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An explicit inversion formula for the exponential Radon transform using data from 180°

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Abstract

We derive a direct inversion formula of convolution-backprojection type for the exponential Radon transform. Our formula requires only the values of the transform over an 180° range of angles. It is an explicit formula, except that it involves a holomorphic function for which an explicit expression has not been found. In practice, this function can be approximated by an easily computed polynomial of rather low degree.

1 Introduction.

Background. Let f be a smooth, compactly supported function in the plane $\mathbf{R}^2$ and let μ be a real number. The set of all oriented lines in the plane can be identified with the product space $S^1 \times \mathbf{R}$ by associating the pair $(\theta, s) \in S^1 \times \mathbf{R}$ with the line $\{x \in \mathbf{R}^2; x \cdot \theta = s\}$. A parametrization of this line is then given by the mapping $t \mapsto s\theta + t\theta^\perp$, where $\theta^\perp$ is obtained by rotating θ counterclockwise through a right angle.

The exponential Radon transform of f is defined to be the function $R_\mu f$ on $S^1 \times \mathbf{R}$ given by the integral

$$R_\mu f(\theta, s) = \int_{-\infty}^{\infty} f(s\theta + t\theta^\perp) e^{\mu t}\, dt. \tag{1}$$

The exponential Radon transform arises in the medical imaging techique known as Single Photon Emission Computed Tomography or SPECT. The objective is to determine the distribution of a radioactive substance in an organ, by measuring the intensity of radiation emanating from the body in various directions. Consider a plane cross section of the organ being investigated, and suppose that the function f represents the concentration of radioactive material in this plane. Assume a detector sensitive to gamma photons, together with a collimator, are set up to register photons travelling along some line in this plane, represented by a pair (θ, s). The signal measured by the detector will then be a measure of the amount of radioactive substance along that line. Since some of the photons are absorbed before they reach the detector, radioactivity at a greater distance from the detector will contribute less to the signal. Assuming that all tissue absorbs photons with the same probability, and ignoring the possibility that photons may be scattered, the signal measured by the detector is proportional to $R_\mu f(\theta, s)$ where the attenuation μ is a material constant of the tissue, related to the probability that photons will be absorbed.

By using an array of detectors, which are rotated around the body, it is therefore possible to obtain a sampling of the Radon transform $R_\mu f$, from which f must then be reconstructed. For practical reasons, it is often possible to measure $R_\mu f(\theta, s)$ only for θ on half of the circle S^1.

The problem. Our problem is to find an explicit formula for reconstructing f, given the values of $R_\mu f$ (for a known μ) on the set $S^1_+ \times \mathbf{R}$, where $S^1_+ = \{\theta \in S^1; \theta_1 \geq 0\}$ is the right half of the unit circle. We also assume that we are given a compact set D which is known to contain the support of f.

Results on related problems. Several results on similar problems have previously been obtained. When the values of $R_\mu f$ are known on all of $S^1 \times \mathbf{R}$, the problem is solved by the famous Tretiak-Metz fromula given in [6]. An iterative algorithm, but not an explicit formula, for the problem at hand was recently obtained by Noo and Wagner [4]. A generalization of the Tretiak-Metz formula, which handles non-constant attenuation has been discovered by Novikov [5], see also [3].

2 Notation and definitions.

The following is a summary of various notations and definitions used in this note.

The exponential Radon transform of a function f, defined by (1) above, is denoted $R_\mu f$. The dual of the Radon transform is an operator R_μ^* mapping distributions on $S^1 \times \mathbf{R}$ to distributions on $\mathbf{R}^2$. It is defined by the relation

$$\langle R_\mu^* u, f\rangle = \langle u, R_\mu f\rangle \tag{2}$$

where $\langle \ , \ \rangle$ denotes the pairing of distributions and test functions, either on $\mathbf{R}^2$ or on $S^1 \times \mathbf{R}$. If g is a locally integrable (or smooth) function on $S^1 \times \mathbf{R}$, then $R_\mu^* g$ is actually a locally integrable (or smooth) function given by the formula

$$R_\mu^* g(x) = \int_{S^1} g(\theta, x\cdot\theta) e^{\mu x\cdot\theta^\perp}\, d\theta \tag{3}$$

where $d\theta$ denotes linear Lebesgue measure on the circle S^1.

If u and v are compactly supported smooth functions on $S^1 \times \mathbf{R}$, we write $u *_s v$ for the convolution of u and v in the second variable:

$$(u *_s v)(\theta, s) = \int_{-\infty}^{\infty} u(\theta, t) v(\theta, s-t)\, dt. \tag{4}$$

This definition is extended by continuity to the case where u is a distribution; $u *_s v$ is then also a distribution in general.

