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A natural interpretation of classical proofs

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Abstract

We extend the double negation interpretation from formulas to derivations. The resulting interpretation is compositional and respects the most obvious conversion rules of the Gentzen sequent calculus for classical predicate logic. It furthermore induces a notion of classical proof similar to that of intuitionistic logic. The interpretation agrees with the Kolmogorov interpretation for formulas not containing implication.

1 Introduction

It is the topic of this paper to extend the double negation interpretation from formulas to derivations in such a way that the interpretation respects the structure of classical predicate logic. The plan is to interpret Gentzen's sequent calculus for classical predicate logic in terms of the minimal logic fragment of his calculus of natural deduction for intuitionistic predicate logic, extended with a sign for falsity in the sense of minimal logic. The presence of the sign has to do with our interpretation of sequents in terms of tableau sequents. The choice to interpret Gentzen's sequent calculus in what essentially amounts to his calculus of natural deduction is mainly that we want to facilitate comparisons between this and other interpretations, but also to see how far Gentzen's calculi will take us.

We hold Gentzen's sequent calculus to provide a natural formalization of classical predicate logic and we want the interpretation to reflect the structure of classical logic. Thus we analyze and factor Gentzen's sequent

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calculus over a novel calculus of natural deduction with a structure similar to that of Gentzen's sequent calculus. This novel calculus is intuitionistic in the sense that it is built up from introduction and elimination rules that suggest conversion rules similar to those of intuitionistic calculi. Furthermore, these conversion rules are normalizing and have the Church-Rosser property. We introduce the corresponding notion of β-equivalence. It is then, from an intuitionistic standpoint, natural to say that two classical derivations denote one and the same proof provided that they are β-equivalent in this sense.

We use the introduction rules of the novel calculus to determine the meaning of the symbols concerned. From this follows the interpretation of symbols, and thus formulas, up to equivalence. We then interpret the inference figures of the novel calculus. This completes the interpretation.

The interpretation turns out to agree with Kolmogorov's interpretation [6] for formulas not containing implication. It also respects the conversion rules for the novel calculus.

2 Calculi

We present three classical calculi and one intuitionistic calculus. The classical calculi derives from the Gentzen sequent calculus for classical predicate logic, while the intuitionistic calculus derive from the Gentzen natural deduction calculus for intuitionistic predicate logic. We also indicate how to prove normalization and normal form theorems for the classical calculi.

All four calculi make use of signs for truth and falsity, the latter interpreted in the sense of minimal logic. These signs result from expressing the Gentzen sequent calculus for classical predicate logic in terms of tableau sequents instead of Gentzen sequents, thus removing multiple conclusions from the calculus. The signs then persist throughout the interpretation to the intuitionistic calculus, in which they serve to distinguish the truth of $\neg A$ from the falsity of A.

This section is based on two ideas. First, to use tableau sequents and to mark signed formulas with variables analogous to the marks placed upon discharged assumptions in natural deduction. This reduces bureaucracy while preserving the structure needed for the passage to natural deduction; marking signed formulas with variables makes it possible to distinguish derivations such as

$$\dfrac{\dfrac{\overset{x}{\mathsf{F}A}, \overset{\beta}{\mathsf{F}A\&B} \quad \overset{y}{\mathsf{F}B}, \overset{\beta}{\mathsf{F}A\&B}}{\overset{\alpha}{\mathsf{F}A\&B}, \overset{\beta}{\mathsf{F}A\&B}}\,{\scriptstyle \mathsf{F}\&(x,y;\alpha)} \qquad \overset{z}{\mathsf{F}C}}{\overset{\beta}{\mathsf{F}A\&B}, \overset{\gamma}{\mathsf{F}(A\&B)\&C}}\,{\scriptstyle \mathsf{F}\&(\alpha,z;\gamma)}$$

and

$$\dfrac{\dfrac{\overset{x}{\mathsf{F}A}, \overset{\beta}{\mathsf{F}A\&B} \quad \overset{y}{\mathsf{F}B}, \overset{\beta}{\mathsf{F}A\&B}}{\overset{\alpha}{\mathsf{F}A\&B}, \overset{\beta}{\mathsf{F}A\&B}}\,{\scriptstyle \mathsf{F}\&(x,y;\alpha)} \qquad \overset{z}{\mathsf{F}C}}{\overset{\alpha}{\mathsf{F}A\&B}, \overset{\gamma}{\mathsf{F}(A\&B)\&C}}\,{\scriptstyle \mathsf{F}\&(\beta,z;\gamma)} \quad .$$

Second, the construction of the third calculus: We want to embed the tableau calculus in a natural deduction calculus in such a way that the conversion rules of the latter induce plausible conversion rules on the former calculus. The reason is that we only have partial knowledge of the conversion rules for the classical sequent calculus; the Gentzen conversion rules are not confluent and consequently we only trust the most obvious rules. That normalization according to Gentzen's Hauptsats fails to be confluent is mentioned already in [3]. Nevertheless we want a natural deduction calculus that reflects these rules. These considerations determine the third calculus.

2.1 Calculus *C1*

We shall consider the Gentzen [3, pp. 81–85] sequent calculus for classical predicate logic, also known as *LK*, but express it in terms of tableau sequents instead of Gentzen sequents. We call this version *C1*.

2.1.1 Preliminary matters

We define a tableau sequent as a sequent of marked assumptions, where an assumption is a signed formula marked by a variable, and consider two sequents to be equal provided they are equal up to interchange of assumptions. Thus we do not need the structural inference figure of interchange. We take the inductive definitions of formulas and signed formulas for granted and consider two formulas respectively two signed formulas as equal provided they are syntactically equal up to a change of bound variables. Thus we do not need to incorporate change of bound variables into the quantifier inference figures.

Formally, we shall consider contexts as extensional sets of assumptions and take $\overset{x}{\alpha}, \Gamma$ to mean $\{\overset{x}{\alpha}\} \cup \Gamma$, where $\overset{x}{\alpha}$ is an assumption and Γ is a context. Thus we do not need the structural inference figure of contraction. We write tableau sequents as $\Gamma \vdash \psi$, where Γ is a context, but suppress $\vdash \psi$ in the inference rules for tableau sequents, and write them like contexts. The symbol ψ stands for the empty succedent and derives from the Greek word $\psi\epsilon\tilde{\upsilon}\delta o\varsigma$, meaning falsehood.

We incorporate weakening into the $\mathsf{F}\&$-inference figure

$$\frac{\mathsf{F}\overset{x}{A}, \Gamma_1 \quad \mathsf{F}\overset{y}{B}, \Gamma_2}{\mathsf{F}A\overset{z}{\&}B, \Gamma}\ \mathsf{F}\&(x,y;z)$$

by $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma$. When $\Gamma_1 = \Gamma$ and $\Gamma_2 = \Gamma$ we recover the $\mathsf{F}\&$-inference figure of *LK*. We similarly incorporate weakening into the other inference figures. Thus we do not need the structural inference figure of weakening.

2.1.2 *C1* inference figures

The *C1* inference figures can be found on page 5. These are subject to restrictions on contexts and variables. First, to make an inference, the principal assumption of a premiss must not occur in the context of that premiss, e.g. to make a $\mathsf{F}\&$-inference we must have $\mathsf{F}\overset{x}{A} \notin \Gamma_1$ and $\mathsf{F}\overset{y}{B} \notin \Gamma_2$, but may have $\mathsf{F}A\overset{z}{\&}B \in \Gamma$. Second, we have the usual restrictions on variables; the individual variable bound by a $\mathsf{F}\forall$-inference respectively a $\mathsf{T}\exists$-inference must not occur free in any assumption of the context of the corresponding premiss.

The way the structural rules are implicit in the inference figures makes *C1* resemble natural deduction. The calculus shares this resemblance with the two sequent calculi presented in [13].

2.2 Calculus *C2*

We can express every *C1*-derivation as a natural deduction style derivation. The *C1* inference figures then induce a natural deduction style calculus of classical predicate logic, which we call *C2*.

