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Tilahun Abebaw

Contents

Notations

- We use the standard notation $\mathbb{C}$ for the field of complex numbers,$\mathbb{Z}$ for the ring of integers and $\mathbb{N}$ for the set of natural numbers.
- $A_n = \mathbb{C} < x_1, ..., x_n, \partial_{x_1}, ..., \partial_{x_n} >$, the ring of differential operators of the polynomial ring $\mathbb{C}[x_1, ..., x_n]$.
- For multi-indices $\alpha = (\alpha_1, ..., \alpha_n), \beta = (\beta_1, ..., \beta_n) \in \mathbb{N}^n$, by $x^\alpha \partial^\beta$ we mean $x_1^{\alpha_1}...x_n^{\alpha_n}\partial_{x_1}^{\beta_1}...\partial_{x_n}^{\beta_n}$ and the degree of $x^\alpha \partial^\beta$ is $\mathrm{deg(x^\alpha \partial^\beta) = \alpha_1 + ... + \alpha_n + \beta_1 + ... + \beta_n}$.
- For $P = \sum_{\alpha,\beta \in \mathbb{N}^n} a_{\alpha,\beta} x^\alpha \partial^\beta \in A_n$, the degree of P is $\mathrm{degP = max\{deg(x^\alpha \partial^\beta) : a_{\alpha,\beta} \neq 0\}}$.

1 Introduction

Let $\alpha_1, ..., \alpha_m$ be linear forms defined on $\mathbb{C}^n$ and $X = \mathbb{C}^n \setminus \cup_{i=1}^m V(\alpha_i)$, where $V(\alpha_i) = \{p \in \mathbb{C}^n : \alpha_i(P) = 0\}$. Then the coordinate ring O_X of X is the localization $\mathbb{C}[x]_\alpha$, where $\alpha = \prod_{i=1}^m \alpha_i$. The ring O_X is a holonomic A_n-module, where A_n is the n-th Weyl algebra and since holonomic A_n-modules have finite length, O_X has finite length. We consider a "twisted" variant of this A_n-module. Defining M_α^β to be the free rank 1 $\mathbb{C}[x]_\alpha$-module on the generator α^β, where $\alpha^\beta = \alpha_1^{\beta_1}...\alpha_m^{\beta_m}$ and the multi-index $\beta = (\beta_1, ..., \beta_m) \in \mathbb{C}^m$, we can give it a structure as an A_n-module in the following way. Define the actions of the generators of A_n as follows:

$$x_i \bullet \frac{p}{\alpha^r}\alpha^\beta = \frac{x_i p}{\alpha^r}\alpha^\beta$$

for $i = 1, 2, ..., n$ and

$$\partial_j \bullet \frac{p}{\alpha^r}\alpha^\beta = \partial_j(\frac{p}{\alpha^r})\alpha^\beta + \frac{p}{\alpha^r}\partial_j(\alpha^\beta)$$

where

$$\partial_j(\alpha^\beta) = \sum_{i=1}^m \beta_i \frac{\partial_j(\alpha_i)}{\alpha_i}\alpha^\beta$$

for $j = 1, 2, ..., n$. Clearly these relations mean that α^β behaves as the corresponding complex function is defined on the complement of the union of the hyperplanes.

The A_n-module M_α^β is a holonomic module (Theorem 1) and hence it has finite length with decomposition factors that have support on the intersection of the hyperplanes defined by the linear forms (Proposition 9). It seems difficult to calculate the number of these decomposition factors in general. It has been done for the case $\beta \in \mathbb{Z}^m$, (see [5]) and our main result in this paper is a computation in the case $n = 2$. Our methods are algebraic, in particular we calculate the A_2-annihilator of α^β. Along the way we prove that the module is irreducible in the generic situation.

2 Preliminaries

2.1 Definition of the module M_α^β

Let $\alpha_i : \mathbb{C}^n \longrightarrow \mathbb{C}$, $i = 1, 2, ..., m$ such that,

$$\alpha_i(x_1, ..., x_n) = \sum_{j=1}^n \alpha_{ij}x_j, \alpha_{ij} \in \mathbb{C}$$

be linear forms and H_i be the hyperpane in $\mathbb{C}^n$ defined by α_i, that is, $H_i = \{P \in \mathbb{C}^n : \alpha_i(P) = 0\}$. If we let $X = \mathbb{C}^n \setminus \cup_{i=1}^m H_i$, then the coordinate ring of X is the localization $\mathbb{C}[x_1, \ldots, x_n]_\alpha$, where $\alpha = \prod_{i=1}^{m} \alpha_i$, that is, the ring of rational functions of the form $\frac{p}{\alpha^r}$, where p is a polynomial in $\mathbb{C}[x_1, \ldots, x_n]$. Since rational functions are preserved by partial differentiation and multiplication by polynomials, $\mathbb{C}[x_1, \ldots, x_n]_\alpha$ is an A_n-module, where A_n is the n-th Weyl Algebra. Consider for varying values of the complex parameters $\beta_1, ..., \beta_m$, the function

$$\alpha^\beta = \alpha_1^{\beta_1} ... \alpha_m^{\beta_m}.$$

Here $\beta = (\beta_1, ..., \beta_m)$ and we will throughout this paper use the above multi-index notation. Also we will use $\mathbb{C}[x]$ instead of $\mathbb{C}[x_1, ..., x_n]$.

Definition 1. The module M_α^β is the free rank 1 $\mathbb{C}[x]_\alpha$-module on the generator α^β. We can give M_α^β a structure as an A_n-module in the following way. Define the actions of the generators of A_n as follows:

$$x_i \bullet \frac{p}{\alpha^r}\alpha^\beta = \frac{x_i p}{\alpha^r}\alpha^\beta$$

for $i = 1, 2, ..., n$ and

$$\partial_j \bullet \frac{p}{\alpha^r}\alpha^\beta = \partial_j(\frac{p}{\alpha^r})\alpha^\beta + \frac{p}{\alpha^r}\partial_j(\alpha^\beta)$$

where

$$\partial_j(\alpha^\beta) = \sum_{i=1}^{m} \beta_i \frac{\partial_j(\alpha_i)}{\alpha_i}\alpha^\beta$$

for $j = 1, 2, ..., n$.

The verification that M_α^β is an A_n-module is left to the reader. The problem which we consider in this paper, and solve in some cases is to find the number of the decomposition factors of M_α^β. We will throughout this paper use the notations $\mathrm{DF}(\mathrm{M}_\alpha^\beta)$ for the set of decomposition factors of M_α^β and $c(M_\alpha^\beta)$ for the number of decomposition factors of M_α^β.

2.2 The simplest example

This is clearly the A_1-module $M_\alpha^\beta = \mathbb{C}[x]_x x^\beta$, that is the case where $m = n = 1$. We have the following result, which we do in detail as a preparation for later results.

Proposition 1. *(i) If $\beta \in \mathbb{Z}$, then $c(M_\alpha^\beta) = 2$.*
(ii) If $\beta \in \mathbb{C} \setminus \mathbb{Z}$, then M_α^β is an irreducible A_1-module, so $c(M_\alpha^\beta) = 1$.

Proof. By definition $M_\alpha^\beta = \mathbb{C}[x]_x x^\beta \cong \oplus_{i \in \mathbb{Z}} \mathbb{C}x^{\beta+i}$.
(i) If $\beta \in \mathbb{Z}$, then clearly $M_\alpha^\beta \cong \mathbb{C}[x]_x$. Consider the submodule $\mathbb{C}[x]$. First we are going to show that $\mathbb{C}[x]$ is irreducible. Suppose $0 \neq f \in \mathbb{C}[x]$ and

consider the submodule $A_1 f$ of $\mathbb{C}[x]$. Let m be the degree of f and a be its coefficient. Then $\partial_x^m f = m!a$ is a non-zero constant in the submodule generated by f. Since a non-zero constant generates $\mathbb{C}[x]$, then $\mathbb{C}[x] \subset A_1 f$. But f was an arbitrary element. This means $\mathbb{C}[x]$ is irreducible. Again we consider the A_1-module $\mathbb{C}[x]_x/\mathbb{C}[x]$ and show that it is irreducible. Clearly the module is generated as an A_1-module by the class of x^{-1} modulo $\mathbb{C}[x]$. Let $0 \neq g \in \mathbb{C}[x]_x/\mathbb{C}[x]$. Then we may assume that all the terms of g have negative degree. Let h be such that $-h$ is the minimum of the degrees of the terms of g. Then $x^{h-1}g = bx^{-1}$, where b is the coefficient of the term with degree $-h$. Since bx^{-1} generates $\mathbb{C}[x]_x/\mathbb{C}[x]$, then $\mathbb{C}[x]_x/\mathbb{C}[x] \subset A_1 g$. But g was an arbitrary element. This means that $\mathbb{C}[x]_x/\mathbb{C}[x]$ is irreducible. So we have a composition series $0 \subset \mathbb{C}[x] \subset \mathbb{C}[x]_x$ of M_α^β, and hence $c(M_\alpha^\beta) = 2$. This proves (i).

(ii) Suppose $\beta \in \mathbb{C} \setminus \mathbb{Z}$. We have the formula

$$(x\partial_x - (\beta + i))x^{\beta+j} = (j - i)x^{\beta+j}$$

If $f = \sum\limits_{i=0}^{k} \alpha_i x^{\beta+i} \in M_\alpha^\beta$ where $\alpha_k \neq 0$, then

$$\prod_{i=0}^{k-1}(x\partial_x - (\beta + i))f = \alpha_k k! x^{\beta+k}.$$

So the monomial $x^{\beta+k} \in A_1 f$. Now use the formulas

$$\partial_x^i x^{\beta+k} = (\beta + k)...(\beta + k - i)x^{\beta+k-i} \tag{2.1}$$

and

$$x^i x^{\beta+k} = x^{\beta+k+i} \tag{2.2}$$

If now $\beta \in \mathbb{C} \setminus \mathbb{Z}$, then the coefficient in (2.1) is non-zero for all $i \geq 0$ and hence $x^{\beta+k-i} \in A_1 f$. (2.2) gives that $x^{\beta+k+i} \in A_1 f$ for all $i \geq 0$, and so $M_\alpha^\beta \subset A_1 f$. But f was an arbitrary element. This means that M_α^β is irreducible. This concludes the proof. □

2.3 A basic property of the module M_α^β

In the following proposition we are going to prove a basic property of the module M_α^β, which we will use later on.

Proposition 2. *(i)* $\mathrm{M}_\alpha^\beta \cong \mathrm{M}_\alpha^\gamma$, *if* $\beta \equiv \gamma \ (\mathrm{mod}\mathbb{Z}^\mathrm{m})$.
(ii) $\mathrm{M}_\alpha^\beta \cong \mathbb{C}[\mathrm{x}]_\alpha$, *if* $\beta \in \mathbb{Z}^m$.

Proof. (ii) is a special case of (i). Suppose that $\beta = \gamma + \tau$, $\tau \in \mathbb{Z}^m$.
Define $\theta : \mathrm{M}_\alpha^\beta \longrightarrow \mathrm{M}_\alpha^\gamma$ by:

$$\theta(\frac{p}{\alpha^r}\alpha^\beta) = \frac{p}{\alpha^r}\alpha^\tau\alpha^\gamma$$

Clearly this is a 1-1, onto map and it is an easy exercise to show that it is an A_n-module homomorphism. $\square$

2.4 M_α^β is a holonomic module

We are now going to show that our module M_α^β is a holonomic module and hence has finite length. For this we need the following definitions and results. For details see [4].

Definition 2. Let M be a left A_n-module. A family $\Gamma = \{\Gamma_i\}_{i\geq 0}$ of $\mathbb{C}$-vector spaces is a filtration of M with respect to the *Bernstein* filtration $\mathcal{B}$ of A_n if it satisfies:

- $\Gamma_0 \subset \Gamma_1 \subset \cdots \subset M$,
- $M = \cup_{i\geq 0}\Gamma_i$,
- $B_i\Gamma_j \subseteq \Gamma_{i+j}$, where B_i is the set of all operators of A_n of degree less than or equal to i and
- Γ_i is a finite dimensional vector space.

It is known that a finitely generated A_n-module M has a filtration of the above type such that $gr^\Gamma M$ is a finitely generated $gr^\mathcal{B} A_n$-module.

Definition 3. The dimension of the A_n-module M is

$$d(M) = \mathrm{dim}_{\mathrm{gr}^\mathcal{B}\mathrm{A_n}}\mathrm{gr}^\Gamma\mathrm{M}$$

for any filtration Γ such that $gr^\Gamma M$ is a finitely generated $gr^\mathcal{B} A_n$-module. Similarly the multiplicity $m(M)$ of M is the multiplicity of $gr^\Gamma M$ as $gr^\mathcal{B} A_n$-module. The A_n- module M is called *holonomic* if $d(M) = n$ or $M = 0$.

Since $gr^\mathcal{B} A_n$ is polynomial algebra on $2n$ variables, this means that the dimension $d(M)$ of M is less than or equal to $2n$. *Bernstein′s* inequality says that there is also a lower bound: $d(M) \geq n$.

Example 1. Since the dimension of $\mathbb{C}[x]$ as A_n-module is n, it is a holonomic module. The dimension of A_n as a left A_n-module is $2n$, so A_n is not a holonomic module.

Proposition 3 ([2, 4])**.** *(i) Submodules, quotients and finite sums of holonomic A_n-modules are holonomic.*
(ii) Holonomic modules are finitely generated and have finite length.

We will use the definition in the following form. The proof of the following Lemma can be found in [4].

Lemma 1. *Let M be a left A_n-module with filtration Γ with respect to the Bernstein filtration $\mathcal{B}$ of A_n. Suppose that there exist constants c_1, c_2 such that for $j \succ\succ 0$*

$$\mathrm{dim}_{\mathbb{C}}\Gamma_{\mathrm{j}} \leq \frac{\mathrm{c_1 j^n}}{\mathrm{n!}} + \mathrm{c_2 j^{n-1}}.$$

Then M is a holonomic A_n-module whose multiplicity cannot exceed c_1. In particular M is finitely generated, and has finite length.

We are now in a position to prove that M_α^β is a holonomic module.