For convenience, we will use the notation

$$\mathrm{ch}_\mu(t) = \frac{\cosh(\mu t)}{t}. \tag{5}$$

We will use ch_μ both as a meromorphic function, and as a distribution on $\mathbf{R}$. In the latter case, the singularity should be taken as a principal value.

We define a linear mapping U from compactly supported distributions on $\mathbf{R}$ to distributions on $S^1 \times \mathbf{R}$ as follows. If ρ is a distribution on $\mathbf{R}$ and g is a smooth function on $S^1 \times \mathbf{R}$, we let

$$\langle U(\rho), g\rangle = \int_{S^1_+} \langle \rho, e^{-t\mu\theta_2} g_s'(\theta, t\theta_1)\rangle\, d\theta. \tag{6}$$

Here g_s' denotes the partial derivative of g with respect to the second variable, and for every θ, the distribution ρ acts on $e^{-t\mu\theta_2} g_s'(\theta, t\theta_1)$ as a function of t. We use $S^1_+ = \{\theta \in S^1; \theta_1 \geq 0\}$ to denote the right half of the unit circle. Note that $U(\rho)$ is supported in $S^1_+ \times \mathbf{R}$.

Let ψ be a function, holomorphic in all of the complex plane except possibly on the real line. We define $B_+\psi$ and $B_-\psi$ to be the boundary values of ψ on the real line from above and from below:

$$B_\pm \psi(t) = \lim_{\epsilon \to +0} \psi(t \pm i\epsilon) \tag{7}$$

provided that the limits exist, and let $B_\Delta \psi = B_+\psi - B_-\psi$.

Finally, if φ and ψ are holomorphic outside a compact set in the complex plane, we define an entire function, which we denote $[\varphi,\psi]$, by the formula

$$[\varphi,\psi](z) = \frac{1}{2\pi i}\int_{|\zeta|=R} \varphi(\zeta)\psi(z-\zeta)\, d\zeta. \tag{8}$$

Here R is a positive number, depending on z, chosen so that the integrand is holomorphic for $|\zeta| \geq R$. Note that $[\varphi,\psi]$ depends bilinearly and antisymmetrically on φ and ψ.

3 Statement of results.

We begin by making the following simple observation.

Lemma 1. *Let f be a compactly supported smooth function on $\mathbf{R}^2$ and let u be a distribution on $S^1 \times \mathbf{R}$. Then*

$$R^*_{-\mu}(u *_s R_\mu f) = (R^*_{-\mu}u) * f. \tag{9}$$

This result is proved in [2, Theorem II 1.3] when $\mu = 0$ and the general case follows along the same lines.

Corollary 1. *If $\operatorname{supp} f \subset D$ and the restriction of $R^*_{-\mu}u$ to the set $D-D = \{x-y; x,y \in D\}$ equals δ_0, the Dirac measure at the origin, then f is the restriction to D of $R^*_{-\mu}(u *_s R_\mu f)$.*

Hence our problem is solved by a convolution-backprojection type formula if we can find a distribution u supported in $S^1_+ \times \mathbf{R}$ such that $R^*_{-\mu}u$ restricted to $D - D$ equals δ_0. The following result reduces the problem to finding a distribution on the real line.

Theorem 1. *Let ρ be a compactly supported distribution on $\mathbf{R}$ and let $u = U(\rho)$ where $U(\rho)$ is defined by (6). Then*

$$R^*_{-\mu}u(x) = 2(\rho * \mathrm{ch}_\mu)(x_1)\, \delta_0(x_2). \tag{10}$$

To complete the solution we must find a distribution ρ such that $\rho * \mathrm{ch}_\mu$ restricted to a given compact set equals $\delta_0/2$. The next result shows that such ρ exist and the proof indicates how a solution can be computed.

Theorem 2. *Let $0 < r < R$, let ψ be holomorphic in $\mathbf{C} \setminus [-R,R]$ and let $\alpha = B_\Delta \psi$. Suppose that ψ and α satisfy the following properties:*

- *$\psi(z)$ is sufficiently well behaved near $[-R,R]$, so that α is an integrable function and smooth in the open interval $(-r,r)$,*
- *$B_+\psi(t) = -B_-\psi(t)$ when $-r \leq t \leq r$,*
- *$\psi(z)$ is uniformly bounded for large $|z|$,*
- *$\alpha(0) \neq 0$.*

Then except possibly for some small set of exceptional μ and ψ, there exists an entire holomorphic function h such that

$$\rho(t) = \frac{h(t)\alpha(t)}{t} \tag{11}$$

satisfies $\rho * \mathrm{ch}_\mu = \delta_0/2$ in $[-r,r]$. Here the singularity of ρ is taken as a principal value.