2.2.1 *C2* inference figures

The *C2* inference figures can be found on page 7. They are subject to the usual restrictions on variables; the individual variable bound by a $\mathsf{F}\forall$-

$$\frac{}{\overset{u}{\mathsf{T}}A, \overset{x}{\mathsf{F}}A, \Gamma}\ \mathit{axiom}(u,x)$$

$$\frac{\overset{u}{\mathsf{T}}A, \Gamma_1 \quad \overset{x}{\mathsf{F}}A, \Gamma_2}{\Gamma}\ \mathit{cut}(u,x)$$

$$\frac{\overset{u}{\mathsf{T}}A, \Gamma_1}{\overset{w}{\mathsf{T}}A \& B, \Gamma}\ \mathsf{T\&1}(u;w)$$

$$\frac{\overset{v}{\mathsf{T}}B, \Gamma_2}{\overset{w}{\mathsf{T}}A \& B, \Gamma}\ \mathsf{T\&2}(v;w)$$

$$\frac{\overset{x}{\mathsf{F}}A, \Gamma_1 \quad \overset{y}{\mathsf{F}}B, \Gamma_2}{\overset{z}{\mathsf{F}}A \& B, \Gamma}\ \mathsf{F\&}(x,y;z)$$

$$\frac{\overset{u}{\mathsf{T}}A, \Gamma_1 \quad \overset{v}{\mathsf{T}}B, \Gamma_2}{\overset{w}{\mathsf{T}}A \vee B, \Gamma}\ \mathsf{T}\vee(u,v;w)$$

$$\frac{\overset{x}{\mathsf{F}}A, \Gamma_1}{\overset{z}{\mathsf{F}}A \vee B, \Gamma}\ \mathsf{F}\vee\mathsf{1}(x;z)$$

$$\frac{\overset{y}{\mathsf{F}}B, \Gamma_2}{\overset{z}{\mathsf{F}}A \vee B, \Gamma}\ \mathsf{F}\vee\mathsf{2}(y;z)$$

$$\frac{\overset{x}{\mathsf{F}}A, \Gamma_1 \quad \overset{v}{\mathsf{T}}B, \Gamma_2}{\overset{w}{\mathsf{T}}A \supset B, \Gamma}\ \mathsf{T}\supset(x,v;w)$$

$$\frac{\overset{u}{\mathsf{T}}A, \overset{y}{\mathsf{F}}B, \Gamma_1}{\overset{z}{\mathsf{F}}A \supset B, \Gamma}\ \mathsf{F}\supset(u,y;z)$$

$$\frac{\overset{x}{\mathsf{F}}A, \Gamma_1}{\overset{w}{\mathsf{T}}{\sim}A, \Gamma}\ \mathsf{T}{\sim}(x;w)$$

$$\frac{\overset{u}{\mathsf{T}}A, \Gamma_1}{\overset{z}{\mathsf{F}}{\sim}A, \Gamma}\ \mathsf{F}{\sim}(u;z)$$

$$\frac{\overset{u}{\mathsf{T}}A(t/x), \Gamma_1}{\overset{w}{\mathsf{T}}(\forall x)A, \Gamma}\ \mathsf{T}\forall(v;w)$$

$$\frac{\overset{y}{\mathsf{F}}A, \Gamma_1}{\overset{z}{\mathsf{F}}(\forall x)A, \Gamma}\ \mathsf{F}\forall(x,y;z)$$

$$\frac{\overset{u}{\mathsf{T}}A, \Gamma_1}{\overset{w}{\mathsf{T}}(\exists x)A, \Gamma}\ \mathsf{T}\exists(x,v;w)$$

$$\frac{\overset{y}{\mathsf{F}}A(t/x), \Gamma_1}{\overset{z}{\mathsf{F}}(\exists x)A, \Gamma}\ \mathsf{F}\exists(y;z)$$

Table 1: The *C1* inference figures.

inference respectively a $\mathsf{T}\exists$-inference must not occur free in any assumption on which the conclusion depends.

2.2.2 *C2* α-equivalence

A *C2*-derivation of a tableau sequent $\Gamma \vdash \psi$ is a *C2*-derivation of ψ from the assumptions of Γ. We consider two *C2*-derivations of one and the same tableau sequent to be equal provided they are syntactically equal up to a change of bound variables, including variables which become bound when assumptions are discharged.

2.2.3 Translating *C1* to *C2*

By induction on the height of a derivation, we can define a translation, f, that takes a *C1*-derivation of a tableau sequent to a *C2*-derivation of the same tableau sequent. We exemplify the translation by the case

$$\left.\dfrac{\begin{array}{cc} X & Y \\ \overset{x}{\mathsf{F}A},\Gamma_1 & \overset{y}{\mathsf{F}B},\Gamma_2 \end{array}}{\overset{z}{\mathsf{F}A\&B},\Gamma}\ \mathsf{F}\&(x,y;z)\ \right\} \mapsto \left\{\ \dfrac{\begin{array}{ccc} & [\overset{x}{\mathsf{F}A}] & [\overset{y}{\mathsf{F}B}] \\ & f(X) & f(Y) \\ \overset{z}{\mathsf{F}A\&B} & \psi & \psi \end{array}}{\psi}\ \mathsf{F}\&(x,y)\right. .$$

The remaining cases are treated in an analogous way.

2.2.4 *C1* α-equivalence

We consider two *C1*-derivations of one and the same tableau sequent to be equal provided their f-translations are equal. This make the two calculi *C1* and *C2* isomorphic.

By induction on the height of a derivation, we can define an inverse translation, f_Γ^{-1}, that takes a *C2*-derivation of the tableau sequent $\Gamma \vdash \psi$ to a *C1*-derivation of the same sequent. We exemplify the translation by the case

$$\left.\dfrac{\begin{array}{ccc} & [\overset{x}{\mathsf{F}A}] & [\overset{y}{\mathsf{F}B}] \\ & X & Y \\ \overset{z}{\mathsf{F}A\&B} & \psi & \psi \end{array}}{\psi}\ \mathsf{F}\&(x,y)\ \right\} \mapsto \left\{\ \dfrac{\begin{array}{cc} f_{\Gamma_A}^{-1}(X) & f_{\Gamma_B}^{-1}(Y) \\ \overset{x}{\mathsf{F}A},\Gamma & \overset{y}{\mathsf{F}B},\Gamma \end{array}}{\Gamma}\ \mathsf{F}\&(x,y;z)\right. ,$$

where $\Gamma_A = \overset{x}{\mathsf{F}A},\Gamma$, $\Gamma_B = \overset{y}{\mathsf{F}B},\Gamma$, and $\overset{z}{\mathsf{F}A\&B} \in \Gamma$. The remaining cases are treated in an analogous way.

$$\frac{\mathsf{F}A \quad \mathsf{T}A}{\psi}\ \mathit{axiom} \qquad \frac{\begin{matrix}\overset{u}{[\mathsf{T}A]} & \overset{x}{[\mathsf{F}A]}\\ \psi & \psi\end{matrix}}{\psi}\ \mathit{cut}(u,x)$$

$$\frac{\mathsf{T}A\&B \quad \begin{matrix}\overset{u}{[\mathsf{T}A]}\\ \psi\end{matrix}}{\psi}\ \mathsf{T}\&1(u) \qquad \frac{\mathsf{F}A\&B \quad \begin{matrix}\overset{x}{[\mathsf{F}A]} & \overset{y}{[\mathsf{F}B]}\\ \psi & \psi\end{matrix}}{\psi}\ \mathsf{F}\&(x,y)$$

$$\frac{\mathsf{T}A\&B \quad \begin{matrix}\overset{v}{[\mathsf{T}B]}\\ \psi\end{matrix}}{\psi}\ \mathsf{T}\&2(v)$$

$$\frac{\mathsf{F}A\vee B \quad \begin{matrix}\overset{x}{[\mathsf{F}A]}\\ \psi\end{matrix}}{\psi}\ \mathsf{F}\vee1(x)$$

$$\frac{\mathsf{T}A\vee B \quad \begin{matrix}\overset{u}{[\mathsf{T}A]} & \overset{v}{[\mathsf{T}B]}\\ \psi & \psi\end{matrix}}{\psi}\ \mathsf{T}\vee(u,v)$$

$$\frac{\mathsf{F}A\vee B \quad \begin{matrix}\overset{y}{[\mathsf{F}B]}\\ \psi\end{matrix}}{\psi}\ \mathsf{F}\vee2(y)$$

$$\frac{\mathsf{T}A\supset B \quad \begin{matrix}\overset{x}{[\mathsf{F}A]} & \overset{v}{[\mathsf{T}B]}\\ \psi & \psi\end{matrix}}{\psi}\ \mathsf{T}\supset(x,v) \qquad \frac{\mathsf{F}A\supset B \quad \begin{matrix}\overset{u}{[\mathsf{T}A]}, \overset{y}{[\mathsf{F}B]}\\ \psi\end{matrix}}{\psi}\ \mathsf{F}\supset(u,y)$$

$$\frac{\mathsf{T}{\sim}A \quad \begin{matrix}\overset{x}{[\mathsf{F}A]}\\ \psi\end{matrix}}{\psi}\ \mathsf{T}{\sim}(x) \qquad \frac{\mathsf{F}{\sim}A \quad \begin{matrix}\overset{u}{[\mathsf{T}A]}\\ \psi\end{matrix}}{\psi}\ \mathsf{F}{\sim}(u)$$

$$\frac{\mathsf{T}(\forall x)A \quad \begin{matrix}\overset{v}{[\mathsf{T}A(t/x)]}\\ \psi\end{matrix}}{\psi}\ \mathsf{T}\forall(v) \qquad \frac{\mathsf{F}(\forall x)A \quad \begin{matrix}\overset{y}{[\mathsf{F}A]}\\ \psi\end{matrix}}{\psi}\ \mathsf{F}\forall(x,y)$$

$$\frac{\mathsf{T}(\exists x)A \quad \begin{matrix}\overset{v}{[\mathsf{T}A]}\\ \psi\end{matrix}}{\psi}\ \mathsf{T}\exists(x,v) \qquad \frac{\mathsf{F}(\exists x)A \quad \begin{matrix}\overset{y}{[\mathsf{F}A(t/x)]}\\ \psi\end{matrix}}{\psi}\ \mathsf{F}\exists(y)$$

Table 2: The *C2* inference figures.

