ISSN: 1401-5617

Integral representation formulas associated with toric varieties

Alexey Shchuplev

Research Reports in Mathematics
Number 1, 2005

Department of Mathematics
Stockholm University

Electronic versions of this document are available at
http://www.math.su.se/reports/2005/1

Date of publication: January 17, 2005
2000 Mathematics Subject Classification: Primary 32A26, Secondary 14M25.
Keywords: integral representations, toric varieties, differential forms.

Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.math.su.se/
info@math.su.se

Integral representation formulas associated with toric varieties

Alexey Shchuplev

Abstract

A finite family $\{Z_\nu\}$ of planes in $\mathbb{C}^d$ is called atomic if the top non trivial homology group $H_k\left(\mathbb{C}^d \setminus \bigcup_\nu Z_\nu, \mathbb{Z}\right)$ is generated by a single element. One shows that families of coordinate planes giving rise to the concept of a toric variety are atomic. For this class of atomic families, the thesis presents a construction of a cycle γ and of a differential form η that generate the indicated homology group and the dual de Rham cohomology group, respectively. New integral formulas of the Bochner-Ono type with kernels η for holomorphic functions in special bounded domains of $\mathbb{C}^d$ are obtained.

Contents

1. Introduction

Integral representations and multidimensional residues are among the most important tools in several complex variables. Many significant results of theoretical and practical importance have been proved by means of them.

The Cauchy formula for product domains was first obtained in the pioneering work by Poincaré (1887). This has given rise to the theory of multidimensional resudues in its classical shape as the integration of closed differential forms over cycles. This formula has allowed to prove basic properties of holomorphic functions in several dimensions, such as the series expansion and the uniqueness of analytic continuation. Later, using the Cauchy formula, Hartogs in 1906 was able to prove that there exist certain domains in $\mathbb{C}^d$, $d > 1$ such that any holomorphic function on these domains admits an analytic continuation to some larger domain.

There are also other general integral representations that have played a significant role in the development of complex analysis. The Bochner-Martinelli formula (1938) has allowed to prove Hartogs' theorem about removal of compact singularities of holomorphic functions and its generalizations. By using the more complicated Bergman-Weil formula the multidimensional Runge theorem on polynomial approximation of holomorphic functions as well as solutions to the Poincaré and Cousin problems have been obtained. The works of Leray and Lelong have brought new ideas into multidimensional residue theory and stimulated the investigation of residue currents.

1.1. The Bochner-Martinelli and Cauchy formulas

There is a good number of various formulas for integral representation of holomorphic functions apart from the ones mentioned above. The Bochner-Martinelli and the Cauchy formulas are standard among them, and many others have been deduced from them.

Let us take a close look at them. At the first glance these formulas look quite different. Indeed, provided a function f is holomorphic in the closure of a bounded domain D in $\mathbb{C}^d$ with piece-wise smooth boundary ∂D, the Bochner-Martinelli formula is

$$f(z) = \frac{(d-1)!}{(2\pi i)^d} \int_{\partial D} f(\zeta)\eta_{BM}(\zeta - z),$$

where the form η_{BM} in the standard multi-index notation

$$|\zeta - z|^2 = |\zeta_1 - z_1|^2 + \cdots + |\zeta_d - z_d|^2,$$

$d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_d$, and $d\zeta_{[k]}$ denoting the same product as $d\zeta$ but with the k-th differential omitted is

$$\eta_{BM}(\zeta - z) = \frac{\sum_{k=1}^{d}(-1)^{k-1}(\bar{\zeta}_k - \bar{z}_k)\, d\bar{\zeta}_{[k]} \wedge d\zeta}{|\zeta - z|^{2d}},$$

and represents values $f(z)$ at every point z of D. The Cauchy formula looks simpler

$$f(z) = \frac{1}{(2\pi i)^d} \int_{\Gamma} f(\zeta)\eta_C(\zeta - z), \quad \text{where}$$

$$\eta_C(\zeta - z) = \frac{d\zeta_1}{\zeta_1 - z_1} \wedge \cdots \wedge \frac{d\zeta_d}{\zeta_d - z_d}$$

but applies only if the domain in question is a polydisc $D = \{z \in \mathbb{C}^d \colon |\zeta_1 - z_1| < r_1, \ldots, |\zeta_d - z_d| < r_d\}$. The integration set in this case is the distinguished boundary of the polydisc $\Gamma = \{z \in \mathbb{C}^d \colon |\zeta_1 - z_1| = r_1, \ldots, |\zeta_d - z_d| = r_d\}$. What do these formulas have in common that makes them very powerful and how one can clarify the connection between them?

To answer this question, August Tsikh [15] proposed to consider the singularities of *kernels* η_{BM} and η_C from a new angle. In our examples, the singularity of the Cauchy kernel centered at the origin consists of all coordinate hyperplanes $Z_C = \bigcup_{k=1}^{d}\{z \in \mathbb{C}^d \colon z_k = 0\}$, while the singular set of the Bochner-Martinelli kernel (also centered at the origin) is a single point $Z_{BM} = \{0\}$. These two set possess one common property, which we shall mark out, namely, they are unions of linear subspaces of $\mathbb{C}^d$. But we can say more than this, namely, the complements of Z_C and Z_{BM} are homotopy equivalent to oriented compact real manifolds with the top non-trivial homology groups being generated by a single element:

$$\begin{aligned}
&\mathbb{C}^d \setminus Z_C \simeq \underbrace{S^1 \times \cdots \times S^1}_{d \text{ times}} \text{ and } H_k(\mathbb{C}^d \setminus Z_C, \mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } k = d, \\ 0 \text{ if } k > d; \end{cases} \\
&\mathbb{C}^d \setminus Z_{BM} \simeq S^{2d-1} \text{ and } H_k(\mathbb{C}^d \setminus Z_{BM}, \mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } k = 2d-1, \\ 0 \text{ if } k > 2d-1. \end{cases}
\end{aligned} \tag{1}$$

We shall consider this as a key property providing the link between the Cauchy and the Bochner-Martinelli kernels, and a whole family of integral kernels that we obtain in the present work. Let us give a definition

Definition 1.1 ([15]). *A finite family $\{Z_\nu\}_{\nu\in\mathcal{N}}$ of linear subspaces of $\mathbb{C}^d$ is said to be atomic if the top non trivial homology group $H_k\left(\mathbb{C}^d \setminus \bigcup_\nu Z_\nu, \mathbb{Z}\right)$ is generated by a single element, in other words, such $\{Z_\nu\}_{\nu\in\mathcal{N}}$ that there exists an integer $k_0 \in \mathbb{N}$ such that*

$$H_k\left(\mathbb{C}^d \setminus \bigcup_\nu Z_\nu, \mathbb{Z}\right) = \begin{cases} \mathbb{Z} \textit{ if } \ k = k_0, \\ 0 \textit{ if } \ k > k_0. \end{cases}$$

A generator η of the dual de Rham cohomology class $H^{k_0}\left(\mathbb{C}^d \setminus \bigcup_\nu Z_\nu\right)$ is then said to be a kernel for the atomic family $\{Z_\nu\}_{\nu\in\mathcal{N}}$.

Given an atomic family Z in $\mathbb{C}^d$, the problem is to construct a C^∞ differential form on $\mathbb{C}^d$ with Z as its singular set and then to show that this form is a kernel and produces some integral representation of holomorphic functions in $\mathbb{C}^d$.

1.2. Outline of the thesis

This thesis presents a partial solution to the stated problem. More precisely, we give the construction of kernels for a special but rather wide class of atomic families and prove integral reresentation formulas for holomorphic functions in special bounded domains of $\mathbb{C}^d$.

Let us note that not every family of coordinate subspaces is atomic. For example, the set of three lines

$$\{z_1 = z_2 = 0\} \cup \{z_2 = z_3 = 0\} \cup \{z_1 = z_3 = 0\}$$

in $\mathbb{C}^3$ is not atomic (see [16]). This raises the question of which families of linear subspaces of $\mathbb{C}^d$ are atomic. The answer to this general question is still unknown. There is however a class of such families that are known to be atomic. They appear in the theory of toric varieties.

In the Section 2 we give the definition of toric varieties according to [6] and basic facts of toric geometry that will be used in the text. The construction due to D. Cox is the most suitable for solving the problem because it represents the varieties as quotient spaces of an affine space minus a family of coordinate planes under the action of a group:

$$X = (\mathbb{C}^d \setminus Z)/G.$$

Provided that certain combinatorial conditions are fulfilled, the exceptional set Z is atomic.

Given an atomic family Z coming from the representation of a toric variety, to get a hint how to construct a kernel we turn to the basic example of the Bochner-Martinelli kernel.

Example 1.1.

It is well-known that the differential form

$$\frac{dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n}{(1 + |z_1|^2 + \cdots + |z_n|^2)^{n+1}}$$

coincides up to a constant factor with the volume form on the projective space $\mathbb{P}_n$ written in local coordinates (it is called the Fubini-Study volume form). One can easily verify that the form

$$\frac{dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n}{(1 + |z_1|^2 + \cdots + |z_n|^2)^{n+1}} \wedge \frac{dz_{n+1}}{z_{n+1}}$$

after the change of coordinates

$$z_1 = \frac{\zeta_1}{\zeta_{n+1}}, \ldots, \; z_n = \frac{\zeta_n}{\zeta_{n+1}}, \; z_{n+1} = \frac{1}{\zeta_{n+1}}$$

turns into $(-1)^{1+\cdots+n}\eta_{BM}$. Notice now that this change of coordinates gives the transition functions between two charts of $\mathbb{P}_{n+1}$ such that $\mathbb{P}_n$ turns to the hyperplane at infinite of $\mathbb{P}_{n+1}$. So there is a clear geometric description of the situation. The affine space $\mathbb{C}^{n+1}$ is compactified to $\mathbb{P}_{n+1}$ by gluing $\mathbb{P}_n$ at infinity, and the Bochner-Martinelli kernel on $\mathbb{C}^{n+1}$ is the same form (up to the sign) as the volume form on the hyperplane at infinity multiplied by the Cauchy kernel.

We follow this example in our construction. Thanks to Theorem 4 proved in [16] one can embedd an n-dimensional toric variety $X = (\mathbb{C}^d \setminus Z)/G$ into a certain d-dimensional toric variety $\widetilde{X}$ as the 'skeleton at infinity'. This theorem is the direct analogue of the decomposition $\mathbb{P}_{n+1} = \mathbb{C}^{n+1} \sqcup \mathbb{P}_n$ and in Section 3 we show how to make a kernel η in $\mathbb{C}^d$ with singularity along Z using the volume form ω on the'skeleton at infinity' X. More precisely, the following theorem holds.

