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Introduction

Convexity is an important and fundamental concept and it can be viewed in many different ways. One example how convexity naturally appears in complex analysis is that the domain of convergence for a power series is (logarithmically) convex. There is also the Paley-Wiener theorem which shows a clear connection between a certain class of entire functions and compact convex sets in $\mathbf{R}^n$. All convex domains are domains of holomorphy, but the converse is not true. Instead there exists the weaker concept pseudoconvexity, which can be shown to be equivalent with being a domain of holomorphy. Some, but not all theorems which hold for convex domains, remain valid for pseudoconvex ones. Convexity concepts that are weaker than ordinary convexity but stronger than pseudoconvexity have been studied, for example by Martineau [8] (lineally convex sets) in his work with the Fantappiè transform, and by Benkhe and Peschl [2] (Planarconvexität). For a thorough account of the field of convexity, see [3].

The property of convexity requires that the intersection of the set and real lines should be contractible. An obvious weakening would be to require this only for complex lines. This property lies in between of convexity and pseudoconvexity and one calls it $\mathbb{C}$-convexity. In one complex variable this property is not so interesting, and some theorems about $\mathbb{C}$-convexity apply only in two or more dimensions. As we will shall discuss further in the paper there are theorems that work for $\mathbb{C}$-convex domains but not for general pseudoconvex ones. To have a good working definition of a $\mathbb{C}$-convex *function*, we need it to bee C^2. There is a possible non-smooth geometric definition which we will mention later, but it seems hard to use. In the case of convexity there is an obvious non smooth definition: we just require all chords to lie over the graph. A similar non-smooth definition for $\mathbb{C}$-convex function which is easy to test, is desirable. There is a close relation between convex sets and convex functions. We will see that this is the case also for $\mathbb{C}$-convexity. An important function that can be connected to a set with nonempty boundary is δ, the boundary distance function. Booth the properties of δ and $-\log\delta$ will be important in this paper. Open convex sets are unions of open strictly convex sets. To get a similar theorem for $\mathbb{C}$-convex sets is the main aim of this paper. As the proofs involve approximations of $\mathbb{C}$-convex functions, the results are about rather smooth $\mathbb{C}$-convex sets. We also include some applications concerning the Kobayashi and Carathéodory metrics.

Basic Facts

Given a twice continuously differentiable real valued function u and we write $h = e^{-u}$. Then the definitions of u being a convex (1), $\mathbb{C}$-convex (3) and plurisubharmonic (5) function become that for all non zero $w \in \mathbb{C}^n$ the properties

$$\sum u_{j\bar{k}} w_j \bar{w}_k \geq \left|\sum u_{jk} w_j w_k\right| \tag{1}$$
$$\Leftrightarrow$$
$$\frac{\left|\sum h_j w_j\right|^2}{h} \geq \sum h_{j\bar{k}} w_j \bar{w}_k + \left|\frac{\sum (h_j h_k - h h_{jk}) w_j w_k}{h}\right| \tag{2}$$

$$\sum u_{j\bar{k}} w_j \bar{w}_k \geq \left|\sum (u_{jk} - u_j u_k) w_j w_k\right| \tag{3}$$
$$\Leftrightarrow$$
$$\frac{\left|\sum h_j w_j\right|^2}{h} \geq \sum h_{j\bar{k}} w_j \bar{w}_k + \left|\sum h_{jk} w_j w_k\right| \tag{4}$$

$$\sum u_{j\bar{k}} w_j \bar{w}_k \geq 0 \tag{5}$$
$$\Leftrightarrow$$
$$\frac{\left|\sum h_j w_j\right|^2}{h} \geq \sum h_{j\bar{k}} w_j \bar{w}_k \tag{6}$$

should hold. As we will see further there is a good reason to look at $h = e^{-u}$. If $\geq$ is changed to a $>$ the we put in a "strictly" in front of the property. There is natural inclusion of these properties for sets: if a set is convex, then it is $\mathbb{C}$-convex and then it is also pseudoconvex. With functions the situation is not so simple. If a function is convex or $\mathbb{C}$-convex then it is plurisubharmonic (we will further on show the connection between pseudoconvex sets and plurisubharmonic functions), but there are functions which are convex but not $\mathbb{C}$-convex. This can be shown by looking at