To see that the two calculi *C1* and *C2* are isomorphic, it suffice to show that $f(f_\Gamma^{-1}(X)) =_\alpha X$ for every *C2*-derivations X of $\Gamma \vdash \psi$. That $f_\Gamma^{-1}(f(X)) =_\alpha X$ for every *C1*-derivations X of $\Gamma \vdash \psi$ then follows from the definition of α-equivalence for *C1*, according to which $f_\Gamma^{-1}(f(X)) =_\alpha X$ provided that $f(f_\Gamma^{-1}(f(X))) =_\alpha f(X)$. The proof that $f(f_\Gamma^{-1}(X)) =_\alpha X$ for every *C2*-derivations X of $\Gamma \vdash \psi$ is by induction on the height of X. We refrain from presenting this proof here, convinced that the reader can prove the proposition herself.

2.3 Calculus *C3*

We shall present a natural deduction style calculus, essentially equivalent to *C2*, with conversion rules similar to those of intuitionistic calculi of natural deduction. We call this calculus *C3*. It emerges from the following considerations.

2.3.1 Considerations underlying *C3*

We want to embed *C2* in a natural deduction calculus in such a way that the conversion rules of the latter induce plausible conversion rules on the former calculus. To arrive at a natural deduction calculus that induces the conversion rule

$$\dfrac{\dfrac{\overset{w}{[\mathsf{T}A\&B]} \quad \begin{array}{c}\overset{u}{[\mathsf{T}A]}\\ U\\ \psi\end{array}}{\psi}\,{\scriptstyle \mathsf{T}\&1(u)} \qquad \dfrac{\overset{z}{[\mathsf{F}A\&B]} \quad \begin{array}{c}\overset{x}{[\mathsf{F}A]}\\ X\\ \psi\end{array} \quad \begin{array}{c}\overset{y}{[\mathsf{F}B]}\\ Y\\ \psi\end{array}}{\psi}\,{\scriptstyle \mathsf{F}\&(x,y)}}{\psi}\,{\scriptstyle cut(w,z)} \qquad \text{conv} \qquad \dfrac{\begin{array}{c}\overset{u}{[\mathsf{T}A]}\\ U\\ \psi\end{array} \quad \begin{array}{c}\overset{x}{[\mathsf{F}A]}\\ X\\ \psi\end{array}}{\psi}\,{\scriptstyle cut(u,x)},$$

we note that, in intuitionistic sequent calculi, cut is nothing but explicit substitution, and so we replace the cut of the above left hand member with a substitution and the weak principle of reductio ad absurdum,

$$\dfrac{\dfrac{\dfrac{\overset{w}{[\mathsf{T}A\&B]} \quad \begin{array}{c}\overset{u}{[\mathsf{T}A]}\\ U\\ \psi\end{array}}{\psi}\,{\scriptstyle \mathsf{T}\&1(u)}}{\mathsf{F}A\&B}\,{\scriptstyle WRAA(w)} \quad \begin{array}{c}\overset{x}{[\mathsf{F}A]}\\ X\\ \psi\end{array} \quad \begin{array}{c}\overset{y}{[\mathsf{F}B]}\\ Y\\ \psi\end{array}}{\psi}\,{\scriptstyle \mathsf{F}\&(x,y)}.$$

To achieve the above conversion, we want to bring the $\mathsf{T}\&1$- and $\mathsf{F}\&$-inference, or rather their premiss derivations, together. We can for example divide the F&-inference of the above derivation into an axiom and another inference,

$$\dfrac{\dfrac{\dfrac{\overset{w}{[\mathsf{T}A\&B]} \qquad \begin{matrix}\overset{u}{[\mathsf{T}A]}\\ U\\ \psi\end{matrix}}{\psi}\ \mathsf{T}\&1(u)}{\mathsf{F}A\&B}\ WRAA(w) \qquad \dfrac{\begin{matrix}\overset{x}{[\mathsf{F}A]}\\ X\\ \psi\end{matrix} \quad \begin{matrix}\overset{y}{[\mathsf{F}B]}\\ Y\\ \psi\end{matrix}}{\mathsf{T}A\&B}\ (x,y)}{\psi}\ \mathit{axiom},$$

apply β-conversion to arrive at

$$\dfrac{\dfrac{\begin{matrix}\overset{x}{[\mathsf{F}A]}\\ X\\ \psi\end{matrix} \quad \begin{matrix}\overset{y}{[\mathsf{F}B]}\\ Y\\ \psi\end{matrix}}{\mathsf{T}A\&B}\ (x,y) \qquad \begin{matrix}\overset{u}{[\mathsf{T}A]}\\ U\\ \psi\end{matrix}}{\psi}\ \mathsf{T}\&1(u),$$

and then use a new rule to convert the latter derivation to the above right hand member. We could also divide the T&-inference into an axiom and another inference,

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\overset{u}{[\mathsf{T}A]}\\ U\\ \psi\end{matrix}}{\mathsf{F}A\&B}\ (u) \qquad \overset{w}{[\mathsf{T}A\&B]}}{\psi}\ \mathit{axiom}}{\mathsf{F}A\&B}\ WRAA(w) \qquad \begin{matrix}\overset{x}{[\mathsf{F}A]}\\ X\\ \psi\end{matrix} \quad \begin{matrix}\overset{y}{[\mathsf{F}B]}\\ Y\\ \psi\end{matrix}}{\psi}\ \mathsf{F}\&(x,y),$$

apply η-conversion to arrive at

$$\dfrac{\dfrac{\begin{matrix}\overset{u}{[\mathsf{T}A]}\\ U\\ \psi\end{matrix}}{\mathsf{F}A\&B}\ (u) \qquad \begin{matrix}\overset{x}{[\mathsf{F}A]}\\ X\\ \psi\end{matrix} \quad \begin{matrix}\overset{y}{[\mathsf{F}B]}\\ Y\\ \psi\end{matrix}}{\psi}\ \mathsf{F}\&(x,y),$$

and then use another new rule to convert the latter derivation to the above right hand member. We could also divide both inferences.

There are other, but from an intuitionistic standpoint doubtful, ways to achieve the above conversion. We could for example replace the cut of the above left hand side member with a substitution, but use the strong principle of reductio ad absurdum instead of $WRAA$,

$$\dfrac{\dfrac{\dfrac{\overset{z}{[\mathsf{F}A\&B]} \quad \begin{matrix}\overset{x}{[\mathsf{F}A]}\\ X\\ \psi\end{matrix} \quad \begin{matrix}\overset{y}{[\mathsf{F}B]}\\ Y\\ \psi\end{matrix}}{\psi}\,\mathsf{F}\&(x,y)}{\mathsf{T}A\&B}\,SRAA(z) \qquad \begin{matrix}\overset{u}{[\mathsf{T}A]}\\ U\\ \psi\end{matrix}}{\psi}\,\mathsf{T}\&1(u),$$

and then mimic the previous constructions.

We prefer the construction where the $\mathsf{F}\&$-inference is divided into an axiom and another inference on the ground that the latter inference can be taken as an introduction rule in Martin-Löf type theory and, moreover, that the $\mathsf{T}\&1$-inference instantiates the corresponding elimination rule. Applying the construction to the *C2* inference figures yields the *C3* inference figures.

