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Interpreting Descriptions in Intensional Type Theory

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Abstract

Calculi of indefinite and definite descriptions are presented, and interpreted in Martin-Löf's intensional type theory. The interpretations are formalizations of the implicit ideas found in the literature of constructive mathematics: if we have proved that an element with a certain property exists, we speak of 'the element such that the property holds' and refer by that phrase to the element constructed in the existence proof. In particular, we deviate from the practice of interpreting descriptions by contextual definitions.

1 Introduction

There are two kinds of descriptions to be considered in this paper: $\varepsilon x A(x)$ is *an* element such that $A(x)$ and $\imath x A(x)$ is *the* element such that $A(x)$, thus requiring uniqueness. Formally, $\varepsilon x A(x)$ is an individual if $\exists x A(x)$ is true, while $\imath x A(x)$ is an individual only if $\forall x \forall y (A(x) \; \& \; A(y) \supset x = y)$ is true as well. This difference is the reason for us to distinguish between $\varepsilon x A(x)$ and $\imath x A(x)$. We say that they are *indefinite* and *definite* descriptions, respectively. It is not quite correct to expect that this difference should always be reflected in English by the articles *a/an* and *the*. We say things like 'I met a man. The man was tall', and refer by 'the man' to the man we met, even if there are more than one man in the world [8]. Hence this occurrence of 'The man' is best represented by $\varepsilon x A(x)$, rather than by $\imath x A(x)$.

We also vary our phrases by saying 'I met a man, who was tall' or 'I met a man. He was tall'. We will do no attempts to analyze such grammatical variants, but view them as synonymous to 'I met a man. The man was tall'. Thus the representation in natural language of $\varepsilon x A(x)$ can be 'he', 'who', 'it' etc., as well as the typical one: 'an element such that $A(x)$'.

Our main goal is to understand *definite* descriptions intensionally. That is, we are not satisfied with a proof that one can live without definite descriptions by eliminating them, we actually want to carry out an interpretation of them which explains what they *mean*. Among constructivists, it is often believed that this can be done and the aim of this paper is to show that this belief is indeed correct; but also to investigate *how* such an interpretation really works, intensionally.

Therefore, we introduce a $\imath$-calculus and give a formal interpretation into Martin-Löf's intensional type theory. However, the hope that this should be a *trivial* task is immediately turned down, as we will demonstrate. We simplify our task by proceeding in two steps. First, we introduce an ε-calculus and show

how it can be interpreted into intensional type theory. Then, we introduce the ℩-calculus and show that it can be interpreted into the ε-calculus. Interestingly, the first step relies quite heavily on type-theoretical choice and the strong disjunction elimination which is present in Martin-Löf's type theory. It is an open problem whether a similar interpretation could be carried out in weaker type theories, like logic-enriched type theory [2].

$$℩ \longrightarrow \varepsilon \longrightarrow \text{MLTT}$$

Being more general than the ℩-calculus, but still interpretable in type theory, the ε-calculus would be very interesting, was it not the case that it suffers from some unfamiliar restrictions: modus ponens and existence introduction are not valid in general. We will explain how this fact is dealt with, and why it is not a problem in the ι-calculus.

2 The Structure of the Paper

We relate our work to previous work on descriptions in section 3 and discuss some properties of our calculi. In section 4, we introduce the ε-calculus, which is extended to include equality in section 5. We then show how this calculus is interpreted into Martin-Löf's type theory in section 6, and give some examples in section 7. Then we introduce the ℩-calculus in section 8 and explain how it is translated into the ε-calculus. We end by some comments about how restricted quantifiers are interpreted (sect. 9) and how the calculus can be used to treat partial functions (sect. 10).

3 Background

Russell [9, 10] proposed a contextual definition of descriptions. It interprets the sentence 'the father of Charles II. was executed' as

> It is not always false of x that x begat Charles II. and that x was executed and that "if y begat Charles II., y is identical with x" is always true of y. [9, p. 482]

The theory makes every proposition of the form P(the present King of France) false, a fact that Russell considers 'a great advantage in the present theory' [p. 482]. In particular, propositions of the form $x = x$ are false according to Russell if 'the present King of France' is substituted for x, and so its negation $\sim(x = x)$ is instead true.

In [10, pp. 167–180], Russell considers examples like 'I met a man' and 'I met a unicorn' and argues that the latter one is as meaningful as the first one, even if we assume that unicorns do not exist, because it is perfectly clear what the speaker is trying to communicate. Hence, Russell argues, it must be admitted that descriptive phrases may be meaningful even if they don't refer to anything.

In mathematics, however, it is very unusual to use descriptive phrases that do not refer, even though descriptions as such are very common. At least, it seems to be practice to require a *hypothetical* reference, that is, that the description refer under some condition. This is the case when multiplicative inverses are defined by descriptions, as in done by Mines, Richman and Ruitenburg, among several others:

> If a and b are elements of a monoid, and $ab = 1$, then we say that a is a *left inverse* of b and b is a *right inverse* of a. If b has a left inverse a and a right inverse c, then $a = a(bc) = (ab)c = c$; in this case we say that a is the *inverse* of b and write $a = b^{-1}$. If b has an inverse we say that b is a *unit*, or that b is *invertible*. [5, p. 36]

According to this passage, the expression 'b^{-1}' is used only when b has an inverse, and it then refers to this inverse. It is, it seems, not even allowed to *use* the expression unless b has an inverse. This is indeed the usual attitude taken by most mathematicians, in contrast to the attitude advocated by Russell, who seems to claim (when we have translated his examples to more mathematical ones) that we need to say things like '0^{-1} does not exist' and that we have to consider an expression like '$\sim(0^{-1} = 2)$' to be a true proposition. It is rather mathematical practice to consider such expressions meaningless—as ill-formed propositions. More precisely, a term is considered to make sense only when it refers to some individual. In order for '$\sim(0^{-1} = 2)$' to be accepted as a legitimate proposition, '0^{-1}' must make sense, which it does not unless the ring under consideration is trivial (a case in which '$\sim(0^{-1} = 2)$' actually turns out to be *false*).

It is also doubtful whether Russell's argument is acceptable in every-day life. Russell tries to understand the claim 'I met a unicorn' under the assumption that there are no unicorns. But the claim is clearly inconsistent with that assumption. It would not make sense to claim that 'there are no unicorns, but I met one'. Indeed, the man who thinks that he met a unicorn must be a different one than the person who believes that there are no unicorns. It therefore appears, that we must understand the term 'a unicorn' under the *assumption* that there are unicorns. Further, if the man continues with the sentence 'Unfortunately, the unicorn ran away before anyone else saw it', it is quite obvious that he refers by 'the unicorn' to *the unicorn he claims that he met*, rather than to any other unicorn or to the *concept* of unicorn, as Russell proposes.

Our idea is simply, following Stenlund [12], that the *reference* of a description may depend on an *assumption*. In mathematics, the necessary assumptions are made explicit beforehand, while it is common in every-day life not to spell out such assumptions.

Let us analyze the claim 'I met a unicorn'. Write '$P(x)$' for 'I met x' and '$Q(x)$' for 'x is a unicorn'. We can then write 'I met a unicorn' as '$P(\varepsilon x Q(x))$'. This expression will be admitted as a proposition (under some assumptions) in our system only if $\exists x Q(x)$ can be proved (under the same assumptions). Hence, the inference rule

$$\frac{P(\varepsilon x Q(x))}{\exists x Q(x)}$$

will be *admissible* in the sense that whenever the premise can be derived from certain assumptions, so can the conclusion. But we cannot derive $P(\varepsilon x Q(x)) \supset \exists x Q(x)$, unless we can derive $\exists x Q(x)$ (possibly under assumptions).

There is, however, an alternative formalization of 'I met a unicorn'. If we regard it as a short form of the proposition 'I met something, which was a unicorn', or 'I met something, and the thing was a unicorn', then the natural formalization is $\exists x P(x)$ & $Q(\varepsilon x P(x))$. This can be proved to be a proposition in the calculus we are about to study, and it implies $\exists x Q(x)$. Furthermore, the interpretation we will propose will interpret $\varepsilon x Q(x)$ as referring to the thing which the man claims he met, in correspondence with how we understand *him* when he speaks about 'the unicorn'. We will return to this example in section 7.

This view has some consequences for Russell's examples. All talk about 'the present King of France' presupposes the existence of a present King of France, not necessarily in our ordinary world, but in an imagined context of the utterances. In this context, it is true that 'the present King of France is the present King of France', contrary to Russell's proposal. In the same way, if '$\sim(0^{-1} = 2)$' is to be accepted as a legitimate proposition, we have to presuppose that 0 has an inverse.

The attitude we will take towards descriptions is thus that they are used only when they refer to individuals (possibly under assumptions). Formal systems based on this idea were introduced by Stenlund [12, 13], who also argued philosophically for this view in a much more elaborate way than we have done above. We will introduce systems that are similar to Stenlund's intuitionistic one, with the aim of interpreting them in Martin-Löf's type theory.

The idea behind the interpretations is to follow the practice of constructivists, like Mines, Richman and Ruitenburg, quoted above: when we have a constructive existence proof that b is invertible, that is, we have an a with $ab = ba = 1$, we allow ourselves to use the notation b^{-1}, and refer by this notation to the individual a.