Theorem 5. *The differential $(d,\, n)$-form*

$$\eta = (-1)^n \omega([\zeta]) \wedge \frac{d\zeta_{n+1}}{\zeta_{n+1}} \wedge \cdots \wedge \frac{d\zeta_d}{\zeta_d}$$

in $\mathbb{C}^d$ is a kernel for the atomic family Z.

Here the volume form ω is written in the homogeneous coordinates of X. We call this kernel associated with the toric variety X.

Having obtained a kernel, we prove the formula of integral representation of holomorphic functions in special bounded domains of $\mathbb{C}^d$.

Theorem 6. *Let f be holomorphic in the closure of a polyhedron U_ρ defined by the system of $r = d - n$ inequalities $a_{j1}|\zeta_1|^2 + \cdots + a_{jd}|\zeta_d|^2 < \rho_j$, $j = 1, \ldots, r$ with all a_{ji} non-negative and $\rho \in K \subset \mathbb{R}^r_+$, and γ be the distinguished boundary of U_ρ defined by equalities instead of inequalities in the system. Then for every point z of a certain polyhedron $D \subset U_\rho$*

$$f(z) = \frac{1}{(2i)^d \pi^r \mathrm{Vol}(X_\Sigma)} \int\limits_\gamma f(\zeta)\eta(\zeta - z).$$

The integral represent the values of the function in the subdomain $D \subset U_\rho$, so we get an integral representation of the Bochner-Ono type [1]. Also, in the proof a precise description for a canonical generating cycle of the top non-trivial homology group of $\mathbb{C}^d \setminus Z$ is given.

Section 4 presents a class of natural volume forms for projective complete simplicial toric varieties (Definition 4.1), followed by several examples of integral kernels associated with them in Section 5. Moreover, we compute the volumes of varieties with respect to these natural forms in terms of volume of polytopes Δ in $\mathbb{R}^n$ associated with the varieties.

Proposition 4.1.

$$\mathrm{Vol}(X_\Sigma) = \pi^n \mathrm{Vol}(\Delta).$$

For this particular choice of volume forms we reformulate Theorem 6.

Theorem 6′. *Let f be holomorphic in the closure of the domain U_ρ and γ be the domain's distinguished boundary. Then for every $z \in D \subset U_\rho$*

$$f(z) = \frac{1}{(2\pi i)^d \mathrm{Vol}(\Delta)} \int\limits_\gamma f(\zeta)\eta(\zeta - z).$$

The preliminary results of the thesis have been published in [17] and [18].

I would like to thank my supervisors August Tsikh and Mikael Passare for the inspiration and many fruitful discussions.

2. Toric varieties

Toric varieties are algebraic varieties and they are generalizations of both affine and projective spaces. Additionally, the class of toric varieties includes all their products and many other. They are almost as simple to study but appear to be more convenient in many cases. It seems that the first definition of a toric variety is due to M. Demazure and says that an n-dimensional toric variety is a variety on which the action of the *algebraic torus* $\mathbb{T}^n = (\mathbb{C} \setminus \{0\})^n$ on itself extends to an action on the whole variety. The algebraic torus, called also the *complex* torus, acts on itself by the component-wise multiplication. A toric variety then is a disjoint union of the 'big' torus $\mathbb{T}^n$ and *something*, which is invariant under the extended action.

In the present work we use an equivalent definition of toric variety that is suitable for our construction. Moreover, the statement that toric varieties are generalizations of projective spaces becomes clearer.

2.1. The definition and construction

To start with, let us first recall the construction and basic facts of the projective spaces.

Example 2.2. The projective space.

Usually, the projective space $\mathbb{CP}_n$ is defined as the set of all lines passing through the origin in $\mathbb{C}^{n+1}$. The same definition can be reformulated as follows. Consider the equivalence relation $\sim$ on the set of non-zero points $\mathbb{C}^{n+1} \setminus \{0\}$ defined by

$$x \sim y \text{ iff } y = (\lambda x_1, \dots, \lambda x_{n+1}) \text{ for } \lambda \in \mathbb{C} \setminus \{0\}.$$

Then the projective space is the set of all equivalence classes.

Every point $x = (x_1, \dots, x_{n+1}) \neq 0$ determines an element of the projective space, namely the line passing through the points x and 0. This line is the equivalence class of all points proportional to x. As only the ratio of coordinates is then of interest, the equivalence class is commonly denoted by the $(n+1)$-tuple of *homogeneous coordinates* $(x_1 : \dots : x_{n+1})$. It is useful sometimes to interprete homogeneous coordinates of a point in $\mathbb{P}_n$ as the Cartesian coordinates of a point in $\mathbb{C}^{n+1}$ (lying in the corresponding equivalence class).

Note that a subset of $\mathbb{P}_n$ defined by $x_i \neq 0$ is homeomorphic to $\mathbb{C}^n$ via

$$(x_1\colon \dots : x_{n+1}) \mapsto \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i}\right). \tag{2}$$

Therefore the projective space $\mathbb{P}_n$ has a canonical covering $\mathcal{U}$ by $(n+1)$ open sets $U_i = \{x_i \neq 0\}$. Introducing local coordinates in every U_i according to (2), we endow the projective space with the structure of complex manifold. Indeed, the transition functions between the chart U_i with local coordinates u and the chart U_j $(i < j)$ with local coordinates v

$$u_1 = \frac{v_1}{v_i}, \dots, u_{i-1} = \frac{v_{i-1}}{v_i}, u_i = \frac{v_{i+1}}{v_i}, \dots, u_{j-1} = \frac{1}{v_i}, u_j = \frac{v_j}{v_i}, \dots, u_d = \frac{v_n}{v_i}$$

are analytic in $U_i \cap U_j$. But what is more important is that they are monomial. It is this feature that allows to use algebraic methods while studying analytic properties of projective spaces and vice versa. The class of toric varieties is a generalization of projective spaces preserving the monomiality of transition functions (see [7]).

There are several approaches to the notion of a toric variety and several definitions ([9, 12, 6]). They give different constructions but all of them based on the fact that all analytic or algebraic properties of toric varieties, thanks to the monomiality of the transition functions, can be expressed in a purely combinatorial way. The combinatorial object associated with a toric variety is a so-called *fan*. We start with definitions.

A subset σ of $\mathbb{R}^n$ is called a *strongly convex rational polyhedral cone* if there exists a finite number of elements $v_1, \dots, v_s$ in the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ (*integral generators*) such that σ is generated by them, i.e.

$$\sigma = \{a_1 v_1 + \dots + a_s v_s \colon a_i \in \mathbb{R}, a_i \geq 0\},$$

and σ does not contain any line. We say that a subset τ of σ given by some a_i being equal to zero is a *proper face* of σ and write $\tau < \sigma$. Faces of a cone are cones also. The *dimension* of a cone σ is, by definition, the dimension of a minimal subspace of $\mathbb{R}^n$ containing σ. A cone σ is called *simplicial* if its generators can be chosen to be linearly independent. An n-dimensional simplicial cone is said to be *primitive* if its n generators form a basis of the lattice $\mathbb{Z}^n$.

Definition 2.1. *A (n-dimensional) fan in $\mathbb{R}^n$ is a non-empty collection Σ of strongly convex rational polyhedral cones in $\mathbb{R}^n$ satisfying the following conditions:*

1. *Every face of any σ in Σ is contained in Σ.*
2. *For any σ, σ' in Σ, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.*

The set $|\Sigma| = \bigcup\limits_{\sigma \in \Sigma}$ is called the support of Σ.

A fan is also called a *rational polyhedral decomposition.* The *dimension* of a fan is the maximal dimension of its cones. An n-dimensional fan is *simplicial* (*primitive*) if all its n-dimensional cones are simplicial (primitive). In the case $|\Sigma| = \mathbb{R}^n$, the fan in $\mathbb{R}^N$ is called *complete.*

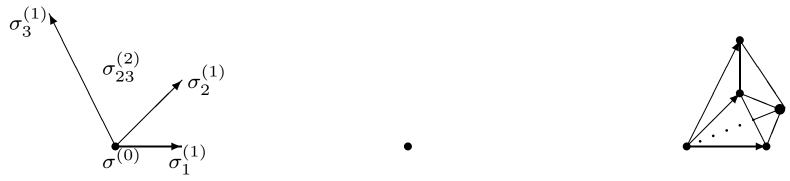


Figure 1: Examples of fans.

Example 2.3. Examples of fans in $\mathbb{R}^2$ and $\mathbb{R}^3$.

The first fan on Fig. 1 in $\mathbb{R}^2$ consists of one zero-dimensional cone, three one-dimensional cones and one two-dimensional generated by $\sigma_2^{(1)}$ and $\sigma_3^{(1)}$. It is obviously not complete. The second fan consisting only of the origin can be seen as a fan of any dimension and the corresponding variety depends on this choice. The third fan comprises of two contiguous three-dimensional cones and all their faces.

For any given n-dimensional fan Σ one can construct an n-dimensional toric variety X_Σ. Let the cones of Σ be generated by d *integral generators* $v_1, \dots, v_d$ (we may think of them as integral vectors). Assign a variable ζ_i to each generator v_i. For every n-dimensional cone $\sigma \in \Sigma$, let $\zeta_{\hat\sigma}$ be the monomial

$$\zeta_{\hat\sigma} := \prod_{\substack{j \in \{1,\dots,d\} \\ v_j \notin \sigma}} \zeta_j$$

and $Z(\Sigma) \subset \mathbb{C}^d$ be the zero set of the ideal generated by such monomials $\zeta_{\hat\sigma}$ in $\mathbb{C}[\zeta_1, ..., \zeta_d]$, i.e.

$$Z(\Sigma) = \{\zeta \in \mathbb{C}^d : \zeta_{\hat\sigma} = 0 \text{ for all } n\text{-dimensional cones } \sigma \text{ in } \Sigma\}. \tag{3}$$

Evidently, the set Z consists of coordinate planes, in general, of different dimensions.