Theorem 1. *The A_n-module M_α^β is holonomic.*

Proof. Let m be the degree of α. Set

$$\Gamma_k = \{\frac{q}{\alpha^k}\alpha^\beta : q \in \mathbb{C}[x], \mathrm{degq} \leq \mathrm{(m+1)k}\}.$$

We first check, in detail, that $\Gamma = \{\Gamma_k\}_{k \geq 0}$ is a filtration for M_α^β. Let $\frac{q}{\alpha^k}\alpha^\beta$ be an element of M_α^β, and assume that q has degree s. Then

$$\frac{q}{\alpha^k}\alpha^\beta = \frac{q\alpha^s}{\alpha^{k+s}}\alpha^\beta.$$

But $q\alpha^s$ has degree $s(m+1) \leq (m+1)(s+k)$, and hence

$$\frac{q}{\alpha^k}\alpha^\beta \in \Gamma_{s+k}.$$

It follows that M_α^β is the union of all Γ_k for $k \geq 0$.
Next suppose that $\frac{q}{\alpha^k}\alpha^\beta \in \Gamma_k$. Equivalently $\mathrm{degq} \leq \mathrm{(m+1)k}$. Multiplying by x_i increases the degree of q by 1. Thus

$$x_i\frac{q}{\alpha^k}\alpha^\beta = \frac{x_i q}{\alpha^k}\alpha^\beta = \frac{x_i q\alpha}{\alpha^{k+1}}\alpha^\beta \in \Gamma_{k+1}. \tag{2.3}$$

On the other hand

$$\partial_i(\frac{q}{\alpha^k}\alpha^\beta) = \partial_i(\frac{q}{\alpha^k})\alpha^\beta + \frac{q}{\alpha^k}\partial_i(\alpha^\beta), \tag{2.4}$$

and

$$\partial_i(\frac{q}{\alpha^k})\alpha^\beta = \frac{\alpha\partial_i(q) - kq\partial_i(\alpha)}{\alpha^{k+1}}\alpha^\beta. \tag{2.5}$$

The numerator in (2.5) has degree less than or equal to $(m+1)k + (m-1) \leq (m+1)(k+1)$, so that

$$\partial_i(\frac{q}{\alpha^k})\alpha^\beta \in \Gamma_{k+1} \tag{2.6}$$

On the other hand

$$\frac{q}{\alpha^k}\partial_i(\alpha^\beta) = \sum_{j=1}^{m}\frac{q}{\alpha^k}\beta_j\frac{\partial_i(\alpha_j)}{\alpha_j}\alpha^\beta.$$

Now consider

$$\frac{q}{\alpha^k}\beta_j\frac{\partial_i(\alpha_j)}{\alpha_j}\alpha^\beta.$$

Then

$$\beta_j\frac{q\partial_i(\alpha_j)}{\alpha^k\alpha_j}\alpha^\beta=\beta_j\frac{q\partial_i(\alpha_j)\alpha_1\ldots\widehat{\alpha_j}\ldots\alpha_m}{\alpha^{k+1}}\alpha^\beta.$$

The numerator has degree less than or equal to $(m+1)k+(m-1)\leq(m+1)(k+1)$. This implies

$$\frac{q}{\alpha^k}\beta_j\frac{\partial_i(\alpha_j)}{\alpha_j}\alpha^\beta\in\Gamma_{k+1}$$

and then

$$\sum_{j=1}^m\frac{q}{\alpha^k}\beta_j\frac{\partial_i(\alpha_j)}{\alpha_j}\alpha^\beta\in\Gamma_{k+1}.$$

Hence

$$\frac{q}{\alpha^k}\partial_i(\alpha^\beta)\in\Gamma_{k+1}\tag{2.7}$$

So (2.7) together with (2.6) means that

$$\partial_i(\frac{q}{\alpha^k}\alpha^\beta)\in\Gamma_{k+1}$$

if $\frac{q}{\alpha^k}\alpha^\beta\in\Gamma_k$. This may be summed up as: $B_1\Gamma_k\subseteq\Gamma_{k+1}$. Since $B_i=B_1^i$, we also have that $B_i\Gamma_k\subseteq\Gamma_{i+k}$.

Finally, the dimension of Γ_k cannot exceed the dimension of the vector space of polynomials of degree $(m+1)k$. This concludes the proof that $\Gamma=\{\Gamma_k\}_{k\geq0}$ is a filtration of M_α^β and shows that

$$\mathrm{dim}_{\mathbb{C}}\Gamma_{\mathrm{k}}\leq\begin{pmatrix}(m+1)k+n\\ n\end{pmatrix}.$$

Since the term of highest degree in k of this binomial number is $(m+1)^nk^n/n!$ it follows that

$$\mathrm{dim}_{\mathbb{C}}\Gamma_{\mathrm{k}}\leq\frac{(\mathrm{m}+1)^{\mathrm{n}}\mathrm{k}^{\mathrm{n}}}{\mathrm{n}!}+\mathrm{ck}^{\mathrm{n}-1}$$

for very large values of k. By Lemma 1, M_α^β must be holonomic module of multiplicity less than or equal to $(m+1)^n$, and has finite length. □

2.5 External products

In this subsection we will give the definition of external product of modules which we will use later. We will start by considering external product of algebras. For more details and some of the proofs see [4].

2.5.1 External products of algebras

Let K be a field of characterstic zero and A, B be K-algebras. The extenal product $A\widehat{\otimes}B$ is the tensor product $A\otimes_K B$ on wich we define a multiplication. For $a, a^{'}\in A$ and $b, b^{'}\in B$, let

$$(a\otimes b)(a^{'}\otimes b^{'}) = aa^{'}\otimes bb^{'}$$

It is easy to check that $A\otimes_K B$ with this product is a K-algebra Let $K[x] = k[x_1, ..., x_n]$ and $k[y] = K[y_1, ..., y_m]$ be polynomial rings. Write $K[x, y]$ for the polynomial ring on $x_1, ..., x_n, y_1, ..., y_m$. Let A_n be the Weyl algebra generated by $x_1, ..., x_n, \partial_{x_1}, ..., \partial_{x_n}$ and A_m the Weyl algebra generated by $y_1, ..., y_m, \partial_{y_1}, ..., \partial_{y_m}$. Both are subalgebras of A_{m+n}, the Weyl algebra generated by $x_1, ..., x_n, \partial_{x_1}, ..., \partial_{x_n}, y_1, ..., y_m, \partial_{y_1}, ..., \partial_{y_m}$. Then the following isomorphisms are induced by the multiplication map:

- $K[x]\widehat{\otimes}K[y]\cong K[x, y]$,
- $A_m\widehat{\otimes}A_n\cong A_{m+n}$.

2.5.2 External product of modules

LetK be a field of characterstic zero and A, B be K-algebras. Suppose that M is a left A-module and N is a left B-module. Then we may turn the K-vector space $M\otimes_K N$ into an $A\widehat{\otimes}B$-module $M\widehat{\otimes}N$. The action of $a\otimes b\in A\widehat{\otimes}B$ on $u\otimes v\in M\otimes_K N$ is given by the formula

$$(a\otimes b)(u\otimes v) = au\otimes bv.$$

Definition 4. The $A\widehat{\otimes}B$-module $M\widehat{\otimes}N$, which is defined above, is called the external product of M and N.

We have the following Lemma on the dimension and multiplicity of external product of modules. The proof can be found [4].

Lemma 2. *Let M be a finitely generated left A_m-module and N be a finitely generated left A_n-module. Then:*
(i) $M\widehat{\otimes}N$ is finitely generated $A\widehat{\otimes}B$-module
(ii) $d(M\widehat{\otimes}N) = d(M) + d(N)$
(iii) $m(M\widehat{\otimes}N)\leq m(M)m(N)$
(iv) If M is a holonomic A_m-module and N is a holonomic A_n-module, then $M\widehat{\otimes}N$ is a holonomic A_{m+n}-module.

The proof of the second part of the following Lemma can be found [8].

Lemma 3. *Let M be a simple A_n-module.*
(i) The set of endomorphisms $\mathrm{End}_{\mathrm{A_n}}\mathrm{M}$ *is a skew field.*
(ii) If $\phi\in End_{A_n}M$, then ϕ is algebraic over $\mathbb{C}$.
(iii) $\mathrm{Hom}_{\mathrm{A_n}}(\mathrm{M}, \mathrm{M}) = \mathbb{C}$.

Proof. The first statement is Schur's Lemma. For the second let $A = \mathbb{C}[\phi]$, the subalgebra of $\mathrm{End_{A_n}M}$ generated by 1 and ϕ. Assume that ϕ is transcendent over $\mathbb{C}$. Then A is identified with a polynomial algebra over $\mathbb{C}$ in one variable.
Let $D = A \otimes_{\mathbb{C}} A_n$. Then there is a unique structure of D-module on M such that $(a \otimes u)m = aum = uam$ for $a \in A, u \in A_n$ and $m \in M$. Choose a non-zero element $m_0 \in M$. We have $M = Dm_0$ since M is simple. Put $D_k = A \otimes B_k$ and $M_k = D_k m_0$, where $\{B_k\}$ is the Bernstein filtration of A_n. The vector space $grM = \oplus M_{k+1}/M_k$ is a finitely generated (cyclic) module over the graded algebra grD and grD is finitely generated over A. Hence by [11, Theorem 24.1] there exists $f \in A - \{0\}$ such that $grM \otimes_A A_f$ is free over A_f. Since A_f is principal ring, every $(M_k/M_{k-1}) \otimes_A A_f$ is free over A_f. Hence $M \otimes_A A_f$ is a successive extension of free A_f-modules and hence free over A_f.
Now let $a \in A - \{0\}$ be an element that does not divide any powers of f. Then the induced multiplication map $A_f \longrightarrow A_f, (b \mapsto ab)$, is not surjective. Using that M is free it follows that the induced mapping $\eta : M \otimes_A A_f \longrightarrow M \otimes_A A_f$ is not surjective. We have $\eta(m \otimes b) = m \otimes ab = am \otimes b$ for $m \in M$ and $b \in A_f$. Since M is simple the mapping $m \mapsto am$ of m is bijective and we reach a contradiction. Hence ϕ is algebraic. Since $\mathbb{C}$ is algebraically closed this implies (iii). □

The following proposition will be one of our main tools.

Proposition 4. *Let M be an irreducible A_n-module and N be an irreducible A_m-module. Then $M\widehat{\otimes}N$ is an irreducible A_{m+n}-module.*

Proof. Clearly $M \otimes_{\mathbb{C}} N = \{\sum_{i=1}^{k} a_i m_i \otimes n_i : m_i \in M, n_i \in N, a_i \in \mathbb{C}\}$. Now, let $f \in M\widehat{\otimes}N$ and $f \neq 0$. Then we want to show that $A_{m+n}f = M\widehat{\otimes}N$. We will prove this in two steps.

Step I

Let $f = m_0 \otimes n_0, m_0 \in M \setminus \{0\}, n_0 \in N \setminus \{0\}$. We know that $A_n m_0 = M$ and $A_m n_0 = N$ and if $m_1 \otimes n_1 \in M \otimes N$, then $m_1 = am_0$ and $n_1 = bn_0$ for some $a \in A_n, b \in A_m$. This implies that

$$m_1 \otimes n_1 = (am_0) \otimes (bn_0) = ab(m_0 \otimes n_0) \in A_{m+n}f.$$

If $g = \sum\limits_{i=1}^{k} m_i \otimes n_i \in M \otimes N$, then $m_i = a_i m_0$ and $n_i = b_i n_0, a_i \in A_n$ and $b_i \in A_m$ and hence

$$g = \sum_{i=1}^{k} a_i m_0 \otimes b_i n_0 = \sum_{i=1}^{k} a_i b_i (m_0 \otimes n_0) = (\sum_{i=1}^{k} a_i b_i) m_0 \otimes n_0 = cm_0 \otimes n_0,$$

where $c = \sum_{i=1}^{k} a_i b_i \in A_{m+n}$. This implies that $g \in A_{m+n}f$. Therefore $A_{m+n}f = M\widehat{\otimes}N$.

Step II

Let $f = \sum_{i=0}^{k} m_i \otimes n_i, m_i \in M, n_i \in N$. We will proceed by induction on k. We already proved the result for $k = 1$ in the preceding step. First we will consider the case $k = 2$. Suppose $f = m_0 \otimes n_0 + m_1 \otimes n_1$, where $m_0 \otimes n_0 \neq 0$, and $m_1 \otimes n_1 \neq 0$. We know that $(a \otimes 1)f = am_0 \otimes n_0 + am_1 \otimes n_1$. Suppose $am_0 = 0$ and $am_1 \neq 0$. Then $(a \otimes 1)f = am_1 \otimes n_1 \neq 0$. By the first case, $am_1 \otimes n_1$ generates $M \otimes N$, and hence $A_{m+n}f = M\widehat{\otimes}N$. So we should check if there are elements $a \in A_n$ such that $am_0 = 0$ but $am_1 \neq 0$.
Let us answer the question, *do we have* $a \in A_n - \{0\}$ *and* $am_0 = 0$ *and* $am_1 \neq 0$*?*

Lemma 4. *If M is an irreducible A_n-module and $m \in M, m \neq 0$, then* $\mathrm{Ann(m)} \neq 0$.

Proof. Consider the map

$$\phi : A_n \longrightarrow M$$

defined by $\phi(a) = am \in M$. Since M is irreducible and $m \neq 0$, ϕ is a surjective map. If $\mathrm{Ker}\phi = \{a \in A_n : am = 0\} = 0$, then $A_n \cong M$ and hence A_n is irreducible which is a contradiction. Hence $\mathrm{Ann(m)} = \mathrm{Ker}\phi \neq 0$. □

Let us continue the proof of Proposition 4 step II. Let $J_0 = \mathrm{Ann(m_0)}$ and $J_1 = \mathrm{Ann(m_1)}$. If $J_1 \subsetneq J_0$, then we can apply the argument above. If $J_0 \subsetneq J_1$, then we can apply the first case, because we have $a \in A_n$ such that $am_0 = 0$ and $am_1 \neq 0$ and hence

$$am_0 \otimes n_0 + am_1 \otimes n_1 = am_1 \otimes n_1 \neq 0$$

and $A_{m+n}(am_1 \otimes n_1) = A_{m+n}f = M\widehat{\otimes}N$. So $J_0 = J_1$ is the only case which the argument does not work. So suppose this is the case. Then consider the isomorphisms

$$\phi_0 : A_n/J_0 \longrightarrow M$$

$$a + J_0 \longmapsto am_0$$

and

$$\phi_1 : A_n/J_1 \longrightarrow M$$

$$a + J_1 \longmapsto am_1.$$

We have $M \overbrace{\overset{\phi_0^{-1}}{\to} A_n/J_0(=J_1) \overset{\phi_1}{\to}}^{\eta} M$. That is $\eta = \phi_1 o \phi_0^{-1}$. Then by Lemma 3, $\eta(m) = \alpha m$ for some $\alpha \in \mathbb{C}$. This implies $\eta(m_0) = \alpha m_0 = m_1$ and hence

$$f = m_0 \otimes n_0 + m_1 \otimes n_1 = m_0 \otimes n_0 + \alpha m_0 \otimes n_1 = m_0 \otimes n_0 + m_0 \otimes \alpha n_1 = m_0 \otimes (n_0 + \alpha n_1) = m_0 \otimes n_2,$$

where $n_2 = n_0 \otimes \alpha n_1$. This implies $f = m_0 \otimes n_2$ and then by the first case above, $A_{m+n} f = M \otimes N$. The case $k > 2$ is treated in the same way. By the above argument, if $f = \sum_{i=0}^{k} m_i \otimes n_i$, either there exists $a \otimes 1$ such that $0 \neq (a \otimes 1) f = \sum_{i=0}^{k-1} a m_i \otimes n_i$ and hence by induction f generates $M \widehat{\otimes} N$, or we use Lemma 4 in the same way as above to see that $f = \sum_{i=0}^{k-1} \tilde{m}_i \otimes \tilde{n}_i$ and again by induction generates $M \widehat{\otimes} N$. ☐

Proposition 5. *Let M be an A_n-module with a composition series*

$$0 = M_0 \subset M_1 \subset \ldots M_r = M$$

and N be an irreducible A_m-module. Then

$$0 = M_0 \widehat{\otimes} N \subset M_1 \widehat{\otimes} N \subset \ldots \subset M_r \widehat{\otimes} N = M \widehat{\otimes} N$$

is a composition series of $M \widehat{\otimes} N$.