Martin-Löf [4, p. 45] noted that, in his type theory, there is a natural way of making sense of indefinite descriptions by observing that the rules

$$\frac{\exists x A(x)}{\varepsilon x A(x) : I} \qquad \frac{\exists x A(x)}{A(\varepsilon x A(x))}$$

can be viewed as special cases of the rules

$$\frac{p : \exists x A(x)}{\pi_\ell(p) : I} \qquad \frac{p : \exists x A(x)}{\pi_r(p) : A(\pi_\ell(p))}$$

if we make the definition $\varepsilon x A(x) \stackrel{\text{def}}{=} \pi_\ell(p)$, where $\pi_\ell(p)$ is the left projection of the existence proof p. This is the idea behind the interpretation. But unfortunately, it is not as easy as it might seem at a first sight. The reason is that in replacing '$\pi_\ell(p)$' by '$\varepsilon x A(x)$', we remove the p, which can be crucial. For instance, consider the following derivation, which looks quite harmless:

$$\dfrac{P(a) \vee P(b) \qquad \dfrac{\dfrac{[P(a)]}{\exists x P(x)}}{P(\varepsilon x P(x))} \qquad \dfrac{\dfrac{[P(b)]}{\exists x P(x)}}{P(\varepsilon x P(x))}}{P(\varepsilon x P(x)).}$$

When we replace all ε-terms by the left projections of the corresponding existence proofs, we get the following (after reduction):

$$\dfrac{P(a) \vee P(b) \qquad \dfrac{\dfrac{[P(a)]}{\exists x P(x)}}{P(a)} \qquad \dfrac{\dfrac{[P(b)]}{\exists x P(x)}}{P(b)}}{?}$$

So we don't get the same proposition twice above the final line, as we want to. This problem remains if we have a unique element satisfying P: consider for instance natural numbers and take $a \stackrel{\text{def}}{=} n + 0$ and $b \stackrel{\text{def}}{=} 0 + n$. Then $P(n + 0)$ and $P(0 + n)$ are equivalent, but they are not definitionally equal propositions according to intensional type theory. Fortunately, it turns out that this problem can be solved, as will be shown in section 6.

We should finally mention a work by Abadi, Gonthier and Werner [1], who took the approach of *extending* type theory with indefinite descriptions with a special operational semantics. Our aim is instead to keep the type theory and its semantics as it is, and show that a first order proof involving descriptions can be interpreted in it.

4 Indefinite Descriptions

We will set up a natural deduction system for first order logic with indefinite descriptions, very much like the systems for definite descriptions introduced by Stenlund [12, 13]. Characteristic for all these systems is the idea that terms containing descriptions may refer to individuals only under some assumptions, and that they are allowed in the system only when they refer (possibly depending on currently open assumptions). Hence we have to incorporate in the system a possibility to judge that a term t refers to an individual. Such a judgement will read '$t : I$' in our system. Further, we allow as propositions only formulas in which all terms refer, so we need also the possibility to judge that a formula A is admitted as a proposition, which we do by saying '$A : \mathsf{prop}$'.[1] We have also a third form of judgement: the usual one, that a proposition is true (under the open assumptions). As usual in first order logic, we write simply 'A' instead of 'A true' for this kind of judgements. Because of these three different kinds of judgements, our system is more complicated than natural deduction for ordinary first order logic. It looks in fact already like a piece of type theory.

The rules for forming individuals and propositions are as follows.[2]

$$\frac{t_1 : I \quad \cdots \quad t_n : I}{f(t_1, \ldots, t_n) : I}\, fF \qquad \frac{t_1 : I \quad \cdots \quad t_n : I}{P(t_1, \ldots, t_n) : \mathsf{prop}}\, PF \qquad \frac{\exists x A(x)}{\varepsilon x A(x) : I}\, \varepsilon F$$

$$\frac{A : \mathsf{prop} \quad \begin{array}{c}[A] \\ \vdots \\ B : \mathsf{prop}\end{array}}{A \,\&\, B : \mathsf{prop}}\, \& F \qquad \frac{A : \mathsf{prop} \quad B : \mathsf{prop}}{A \vee B : \mathsf{prop}}\, \vee F \qquad \frac{}{\bot : \mathsf{prop}}\, \bot F$$

[1] Stenlund used the dichotomy 'formula expression' vs. 'formula'.

[2] Per Martin-Löf proposed to me that also the formation rules for functions and predicates should allow partial ones. This is indeed a good idea, but makes the calculus more complicated. For many applications, like set theory and algebra, it is sufficient with total primitive functions and relations, because the partial ones can be defined with descriptions.

$$\frac{A : \mathsf{prop} \qquad \begin{matrix}[A]\\ \vdots \\ B : \mathsf{prop}\end{matrix}}{A \supset B : \mathsf{prop}}\supset F \qquad \frac{\begin{matrix}[x : I]\\ \vdots \\ A(x) : \mathsf{prop}\end{matrix}}{\forall x A(x) : \mathsf{prop}}\forall F \qquad \frac{\begin{matrix}[x : I]\\ \vdots \\ A(x) : \mathsf{prop}\end{matrix}}{\exists x A(x) : \mathsf{prop}}\exists F$$

The rules fF and PF are schemata: every primitive n-ary function f and every primitive n-ary predicate P needs such a rule. In particular, we admit the case $n = 0$, giving us constants. (In this case we have zero premises.)

We have the following rules for introducing and eliminating logical operations.[3]

$$\frac{A \quad B}{A \,\&\, B}\&I \qquad \frac{A \,\&\, B}{A}\&E\ell \qquad \frac{A \,\&\, B}{B}\&Er \qquad \frac{A : \mathsf{prop} \qquad \bot}{A}\bot E$$

$$\frac{A \quad B : \mathsf{prop}}{A \vee B}\vee I\ell \qquad \frac{A : \mathsf{prop} \quad B}{A \vee B}\vee Ir \qquad \frac{A \vee B \qquad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}\vee E$$

$$\frac{A : \mathsf{prop} \quad \begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \supset B}\supset I \qquad \frac{A \quad A \supset B}{B}\supset E \qquad \frac{\begin{matrix}[x : I]\\ \vdots \\ A(x)\end{matrix}}{\forall x A(x)}\forall I \qquad \frac{\forall x A(x) \quad t : I}{A(t)}\forall E$$

$$\frac{\begin{matrix}[x : I]\\ \vdots \\ A(x) : \mathsf{prop}\end{matrix} \quad t : I \quad A(t)}{\exists x A(x)}\exists I \qquad \frac{\exists x A(x) \quad \begin{matrix}[x : I][A(x)]\\ \vdots \\ B\end{matrix}}{B}\exists E \qquad \frac{\exists x A(x)}{A(\varepsilon x A(x))}\varepsilon$$

Two kinds of assumptions are allowed: that a variable x is an individual (written '$x : I$') and that a proposition A is true (written 'A'). The usual variable restrictions apply.[4] Any assumption A which is not discharged anywhere in the derivation tree must come together with a derivation of $A : \mathsf{prop}$ (possibly with other open assumptions, which in turn come together with derivations, etc.). We introduce also some unusual restrictions on $\supset E$ (*modus ponens*) and

[3]One could consider to have more premises in these rules. For instance, conjunction introduction could look as follows:

$$\frac{A : \mathsf{prop} \qquad \begin{matrix}[A]\\ \vdots \\ B : \mathsf{prop}\end{matrix} \qquad A \qquad B}{A \,\&\, B.}$$

We discuss this alternative in an appendix.

[4]To be precise: a rule which discharges an assumption of the form $x : I$ is allowed only if x does not occur free in any non-discharged assumption in the same sub-derivation, nor in the conclusion of the rule. The notation $A(t)$, of course, is intended to presuppose that t is free for x in $A(x)$.

$\exists I$. We will soon discuss these restrictions, but first, let us consider an example of a derivation in the system, namely of a familiar property of fields:

$$\forall x(x \neq 0 \supset xx^{-1} = 1)\,.$$

To this end, we introduce some abbreviations

$$\begin{aligned} a^{-1} &\stackrel{\text{def}}{=} \varepsilon x(ax = 1 \mathbin{\&} xa = 1) \\ U(a) &\stackrel{\text{def}}{=} \exists x(ax = 1 \mathbin{\&} xa = 1) \\ \text{FIELD} &\stackrel{\text{def}}{=} \forall x(x \neq 0 \supset U(x))\,, \end{aligned}$$

which let us state two useful special cases of the rules εF and ε:

$$\frac{U(a)}{a^{-1} : I}\,\varepsilon F \qquad \frac{U(a)}{aa^{-1} = 1 \mathbin{\&} a^{-1}a = 1.}\,\varepsilon$$

We can now make the following formal derivation in the system:

$$\cfrac{\cfrac{\cfrac{[x : I]^2 \qquad \overline{0 : I}}{x \neq 0 : \mathsf{prop}} \qquad \cfrac{\cfrac{\cfrac{[x \neq 0]^1 \qquad \cfrac{\text{FIELD} \qquad [x : I]^2}{x \neq 0 \supset U(x)}}{U(x)}}{xx^{-1} = 1 \mathbin{\&} x^{-1}x = 1}}{xx^{-1} = 1}}{x \neq 0 \supset xx^{-1} = 1}\,{}^{1}}{\forall x(x \neq 0 \supset xx^{-1} = 1).}\,{}^{2}$$

This illustrates how the calculus can be used to define and reason about functions (like x^{-1}) that are only partially defined. We devote section 10 to some comments about such applications.