Remarks. 1. The function ψ can be chosen, for example, as $\psi(z) = 1/\sqrt{a-z^2}$, with a branch cut along $[-\sqrt{a},\sqrt{a}]$ where $r^2 \leq a \leq R^2$, or as a linear combination of such functions for different values of a.

2. Numerical experiments suggest that the exceptional cases in Theorem 2 actually never occur. In any case, they can always be avoided by rescaling ψ slightly.

3. The proof of Theorem 2 indicates how a polynomial approximating h can be computed.

4 Proofs.

Proof of Theorem 1. We may assume that ρ is a smooth function since the general case then follows by passing to a limit. Write $\theta(\phi) = (\cos\phi, \sin\phi)$ and notice that $u = U(\rho) = \lim_{\epsilon\to 0} u_\epsilon$ where

$$\begin{aligned}
\langle u_\epsilon, g\rangle &= \int_{-\pi/2+\epsilon}^{\pi/2-\epsilon}\int_{-\infty}^{\infty} \rho(t) e^{-t\mu\sin\phi} g_s'(\theta(\phi), t\cos\phi)\,dt\,d\phi \\
&= \int_{-\pi/2+\epsilon}^{\pi/2-\epsilon}\int_{-\infty}^{\infty} \frac{1}{\cos\phi}\rho\left(\frac{s}{\cos\phi}\right) e^{-s\mu\tan\phi} g_s'(\theta(\phi), s)\,ds\,d\phi \\
&= -\int_{-\pi/2+\epsilon}^{\pi/2-\epsilon}\int_{-\infty}^{\infty} \frac{1}{\cos^2\phi}\left(-\mu\sin\phi\,\rho\left(\frac{s}{\cos\phi}\right) + \rho'\left(\frac{s}{\cos\phi}\right)\right) e^{-s\mu\tan\phi} g(\theta(\phi), s)\,ds\,d\phi.
\end{aligned} \tag{12}$$

Since the distribution u_ϵ is defined by the integrable function appearing in the last integral above, we may apply the formula (3) to compute $R^*_{-\mu}u_\epsilon$. This integral may be computed explicitly by making the change of variables $t = \tan\phi$:

$$\begin{aligned}
R^*_{-\mu}u_\epsilon(x) &= -\int_{-\pi/2+\epsilon}^{\pi/2-\epsilon} \frac{1}{\cos^2\phi}\left(-\mu\sin\phi\rho\left(\frac{x_1\cos\phi + x_2\sin\phi}{\cos\phi}\right) + \rho'\left(\frac{x_1\cos\phi + x_2\sin\phi}{\cos\phi}\right)\right) \\
&\qquad e^{-\mu(x_1\cos\phi + x_2\sin\phi)\tan\phi} e^{-\mu(-x_1\sin\phi + x_2\cos\phi)}\,d\phi \\
&= -\int_{-\pi/2+\epsilon}^{\pi/2-\epsilon} \frac{1}{\cos^2\phi}\left(-\mu\sin\phi\rho(x_1 + x_2\tan\phi) + \rho'(x_1 + x_2\tan\phi)\right) e^{-\mu x_2/\cos\phi}\,d\phi \\
&= -\int_{-T}^{T}\left(-\frac{\mu t}{\sqrt{1+t^2}}\rho(x_1 + x_2 t) + \rho'(x_1 + x_2 t)\right) e^{-\mu x_2\sqrt{1+t^2}}\,dt \\
&= -\left[\frac{1}{x_2} e^{-\mu x_2\sqrt{1+t^2}}\rho(x_1 + x_2 t)\right]_{t=-T}^{T} \\
&= \frac{1}{x_2} e^{-\mu x_2\sqrt{1+T^2}}\left(\rho(x_1 - x_2 T) - \rho(x_1 + x_2 T)\right).
\end{aligned} \tag{13}$$