Proof. It suffices to note that $M_i \widehat{\otimes} N / M_{i-1} \widehat{\otimes} N \cong M_i / M_{i-1} \widehat{\otimes} N$ is irreducible by Proposition 4. ☐

2.6 Decomposition factors of modules

Let R be a ring and M be an R-module. If $0 = M_0 \subset M_1 \subset \ldots M_r = M$ is a composition series of M, then the set

$$\mathrm{DF(M) := \{M_i/M_{i-1}\}_{i=1}^{r}}$$

of simple R-modules is the set of decomposition facors of M.
We have the following Proposition on the decomposition factors of R-modules.

Proposition 6. *Let M be an R-module .*
(i) Let N be a submodule of M. Consider the exact sequence of R-modules $N \subset M \overset{\phi}{\to} M/N$. Then,
(a) $\mathrm{DF(M) = DF(N) \cup DF(M/N)}$ *and*
(b) $c(M) = c(N) + c(M/N)$.
(ii) If $M = M_k \supset M_{k-1} \supset \ldots \supset M_0$ is a sequence of R-modules, then

$$\mathrm{DF(M) = \bigcup_{i=1}^{k} DF(M_i/M_{i-1})}.$$

Proof. Once we have proved (i), (ii) can easily be proved by induction on k. To prove (i) consider

$$M \xrightarrow{\phi} M/N = F_k \supset F_{k-1} \supset ... \supset F_1 \supset F_0 = 0$$

Then $F_j = M_j/N$, where $M \supset M_j = \phi^{-1}(F_j)$ for $j = 0, 1, ..., k$. But

$$M_j/M_{j-1} = \phi^{-1}(F_j)/\phi^{-1}(F_{j-1}) \cong M_j/N/M_{j-1}/N = F_j/F_{j-1}.$$

Hence if F_j/F_{j-1} are irreducible, then M_j/M_{j-1} also are irreducible. Suppose $0 = N_0 \subset N_1 \subset ... \subset N_s = N$ is a composition series of N. Then

$$N_0 \subset N_1 \subset ... \subset N_s = N = M_0 \subset M_1 \subset ... \subset M_k = M$$

is a composition series of M. Therefore

$$\{N_j/N_{j-1}\}_{j=0}^{s} \cup \{F_i/F_{i-1} \cong M_i/M_{i-1}\}_{i=0}^{k}$$

is the set of decomposition factors of M. □

Corollary 1. *Let* $0 = M_0 \subset M_1 \subset ... \subset M_k = M$ *be a composition series of an* A_n *module* M *and* $0 = N_0 \subset N_1 \subset ... \subset N_l = N$ *be a composition series of an* A_m*-module* N*. Then*

$$\mathrm{DF}(\mathrm{M}\widehat{\otimes}\mathrm{N}) = \{\mathrm{M_i/M_{i-1}}\widehat{\otimes}\mathrm{N_j/N_{j-1}}\}_{\mathrm{i=1,j=1}}^{\mathrm{k,l}}$$

and hence $c(M\widehat{\otimes}N) = c(M)c(N)$.

Proof. It is an easy consquence of Proposition 6. □

3 The module M_α^β, where $\beta \in \mathbb{Z}^m$

By Proposition 2, in the case where $\beta \in \mathbb{Z}^m$, $M_\alpha^\beta \cong \mathbb{C}[x]_\alpha$. Our aim in this section is to find the number of decomposition factors of $\mathbb{C}[x]_\alpha$. This will turn out to be equivalent to analyzing expressions in partial fractions for functions in $\mathbb{C}[x]_\alpha$. Let us proceed in the following way.

- To every subset

$$S = \{\alpha_{i_1}, \dots, \alpha_{i_d}\} \subset \triangle = \{\alpha_1, \dots, \alpha_m\}$$

  that consists of linearly independent forms, choose coordinates $z_{d+1}, \dots, z_n$ such that $\alpha_{i_1}, \dots, \alpha_{i_d}, z_{d+1}, \dots, z_n$ are linear coordinates in space.

- In order to simplify the notations let us denote $\alpha_{i_k} = z_k, k = 1, 2, ..., d$.

- Let $A_S = \mathbb{C}[z_{d+1}, \dots, z_n]$ be the corresponding ring of polynomials.

- Define $R_S = \{h \in \mathbb{C}[x]_\alpha : h = \frac{g}{\prod_{j=1}^d z_j^{m_j}}; g \in A_S, m_j > 0, \forall j\}$.
- We will just use these modules for certain subsets S called no-broken circuits defined below.

Consider the following sequence of A_n-modules

$$0 \subset R_0 (= \mathbb{C}[x]) \subset R_1 \subset \cdots \subset R_r = \mathbb{C}[x]_\alpha,$$

where $r \leq n$ and R_k is the subspace of $\mathbb{C}[x]_\alpha$ which is generated by monomials in $x_1, ..., x_n, \alpha_1^{-1}, ..., \alpha_m^{-1}$ such that at most k of $\alpha_1, ..., \alpha_m$ have strictly negative exponents. Clearly R_k is an A_n-submodule of $\mathbb{C}[x]_\alpha$. The main theorem in this section is the following.

Theorem 2.

$$R_k / R_{k-1} = \oplus_W \oplus_S R_S$$

where W runs over the subspaces of dimension k generated by elements of $\triangle$ and S runs over certain subsets of k elements of $\triangle$ (the so called no-broken circuits, see definition below) which generate W.

The proof of Theorem 2 can be found in [5], whose exposition we follow. We will indicate some parts of it below.

3.1 Basic Lemma

Lemma 5 ([5]). *Let $\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_{k+1}$ be non-zero linear forms with $\alpha_1 = \sum_{j=2}^{k+1} c_j \alpha_j$. Then we have*

$$\frac{1}{\prod_{j=1}^{k+1} \alpha_j} = \sum_{j=2}^{k+1} c_j \frac{1}{\alpha_1^2 \prod_1^{j-1} \alpha_i \prod_{i=j+1}^{k+1} \alpha_i}$$

Proof.

$$\frac{1}{\prod_{j=1}^{k+1} \alpha_j} = \frac{\alpha_1}{\alpha_1^2 \prod_{j=2}^{k+1} \alpha_j} = \sum_{j=2}^{k+1} c_j \frac{\alpha_j}{\alpha_1^2 \prod_{j=2}^{k+1} \alpha_j}.$$

□

Given non-zero linear forms $\alpha_1, \alpha_2, \ldots, \alpha_m$, let d be the dimension of the vector space they generate.

Proposition 7 ([5]). *Every expression $\frac{1}{\prod_{j=1}^{m} \alpha_i^{h_j}}$ can be expressed as linear combinations of expressions $\frac{1}{\prod_{j=1}^{d} \alpha_{i_j}^{m_j}}$ with $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_d}$ linearly independent and $\sum_{j=1}^{d} m_j = \sum_{i=1}^{m} h_i$.*

Proof. Let us apply reduction and an induction on the vector of exponents $(h_1, \dots, h_m)$ in the following way.

- Using the given ordering we can take the first linearly dependent elements that appear in the product with non-zero exponents.
- Using Lemma 5 we can substitute the product of these terms with a sum in which developing the vector of exponents is increased in the lexicographical order maintaining the same sum.
- In each term the space generated by the factors remains the same.
- Clearly this recursive procedure terminates after a finite number of steps, when all the summands are of the required type.

$\square$

3.2 No-broken circuits

We will systemize the procedure in the proof of the preceding proposition.

Definition 5. Let $\alpha_1, \dots, \alpha_m$ be non-zero linear forms. Let $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_h}$, $i_1 < i_2 < ... < i_h$ be an ordered sublist of linearly independent elements. We say that the sublist is a broken circuit if there exists an integer $k \leq h$ and an integer $i < i_k$ such that the elements $\alpha_i, \alpha_{i_k}, \dots, \alpha_{i_h}$ are linearly dependent, otherwise it is called no-broken circuit.

Lemma 6. *If $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_h}$ is a broken circuit, then $\frac{1}{\prod\limits_{j=1}^{h} \alpha_{i_j}}$ is a linear combination of expressions $\frac{1}{\prod\limits_{j=1}^{m} \alpha_j^{h_j}}$ with the vector of exponents lexicographically bigger than the vector of exponents of $\frac{1}{\prod\limits_{j=1}^{h} \alpha_{i_j}}$.*

Proof. From the given hypothesis we have $\alpha_i = \sum\limits_{j=k}^{h} c_j \alpha_{i_j}$, with $i < i_k$. Let us substitute and simplify:

$$\frac{1}{\prod_{j=1}^{h} \alpha_{i_j}} = \frac{\alpha_i}{\alpha_i \prod_{j=1}^{h} \alpha_{i_j}} = \frac{c_k \alpha_{i_k} + \cdots + c_h \alpha_{i_h}}{\alpha_i \prod_{j=1}^{h} \alpha_{i_j}}$$

Simplifying every term in the numerator with the corresponding factor in the denominator we get the desired expressions. $\square$

Theorem 3. *Every expession* $\frac{1}{\prod\limits_{j=1}^{m} \alpha_j^{h_j}}$ *can be expressed as a linear combination of expressions* $\frac{1}{\prod\limits_{j=1}^{d} \alpha_{i_j}^{m_j}}$, *with* $\alpha_{i_1}, \ldots, \alpha_{i_d}$ *a no-broken circuit and* $\sum\limits_{j=1}^{d} m_j = \sum_{i=1}^{m} h_i$.

Proof. The fact that an expression of the given type can be written as a linear combination of expressions relative to no-broken circuits can be proved by induction on the lexicographic order of the vector exponents as in Proposition 7 and repeatedly using Lemma 6. □

Corollary 2. *The space* R_S *has basis the monomials* $\prod\limits_{i=1}^{n} z_i^{h_i}$ *such that* $h_i \geq 0$ $\forall i > d$, $h_i < 0$ $\forall i \leq d$ *and* $\mathbb{C}[x]_\alpha = \sum_S R_S$ *as* S *varies among the no-broken circuits.*

Proof.

- The elements $z_1, z_2, \ldots, z_n$ are linear coordinates in space and R_S is contained in the ring of Laurent polynomials in these variables. These polynomials have as basis all the monomials in the variables with integer exponents. The proposed monomials are thus part of these basis and so linearly independent.
- From Theorem 3 it follows immediately that every function f in R can be written as a linear combination of expessions
$$f = \frac{g}{\prod_{j=1}^{d} \alpha_{i_j}^{m_j}}$$
such that $g \in \mathbb{C}[x]$, $m_j > 0$, $\forall j$ and $S = \alpha_{i_1}, \ldots, \alpha_{i_d}$ a no-broken circuit.
- We write f as a polynomial in the variables $\alpha_{i_1}, \ldots, \alpha_{i_d}, z_{d+1}, \ldots, z_n$. Simplify the α_i that appear in the numerator and the denominator. Thus with as easy induction we can prove that every element in R is a sum of elements of the spaces R_S.

□

Corollary 3. *The number of decomposition factors of* $\mathbb{C}[x]_\alpha$ *equals the number of no-broken circuits.*

3.3 The plane case

Consider the A_2-module $M_\alpha^\beta = \mathbb{C}[x,y]_\alpha \alpha^\beta$, where $\alpha^\beta = x^{\beta_1} y^{\beta_2} (x + c_3 y)^{\beta_3} ... (x + c_m y)^{\beta_m}$. If $\beta_1, ..., \beta_m \in \mathbb{Z}$, then by Proposition 2, $M_\alpha^\beta \cong \mathbb{C}[x,y]_{xy \prod_{i=3}^{m} (x + c_i y)}$. We have the following sequence of A_2-modules
$$0 \to R_0 (= \mathbb{C}[x,y]) \subset R_1 \subset R_2 = M_\alpha^\beta,$$

where R_1 is the subspace of $\mathbb{C}[x,y]_{xy\prod_{i=3}^{m}(x+c_iy)}$ which is generated by the monomials $x^{\pm 1}y^{\pm 1}, (x+c_3y)^{-1}, ..., (x+c_my)^{-1}$ such that atmost one of $x, y, x+c_3y, ..., x+c_my$ has stricly negative exponent. Then

$$R_1/R_0 = \oplus_{j=1}^{m} R_{S_j},$$

where R_{S_1} and R_{S_2} isomorphic to the submodules generated by $e_{S_1} = \frac{1}{x}$ and $e_{S_2} = \frac{1}{y}$ modulo R_0 respectively and R_{S_j} is isomorphic to the submodule generated by $e_{S_j} = \frac{1}{z_i}$ modulo R_0, where $z_i = x + c_iy, i = 3, ..., m$ and each $R_{S_j}, j = 1, ..., m$ is irreducible. On the other hand

$$R_2/R_1 = \oplus_{i=2}^{m} R_{S_i},$$

where R_{S_2} is is isomorphic to the submodule generated by $e_{S_2} = \frac{1}{xy}$ modulo R_1, R_{S_i} for $i = 3, ..., m$ is is isomorphic to the submodule generated by $e_{S_i} = \frac{1}{xz_i}$ modulo R_1, where $z_i = x + c_iy, i = 3, ..., m$ and each R_{S_i} is irreducible. Hence $c(R_2/R_1) = m-1$, $c(R_1/R_0) = m$ and $c(R_0) = 1$. We know that

$$\mathrm{DF}(\mathrm{M}_\alpha^\beta) = \mathrm{DF}(\mathrm{R}_0) \cup \mathrm{DF}(\mathrm{R}_1/\mathrm{R}_0) \cup \mathrm{DF}(\mathrm{R}_2/\mathrm{R}_1),$$

and

$$c(M_\alpha^\beta) = c(R_0) + c(R_1/R_0) + c(R_2/R_1).$$

Therefore $c(M_\alpha^\beta) = 2m$.

Remark 1. Observe that the set of no-broken circuits of the set $\{x, y, x+c_3y, ..., x+c_my\}$ of the linear forms is

$$\{\emptyset, \{x\}, \{y\}, \{x+c_3y\}, ..., \{x+c_my\}, \{x,y\}, \{x, x+c_3y\}, ..., \{x, x+c_my\}\}.$$

4 On the support of modules

Let X be a smooth affine algebraic variety. (X will be $\mathbb{C}^n$ or an open subset of $\mathbb{C}^n$ which is the complement of a union of hyperplanes defined by forms). We denote by D_X the ring of differential operators on X and if $X = \mathbb{C}^n$ this is the same as A_n. If X is an affine open subset of $\mathbb{C}^n$ defined by $0 \neq f \in \mathbb{C}[x]$, then $D_X = \mathbb{C}[x]_f \otimes_{\mathbb{C}[x]} A_n$. We will use the notation $O_X = \mathbb{C}[x]_f$ in this case.
If M is a D_X-module then it can be viewed as an O_X-module and hence has an annihilator, $Ann_{O_X}M$.