Let us now turn to the restrictions on $\supset E$ and $\exists I$. They are motivated by the fact that we have similar restrictions in natural language, restrictions which seem to be unavoidable when references of descriptions depend on contexts.

If the author of a novel lets the detective conclude ‘if the man was in the room, the murder didn’t take place there’, and later on ‘the man was in the room’, it would be correct to conclude ‘the murder didn’t take place there’ only if it is obvious that the term ‘the man’ refers to the same individual both times.

So we need a restriction like the following one:

In $\supset E$, ε-terms occurring in A
refer to the same individual in both occurrences of A.

Unfortunately, this restriction is semantical, which is not satisfying in a formal system, which should allow correctness of proofs to be automatically and syntactically checked. Moreover, it is not very precise unless we define carefully what ‘refer to the same individual’ means. Finally, it is in general not decidable. We therefore replace this condition by a syntactical and decidable one which is met only when the semantical one is too.

One such syntactical restriction would be not to allow ε-terms at all in A. This would be sufficient, but far too restrictive. Our solution uses the fact that the *derivation* of $t : I$ determines the individual to which t refers. Hence, we may check that two occurrences of t refer to the same individual by checking that they have equal derivations of $t : I$.

The restriction we will choose is to require that both occurrences of A in $\supset E$ is proved to be propositions in the same ways, after reduction. Consider the situation

$$\dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2 \\ A & A \supset B\end{matrix}}{B}\,{\scriptstyle \supset E}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are derivations, possibly involving hypotheses (we assume that they include also the end formulas A and $A \supset B$, respectively; but we display these for enhanced clarity). We will define derivations $\mathcal{D}_1^*$ and $\mathcal{D}_2^*$ of $A : \mathsf{prop}$ and $A \supset B : \mathsf{prop}$, respectively, in a moment. By inspection of the formation rules, we see that $\mathcal{D}_2^*$ must look as follows:

$$\dfrac{\begin{matrix} & [A] \\ \vdots & \vdots \\ A : \mathsf{prop} & B : \mathsf{prop}\end{matrix}}{A \supset B : \mathsf{prop}}$$

and so we can pick out the derivation of $A : \mathsf{prop}$. We reduce it and compare the result syntactically with the reduced form of $\mathcal{D}_1^*$, and require them to be equal. This is our condition for accepting a use of $\supset E$.

For the existence introduction rule, we have the following situation:

$$\dfrac{\begin{matrix}[x : I] & & \\ \mathcal{D}_3 & \mathcal{D}_4 & \mathcal{D}_5 \\ A(x) : \mathsf{prop} & t : I & A(t)\end{matrix}}{\exists x A(x).}\,{\scriptstyle \exists I}$$

We define the derivation $\mathcal{D}_5^*$ of $A(t) : \mathsf{prop}$ and compare it with the one obtained when t is substituted for x in the derivation we had of $A(x) : \mathsf{prop}$,[5]

$$\begin{matrix}\mathcal{D}_4 \\ t : I \\ \mathcal{D}_3[t/x] \\ A(t) : \mathsf{prop}.\end{matrix}$$

The restriction on $\exists I$ is equally needed because an unrestricted use of it would let us derive the unrestricted version of $\supset E$ from the restricted one, assuming the domain I is inhabited:

$$\dfrac{\dfrac{\begin{matrix}\vdots & & \\ A : \mathsf{prop} & t : I & A\end{matrix}}{\exists x A}\,{\scriptstyle \exists I} \qquad \dfrac{[A] \quad A \supset B}{B}\,{\scriptstyle \supset E}}{B.}\,{\scriptstyle \exists E}$$

Let us turn to the definition of $\mathcal{D}^*$. It is given in the proof of the following theorem, which is a sharpening of a theorem by Stenlund [12, Theorem 3.2.6.].

Theorem 1. *If there is a derivation $\mathcal{D}$ of A from some assumptions, there is a derivation $\mathcal{D}^*$ of $A : \mathsf{prop}$ from the* same *assumptions.*

[5]It might be necessary to change variables if t is not free for x in $\mathcal{D}_3$, but this is done in the usual way.

Let us first isolate a lemma, to be used when $\mathcal{D}$ includes a use of $\vee E$. It will also play a crucial role in the interpretation.

Lemma 2. *If we have a derivation of $C : \mathsf{prop}$ from an assumption A (and possibly other assumptions Γ) as well as a derivation of $C : \mathsf{prop}$ from the assumption B (and Γ), we can construct a derivation of $C : \mathsf{prop}$ from the assumption $A \vee B$ (and Γ).*

Proof of the lemma. Say the derivations are $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. By inspection of the formation-rules, we see that $\mathcal{D}_1$ and $\mathcal{D}_2$ must *end* in the same ways: the only possible differences occur *before* some application of εF. If there is no such application, the two proofs must be identical, and so in particular, they do not use the assumptions A and B unless they are found in Γ. Hence we are done in that case. Now, assume there is at least one application of εF. There has, then, to be some application of εF below which $\mathcal{D}_1$ and $\mathcal{D}_2$ are identical. So they look as follows:

$$\begin{array}{c} A \\ \vdots \\ \dfrac{\exists x D(x)}{\varepsilon x D(x) : I}\,\varepsilon F \\ \vdots \\ C : \mathsf{prop} \end{array} \qquad \begin{array}{c} B \\ \vdots \\ \dfrac{\exists x D(x)}{\varepsilon x D(x) : I}\,\varepsilon F \\ \vdots \\ C : \mathsf{prop} \end{array}$$

with the lower dotted parts identical. We now use $\vee E$ and get

$$\begin{array}{c} \dfrac{\dfrac{A \vee B \quad \begin{array}{c}[A] \\ \vdots \\ \exists x D(x)\end{array} \quad \begin{array}{c}[B] \\ \vdots \\ \exists x D(x)\end{array}}{\exists x D(x)}\,\vee E}{\varepsilon x D(x) : I}\,\varepsilon F \\ \vdots \\ C : \mathsf{prop}. \end{array}$$

This procedure is repeated if necessary. The result is a derivation with the required properties. $\square$

Proof of the theorem. We define $\mathcal{D}^*$ by induction. In the base case, $\mathcal{D}$ is nothing but an assumption 'A'. If $\mathcal{D}$ is an *open* assumption in a bigger derivation tree, a derivation of $A : \mathsf{prop}$ is required in order for the tree to be a valid derivation; and we take $\mathcal{D}^*$ to be this derivation of $A : \mathsf{prop}$. If $\mathcal{D}$ is an assumption which is *discharged* somewhere below in the tree, then A occurs also as a premise in the discharging rule (see the derivation rules in which assumptions are discharged), and $\mathcal{D}^*$ is defined with respect to this occurrence.

As induction steps, we have many cases. Here are some of them, the rest are similar.

1. $\mathcal{D}$ ends with $\& I$:

$$\begin{array}{c} \mathcal{D}_1 \quad \mathcal{D}_2 \\ \dfrac{A \quad B}{A \,\&\, B.}\,\& I \end{array}$$

By the induction hypothesis, $\mathcal{D}_1^*$ and $\mathcal{D}_2^*$ are defined, and so we let $\mathcal{D}^*$ be the derivation

$$\dfrac{\begin{array}{c}\mathcal{D}_1^* \\ A : \mathsf{prop}\end{array} \quad \begin{array}{c}\mathcal{D}_2^* \\ B : \mathsf{prop}\end{array}}{A \,\&\, B : \mathsf{prop}.}\,\&I$$

2. $\mathcal{D}$ ends with $\&E\ell$. $\mathcal{D}$ is then of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1 \\ A \,\&\, B\end{array}}{A.}\,\&E\ell$$

But $\mathcal{D}_1^*$ must be of the form

$$\dfrac{\begin{array}{c} \\ \vdots \\ A : \mathsf{prop}\end{array} \quad \begin{array}{c}[A] \\ \vdots \\ B : \mathsf{prop}\end{array}}{A \,\&\, B : \mathsf{prop}.}\,\&F$$

and we let $\mathcal{D}^*$ be the piece ending with $A : \mathsf{prop}$.

3. $\mathcal{D}$ ends with $\&Er$. Then it is of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1 \\ A \,\&\, B\end{array}}{B.}\,\&Er$$

But $\mathcal{D}_1^*$ must be as in the previous case and we let $\mathcal{D}^*$ be the tree

$$\begin{array}{c}\dfrac{\begin{array}{c}\mathcal{D}_1 \\ A \,\&\, B\end{array}}{A}\,\&E\ell \\ \vdots \\ B : \mathsf{prop}.\end{array}$$

4. $\mathcal{D}$ ends with $\bot E$. This case is obvious but our form of the rule $\bot E$ is essential; the theorem is not true with Stenlund's rule, in which the premise $A : \mathsf{prop}$ is left out. In Stenlund's proof sketch of the corresponding theorem [13, p. 206], this rule is forgotten. Stenlund has agreed that the new form of the rule is probably the right one (private communication, Jan. 30, 2003).