Here $T = \tan(\pi/2-\epsilon)$. Since ρ has compact support, this expression converges to 0 uniformly on compact sets outside the line $x_2 = 0$, when $\epsilon \to 0$. This means that the limit $R^*_{-\mu}u$ is supported on the x_1-axis. Moreover, an easy estimate shows that $\langle R^*_{-\mu}u_\epsilon, f\rangle \to 0$ when $\epsilon \to 0$ if $f(x_1, 0) \equiv 0$, so $\langle R^*_{-\mu}u, f\rangle$ does not depend on derivatives of f in the x_2 direction. To determine $R^*_{-\mu}u$ it suffices therefore to compute the limit

$$\begin{aligned}
\lim_{\epsilon\to+0}\int_{-\infty}^{\infty} R^*_{-\mu}u_\epsilon(x)\,dx_2 &= \lim_{T\to+\infty}\int_{-\infty}^{\infty}\frac{1}{x_2}e^{-\mu x_2\sqrt{1+T^2}}\left(\rho(x_1 - x_2T) - \rho(x_1 + x_2T)\right)dx_2 \\
&= \lim_{T\to+\infty}\int_{-\infty}^{\infty}\frac{1}{v}e^{-\mu v\sqrt{1+T^2}/T}\left(\rho(x_1 - v) - \rho(x_1 + v)\right)dv \\
&= \int_{-\infty}^{\infty}\frac{1}{v}e^{-\mu v}\left(\rho(x_1 - v) - \rho(x_1 + v)\right)dv \\
&= 2\int_{\mathbf{R}}\frac{\cosh(\mu v)}{v}\rho(x_1 - v)\,dv \\
&= 2(\rho * \mathrm{ch}_\mu)(x_1).
\end{aligned} \tag{14}$$

This proves Theorem 1. ■

Proof of Theorem 2. If h is any entire function, ρ is defined by (11) and $0 < |t| < r$, then it follows from the definition (8) that

$$(\rho * \mathrm{ch}_\mu)(t) = -2\pi i \left[\mathrm{ch}_\mu, \frac{h\psi}{z}\right](t). \tag{15}$$

This can be seen by shrinking the contour of integration in the definition of the right hand side to an infinitessimal neighborhood of the interval $[-R, R]$. The condition $B_+\psi(t) = -B_-\psi(t)$ ensures that the singularities in ρ and ch_μ are both counted as principal values. Therefore the condition $(\rho * \mathrm{ch}_\mu)(t) = 0$ for $0 < |t| < r$ is equivalent to

$$\left[\mathrm{ch}_\mu, \frac{h\psi}{z}\right] = 0. \tag{16}$$

Write $h(z) = h_0 + zh_1(z)$. Then $\rho * \mathrm{ch}_\mu$ has a point mass at the origin of size $-\pi^2 h_0\alpha(0)$, so we must take $h_0 = -1/(2\pi^2\alpha(0))$. This works fine since $\alpha(0) \neq 0$.

Rewrite the condition (16) as

$$\left[\frac{1}{z}, h_1\psi\right] = -\left[\mathrm{ch}_\mu, \frac{h_0\psi}{z}\right] - \left[\frac{\cosh(\mu z) - 1}{z}, h_1\psi\right]. \tag{17}$$

Notice that if φ is holomorphic outside a compact set, then $[z^{-1}, \varphi]$ is the unique entire function such that $[z^{-1}, \varphi] - \varphi \to 0$ at infinity. From this it follows, by using the assumption that $\psi(z)$ is bounded for large z, that if φ_1 and φ_2 are entire functions, then the equality $\varphi_1 = [z^{-1}, \varphi_2/\psi]$ implies that $[z^{-1}, \varphi_1\psi] = \varphi_2$. Hence (17) will be satisfied if we can find h_1 such that

$$\begin{aligned} h_1 &= -\left[\frac{1}{z}, \frac{1}{\psi}\left[\mathrm{ch}_\mu, \frac{h_0\psi}{z}\right]\right] - \left[\frac{1}{z}, \frac{1}{\psi}\left[\frac{\cosh(\mu z) - 1}{z}, h_1\psi\right]\right] \\ &= F - \Phi(h_1). \end{aligned} \tag{18}$$

where F and $\Phi(h_1)$ are defined by the two expressions on the line above.

Let K be a compact set containing $[-R, R]$ in its interior and let $A(K)$ be the Banach space of functions continuous in K and holomorphic in the interior of K. Then Φ can be extended to an operator on all of K, since $(\cosh(\mu z) - 1)/z$ is an entire function, and this operator is compact by the Ascoli-Arzela theorem. Therefore the equation $h_1 = F - \Phi(h_1)$ has a solution $h_1 \in A(K)$ unless -1 is an eigenvalue of Φ. Since F and $\Phi(h_1)$ are both entire, it follows that h_1 is also.