2.3.2 *C3* inference figures

The *C3* inference figures can be found on page 11. They are subject to the usual restrictions on variables; the individual variable bound by a $\forall$-introduction respectively a $\exists$-elimination must not occur free in any assumption on which the conclusion depends.

Note that the *C3* elimination rules, except $\supset$- and $\sim$-elimination, are formal instances of the corresponding intuitionistic general elimination rules [16], with their conclusions specialized to ψ. There is also a close relationship between the *C3* introduction and elimination rules. They relate like introduction and elimination rules of intuitionistic natural deduction, where the introduction rules determine the meaning of the symbols concerned, and the elimination rules express the consequences of the former.

2.3.3 *C3* α-equivalence

A *C3*-derivation of a tableau sequent $\Gamma \vdash \alpha$ is a *C3*-derivation of α from the assumptions of Γ. We consider two *C3*-derivations of one and the same tableau sequent to be equal provided they are syntactically equal up to changes of bound variables in formulas and changes of variables marking discharged assumptions in derivations.

$$\dfrac{\begin{array}{c}\overset{u}{[\mathsf{T}A]}\\ \psi\end{array}}{\mathsf{F}A}\ \mathit{WRAA}(u) \qquad \dfrac{\mathsf{F}A \quad \mathsf{T}A}{\psi}\ \mathit{axiom}$$

$$\dfrac{\begin{array}{cc}\overset{x}{[\mathsf{F}A]} & \overset{y}{[\mathsf{F}B]}\\ \psi & \psi\end{array}}{\mathsf{T}A\&B}\ \&I(x,y) \qquad \dfrac{\mathsf{T}A\&B \quad \begin{array}{c}\overset{u}{[\mathsf{T}A]}\\ \psi\end{array}}{\psi}\ \&E1(u) \qquad \dfrac{\mathsf{T}A\&B \quad \begin{array}{c}\overset{v}{[\mathsf{T}B]}\\ \psi\end{array}}{\psi}\ \&E2(v)$$

$$\dfrac{\begin{array}{c}\overset{x}{[\mathsf{F}A]}\\ \psi\end{array}}{\mathsf{T}A\vee B}\ \vee I1(x) \qquad \dfrac{\begin{array}{c}\overset{y}{[\mathsf{F}B]}\\ \psi\end{array}}{\mathsf{T}A\vee B}\ \vee I2(y) \qquad \dfrac{\mathsf{T}A\vee B \quad \begin{array}{c}\overset{u}{[\mathsf{T}A]}\\ \psi\end{array} \quad \begin{array}{c}\overset{v}{[\mathsf{T}B]}\\ \psi\end{array}}{\psi}\ \vee E(u,v)$$

$$\dfrac{\begin{array}{c}\overset{u}{[\mathsf{T}A]}, \overset{y}{[\mathsf{F}B]}\\ \psi\end{array}}{\mathsf{T}A\supset B}\ \supset I(u,y) \qquad \dfrac{\mathsf{T}A\supset B \quad \begin{array}{c}\overset{x}{[\mathsf{F}A]}\\ \psi\end{array} \quad \begin{array}{c}\overset{v}{[\mathsf{T}B]}\\ \psi\end{array}}{\psi}\ \supset E(x,v)$$

$$\dfrac{\begin{array}{c}\overset{u}{[\mathsf{T}A]}\\ \psi\end{array}}{\mathsf{T}{\sim}A}\ {\sim}I(u) \qquad \dfrac{\mathsf{T}{\sim}A \quad \begin{array}{c}\overset{u}{[\mathsf{F}A]}\\ \psi\end{array}}{\psi}\ {\sim}E$$

$$\dfrac{\begin{array}{c}\overset{y}{[\mathsf{F}A]}\\ \psi\end{array}}{\mathsf{T}(\forall x)A}\ \forall I(x,y) \qquad \dfrac{\mathsf{T}(\forall x)A \quad \begin{array}{c}\overset{v}{[\mathsf{T}A(t/x)]}\\ \psi\end{array}}{\psi}\ \forall E(v)$$

$$\dfrac{\begin{array}{c}\overset{y}{[\mathsf{F}A(t/x)]}\\ \psi\end{array}}{\mathsf{T}(\exists x)A}\ \exists I(y) \qquad \dfrac{\mathsf{T}(\exists x)A \quad \begin{array}{c}\overset{v}{[\mathsf{T}A]}\\ \psi\end{array}}{\psi}\ \exists E(x,v)$$

Table 3: The *C3* inference figures.

2.3.4 Law of the excluded middle

Since it is our intension to interpret classical calculi in *C3*, one may ask for a derivation of the law of the excluded middle. However, note that the $\lor$-introduction rules makes disjunction decidable, and so there can be no *C3*-derivation of $\vdash \mathsf{T}A\lor\sim A$. This makes our task look impossible, but it is not, because the Gentzen sequent $\rightarrow A\lor\sim A$ corresponds to a tableau sequent $\overset{z}{\mathsf{F}A\lor\sim A}\vdash \psi$, and the latter is derivable in *C3*, e.g.

$$\dfrac{\overset{z}{\mathsf{F}A\lor\sim A} \qquad \dfrac{\dfrac{\overset{y}{[\mathsf{F}\sim A]} \qquad \dfrac{\dfrac{\overset{z}{\mathsf{F}A\lor\sim A}\qquad \dfrac{\dfrac{\overset{x}{[\mathsf{F}A]}\quad \overset{u}{[\mathsf{T}A]}}{\psi}\,{\scriptstyle \mathit{axiom}}}{\mathsf{T}A\lor\sim A}\,{\scriptstyle \lor I1(x)}}{\psi}\,{\scriptstyle \mathit{axiom}}}{\mathsf{T}\sim A}\,{\scriptstyle \sim I(u)}}{\psi}\,{\scriptstyle \mathit{axiom}}}{\mathsf{T}A\lor\sim A}\,{\scriptstyle \lor I2(y)}}{\psi}\,{\scriptstyle \mathit{axiom}} \quad .$$

We conclude that the law of the excluded middle is derivable in *C3*, but only because of our interpretation of the consequence relation of classical logic.

2.3.5 Translating *C2* to *C3*

By induction on the height of a derivation, we can define a translation, g, that takes a *C2*-derivation of a tableau sequent to a *C3*-derivation of the same tableau sequent. We exemplify the translation by the four cases

$$\left.\dfrac{\mathsf{F}A \quad \mathsf{T}A}{\psi}\,{\scriptstyle \mathit{axiom}}\right\} \quad \mapsto \quad \left\{\dfrac{\mathsf{F}A \quad \mathsf{T}A}{\psi}\,{\scriptstyle \mathit{axiom}}\right. ,$$

$$\left.\dfrac{\begin{matrix}\overset{u}{[\mathsf{T}A]} & \overset{x}{[\mathsf{F}A]}\\ U & X\\ \psi & \psi\end{matrix}}{\psi}\,{\scriptstyle cut(u,x)}\right\} \quad \mapsto \quad \left\{\begin{matrix}\dfrac{\begin{matrix}\overset{u}{[\mathsf{T}A]}\\ g(U)\\ \psi\end{matrix}}{\mathsf{F}A}\,{\scriptstyle WRAA(u)}\\ g(X)\\ \psi\end{matrix}\right. ,$$

$$\left.\dfrac{\mathsf{T}A\&B \qquad \begin{matrix}\overset{u}{[\mathsf{T}A]}\\ U\\ \psi\end{matrix}}{\psi}\,{\scriptstyle \mathsf{T}\&1(u)}\right\} \quad \mapsto \quad \left\{\dfrac{\mathsf{T}A\&B \qquad \begin{matrix}\overset{u}{[\mathsf{T}A]}\\ g(U)\\ \psi\end{matrix}}{\psi}\,{\scriptstyle \&E1(u)}\right. ,$$

and

$$\left.\dfrac{\mathsf{F}A\&B \qquad \begin{array}{c}[\overset{x}{\mathsf{F}A}]\\ X\\ \psi\end{array} \qquad \begin{array}{c}[\overset{y}{\mathsf{F}B}]\\ Y\\ \psi\end{array}}{\psi}\,{\scriptstyle \mathsf{F}\&(x,y)}\right\} \quad\mapsto\quad \left\{\dfrac{\mathsf{F}A\&B \qquad \dfrac{\begin{array}{c}[\overset{x}{\mathsf{F}A}]\\ g(X)\\ \psi\end{array} \quad \begin{array}{c}[\overset{y}{\mathsf{F}B}]\\ g(Y)\\ \psi\end{array}}{\mathsf{T}A\&B}\,{\scriptstyle \&I(x,y)}}{\psi}\,{\scriptstyle axiom}\right. .$$

The remaining cases are treated in an analogous way.