5. $\mathcal{D}$ ends with $\vee E$,

$$\dfrac{\begin{array}{c} \\ \vdots \\ A \vee B\end{array} \quad \begin{array}{c}[A] \\ \vdots \\ C\end{array} \quad \begin{array}{c}[B] \\ \vdots \\ C\end{array}}{C.}\,\vee E$$

Let us call the sub-derivations $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, respectively. By induction, we have derivations $\mathcal{D}_2^*$ and $\mathcal{D}_3^*$ of $C : \mathsf{prop}$ from the assumptions A and B, respectively. According to the lemma, we get a derivation of $C : \mathsf{prop}$ from the assumption $A \vee B$. But we have also a derivation $\mathcal{D}_1$ of $A \vee B$. This derivation, followed by the one proving $C : \mathsf{prop}$ from $A \vee B$ is taken to be $\mathcal{D}^*$.

6. $\mathcal{D}$ ends with $\exists E$,

$$
\dfrac{\begin{array}{cc} & [x:I][A(x)] \\ \vdots & \vdots \\ \exists x A(x) & B \end{array}}{B.}\ \exists E
$$

By induction hypothesis, we get a derivation of B : prop using the assumptions $x : I$ and $A(x)$, which we now have to get rid of. First notice that the derivation cannot make use of such assumptions unless εF is used. If it is not, we can take $\mathcal{D}^*$ to be the derivation we have. Otherwise, the derivation has the following form:

$$
\begin{array}{c}
x : I \quad A(x) \\
\vdots \\
\dfrac{\exists y D(y)}{\varepsilon y D(y) : I}\ \varepsilon F \\
\vdots \\
B : \mathsf{prop}
\end{array}
$$

where the last dots represent a part where only formation rules are used. We now transform this tree using $\exists E$:

$$
\begin{array}{c}
\dfrac{\begin{array}{cc} & [x:I][A(x)] \\ \vdots & \vdots \\ \exists x A(x) & \exists y D(y) \end{array}}{\dfrac{\exists y D(y)}{\varepsilon y D(y) : I}\ \varepsilon F}\ \exists E \\
\vdots \\
B : \mathsf{prop}
\end{array}
$$

(the variable restrictions are met: since x does not occur free in B, it cannot occur free in $\exists y D(y)$ either). We repeat this procedure if necessary and the result is $\mathcal{D}^*$.

7. $\mathcal{D}$ ends with the ε-rule:

$$
\begin{array}{c}
\vdots \\
\dfrac{\exists x A(x)}{A(\varepsilon x A(x))}\ \varepsilon
\end{array}
$$

By induction hypothesis, we have a derivation of $\exists x A(x)$: prop, hence a derivation of $A(x)$: prop from the assumption $x : I$. By substituting $\varepsilon x A(x)$ for x in this one, we get $\mathcal{D}^*$:

$$
\begin{array}{c}
\vdots \\
\dfrac{\exists x A(x)}{\varepsilon x A(x) : I}\ \varepsilon F \\
\vdots \\
A(\varepsilon x A(x)) : \mathsf{prop}.
\end{array}
$$

$\square$

Derivations are said to be equal if they have identical normal forms:

Definition 3. *The relation $\approx$ is the smallest equivalence relation between derivations such that*

- *if a derivation ends by a redex in the sense of Prawitz* [7, II. § 2., pp. 35–38][6]*, they are $\approx$-equal,*
- *if derivations are composed by $\approx$-equal subderivations, then they are themselves $\approx$-equal.*

The relation $\approx$ is decidable, since one can normalize and compare the normal forms, which are $\approx$-equal if and only if they are syntactically equal. We are now ready to state precisely what restrictions we put on $\supset E$ and $\exists I$.

Restriction on $\supset E$. The inference

$$\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ A & A \supset B\end{array}}{B}\supset E$$

is allowed if $\mathcal{D}_2^*$, which is of the form

$$\dfrac{\begin{array}{cc} & [A]\\ \mathcal{D}_3 & \mathcal{D}_4\\ A : \mathsf{prop} & B : \mathsf{prop}\end{array}}{A \supset B : \mathsf{prop},}\supset F$$

satisfies $\mathcal{D}_3 \approx \mathcal{D}_1^*$.

Restriction on $\exists I$. The inference

$$\dfrac{\begin{array}{ccc}[x : I] & & \\ \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3\\ A(x) : \mathsf{prop} & t : I & A(t)\end{array}}{\exists x A(x)}\exists I$$

is allowed if the following derivation is $\approx$-equal to D_3^*.

$$\begin{array}{c}\mathcal{D}_2\\ t : I\\ \mathcal{D}_1[t/x]\\ A(t) : \mathsf{prop}\end{array}$$

[6] Per Martin-Löf suggested that one should have also the reductions

$$\dfrac{\dfrac{\begin{array}{ccc}[x : I] & & \\ \vdots & \vdots & \vdots\\ A(x) : \mathsf{prop} & t : I & A(t)\end{array}}{\exists x A(x)}\exists I}{\varepsilon x A(x) : I}\varepsilon F \quad\rightsquigarrow\quad \begin{array}{c}\vdots\\ t : I\end{array} \qquad\qquad \dfrac{\dfrac{\begin{array}{ccc}[x : I] & & \\ \vdots & \vdots & \vdots\\ A(x) : \mathsf{prop} & t : I & A(t)\end{array}}{\exists x A(x)}\exists I}{A(\varepsilon x A(x))}\varepsilon \quad\rightsquigarrow\quad \begin{array}{c}\vdots\\ A(t)\end{array}$$

Indeed, these reductions will be justified by the translation we are about to define in the next section, but they violate the classical interpretation where $\varepsilon x A(x)$ does not depend on the proof of existence of an object with the property A. We therefore omit these reductions, which we do not need for the present purpose.

It is thus a trivial but not in general convenient task to decide if an instance of $\supset E$ or $\exists I$ meets the requirement. It seems that human intuition is quite good at guessing right (we know what we refer to by descriptions), so that a human informal derivation often includes only acceptable instances, but it would be desirable to let a computer check the derivations. We will return to these issues in section 6. Notice however that the conditions are always satisfied when there are no ε-terms at all involved, because then there is, for each formula, at most one way of deriving that it is a proposition.

In section 8 we will show that if we use *definite* descriptions only, no restrictions will be necessary, except for the usual variable restrictions. In fact, we give a process which repairs all illegal applications of $\supset E$ and $\exists I$. This process works also partially for indefinite descriptions, but not in general.

Observe that $\vee E$ (only!) is similar to $\supset E$ and $\exists I$ in that the same proposition occurs twice among the premises. Surprisingly, no restriction is needed in this case, as the translation given in section 6 will show.

5 Equality

We have omitted rules for equality, because ε-terms make it possible to define functions that do not preserve equality ('non-extensional' functions). Hence equality is not very natural in the system. However, equality can be introduced as any binary predicate (write '$t = s$' for $P(t, s)$). Reasoning with equality is then performed by using an axiom[7] $\forall x(x = x)$ for reflexivity, and for each primitive predicate $P(x_1, \ldots, x_n)$ (with $n \geq 1$), including the binary primitive predicate $=$, an axiom

$$\forall x_1 \cdots \forall y_n\big((x_1 = y_1 \;\&\; \cdots \;\&\; x_n = y_n) \supset (P(x_1, \ldots, x_2) \supset P(y_1, \ldots, y_2))\big)\,, \tag{P ext}$$

and, finally, for each n-ary primitive function (with $n \geq 1$) an axiom

$$\forall x_1 \cdots \forall y_n\big((x_1 = y_1 \;\&\; \cdots \;\&\; x_n = y_n) \supset (f(x_1, \ldots, x_2) = f(y_1, \ldots, y_2))\big)\,. \tag{f ext}$$

The rules for symmetry and transitivity can then be derived, but the replacement rule will *not* in general be justified by the translation into type theory (but see section 8).

6 The Translation into Type Theory

We now turn to the translation of the ε-calculus into intensional type theory, as presented in [6], where also the translation of first order logic is explained. We concentrate here on the things that have to be changed rather than defining the translation from the beginning.

First of all, it is necessary to understand that the translation of a proposition A will be determined, not by its syntactical form, but by the *derivation* of

[7] An *axiom* is here formally the same as an assumption. There is, however, a difference as regards to their interpretations (Sect. 6): axioms are interpreted as *proved propositions*, while assumptions are interpreted as assumptions also in type theory. Therefore, axioms can always be used at no cost, unlike assumptions.

$A : \mathsf{prop}$. For example, consider the derivations

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[x:I] \\ \vdots \\ P(x):\mathsf{prop}\end{matrix} \quad \begin{matrix}\vdots \\ a:I\end{matrix} \quad \begin{matrix}\vdots \\ P(a)\end{matrix}}{\exists x P(x)}}{\varepsilon x P(x) : I}}{P(\varepsilon x P(x)) : \mathsf{prop}} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}[x:I] \\ \vdots \\ P(x):\mathsf{prop}\end{matrix} \quad \begin{matrix}\vdots \\ b:I\end{matrix} \quad \begin{matrix}\vdots \\ P(b)\end{matrix}}{\exists x P(x)}}{\varepsilon x P(x) : I}}{P(\varepsilon x P(x)) : \mathsf{prop}.}$$

The end formula '$P(\varepsilon x P(x))$' will in the left case be translated into a proposition which is definitionally equal to the translation of $P(a)$, while in the right case it will be translated into a proposition which is definitionally equal to the translation of $P(b)$. However, we will have the following facts.