In the case Σ is an n-dimensional complete simplicial fan, there is an equivalent construction of the same set, due to V. Batyrev [3]. A subset of generators $\mathcal{P} = \{v_{i_1}, \dots, v_{i_k}\}$ is called a *primitive collection* if they do not generate any cone of Σ but so does every proper subset of them. Then $Z(\Sigma)$ coincides with the union of coordinate planes

$$Z(\Sigma) = \bigcup_{\mathcal{P}} \{\zeta_{i_1} = \dots = \zeta_{i_k} = 0\},$$

where the union is taken over all primitive collections.

To define a group acting on $\mathbb{C}^d \backslash Z$, one considers a *lattice of relations* between generators of one-dimensional cones of Σ. In other words, one considers $r = d - n$ independent linear relations over $\mathbb{Z}$ between $v_1, \dots, v_d$:

$$\begin{cases} a_{11}v_1 + \dots + a_{1d}v_d = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{r1}v_1 + \dots + a_{rd}v_d = 0. \end{cases} \tag{4}$$

The group G is then an r-dimensional surface

$$G = \{(\lambda_1^{a_{11}} \dots \lambda_r^{a_{r1}}, \dots, \lambda_1^{a_{1d}} \dots \lambda_r^{a_{rd}}) \colon \lambda_i \in \mathbb{T}\} \subset \mathbb{T}^d. \tag{5}$$

and is therefore isomorphic to $\mathbb{T}^r$. The action of G on $\mathbb{C}^d \setminus Z$ defines an equivalence relation

$$\xi \sim \zeta \text{ iff } \exists \lambda \in \mathbb{T}^r : \xi = G \cdot \zeta = (\lambda_1^{a_{11}} \ldots \lambda_r^{a_{r1}} \zeta_1, \ldots, \lambda_1^{a_{1d}} \ldots \lambda_r^{a_{rd}} \zeta_d). \quad (6)$$

Then it follows from [6], although many worked in this direction, that for a simplicial fan Σ the quotient space

$$X_\Sigma = (\mathbb{C}^d \setminus Z) \big/ G, \quad (7)$$

with Z and G constructed as above, is well-defined. We take this representation as the definition of *simplicial toric variety*.

Definition 2.2. *Let Σ be a simplicial fan in $\mathbb{R}^n$. Then the simplicial toric variety X_Σ associated to the fan Σ is the quotient space* (7).

The d-dimensional torus $\mathbb{T}^d$ acts by component-wise multiplication on $\mathbb{C}^d \setminus Z$. This action descends to an action of

$$T \simeq \mathbb{T}^d / G \simeq \mathbb{T}^n$$

on X_Σ. The image of the subset $\mathbb{T}^d \subset (\mathbb{C}^d \setminus Z)$ in X_Σ is homeomorphic to $\mathbb{T}^n$. The torus T acts naturally on $\mathbb{T}^n \subset X_\Sigma$ ('big' torus of X_Σ) and this action extends to the action of T on the whole X_Σ. The rest $X_\Sigma \setminus \mathbb{T}^n$ is the image of $(\mathbb{C}^d \setminus Z) \setminus \mathbb{T}^d$, which is invariant under the action of $\mathbb{T}^d$. Therefore, the dimension of $X_\Sigma \setminus \mathbb{T}^n$ is less than n and $X_\Sigma \setminus \mathbb{T}^n$ is T-invariant. So, this definition is compatible with that given at the beginning.

Example 2.4. The projective space as a toric variety.

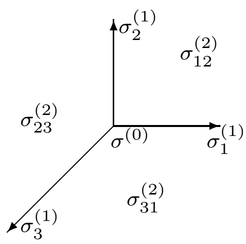


Figure 2: The fan of the projective plane.

The complete fan on the Fig. 2 is the polyhedral decomposition of $\mathbb{R}^2$ into three two-dimensional simplicial cones, three one-dimensional and one zero-dimensional, the origin, and it corresponds to the projective plane.

In the general case, the fan corresponding to $\mathbb{P}_n$ is formed by $n+1$ cones of the maximal dimension in $\mathbb{R}^n$. Let us describe this fan in detail. Fix first the basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. Then the one-dimensional generators of the cones are

$$v_1 = e_1, \ldots, v_n = e_n, \, v_{n+1} = -e_1 - \cdots - e_n.$$

There are $n+1$ simplicial n-dimensional cones σ_0 generated by $\{v_1, \ldots, v_n\}$ that coincides with the positive orthant $\mathbb{R}^n_+$ and σ_i, $i = 1, \ldots, n$ with lists of generators

$$\{v_1, \ldots, v_{i-1}, v_{n+1}, v_{i+1}, \ldots, v_n\}.$$

It is easy to see that these cones with all their faces form a complete fan.

According to (3), the exceptional set Z in the case of the projective space is the common zero of the monomials $\zeta_1, \dots, \zeta_{n+1}$, which is the origin. The integral generators are, of course, linearly dependent, but there is only one identity of the kind (4), namely,

$$v_1 + \dots + v_{n+1} = 0.$$

Therefore the group G is the algebrac torus $\mathbb{T}$ acting on $\mathbb{C}^{n+1} \setminus \{0\}$ by the component-wise multiplication, and the representation (7) coincides with the regular definition of the projective space.

This example shows that ζ_i assigned to the one-dimensional generators of the fan are nothing else but homogeneous coordinates, and this term has been preserved for toric varieties. The meaning of these coordinates is exactly the same as in the case of projective space, the homogeneous coordinates of a point of X_Σ are the coordinates of a point in $\mathbb{C}^d$ from the corresponding equivalence class. The equivalence classes now are not lines passing the origin but the orbits of the action of group G that are r-dimensional surfaces in $\mathbb{C}^d$. The exceptional set becomes something larger than the origin also. To figure out what the lattice of relations (4) means, let us turn to the projective space again.

Example 2.5. Monomial functions on the projective space.

As far as one has local coordinates in charts of the projective space, one can define a monomial $u^\alpha = u_1^{\alpha_1} \dots u_n^{\alpha_n}$ in terms of local coordinates in one of the charts. This monomial function extends to the whole space and one can determine how it looks in other charts by means the transition functions. On the other hand, one can rewrite the monomial in homogeneous coordinates to get a globally defined function in $\mathbb{C}^{n+1}$. Of course, one has to use (2) to study the function in different charts.

But not every globally defined monomial in $\mathbb{C}^{n+1}$ gives a function on the projective space, it is subject to the special condition. Indeed, consider a monomial in homogeneous coordinates $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n} x_{n+1}^{\alpha_{n+1}}$. To rewrite it in the local coordinates of, say, U_{n+1} we rearrange it and get

$$x^\alpha = \left(\frac{x_1}{x_{n+1}}\right)^{\alpha_1} \dots \left(\frac{x_n}{x_{n+1}}\right)^{\alpha_n} x_{n+1}^{\alpha_1 + \dots + \alpha_{n+1}}.$$

It is clear that $\alpha_1 + \dots + \alpha_{n+1}$ must equal zero for this monomial to define a function on the projective space.

So, the group G acting on $\mathbb{C}^d \setminus Z$ acts naturally on the *homogeneous coordinate ring* $\mathbb{C}[\zeta_1, \dots, \zeta_d]$ and the relations (4) define homogeneous monomials (and the Laurent polynomials) that give rise to well-defined functions on the variety X_Σ.

Note that the exceptional set Z involved in the definition is the union of coordinate planes and most likely is an atomic family. However, one can assert that only in the case of complete fans.

Theorem 1 ([16]). *Let Σ be a complete simplicial fan in $\mathbb{R}^n$. Then $Z(\Sigma)$ is an atomic family of coordinate planes.*

Roughly speaking, the proof of the assertion follows from the fact that a toric variety corresponding to a complete fan is compact and homotopy equivalent to an oriented compact real manifold (compare with (1)).

Furthermore, complete fans have the following property that we use in the construction.

Proposition 2.1. *Let Σ be a complete fan in $\mathbb{R}^n$ with d generators. Then all coefficients a_{ij} in the lattice of relations* (4) *can be chosen non-negative.*

This fact seems to be well-known. Here we give a simple proof.

Proof. Let the coefficient a_{jk} in the identity $a_{j1}v_1 + \cdots + a_{jd}v_d = 0$ be negative. Consider then the vector $-v_k$. For the fan is complete, this vector lies in some cone generated by the vectors $v_{i_1}, \ldots, v_{i_m}$ and therefore can be represented as linear combination of them $-v_k = b_{i_1}v_{i_1} + \cdots + b_{i_m}v_{i_m}$ with all non-negative coefficients. The identity we started with is equivalent then to $a_{j1}v_1 + \cdots + a_{jd}v_d + |a_{ij}|(v_k + b_{i_1}v_{i_1} + \cdots + b_{i_m}v_{i_m}) = 0$ with the coefficient at v_k being equal zero and containing no new negative coefficients. Proceeding in this way we get all a_{ij} being non-negative. $\square$

2.2. Projective toric varieties

There are many reasons to consider compact *projective* simplicial toric varieties, i.e., those that can be embedded into some projective space and the construction of integral kernels given in the present work involves only them. The criterion for a toric variety to be projective can, of course, be expressed in terms of its fan or in terms of the polytope Δ dual to the fan.

Let Σ be a complete simplicial fan in $\mathbb{R}^n$. A real valued function $h : |\Sigma| \to \mathbb{R}$ on the support of the fan Σ is said to be a *Σ-linear strictly convex support function* if

1. for each $\sigma \in \Sigma$, there exists $m_\sigma \in \mathbb{Z}^n$ such that $h(x) = \langle m_\sigma, x\rangle$ for $x \in \sigma$;
2. $\langle m_\sigma, x\rangle = \langle m_\tau, x\rangle$ whenever $x \in \tau < \sigma$;
3. $h(x) + h(y) \geq h(x + y)$ for $x, y \in |\Sigma|$;
4. $m_\sigma \neq m_{\sigma'}$ for different n-dimensional cones of Σ.

The convex set

$$\Delta = \{m \in \mathbb{Z}^n \colon \langle m, x\rangle \geq h(x), \forall x \in \mathbb{R}^n\}$$

is an n-dimensional (non-empty) polytope with integer vertices $\{m_\sigma\}$, for all n-dimensional cones σ.

Definition 2.3. *The convex hull of a finite number of points in $\mathbb{Z}^n$ is called a simple integral polytope if it is n-dimensional, and each of its vertex is a point of the lattice $\mathbb{Z}^n$ and belongs to exactly n edges. The simple polytope is absolutely simple if, in addition, minimal integer vectors on n edges meeting at a vertex generate the lattice $\mathbb{Z}^n$.*

It turns out that for a fan Σ to possess a dual polytope Δ is equivalent to the condition that X_Σ can be embedded into the projective space as a closed subvariety.