$$u(z_1, z_2) = k(z_1\bar{z}_1 + z_2\bar{z}_2)$$

for rather large values of the constant k. This has to do with the fact that condition 3 is non linear, which the other conditions on u are. But since 1 and 5 *are* linear, it is easy to use distribution theory to make definitions even in the non differentiable case. This is not the case for $\mathbb{C}$-convexity, since there is no general method of taking the product of two distributions (in our case $u_j u_k$). This makes it hard to define a continuous $\mathbb{C}$-convex function. Using the forthcoming Propositions 1 and 2 it is easy to realize that there are functions that are $\mathbb{C}$-convex but not convex. They reduce the problem to find a domain with C^2-boundary which is $\mathbb{C}$-convex but not convex. For example let $t \in [0, 1]$, then the set

$$E_t = \{(z, w) : |w|^2 < 1 - |z|^2 - t(Re)(z^2)^2\}$$

is real analytic and $\mathbb{C}$-convex for all t, but convex only for $t \in [0, 3/4]$. This and more examples of this situation are given in [1].

The following three propositions show the similarities between convexity, $\mathbb{C}$-convexity and pseudoconvexity.

Proposition 1 *Let $D = \{z : \rho(z) < 0\}$ be a domain in $\mathbb{C}^n$ where ρ is a C^2 defining function whose gradient is different from zero on ∂D. Then the following are equivalent:*

i) D is convex.

ii) The Hessian of ρ is positive semidefinite when restricted to real tangent space at any point on ∂D.

iii) The function $u = -\log \delta^2$ is convex near the boundary.

Proof. First we do $i) \iff iii)$. Assume that D is convex, then at each point $x \in \partial D$ there exist a real tangent plane T_x which does not intersect D. We have

$$\delta(z) \leq \delta_{T_x}(z) = |z - T_x|$$
$$\delta(z) = \inf_{x \in \partial D} \delta_{T_x}(z).$$

The boundary function δ_{T_x} is affine and linear. The infimum of such functions is concave. Hence δ is concave. Since log is increasing, $\log \delta$ is concave, and therefore $-\log \delta^2 = -2 \log \delta$ is convex. Now to the converse. Assume that there exist a non constant convex function $f : D \to \mathbf{R}$ such that

$$D_t = \{x \in D : f(x) \leq t\}$$

is compact for all $t < m = \sup_{x \in D} f(x)$. Now take $x_1, x_2, x_3 \in D$ and assume $[x_1, x_2]$, $[x_2, x_3] \subset D$. Let $X = \{x \in [x_1, x_2] : [x, x_3] \subset D\}$ and let t_0 be the supremum of f on the union of the line segments $[x_1, x_2]$, $[x_2, x_3]$. Note that t_0 is smaller than the supremum of f on D. Indeed, since D is open and f is convex and non constant, it cannot attain its supremum at any point of D. Otherwise it would be constant on every line going through that point and since D is assumed to be connected f would be constant on the whole of D. Now since D is open, X is open relatively $[x_1, x_2]$. But if $x \in [x_1, x_2]$ then

$$[x, x_3] \subset D \Rightarrow [x, x_3] \subset D_{t_0}$$

since t_0 was chosen to be the supremum of f on $[x_1, x_2]$, $[x_2, x_3]$ and f attains its maximum value on line segments at one of the endpoints. But D_{t_0} is by assumption compact whence X is also closed relatively $[x_1, x_2]$. Of course $x_2 \in X$, so X is not empty which implies $X = [x_1, x_2]$ and D is convex. Now assume that D is such that $-\log \delta^2$ is convex. Then $f = -\log \delta^2 + |x|^2$ is also convex.

Since its level sets stays away from the boundary, D_t is closed. The $|x|^2$ term makes D_t bounded.