Proposition 4. *If $\mathcal{D}_1$ and $\mathcal{D}_2$ are derivations of $A : \mathsf{prop}$ and $\mathcal{D}_1 \approx \mathcal{D}_2$, then both occurrences of A are translated into definitionally equal propositions.*

Proof. Each derivation will be translated to a derivation in type theory, and each redex will correspond to a redex in type theory. □

Proposition 5. *If $\mathcal{D}_1$ and $\mathcal{D}_2$ are derivations of A and $\mathcal{D}_1 \approx \mathcal{D}_2$, then both occurrences of A are translated into definitionally equal propositions.*

Proof. Both derivations will be translated into derivations in type theory of $p : A_1$ and $q : A_2$, respectively. Since p and q must be definitionally equal, also A_1 and A_2 must be definitionally equal, by the monomorphic property of type theory [6]. □

Proposition 6. *If $\mathcal{D}_1$ is a derivation of A, then it is translated into a derivation of $p : A_1$, and $\mathcal{D}^*$ is translated into a derivation of $A_1 : \mathsf{prop}$.*

Proof. This follows immediately from the definition of $\mathcal{D}^*$ and the translation. □

We now give the translation. Fix a set I, which will act as the domain of discourse, i.e., as the interpretation of the symbol 'I'. For each n-ary function symbol f of the calculus there has to correspond an n-ary function on I. We denote it by the same symbol, i.e., we write $f : (I, \ldots, I)I$. Correspondingly, for each n-ary predicate symbol P there has to correspond an n-ary propositional function $P : (I, \ldots, I)\mathsf{prop}$. These requirements in themselves justify the rules fF and PF in the interpretation.

The equality is supposed to be interpreted as an equivalence relation $=_I$ on I. The axioms for equality are interpreted in the obvious ways. For instance, the reflexivity axiom $\forall x(x = x)$ is interpreted as $\lambda(\text{refl}) : (\forall x : I)(x =_I x)$. This is the reason why we call them *axioms*, rather than *assumptions*: they are interpreted as *proved* propositions in contrast to assumptions, which must in general be interpreted as assumptions also in type theory; i.e., an assumption A is interpreted as A true, or rather, as $p : A$, where p is a fresh variable.

The rules εF and ε are interpreted as Martin-Löf [4, p. 45] proposed:

$$\frac{p : \exists x A(x)}{\pi_\ell(p) : I} \qquad \frac{p : \exists x A(x)}{\pi_r(p) : A(\pi_\ell(p)).}$$

The rule $\&F$ is interpreted as the general rule of Σ-formation

$$\frac{A : \mathsf{prop} \quad B(x) : \mathsf{prop}\ (x : A)}{\Sigma(A, B) : \mathsf{prop},}$$

rather than the specialized one where $B(x)$ is not allowed to depend on x. Likewise for the rule $\supset F$, which is interpreted as the general rule of Π-formation:

$$\frac{A : \mathsf{prop} \quad B(x) : \mathsf{prop}\ (x : A)}{\Pi(A, B) : \mathsf{prop}.}$$

The other formation rules are interpreted as usual [6].

Among the introduction rules, the only one needing a new idea is $\exists I$. The reason is that this rule has two occurrences of A and t among the premises, and we cannot be sure that they have been interpreted in the same ways. We may therefore be faced with a situation where we would need a rule like this one:

$$\frac{A_1(x) : \mathsf{prop}\ (x : I) \quad t_1 : I \quad A_2(t_2) : \mathsf{prop} \quad p : A_2(t_2)}{(\exists x : I)A_1(x) \text{ true},}$$

which is *not* a valid rule in type theory. However, the restriction put on $\exists I$ in section 4 gives us $A_2(t_2) = A_1(t_1) : \mathsf{prop}$, so that we get

$$\frac{A_1(x) : \mathsf{prop}\ (x : I) \quad t_1 : I \quad \dfrac{p : A_2(t_2) \quad A_2(t_2) = A_1(t_1) : \mathsf{prop}}{p : A_1(t_1)}}{(t_1, p) : (\exists x : I)A_1(x)},$$

which *is* a valid derivation in type theory.

Among the elimination rules, there are two that need some care: $\supset E$ and $\vee E$. Let us begin with the former one. After having interpreted the premises, we might need a rule like this one:

$$\frac{p : A_1 \quad q : \Pi(A_2, B)}{B(p) \text{ true}},$$

which we do not have. Again, the restrictions put on $\supset E$ in section 4 gives us $A_1 = A_2 : \mathsf{prop}$ and we may use the following derivation, which is indeed valid:

$$\frac{\dfrac{p : A_1 \quad A_1 = A_2 : \mathsf{prop}}{p : A_2} \quad q : \Pi(A_2, B)}{\mathsf{app}(q, p) : B(p).}$$

Finally, we should interpret $\vee E$. Assume we are interpreting a derivation $\mathcal{D}$ which ends with $\vee E$. We have the following situation, after having interpreted the premises:

$$\frac{c : A \vee B \quad \begin{matrix}[x : A] \\ d(x) : C_1\end{matrix} \quad \begin{matrix}[y : B] \\ e(y) : C_2\end{matrix}}{?.}$$

It might well be, as an example on page 4 showed, that C_1 and C_2 are different propositions, so it is not obvious what should be put as conclusion. Looking at the corresponding rule in type theory we see that one premise is lacking:

$$\frac{c : A \vee B \quad \begin{matrix}[z : A \vee B] \\ C(z) : \mathsf{prop}\end{matrix} \quad \begin{matrix}[x : A] \\ d(x) : C(\mathsf{inl}(x))\end{matrix} \quad \begin{matrix}[y : B] \\ e(y) : C(\mathsf{inr}(y))\end{matrix}}{\mathsf{when}(c, d, e) : C(c)}.$$

Hence we have to come up with a propositional function C with $C(\mathsf{inl}(x)) = C_1$ and $C(\mathsf{inr}(y)) = C_2$, and such that $C(c)$ is the proposition we get from the interpretation of the derivation $\mathcal{D}^*$ of $C : \mathsf{prop}$, defined in the proof of Theorem 1.

By this theorem, we have derivations of $C : \mathsf{prop}$ from the assumptions A and B, respectively. By Lemma 2, this gives us a derivation of $C : \mathsf{prop}$ from the assumption $A \vee B$. Interpreting this derivation, we get a propositional function $C(z) : \mathsf{prop}$ $(z : A \vee B)$. Now, the derivation $\mathcal{D}^*$ was defined by substituting the derivation of $A \vee B$ for the assumption (see the proof of Theorem 1). We were assuming that the derivation of $A \vee B$ had already been interpreted, yielding $c : A \vee B$. Thus, $\mathcal{D}^*$ corresponds precisely to $C(c)$, as we required.

Further, $C(\mathsf{inl}(x))$ corresponds to replacing the assumption $A \vee B$ by the derivation

$$\frac{A \quad B : \mathsf{prop}}{A \vee B.}\ \vee I\ell$$

in the derivation of $C : \mathsf{prop}$. Reducing the resulting derivation, we get the original derivation from A to $C : \mathsf{prop}$ back. Hence, by Proposition 4, $C(\mathsf{inl}(x)) = C_1$. An analogous argument shows that $C(\mathsf{inr}(y)) = C_2$.

It probably helps to consider an example: the 'problematic' derivation mentioned before (p. 4) is interpreted as in the figure on page 17.

7 The Man and the Unicorn

It was promised in the introduction that we return to the example of the unicorn after having defined the translation. Recall that '$P(x)$' means 'I met x' and '$Q(x)$' means 'x was a unicorn'. Hence 'I met a unicorn' is formalized as '$P(\varepsilon x Q(x))$'. In order for this to be a proposition, we must have $\varepsilon x Q(x) : I$, hence we must assume, or prove, $\exists x Q(x)$. So if we can prove that $P(\varepsilon x Q(x))$ is a proposition, we must have assumptions enough for proving also $\exists x Q(x)$.