Theorem 2 ([8, 7, 12]). *An n-dimensional compact simplicial toric variety X_Σ is projective if and only if the simple polytope Δ dual to Σ is n-dimensional and $m_\sigma \neq m_{\sigma'}$ for different n-dimensional cones of Σ.*

Let $A = \Delta \cap \mathbb{Z}^n = \{\alpha_1, \dots, \alpha_N\}$, then the map

$$f : X_\Sigma \to \mathbb{P}_{N-1} \quad z \mapsto (z^{\alpha_1} : \dots : z^{\alpha_N}) \tag{8}$$

is a closed embedding.

Note that starting from a simple polytope Δ we can construct its normal fan such that the condition of the theorem is satisfied and the corresponding complete simplicial toric variety is projective.

Consider the case of smooth compact projective toric varieties in more details because there is even more information one can recover from the fan. The fan associated to such a variety is complete and primitive [12, Theorem 1.10] and the dual integral polytope is, according to Theorem 2, absolutely simple. To continue, we need some facts from symplectic geometry.

Let M be a smooth complex manifold endowed with a closed nondegenerated differential form $\omega \in \bigwedge^2 \mathsf{T}^* M$, which makes (M, ω) into a symplectic manifold. The canonical example is a complex plane $\mathbb{C}$ with the form $\omega = -\frac{1}{2i} d\zeta \wedge d\bar{\zeta}$. If M is equipped with a Hermitian metric H and the associated differential form $\omega = -\mathrm{Im}(H)$ is closed then M is called *Kähler* manifold and ω *Kähler* form.

Let G be a Lie group acting on (M, ω) by diffeomorphisms $g \in G\colon \zeta \mapsto g \cdot \zeta$. A group action is called *symplectic* if every diffeomorphism $g \in G$ preserves the symplectic form ω. For every $\zeta \in M$, define a map

$$f_\zeta : G \to M, \; f_\zeta(g) = g \cdot \zeta$$

such that the image of G under this mapping $G \cdot \zeta$ is a *flow* or an *orbit* of the group actions. Its differential map at the point $1 \in G$ is a linear map

$$\mathsf{T}_1 f_\zeta : \mathfrak{g} \to \mathsf{T}_\zeta M$$

associating a tangent vector $\mathsf{T}_1 f_\zeta(X) = \underline{X}_\zeta \in \mathsf{T}_\zeta M$ to every direction $X \in \mathfrak{g}$. When ζ varies in M, we get a vector field $\underline{X}$ called the *fundamental field* associated with X.

Definition 2.4. *A vector field X on a symplectic manifold (M, ω) is called Hamiltonian if $\imath_X \omega$ is exact and locally Hamiltonian if it is closed. One writes $\mathcal{H}(M)$ and $\mathcal{H}_{loc}(M)$ for the spaces of Hamiltonian and locally Hamiltonian vector fields on M, respectively.*

Obviously, there is an exact sequence

$$0 \longrightarrow \mathcal{H}(M) \longrightarrow \mathcal{H}_{loc}(M) \longrightarrow H^1(M; \mathbb{R})$$

As ω is non-degenerate, every C^∞-function f defines a Hamiltonian vector field X_f via $\imath_{X_f} = df$. If the group action is symplectic, then all fundamental vector fields are locally Hamiltonian [2, Prop. 3.1.1.]. Combining this with the sequence, we get the following diagram

$$\begin{array}{ccccccc} & & C^\infty(M) & & \mathfrak{g} & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{H}(M) & \longrightarrow & \mathcal{H}_{loc}(M) & \longrightarrow & H^1(M; \mathbb{R}) \end{array}$$

Definition 2.5. *A symplectic action of G on M is Hamiltonian if there exists a linear map (morphism of Lie algebras) $\widetilde{\mu} : \mathfrak{g} \to C^\infty(M)$ making the diagram commute.*

By duality, there is an associated map μ of dual spaces

$$\mu : (C^\infty(M))^* = M \longrightarrow \mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, \mathbb{R})$$

defined by

$$\mu : \zeta \mapsto (X \mapsto \widetilde{\mu}_X(\zeta))$$

called the *moment map.*

In our case, the set$\mathbb{C}^d \setminus Z$ endowed with the form $\omega = -\frac{1}{2i}\sum_{j=1}^{d} d\zeta_j \wedge d\bar{\zeta}_j$ is a symplectic manifold. Consider the action of the maximal compact subgroup $G_{\mathbb{R}}$ of the group G defined in (5)

$$G_{\mathbb{R}} = \{(\lambda_1^{a_{11}} \dots \lambda_r^{a_{r1}}, \dots, \lambda_1^{a_{1d}} \dots \lambda_r^{a_{rd}}) \colon \lambda_i \in S^1 \subset \mathbb{T}\}$$

The action of $G_{\mathbb{R}}$ is clearly symplectic. The Lie algebra $\mathfrak{t}$ of $G_{\mathbb{R}}$ is isomorphic to $\mathbb{R}^r$ as well as its dual. Denoting the columns of coefficients in (4) by $a^k = (a_{1k}, \dots, a_{rk})$, $k = 1, \dots, d$, we can write down the fundamental field for every $X = (x_1, \dots, x_r) \in \mathbb{R}^r$.

$$\underline{X} = -i\sum_{k=1}^{d}\langle a^k, X\rangle \left(\bar{\zeta}_k\frac{\partial}{\partial\bar{\zeta}_k} - \zeta_k\frac{\partial}{\partial\zeta_k}\right)$$

The interior product of $\underline{X}$ with the symplectic form ω is then

$$\imath_{\underline{X}}\omega = \frac{1}{2}\sum_{k=1}^{d}\langle a^k, X\rangle \left(\zeta_k d\bar{\zeta}_k + \bar{\zeta}_k d\zeta_k\right),$$

which is the full differential with respect to ζ of the function

$$\widetilde{\mu}_X(\zeta) = \frac{1}{2}\sum_{k=1}^{d}\left(\sum_{j=1}^{r} a_{jk}x_j|\zeta_k|^2\right) = \frac{1}{2}\sum_{j=1}^{r}\langle a_j, |\zeta|^2\rangle x_j,$$

where $a_j = (a_{j1}, \dots, a_{jd})$, $j = 1, \dots, r$ are the rows of the coefficients and $|\zeta|^2 = (|\zeta_1|^2, \dots, |\zeta_d|^2)$. So, for every $\zeta \in \mathbb{C}^d \setminus Z$, the image $\mu(\zeta)$ is a point $(\rho_1, \dots, \rho_r)$ in $\mathfrak{t}^* \simeq \mathbb{R}^r$ with coordinates

$$\begin{cases} a_{11}|\zeta_1|^2 + \dots + a_{1d}|\zeta_d|^2 = \rho_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{r1}|\zeta_1|^2 + \dots + a_{rd}|\zeta_d|^2 = \rho_r. \end{cases} \tag{9}$$

For $\rho \in \mu(\mathbb{C}^d \setminus Z) \subseteq \mathfrak{t}^*$, the cycle $\mu^{-1}(\rho)$ is a smooth manifold, but the restriction of the symplectic form ω to $\mu^{-1}(\rho)$ will fail to be symplectic as it will be degenerate. However, it is degenerate only along the orbits of action of $G_{\mathbb{R}}$, then the restriction of ω descends to the quotient $\mu^{-1}(\rho)/G_{\mathbb{R}}$ as a symplectic form. This process is called *symplectic reduction.* In such a way we get the representation of $\mathbb{P}_n$ as the quotient S^{2n+1}/S^1. It turns out that $\mu^{-1}(\rho)/G_{\mathbb{R}}$ is another representation of X_Σ.

Theorem 3 ([5]). *Let Σ be a complete primitive fan in $\mathbb{R}^n$ with d integral generators and $\rho \in \mu(\mathbb{C}^d \setminus Z)$. Then the map*

$$\mu^{-1}(\rho)/G_{\mathbb{R}} \longrightarrow (\mathbb{C}^d \setminus Z)/G = X_\Sigma$$

is a diffeomorphism.

Observe now that $\mathfrak{t}^* \simeq H^2(X_\Sigma; \mathbb{R}) \simeq \mathbb{R}^r$ [2, Prop. 4.3.2] and the set $K_\Sigma = \mu(\mathbb{C}^d \setminus Z) \subset \mathfrak{t}^*$ is the *Kähler cone* of X_Σ [5], i.e. the cone of cohomology classes of Kähler form on X_Σ, which is not empty [3, Theorem 4.5]. Therefore, we come to the following conclusion: for every absolutely simple polytope $\Delta \subset \mathbb{R}^n$ there is a complete primitive fan Σ in $\mathbb{R}^n$ and a strictly convex support function h such that Δ is dual to the fan Σ. Then $X_\Sigma = (\mathbb{C}^d \setminus Z)/G$ is a smooth compact projective simplicial toric variety, that means that the Kähler cone K_Σ is not empty and for every $\rho \in K_\Sigma$ there is a diffeomorphism $\mu^{-1}(\rho)/G_\mathbb{R} \to X_\Sigma$.

There is a recipe for the description of the Kähler cone of X_Σ (see [3]). Let $\mathcal{P}_I = \{v_{i_1}, \dots, v_{i_k}\}$ be a primitive collections for the fan Σ. For the fan is complete, the sum $\sum_{i\in I} v_i$ belongs to some cone of Σ generated by $\{v_j\}$, $j \in J$, so

$$\sum_{i\in I} v_i = \sum_{j\in J} c_j v_j$$

with all c_j being positive rational numbers. Since the relations (4) are the basis of all relations between generators, this relation can be rewritten as

$$\sum_{i\in I} v_i - \sum_{j\in J} c_j v_j = t_1^I(a_{11}v_1 + \dots + a_{1d}v_d) + \dots + t_r^I(a_{r1}v_1 + \dots + a_{rd}v_d) \quad (10)$$

Then the system $l_I(\rho) = t_1^I\rho_1 + \dots + t_r^I\rho_r > 0$ for all primitive collections of Σ defines the Kähler cone of X_Σ in $\mathbb{R}^r$.

Remark. We have not used other properties of the generators except for the lattice of relations (4) to obtain relations (10). Therefore they are valid for any symbols satisfying the lattice of relations, including $|\zeta_i|^2$.

Example 2.6. The case of the projective space.