2.3.6 Conversion rules for *C3*

The conversion rules for *C3* suggest themselves from the *C3* introduction and elimination rules in the same way as do the conversion rules for intuitionistic calculi.

F-conversion

$$\dfrac{\dfrac{\begin{array}{c}[\overset{u}{\mathsf{T}A}]\\ X\\ \psi\end{array}}{\mathsf{F}A}\,{\scriptstyle WRAA(u)} \qquad \begin{array}{c}U\\ \mathsf{T}A\end{array}}{\psi}\,{\scriptstyle axiom} \quad \text{conv} \quad \begin{array}{c}U\\ \mathsf{T}A\\ X\\ \psi\end{array},$$

where no assumption of U may occur discharged in X.

&-conversion

$$\dfrac{\dfrac{\begin{array}{c}[\overset{x}{\mathsf{F}A}]\\ X\\ \psi\end{array} \quad \begin{array}{c}[\overset{y}{\mathsf{F}B}]\\ Y\\ \psi\end{array}}{\mathsf{T}A\&B}\,{\scriptstyle \&I(x,y)} \qquad \begin{array}{c}[\overset{u}{\mathsf{T}A}]\\ U\\ \psi\end{array}}{\psi}\,{\scriptstyle \&E1(u)} \quad \text{conv} \quad \begin{array}{c}\dfrac{\begin{array}{c}[\overset{u}{\mathsf{T}A}]\\ U\\ \psi\end{array}}{\mathsf{F}A}\,{\scriptstyle WRAA(u)}\\ X\\ \psi\end{array},$$

where no assumption of U, except $\overset{u}{\mathsf{T}A}$, may occur discharged in X. The case of the second &-elimination rule is similar.

$\vee$-conversion

$$\dfrac{\dfrac{\begin{array}{c}[\overset{x}{\mathsf{F}A}]\\ X\\ \psi\end{array}}{\mathsf{T}A\vee B}\,{\scriptstyle \vee I1(x)} \qquad \begin{array}{c}[\overset{u}{\mathsf{T}A}]\\ U\\ \psi\end{array} \quad \begin{array}{c}[\overset{v}{\mathsf{T}B}]\\ V\\ \psi\end{array}}{\psi}\,{\scriptstyle \vee E(u,v)} \quad \text{conv} \quad \begin{array}{c}\dfrac{\begin{array}{c}[\overset{u}{\mathsf{T}A}]\\ U\\ \psi\end{array}}{\mathsf{F}A}\,{\scriptstyle WRAA(u)}\\ X\\ \psi\end{array},$$

where no assumption of U, except $\overset{u}{\mathsf{T}}A$, may occur discharged in X. The case of the second $\vee$-introduction rule is similar.

$\supset$-conversion

$$\dfrac{\dfrac{\begin{matrix}[\overset{u}{\mathsf{T}}A],[\overset{y}{\mathsf{F}}B]\\ UY\\ \psi\end{matrix}}{\mathsf{T}A\supset B}\,{\scriptstyle \supset I(u,y)} \qquad \begin{matrix}[\overset{x}{\mathsf{F}}A]\\ X\\ \psi\end{matrix} \quad \begin{matrix}[\overset{v}{\mathsf{T}}B]\\ V\\ \psi\end{matrix}}{\psi}\,{\scriptstyle \supset E(v)} \qquad \text{conv} \qquad \begin{matrix} \\ \\ \\ \dfrac{\begin{matrix} \\ \\ \\ \\ [\overset{u}{\mathsf{T}}A], \end{matrix}\dfrac{\begin{matrix}[\overset{v}{\mathsf{T}}B]\\ V\\ \psi\end{matrix}}{\mathsf{F}B}\,{\scriptstyle WRAA(v)}}{} \end{matrix}$$

$$\dfrac{\begin{matrix}[\overset{u}{\mathsf{T}}A],\ \dfrac{\begin{matrix}[\overset{v}{\mathsf{T}}B]\\ V\\ \psi\end{matrix}}{\mathsf{F}B}\,{\scriptstyle WRAA(v)}\\ UY\\ \psi\end{matrix}}{\begin{matrix}\mathsf{F}A\\ X\\ \psi\end{matrix}}\,{\scriptstyle WRAA(u)} \qquad ,$$

where no assumption of V, except $\overset{v}{\mathsf{T}}B$, may occur discharged in either UY or X, and no assumption of UY, except $\overset{u}{\mathsf{T}}A$ and $\overset{y}{\mathsf{F}}B$, may occur discharged in X.

$\sim$-conversion

$$\dfrac{\dfrac{\begin{matrix}[\overset{u}{\mathsf{T}}A]\\ U\\ \psi\end{matrix}}{\mathsf{T}{\sim}A}\,{\scriptstyle \sim I(u)} \qquad \begin{matrix}[\overset{x}{\mathsf{F}}A]\\ X\\ \psi\end{matrix}}{\psi}\,{\scriptstyle \sim E} \qquad \text{conv} \qquad \dfrac{\begin{matrix}[\overset{u}{\mathsf{T}}A]\\ U\\ \psi\end{matrix}}{\begin{matrix}\mathsf{F}A\\ X\\ \psi\end{matrix}}\,{\scriptstyle WRAA(u)} \qquad ,$$

where no assumption of U, except $\overset{u}{\mathsf{T}}A$, may occur discharged in X.

$\forall$-conversion

$$\dfrac{\dfrac{\begin{matrix}[\overset{y}{\mathsf{F}}A]\\ Y\\ \psi\end{matrix}}{\mathsf{T}(\forall x)A}\,{\scriptstyle \forall I(y)} \qquad \begin{matrix}[\mathsf{T}\overset{v}{A}(t/x)]\\ V\\ \psi\end{matrix}}{\psi}\,{\scriptstyle \forall E(v)} \qquad \text{conv} \qquad \dfrac{\begin{matrix}[\mathsf{T}\overset{v}{A}(t/x)]\\ V\\ \psi\end{matrix}}{\begin{matrix}\mathsf{F}A(t/x)\\ Y(t/x)\\ \psi\end{matrix}}\,{\scriptstyle WRAA(v)} \qquad ,$$

where no assumption of V, except $\mathsf{T}\overset{v}{A}(t/x)$, may occur discharged in Y.

∃-conversion

$$
\dfrac{\dfrac{\begin{matrix}\overset{y}{[\mathsf{F}A(t/x)]}\\ Y\\ \psi\end{matrix}}{\mathsf{T}(\exists x)A}\,\exists I(y) \qquad \begin{matrix}\overset{v}{[\mathsf{T}A]}\\ V\\ \psi\end{matrix}}{\psi}\,\exists E(v) \quad \text{conv} \quad \begin{matrix}\dfrac{\begin{matrix}\overset{v}{[\mathsf{T}A(t/x)]}\\ V(t/x)\\ \psi\end{matrix}}{\mathsf{F}A(t/x)}\,WRAA(v)\\ Y\\ \psi\end{matrix} \quad ,
$$

where no assumption of V, except $\overset{v}{\mathsf{T}A(t/x)}$, may occur discharged in Y.

2.3.7 *C3* β-equivalence

We say that two *C3*-derivations of one and the same sequent are β-equivalent provided the equivalence relation, which the *C3* conversion rules generate, relates them.

2.3.8 *C2* β-equivalence

We say that two *C2*-derivations of one and the same tableau sequent are β-equivalent provided their g-translations are equivalent.

2.3.9 *C1* β-equivalence

We say that two *C1*-derivations of one and the same tableau sequent are β-equivalent provided their f-translations are equivalent.

2.3.10 A notion of classical proof

It is from an intuitionistic standpoint natural to say that two classical derivations denote one and the same proof provided that they are β-equivalent in the sense of Sect. 2.3.7–2.3.9. We adopt this notion of classical proof. For a discussion of the identity problem for proofs, including the identity problem for classical proofs, see [17, pp. 9–22].

2.4 Normalization and normal form theorems

One can use the notion of computability (or convertibility) invented by Tait [15] to prove normalization for Gödel's theory [4] of functionals of finite type, and carried over to proofs via the Curry-Howard correspondence by Martin-Löf [7, 8, 10], to prove normalization for the *C3* conversion rules. One can, moreover, prove the Church-Rosser property for the *C3* conversion rules using the method developed by Church [2] and Rosser for combinatory logic, and later perfected by Tait and Martin-Löf [9]. For the Martin-Löf form of the proof of the Church-Rosser property for λ-calculus, see also Barendregt [1, p. 128] and Hindley, Lercher, and Seldin [5, p. 139]. We shall not prove normalization and the Church-Rosser property for the *C3* conversion rules, but take these theorems for granted, convinced that the reader can prove them herself. Thus we take for granted that for every *C3*-derivation there exists a unique equivalent normal *C3*-derivation.