More interesting is the alternative way of formalizing 'I met a unicorn', namely as 'I met something, and the thing was a unicorn'. Formally, this is $\exists x P(x)\ \&\ Q(\varepsilon x P(x))$. It can be proved to be a proposition in the following way:

$$\frac{\dfrac{\dfrac{[x : I]}{P(x) : \mathsf{prop}}\ PF}{\exists x P(x) : \mathsf{prop}}\ \exists F \qquad \dfrac{\dfrac{[\exists x P(x)]}{\varepsilon x P(x) : I}\ \varepsilon F}{Q(\varepsilon x P(x)) : \mathsf{prop}}\ QF}{\exists x P(x)\ \&\ Q(\varepsilon x P(x)) : \mathsf{prop}.}\ \&F$$

This, as may be checked by the reader, makes $\exists x P(x)\ \&\ Q(\varepsilon x P(x))$ interpreted in type theory as a derivation of

$$(\exists p : (\exists x : I)P(x))Q(\pi_\ell(p))\,.$$

We may, from this proposition, derive that there are unicorns (notice that the restriction on $\exists I$ is met!):

$$\frac{\dfrac{[x : I]}{Q(x) : \mathsf{prop}}\ QF \qquad \dfrac{\dfrac{\exists x P(x)\ \&\ Q(\varepsilon x P(x))}{\exists x P(x)}\ \&E}{\varepsilon x P(x) : I}\ \varepsilon F \qquad \dfrac{\exists x P(x)\ \&\ Q(\varepsilon x P(x))}{Q(\varepsilon x P(x))}\ \&E}{\exists x Q(x)}\ \exists I$$

$$\dfrac{c : P(a) \vee P(b) \qquad \begin{array}{c}[y : P(a) \vee P(b)] \\ \vdots \\ P(\pi_\ell(\mathsf{when}(y, (z)(a, z), (z)(b, z)))) : \mathsf{prop}\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}[x : I] \\ P(x) : \mathsf{prop}\end{array} \;\; a : I \;\; [z : P(a)]}{(a, z) : (\exists x : I)P(x)}}{z : P(a)} \qquad \dfrac{\dfrac{\begin{array}{c}[x : I] \\ P(x) : \mathsf{prop}\end{array} \;\; b : I \;\; [z : P(b)]}{(a, z) : (\exists x : I)P(x)}}{z : P(b)}}{\mathsf{when}(c, (z)z, (z)z) : P(\pi_\ell(\mathsf{when}(c, (z)(a, z), (z)(b, z))))}$$

The interpretation of the ‘problematic’ derivation mentioned on page 4.

Hence, $\varepsilon x Q(x)$ is interpreted as

$$\pi_\ell(\pi_\ell(\pi_\ell(q)), \pi_r(q)) : I \ (q : (\exists p : (\exists x : I)P(x))Q(\pi_\ell(p)))\,,$$

which reduces to

$$\pi_\ell(\pi_\ell(q)) : I \ (q : (\exists p : (\exists x : I)P(x))Q(\pi_\ell(p)))\,.$$

In particular, if $a : I$, $b : P(a)$, and $c : Q(a)$, we may take $q \stackrel{\text{def}}{=} ((a, b), c)$ and so $\varepsilon x Q(x)$ is interpreted as something which is definitionally equal to a. This is to say, if it is in fact true that the man met an individual which was a unicorn, $\varepsilon x Q(x)$ refers to that individual.

8 Definite Descriptions

We now extend our system with definite descriptions, obtaining a system very similar to Stenlund's [12, 13], except for some minor changes, and the fact that we can have ε-terms simultaneous in the system.

When no ε-terms are involved, but we have definite descriptions only, some new inference rules can be justified, for instance unrestricted modus ponens, as we will show. One could therefore introduce these rules in the system if ε-terms were abandoned. This would, however, make it necessary to extend also the translation, which is much more complicated than one would expect. We will therefore instead *derive* the new rules. In other words, we show that a ι-calculus *with* the new rules formally added, can be interpreted in the ε-calculus. The translation from the $\imath$-calculus to the ε-calculus then works by removing all applications of the new rules, replacing them by derivations of these rules, and replacing also all $\imath$ by ε.

Formally, we add the following two rules only:

$$\frac{\exists x A(x) \qquad \forall x \forall y (A(x) \ \& \ A(y) \supset x = y)}{\imath x A(x) : I}\ \imath F$$

$$\frac{\exists x A(x) \qquad \forall x \forall y (A(x) \ \& \ A(y) \supset x = y)}{A(\imath x A(x)).}\ \imath$$

Equality is treated and interpreted as in section 5. In particular, all primitive function symbols are supposed to be interpreted as equality-preserving functions. Hence we assume that we have the axioms for reflexivity and extensionality (P ext) (f ext) at hand, so that we need no special inference rules for equality. When we use 'Γ' to denote an arbitrary set of assumption formulas, we will always assume that it contains the necessary axioms for equality, because we can do so at no cost, keeping in mind that we know how to interpret axioms.

The definition of $\mathcal{D}^*$ is extended in the obvious way: a derivation that ends with the $\imath$-rule is replaced by the same derivation but ending with the $\imath F$-rule, followed by the derivation from $\imath x A(x) : I$ to $A(\imath x A(x)) : \mathsf{prop}$ (for details, see the case of the ε-rule in the definition of $\mathcal{D}^*$, Theorem 1).

Terms of the form $\imath x A(x)$ are interpreted exactly as $\varepsilon x A(x)$. In particular, the interpretations of the rules $\imath$ and $\imath F$ do not make use of the premises about uniqueness. These premises are there solely because they allow us to prove the following meta-mathematical result.

Theorem 7. *The following rules are derivable:*

1. *Unrestricted modus ponens*

$$\frac{A \qquad A \supset B}{B}$$

when A does not contain any ε-term.

2. *Unrestricted*[8] *existence introduction*

$$\frac{\begin{array}{c}[x : I] \\ \vdots \\ A(x) : \mathsf{prop}\end{array} \qquad t : I \quad A(t)}{\exists x A(x)}$$

when $A(t)$ does not contain any ε-term.

3. *The replacement rule*

$$\frac{A(t) \quad t = s}{A(s)}$$

when $A(t)$ does not contain any ε-term.

We need a number of lemmas.

Lemma 8. *Assume A does not contain any ε-term. If $\mathcal{D}_1, \mathcal{D}_2$ are derivations $\Gamma \vdash A : \mathsf{prop}$, then there is a derivation $\mathcal{D} : \Gamma \vdash A \supset A$ such that $\mathcal{D}^*$ is*

$$\frac{\begin{array}{c}\mathcal{D}_1 \\ A : \mathsf{prop}\end{array} \quad \begin{array}{c}\mathcal{D}_2 \\ A : \mathsf{prop}\end{array}}{A \supset A : \mathsf{prop}.} \supset F$$

Proof. First note that if A does not contain any $\imath$-term, the lemma is trivial (since, by assumption, A does not contain ε-terms either). We need to take care of the case when A contains some $\imath$-term(s). We will do that by making a simultaneous induction over terms and formulas.

Define the following measure:

$$\begin{aligned}
\mu(x) &= 0 && \text{for variables } x \\
\mu(f(t_1, \ldots, t_n)) &= \max \mu(t_i) + 1 \\
\mu(\imath x A(x)) &= \mu(A(x)) + 1 \\
\mu(P(t_1, \ldots, t_n)) &= \max \mu(t_i) + 1 && \text{`='} \text{ included} \\
\mu(A \,\&\, B) &= \max(\mu(A), \mu(B)) + 1 \\
\mu(A \vee B) &= \max(\mu(A), \mu(B)) + 1 \\
\mu(A \supset B) &= \max(\mu(A), \mu(B)) + 1 \\
\mu(\bot) &= 1 \\
\mu(\forall x A(x)) &= \mu(A(x)) + 1 \\
\mu(\exists x A(x)) &= \mu(A(x)) + 1
\end{aligned}$$

[8]By *unrestricted*, we mean without restrictions on how $A(x) : \mathsf{prop}$ and $t : I$ are derived. It is still, of course, necessary to have variable restrictions.

Base case: $\mu(A)=1$. Then the lemma is obvious, because there is no ι-term in A.

Suppose the lemma is proved for $\mu(A)<N$ (with $N\geq 2$), we shall prove it when $\mu(A)=N$. There are a number of cases.

1. A is of the form $P(t_1,\dots,t_n)$, with $n\geq 1$. Use (P ext) to reduce the problem to proving that for given derivations $\mathcal{D}_3$, $\mathcal{D}_4$ of $\Gamma\vdash t:I$, there is a derivation $\mathcal{D}$ of $\Gamma\vdash t=t$ such that $\mathcal{D}^*$ is

$$\dfrac{\begin{array}{cc}\mathcal{D}_3 & \mathcal{D}_4\\ t:I & t:I\end{array}}{t=t:\mathsf{prop.}}\,{=}F$$

If t is a variable, or a constant, it is trivial. If t is $f(t_1,\dots,t_n)$, with $n\geq 1$, we reduce the problem to deriving $t_i=t_i$ in an appropriate way. Because

$$\mu(t_i=t_i)=\mu(t_i)+1\leq\mu(t)<N\,,$$

it follows by the induction hypothesis that $t_i=t_i\supset t_i=t_i$ can be derived in an appropriate way, so that the following derivation solves the problem:

$$\dfrac{\dfrac{\forall x(x=x)\quad \begin{array}{c}\vdots\\ t_i:I\end{array}}{t_i=t_i}\,\forall E \qquad \begin{array}{c}\vdots\\ t_i=t_i\supset t_i=t_i\end{array}}{t_i=t_i.}\,\supset E$$

It remains the case when t is of the form $\iota xB(x)$ (the form $\varepsilon xB(x)$ is excluded by assumption). Then $\mathcal{D}_3$ and $\mathcal{D}_4$ end by the rule ιF. Let $\mathcal{D}_3'$ and $\mathcal{D}_4'$ be the derivations of $B(t)$ obtained by replacing the final rule ιF by the rule ι in $\mathcal{D}_3$ and $\mathcal{D}_4$, respectively. Since there is a derivation of $\forall x\forall y(B(x)\ \&\ B(y)\supset x=y)$ in $\mathcal{D}_3$ (and $\mathcal{D}_4$), it suffices to derive $B(t)\ \&\ B(t)$ in a way which makes it possible to apply $\supset E$. This can be done as follows (the careful reader will notice that this derivation is not reduced, simply because the argument will be easier).