Now to i) $\iff$ ii). Take a $x \in \partial D$, we must show that $T_x \cap D = \emptyset$ precisely if the Hessian H_ρ, defined in equation 7, is non negative along any direction w in T_x. Since he defining function ρ is defined for values slightly bigger than zero we can Taylor expand it, which after some calculation yields

$$\rho(x+w) = H_p(x,w) + R_x(w)$$

where $R_x(w)/|w|^2 \to 0$ when $|w| \to 0$. This shows that the condition on the Hessian is necessary. Now assume that $H_\rho \geq 0$. Then it is possible to find (not so easy) $\rho_n \searrow \rho$ with $H_{\rho_n} > 0$ which will define a domain D_n. Now D_n will be convex and increasing and since D is the union of those, it will be convex. ■

Proposition 2 *Let $D = \{z : \rho(z) < 0\}$ be a domain in $\mathbb{C}^n, n > 1$ where ρ is a C^2 defining function whose gradient is non zero on ∂D. Then the following are equivalent:*

i) D is $\mathbb{C}$-convex.

ii) The Hessian of ρ is positive semidefinite when restricted to complex tangent space at any point on ∂D.

iii) The function $u = -\log \delta^2$ is $\mathbb{C}$-convex near the boundary.

Proof. This is Theorem 2.5.16 in [1]. ■

Proposition 3 *Let $D = \{z : \rho(z) < 0\}$ be a domain in $\mathbb{C}^n, n > 1$ where ρ is a C^2 defining function whose gradient is different from zero on ∂D. Then the following are equivalent:*

i) D is a domain of holomorphy.

ii) The Levi form of ρ is positive semidefinite when restricted to complex tangent space at any point on ∂D.

iii) The function $u = -\log \delta^2$ is plurisubharmonic near the boundary.

Proof. These are Theorems 4.1.19, 4.1.27 and corollary 4.2.8 in [3]. ■

These propositions show that $u = -\log \delta^2$ is an important function. To study the boundary distance function we also look at the properties of $\delta^2 = h = e^{-u}$. The definition of a strictly convex C^2 domain, is that the Hessian of any defining function restricted to the real tangent plane is positive definite as supposed to positive semidefinite. In analogy with this, we say that a domain is strictly $\mathbb{C}$-convex if the Hessian

$$H_\rho(z,w) = \sum \rho_{j\bar{k}}(z) w_j \bar{w}_k - \mathrm{Re}\left(\sum \rho_{jk}(z) w_j w_k\right) \tag{7}$$

of any defining C^2-function ρ is positive definite when restricted to complex tangent plane at any point z on the boundary. Since this is a geometric requirement, it is clearly independent of the choice of ρ. There is an other possible definition of a $\mathbb{C}$-convex function that could be interesting. Let $u : D \to \mathbf{R}$ be a C^2-function, where D is a one dimensional complex disc. If $h = e^{-u}$, then the Hartogs domain

$$H_D^u = \{(z, z') \in D \times \mathbb{C} : |z'| < h(z)\}$$

is $\mathbb{C}$-convex if and only if u is $\mathbb{C}$-convex according to our previous definitions. For a proof see [1]. Therefore one could define a function $u : \mathbb{C} \to \mathbf{R}$ to be $\mathbb{C}$-convex in a neighborhood z if there exists an $\epsilon > 0$ such that H_D^u is $\mathbb{C}$-convex for all discs D centered at z with radius smaller than ϵ. If u depends of several variables we require to have this property for all complex lines passing trough the given point. This would give a definition that is equivalent with the old one in the C^2 case. However it is not easy to test whether a non-smooth Hartogs domain is $\mathbb{C}$-convex or not. It is also not known whether $i) \iff ii)$ in Proposition 2 stays valid if the C^2-condition is removed and the extended definition of $\mathbb{C}$-convex function is used. Proposition 2 is crucial for this paper.