Example 2.4 showes that the group G acting on $\mathbb{C}^{n+1} \setminus \{0\}$ is the algebraic torus $\mathbb{T}$, so the dual Lie algebra is just $\mathbb{R}$. The moment map has only one component

$$|\zeta_1|^2 + \dots + |\zeta_{n+1}|^2 = \rho.$$

There is only one primitive collection consisting of all integral generators of the fan. Their sum is identically zero and is the only relation in the lattice (4). Therefore, the Kähler cone is defined by $\rho > 0$.

3. The formula of integral representation

Let Δ be an n-dimensional absolutely simple integral polytope in $\mathbb{R}^n$. The dual fan Σ is then simplicial, complete, and primitive. Assume that it is generated by d integral generators. Then the toric variety $X_\Sigma = (\mathbb{C}^d \setminus Z)/G$, where Z is atomic, is smooth, complete and projective. The last property enables us to define the volume form ω on X_Σ, and now we are at the same position as in Example 1.1.

It turns out that X_Σ can be embedded into a larger compact toric variety $\widetilde{X}_\Sigma$ almost in the same way as $\mathbb{P}_d$ compactifies $\mathbb{C}^{d+1}$ to $\mathbb{P}_{d+1}$ and becomes the infinite hypersurface. The difference of the general toric case is that X_Σ does not compactify $\mathbb{C}^d$ but it is, however, the 'skeleton of infinity', i.e., the complete intersection of some of toric hypersurfaces that compactify $\mathbb{C}^d$ to $\widetilde{X}_\Sigma$. Moreover, the homogeneous coordinates of X_Σ, being coordinates of points on orbits of the group action, become naturally local coordinates in $\widetilde{X}_\Sigma$. This fact is the content of the main theorem of [16].

Theorem 4 ([16]). *Let Σ be a simplicial complete fan in $\mathbb{R}^n$ with d integral generators and $Z(\Sigma)$ the corresponding atomic family of coordinate planes in $\mathbb{C}^d$. There is a d-dimensional simplicial and complete toric variety $\widetilde{X}_\Sigma$ together with a proper map $\pi\colon \widetilde{X}_\Sigma \longrightarrow \mathbb{C}^d$ such that π realizes a blow-up of $Z(\Sigma) \subset \mathbb{C}^d$ into a family of toric hypersurfaces $\mathcal{X}_1, ..., \mathcal{X}_{d-n}$ of $\widetilde{X}_\Sigma$, for which $\mathcal{X}_1 \cap \cdots \cap \mathcal{X}_{d-n} \simeq X_\Sigma$.*

Let ζ be the local coordinates of $\widetilde{X}_\Sigma$ such that the hypersurfaces $\mathcal{X}_i$ are defined by $\{\zeta_{i+n} = 0\}$, $i = 1, \dots, d-n$. Write the volume form ω in the homogeneous coordinates of X_Σ and consider the differential form

$$\eta = (-1)^n \omega(\zeta) \wedge \frac{d\zeta_{n+1}}{\zeta_{n+1}} \wedge \cdots \wedge \frac{d\zeta_d}{\zeta_d} \tag{11}$$

Writing $\omega(\zeta)$, we mean that the volume form is written in the homogeneous coordinates of X_Σ, but we think of them as local coordinates of $\widetilde{X}_\Sigma$. If $z = h(\zeta)$ the local coordinates expressed in homogeneous, then $\omega(\zeta) = h^*(\omega(z))$. In particular, this means that $\omega(\zeta)$ is closed, and consequently so is η.

We state that the following theorem holds.

Theorem 5. *The differential (d, n)-form η in $\mathbb{C}^d$ is a kernel for the atomic family $Z(\Sigma)$.*

Proof. According to Theorem 4 we regard X_Σ as a complete intersection of toric hypersurfaces $\mathcal{X}_1, \dots, \mathcal{X}_{d-n}$ in $\widetilde{X}_\Sigma$. The differential form (11) is a well-defined form in $\mathbb{C}^d$. Furthemore, it is a closed semimeromorphic form with polar singularity of the first order along each of $r = d - n$ hypersurfaces $\mathcal{X}_i$. This is the only singularity of the form due to the smoothness of X_Σ. This means that the assumptions of the Leray theorem (see e.g. [1]) are satisfied and it follows that

$$\int_\gamma \eta = (2\pi i)^r \int_{X_\Sigma} \mathrm{Res}^r(\eta)$$

The residue-form $\mathrm{Res}^r(\eta)$ is precisely the volume form ω on X_Σ multiplied by $(-1)^n$. Assuming that the volume of X_Σ is given by $\left(\frac{i}{2}\right)^n \int_{X_\Sigma} \omega$, we obtain that the right-hand side equals $(2\pi i)^r (2i)^n \mathrm{Vol}(X_\Sigma)$.

The integration set γ in the left-hand side is the Leray coboundary of $\delta^r(\mathcal{X}_1 \cap \cdots \cap \mathcal{X}_r)$ that is a locally-trivial bundle with the base $\mathcal{X}_1 \cap \cdots \cap \mathcal{X}_r$ and the fiber homeomorphic to $\underbrace{S^1 \times \cdots \times S^1}_{r\text{times}}$. It can be constructed in the following way. For every point $\zeta \in X_\Sigma = \mathcal{X}_1 \cap \cdots \cap \mathcal{X}_r$ choose an r-dimensional surface transversal to X_Σ at ζ, and then choose a cycle in it separating the hypersurfaces $\mathcal{X}_i$. This cycle is necessary homeomorphic to $S^1 \times \cdots \times S^1$. One can choose these cycles to get a real $(2n + r)$-dimensional smooth cycle γ.

One can even choose the orbits of the action of G on $\mathbb{C}^d \setminus Z$ as those r-dimensional surfaces, for they satisfy the transversality condition. The real torus $S^1 \times \cdots \times S^1$ in the orbit of G is homeomorphic to the orbit of the action of $G_\mathbb{R}$ on $\mu^{-1}(\rho)$ (see Theorem 3). Note that we use the smoothness of X_Σ here.

Thus, the cycle γ is homeomorphic to the cycle $\mu^{-1}(\rho)$ for any choice of ρ from the Kähler cone K_Σ. Finally, we get

$$\int_{\mu^{-1}(\rho)} \eta = (2i)^d \pi^r \mathrm{Vol}(X_\Sigma).$$

Theorem 3 says also that $\mathbb{C}^d \setminus Z$ is homotopy equivalent to $\mu^{-1}(\rho)$, $\rho \in K_\Sigma$, so this cycle generates the top non-trivial homology group of $\mathbb{C}^d \setminus Z$ and η is the dual differential form. Therefore, it is a kernel for the atomic family Z. □

Having constructed the kernel η for an atomic family, we can try to prove that this differential form is a kernel of integral representation that we call *associated with the toric variety* X_Σ. The first step in this direction is the following proposition.

Proposition 3.1. *Let $\rho \in K_\Sigma$ and U_ρ be the polyhedron*

$$\begin{cases} a_{11}|\zeta_1|^2 + \cdots + a_{1d}|\zeta_d|^2 < \rho_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{r1}|\zeta_1|^2 + \cdots + a_{rd}|\zeta_d|^2 < \rho_r, \end{cases}$$

with distinguished boundary $\gamma = \mu^{-1}(\rho)$. Then for every function f holomorphic in $\overline{U}_\rho$

$$f(0) = \frac{1}{(2i)^d \pi^r \mathrm{Vol}(X_\Sigma)} \int_\gamma f(\zeta)\eta(\zeta).$$

Proof. Let f admits a Taylor series expansion $\sum\limits_\beta a_\beta \zeta^\beta$ about the origin that converges in a polydisc V. The cycle γ is homologous to every cycle $\mu^{-1}(\rho)$ if ρ is taken from the Kähler cone K_Σ, and the form $f(\zeta)\,\eta(\zeta)$ is closed. Therefore the integration set can be replaced by a cycle $\gamma' = \mu^{-1}(\rho') \Subset V$. The series converges then absolutely and uniformly on γ' and one can integrate it term by term. Let us show that

$$\int\limits_{\gamma'} \zeta^\beta \eta(\zeta) = 0 \text{ if } \beta \neq 0.$$

Notice that the following change of variables

$$\begin{cases} \zeta_1 \mapsto e^{i(a_{11}t_1 + \cdots + a_{r1}t_r)}\zeta_1, \\ \dots\dots\dots\dots\dots\dots \\ \zeta_d \mapsto e^{i(a_{1d}t_1 + \cdots + a_{rd}t_r)}\zeta_d \end{cases}$$

with all t_j being real, preserves the integration set and the kernel as the latter is homogeneous with respect to this action (see (6)); but the integrand gets a coefficient $e^{i(a_{11}t_1 + \cdots + a_{r1}t_r)\beta_1 + \dots (a_{1d}t_1 + \cdots + a_{rd}t_r)\beta_d)}$. The rank of the matrix $A = (a_{ij})$ is r, so the image of the linear mapping given by A is $\mathbb{R}^r$. Therefore for any $\beta \neq 0$ one can choose $t = (t_1, \dots, t_r)$ such that the coefficient is not equal to 1, so the integral must equal 0.

The statement follows now from Theorem 5. □

Recall that the Kähler cone of X_Σ defined by the system of linear inequalities $l_I(\rho) > 0$ (see page 13). For a fixed ρ, define a domain D of $\mathbb{C}^d$ by the system

$$|\zeta_{i_1}|^2 + \cdots + |\zeta_{i_k}|^2 < t_1^I \rho_1 + \cdots + t_r^I \rho_r \tag{12}$$

for all primitive collections $\mathcal{P}_I$ of Σ.

Proposition 3.2. *The domain D is a subdomain of U_ρ.*

Proof. Note that the rational vectors $t^I = (t_1^I, \dots, t_r^I)$ are the interior normal vectors to the faces of the Kähler cone. Therefore they generate the dual cone $b_1 t^{I_1} + \dots + b_s t^{I_s}$ where $b_j \in \mathbb{R}^r$, $b_j \geq 0$. Since the Kähler cone is not empty and contained in the positive orthant $\mathbb{R}_+^r$, the dual cone is also non-empty and contains the positive orthant. This means that every basis vector e_i of $\mathbb{R}^r$ can be expressed as a linear combination of $\{t^I\}$ with non-negative rational coefficients. So we can sum the inequalities (12) multiplied by these coefficients to get ρ_i on the right side and

$$a_{i1}|\zeta_1|^2 + \dots + a_{id}|\zeta_d|^2 + b_1 \left(\sum_{j \in J_1} c_j |\zeta_j|^2 \right) + \dots + b_s \left(\sum_{j \in J_s} c_j |\zeta_j|^2 \right)$$

on the left with the same inequality sign. So, $a_{i1}|\zeta_1|^2 + \dots + a_{id}|\zeta_d|^2 < \rho_i$ and the proposition is proved. □

Now we extend the representation of the function at the origin (Proposition 3.1) to the representation in a domain. The formula we shall obtain is of the Bochner-Ono type [1] as it represents values of a function in a subdomain of a domain where f is holomorphic.