Since the translation of Sect. 2.3.5 is injective for cut free derivations, we can conclude that for every *C2*-derivation there exists a unique equivalent cut free *C2*-derivation and, moreover, since the translation of Sect. 2.2.3 is injective for cut free derivations, we can also conclude that for every *C1*-derivation there exists a unique equivalent cut free *C1*-derivation. Note that a cut free derivation obtained by means of Gentzen's Hauptsatz generally fails to be unique.

2.5 Calculus *NJ*

We shall extend the Gentzen [3, pp. 74–81] natural deduction style calculus of intuitionistic predicate logic, also known as *NJ*, with signs and ψ, governed by the weak principle of reductio ad absurdum, the axiom inference figure, the *C3* F-conversion rule, and a propositional constant Ψ such that $\psi = \mathsf{T}\Psi$. However, we keep the name of the original calculus.

The reader may consider *NJ* a fragment of Martin-Löf type theory with $\mathsf{T}A = \mathsf{Proof}(A)$ *type* (A *prop*) and $\mathsf{F}A = (\mathsf{T}A)\psi$ *type* (A *prop*), analogous to the intuitionistic notions of truth and falsity [12]. We shall henceforth suppress the sign T for *NJ*-formulas, and just write A for $\mathsf{T}A$. However, we still write $\mathsf{T}A$ for *C1*-, *C2*-, and *C3*-formulas.

2.5.1 *NJ* inference figures

The *NJ* inference figures can be found on page 17. They are subject to the usual restrictions: the individual variable bound by a $\forall$-introduction respectively a $\exists$-elimination must not occur free in any assumption on which the conclusion depends.

2.5.2 *NJ* α-equivalence

A *NJ*-derivation of a tableau sequent $\Gamma \vdash \alpha$ is a *NJ*-derivation of α from the assumptions of Γ. We consider two *NJ*-derivations of one and the same

$$\frac{\begin{array}{c}\overset{u}{[A]}\\ \psi\end{array}}{\mathsf{F}A}\ \mathit{WRAA}(u) \qquad\qquad \frac{\mathsf{F}A \quad A}{\psi}\ \mathit{axiom}$$

$$\frac{A \quad B}{A\&B}\ \&I \qquad\qquad \frac{A\&B}{A}\ \&E1 \qquad \frac{A\&B}{B}\ \&E2$$

$$\frac{A}{A\vee B}\ \vee I1 \qquad \frac{B}{A\vee B}\ \vee I2 \qquad\qquad \frac{A\vee B \quad \begin{array}{c}\overset{u}{[A]}\\ C\end{array} \quad \begin{array}{c}\overset{v}{[B]}\\ C\end{array}}{C}\ \vee E(u,v)$$

$$\frac{\begin{array}{c}\overset{u}{[A]}\\ B\end{array}}{A\supset B}\ \supset I(u) \qquad\qquad \frac{A\supset B \quad A}{B}\ \supset E$$

$$\frac{\begin{array}{c}\overset{u}{[A]}\\ \perp\end{array}}{\sim A}\ \sim I(u) \qquad\qquad \frac{\sim A \quad A}{\perp}\ \sim E$$

$$\frac{\perp}{A}\ \perp E$$

$$\frac{A}{(\forall x)A}\ \forall I(x) \qquad\qquad \frac{(\forall x)A}{A(t/x)}\ \forall E$$

$$\frac{A(t/x)}{(\exists x)A}\ \exists I \qquad\qquad \frac{(\exists x)A \quad \begin{array}{c}\overset{v}{[A]}\\ C\end{array}}{C}\ \exists E(x,v)$$

Table 4: The *NJ* inference figures.

tableau sequent to be equal provided they are syntactically equal up to a change of bound variables and variables of discharged assumptions.

2.5.3 Conversion rules for *NJ*

The conversion rules for *NJ* are those of the original calculus together with the *C3* F-conversion rule.

2.5.4 *NJ* β-equivalence

We say that two *NJ*-derivations of one and the same sequent are β-equivalent provided the equivalence relation, which the *NJ* conversion rules generate, relate them. Note that two β-equivalent *NJ*-derivations denote one and the same proof in the sense of Martin-Löf type theory, see [11, pp. 11–13] and [14, pp. 9–12].

3 Interpretation

We present the interpretation and some of its properties. The interpretation unfolds from the *C3* introduction rules: These determine the meaning of the symbols concerned in the same way as intuitionistic introduction rules do. From this follows the interpretation of symbols, and thus formulas, up to equivalence. The interpretation turns out to agree with Kolmogorov's interpretation for formulas not containing implication.

The interpretation of inference figures, and thus derivations, follows from that of symbols; the *C3* inference figures are interpreted by compositions of *NJ* inference figures. For every *C3* conversion rule, the interpretation of the left member converts to the interpretation of the right member. Consequently, the interpretation respects convertibility.

3.1 Interpreting symbols

The *C3* introduction rules can be taken to determine the meaning of the symbols concerned in the same way as the intuitionistic introduction rules do, given that we handle the signs first and then the logical symbols. We shall, accordingly, interpret the signs by their intuitionistic counterparts, governed by the usual introduction and elimination rules, and the logical symbols by propositional expressions determined, up to equivalence, by the corresponding introduction rules. We have chosen to reduce the classical logical operations to the usual intuitionistic ones, instead of defining them directly in Martin-Löf type theory, partly out of convenience, and partly out of a wish to limit the use of higher order function types. Let S^* denote the interpretation of the symbol S. We put

$$\begin{array}{rcl}
\mathsf{T}^*A & = & A, \\
\mathsf{F}^*A & = & \mathsf{F}A, \\
A\&^*B & = & \neg\neg A\&\neg\neg B, \\
A\vee^*B & = & \neg\neg A\vee\neg\neg B, \\
A\supset^*B & = & A\supset\neg\neg B, \\
\sim^*A & = & \neg A, \\
(\forall^*x)A & = & (\forall x)\neg\neg A, \\
(\exists^*x)A & = & (\exists x)\neg\neg A,
\end{array}$$

where $\neg A = A\supset\Psi$. That is, we place a double negation at every positive position relative the logical symbol at hand. We extend the interpretation to signed formulas by interpreting atomic formulas as themselves.

Note that for tableau sequents of the form $\overset{x}{\mathsf{F}A}\vdash\psi$, where the formula contains no implications, the interpretation reduces to the Kolmogorov [6] interpretation because of the equivalence with tableau sequents of the form $\overset{u}{\mathsf{T}{\sim}A}\vdash\psi$ and thus sequents of the form $\vdash\mathsf{T}{\sim}{\sim}A$. On the other hand, when a formula contains an implication the interpretations disagree; for example, in the case of the formula $A\supset B$ we get the sequent $\vdash\neg\neg(A\supset\neg\neg B)$ while the Kolmogorov interpretation gets the sequent $\vdash\neg\neg(\neg\neg A\supset\neg\neg B)$. Furthermore, the interpretation is compositional in the sense that $(C(B/A))^* = C^*(B^*/A^*)$, contrary to the Kolmogorov interpretation.

3.1.1 On propositional constants

We generally consider, for both intuitionistic and classical logic, the true formula, $\top$, as an empty conjunction of formulas, and likewise, the false formula, $\bot$, as an empty disjunction of formulas. Note that the two cases $A_1\&^*...\&^*A_m = \neg\neg A_1\&...\&\neg\neg A_m$ and $A_1\vee^*...\vee^*A_m = \neg\neg A_1\vee...\vee\neg\neg A_m$ specialize to $\top^* = \top$ and $\bot^* = \bot$ for $m = 0$, which, moreover, agrees with interpreting atomic formulas as themselves. Thus we should take the true and the false formula of classical logic to be equal to their intuitionistic counterparts.

3.2 Interpreting inference figures

We take the interpretation of Sect. 3.1 seriously and define the signs and logical symbols of *C3* accordingly. We identify the weak principle of reductio ad absurdum and the axiom inference figure with the corresponding *NJ* inference figures, and, moreover, define the *C3* logical symbol inference figures in terms of the *NJ* inference figures. The latter works essentially because we have taken the introduction rules to determine the meaning of the logical symbols concerned and, moreover, because of the close relationship between the *C3* introduction and elimination rules.