Results

Here we will find a method of approximating $\mathbb{C}$-convex functions with strictly $\mathbb{C}$-convex functions. This will allow us to approximate $\mathbb{C}$-convex sets with strictly $\mathbb{C}$-convex sets. We begin with:

Lemma 4 *Let Ω be a bounded set and $u : \Omega \to \mathbf{R}$ is a bounded C^2 function. Then the sequence*

$$\{u^\epsilon : u^\epsilon = -\log\left(e^{-u} - \epsilon(1 + |z|^2)\right), 0 < \epsilon < \inf \frac{e^{-u(z)}}{1 + |z|^2}\}$$

decreases uniformly to u on Ω and is strictly $\mathbb{C}$-convex whenever u is $\mathbb{C}$-convex.

Proof. The inequality involving ϵ guaranties that that u^ϵ is well defined. Since u is $\mathbb{C}$-convex we put $h = e^{-u}$, use expression 4 and write

$$\frac{\left|\sum h_j w_j\right|^2}{h} \geq \sum h_{j\bar{k}} w_j \bar{w}_k + \left|\sum h_{jk} w_j w_k\right|$$

Now define $g^\epsilon(z) = \epsilon(1+|z|^2)$ and put $h^\epsilon = h - g^\epsilon$. This leads to

$$\sum h_{j\bar{k}} w_j \bar{w}_k - \sum h^\epsilon_{j\bar{k}} w_j \bar{w}_k = \epsilon \sum \frac{\partial^2}{\partial z_j \partial \bar{z}_k}(1 + \langle z, \bar{z}\rangle) w_j \bar{w}_k = \epsilon |w|^2$$

$$\sum h_{jk} w_j w_k - \sum h^\epsilon_{jk} w_j w_k = \epsilon \sum \frac{\partial^2}{\partial z_j \partial z_k}(1 + \langle z, \bar{z}\rangle) w_j w_k = 0$$

$$\begin{aligned} \frac{|\sum h_j w_j|^2}{h} &= \frac{\mathrm{Re}\,(\sum h^\epsilon_j w_j + \sum g^\epsilon_j w_j)^2 + \mathrm{Im}\,(\sum h^\epsilon_j w_j + \sum g^\epsilon_j w_j)^2}{h^\epsilon + g^\epsilon} \\ &\le \frac{\mathrm{Re}\,(\sum h^\epsilon_j w_j)^2 + \mathrm{Im}\,(\sum h^\epsilon_j w_j)^2}{h^\epsilon} + \frac{\mathrm{Re}\,(\sum g^\epsilon_j w_j)^2 + \mathrm{Im}\,(\sum g^\epsilon_j w_j)^2}{g^\epsilon} \\ &= \frac{|\sum h^\epsilon_j w_j|^2}{h^\epsilon} + \frac{|\sum g^\epsilon_j w_j|^2}{g^\epsilon} = \frac{|\sum h^\epsilon_j w_j|^2}{h^\epsilon} + \frac{\epsilon \langle \bar{z}, w\rangle \langle z, \bar{w}\rangle}{1+|z|^2} \end{aligned}$$

where we in the inequality used that the function $f(t_0, t_1, t_2) = (t_1^2 + t_2^2)/t_0$ is subadditive in the half plane $t_0 > 0$. Now we get

$$\frac{\left|\sum h^\epsilon_j w_j\right|^2}{h^\epsilon} - \epsilon \left(|w|^2 - \frac{\langle \bar{z}, w\rangle \langle z, \bar{w}\rangle}{1+|z|^2} \right) \ge \sum h^\epsilon_{j\bar{k}} w_j \bar{w}_k + \left| \sum h^\epsilon_{jk} w_j w_k \right|.$$

Using the Cauchy–Schwarz inequality we see that $|w|^2(1+|z|^2) > \langle \bar{z}, w\rangle\langle z, \bar{w}\rangle$ for all $w \neq 0$. Therefore $u^\epsilon = -\log h^\epsilon$ is strictly $\mathbb{C}$-convex. Since Ω is bounded g^ϵ goes uniformly to zero and $u^\epsilon \searrow u$ uniformly. ∎

This lemma is essential when we want to prove:

Theorem 5 *Let D be a bounded $\mathbb{C}$-convex domain with C^2 boundary. Then there exists a strictly increasing sequence $\{D_n\}$ of strictly $\mathbb{C}$-convex C^∞ domains whose union is equal to D.*