Definition 6. $V(Ann_{O_X}M)$ is called the support of M, and is denoted by SuppM. (With $V(I)$ for an ideal $I \subset O_X$ means the closed subvariety of zeroes defined by I.)

We have the following examples.

- For $M_1 = \mathbb{C}[x,y]_{xy}/(\mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y)$,
  $\mathrm{Supp} M_1 = V(x,y) = (0,0)$.
- For $M_2 = \mathbb{C}[x,y]_x/\mathbb{C}[x,y]$,
  $\mathrm{Supp} M_2 = V(x) = \{(0,y) : y \in \mathbb{C}\}$.
- $M_3 = \mathbb{C}[x,y]$,
  $\mathrm{Supp} M_3 = V(0) = \mathbb{C}^2$.

4.1 Basic properties

Proposition 8. *If M is an irreducible D_X-module and $U \subset X$ an affine open subset, then $M_{|U} =: O_U \otimes_{O_X} M$ is an irreducible D_U-module.*
If N is a D_X-module, then $c(N_{|U}) \leq c(N)$.

Proof. Suppose $U = X - V(s)$ and $0 \neq f, g \in M_{|U}$. Then $f = \frac{f'}{s^j}$, $g = \frac{g'}{s^k}$, $f', g' \in M$. By the assumption that M is irreducible, there exists $P \in D_X$ such that $Pf' = g'$. This implies $(s^{-k}Ps^j)(\frac{f'}{s^j}) = \frac{g'}{s^k}$. This gives the result since clearly $s^{-k}Ps^j \in D_U$. □

Proposition 9. *Consider M_α^β and a decomposition factor M_i. It has support on an intersection of hyperplanes H_S for some $S \subset \{1, 2, ..., m\}$.*

$$H_S = \{p \in \mathbb{C}^n : \alpha_i(p) = 0, i \in S\}.$$

The proof of the proposition will be given below after some preliminaries.

Definition 7. Suppose that θ is an automorphism of D_X. If M is a D_X-module, $\theta^* M$ is defined to be the D_X-module which consists of the same elements as M, but on which D_X acts by θ: if $P \in D_X$, $m \in \theta^* M$, then $Pm = \theta(P)m$.

The following Lemma is clear.

Lemma 7. *If $\theta : D_X \longrightarrow D_X$ is an automorphism such that it is the identity on O_X and M has decomposition factors M_i, i=1,...,l, then $\theta^* M$ has decomposition factors $\theta^* M_i$, $i = 1, ..., l$. In particular $c(M) = c(\theta^* M)$. The support of $\theta^* M_i$ equals the support of M_i.*

We will apply this Lemma to the following Proposition.

Proposition 10. *Suppose that $U = X - V(\alpha_1, ..., \alpha_l)$. Then $c(M_{\alpha\,|U}^\beta) = c(M_{\alpha'\,|U}^{\beta'})$, where $\alpha' = \alpha_{l+1}...\alpha_m$ and $\beta' = (\beta_{l+1}, ..., \beta_m)$.*

Proof. It is enough to assume by induction that $U = X - V(\alpha_1)$. Put $\alpha^\beta = \alpha_1^{\beta_1}\tilde{\alpha}^{\tilde{\beta}}$, where $\tilde{\alpha} = \alpha_2...\alpha_m$ and $\tilde{\beta} = (\beta_2, ..., \beta_m)$. Then $M_{\alpha\,|U}^\beta = \mathbb{C}[x]_\alpha \alpha_1^{\beta_1}\tilde{\alpha}^{\tilde{\beta}}$ and the point is that α_1 is invertible here. Now define $\theta : D_U \longrightarrow D_U$ in the following way.

- If $D \in Der_{\mathbb{C}}(\mathbb{C}[x]) \subset D_U$, $\theta(D) = D + \frac{\beta_1 D(\alpha_1)}{\alpha_1}$.
- If $r \in O_U$, then $\theta(r) = r$.

Extending this inductively gives an automorphism of D_U, since it has the inverse $\theta^{-1} : D \mapsto D - \frac{\beta_1 D(\alpha_1)}{\alpha_1}$. We claim that the map

$$\rho : \theta^* M^{\tilde{\beta}}_{\tilde{\alpha}\,|U} \longrightarrow M^{\beta}_{\alpha\,|U}$$

defined by $\rho : r\tilde{\alpha}^{\tilde{\beta}} \mapsto r\alpha^{\beta}$ is a D_U-isomorphism. It suffices to check that $\rho(D(r\tilde{\alpha}^{\tilde{\beta}})) = D(\rho(r\tilde{\alpha}^{\tilde{\beta}})$, i.e. $\rho(\theta(D)(r\tilde{\alpha}^{\tilde{\beta}})) = D(r\alpha^{\beta})$, if $D \in Der_{\mathbb{C}}(\mathbb{C}[x])$ and $r \in \mathbb{C}[x]_{\alpha}$. But

$$\theta(D)r\tilde{\alpha}^{\tilde{\beta}} = (D + \frac{\beta_1 D(\alpha_1)}{\alpha_1})r\tilde{\alpha}^{\tilde{\beta}} = (\sum_{i=1}^{m} \frac{\beta_i D(\alpha_i)}{\alpha_i})r\tilde{\alpha}^{\tilde{\beta}} + D(r)\tilde{\alpha}^{\tilde{\beta}}.$$

Since $D(r\alpha^{\beta}) = (\sum_{i=1}^{m} \frac{\beta_i D(\alpha_i)}{\alpha_i})r\alpha^{\beta} + D(r)\alpha^{\beta}$, the statement is clear. Hence the proposition is clear by the preceeding lemma. □

Lemma 8. *Let $U \subset X$ be an affine open subset.*
(i) $\mathrm{SuppM}_{|\mathrm{U}} = \mathrm{U} \cap \mathrm{SuppM}$.
(ii)$M_{|U} = 0 \Leftrightarrow \mathrm{SuppM} \subset \mathrm{X} - \mathrm{U} =: \mathrm{Z}$.
(iii)If M is irreducible, then SuppM is irreducible as a variety.

Proof. (i) is clear by definition. Let I be the ideal of Z. For any O_X-module M there exists an exact sequence

$$\Gamma_{\mathrm{Z}} M \subset M \longrightarrow M_{|U},$$

where $\Gamma_{\mathrm{Z}} M = \{m \in M : \exists r, I^r m = 0\}$. If $M_{|U} = 0$, then $\Gamma_{\mathrm{Z}} M = M$ and this proves (ii) in one direction. The other direction is a consquence of (i) and the fact that the only module with $\mathrm{SuppM} = \emptyset$ is the the zero module. The proof of (iii) may be found in [2]. □

Corollary 4. *(i)* $\mathrm{DF}(\mathrm{M}_{|\mathrm{U}}) = \{\mathrm{M_i} \in \mathrm{DF(M)} : \mathrm{SuppM_i} \cap \mathrm{U} \neq \emptyset\}$.
(ii) $c(M_{|U}) \leq c(M)$.

4.2 Proof of Proposition 9

We are going to prove the proposition in a more general setting, by letting X be possibly the complement of a union of hyperplanes $V(\alpha_i)$ $i = 1, ..., m$. So the statement to be proved is the following: consider M^{β}_{α} as a D_X-module, the the support of a decomposition factor is an intersection of some of the hyperplanes $V(\alpha_i)$ $i = 1, ..., m$. Make induction on the number of $\alpha_i, i = 1, ..., m$ that are not invertible in O_X. Suppose these are $\alpha_{l+1}, ..., \alpha_m$, and the ones which are invertible are $\alpha_1, ..., \alpha_l$. Then as in the proof of Proposition 10,

$$M_\alpha^\beta \cong \theta^* M_{\alpha'}^{\beta'} \tag{4.1}$$

where ${\alpha'}^{\beta'} = \alpha_{l+1}^{\beta_{l+1}} ... \alpha_m^{\beta_m}$, as D_X-modules. θ being the identity on functions preserves support, so it suffices to prove the proposition for $M_{\alpha'}^{\beta'}$. If the number of non-invertible α_i is zero, then $m = l$, $M_{\alpha'}^{\beta'} \cong O_X$ and we are done by Section 3 and Lemma 8 (i). Now for the induction step, assume that the statement is known for $m - l = p$. Assume first that $X = \mathbb{C}^n$, and

$$M_\alpha^\beta = \mathbb{C}[x, \alpha_1^{-1}, ..., \alpha_m^{-1}]{\alpha'}^{\beta'}.$$

Let N be a decomposition factor of M_α^β with SuppN = Z. Assume first that Z is contained in all the hyperplanes $V(\alpha_i)$, j=1,..,m. (They do not have to intersect in the origin.) Then $Z \subset \cap_{j=1}^m V(\alpha_j) =: H$. If H is the origin we are done. Otherwise choose a decomposition $\mathbb{C}^n \cong \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ with coordinates $\tilde{x_1}, ..., \tilde{x_{n_1}}, \tilde{y_1}, ..., \tilde{y_{n_2}}$ such that $\alpha_i(x) = \sum_{j=1}^{n_1} \alpha_i^j \tilde{x_j}$. This is always possible, letting $\mathbb{C}^{n_2} = H$ and $\mathbb{C}^{n_1}$ a complement. Then $M_\alpha^\beta \cong \mathbb{C}[\tilde{y}]\widehat{\otimes}\mathbb{C}[\tilde{x}]_{\alpha'}^{\beta'}$. All the decomposition factors of this module have the form $\mathbb{C}[\tilde{y}]\widehat{\otimes}\tilde{N}$, (see Section 2) where $\tilde{N}$ is a decomposition factor of $\mathbb{C}[\tilde{x}]_{\alpha'}^{\beta'}$. Since $\mathrm{Supp}(\mathbb{C}[\tilde{y}]\widehat{\otimes}\tilde{N}) = \mathbb{C}^{n_1} \times \mathrm{Supp}\tilde{N}$, we are reduced to proving the proposition for $\mathbb{C}[\tilde{x}]_{\alpha'}^{\beta'}$. This means that we may assume WLOG that $\cap_{i=1}^j V(\alpha_i)$ is the origin. In that case there is some hyperplane, $V(\alpha_1)$ say, which does not contain $Z = \mathrm{SuppN}$. Then $N_{|U_1}$, where $U_1 = X - V(\alpha_1)$ is a non-trivial decomposition factor of $M_{\alpha}^\beta{}_{|U_1}$ with support $U_1 \cap Z \neq 0$. Hence, since α_1 is invertible on U_1, by induction $U_1 \cap Z$ is an intersection $H_S \cap U_1$ of hyperplanes.
Since Z is irreducible, $Z = \overline{U_1 \cap Z} = H_S$, so the result follows.
It remains to see that the inductive hypothesis is true for an arbitrary X that is a complement of a union of hyperplane sections. By the procedure of (4.1) we may assume that $M_\alpha^\beta = (O_X)_\alpha \alpha^\beta$, $\alpha_1, ..., \alpha_m$, for some $m \leq p + 1$ are not invertible. Hence,$(O_X)_\alpha \alpha^\beta = (O_{\mathbb{C}^n})_\alpha \alpha^\beta$, and the proposition follows from Lemma 8 (i) and the preceding discussion for the case $\mathbb{C}^n$, since $(O_X)_\alpha \alpha^\beta = \mathbb{C}[x, \alpha^{-1}]\alpha_{|X}^\beta$.

5 Normal Crossings

In this section, we restrict ourselves to the normal crossings, that is, all the linear forms are some of the coordinate axes. Any module M_α^β where $\alpha_1, ..., \alpha_m$ are linearly independent on $\mathbb{C}^n$ and $m \leq n$ is isomorphic to such a module by change of coordinates. Let $\alpha_1 = x_1, ..., \alpha_m = x_m$, $m \leq n$. Then $M_\alpha^\beta = \mathbb{C}[x_1, ..., x_n]_{x_1..x_m}\alpha^\beta$, where $\alpha^\beta = x_1^{\beta_1}...x_m^{\beta_m}$, $\beta_1, ..., \beta_m \in \mathbb{C}$. Recall that, in section 2 of this paper, we considered the case $m = n = 1$. We are going to start this section by considering the case $m = n = 2$ and then at last we will treat the general case.

5.1 The module M_α^β, where $n = m = 2$

Clearly this is the module $M_\alpha^\beta = \mathbb{C}[x, y]_{xy}x^{\beta_1}y^{\beta_2}$. Then the multiplication map induces the following isomorphism, $\mathbb{C}[x, y]_{xy}x^{\beta_1}y^{\beta_2} \cong \mathbb{C}[x]_x x^{\beta_1}\widehat{\otimes}\mathbb{C}[y]_y y^{\beta_2}$.

Theorem 4. *(i) If $\beta_1, \beta_2 \in \mathbb{C} \setminus \mathbb{Z}$, then M_α^β is an irreducible A_2-module.*
(ii) If $\beta_1, \beta_2 \in \mathbb{Z}$, then $c(\mathbb{C}[x, y]_{xy}) = 4$.
(iii) If $\beta_1 \in \mathbb{Z}$ and $\beta_2 \in \mathbb{C} \setminus \mathbb{Z}$, then $c(M_\alpha^\beta) = 2$.

Proof. (i) By Proposition 1, if $\beta_1, \beta_2 \in \mathbb{C} \setminus \mathbb{Z}$, then $\mathbb{C}[x]_x x^{\beta_1}$ is irreducible $\mathbb{C} < x, \partial_x >$-module and $\mathbb{C}[y]_y y^{\beta^2}$ is an irreducible $\mathbb{C} < y, \partial_y >$-module. Hence by Proposition 4, $\mathbb{C}[x]_x x^{\beta_1}\widehat{\otimes}\mathbb{C}[y]_y y^{\beta_2} \cong M_\alpha^\beta$ is irreducible A_2-module.

(ii) By Proposition 2, if $\beta_1, \beta_2 \in \mathbb{Z}$, then $M_\alpha^\beta \cong \mathbb{C}[x, y]_{xy}$ and $\mathbb{C}[x, y]_{xy} \cong \mathbb{C}[x]_x\widehat{\otimes}\mathbb{C}[y]_y$. By Proposition 1, $c(\mathbb{C}[x]_x) = 2$ and $c(\mathbb{C}[y]_y) = 2$. Therefore $c(M_\alpha^\beta) = c(\mathbb{C}[x]_x).c(\mathbb{C}[y]_y) = 2(2) = 4$, this proves (ii).

(iii) If $\beta_1 \in \mathbb{Z}$ and $\beta_2 \in \mathbb{C}\setminus\mathbb{Z}$, then $\mathbb{C}[x, y]_{xy}x^{\beta_1}y^{\beta_2} \cong \mathbb{C}[x]_x x^{\beta_1}\widehat{\otimes}\mathbb{C}[y]_y y^{\beta_2}$ and $\mathbb{C}[x]_x x^{\beta_1} \cong \mathbb{C}[x]_x$. So we have $M_\alpha^\beta \cong \mathbb{C}[x]_x\widehat{\otimes}\mathbb{C}[y]_y y^{\beta_2}$. But $c(\mathbb{C}[x]_x) = 2$ and $\mathbb{C}[y]_y y^{\beta_2}$ is an irreducible $\mathbb{C} < y, \partial_y >$-module. Therefore $c(M_\alpha^\beta) = c(\mathbb{C}[x]_x)c(\mathbb{C}[y]_y y^{\beta_2}) = 2(1) = 2$. This completes the proof. □

5.2 The general case, $m \leq n$

The module is $M_\alpha^\beta = \mathbb{C}[x]_{x_1...x_m}x_1^{\beta_1}...x_m^{\beta_m}$. We are now going to consider the module in the following cases.