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_3'\\ B(t)\end{array}\quad \dfrac{\dfrac{\begin{array}{c}[x:I]\\ \vdots\\ B(x)\supset B(x)\end{array}}{\forall x(B(x)\supset B(x))}\,\forall I \quad \begin{array}{c}\mathcal{D}_3\\ t:I\end{array}}{B(t)\supset B(t)}\,\forall E}{B(t)}\,\supset E \qquad \dfrac{\begin{array}{c}\mathcal{D}_4'\\ B(t)\end{array}\quad \dfrac{\dfrac{\begin{array}{c}[x:I]\\ \vdots\\ B(x)\supset B(x)\end{array}}{\forall x(B(x)\supset B(x))}\,\forall I \quad \begin{array}{c}\mathcal{D}_4\\ t:I\end{array}}{B(t)\supset B(t)}\,\forall E}{B(t)}\,\supset E}{B(t)\ \&\ B(t).}\,\&I$$

It is straightforward to check that both applications of $\supset E$ are allowed if the dotted parts are filled in appropriately, which is possible to do according to the induction hypothesis, because $\mu(B(x))<\mu(t)<N$.

2. A is of the form $B\ \&\ C$. The derivation then looks as follows:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ B\ \&\ C:\mathsf{prop}\end{array}\qquad \dfrac{\dfrac{\dfrac{[B\ \&\ C]}{B}\quad \begin{array}{c}\vdots\\ B\supset B\end{array}}{B}\qquad \dfrac{\dfrac{[B\ \&\ C]}{C}\quad \begin{array}{c}\vdots\\ C\supset C\end{array}}{C}}{B\ \&\ C}}{B\ \&\ C\supset B\ \&\ C}$$

where the derivations of $B \supset B$ and $C \supset C$ exist by the induction hypothesis. The derivation of $C \supset C$ may depend on assumptions B, but these can be derived, in appropriate ways, from $B \,\&\, C$ (which is currently open and hence available) and $B \supset B$. It is easy to verify that $\mathcal{D}^*$ has the stated form.

3. A is of the form $B \vee C$. This case is easy and left out.

4. A is of the form $B \supset C$. This case is similar to &.

5. A is of the form $\bot$. Trivial. (And, in fact, included in the base case.)

6. A is of the form $\forall x B(x)$. Easy.

7. A is of the form $\exists x B(x)$. Easy.

□

This lemma is enough for proving the first two parts of the theorem.

Proof of Theorem 7, parts 1–2. For part 1, use the transformation

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ A\end{matrix}\quad\begin{matrix}\mathcal{D}_2\\ A\supset B\end{matrix}}{B} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ A\end{matrix}\quad\begin{matrix}\vdots\\ A\supset A\end{matrix}}{A}{\scriptstyle \supset E}\qquad \begin{matrix}\mathcal{D}_2\\ A\supset B\end{matrix}}{B}{\scriptstyle \supset E}$$

and the lemma to conclude that the dots can be filled in.

For part 2, use the transformation

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ A(x):\mathsf{prop}\end{matrix}\quad\begin{matrix}\mathcal{D}_2\\ t:I\end{matrix}\quad\begin{matrix}\mathcal{D}_3\\ A(t)\end{matrix}}{\exists x A(x)} \quad\rightsquigarrow\quad \dfrac{\begin{matrix}\mathcal{D}_1\\ A(x):\mathsf{prop}\end{matrix}\quad\begin{matrix}\mathcal{D}_2\\ t:I\end{matrix}\quad\dfrac{\begin{matrix}\mathcal{D}_3\\ A(t)\end{matrix}\quad\begin{matrix}\vdots\\ A(t)\supset A(t)\end{matrix}}{A(t)}{\scriptstyle \supset E}}{\exists x A(x).}{\scriptstyle \exists I}$$

□

In the following, we will therefore freely use modus ponens as a derived rule. In particular, to prove the third part of the theorem, we need only prove that $A(t) \supset A(s)$ is derivable from Γ if $A(t)$ and $t = s$ are derivable from Γ. To do that, we need a couple of more lemmas.

Lemma 9. *Assume $A(t)$ does not contain any ε-term, nor does s, and that $\Gamma \vdash A(t) : \mathsf{prop}, A(s) : \mathsf{prop}, t = s$. Then $\Gamma \vdash A(t) \supset A(s)$.*

Proof. By induction on μ, as defined in the proof of the previous lemma. We assume that the lemma is true whenever $\mu(A(t) \supset A(s)) < N$, and prove it in the case $\mu(A(t) \supset A(s)) = N$.

If $A(t)$ is atomic, (P ext) reduces the problem to deriving $t_i(t) = t_i(s)$. If t_i is a variable or a constant, it is trivial. If t_i is of the form $f(\ldots)$, (f ext) reduces the problem to the arguments of f. Finally, if $t_i(t)$ is of the form $\imath x B(t, x)$ we need to derive $\imath x B(t, x) = \imath x B(s, x)$.

Because $\Gamma \vdash \imath x B(s,x) : I$, we have $\Gamma \vdash \forall x \forall y (B(s,x) \mathbin{\&} B(s,y) \supset x = y)$, so it suffices to derive $\Gamma \vdash B(s, \imath x B(t,x)) \mathbin{\&} B(s, \imath x B(s,x))$. The second conjunct is easy to derive: just change the last rule in the derivation of $\Gamma \vdash \imath x B(s,x) : I$. In order to derive $\Gamma \vdash B(s, \imath x B(t,x))$, use that $\mu(B(t,x) \supset B(s,x)) < \mu(t_i(t)) + \mu(t_i(s)) < \mu(A(t)) + \mu(A(s)) < \mu(A(t) \supset A(s))$, so the induction hypothesis gives $x : I, \Gamma \vdash B(t,x) \supset B(s,x)$. Now substitute $\imath x B(t,x)$ for x in this derivation.

We now have to treat the cases when $A(t)$ is a composite formula. These cases are similar to each other. We exemplify with conjunction.

$$
\begin{prooftree}
\AxiomC{$\mathcal{D}_1$}
\noLine
\UnaryInfC{$B(s) \mathbin{\&} C(s) : \mathsf{prop}$}
\AxiomC{$[B(s) \mathbin{\&} C(s)]$}
\UnaryInfC{$B(s)$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$B(s) \supset B(t)$}
\BinaryInfC{$B(t)$}
\AxiomC{$[B(s) \mathbin{\&} C(s)]$}
\UnaryInfC{$C(s)$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$C(s) \supset C(t)$}
\BinaryInfC{$C(t)$}
\BinaryInfC{$B(t) \mathbin{\&} C(t)$}
\BinaryInfC{$B(s) \mathbin{\&} C(s) \supset B(t) \mathbin{\&} C(t)$}
\end{prooftree}
$$

The derivations of $B(s) \supset B(t)$ and $C(s) \supset C(t)$ exist by the induction hypothesis. In the latter case, the derivation may depend on assumptions $B(s)$ and $B(t)$, but these can be derived from $B(s) \mathbin{\&} C(s)$ (which is currently open and hence available) and $B(s) \supset B(t)$. $\square$

The next lemma shows that the assumption $\Gamma \vdash A(s) : \mathsf{prop}$ can be removed from the previous lemma.

Lemma 10. *Assume $A(t)$ does not contain any ε-term, nor does s. If $\Gamma \vdash A(t) : \mathsf{prop}\, t = s$, then $\Gamma \vdash A(s) : \mathsf{prop}$.*

Proof. Induction on μ again. We assume this time that the lemma is true for $\mu(A(t)) < N$ and prove it in the case $\mu(A(t)) = N$.

If $A(t)$ is atomic, say $P(t_1(t), \ldots, t_n(t))$, then PF reduces the problem to deriving $t_i(s) : I$. If t_i is a variable or constant, this is trivial. If t_i is $f(\ldots)$, it reduces, by fF, to the arguments of f. If $t_i(t)$ is $\imath x B(t,x)$, then $\Gamma,\, x : I \vdash B(t,x) : \mathsf{prop}$ and hence, by the induction hypothesis, $\Gamma,\, x : I \vdash B(s,x) : \mathsf{prop}$. Hence, by the previous lemma, $\Gamma,\, x : I \vdash B(t,x) \supset B(s,x)$, so, since $\Gamma \vdash \exists x B(t,x)$, we conclude $\Gamma \vdash \exists x B(s,x)$. A similar argument applies to the uniqueness premise of the $\imath F$-rule. Hence $\Gamma \vdash \imath x B(s,x) : I$.