3.2.1 Logical symbol inference figures

&*-introduction

$$\left.\dfrac{\begin{array}{cc}\overset{x}{[\mathsf{F}A]} & \overset{y}{[\mathsf{F}B]}\\ X & Y\\ \psi & \psi\end{array}}{\mathsf{T}A\&^*B}\,{\scriptstyle \&^*I(x,y)}\right\} = \left\{\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\overset{x}{[\neg A]}\quad\overset{u}{[A]}}{\psi}\,{\scriptstyle\supset E}}{\mathsf{F}A}\,{\scriptstyle WRAA(u)}\\ X\\ \psi\end{array}}{\neg\neg A}\,{\scriptstyle\supset I(x)}\qquad\dfrac{\begin{array}{c}\dfrac{\dfrac{\overset{y}{[\neg B]}\quad\overset{v}{[B]}}{\psi}\,{\scriptstyle\supset E}}{\mathsf{F}B}\,{\scriptstyle WRAA(v)}\\ Y\\ \psi\end{array}}{\neg\neg B}\,{\scriptstyle\supset I(y)}}{\neg\neg A\&\neg\neg B}\,{\scriptstyle\&I}\right. .$$

&*-elimination

$$\left.\dfrac{\mathsf{T}A\&^*B\qquad\begin{array}{c}\overset{u}{[\mathsf{T}A]}\\ U\\ \psi\end{array}}{\psi}\,{\scriptstyle\&^*E1(u)}\right\} = \left\{\dfrac{\dfrac{\neg\neg A\&\neg\neg B}{\neg\neg A}\,{\scriptstyle\&E1}\qquad\dfrac{\begin{array}{c}\overset{u}{[A]}\\ U\\ \psi\end{array}}{\neg A}\,{\scriptstyle\supset I(u)}}{\psi}\,{\scriptstyle\supset E}\right. .$$

We handle the case of the second &*-elimination rule analogously.

$\vee^*$-introduction

$$\left.\dfrac{\begin{array}{c}\overset{x}{[\mathsf{F}A]}\\ X\\ \psi\end{array}}{\mathsf{T}A\vee^*B}\,{\scriptstyle\vee^*I1(x)}\right\} = \left\{\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\overset{x}{[\neg A]}\quad\overset{u}{[A]}}{\psi}\,{\scriptstyle\supset E}}{\mathsf{F}A}\,{\scriptstyle WRAA(u)}\\ X\\ \psi\end{array}}{\neg\neg A}\,{\scriptstyle\supset I(x)}}{\neg\neg A\vee\neg\neg B}\,{\scriptstyle\vee I1}\right. .$$

We handle the case of the second $\vee^*$-introduction rule analogously.

$\vee^*$**-elimination**

$$\left.\dfrac{\mathsf{T}A\vee^*B \qquad \begin{matrix}\overset{u}{[\mathsf{T}A]}\\ U\\ \psi\end{matrix} \qquad \begin{matrix}\overset{v}{[\mathsf{T}B]}\\ V\\ \psi\end{matrix}}{\psi}\,{\scriptstyle \vee^*E(u,v)}\right\}$$

$$= \left\{\dfrac{\neg\neg A\vee\neg\neg B \qquad \dfrac{\overset{x}{[\neg\neg A]} \quad \dfrac{\begin{matrix}\overset{u}{[A]}\\ U\\ \psi\end{matrix}}{\neg A}\,{\scriptstyle\supset Iu}}{\psi}\,{\scriptstyle\supset E} \qquad \dfrac{\overset{y}{[\neg\neg B]} \quad \dfrac{\begin{matrix}\overset{v}{[B]}\\ V\\ \psi\end{matrix}}{\neg B}\,{\scriptstyle\supset I(v)}}{\psi}\,{\scriptstyle\supset E}}{\psi}\,{\scriptstyle\vee E(x,y)}\right. .$$

$\supset^*$**-introduction**

$$\left.\dfrac{\begin{matrix}\overset{u}{[\mathsf{T}A]},\overset{y}{[\mathsf{F}B]}\\ UY\\ \psi\end{matrix}}{\mathsf{T}A\supset^*B}\,{\scriptstyle\supset^*I(u,y)}\right\} = \left\{\dfrac{\dfrac{\begin{matrix}\overset{u}{[A]}, \quad \dfrac{\dfrac{\overset{y}{[\neg B]} \quad \overset{v}{[B]}}{\psi}\,{\scriptstyle\supset E}}{\mathsf{F}B}\,{\scriptstyle WRAA(v)}\\ UY\\ \psi\end{matrix}}{\neg\neg B}\,{\scriptstyle\supset I(y)}}{A\supset\neg\neg B}\,{\scriptstyle\supset I(u)}\right. .$$

$\supset^*$**-elimination**

$$\left.\dfrac{\mathsf{T}A\supset^*B \qquad \begin{matrix}\overset{x}{[\mathsf{F}A]}\\ X\\ \psi\end{matrix} \qquad \begin{matrix}\overset{v}{[\mathsf{T}B]}\\ V\\ \psi\end{matrix}}{\psi}\,{\scriptstyle\supset^*E(x,v)}\right\} = \left\{\begin{matrix}\dfrac{\dfrac{\dfrac{A\supset\neg\neg B \quad \overset{u}{[A]}}{\neg\neg B}\,{\scriptstyle\supset E} \qquad \dfrac{\begin{matrix}\overset{v}{[B]}\\ V\\ \psi\end{matrix}}{\neg B}\,{\scriptstyle\supset I(v)}}{\psi}\,{\scriptstyle\supset E}}{\mathsf{F}A}\,{\scriptstyle WRAA(u)}\\ X\\ \psi\end{matrix}\right. .$$

$\sim^*$**-introduction**

$$\left.\dfrac{\begin{matrix}\overset{u}{[\mathsf{T}A]}\\ U\\ \psi\end{matrix}}{\mathsf{T}\sim^*A}\,{\scriptstyle\sim^*I(u)}\right\} = \left\{\dfrac{\begin{matrix}\overset{u}{[A]}\\ U\\ \psi\end{matrix}}{\neg A}\,{\scriptstyle\supset I(u)}\right. .$$

$\sim^*$-elimination

$$\left.\dfrac{\mathsf{T}{\sim^*}A \qquad \begin{array}{c}[\mathsf{F}\overset{x}{A}]\\ X\\ \psi\end{array}}{\psi}\;{\scriptstyle \sim^*E}\right\} \;=\; \left\{\begin{array}{c} \dfrac{\dfrac{\neg A \quad [\overset{u}{A}]}{\psi}\;{\scriptstyle \supset E}}{\mathsf{F}A}\;{\scriptstyle WRAA(u)} \\ X \\ \psi \end{array}\right. .$$

$\forall^*$-introduction

$$\left.\dfrac{\begin{array}{c}[\mathsf{F}\overset{y}{A}]\\ Y\\ \psi\end{array}}{\mathsf{T}(\forall^* x)A}\;{\scriptstyle \forall^*I(x,y)}\right\} \;=\; \left\{\begin{array}{c} \dfrac{\dfrac{[\overset{y}{\neg A}] \quad [\overset{v}{A}]}{\psi}\;{\scriptstyle \supset E}}{\mathsf{F}A}\;{\scriptstyle WRAA(v)} \\ Y \\ \dfrac{\dfrac{\psi}{\neg\neg A}\;{\scriptstyle \supset I(y)}}{(\forall x)\neg\neg A}\;{\scriptstyle \forall I(x)} \end{array}\right. .$$

$\forall^*$-elimination

$$\left.\dfrac{\mathsf{T}(\forall^* x)A \qquad \begin{array}{c}[\mathsf{T}A(\overset{v}{t}/x)]\\ V\\ \psi\end{array}}{\psi}\;{\scriptstyle \forall^*E(v)}\right\} \;=\; \left\{ \dfrac{\dfrac{(\forall x)\neg\neg A}{\neg\neg A(t/x)}\;{\scriptstyle \forall E} \qquad \dfrac{\begin{array}{c}[A(\overset{v}{t}/x)]\\ V\\ \psi\end{array}}{\neg A(t/x)}\;{\scriptstyle \supset I(v)}}{\psi}\;{\scriptstyle \supset E}\right. .$$