Proof. Take a compact subset K. It will bee enough to find a set with the required properties that contains K. If K is large we can, according to Proposition 2 and with notations as before, assume that the complement of K is contained in N_D. The set where $u = -\log \delta^2$ is $\mathbb{C}$-convex. Let $\Delta^\epsilon = \delta^2 - \epsilon(1+|z|^2)$ and $u^\epsilon = -\log \Delta^\epsilon$. This function is defined on the set $\{z : \delta(z) > \epsilon(1+|z|^2)\}$. Let

$$D^{\epsilon'}_{u^\epsilon} = \{z : u^\epsilon < 1/\epsilon'\} = \{z : \delta^2 > e^{-1/\epsilon'} + \epsilon(1+|z|^2)\} \subset D^\epsilon = \{z : \delta^2 > \epsilon\}.$$

Note that u will be bounded on $D^{\epsilon'}_{u^\epsilon}$. By lemma 4 u^ϵ is strictly $\mathbb{C}$-convex on $N_D \cap D^{\epsilon'}_{u^\epsilon}$. It is easy to realize that u^ϵ will be strictly $\mathbb{C}$-convex also on the closure of this set. We will use this function to create the domain. But we want it to be C^∞, so we will convolve it with a symmetric modifier. Therefore take a non negative $\phi \in C^\infty_0(\mathbb{C}^n)$ with total mass one and let $v^\epsilon = u^\epsilon * \phi$ be a function from $D^{\epsilon'}_{u^\epsilon}$. The convergence of

$$v^\epsilon \to u^\epsilon$$

in $C^2(D^\epsilon_{u^\epsilon})$-norm when $\mathrm{supp}(\phi)$ tends to zero will be uniform. Therefore, for some ϕ whose support is small enough, v^ϵ will be strictly $\mathbb{C}$-convex on $N_D \cap D^{\epsilon'}_{u_\epsilon}$.

Let $D^{\epsilon'}_{v^\epsilon}$ be defined in the obvious way then

$$K \Subset D^{\epsilon'}_{v^\epsilon} \Subset D^{\epsilon'}_{u^\epsilon}.$$

for some good parameters ϵ and ϵ'. That K can be made a subset of the other two is obvious. The other inclusion can be seen from using that outside K, u^ϵ is strictly plurisubharmonic, since it is strictly $\mathbb{C}$-convex. And since v^ϵ is the convolution of a u^ϵ with the symmetric function ϕ, we will have $u^\epsilon < v^\epsilon$ outside K. We conclude that the inclusions above hold for a fix ϵ, ϕ and for all ϵ' in some interval. Now we want to apply Proposition 2 and show that the open *set* $D^{\epsilon'}_{v^\epsilon}$ is strictly $\mathbb{C}$-convex. One way is to show that:

i) it is a domain, i.e. it is connected.

ii) the gradient of the defining function v^ϵ is non zero at the boundary.

iii) the Hessian of the defining function v^ϵ restricted to the complex tangent plane is positive definite.

For i) we a priori do not know whether $D^{\epsilon'}_{v^\epsilon}$ is connected or not. For the result it is not even important since we can take the component that contain K. But we will show that it is connected if K is large. Since the boundary of D is C^2 we can use lemma 2.1.29 in [3]. Thus we know that for all points z near the boundary of D there exists a unique point z' on ∂D that is the closest one. Moreover, z lies on the inward pointing normal $n_{z'}$ at this nearest point. Hence ∂D^{ϵ_0} and therefore also D^{ϵ_0} will be connected for all ϵ_0 small enough. This is because we have a continuous function from ∂D to ∂D^{ϵ_0}; namely the function which sends a point the distance ϵ_0 along the inward pointing normal. Take $z \in \partial D^{\epsilon_0}$ and the $z' \in \partial D$ its nearest boundary point. We claim that

$$\#\{\zeta : \zeta \in \partial D^{\epsilon'}_{u^\epsilon}, \zeta \in [z, z']\} = 1.$$