Theorem 6. *Let f be holomorphic in the closure of a domain U_ρ, $\rho \in K_\Sigma$ with distinguished boundary γ. Then for every $z \in D \subset U_\rho$*

$$f(z) = \frac{1}{(2i)^d \pi^r \mathrm{Vol}(X_\Sigma)} \int_\gamma f(\zeta) \eta(\zeta - z).$$

Proof. Let ρ be a point from the Kähler cone of X_Σ and $z \in D$. Consider the homotopy $\Gamma(t)$ of the cycle $\mu^{-1}(\rho) = \gamma$

$$\begin{cases} a_{11}|\zeta_1 - tz_1|^2 + \dots + a_{1d}|\zeta_d - tz_d|^2 = R_1(t, z, \rho), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{r1}|\zeta_1 - tz_1|^2 + \dots + a_{rd}|\zeta_d - tz_d|^2 = R_r(t, z, \rho). \end{cases}$$

Assume that one can choose a smooth curve $R(t) = (R_1(t, z, \rho), \dots, R_r(t, z, \rho))$ in $\mathbb{R}^r$ such that for all $t \in [0, 1]$ the cycles $\Gamma(t)$ lie in the domain U_ρ and do not intersect the set $Z + z = \{\zeta \in \mathbb{C}^d : \zeta - z \in Z\}$. By the Stokes theorem

$$\int_\gamma f(\zeta)\, \eta(\zeta - z) = \int_{\Gamma(1)} f(\zeta)\, \eta(\zeta - z).$$

The change of variables $\zeta \mapsto \zeta + z$ in the integral gives

$$\int_{\gamma'} f(\zeta + z)\, \eta(\zeta)$$

where the cycle γ' is $\mu^{-1}(R(1))$. Assuming that $R(1) \in K_\Sigma$ and applying Proposition 3.1, we get the statement proved. It is left to show that it is always possible to choose an appropriate curve $R(t)$ with $R(1) \in K_\Sigma$.

Note first that since all the coefficients a_{ij} in (4) are non-negative, the expression $a_{i1}|\zeta_1 - tz_1|^2 + \dots + a_{id}|\zeta_d - tz_d|^2$ is the absolut value of the vector

$\left(\sqrt{a_{i1}}(\zeta_1 - tz_1), \dots, \sqrt{a_{id}}(\zeta_d - tz_d)\right)$ being seen as a vector in $\mathbb{R}^{2d}$. This vector is the sum

$$(\sqrt{a_{i1}}\zeta, \dots, \sqrt{a_{id}}\zeta_d) - t\,(\sqrt{a_{i1}}z_1, \dots, \sqrt{a_{id}}z_d)$$

of two vectors in $\mathbb{R}^{2d}$ and therefore is the subject to the triangle inequality in the standard metric of $\mathbb{R}^{2d}$. So,

$$\begin{aligned}\left(a_{i1}|\zeta_1 - tz_1|^2 + \dots + a_{id}|\zeta_d - tz_d|^2\right)^{1/2} &\geq \\ \geq \left(a_{i1}|\zeta_1|^2 + \dots + a_{id}|\zeta_d|^2\right)^{1/2} &- t\left(a_{i1}|z_1|^2 + \dots + a_{id}|z_d|^2\right)^{1/2}.\end{aligned}$$

Denote the image $\mu(z)$ by μ. Then the inequality obtained means that

$$\left(a_{i1}|\zeta_1|^2 + \dots + a_{id}|\zeta_d|^2\right)^{1/2} \leq \sqrt{R_i(t)} + t\sqrt{\mu_i} \text{ for all } \zeta \in \Gamma(t).$$

Therefore, to satisfy the first condition for $\Gamma(t)$ it is enough to require

$$0 < R_i(t) \leq (\sqrt{\rho_i} - t\sqrt{\mu_i})^2. \tag{13}$$

The singularity set of the form $f(\zeta)\,\eta(\zeta - z)$ is the union

$$Z + z = \bigcup\{\zeta_{i_1} = z_{i_1}, \dots, \zeta_{i_k} = z_{i_k}\}$$

over all primitive collections I_j, $j = 1, \dots, s$. For all $\zeta \in \Gamma(t)$, we have

$$l_I(R(t)) = \sum_{i \in I} |\zeta_i - tz_i|^2 - \sum_{j \in J} c_j|\zeta_j - tz_j|^2.$$

Substituting any point of $Z + z$ into this identity, we get

$$-\sum_{j \in J} c_j|\zeta_j - tz_j|^2 = l_I(R(t)) - (1-t)^2 \sum_{i \in I} |z_i|^2.$$

Therefore, the cycle $\Gamma(t)$ does not intersect the singularity set if the right-hand side is greater than zero. Since by definition $\sum_{i \in I} |z_i|^2 < l_I(\rho)$, the second condition for the cycle is satisfied if

$$l_I(R(t)) \geq (1-t)^2 l_I(\rho) \tag{14}$$

One can show that conditions (13) and (14) define a connected set in $\mathbb{R}^r \times [0, 1]$, so it is always possible to choose a smooth curve $R(t)$. But we can provide the exact construction.

Note that for $t = 1$ the conditions become

$$\begin{gathered}0 < R_i(1) \leq (\sqrt{\rho_i} - \sqrt{\mu_i})^2,\ i = 1, \dots, r \\ l_I(R(1)) \geq 0, \text{ for all primitive collections } I.\end{gathered}$$

The first series of conditions defines an r-dimensional parallelepiped in $\mathbb{R}^r_+$ with faces parallel to the coordinate hyperplanes. The second defines the Kähler cone, which as we know is non-empty. Therefore, their intersection is non-empty and we can choose a point ε from this intersection. Let us show that the homotopy given by the curve $R_i(t) = \rho_i(1-t)^2 + \varepsilon_i t$ satisfy the conditions (13) and (14).

Indeed, $R_i(t)$ is obviously greater than 0 and

$$\begin{aligned}(\sqrt{\rho_i} - t\sqrt{\mu_i})^2 - R_i(t) &= t(\sqrt{\rho_i} - \sqrt{\mu_i})\,(2\sqrt{\rho_i} - t(\sqrt{\rho_i} + \sqrt{\mu_i})) - t\varepsilon_i > \\ &> t(\sqrt{\rho_i} - \sqrt{\mu_i})\,(2\sqrt{\rho_i} - t(\sqrt{\rho_i} + \sqrt{\mu_i})) - t(\sqrt{\rho_i} - \sqrt{\mu_i}) = \\ &= t(1-t)(\sqrt{\rho_i} - \sqrt{\mu_i})(\sqrt{\rho_i} + \sqrt{\mu_i}) \geq 0,\end{aligned}$$

because $\sqrt{\rho_i} - \sqrt{\mu_i} > 0$ according to Proposition 3.2. As for the condition (14), we have

$$l_I(R(t)) = (1-t)^2 l_I(\rho) + t l_I(\varepsilon) \geq (1-t)^2 l_I(\rho)$$

for $\varepsilon \in K_\Sigma$.

Thus, the curve $R(t)$ defines a necessary homotopy. Moreover, its end $R(1) = \varepsilon$ lies in K_Σ, so $\gamma' = \mu^{-1}(\varepsilon)$.

□

4. The volume form of a toric variety

There is a natural construction for volume forms on projective toric varieties. Let Δ be an n-dimensional simple integral polytope $\Delta \subset \mathbb{R}^n$. Given Δ, there is a complete simplicial toric variety X_Σ associated to the fan Σ dual to Δ. The variety X_Σ constructed in this way admits a closed embedding into the projective space (Theorem 2). Let us modify the embedding (8) as follows.

Let $P(z) = \sum\limits_{\alpha \in \Delta \cap \mathbb{Z}^n} c_\alpha z^\alpha$ be a Laurent polynomial in the torus $\mathbb{T}^n$ with all non-negative coefficients c_α such that its Newton polytope N_P coincides with Δ. Put elements of $\Delta \cap \mathbb{Z}^n$ in an order $\alpha_1 \ldots, \alpha_N$ and define an embedding of a 'big' torus $f\colon \mathbb{T}^n \longrightarrow \mathbb{P}_{N-1}$ by

$$(z_1, \ldots, z_n) \longmapsto (\sqrt{c_{\alpha_1}} z^{\alpha_1} : \ \ldots : \sqrt{c_{\alpha_N}} z^{\alpha_N}).$$

The closure $\overline{f(\mathbb{T}^n)}$ is then the image of X_Σ, which can have singularities, but observe that the $f(\mathbb{T}^n) \subset f(X_\Sigma)$ is always smooth.

On the projective space $\mathbb{P}_{N-1}$, there is a globally defined *Fubini-Study* differential form associated with the Fubini-Study metric on the projective space. In the homogeneous coordinates ξ the form can be written down as

$$\omega_{FS} = \frac{i}{2|\xi|^4}\left(\sum_{k=1}^N |\xi_k|^2 \sum_{k=1}^N d\xi_k \wedge d\bar{\xi}_k - \sum_{k=1}^N \bar{\xi}_k d\xi_k \wedge \sum_{k=1}^N \xi_k d\bar{\xi}_k\right) =$$

$$= \partial\bar{\partial} \log |\xi|^2 = \frac{1}{2i} \mathrm{dd^c} |\xi|^2;$$

here $\mathrm{d} = \partial + \bar{\partial}$ and $\mathrm{d^c} = i(\bar{\partial} - \partial)$.