If $A(t)$ is composite, it is an easy exercise to prove the lemma using the induction hypothesis. The most complicated case is conjunction (or implication, which requires precisely the same argument). If $A(t)$ is $B(t) \mathbin{\&} C(t)$, we have $\Gamma \vdash B(t) : \mathsf{prop}$ and $\Gamma, B(t) \vdash C(t) : \mathsf{prop}$. By induction hypothesis, $\Gamma \vdash B(s) : \mathsf{prop}$ and $\Gamma, B(t) \vdash C(s) : \mathsf{prop}$. Since $\Gamma \vdash B(s) : \mathsf{prop}$, we have $\Gamma \vdash B(s) \supset B(t)$ (the previous lemma), hence $\Gamma, B(s) \vdash C(s) : \mathsf{prop}$. Hence $\Gamma \vdash B(s) \mathbin{\&} C(s) : \mathsf{prop}$. $\square$

Proof of Theorem 7, part 3. First notice that it is sufficient to prove the theorem when s is a fresh variable z, because if this has been done, we know that

$$\Gamma,\, z : I,\, t = z \vdash A(z)\,,$$

so we can, by replacing all z by s, and each assumption $z : I$ by a derivation of $s : I$ (which exists because $\Gamma \vdash t = s$), and each assumption $t = z$ by a derivation of $t = s$, derive $A(s)$ from Γ. We therefore assume in the following that s is a variable.

Now the previous lemma gives $\Gamma \vdash A(s) : \mathsf{prop}$. Hence, by Lemma 9, $\Gamma \vdash A(t) \supset A(s)$. Using that modus ponens is derivable, we conclude $\Gamma \vdash A(s)$. $\square$

9 Restricted Quantifiers

The translation interprets propositions $U(x)$, where x is a free variable, as propositional functions U. Taking the subsets-as-propositional-functions attitude towards subsets in type theory [6, 11, 3], and writing $x \mathrel{\varepsilon} U$ for $U(x)$, we are justified in saying that propositions depending on variables are interpreted as subsets, and restricted quantifiers are interpreted as follows:

$$\begin{aligned}\forall x(U(x) \supset A(x)) &\rightsquigarrow (\forall x : I)(\forall p : x \mathrel{\varepsilon} U)A_p(x)\\ \exists x(U(x) \mathbin{\&} A(x)) &\rightsquigarrow (\exists x : I)(\exists p : x \mathrel{\varepsilon} U)A_p(x)\,,\end{aligned}$$

where $A_p(x)$ may depend on p. Using the notation in [3], we can write

$$\begin{aligned}\forall x(U(x) \supset A(x)) &\rightsquigarrow (\forall x_p \mathrel{\varepsilon} U)A_p(x)\\ \exists x(U(x) \mathbin{\&} A(x)) &\rightsquigarrow (\exists x_p \mathrel{\varepsilon} U)A_p(x)\,.\end{aligned}$$

Thus, restricted quantifiers are indeed interpreted as restricted quantifiers in the sense of [3]. The notions of restricted quantifiers in [6, 11] are however in general too restrictive, because they do not allow $A_p(x)$ to depend on p. For example, the field property, derived on page 7, is interpreted as follows:

$$\forall x\big(U(x) \supset x \cdot x^{-1} = 1\big) \rightsquigarrow (\forall x_p \mathrel{\varepsilon} U)(x \cdot x_p^{-1} = 1)\,,$$

where x_p^{-1} is the left projection of p [3]. In this case, it is obvious that A_p depends on p in a non-trivial way.

10 Partial Functions

A common use of definite descriptions is for defining (partial) functions. Let $R(x, y)$ be a binary partial functional relation, i.e., $x : I$, $y : I$, $\Gamma \vdash R(x, y) : \mathsf{prop}$ and $\Gamma \vdash \forall x \forall y \forall z(R(x, y) \mathbin{\&} R(x, z) \supset y = z)$. Let '$x \mathrel{\varepsilon} D$' denote the interpretation in type theory of $\exists y R(x, y)$. Then

$$\dfrac{\exists y R(x,y) \qquad \dfrac{\begin{array}{c}\vdots\\ \forall x\forall y\forall z(R(x,y) \mathbin{\&} R(x,z) \supset y = z) \qquad x : I\end{array}}{\forall y\forall z(R(x,y) \mathbin{\&} R(x,z) \supset y = z)}\,\forall E}{\imath y R(x,y) : I}\,\imath$$

is interpreted as

$$\pi_\ell(p) : I \;\; (\ldots,\, x : I,\, p : x \mathrel{\varepsilon} D)\,,$$

hence as a partial function from D to $I/{=_I}$ in the sense of [3]. It is not in general a partial function in the sense of [11], because $(\forall p, q : x \mathrel{\varepsilon} D)\,\mathsf{Id}(I, \pi_\ell(p), \pi_\ell(q))$ is not in general true.

Suppose, on the other hand, that we in type theory have a partial function on I, that is, an extensional domain of definition $D \subseteq I$

$$x \;\varepsilon\; D : \mathsf{prop} \; (x : I)$$

and a partial function, which we in the notation of [3] can write

$$f(x_p) : I \; (x_p \;\varepsilon\; D)\,,$$

and which satisfies $(\forall x_p \;\varepsilon\; D, q : x \;\varepsilon\; D)(f(x_p) =_I f(x_q))$ as was required in [3]. Consider our ℩-calculus with a unary primitive predicate symbol $D(x)$, whose interpretation is supposed to be $x \;\varepsilon\; D$, and a binary primitive relation symbol $R(x, y)$, whose interpretation is supposed to be $(\exists p : x \;\varepsilon\; D)(f(x_p) =_I y)$. Then the following axioms are justified, in the sense that their interpretations can be proved in type theory:

$$\begin{array}{l}\forall x(D(x) \supset \exists y R(x, y)) \\ \forall x \forall y \forall z(R(x, y) \;\&\; R(x, z) \supset y = z)\,.\end{array}$$

Also the required extensionality properties for D and R can be proved. Now, introduce $f(x)$ as a notation for $\text{℩} y R(x, y)$. We can then derive the following:

$$\dfrac{\dfrac{D(x) \qquad \dfrac{\forall x(D(x) \supset \exists y R(x, y)) \quad x : I}{D(x) \supset \exists y R(x, y)}\,\forall E}{\exists y R(x, y)}\,{\supset} E \qquad \begin{array}{c}\vdots \\ \forall y \forall z(R(x, y) \;\&\; R(x, z) \supset y = z)\end{array}}{f(x) : I.}\,\text{℩} F$$

If the axiom $\forall x(D(x) \supset \exists y R(x, y))$ is interpreted as

$$\lambda x \lambda p(f(x_p), (p, \mathrm{refl}(f(x_p)))) : (\forall x_p \;\varepsilon\; D)(\exists y : I)(\exists q : x \;\varepsilon\; D)(f(x_p) =_I y)\,,$$

where refl is the reflexivity proof of $=_I$, then $f(x) : I$, as derived above, will be interpreted as the type-theoretical $f(x_p) : I$, after reduction. This shows that the interpretation indeed captures the intended meaning of f, up to definitional equality.

11 Summary and Conclusion

We have given calculi of indefinite and definite descriptions and interpreted them into intensional type theory. The calculus of indefinite descriptions suffers from unusual restrictions on modus ponens and existence introduction, while the unrestricted versions of these rules can be derived for the calculus of definite descriptions. In fact, we had in the end just *one* calculus, in which both indefinite and definite descriptions could coexist, the unrestricted versions of the rules being derivable for formulas without ε-terms.

In the translation, we used strong disjunction elimination in an essential way (p. 15), and it is therefore unclear if type theories without this rule would work. Moreover, the type-theoretical axiom of choice was implicitly used it the interpretation of descriptions, in the form of left projections of existence proofs (p. 14).

It seems that the interpretation follows closely what constructive mathematicians have in mind when they speak about descriptions. One conclusion is therefore that strong disjunction elimination and the intensional ('type-theoretical') axiom of choice seem to be important for a natural formalization of constructive mathematics with descriptions.

A conclusion from sections 9–10 is that restricted quantifiers and partial functions in the sense of first order logic with descriptions correspond to restricted quantifiers and partial functions in the sense of [3], rather than in the sense of [6, 11].

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A Appendix

In the logical inference rules, we could have, among the premises, some premises that express that all formulas involved are indeed propositions. Conjunction introduction, for instance, would then look as follows:

$$\frac{A : \mathsf{prop} \quad \begin{matrix}[A] \\ \vdots \\ B : \mathsf{prop}\end{matrix} \quad A \quad B}{A \,\&\, B}$$

and disjunction elimination as follows:

$$\frac{A : \mathsf{prop} \quad B : \mathsf{prop} \quad \begin{matrix}[A \vee B] \\ \vdots \\ C : \mathsf{prop}\end{matrix} \quad A \vee B \quad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C.}$$

This was the form my ε-calculus had originally, but Theorem 1 showed that it was conservative to simplify it to the present one.

Although the more verbose ε-calculus is in a sense more natural, it is very heavy to produce derivations in it, and, contrary to what one might first think, the translation into type theory is more difficult. The reason is that every rule except $\bot E$ has the undesirable property that the same proposition occurs several times among the premises. Since the same propositions could be interpreted in different ways in different places, we would need to introduce restrictions on every rule except $\bot E$ in order to get the interpretation through. Alternatively, we could perform the translation by just throwing all extra premises away and proceed as in this paper, but it is then unclear what would be gained by having them in the first place.

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