Indeed, let ζ be such a point. We can then calculate

$$\frac{\partial \Delta^\epsilon}{\partial n_{z'}}(\zeta) = 2\sqrt{e^{-1/\epsilon'} + \epsilon(1 + |\zeta|^2)} - \frac{\partial \epsilon(1 + |z|^2)}{\partial n_{z'}}(\zeta) > 0.$$

If D^{ϵ_0} is large (and we choose K so large that it contain D^{ϵ_0}) there exist a small a such that

$$\frac{\partial u^\epsilon}{\partial n_{z'}}(\zeta) = \frac{\partial - \log \Delta^\epsilon}{\partial n_{z'}}(\zeta) < a < 0$$

for all ζ outside D^{ϵ_0}. But v^ϵ is the convolution of the C^2 function u^ϵ with ϕ, and if the support of ϕ is small, the partial derivative $\partial_{n_{z'}} v^\epsilon(\zeta) < 0$. Since v^ϵ is the defining function we will have

$$\#\{\zeta : \zeta \in \partial D^{\epsilon'}_{v^\epsilon}, \zeta \in [z, z']\} = 1$$

and $D^{\epsilon'}_{v^\epsilon}$ is connected. Now to *ii*). According to Sard's theorem, the level sets $v^\epsilon < c$ will be non singular C^∞ for almost all c in the range of v^ϵ. So we can choose ϵ' so that the gradient is non zero on the boundary. For *iii*) take $z \in \partial D^{\epsilon'}_{v^\epsilon}$ and w in the complex tangent plane at z. Since v^ϵ is strictly $\mathbb{C}$-convex we can write

$$\sum v^\epsilon_{j\bar{k}}(z)\, w_j \bar{w}_k > |\sum (v^\epsilon_{jk}(z) - v^\epsilon_j(z) v^\epsilon_k(z))\, w_j w_k| = |\sum v^\epsilon_{jk}(z)\, w_j w_k|$$

and conclude that $D^{\epsilon'}_{v^\epsilon}$ is strictly $\mathbb{C}$-convex. ■

Applications

We will now use these facts to generalize some discoveries made by L. Lempert. It concerns the relationship between the Carathéodory metric and the Kobayashi metric. Let $D \subset C^n$ be a domain, then the Carathéodory metric is defined by

$$c_D(z, w) = \sup\{\delta_h(F(z), F(w)) : F \in \mathrm{Hol}(D, U)\}$$

where U denotes the open unit disk in $\mathbb{C}$ and δ_h is the Poincaré hyperbolic metric in U. We can also define another, similar function

$$k_D(z, w) = \inf\{\delta_h(\zeta, \omega) : f \in \mathrm{Hol}(U, D), f(\zeta) = z, f(\omega) = w\}.$$

The problem is that k_D is not always a metric since it for some D doesn't satisfy the triangle inequality. This can be solved by introducing the so called Kobayashi metric

$$k'_D(z, w) = \inf\{k_D(z, a_1) + k_D(a_1, a_2) + \cdots + k_D(a_n, w) : a_i \in D, n \in \mathbf{N}\}.$$

Now take $f \in \mathrm{Hol}(U, D)$ and $F \in \mathrm{Hol}(D, U)$ and let

$$\begin{aligned} c_F(z, w) &= \delta_k(F(z), F(w)) \\ k_f(z, w) &= \inf\{\delta_h(\zeta, \omega) : f(\zeta) = z, f(\omega) = w\} \end{aligned}$$

But $g = F \circ f \in \mathrm{Hol}(U, U)$ which means that $c_F(z, w) = \delta_h(g(\zeta), g(\omega))$. But the Schwarz lemma [6] says that $\delta_h(g(\zeta), g(\omega)) \leq \delta_h(\zeta, \omega)$ for every $g \in \mathrm{Hol}(U, U)$. Therefore we have

$$c_D(z, w) \leq k'_D(z, w) \leq k_D(z, w)$$

for all domains D. Now we come to the result from Lempert which says that if D is a strictly $\mathbb{C}$-convex domain with C^∞ boundary, then $c_D(z, w) = k'_D(z, w) = k_D(z, w)$. This implies that k_D really is a metric. This, that is Theorem 1 in [7], does not hold for general pseudoconvex lines. To be able to extend this results we need the following lemma:

Lemma 6 *If $D_1 \subset D_2 \subset \cdots \subset D$ are domains with $\cup D_n = D$, then we have*

$$\begin{aligned} c_{D_n} &\searrow c_D \\ k_{D_n} &\searrow k_D. \end{aligned}$$

Proof. Starting with k_D. We know that $k_D \leq k_{D_n}$ since k_D is an infimum of a larger set. Therefore take a $f \in \mathrm{Hol}(U, D)$ which is almost realizes the infimum (for some choice of z an w). Now for all $0 < r < 1$, define $f_r(\zeta) = f(r\zeta)$ which is a holomorphic function whose closure lies in D. Hence there exists an N such that $f_r \in \mathrm{Hol}(U, D_n)$ for all $n \geq N$. By choosing r close enough to one we conclude that k_D must be the limit of k_{D_n}

Now to c_D. We know that $c_D \leq c_{D_n}$ because c_D is a supremum of a smaller set. Now take a $F_n \in \mathrm{Hol}(D_n, U)$ which is closer than $1/n$ to the supremum (for some choice of z and w). Now take open compactly contained subset D' of D, with $z, w \in D'$. We can assume that $D' \subset D_1$. So we have a sequence of holomorphic functions F_n from D' to the unit disc, which obviously makes the sequence uniformly bounded. Therefore we can use Vitali's theorem and conclude that there exists a subsequence F_{n_k} and a holomorphic function, which is the subsequence's limit function. Since this was valid for any open compactly contained subset D', we can extend the function by analytic continuation to the whole of D. Then we get a holomorphic function $\overline{F} \in \mathrm{Hol}(D, \overline{U})$ which is the limit function of F_{n_k}. But now we can take $0 < r < 1$ and define $F_r(z) = r\overline{F}(z) \in \mathrm{Hol}(D, U)$. By choosing r close enough to one we conclude that c_D must be the limit of c_{D_n}. ■

Combining Theorem 5 and lemma 6 we get

Theorem 7 *If a domain D is a union of increasing strictly $\mathbb{C}$-convex C^∞ domains (for example if $\log \delta^2$ is $\mathbb{C}$-convex near the C^2 boundary), then*

$$c_D = k'_D = k_D.$$

In the next section we will introduce a (possibly) bigger class of sets which can be exhausted by strictly $\mathbb{C}$-convex domains with smooth boundary. And for which Theorem 7 obviously is true.

If D is strictly $\mathbb{C}$-convex with C^∞ boundary, then for all $z, w \in D$, there will be a so called extremal function $f \in \mathrm{Hol}(U, D)$, realizing the infimum. One calls the image $f(U)$ an extremal disc in D. To every two points in D there will be a unique extremal disc passing trough the points. An extremal mapping f can be extended into a C^∞ mapping $\tilde{f}$ which will be a C^∞ embedding of $\bar{U}$ onto $\tilde{f}(\bar{U})$ with $\tilde{f}(\partial U) \subset \partial D$. Moreover, any two extremal discs will meet in at most one point and f will be extremal with respect to any two points in the extremal disc. These facts are immediate consequences of Theorems 2, 3 and 4 in [7].

We will now show almost this in the weaker case. Therefore let the domain D be the exhausted by $D_1 \subset D_2 \subset ... \subset D$ which are smooth and strictly $\mathbb{C}$-convex. Pick two points z and w which can be assumed to belong to D_1. Then for each n there will be an extremal function $f_n \in \mathrm{Hol}(U, D_n)$. Since the metric δ_h is invariant under projective automorphisms of U we can assume that $f_n(0) = z$ and $f_n(r_n) = w$ where r_n is a positive real number. Since f_n is injective we have $k_n(z, w) = \delta_h(0, r_n)$ (we are writing k for k_D and k_n for k_{D_n}). Since D is bounded, we can as in the proof lemma 6, use Vitali's theorem and conclude that there is a subsequence f_{n_k} converging to a function $f \in \mathrm{Hol}(U, \bar{D})$. To show that this is an extremal function for $z, w \in D$, we must first show that the image does not intersect the boundary of D. For this we need

Lemma 8 *If $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega$ are domains with $\cup \Omega_n = \Omega$ where Ω_n is pseudoconvex and has C^1 boundary, then there exists a negative plurisubharmonic function $\Phi : \Omega \to \mathbf{R}$ such that*

$$\Phi(z) \to 0$$

when $z \to \partial\Omega$.