The form ω_{FS} is closed (in every chart), what makes $(\mathbb{P}_{N-1}, \omega_{FS})$ into a Kähler manifold. The essential advantage of Kähler geometry is that the Kähler form measures volumes of all complex subsets of arbitrary dimensions. More precisely, if $A \subset \mathbb{P}_{N-1}$ is a complex subset of pure dimension k then the volume of A with respect to the measure defined by the Kähler form is given by the integral

$$\mathrm{Vol}(A) = \frac{1}{k!} \int_A \omega_{FS}^k.$$

We introduce a differential (n, n)-form ω on the torus $\mathbb{T}^n$ as the pullback image of the Fubini-Study volume form ω_{FS}^n:

$$\omega = \frac{1}{n!} f^*(\omega_{FS}^n) = \frac{1}{n!} \left(\mathrm{dd^c} \ln P(|z_1|^2, \ldots, |z_n|^2)\right)^n.$$

The form ω is well-defined since f is a finite covering of $f(\mathbb{T}^n)$; it is positive in the torus $\mathbb{T}^n \subset X_\Sigma$ as it inherits this property from ω_{FS}, but the form can vanish or be not defined in other points of the variety, however that does not affect the value of the integral

$$\int\limits_{\mathrm{reg}X_\Sigma} \omega = \int\limits_{\mathbb{T}^n} \omega.$$

Definition 4.1. *We call* $\omega = \frac{1}{n!} f^*(\omega_{FS}^n)$ *the volume form of a compact simplicial projective toric variety* X_Σ *and define the volume of the variety with respect to this measure as*

$$\mathrm{Vol}(X_\Sigma) = \left(\frac{i}{2}\right)^n \int\limits_{\mathbb{T}^n} \omega. \tag{15}$$

The following simple proposition gives the exact value of the volume.

Proposition 4.1.

$$\mathrm{Vol}(X_\Sigma) = \pi^n \mathrm{Vol}(\Delta).$$

Proof. The obvious change of variables in the integral give the following

$$\left(\frac{i}{2}\right)^n \int\limits_{\mathbb{T}^n} \omega = \frac{1}{n!}\left(\frac{i}{2}\right)^n \int\limits_{\mathbb{T}^n} f^*(\omega_{FS}^n) = \frac{1}{n!}\left(\frac{i}{2}\right)^n \int\limits_{f(\mathbb{T}^n)} \omega_{FS}^n.$$

The formula for the value of the last integral the remarkable fact of projective geometry as it relates two quantity of different nature, namely, the volume of an *algebraic* subset is expressed in terms of degree of the mapping [11].

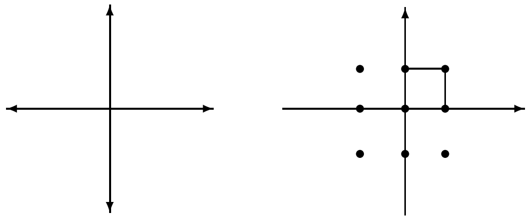
$$\frac{1}{n!}\left(\frac{i}{2}\right)^n \int\limits_{f(\mathbb{T}^n)} \omega_{FS}^n = \frac{\pi^n}{n!}\deg(f)$$

It is left to compute the degree of the embedding f. By definition, it is equal to the number of intersection points of $f(\mathbb{T}^n)$ with a generic plane of codimension n. Let such a plane be defined as a zero locus of n homogeneous linear forms $l_j(\xi)$, $j = 1, \ldots, n$. Then the degree of f equals the number of solutions to the system $l_j(\xi)\Big|_{f(\mathbb{T}^n)}$. In general, the number of solutions to the system of k algebraic equations having only isolated zeros in $\mathbb{P}_k$ is given by the Bernstein theorem [4] and equal to $n!$ multiplied by the normalized volume of the Minkowski sum (see [14] for the definition) $\Delta_1 + \cdots + \Delta_k$ of the Newton polytopes Δ_i of all equations in the system. In our case, all equations have the same Newton polytope Δ, so $\deg(f) = n!\mathrm{Vol}(n\Delta) = n!\mathrm{Vol}(\Delta)$. Throughout here, $\mathrm{Vol}(\Delta)$ denote the the volume of Δ in $\mathbb{R}^n$ normalized by the condition that the standard n-dimensional simplex has the volume $\frac{1}{n!}$. □

Example 4.7. The volume form of $\mathbb{P}_1 \times \mathbb{P}_1$.

The product of two copies of the Riemann spheres is a toric variety and is associated with the two-dimensional complete fan Σ on Fig. 3(a). Let P be a polynomial $P(z_1, z_2) = 1 + z_1 + z_2 + az_1z_2$ where the coefficient a is positive. Its Newton polytope N_P is a unit square in $\mathbb{R}^2$ (Fig. 3(b)) and is obviously dual to the fan Σ. Following the construction, we define a differential form on $\mathbb{T}^2 \subset \mathbb{P}_1 \times \mathbb{P}_1$ as the pull-back of ω_{FS}^2 under the mapping $f\colon (z_1, z_2) \mapsto (1\colon z_1\colon z_2\colon \sqrt{a}z_1z_2)$

$$\omega = \frac{1}{2!} f^*(\omega_{FS}^2) = \frac{1 + a|z_1|^2 + a|z_2|^2 + a|z_1|^2|z_2|^2}{(1 + |z_1|^2 + |z_2|^2 + a|z_1|^2|z_2|^2)^3}\, dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$$



(a) (b)

Figure 3: (a) the fan; (b) the Newton polytope.

The volume of $\mathbb{P}_1 \times \mathbb{P}_1$ with respect to this measure is equal to $-4\pi^2$. Note that the volume form does not coincide with the product of two volume forms on copies of $\mathbb{P}_1$ (it happens only if $a = 1$), although it gives the same volume.

In fact, in the polar coordinate system the differential form in (15) can be easily integrated with respect to angular coordinates. Then the proposition provides a new proof of the Passare formula, which represents the volume of a polytope as an integral of a rational form over the positive orthant $\mathbb{R}^n_+$ [13].

For the volume forms constructed we rephrase Theorem 6 as follows.

Theorem 6′. *Let f be holomorphic in the closure of the domain U_ρ and γ be the domain's distinguished boundary. Then for every $z \in D \subset U_\rho$*

$$f(z) = \frac{1}{(2\pi i)^d \mathrm{Vol}(\Delta)} \int\limits_\gamma f(\zeta)\eta(\zeta - z).$$

5. Examples

Example 5.1. Integral representations associated with projective spaces

Let Δ be the standard simplex in $\mathbb{R}^n$, which is an absolutely simple polytope. Its dual fan Σ with $(n+1)$ generators is then the fan of the projective space $\mathbb{P}_n$. The volume form ω defined on it coincides with the Fubini-Study volume form ω^n_{FS}, $\mathbb{P}_n$ is embedded into $\mathbb{P}_{n+1}$ and the form η is the Bochner-Martinelli kernel η_{BM} in $\mathbb{C}^{n+1}$(see Example 1.1). The integration cycle is a sphere S^{2n+1} with the radius $\sqrt{\rho}$ (see Example 2.6). Thus, the following corollary from Theorem 6′ holds:

Let f be holomorphic in the closed ball B^{2n+2}_ρ with radius ρ and $S^{2n+1} = \partial B^{2n+2}_\rho$. Then for every $z \in B^{2n+2}_\rho$

$$f(z) = \frac{n!}{(2\pi i)^{n+1}} \int\limits_{S^{2n-1}} f(\zeta)\eta_{BM}(\zeta - z).$$

Afterwards, using analytic methods one proves this formula for any bounded domain in $\mathbb{C}^{n+1}$ with appropriate boundary.

Example 5.2. Integral representations associated with $\mathbb{P}_1 \times \mathbb{P}_1$

Let Δ be a unit square in $\mathbb{R}^2$ and $P(z) = 1 + z_1 + z_2 + az_1z_2$ as in example 4.8. The volume form computed there produces an integral kernel

$$\eta = \frac{(\bar\zeta_3 d\bar\zeta_1 - \bar\zeta_1 d\bar\zeta_3) \wedge (\bar\zeta_2 d\bar\zeta_4 - \bar\zeta_4 d\bar\zeta_2)}{(|\zeta_1|^2|\zeta_4|^2 + |\zeta_3|^2|\zeta_4|^2 + |\zeta_2|^2|\zeta_3|^2 + a|\zeta_1|^2|\zeta_2|^2)^3} d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_3 \wedge d\zeta_4$$

with singularity along $\{z_1 = z_3 = 0\} \cup \{z_2 = z_4 = 0\}$.

The domain U_ρ in this case is the product of two balls $B^4_{\rho_1} \times B^4_{\rho_2}$ in $\mathbb{C}_{z_1,\, z_3} \times \mathbb{C}_{z_2,\, z_4}$. The Kähler cone coincides with $\mathbb{R}^2_+$ so the integral represents values of a holomorphic function at every point z of U_ρ:

$$f(z) = \frac{1}{(2\pi i)^4} \int_{\partial B^4_{\rho_1} \times \partial B^4_{\rho_2}} f(\zeta)\eta(\zeta - z).$$

Example 5.3. Integral representation associated with the blow-up of $\mathbb{P}_1 \times \mathbb{P}_1$ at the origin

Let $P(z) = 1 + z_1^2 + z_2^2 + z_1^2 z_2 + z_1 z_2^2$ with the Newton polytope depicted on Fig. 5(a). The dual fan Σ has five integral generators and the corresponding toric variety is the blow-up of the product $\mathbb{P}_1 \times \mathbb{P}_1$ at the origin.

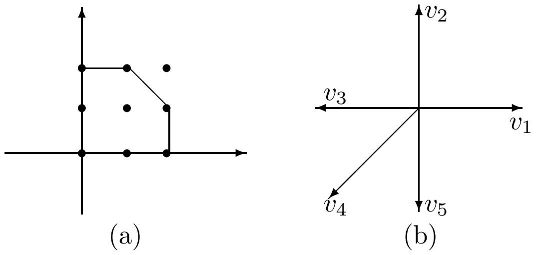


Figure 4: (a) the Newton polytope; (b) the fan.

