{0},X_{\bullet k},\ldots,X_{\bullet m})\big)}\to\frac{\mathbb{F}_p[X_{i0},\ldots,X_{i,N-k}]_{i=1}^n}{\big(\Delta_0(X_{\bullet 0}),\ldots,\Delta_{m-dk}(X_{\bullet 0},\ldots,X_{\bullet m-dk})\big)^{p^{(d-1)k}}}$$

and so we get $[S_{N+1}]\cdot\mathbb{L}^{nk}=[T_{N+1}]$ and hence

$$\begin{aligned}
\operatorname{vol}(\operatorname{ord}\Delta\geq m-dk\ \wedge\ \operatorname{ord}X_i\geq 0) &= \lim_{N\to\infty}\frac{[T_{N+1}]}{\mathbb{L}^{n(N+1)}}\\
&= \lim_{N\to\infty}\frac{[S_{N+1}]\cdot\mathbb{L}^{nk}}{\mathbb{L}^{n(N+1)}}\\
&=\mathbb{L}^{nk}\lim_{N\to\infty}\frac{[S_{N+1}]}{\mathbb{L}^{n(N+1)}}\\
&=\mathbb{L}^{nk}\operatorname{vol}(\operatorname{ord}\Delta\geq m\ \wedge\ \operatorname{ord}X_i\geq k).
\end{aligned}$$

□

Both (3.10) and (3.11) will be consequences of the following:

Corollary 3.4.2. *Let s, t and k be non-negative integers and set $V_k=\{\operatorname{ord}X_i=k, i=1,\ldots,n\}$. Let $e=e(n,s,t)=ns+t+n(n+1)/2$. Then*

$$\int_{V_k}\left|\Big(\prod_{i=1}^n X_i\Big)^s X_n^t\Delta_n\right|dX_1\ldots dX_n=\mathbb{L}^{-ek}\int_{V_0}|\Delta_n|dX_1\ldots dX_n$$

Proof. We have

$$\begin{aligned}
\int_{V_k}\left|\Big(\prod_{i=1}^n X_i\Big)^s X_n^t\Delta_n\right|dX_1\cdots dX_n &= \sum_{\xi\geq 0}\mathbb{L}^{-\xi}\operatorname{vol}\left(\operatorname{ord}\Big(\prod_{i=1}^n X_i\Big)^s X_n^t\Delta_n=\xi\ \wedge\ \operatorname{ord}X_i=k\right)\\
&=\sum_{\xi\geq 0}\mathbb{L}^{-\xi}\operatorname{vol}(\operatorname{ord}\Delta_n=\xi-(ns+t)k\ \wedge\ \operatorname{ord}X_i=k).
\end{aligned}$$

By the lemma this equals

$$\sum_{\xi\geq 0}\mathbb{L}^{-\xi}\mathbb{L}^{-kn}\operatorname{vol}\big(\operatorname{ord}\Delta_n=\xi-(ns+t)k-\tfrac{n(n-1)}{2}k\ \wedge\ \operatorname{ord}X_i=0\big)$$

Let $\xi' = \xi - (ns+t)k - \frac{n(n-1)}{2}k$. Since $\operatorname{ord}\Delta_n \geq 0$, we have $\operatorname{vol}(\operatorname{ord}\Delta_n = \xi' \wedge \operatorname{ord} X_i = 0) = 0$ when $\xi' < 0$ so our expression becomes

$$\mathbb{L}^{-(ns+t)k-\frac{n(n-1)}{2}k}\mathbb{L}^{-nk}\sum_{\xi'\geq 0}\mathbb{L}^{-\xi'}\operatorname{vol}(\operatorname{ord}\Delta_n = \xi' \wedge \operatorname{ord} X_i = 0)$$

so we are through. ☐

Now (3.11) is immediate.

Proof of (3.11). Using the corollary we get

$$\begin{aligned} V^n_{(n)}(s,t) =& \sum_{\xi\geq 0}\int_{V_\xi}\left|\Big(\prod_{i=1}^n X_i\Big)^s X_n^t \Delta_n\right| dX_1\dots dX_n \\ \overset{3.4.2}{=}& \sum_{\xi\geq 0}\mathbb{L}^{-e(n,s,t)\xi}\int_{V_0}|\Delta_n| dX_1\dots dX_n \\ =& \frac{1}{1-\mathbb{L}^{-e(n,s,t)}}\int_{V_0}|\Delta_n| dX_1\dots dX_n \end{aligned}$$

and in particular, putting $s = t = 0$ so that $e(n,s,t) = n(n+1)/2$,

$$V^n_{(n)} = \frac{1}{1-\mathbb{L}^{-n(n+1)/2}}\int_{V_0}|\Delta_n| dX_1\dots dX_n.$$

So (3.11) follows by putting these two equations together. ☐

Finally we prove (3.10):

Proof of (3.10). Let α be a tuple of positive integers with sum n, so (α,m) is a tuple with sum $n+m$. Then

$$V^{n+m}_{(\alpha,m)}(s,t) = \frac{1}{\alpha!m!}\int_{U_{(\alpha,m)}}\left|\Big(\prod_{1\leq i\leq n+m} X_i\Big)^s X^t_{n+m}\Delta_{n+m}\right| dX_1\dots dX_{n+m}$$

Now by proposition 3.3.9 we may replace $X_i - X_j$ with X_i for i,j such that $1 \leq i \leq n < j \leq n+m$. Put $\Delta'_m = \prod_{n<i<j\leq n+m}(X_i - X_j)$. Then

$$V^{n+m}_{(\alpha,m)}(s,t) = \frac{1}{\alpha!m!}\int_{U_{(\alpha,m)}}\left|\Big(\prod_{1\leq i\leq n} X_i\Big)^{s+m}\Delta_n\Big(\prod_{n<i\leq n+m} X_i\Big)^s X^t_{n+m}\Delta'_m\right| dX_1\dots dX_{m+n}. \tag{3.12}$$