- $\beta_i \in \mathbb{C} - \mathbb{Z}$ for $i = 1, ..., m$,
- $\beta_i \in \mathbb{Z}$ for $i = 1, ..., m$ and
- some of them are integers and some are not.

The following Theorem gives all the results.

Theorem 5. *Let $M_\alpha^\beta = \mathbb{C}[x]_\alpha\alpha^\beta, \alpha = x_1...x_m$ and $m \leq n$. Then:*
(i) If $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{C} \setminus \mathbb{Z}$, then M_α^β is irreducible.
(ii) If $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{Z}$, then $c(M_\alpha^\beta) = 2^m$.
(iii) Suppose that k of the $\beta_1, \beta_2, \ldots, \beta_m$ are integers and the others are elements of $\mathbb{C} \setminus \mathbb{Z}$, then $c(M_\alpha^\beta) = 2^k$.

Proof. (i) If $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{C} \setminus \mathbb{Z}$, then $M_\alpha^\beta = \mathbb{C}[x]_{x_1 \ldots x_m} x_1^{\beta_1} \ldots x_m^{\beta_m}$ and the multiplication map induces the following isomorphisms,

$$\mathbb{C}[x]_{x_1 \ldots x_m} x_1^{\beta_1} \ldots x_m^{\beta_m} \cong \mathbb{C}[x_1, \ldots, x_m]_{x_1 \ldots x_m} x_1^{\beta_1} \ldots x_m^{\beta_m} \widehat{\otimes} \mathbb{C}[x_{m+1}, \ldots, x_n]$$

and

$$\mathbb{C}[x_1, \ldots, x_m]_{x_1 \ldots x_m} x_1^{\beta_1} \ldots x_m^{\beta_m} \cong \mathbb{C}[x_1]_{x_1} x_1^{\beta_1} \widehat{\otimes} \ldots \widehat{\otimes} \mathbb{C}[x_m]_{x_m} x_m^{\beta_m}.$$

By Proposition 1, $\mathbb{C}[x_i]_{x_i} x_i^{\beta_i}$ is an irreducible $\mathbb{C} < x_i, \partial_i >$-module. So, by Proposition 4, $\mathbb{C}[x_1]_{x_1} x_1^{\beta_1} \widehat{\otimes} \ldots \widehat{\otimes} \mathbb{C}[x_m]_{x_m} x_m^{\beta_m}$ is an irreducible A_m-module. On the other hand, $\mathbb{C}[x_{m+1}, \ldots, x_n]$ is an irreducible $\mathbb{C} < x_{m+1}, \ldots, x_n, \partial_{m+1}, \ldots, \partial_n >$-module, [4, Chapter 5, Proposition 1.2]. Since

$$M_\alpha^\beta \cong \mathbb{C}[x_1, \ldots, x_m]_{x_1 \ldots x_m} x_1^{\beta_1} \ldots x_m^{\beta_m} \widehat{\otimes} \mathbb{C}[x_{m+1}, \ldots, x_n],$$

by Proposition 4, M_α^β is irreducible.

(ii) If $\beta_1, \ldots, \beta_m \in \mathbb{Z}$, then

$$M_\beta \cong \mathbb{C}[x_1, \ldots, x_n]_{x_1 \ldots x_m} \cong \mathbb{C}[x_1, \ldots, x_m]_{x_1 \ldots x_m} \widehat{\otimes} \mathbb{C}[x_{m+1}, \ldots, x_n].$$

But $\mathbb{C}[x_1, \ldots, x_m]_{x_1 \ldots x_m} \cong \mathbb{C}[x_1]_{x_1} \widehat{\otimes} \ldots \widehat{\otimes} \mathbb{C}[x_m]_{x_m}$, with $c(\mathbb{C}[x_i]_{x_i}) = 2$ and $\mathbb{C}[x_{m+1}, \ldots, x_n]$ is an irreducible $\mathbb{C} < x_{m+1}, \ldots, x_n, \partial_{m+1}, \ldots, \partial_n >$-module. Hence by Corollary 1, $c(M_\alpha^\beta) = \underbrace{2.2 \ldots 2}_{m-copies} = 2^m$. This proves (ii).

(iii) Suppose some of the $\beta_1, \ldots, \beta_m$ are integers and the others are elements of $\mathbb{C} \setminus \mathbb{Z}$. WLOG assume that $\beta_1, \ldots, \beta_k \in \mathbb{Z}$ and $\beta_{k+1}, \ldots, \beta_m \in \mathbb{C} \setminus \mathbb{Z}$. Then

$$M_\alpha^\beta \cong \mathbb{C}[x_1, \ldots, x_k]_{x_1 \ldots x_k} x_1^{\beta_1} \ldots x_k^{\beta_k} \widehat{\otimes} \mathbb{C}[x_{k+1}, \ldots, x_n]_{x_{k+1} \ldots x_m} x_{k+1}^{\beta_{k+1}} \ldots x_m^{\beta_m}.$$

But

$$\mathbb{C}[x_1, \ldots, x_k]_{x_1 \ldots x_k} x_1^{\beta_1} \ldots x_k^{\beta_k} \cong \mathbb{C}[x_1, \ldots, x_k]_{x_1 \ldots x_k}$$

and $\mathbb{C}[x_{k+1}, \ldots, x_n]_h x_{k+1}^{\beta_{k+1}} \ldots x_m^{\beta_m}$ is an irreducible $\mathbb{C} < x_{k+1}, \ldots, x_n, \partial_{k+1}, \ldots, \partial_n >$-module. But in (ii) above we have shown that $c(\mathbb{C}[x_1, x_2 \ldots, x_k]_{x_1 \ldots x_k}) = 2^k$, and hence

$$c(M_\alpha^\beta) = c(\mathbb{C}[x_1, \ldots, x_k]_{x_1 \ldots x_k}) c(\mathbb{C}[x_{k+1}, \ldots, x_n]_{x_{k+1} \ldots x_m} x_{k+1}^{\beta_{k+1}} \ldots x_m^{\beta_m}) = 2^k(1) = 2^k.$$

This completes the proof. ☐

6 Blowup

6.1 Definition

The blowup of $\mathbb{A}^2$ at the origin is the locus:

$$\tilde{\mathbb{A}^2} = \{(x, y), [W_0, W_1]) : xW_1 = yW_0\} \subset \mathbb{A}^2 \times \mathbb{P}^1$$

together with the map

$$\pi : \tilde{\mathbb{A}^2} \longrightarrow \mathbb{A}^2$$

which is the restriction of the projection of $\mathbb{A}^2 \times \mathbb{P}^1$ onto the first factor.
Let $U_0 \subset \tilde{\mathbb{A}^2}$ be the open subset given by $W_0 \neq 0$. In terms of Euclidian coordinates, $w_1 = \frac{W_1}{W_0}$. We can write:

$$U_0 = \{(x, y), (w_1)) : xw_1 = y\} = \{(x, xw_1, w_1)\} \subset \mathbb{A}^2 \times \mathbb{A}^1.$$

From this description we see that $U_0 \cong \mathbb{A}^2$ with coordinates x, w_1. The map $\pi : U_0 \longrightarrow \mathbb{A}^2$ is given by $\pi(x, w_1) = (x, xw_1)$.
Let $U_1 \subset \tilde{\mathbb{A}^2}$ be the open subset given by $W_1 \neq 0$. In terms of Euclidian coordinates $w_0 = \frac{W_0}{W_1}$.
We can write: $U_1 = \{(x, y), (w_0)) : x = yw_0\} = \{(yw_0, y, w_0)\} \subset \mathbb{A}^2 \times \mathbb{A}^1$. From this description we see that $U_1 \cong \mathbb{A}^2$ with coordinates y, w_0. The map $\pi : U_1 \longrightarrow \mathbb{A}^2$ is given by $\pi(y, w_0) = (yw_0, w_0)$. This implies that $\tilde{\mathbb{A}^2} = U_0 \cup U_1$, so $\{U_0, U_1\}$ is an affine cover of $\tilde{\mathbb{A}^2}$ and also we can see that: $\pi_{|U_0}(x, y) = \pi_1(x, y) = (x, xy)$ and $\pi_{|U_1}(x, y) = \pi_2(x, y) = (xy, y)$.
For these facts, as well as generalization see [9, 10].

6.2 Describing the pullback of the module in the blowup

We are now going to describe the module $M_\alpha^\beta = \mathbb{C}[x, y]_{xyL_3...L_m}\alpha^\beta$, where $\alpha^\beta = x^{\beta_1}y^{\beta_2}L_3^{\beta_3}...L_m^{\beta_m}$ and $L_i = x + c_iy$ for $i = 3, .., m, c_i \neq c_j$ for $i \neq j$ pulled back to the affine blowup. Let us consider the polynomial map

$$\pi_2 : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

defined by $\pi_2(z, w) = (zw, w)$. Then the homomorphism of rings,

$$\pi_2^\sharp : \mathbb{C}[x, y] \to \mathbb{C}[z, w]$$

defined by $\pi_2^\sharp(f) = fo\pi_2$ is called the comorphism of π_2. It gives an isomorphism $\mathbb{C}[x, y] \cong \mathbb{C}[zw, w]$. We have that $\pi_2^\sharp(\alpha) = \alpha''$, where $\alpha'' = w^m z(z + c_3)...(z_m + c_m)$.
The inverse image of M_α^β by π_2 is defined as a $\mathbb{C}[z, w]$-module by $\pi_2^*(M_\alpha^\beta) = \mathbb{C}[z, w] \otimes_{\mathbb{C}[x,y]} M_\alpha^\beta$, which implies that

$$\pi_2^*(M_\alpha^\beta) \cong \mathbb{C}[z, w] \otimes_{\mathbb{C}[x,y]} \mathbb{C}[x, y]_\alpha\alpha^\beta \cong \mathbb{C}[z, w]_{\alpha''}\alpha^\beta.$$

Using the comorphism formally on α^β gives

$$\alpha''^{\beta''} = \pi_2^\sharp(\alpha^\beta) = w^{\sum_{i=1}^m \beta_i} z^{\beta_1}(z + c_3)^{\beta_3}...(z + c_m)^{\beta_m},$$

and hence one would expect

$$\pi_2^*(M_\alpha^\beta) \cong \mathbb{C}[z, w]_{\tilde{\alpha}}\alpha''^{\beta''}$$

as A_2-modules. This is indeed the case. The standard A_2-module structure on the pullback is defined by using the chain rule:

$$\partial_z \mapsto y\partial_x$$

and

$$\partial_w \mapsto \frac{x}{y}\partial_x + \partial_y$$

to induce actions of ∂_z, ∂_w on α^β and then extending. Hence it suffices to see that $\partial_z \alpha^\beta = \partial_z {\alpha''}^{\beta''}$ and $\partial_w \alpha^\beta = \partial_w {\alpha''}^{\beta''}$ which is an easy exercise. The multiplication map gives the isomorphism,

$$\mathbb{C}[z,w]_{\alpha''}{\alpha''}^{\beta''} \cong \mathbb{C}[w]_w w^{\beta_2'} \widehat{\otimes} \mathbb{C}[z]_{z\prod_{i=3}^m (z+c_i)} \tilde{\alpha}^{\tilde{\beta}}$$

where $\tilde{\alpha}^{\tilde{\beta}} = z^{\beta_1}(z+c_3)^{\beta_3}...(z+c_m)^{\beta_m}$ and $\beta_2' = \sum_{i=1}^m \beta_i$. This is an external product, and so we can obtain information on the number of its decomposition factors by the methods of section 2.

6.3 Composition series of the A_1-module $\mathbb{C}[z]_{\tilde{\alpha}}\tilde{\alpha}^{\tilde{\beta}}$

Let us consider the number of decomposition factors of $\mathbb{C}[w]_w w^{\beta_1'}$ as $\mathbb{C} < w, \partial_w >$-module and $\mathbb{C}[z]_{z\prod_{i=3}^m (z+c_i)} {\alpha'}^{\beta'}$ as $\mathbb{C} < z, \partial_z >$-module separately. We know , by Proposition 1, that if $\beta_2' \in \mathbb{C} \setminus \mathbb{Z}$, then $c(\mathbb{C}[w]_w w^{\beta_2'}) = 1$ and if $\beta_2' \in \mathbb{Z}$ $c(\mathbb{C}[w]_w w^{\beta_2'}) = 2$. In this subsection, we will prove that the A_1-module $\mathbb{C}[z]_{\tilde{\alpha}}\tilde{\alpha}^{\tilde{\beta}}$ is irreducible, if $\beta_1, \beta_3, ..., \beta_m \in \mathbb{C} \setminus \mathbb{Z}$. We have the following proposition.

Proposition 11. *If $\beta_1, \beta_3, ..., \beta_m \in \mathbb{C} \setminus \mathbb{Z}$, then the A_1-module $\mathbb{C}[z]_{\tilde{\alpha}}\tilde{\alpha}^{\tilde{\beta}}$ is irreducible.*

Proof. We are going to prove the Proposition in two steps.