$\exists^*$-introduction

$$\left.\dfrac{\begin{array}{c}[\mathsf{F}A(\overset{y}{t}/x)]\\ Y\\ \psi\end{array}}{\mathsf{T}(\exists^* x)A}\;{\scriptstyle \exists^*I(y)}\right\} \;=\; \left\{\begin{array}{c} \dfrac{\dfrac{[\neg A(\overset{y}{t}/x)] \quad [A(\overset{v}{t}/x)]}{\psi}\;{\scriptstyle \supset E}}{\mathsf{F}A(t/x)}\;{\scriptstyle WRAA(v)} \\ Y \\ \dfrac{\dfrac{\psi}{\neg\neg A(t/x)}\;{\scriptstyle \supset I(y)}}{(\exists x)\neg\neg A}\;{\scriptstyle \exists I} \end{array}\right. .$$

$\exists^*$-elimination

$$\left.\dfrac{\mathsf{T}(\exists^* x)A \qquad \begin{array}{c}[\mathsf{T}\overset{v}{A}]\\ V\\ \psi\end{array}}{\psi}\;{\scriptstyle \exists^*E(x,v)}\right\} \;=\; \left\{ \dfrac{(\exists x)\neg\neg A \qquad \dfrac{[\overset{y}{\neg\neg A}] \quad \dfrac{\begin{array}{c}[\overset{v}{A}]\\ V\\ \psi\end{array}}{\neg A}\;{\scriptstyle \supset I(v)}}{\psi}\;{\scriptstyle \supset E}}{\psi}\;{\scriptstyle \exists E(x,y)}\right. .$$

3.2.2 On conversion rules

Note that, for every *C3* conversion rule, the interpretation of the left member converts to the interpretation of the right member, e.g., for the first &-conversion rule, we have

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\dfrac{\dfrac{\overset{x}{[\neg A]}\quad \overset{u}{[A]}}{\psi}{\scriptstyle \supset E}}{\mathsf{F}A}{\scriptstyle WRAA(u)}\\ X\\ \psi\end{matrix}}{\neg\neg A}{\scriptstyle \supset I(x)} \qquad \dfrac{\begin{matrix}\dfrac{\dfrac{\overset{y}{[\neg B]}\quad \overset{v}{[B]}}{\psi}{\scriptstyle \supset E}}{\mathsf{F}B}{\scriptstyle WRAA(v)}\\ Y\\ \psi\end{matrix}}{\neg\neg B}{\scriptstyle \supset I(y)}}{\neg\neg A \& \neg\neg B}{\scriptstyle \& I}}{\neg\neg A}{\scriptstyle \& E1} \qquad \dfrac{\begin{matrix}\overset{u}{[A]}\\ U\\ \psi\end{matrix}}{\neg A}{\scriptstyle \supset I(u)}}{\psi}{\scriptstyle \supset E} \qquad \text{conv} \qquad \begin{matrix}\dfrac{\begin{matrix}\overset{u}{[A]}\\ U\\ \psi\end{matrix}}{\mathsf{F}A}{\scriptstyle WRAA(u)}\\ X\\ \psi\end{matrix} \quad ,
$$

where no assumption of U, except $\overset{u}{A}$, may occur discharged in X. Consequently, the interpretation respects β-equivalence.

3.2.3 On normal derivations

For every *C3* introduction, except $\sim^*$-introductions, its interpretation contains a derivation

$$
\dfrac{\begin{matrix}\dfrac{\dfrac{\overset{x}{[\neg A]}\quad \overset{u}{[A]}}{\psi}{\scriptstyle \supset E}}{\mathsf{F}A}{\scriptstyle WRAA(u)}\\ X\\ \psi\end{matrix}}{\neg\neg A}{\scriptstyle \supset I(x)} \quad .
$$

Note that X must have the form

$$
\begin{matrix}\dfrac{\mathsf{F}A\quad \begin{matrix}X_1\\ A\end{matrix}}{\psi}{\scriptstyle axiom}, \ldots, \dfrac{\mathsf{F}A\quad \begin{matrix}X_n\\ A\end{matrix}}{\psi}{\scriptstyle axiom}\\ X_0\\ \psi\end{matrix} \quad ,
$$

whence

$$
\dfrac{\begin{matrix}\dfrac{\dfrac{\overset{x}{[\neg A]}\quad \overset{u}{[A]}}{\psi}{\scriptstyle \supset E}}{\mathsf{F}A}{\scriptstyle WRAA(u)}\\ X\\ \psi\end{matrix}}{\neg\neg A}{\scriptstyle \supset I(x)} \qquad \text{conv} \qquad \dfrac{\begin{matrix}\dfrac{\overset{x}{[\neg A]}\quad \begin{matrix}X_1\\ A\end{matrix}}{\psi}{\scriptstyle \supset E}, \ldots, \dfrac{\overset{x}{[\neg A]}\quad \begin{matrix}X_n\\ A\end{matrix}}{\psi}{\scriptstyle \supset E}\\ X_0\\ \psi\end{matrix}}{\neg\neg A}{\scriptstyle \supset I(x)}
$$

by means of F-conversion and, moreover, if X is normal then the right derivation is normal. Accordingly, normal *C3*-derivations are generally not interpreted by normal *NJ*-derivations, but their interpretations can be brought to normal form by means of F-conversion. The latter corresponds to β-conversion in type theory, cf. Sect. 2.5.

4 Conclusion

We have extended the double negation interpretation from formulas to derivations. The resulting interpretation is compositional and respects the most obvious conversion rules of the Gentzen sequent calculus for classical predicate logic. It furthermore induces a notion of classical proof similar to that of intuitionistic logic. The interpretation agrees with the Kolmogorov interpretation for formulas not containing implication. The above-mentioned conversion rules were used to guide the construction of the intermediate natural deduction calculus and, to a large degree, by that means determine the interpretation.

The interpretation embeds classical predicate logic in Martin-Löf type theory. This makes it possible to mix classical and intuitionistic logic in the same derivation. However, the axiom of choice remains out of reach due to the interpretation of the consequence relation of classical logic, except for derivations of the form $\Gamma \vdash \mathsf{T}(\exists^* x)A$.

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References

[1] H. P. Barendregt. *Some extensional term models for combinatory logics and λ-calculi.* PhD thesis, University of Utrecht, 1971.

[2] A. Church. *The calculi of lambda-conversion.* Princeton University Press, 1941.

[3] G. Gentzen. Investigations into logical deduction. In M. E. Szabo, editor, *The Collected Papers of Gerhard Gentzen*, pages 68–131. North-Holland Publishing Company, 1969.

[4] K. Gödel. Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. *Dialectica*, 12:280–287, 1958.

[5] J. R. Hindley, B. Lercher, and J. P. Seldin. *Introduction to combinatory logic.* Number 7 in London Mathematical Society Lecture Notes Series. Cambridge University Press, 1972.

[6] A. N. Kolmogorov. On the principle of excluded middle. In J. von Heijenoort, editor, *From Frege to Gödel*, pages 414–437. Harvard University Press, 1967.

[7] P. Martin-Löf. Hauptsatz for the intuitionistic theory of iterated inductive definitions. In J. E. Fernstad, editor, *Proceedings of the Second Scandinavian Logic Symposium*, pages 179–216. North-Holland, 1971.

[8] P. Martin-Löf. Hauptsatz for the theory of species. In J. E. Fernstad, editor, *Proceedings of the Second Scandinavian Logic Symposium*, pages 217–233. North-Holland, 1971.

[9] P. Martin-Löf. A theory of types. The article was withdrawn from publication, 1971.

[10] P. Martin-Löf. Hauptsatz for intuitionistic simple type theory. In P. Suppes, L. Henkin, A. Joja, and Gr. C. Moisil, editors, *Proceedings of the Fourth International Congress for Logic, Methodology and Philosophy of Science*, pages 279–290. North-Holland, 1973.

[11] P. Martin-Löf. *Intuitionistic Type Theory.* Bibliopolis, 1984.

[12] P. Martin-Löf. Verificationism then and now. In W. de Pauli-Schimanovich, E. Köhler, and F. Stadler, editors, *The Foundational Debate: Complexity and Constructivity in Mathematics and Physics*, pages 187–196. Kluwer, 1994.

[13] S. Negri and J. von Plato. Sequent calculus in natural deduction. *The Journal of Symbolic Logic*, 66:1803–1816, 2001.

[14] B. Nordström, K. Petersson, and J. M. Smith. *Programming in Martin-Löf's Type Theory.* Oxford University Press, 1990.

[15] W. W. Tait. Normal form theorem for bar recursive functions of finite type. In J. E. Fernstad, editor, *Proceedings of the Second Scandinavian Logic Symposium*, pages 353–368. North-Holland, 1971.

[16] J. von Plato. Natural deduction with general elimination rules. *Archive of Mathematical Logic*, 40:541–567, 2001.

[17] F. Widebäck. *Identity of proofs.* PhD thesis, Stockholm University, 2001.