We write this as

$$V^{n+m}_{(\alpha,m)}(s,t) = \frac{1}{\alpha!m!}\sum_{k\geq 0}\int_{\substack{U_{(\alpha,m)}\\ \operatorname{ord} X_n = k}}|I| dX_1\dots dX_{m+n} \tag{3.13}$$

where the integrand $I := \left(\prod_{1\leq i\leq n} X_i\right)^{s+m}\Delta_n\left(\prod_{n<i\leq n+m} X_i\right)^s X^t_{n+m}\Delta'_m$ is the same as in (3.12). We take care of each term in this sum separately. For every $k \in \mathbb{N}$, theorem 3.3.7 gives that

$$\begin{aligned} &\frac{1}{\alpha!m!}\int_{\substack{U_{(\alpha,m)}\\ \operatorname{ord} X_n = k}}|I| dX_1\dots dX_{m+n} \\ =& \left(\frac{1}{\alpha!}\int_{\substack{U_\alpha\\ \operatorname{ord} X_n = k}}\left|\Big(\prod_{i=1}^n X_i\Big)^{s+m}\Delta_n\right| dX_1\dots dX_m\right)\cdot\left(\frac{1}{m!}\int_{k<|X_i|=|X_j|}\left|\Big(\prod_{i=n+1}^{n+m} X_i\Big)^s X^t_{n+m}\Delta'_m\right| dX_{n+1}\dots dX_{n+m}\right). \end{aligned}$$

We first take care of the second factor in this product. For every $k \in \mathbb{N}$ we use corollary 3.4.2 in the following computation:

$$\begin{aligned}
&\frac{1}{m!}\int_{k<|X_i|=|X_j|}\left|\Big(\prod_{i=1}^{m} X_i\Big)^s X_m^t \Delta_m\right| dX_1 \dots dX_m \\
&\qquad = \frac{1}{m!}\sum_{\xi=k+1}^{\infty}\int_{V_\xi}\left|\Big(\prod_{i=1}^{m} X_i\Big)^s X_m^t \Delta_m\right| dX_1 \dots dX_m \\
&\qquad \overset{3.4.2}{=} \frac{1}{m!}\sum_{\xi=k+1}^{\infty}\mathbb{L}^{-e(m,s,t)\xi}\int_{V_0}|\Delta_m| dX_1 \dots dX_m \\
&\qquad = \frac{1}{m!}\mathbb{L}^{-e(m,s,t)(k+1)}\sum_{\xi=0}^{\infty}\mathbb{L}^{-e(m,s,t)\xi}\int_{V_0}|\Delta_m| dX_1 \dots dX_m \\
&\qquad \overset{3.4.2}{=} \mathbb{L}^{-e(m,s,t)(k+1)}\frac{1}{m!}\int_{0\le|X_i|=|X_j|}\left|\Big(\prod_{i=1}^{m} X_i^s\Big) X_n^t \Delta_m\right| dX_1 \dots dX_m \\
&\qquad = \mathbb{L}^{-e(m,s,t)(k+1)} V_{(m)}^m(s,t).
\end{aligned}$$

Putting this into equation (3.13) gives

$$V_{(\alpha,m)}^{n+m}(s,t) = V_{(m)}^m(s,t)\mathbb{L}^{-e(m,s,t)}\sum_{k\ge 0}\left(\frac{1}{\alpha!}\int_{\substack{U_\alpha \\ \operatorname{ord} X_n = k}}\left|\prod_{i=1}^{n} X_i^{s+m}\Delta_n\right| dX_1 \dots dX_n\right)\mathbb{L}^{-e(m,s,t)k}.$$

We compute the sum in this expression, with $e := e(m,s,t)$.

$$\begin{aligned}
&\sum_{k\ge 0}\left(\frac{1}{\alpha!}\int_{\substack{U_\alpha \\ \operatorname{ord} X_n = k}}\left|\prod_{i=1}^{n} X_i^{s+m}\Delta_n\right| dX_1 \dots dX_n\right)\mathbb{L}^{-ek} \\
&\qquad = \frac{1}{\alpha!}\sum_{k\ge 0}\sum_{\xi\ge 0}\operatorname{vol}\left(\operatorname{ord}\Big(\prod_{i=1}^{n} X_i^{s+m}\Big)\Delta_n = \xi \wedge U_\alpha \wedge \operatorname{ord} X_n = k\right)\mathbb{L}^{-\xi}\mathbb{L}^{-ek} \\
&\qquad = \frac{1}{\alpha!}\sum_{k\ge 0}\sum_{\xi\ge 0}\operatorname{vol}\left(\operatorname{ord}\Big(\prod_{i=1}^{n} X_i^{s+m}\Big) X_n^e\Delta_n = \xi + ek \wedge U_\alpha \wedge \operatorname{ord} X_n = k\right)\mathbb{L}^{-(\xi+ek)} \\
&\qquad = \frac{1}{\alpha!}\sum_{\tau\ge 0}\operatorname{vol}\left(\operatorname{ord}\Big(\prod_{i=1}^{n} X_i^{s+m}\Big) X_n^e\Delta_n = \tau \wedge U_\alpha\right)\mathbb{L}^{-\tau} \\
&\qquad = V_\alpha(s+m,e).
\end{aligned}$$

Therefore we finally get

$$V_{(\alpha,m)}^{n+m}(s,t) = \mathbb{L}^{-e}V_{(m)}^m(s,t)V_\alpha^n\big(s+m, e(m,s,t)\big).$$

$\square$

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