Step I

In this step we are going to show that $\mathbb{C}[x]_{\tilde{\alpha}}\tilde{\alpha}^{\tilde{\beta}} = A_1(\tilde{\alpha}^{\tilde{\beta}})$.
Let $P \in \mathbb{C}[z]_{\tilde{\alpha}}\tilde{\alpha}^{\tilde{\beta}}$. Then $P = \frac{F}{(z\prod_{i=3}^m (z+c_i))^r}\tilde{\alpha}^{\tilde{\beta}}$, for $r \geq 0$ and $F \in \mathbb{C}[z]$. But $\frac{1}{(z\prod_{i=3}^m (z+c_i))^r}$ can be written as $\frac{q}{z^r}\tilde{\alpha}^{\tilde{\beta}} + \sum_{i=3}^m \frac{q_i}{(z+c_i)^r}\tilde{\alpha}^{\tilde{\beta}}$, for some $q, q_i \in \mathbb{C}(c_3, ..., c_m), i = 3, ..., m$ and hence

$$P = \frac{Fq}{z^r}\tilde{\alpha}^{\tilde{\beta}} + \sum_{i=3}^m \frac{Fq_i}{(z+c_i)^r}\tilde{\alpha}^{\tilde{\beta}}.$$

So it suffices to show that $\frac{1}{z^r}, \frac{1}{(z+c_i)^s} \in A_1(\tilde{\alpha}^{\tilde{\beta}})$, for $r, s \geq 1$ and some $i \in \{3, ..., m\}$. By applying the argument successively on $\beta'' = \tilde{\beta} - (k, 0, ..., 0)$

and $\beta^{''} = \tilde{\beta} - (0, ...s, ..., 0)$ and using induction it suffices to prove that $\frac{1}{z}\tilde{\alpha}^{\tilde{\beta}} \in A_1(\tilde{\alpha}^{\tilde{\beta}})$ and $\frac{1}{z+c_i}\tilde{\alpha}^{\tilde{\beta}} \in A_1(\tilde{\alpha}^{\tilde{\beta}})$. This can be done as follows.
Let

$$\tilde{D} = [\prod_{i=3}^{m}(z + c_i)\partial_z - \sum_{i=3}^{m}\beta_i \prod_{j=3, j\neq i}^{m}(z + c_j)].$$

Clearly $\tilde{D} \in A_1$ and $\tilde{D}(\tilde{\alpha}^{\tilde{\beta}}) = \frac{\beta_1 \prod_{i=3}^{m}(z+c_i)}{x}\tilde{\alpha}^{\tilde{\beta}}$. But

$$\frac{\beta_1 \prod_{i=3}^{m}(z + c_i)}{z}\tilde{\alpha}^{\tilde{\beta}} = P_1\tilde{\alpha}^{\tilde{\beta}} + \frac{C}{z}\tilde{\alpha}^{\tilde{\beta}},$$

for some $P_1 \in \mathbb{C}[z]$ and $C = \beta_1 \prod_3^m c_i$. Since $C \neq 0$, we have that

$$\frac{1}{C}[(\tilde{D} - P_1)(\tilde{\alpha}^{\tilde{\beta}}) = \frac{1}{z}\tilde{\alpha}^{\tilde{\beta}}.$$

Hence $\frac{1}{z}\tilde{\alpha}^{\tilde{\beta}} \in A_1(\tilde{\alpha}^{\tilde{\beta}})$.
Let

$$D^{''} = [\prod_{i=3}^{m}(z + c_i)\partial_z - \beta_1 z\prod_{j=4}^{m}(z + c_j) - \sum_{i=4}^{m}\beta_i z \prod_{j=4, j\neq i}^{m}(z + c_j)].$$

Clearly $D^{''} \in A_1$ and

$$D^{''}(\tilde{\alpha}^{\tilde{\beta}}) = \frac{\beta_3 z \prod_{i=4}^{m}(z + c_i)}{z + c_3}\tilde{\alpha}^{\tilde{\beta}}.$$

But

$$\frac{\beta_3 z \prod_{i=4}^{m}(z + c_i)}{z + c_3}\tilde{\alpha}^{\tilde{\beta}} = R\tilde{\alpha}^{\tilde{\beta}} + \frac{C^{'}}{z + c_3}\tilde{\alpha}^{\tilde{\beta}}$$

for some $R \in \mathbb{C}[x]$ and for some $C^{'}$, $0 \neq C^{'} \in \mathbb{C}[c_3, ..., c_k]$. Therefore

$$\frac{1}{C^{'}}(D^{''} - R)(\tilde{\alpha}^{\tilde{\beta}}) = \frac{1}{z + c_3}\tilde{\alpha}^{\tilde{\beta}}$$

and hence $\frac{1}{z+c_3}\tilde{\alpha}^{\tilde{\beta}} \in A_1(\tilde{\alpha}^{\tilde{\beta}})$. Since $z + c_3$ was arbitrary, $\frac{1}{(z+c_i)}\tilde{\alpha}^{\tilde{\beta}} \in A_1(\tilde{\alpha}^{\tilde{\beta}})$, for some $i = 3, ..., m$.
Therefore, $M_{\tilde{\alpha}}^{\tilde{\beta}} = A_1(\tilde{\alpha}^{\tilde{\beta}})$. This completes the proof of Part I.

Step II

In this step we are going to prove that $A_1(\tilde{\alpha}^{\tilde{\beta}})$ is irreducible. It suffices to show that $A_1(\tilde{\alpha}^N\tilde{\alpha}^{\tilde{\beta}}) = A_1(\tilde{\alpha}^{\tilde{\beta}})$ for some large N. But from above, $M_{\tilde{\alpha}}^{\tilde{\beta}^{''}} = A_1(\alpha^{\tilde{\beta}^{''}})$, $\tilde{\beta}^{''} = \tilde{\beta} + N^{''}, N^{''} \in \mathbb{N}^{m-1}$ and by Proposition 2, $M_{\alpha}^{\beta^{'}} \cong M_{\tilde{\alpha}}^{\tilde{\beta}^{''}}$. Therefore $A_1(\tilde{\alpha}^{\tilde{\beta}}) = A_1(\tilde{\alpha}^{\tilde{\beta}^{''}})$ and hence $A_1(\tilde{\alpha}^{\tilde{\beta}})$ is irreducible. This concludes the proof. □

6.4 Composition series of the A_2-module $\mathbb{C}[z,w]_{\alpha''}\alpha''^{\beta''}$

In this subsection we are going to prove the following Theorem.

Theorem 6. *Let $M_{\alpha''}^{\beta''} = \mathbb{C}[z,w]_{\alpha''}\alpha''^{\beta''}$, where $\beta_1, \beta_2' ..., \beta_m \in \mathbb{C} \setminus \mathbb{Z}$, $\alpha'' = zw(z+c_3)...(z+c_m)$ and $\alpha''^{\beta''} = z^{\beta_1} w^{\beta_2'} (z+c_3)^{\beta_3}...(z+c_m)^{\beta_m}$ such that $c_i \neq c_j$ for $i \neq j$.*
(i) If $\beta_2' \in \mathbb{C} \setminus \mathbb{Z}$, then $M_{\alpha''}^{\beta''}$ is irreducible.
(ii) If $\beta_2' \in \mathbb{Z}$, then $c(M_{\alpha''}^{\beta''}) = 2$.

Proof. (i) From the previous section, we know that,

$$\mathbb{C}[z,z]_{\alpha''}\alpha''^{\beta''} = \mathbb{C}[z]_{\tilde{\alpha}}\tilde{\alpha}^{\tilde{\beta}} \widehat{\otimes} \mathbb{C}[w]_w w^{\beta_2'}$$

and by Proposition 11, $\mathbb{C}[x]_{\tilde{\alpha}}\tilde{\alpha}^{\tilde{\beta}}$ is irreducible $\mathbb{C} < z, \partial_z >$-module, where $\tilde{\alpha} = z\prod_{i=3}^{m}(z+c_i)$ and $\tilde{\beta} = (\beta_1, \beta_3, ..., \beta_m)$ and also by Proposition 1, $\mathbb{C}[w]_w w^{\beta_2'}$ is irreducible $\mathbb{C} < w, \partial_w >$-module. Hence by Proposition 4, $M_{\alpha''}^{\beta''}$ is irreducible A_2-module.

(ii) By Proposition 11, $\mathbb{C}[z]_{\tilde{\alpha}}\tilde{\alpha}^{\tilde{\beta}}$ is irreducible $\mathbb{C} < z, \partial_z >$-module and by Proposition 1, $c(\mathbb{C}[w]_w w^{\beta_2'}) = 2$. Therefore

$$c(M_{\alpha''}^{\beta''}) = c(\mathbb{C}[z]_{\tilde{\alpha}}\tilde{\alpha}^{\tilde{\beta}}) c(\mathbb{C}[w]_w w^{\beta_2'}) = 2.$$

This completes the proof. ☐

7 The A_2-module M_α^β in the plane case where all $\beta_i \in \mathbb{C} \setminus \mathbb{Z}$

In this section, we restrict ourselves to n=2, that is the plane case, and we assume that $\beta_i \in \mathbb{C} \setminus \mathbb{Z}$, i=1,...,m. Then $c(M_\alpha^\beta)$ is 1 or $m-1$ according to whether $|\beta| = \sum_{i=1}^{m} \beta_i \in \mathbb{Z}$ or not. Our module in this case is

$$M_\alpha^\beta = \mathbb{C}[x,y]_\alpha \alpha^\beta,$$

where $\alpha = xy\prod_{i=3}^{m}(x+c_i y)$,$\alpha^\beta = x^{\beta_1} y^{\beta_2} (x+c_3 y)^{\beta_3}...(x+c_m y)^{\beta_m}$ and $c_i \neq c_j$ for $i \neq j$. We generalize this result in the following theorem.

Theorem 7. *Asuume that $\beta_i \in \mathbb{C} \setminus \mathbb{Z}$, i=1,...,m.*
(i) If $|\beta| \in \mathbb{C} \setminus \mathbb{Z}$, then $c(M_\alpha^\beta) = 1$.
(ii) If $|\beta| \in \mathbb{Z}$, then $c(M_\alpha^\beta) = m-1$.

The proof will be done in several steps and we start by proving (i).

7.1 Proof of the first part of Theorem 7

In this subsection we will prove (i) and this will be done in four steps. In steps I-III we will prove, under the given assumption, that α^β generates the module M_α^β and using this in step IV we will prove that M_α^β is irreducible. Let $Q \in M_\alpha^\beta$. Then

$$Q = \frac{F}{(xy\prod_{i=3}^m (x+c_i y))^r}\alpha^\beta,$$

for $F \in \mathbb{C}[x,y]$ and $r \geq 0$. Since by Theorem 3

$$\frac{F}{(xy\prod_{i=3}^m (x+c_i y))^r}\alpha^\beta$$

can be written as a linear combination of

$$\frac{F}{x^{s_1}y^{s_2}}\alpha^\beta, \frac{F}{x^{n_1^3}(x+c_3y)^{n_2^3}}\alpha^\beta, ..., \frac{F}{x^{n_1^m}(x+c_my)^{n_2^m}}\alpha^\beta,$$

where $s_1+s_2 = n_1^3+n_2^3 = ... = n_1^m+n_2^m = mr$, it suffices to show that,

$$\frac{F}{x^{s_1}y^{s_2}}\alpha^\beta, \frac{F}{x^{n_1^3}(x+c_3y)^{n_2^3}}\alpha^\beta, ..., \frac{F}{x^{n_1^m}(x+c_my)^{n_2^m}}\alpha^\beta$$

are all elements of $A_2(\alpha^\beta)$. Let us proceed step by step.

Step I

In this step we are going to show that $\frac{1}{x^k}\alpha^\beta \in A_2(\alpha^\beta)$ for $k \geq 1$. By applying the argument successively on $\tilde{\beta} = \beta - (k, 0, ..., 0)$ and using induction it suffices to prove that $\frac{1}{x}\alpha^\beta \in A_2(\alpha^\beta)$. Let

$$D_1 = \frac{1}{\beta_1}[\prod_{i=3}^m (x+c_iy)\partial_x - \sum_{j=3}^m c_j\beta_j \prod_{i=3, i\neq j}^m (x+c_iy)].$$

Clearly $D_1 \in A_2$ and $D_1(\alpha^\beta) = \frac{\prod_{i=3}^m(x+c_iy)}{x}\alpha^\beta$. But

$$\frac{\prod_{i=3}^m(x+c_iy)}{x}\alpha^\beta = \frac{d_1y^{m-2}}{x}\alpha^\beta + H\alpha^\beta,$$

for some $H \in \mathbb{C}[x,y]$ and $d_1 = \prod_{i=3}^m c_i$. Then we have

$$\frac{1}{d_1}(D_1 - H)\alpha^\beta = \frac{y^{m-2}}{x}\alpha^\beta.$$

On the other hand

$$\partial_y(\frac{y^{m-2}}{x}\alpha^\beta) = \frac{(\beta_2+m-2)y^{m-3}}{x}\alpha^\beta + \sum_{i=3}^m \frac{c_i\beta_iy^{m-2}}{x(x+c_iy)}\alpha^\beta$$

and

$$y^{m-3}\partial_x(\alpha^\beta) = \frac{\beta_1 y^{m-3}}{x}\alpha^\beta + \sum_{i=3}^{m-2} \frac{\beta_i y^{m-3}}{x+c_i y}\alpha^\beta.$$

This implies

$$(\partial_y \frac{1}{d_1}(D_1 - H) + y^{m-3}\partial_x)(\alpha^\beta) = \frac{(|\beta| + m - 2)y^{m-3}}{x}\alpha^\beta.$$

Iterating we find that

$$D_2(\alpha^\beta) = \frac{\prod_{i=1}^{m-2}(|\beta| + i)}{x}\alpha^\beta$$

for some $D_2 \in A_2$. Since $|\beta| \in \mathbb{C} \setminus \mathbb{Z}$, we have $\frac{1}{x}\alpha^\beta \in A_2(\alpha^\beta)$.

Step II

In this step we are going to show that $\frac{1}{x^k y^t}\alpha^\beta \in A_2(\alpha^\beta)$ for $k, t \geq 1$. By applying the argument successively on $\tilde{\beta} = \beta - (0, t, ..., 0)$ and using induction it suffices to prove that $\frac{1}{x^k y}\alpha^\beta \in A_2(\alpha^\beta)$. From step I we know that $\frac{1}{x^k}\alpha^\beta \in A_2(\alpha^\beta)$. Let

$$D_3 = \frac{1}{\beta_2}(\prod_{i=3}^{m}(x + c_i y)\partial_y - \sum_{j=3}^{m} c_j \beta_j \prod_{i=3, i\neq j}(x + c_i y))$$

Clearly $D_3 \in A_2$ and

$$D_3(\frac{1}{x^k}\alpha^\beta) = \frac{\prod_{i=3}^{m}(x + c_i y)}{x^k y}\alpha^\beta.$$

But

$$\frac{\prod_{i=3}^{m}(x + c_i y)}{x^k y}\alpha^\beta = \frac{x^{m-2}}{x^k y}\alpha^\beta + \frac{L}{x^k}\alpha^\beta$$

for some $L \in \mathbb{C}[x, y]$. This implies that

$$(D_3 - L)(\frac{1}{x^k}\alpha^\beta) = \frac{x^{m-2}}{x^k y}\alpha^\beta.$$

On the other hand

$$\partial_x(\frac{x^{m-2}}{x^k y}\alpha^\beta) = \frac{(\beta_1 + m - 2 - k)x^{m-3}}{x^k y}\alpha^\beta + \sum_{i=3}^{m} \frac{\beta_i x^{m-2}}{x^k y(x + c_i y)}\alpha^\beta$$

and

$$x^{m-3}\partial_y(\frac{1}{x^k}\alpha^\beta) = \frac{\beta_2 x^{m-3}}{x^k y}\alpha^\beta + \sum_{i=3}^{m} \frac{c_i \beta_i x^{m-3}}{x^k(x + c_i y)}\alpha^\beta.$$

This implies that

$$(\partial_x D_4 + x^{m-3}\partial_y)(\frac{1}{x^k}\alpha^\beta) = \frac{(|\beta| + m - 2 - k)x^{m-3}}{x^k y}\alpha^\beta,$$

where $D_4 = D_3 - L$. Iterating we find that,

$$D_5(\alpha^\beta) = \frac{\prod_{i=1}^{m-2}(|\beta| - i - k)}{x^k y}\alpha^\beta$$

for some $D_5 \in A_2$. Since $|\beta| \in \mathbb{C} \setminus \mathbb{Z}$, we have $\frac{1}{x^k y}\alpha^\beta \in A_2(\alpha^\beta)$.