Number the integral generators, and when all the calculations are done the integral kernel is

$$\eta = \frac{u(\zeta,\, \overline{\zeta})\overline{E(\zeta)}}{v(\zeta,\, \overline{\zeta})} \wedge d\zeta$$

where

$$\begin{aligned} u(\zeta,\, \overline{\zeta}) = {} & |\zeta_1|^8|\zeta_2|^4|\zeta_5|^4 + 4|\zeta_1|^6|\zeta_2|^4|\zeta_3|^2|\zeta_4|^2|\zeta_5|^4 + 4|\zeta_1|^6|\zeta_3|^2|\zeta_4|^6|\zeta_5|^8 + \\ & + |\zeta_1|^4|\zeta_2|^8|\zeta_3|^4 + 4|\zeta_1|^4|\zeta_2|^6|\zeta_3|^4|\zeta_4|^2|\zeta_5|^2 + 9|\zeta_1|^4|\zeta_2|^4|\zeta_3|^4|\zeta_4|^4|\zeta_5|^4 + \\ & + 16|\zeta_1|^4|\zeta_2|^2|\zeta_3|^4|\zeta_4|^6|\zeta_5|^6 + 16|\zeta_1|^2|\zeta_2|^4|\zeta_3|^6|\zeta_4|^6|\zeta_5|^4 + \\ & + 16|\zeta_1|^2|\zeta_2|^2|\zeta_3|^6|\zeta_4|^8|\zeta_5|^6 + 4|\zeta_2|^6|\zeta_3|^8|\zeta_4|^6|\zeta_5|^2, \end{aligned}$$

$$\begin{aligned} v(\zeta,\, \overline{\zeta}) = {} & (\ |\zeta_3|^4|\zeta_4|^6|\zeta_5|^4 + |\zeta_1|^4|\zeta_4|^2|\zeta_5|^4 + |\zeta_2|^4|\zeta_3|^4|\zeta_4|^2 + \\ & + |\zeta_1|^4|\zeta_2|^2|\zeta_5|^2 + |\zeta_1|^2|\zeta_2|^4|\zeta_3|^2\ )^3, \end{aligned}$$

and

$$\begin{aligned} E(\zeta) = {} & \zeta_3\zeta_4\zeta_5 d\zeta_1 d\zeta_2 - \zeta_2\zeta_3\zeta_5 d\zeta_1 d\zeta_4 - \zeta_2\zeta_3\zeta_4 d\zeta_1 d\zeta_5 + \\ & \zeta_1\zeta_4\zeta_5 d\zeta_2 d\zeta_3 + \zeta_1\zeta_3\zeta_5 d\zeta_2 d\zeta_4 + \zeta_1\zeta_2\zeta_5 d\zeta_3 d\zeta_4 + \\ & \zeta_1\zeta_2\zeta_4 d\zeta_3 d\zeta_5 + \zeta_1\zeta_2\zeta_3 d\zeta_4 d\zeta_5. \end{aligned}$$

The lattice of relations between the integral generators is given by

$$\begin{cases} v_1 + v_3 = 0, \\ v_2 + v_5 = 0, \\ v_1 + v_2 + v_4 = 0; \end{cases}$$

and one can easily check that the primitive collections for Σ are $\{1,\,3\}$, $\{1,\,4\}$, $\{2,\,4\}$, $\{2,\,5\}$, and $\{3,\,5\}$. It follows that the domain U_ρ is given by the inequalities

$$\begin{cases} |\zeta_1|^2 + |\zeta_3|^2 < \rho_1, \\ |\zeta_2|^2 + |\zeta_5|^2 < \rho_2, \\ |\zeta_1|^2 + |\zeta_2|^2 + |\zeta_4|^2 < \rho_3, \end{cases}$$

where $\rho = (\rho_1,\, \rho_2,\, \rho_3)$ lies in the Kähler cone being given by

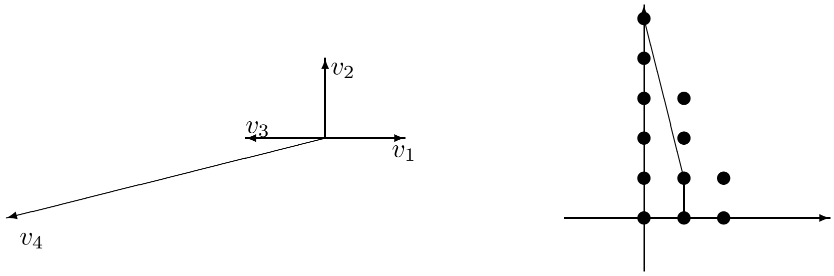
$$\begin{cases} \rho_1 > 0, \\ \rho_3 - \rho_2 > 0, \\ \rho_3 - \rho_1 > 0, \\ \rho_2 > 0, \\ \rho_1 + \rho_2 - \rho_3 > 0. \end{cases}$$

Therefore the subdomain D consists of all points z that satisfy the system

$$\begin{cases} |z_1|^2 + |z_3|^2 < \rho_1, \\ |z_1|^2 + |z_4|^2 < \rho_3 - \rho_2, \\ |z_2|^2 + |z_4|^2 < \rho_3 - \rho_1, \\ |z_2|^2 + |z_5|^2 < \rho_2, \\ |z_3|^2 + |z_5|^2 < \rho_1 + \rho_2 - \rho_3. \end{cases}$$

Example 5.4. Integral representation associated with a Hirzebruch surface

Let the polynomial P be equal $1 + z_1 + z_1 z_2 + z_2^5$. The dual fan to its Newton polytope corresponds to one of the Hirzebruch surfaces.



The volume form on this surface is

$$\omega = \frac{|z_1|^2 + |z_1|^2|z_2|^{10} + 25|z_2|^{10} + 25|z_2|^8}{(1 + |z_1|^2 + |z_1|^2|z_2|^2 + |z_2|^{10})^3}\, dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2,$$

and associated integral kernel is

$$\eta = g(\zeta,\, \overline{\zeta})\overline{E(\zeta)} \wedge d\zeta$$

where

$$g(\zeta,\, \overline{\zeta}) = \frac{|\zeta_1|^2|\zeta_2|^{10} + 25|\zeta_2|^{10}|\zeta_3|^2|\zeta_4|^8 + 25|\zeta_2|^8|\zeta_3|^2|\zeta_4|^{10} + |\zeta_1|^2|\zeta_4|^{10}}{(|\zeta_3|^2|\zeta_4|^{10} + |\zeta_1|^2|\zeta_4|^2 + |\zeta_1|^2|\zeta_2|^2 + |\zeta_2|^{10}|\zeta_3|^2)^3}$$

and

$$E(\zeta) = \zeta_3\zeta_4\, d\zeta_1 \wedge d\zeta_2 - \zeta_2\zeta_3\, d\zeta_1 \wedge d\zeta_4 + \zeta_1\zeta_4\, d\zeta_2 \wedge d\zeta_3 + 4\zeta_1\zeta_3\, d\zeta_2 \wedge d\zeta_4 + \zeta_1\zeta_2\, d\zeta_3 \wedge d\zeta_4.$$

For functions that are holomorphic in

$$\begin{cases} |\zeta_1|^2 + |\zeta_3|^2 < \rho_1, \\ 4|\zeta_1|^2 + |\zeta_2|^2 + |\zeta_4|^2 < \rho_2, \end{cases}$$

where $\rho \in \{\rho_1 > 0,\, \rho_2 - 4\rho_1 > 0\}$, the integral represents values at the points from D given by

$$\begin{cases} |\zeta_1|^2 + |\zeta_3|^2 < \rho_1, \\ |\zeta_2|^2 + |\zeta_4|^2 < \rho_2 - 4\rho_1. \end{cases}$$

Note that the polygon with the integral generators of the fan as vertices in the case is not convex, but this is not an obstacle to construct a kernel. What really matters is the existence of the dual polytope. Similiar formulas of integral representations have been considered by A.A. Kytmanov [10] but his construction is different and does not cover this case.

References

[1] L.A. Aĭzenberg; A.P. Yuzhakov *Integral representations and residues in multidimensional complex analysis.* Translated from the Russian by H. H. McFaden. Translation edited by Lev J. Leifman. Translations of Mathematical Monographs, 58. American Mathematical Society, Providence, RI.

[2] M. Audin *The topology of torus actions on symplectic manifolds.* Translated from the French by the author. Progress in Mathematics, 93. Birkhäuser Verlag, Basel, 1991.

[3] V.V. Batyrev *Quantum cohomology ring of toric manifolds*// Journées de Géométrie Algébrique d'Orsay (Orsay, 1992). Astérisque No. 218 (1993), 9–34.

[4] D.N. Bernstein *The number of roots of a system of equations* (Russian)// Funkcional. Anal. i Priložen. **9** (1975), no. 3, 1–4.

[5] D. Cox *Recent developments in toric geometry.* Algebraic geometry — Santa Cruz 1995, 389 – 436, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[6] D.A. Cox *The homogeneous coordinate ring of a toric variety*// J. Algebraic geometry **4** (1995), 17-50.

[7] V.I. Danilov *The geometry of toric varieties* (Russian)// Uspekhi Mat. Nauk **33** (1978), no. 2(200), 85–134, 247.

[8] W. Fulton *Introduction to toric varieties.* Annals of Mathematics Studies, 131. Princeton University Press, Princeton, NJ.

[9] A.G. Khovanskiĭ *Newton polyhedra (resolution of singularities)* (Russian)// Current problems in mathematics, Vol. 22, 207–239, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1983.

[10] A.A. Kytmanov *On an analogue of the Fubini-Study form for two-dimensional toric manifolds* (Russian) Sibirsk. Mat. Zh. 44 (2003), no. 2, 358–371; translation in Siberian Math. J. 44 (2003), no. 2, 286–297.

[11] D. Mumford *Algebraic geometry. I. Complex projective varieties.* Grundlehren der Mathematischen Wissenschaften, No. 221. Springer-Verlag, Berlin-New York, 1976.

[12] T. Oda *Convex Bodies and Algebraic Geometry.* Ergeb. Math. Grenzgeb. 3. Folge, Bd. 15, Springer-Verlag, Berlin, 1988.

[13] M. Passare *Amoebas, convexity and the volume of integer polytopes*// Advanced Studies in Pure Mathematics, **42** (2004), 263-268.

[14] J.R. Sangwine-Yager *Mixed volumes.* Handbook of convex geometry, Vol. A, 43–71, North-Holland, Amsterdam, 1993.

[15] A.K. Tsikh *Toriska residyer* (Swedish). Proc. Conf. 'Nordan 3' (1999), 16. Stockholm.

[16] A. Tsikh, A. Yger, A. Shchuplev *Some new kernels in residue theory.* (in preparation)

[17] A.V. Shchuplev *On the volume forms of toric varieties and kernels of integral representations.* Proc. of Int. School-Conference 'Geometrical analysis and its applications', Volgograd, 2004, 203 - 205.

[18] A.V. Shchuplev *On the reproducing integral kernels in $\mathbb{C}^d$ and volume forms of toric varieties* (Russian)// (submitted to Uspekhi Mat. Nauk)