Proof. The key is Proposition 1 in [5] which says that there for each Ω_n exists a negative plurisubharmonic C^∞ function Φ_n which approaches zero when z approaches the boundary. Define this Φ_n to be zero outside Ω_n. Then there exists a subsequence Φ_{1n} that converge uniformly on D_1. From this subsequence we can choose a new subsequence Φ_{2n} that converges uniformly on D_2 and so on. Now we look at the diagonal sequence Φ_{nn} which converges to a plurisubharmonic function Φ on Ω. It is obvious that Φ has the required boundary properties.■

It is easy to realize that the sets D and D_n fulfill the conditions in the lemma. Therefore there exists a negative plurisubharmonic function Φ on $\bar{D}$ which is zero on the boundary. But then $g = \Phi \circ f$ will be a non positive subharmonic function, since f is holomorphic. Thus we can use the maximum principle and conclude that since U is open, g will not attain zero, unless it is constantly zero. This cannot be the case because $g(\zeta) < 0$ since $f(\zeta) = z \in D$. Therefore we must have $f \in \mathrm{Hol}(U, D)$. But now we use lemma 6 and conclude that $k(z, w) = \delta_h(0, r)$ where r is the limit point of $\{r_n\}$. We have

$$\lim f_n(r) = f(r) \Rightarrow \lim f_n(r_n) = f(r)$$

since $f_n' \to f'$ and $|f'(r)| < \infty$. Therefore f is an extremal function and $f(U)$ can be called an extremal disc.

Now take two points $\zeta, \omega \in U$ with $f(\zeta) = z$ and $f(\omega) = w$. We intend to show that f is maximal with respect to these points. For this we look at the two sequences

$$\begin{aligned} f_n(\zeta) &= z_n \to z \\ f_n(\omega) &= w_n \to w \end{aligned}$$

and calculate

$$\begin{aligned} |k(z,w) - k_n(z_n, w_n)| &\le \\ |k(z,w) - k_n(z,w)| + |k_n(z,w) - k_n(z_n,w_n)| &\le \\ |k(z,w) - k_n(z,w)| + |\pm(k_n(z,z_n) + k_n(z,w) + k_n(w,w_n) - k(z,w))| &\le \\ |k(z,w) - k_n(z,w)| + |k_1(z,z_n) + k_1(w,w_n)| &\to 0. \end{aligned}$$

But now we have $k_n(z_n, w_n) = \delta_h(\zeta, \omega)$, but then we also have $k(z, w) = \delta_h(\zeta, \omega)$ and are done. Summarizing this we get

Theorem 9 *Let D be a domain which is a union of increasing strictly $\mathbb{C}$-convex C^∞ domains (for example if $\log \delta^2$ is $\mathbb{C}$-convex near the C^2 boundary). Then for all $z, w \in D$ there exists an extremal mapping with respect to the Kobayashi metric. Moreover, this mapping is extremal with respect to any two points in its extremal disc.*

The question about uniqueness is tricky. When D is just any $\mathbb{C}$-convex domain it is even false. This can be shown by considering $D = U \times U$ and look for extremal mappings with respect to $(0, 0)$ and $(p, 0)$ where p is real and positive.

Then $f(z) = (z, h(z))$ will be an extremal mapping for every holomorphic h with $h(U) \subset U$ and $h(0) = h(p) = 0$. But since $U \times U$ is convex we can exhaust it with smooth strictly convex sets. These sets will be $\mathbb{C}$-convex which shows that we cannot hope to have the uniqueness.

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