Step III

In this step we are going to show that $\frac{1}{x^k(x+c_i y)^t}\alpha^\beta \in A_2(\alpha^\beta)$ for $k, t \geq 1$. By using the coordinate function, $\tilde{x} = x$ and $\tilde{y} = x + c_i y$ and step II, we have

$$\frac{1}{\tilde{x}^k \tilde{y}^t}\alpha^\beta = \frac{1}{x^k(x + c_i y)^t}\alpha^\beta \in A_2(\alpha^\beta).$$

By the description of M_α^β recalled at the begning of the proof, we have $M_\alpha^\beta = A_2(\alpha^\beta)$. Then it remains to show that $A_2(\alpha^\beta)$ and hence M_α^β is irreducible to conclude part (i) and we will prove that in the next step.

Step IV

In this step we are going to prove that $A_2(\alpha^\beta)$ and hence M_α^β is irreducible. But, it suffices to show that $M_\alpha^\beta = A_2(\alpha^{\beta+N})$ for any $N \in \mathbb{N}^m$ (See Lemma 12). By step III we know that $M_\alpha^\beta = A_2(\alpha^\beta)$, if $\beta_i \in \mathbb{C} \setminus \mathbb{Z}$ and $|\beta| \in \mathbb{C} \setminus \mathbb{Z}$. Cleary, these conditions are satisfied for $\beta + N$, for any for $N \in \mathbb{N}^m$, as well. Hence $M_\alpha^{\beta+N} = A_2(\alpha^{\beta+N})$. By Proposition 2 , $M_\alpha^\beta \cong M_\alpha^{\beta+N}$, and hence $M_\alpha^\beta = A_2(\alpha^{\beta+N})$. Therefore M_α^β is irreducible. This completes the proof of (i) of Theorem 7.
It remains to prove (ii) of Theorem 7, but before that let us find the annihilator of α^β in the next section which we will use it in the prove of part (ii).

7.2 The annihilator of α^β

In this subsection we are going to find the annihilator of α^β in the Weyl algebra A_2.
Let

$$P = x\partial_x + y\partial_y - (\sum_{i=1}^{m} \beta_i)$$

and

$$Q = y\prod_{i=3}^{m} L_i\partial_y - \beta_2\prod_{i=3}^{m} L_i - \sum_{j=3}^{m}\beta_i y \prod_{i=3, i\neq j}^{m} L_i,$$

where $L_i = x + c_i y$, i=3,...,m and $c_i \neq c_j$ for $i \neq j$. We use the following graded reverse lexicographic order, letting $y > x > \partial_x > \partial_y$,

$$y^i x^j \partial_x^k \partial_y^l > y^{i'} x^{j'} \partial_x^{k'} \partial_y^{l'}$$

if

$$i + j + k + l > i' + j' + k' + l'$$

or

$$i + j + k + l = i' + j' + k' + l'$$

and the last non-zero coordinate of $(i, j, k, l) - (i', j', k', l')$ is negative. The most important point for us with this term order is that there is a $normal form$ algorithm, see [12, Chapter 1] and [7, Chapter 2], with respect to the set $\{P, Q\}$. It inputs an element F of the Weyl algebra and outputs an element R such that there exist S and T in the Weyl algebra with $F = SP + TQ + R$ where the initial term of R is not divisible by initial terms of P and Q. Since the initial term of P is $\underline{x\partial_x}$ and the initial term of Q is $\underline{y^{m-1}\partial_y}$, it follows that

$$A_2 = A_2 P + A_2 Q + N$$

where

$$N = (\oplus_{(i,j,k,l)\in M})\mathbb{C}x^i y^j \partial_x^k \partial_y^l$$

and $M \subset \mathbb{Z}^4_{\geq 0}$, is the set

$$M = \{(i, j, k, l) : ik = 0 \& l \neq 0 \implies j \leq m - 2\}.$$

Hence

$$A_2 = A_2 P + A_2 Q + (\oplus_{i\geq 1} R_i(y, \partial_y)\partial_x^i) + W,$$

where $W = \{\sum_{i=0}^{m-2} y^i S_i(x, \partial_y) : S_i(x, \partial_y) \in \mathbb{C}[x, \partial_y]\}$.
We have the following proposition about the annihilator of α^β.

Proposition 12. *Let* $\mathrm{Ann}_{\mathrm{A}_2}(\alpha^\beta) = \{\mathrm{D} \in \mathrm{A}_2 : \mathrm{D}(\alpha^\beta) = 0\}$. *Then*

$$\mathrm{Ann}_{\mathrm{A}_2}(\alpha^\beta) = \mathrm{A}_2\mathrm{P} + \mathrm{A}_2\mathrm{Q}.$$

Proof. Since $P(\alpha^\beta) = 0$ and $Q(\alpha^\beta) = 0$, then $A_2P + A_2Q \subset \mathrm{Ann}_{\mathrm{A}_2}(\alpha^\beta)$. Let $D \in \mathrm{Ann}_{\mathrm{A}_2}(\alpha^\beta)$. We want to show that $D \in A_2P + A_2Q$. We know that $D = H + T + U$, where $H \in A_2P + A_2Q$, $T = \sum_{i\geq 1} R_i(y, \partial_y)\partial_x^i$ and $U = \sum_{j=0}^{m-2} y^j S_j(x, \partial_y)$. Then $D(\alpha^\beta) = 0$ implies $T(\alpha^\beta) + U(\alpha^\beta) = 0$. Let us now consider poles at $x = 0$. If $m = r(x, y)\alpha^\beta \in M_\alpha^\beta$, define $O_x(m)$ to be the greatest $-k$ such that

$$m = (r_{-k}(x, y)x^{-k} + r_{-k+1}x^{-k+1} + ... + r_s(x, y)x^s)\alpha^\beta$$

where $r_i(x,y), i = -k, -k+1, ...s$ written as a reduced quotient of products of irreducible polynomials containing no x. Then note that if $R(y, \partial_y)$ is a polynomial in just y and ∂_y,

$$O_x(R(y,\partial_y)m) \geq O_x(m).$$

Returning to D, we know that

$$\sum_{i=1}^{k} R_i(y,\partial_y)\partial_x^i(\alpha^\beta) + U(\alpha^\beta) = 0. \tag{7.1}$$

By the above agruement $O_x(U(\alpha^\beta)) \geq 0$. On the other hand

$$\partial_x^i(\alpha^\beta) = \frac{\beta_1(\beta_1-1)...(\beta_1-(i-1))}{x^i}(\alpha^\beta) + r,$$

where $O_x(r) > -i$ and hence $O_x(\partial_x^i(\alpha^\beta)) = -i$ by the assumption that $\beta_1 \notin \mathbb{Z}$. Consider the possible poles in (7.1) and assume that $R_k(y,\partial_y) \neq 0$ and $R_k(y,\partial_y)\alpha^\beta \neq 0$. Then

$$R_k(y,\partial_y)\partial_x^k(\alpha^\beta) = (\frac{\beta_1(\beta_1-1)...(\beta_1-(k-1))R_k(y,\partial_y)}{x^k} + r)\alpha^\beta,$$

where $O_x(r\alpha^\beta) > -k$. For all other terms in (7.1) the contribution to the poles at $x = 0$ will be of order greater than $-k$. This is a contradiction. So $R_k(y,\partial_y)\alpha^\beta = 0$. But $R_k(y,\partial_y) \in \mathbb{C} < x, y, \partial_y >$ and hence we made the desired reduction. In the next step we will prove that this implies $R_k(y,\partial_y) = 0$, which will be a contradiction to the assumption than $R_k(y,\partial_y) \neq 0$. Actually we will prove more generally that if $U \in \mathbb{C} < x, y, \partial_y >$ and $U(\alpha^\beta) = 0$, then $U = 0$. So now we can assume that $U(\alpha^\beta) = 0$ and argue in the same way by considering the poles at the other lines. Write

$$U = \sum_{i=0}^{k} P_i(x,y)\partial_y^i,$$

where $P_k \neq 0$. Consider $U(\alpha^\beta)$ and the order of its pole at L, where L is one of $y, L_3, ..., L_m$. Since $O_L(P_i(x,y)\partial_y^i\alpha^\beta) = O_L(P_i(x,y)\alpha^\beta) - i$, there must be two indices i_1 and i_2 such that

$$O_L(P_{i_1}(x,y)\alpha^\beta) - i_1 = O_L(P_{i_2}(x,y)\alpha^\beta) - i_2 \leq O_L(P_j(x,y)\alpha^\beta) - j$$

for all $0 \leq j \leq k$. In particular there is $i \neq k$, such that $O_L(P_k) - k \geq O_L(P_j) - i$ and hence $O_L(P_k) \geq O_L(P_j) + k - i \geq 1$. Repeating this for all L implies that $yL_3...L_m$ divides P_k. This contradicts that $\mathrm{deg_y P_k \leq m-2}$. Hence U=0. This completes. □

7.3 Proof of the second part of Theorem 7

7.3.1 Preliminaries

For the proof of (ii) of Theorem 7 we need the following Lemmas.

Lemma 9. *Let $\tilde{\beta} = \beta + N$, where $N \in \mathbb{Z}^m$ and $\alpha^{\tilde{\beta}} = x^{\tilde{\beta}_1} y^{\tilde{\beta}_2} ... L_m^{\tilde{\beta_m}} \in M_\alpha^\beta$. Let $|\tilde{\beta}| = \sum_{i=1}^m \tilde{\beta}_i$. Then*

$$A_2 x + \mathrm{Ann}_{\mathrm{A_2}} \alpha^{\tilde{\beta}} = \mathrm{A_2 x + A_2(y\partial_y - (|\tilde{\beta}| + 1)) + A_2(y^{m-2})}.$$

(Recall that, we assumed $\beta_i \in \mathbb{C} - \mathbb{Z}$, i=1,...,m.)

Proof. By Proposition 12,

$$\mathrm{Ann}_{\mathrm{A_2}} \alpha^{\tilde{\beta}} = \mathrm{A_2 P + A_2 Q},$$

where $P = x\partial_x + y\partial_y - |\tilde{\beta}|$ and $Q = y\prod_{i=3}^m L_i \partial_y - \tilde{\beta}_2 \prod_{i=3}^m L_i - \sum_{j=3}^m \tilde{\beta}_i y \prod_{i=3, i\neq j}^m L_i$. But $Q = G.x + C(y^{m-1}\partial_y - \sum_{i=2}^m \tilde{\beta}_i y^{m-2})$ for some $G \in A_2$ and $C = \prod_{i=3}^m c_i$. Hence

$$J := A_2 x + \mathrm{Ann}_{\mathrm{A_2}} \alpha^{\tilde{\beta}} = \mathrm{A_2 x + A_2(y\partial_y - (|\tilde{\beta}|+1)) + A_2(y^{m-1}\partial_y - \sum_{i=2}^{m} \tilde{\beta}_i y^{m-2})}.$$

But $y^{m-1}\partial_y - \sum_{i=2}^m \tilde{\beta}_i y^{m-2} - y^{m-2}(y\partial_y - (|\tilde{\beta}| + 1)) = (\tilde{\beta}_1 + 1) y^{m-2} \in J$. Since $\tilde{\beta}_1 + 1 \neq 0$, by assumption, then $y^{m-2} \in J$. Hence

$$J = A_2 x + A_2(y\partial_y - (|\tilde{\beta}| + 1)) + A_2(y^{m-2}).$$

□

Lemma 10. *Let $A_1 = \mathbb{C} < y, \partial_y >$. Let $J = A_1(y\partial_y - \gamma) + A_1 y^k$ for $k \geq 0$. Then we have the following.*
(i) If $\gamma \notin \{-1, ..., -k\}$, then $J = A_1$.
(ii) If $-k \leq \gamma \leq -1$, then $J = A_1(y\partial_y - \gamma) + A_1 y^{|\gamma|}$. Further more

$$A_1/J \cong \mathbb{C}[y]_y/\mathbb{C}[y]$$

and hence irreducible.

Proof. (i) If $\gamma \notin \{-1, ..., -k\}$, then $j + \gamma \neq 0$, for $j \in \{1, ..., k\}$.

$$\partial_y y^k - y^{k-1}(y\partial_y - \gamma) = (k + \gamma) y^{k-1}.$$

Since $k + \gamma \neq 0$, then $y^{k-1} \in J$. Iterating we find that $1 \in J$, since by assumption $k + \gamma \neq 0, k - 1 + \gamma \neq 0, ..., 1 + \gamma \neq 0$, and hence $J = A_1$.

(ii) If $-k \leq \gamma \leq -1$, still it is clear that $J = A_1(y\partial_y - \gamma) + A_1 y^{|\gamma|}$.

$$A_1 = J + \oplus_{i \geq 0} \mathbb{C}\partial_y^i \oplus \oplus_{j=1}^{|\gamma|-1} \mathbb{C} y^j. \tag{7.2}$$

Let $\theta : A_1 \longrightarrow \mathbb{C}[y]_y/\mathbb{C}[y]$ be the map defined by $\theta(P) = P(\bar{y}^\gamma)$. Clearly $J \subset \mathrm{Ker}\theta$ and θ is surjective and it is a map onto a simple A_1-module. By (7.2), $J = \mathrm{Ker}\theta$. This concludes the proof. □

Lemma 11.
$$A_2/(A_2x + \mathrm{Ann}_{\mathrm{A_2}}\alpha^{\tilde{\beta}}) \cong \mathrm{A_2}\alpha^{\tilde{\beta}}/\mathrm{A_2x}\alpha^{\tilde{\beta}}$$
is a simple A_2-module if and only if $-(m-2) \leq |\tilde{\beta}| + 1 \leq -1$ and zero otherwise.

Proof. By Lemma 9,
$$A_2\alpha^{\tilde{\beta}}/A_2x\alpha^{\tilde{\beta}} \cong A_2/(A_2x + A_2(y\partial_y - (|\tilde{\beta}| + 1)) + A_2(y^{m-2})).$$
The last description makes it clear that the module is the external product
$$\mathbb{C} < x, \partial_x > /\mathbb{C} < x, \partial_x > x \widehat{\otimes} \mathbb{C} < y, \partial_y > /\mathbb{C} < y, \partial_y >< y\partial_y - (|\tilde{\beta}|+1), y^{m-2} > .$$
Hence the result follows by Lemma 10 (ii) and Proposition 4. $\square$

7.3.2 Proof

We are now in a position to prove the last part of Theorem 7. We use the following Lemma as a starting point.

Lemma 12. *(i)There exists N_1 such that $\alpha^{\beta+N_1}$ generates M_α^β.*
(ii)There exists $N_2 > N_1$ such that $A_2\alpha^{\beta+N_3}$ is a simple submodule if $N_3 \in N_2 + \mathbb{N}^m$.

Proof. The first follows directly from the fact that $M = M_\alpha^\beta$ is a holonomic module and hence cyclic see [4]. The second follows from the more difficult fact that M_α^β contains a simple submodule L with support on $\mathbb{C}^2$, the so called Deligne module [3]. This means that M/L has to be torsion as $\mathbb{C}[x]$-module. Since by Proposition 9 all decomposition factors have support on hyperplane intersetions it follows that any element $\overline{n} \in M/L$ is annihilated by a large enough power of α. Take $n = \alpha^{N_1+\beta}$ to be the generator of M/L, from the first statement of the lemma and assume that $\alpha^N\alpha^{N_1+\beta} \in N$ and let $N_2 = N_1 + N$. $\square$

Consider $A_2\alpha^{\beta+N_1}$. Put $\tilde{\beta} = \beta + N_1$. Since, if $\alpha^{\beta+N}$ generates M_α^β, also $x^{-n}\alpha^{\beta+N}$ generates if $n \geq 0$, we may assume $|\tilde{\beta}| \leq -(m-1)$. By Lemma 11, if $|\tilde{\beta}|$ is not one of $-(m-1), ..., -2$, we have that $A_2\tilde{\beta}/A_2x\tilde{\beta} = 0$. Hence $A_2\alpha^{\tilde{\beta}} = A_2x\alpha^{\tilde{\beta}} = ... = A_2\alpha^{\tilde{\beta}_1}$, where $\alpha^{\tilde{\beta}_1} = x^r\alpha^{\tilde{\beta}}$ such that $|\tilde{\beta}_1| = -(m-1)$. Then by Lemma 11
$$A_2\alpha^{\tilde{\beta}_1} \supset A_2x\alpha^{\tilde{\beta}_1} \supset ... \supset A_2x^{m-2}\alpha^{\tilde{\beta}_1},$$
is a chain of strict submodules such that each factor is irreducible and has support at $(0, 0)$. The last submodule, $A_2x^{m-2}\alpha^{\tilde{\beta}_1}$, has the property (again by applying the lemma to $A_2\alpha^N x^{m-2}\alpha^\beta$ for $N \in \mathbb{N}^m$ in succession), that it equals $A_2\alpha^N x^{m-2}\alpha^\beta$ for all $N \in \mathbb{N}^m$, and hence by Lemma 12 is simple. Hence M_α^β has $m-2$ decomposition factors with support at the origin, and one with support on $\mathbb{C}^2$. This concludes the proof.

8 Example

In this section we consider the A_2-module $M_\alpha^\beta = \mathbb{C}[x,y]_{xy(x+y)}\alpha^\beta$, where $\alpha^\beta = x^{\beta_1}y^{\beta_2}(x+y)^{\beta_3}$ and calculate $c(M_\alpha^\beta)$ by considering different cases on $\beta_1, \beta_2, \beta_3$. From section 3 and section 7 we know the following.

- If $\beta_1, \beta_2, \beta_3 \in \mathbb{Z}$, then $c(M_\alpha^\beta) = 6$.
- If $\beta_1, \beta_2, \beta_3, \beta_1 + \beta_2 + \beta_3 \in \mathbb{C} \setminus \mathbb{Z}$, then $c(M_\alpha^\beta) = 1$.
- If $\beta_1, \beta_2, \beta_3 \in \mathbb{C} \setminus \mathbb{Z}$ and $\beta_1 + \beta_2 + \beta_3 \in \mathbb{Z}$, then $c(M_\alpha^\beta) = 2$.

Hence it remains to consider the following two cases.

- Exactly one of $\beta_1, \beta_2, \beta_3$ is in $\mathbb{C} \setminus \mathbb{Z}$.
- Exactly two of $\beta_1, \beta_2, \beta_3$ are in $\mathbb{C} \setminus \mathbb{Z}$.

We generalize the results in the following theorem.

Theorem 8. *(i) If exactly one of $\beta_1, \beta_2, \beta_3$ is in $\mathbb{C} \setminus \mathbb{Z}$, then $c(M_\alpha^\beta) = 3$.*
(ii) If exactly two of $\beta_1, \beta_2, \beta_3$ are in $\mathbb{C} \setminus \mathbb{Z}$ and $\beta_1 + \beta_2 + \beta_3 \in \mathbb{C} \setminus \mathbb{Z}$, then $c(M_\alpha^\beta)$=2.
(iii) If exactly two of $\beta_1, \beta_2, \beta_3$ are in $\mathbb{C} \setminus \mathbb{Z}$ and $\beta_1 + \beta_2 + \beta_3 \in \mathbb{Z}$, then $c(M_\alpha^\beta)$=3.

Proof. (i) WLOG assume $\beta_1 \in \mathbb{C} \setminus \mathbb{Z}$ and $\beta_2, \beta_3 \in \mathbb{Z}$. Then, by Proposition 2, $M_\alpha^\beta \cong \mathbb{C}[x,y]_{xy(x+y)}x^{\beta_1}$. By Proposition 4, $\mathbb{C}[x,y]_x x^{\beta_1}$ and $\mathbb{C}[x,y]_{x(x+y)}x^{\beta_1}/\mathbb{C}[x,y]_x x^{\beta_1}$ are irreducible A_2-modules. Consider the quotient module $N = \mathbb{C}[x,y]_{xy(x+y)}x^{\beta_1}/\mathbb{C}[x,y]_{x(x+y)}x^{\beta_1}$. We want to show that N is irreducible. Let $P \in N \setminus \{0\}$. Assume that $P = \frac{f}{y}x^{\beta_1}$, where $f = \sum\limits_{i=0}^{k} \alpha_i x^i, \alpha_k \neq 0$. That is

$$P = \frac{\sum\limits_{i=0}^{k} \alpha_i x^{\beta_1+i}}{y}.$$

We have the formula

$$(x\partial_i - (\beta_1 + i))\frac{x^{\beta_1+j}}{y} = (j-i)\frac{x^{\beta_1+j}}{y} \tag{8.1}$$

This implies

$$\prod_{i=0}^{k-1}(x\partial_x - (\beta_1 + i))P = \alpha_k k!\frac{x^{\beta_1+k}}{y}$$

and since $\alpha_k \neq 0$ by assumption, $\frac{x^{\beta_1+k}}{y} \in A_2P$. Consider the following two formulas:

$$\partial_x^i \frac{x^{\beta_1+k}}{y} = (\beta_1+k)(\beta_1+k-1)...(\beta_1+k-i)\frac{x^{\beta_1+k-i}}{y} \tag{8.2}$$

and

$$x^i \cdot \frac{x^{\beta_1+k}}{y} = \frac{x^{\beta_1+k+i}}{y}. \tag{8.3}$$

Since $\beta_1 \in \mathbb{C} \setminus \mathbb{Z}$ by assumption, the coefficient in (8.2) is non-zero for all $i \geq 0$ which implies that $\frac{x^{\beta_1+k-i}}{y} \in A_2P$. The formula (8.3) gives that $\frac{x^{\beta_1+k+i}}{y} \in A_2P$ for all $i \geq 0$. Hence $N \subset A_2P$. Since P was arbitrary, this means that N is irreducible. Therefore,

$$\mathbb{C}[x,y]_x x^{\beta_1} \subset \mathbb{C}[x,y]_{xy} x^{\beta_1} \subset \mathbb{C}[x,y]_{xy(x+y)} x^{\beta_1} = M_\alpha^\beta,$$

is a composition series of M_α^β and hence $c(M_\alpha^\beta) = 3$. This completes the proof of (i).

(ii) WLOG (we can change basis) assume that $\beta_2 \in \mathbb{Z}$ and $\beta_1, \beta_3 \in \mathbb{C} \setminus \mathbb{Z}$. By Proposition 2, $M_\alpha^\beta \cong \mathbb{C}[x,y]_{xy(x+y)} x^{\beta_1}(x+y)^{\beta_3}$ and by Proposition 4, $N = \mathbb{C}[x,y]_{x(x+y)} x^{\beta_1}(x+y)^{\beta_3}$ is an irreducible submodule of M_α^β. Let $M = M_\alpha^\beta/N$ be the quotient module. We are going to show that M is irreducible. First let us prove that the module M is generated by the class $\frac{1}{y}\alpha^\beta$ modulo N. Notice that

$$\partial_x(\frac{1}{y}\alpha^\beta) = (\frac{\beta_1}{xy} + \frac{\beta_3}{y(x+y)})\alpha^\beta = (\frac{\beta_1+\beta_3}{xy} - \frac{\beta_3}{x(x+y)})\alpha^\beta \equiv \frac{\beta_1+\beta_3}{xy}\alpha^\beta \mathrm{modN}. \tag{8.4}$$

Using the same decomposition as in (8.4) for $k > 1$ we have that

$$\partial_x^k(\frac{1}{y}\alpha^\beta) = (\beta_1+\beta_3)(\beta_1+\beta_3-1)...(\beta_1+\beta_3-(k-1))\frac{1}{x^k y}\alpha^\beta \in M. \tag{8.5}$$

Since the coefficient in (8.5) is non-zero, by assumption, $\frac{1}{x^k y}\alpha^\beta \in A_2(\frac{1}{y}\alpha^\beta)$ for all $k \geq 0$. On the other hand

$$\partial_y(\frac{1}{x^k y}\alpha^\beta) = \frac{-1}{x^k y^2}\alpha^\beta + \frac{\beta_3}{x^{k+1}y}\alpha^\beta - \frac{\beta_3}{x^{k+1}(x+y)}\alpha^\beta$$

and $\frac{\beta_3}{x^{k+1}(x+y)}\alpha^\beta \in N$. This implies

$$\partial_y(\frac{1}{x^k y}\alpha^\beta) \equiv \frac{-1}{x^k y^2}\alpha^\beta + \frac{\beta_3}{x^{k+1}y}\alpha^\beta \mathrm{modN}.$$

Let $D_k = [\frac{1}{(\beta_1+\beta_3)...(\beta_1+\beta_3-(k-1))}]\partial_x^k$, for $k \geq 1$. Then

$$(\beta_3 D_{k+1} - \partial_y D_k)(\frac{1}{y}\alpha^\beta) \equiv \frac{1}{x^k y^2}\alpha^\beta \mathrm{modN}$$

and hence $\frac{1}{x^k y^2}\alpha^\beta \in A_2(\frac{1}{y}\alpha^\beta)$. Following a similar argument one can easily show that $\frac{1}{x^k y^m}\alpha^\beta \in A_2(\frac{1}{y}\alpha^\beta)$ for $k, m \geq 1$ and hence $M = A_2(\frac{1}{y}\alpha^\beta)$.
Next we are going to show that M is irreducible. Let $P \in M \setminus \{0\}$. By simplifying as before, we can assume that

$$P = \frac{\sum_{i=0}^{k} c_i x^i}{y}\alpha^\beta.$$

Consider the following formulas

$$\prod_{j=0}^{k-1}(x\partial_x - (\beta_1 + \beta_3 + j)(P) = \frac{k! c_k x^k}{y}\alpha^\beta \in M \tag{8.6}$$

and

$$\partial_x^k(\frac{x^k}{y}\alpha^\beta) = (1 + \beta_1 + \beta_3)...(k + \beta_1 + \beta_3)\frac{1}{y}\alpha^\beta \in M \tag{8.7}$$

Since the coefficient of $\frac{1}{y}\alpha^\beta$ in (8.7) is non-zero, by assumption, we have that, $M \subset A_1(P)$. But P was an arbitrary element, so this means M is irreducible. Therefore

$$\mathbb{C}[x, y]_{xy} x^{\beta_1} y^{\beta_2} \subset \mathbb{C}[x, y]_{xy(x+y)} x^{\beta_1} y^{\beta_2} \cong M_\alpha^\beta$$

is a composition series of M_α^β and hence c(M_α^β)=2. This proves (ii).

(iii) WLOG assume that $\beta_2 \in \mathbb{Z}$ and $\beta_1, \beta_3 \in \mathbb{C} \setminus \mathbb{Z}$. By Proposition 2, $M_\alpha^\beta \cong \mathbb{C}[x, y]_{xy(x+y)} x^{\beta_1}(x + y)^{\beta_3}$ and by Proposition 4, $N = \mathbb{C}[x, y]_{x(x+y)} x^{\beta_1}(x+y)^{\beta_3}$ is an irreducible submodule of M_α^β. By Proposition 2, assume that $\beta_1 + \beta_2 + \beta_3 = 0$. Using the arguments in the proof of (ii), one can easily show that the quotient module $M = M_\alpha^\beta/N$ is generated by $\frac{1}{xy}\alpha^\beta$. Clearly $A_2(\frac{1}{y}\alpha^\beta)$ is a submodule of M. First observe that,

$$\partial_x(\frac{1}{y}\alpha^\beta) = (\beta_1 + \beta_3)\frac{1}{xy}\alpha^\beta \equiv 0 \text{modN}.$$

We are now going to show that $A_2(\frac{1}{y}\alpha^\beta)$ is a proper submodule of M. Assume that $\frac{1}{xy}\alpha^\beta \in A_2(\frac{1}{y}\alpha^\beta)$. Then $\frac{1}{xy}\alpha^\beta = D(\frac{1}{y}\alpha^\beta)$, for some $D \in A_2$. For sufficiently large m, $\partial_x^m D(\frac{1}{y}\alpha^\beta) = D'\partial_x(\frac{1}{y}\alpha^\beta)$ for some $D' \in A_2$ and $D'\partial_x(\frac{1}{y}\alpha^\beta) = 0$. But

$$\partial_x^m \frac{1}{xy}\alpha^\beta = \prod_{i=1}^{m-1}(\beta_1 + \beta_3 - i)\frac{1}{x^m y}\alpha^\beta$$

and

$$\gamma = (\beta_1 + \beta_3 - 1)(\beta_1 + \beta_3 - 2)...(\beta_1 + \beta_3 - (m - 1)) \neq 0,$$

which implies $\partial_x^m \frac{1}{xy}\alpha^\beta = \frac{\gamma}{x^m y}\alpha^\beta \neq 0$. This is a contradiction. Therefore $A_2(\frac{1}{y}\alpha^\beta)$ is a proper submodule of M.

Next we want to show that $A_2(\frac{1}{y}\alpha^\beta)$ is an irreducible submodule of M. Let $Q \in A_2(\frac{1}{y}\alpha^\beta) \setminus \{0\}$. Then

$$Q = \frac{\sum\limits_{i=0}^{k} \alpha_i x^i}{y}\alpha^\beta, a_k \neq 0.$$

Using (8.6) and (8.7) we have that $A_2(\frac{1}{y}) \subset A_2 Q$. Since Q was arbitrary, $A_2(\frac{1}{y})$ is irreducible.

It remains to show that $M/A_2(\frac{1}{y})$ is irreducible. Let $R \in M \setminus A_2(\frac{1}{y})$. Then

$$R = \sum_{i,j\geq 1} \frac{a_{ij}}{x^i y^j}\alpha^\beta, a_{ij} \neq 0.$$

We can assume that $i \geq j$. This is possible because otherwise for sufficiently large m, we can take $\partial_x^m R$. Let k be the maximum of all j such that

$$R = \sum_{i,j\geq 1} \frac{a_{ij}}{x^i y^j}\alpha^\beta.$$

Then

$$y^{k-1}R = \sum_{i=k}^{r} \frac{a_i}{x^i}(\frac{1}{y}\alpha^\beta), a_r \neq 0$$

and

$$x^{r-1}\sum_{i=k}^{r} \frac{a_i}{x^i}(\frac{1}{y}\alpha^\beta) = \frac{a_r}{xy}\alpha^\beta.$$

Since $a_r \neq 0$ and R was arbitrary, this implies that $A_2 R = M/A_2(\frac{1}{y})$ is irreducible. This completes the proof. □

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