Now if we let r vary and rescale the function ψ accordingly, then Φ, and hence its spectrum, depends analytically on r. When $r \to 0$, the norm of Φ converges to 0, so -1 is an eigenvalue of Φ for at most a discrete set of r, and the theorem is proved. In fact, numerical experiments seem to suggest that for reasonable choices of ψ, the eigenvalues of Φ are always positive, so that the exceptional set is empty. ■

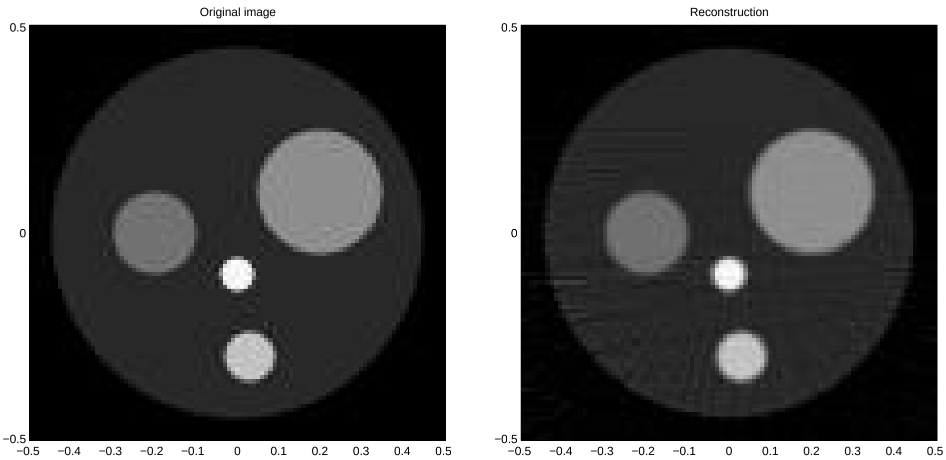
5 Numerical test.

To test our inversion formula numerically, it is necessary to choose a function ψ and compute a polynomial approximating the Taylor expansion of h. In the following test we have used

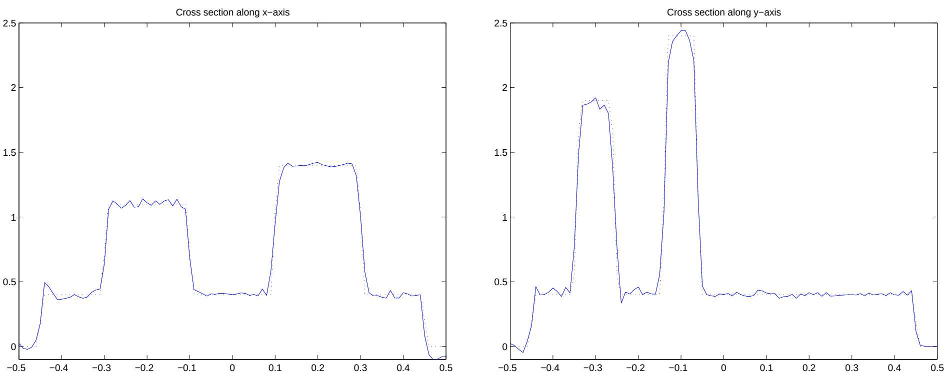
$$\psi(z) = \int_{r^2}^{R^2} \frac{w(a)}{\sqrt{a - z^2}}\, da \tag{19}$$

where $w(a) = (a - r^2)(R^2 - a)$. The reason for using such a function rather than $1/\sqrt{a - z^2}$ is to avoid the singularities at $\pm\sqrt{a}$. Next we truncate the Laurent series expansion of ψ to obtain a Laurent polynomial $\widetilde{\psi}$ approximating ψ in an annulus. Then we solve a system of linear equations to find a polynomial $\widetilde{h}_1$ of given degree such that the Taylor series of $\left[\mathrm{ch}_\mu, h_0\widetilde{\psi}/z + \widetilde{h}_1\widetilde{\psi}\right]$ vanishes to as high degree as possible. Note that since ψ is an even function, we can assume that $\widetilde{h}_1$ is odd.

The following reconstruction was made with the values $r = 1$, $R = 1.1$ and 10 nonzero terms in the polynomial $\widetilde{h}_1$. The test object consists of circular discs, and the Radon transform was sampled at 200 values of θ equally spaced over S^1_+, and 101 values of s equally spaced between -0.5 and 0.5. The width and height of the image are 1 and the attenuation $\mu = 3$.



Exact image and reconstruction obtained using the inversion formula.



Cross section of exact image (dotted) and reconstruction (solid) along the horizontal (left) and vertical (right) lines through the center of the image.

References

[1] P. Kuchment, I. Shneiberg: Some inversion formulae in the single photon emission computed tomography, *Appl. Anal.* **53** (1994), 221 – 231.

[2] F. Natterer: *The mathematics of computerized tomography*, Wiley, 1986.

[3] F. Natterer: Inversion of the attenuated Radon transform, *Inverse Problems* **17** (2001), 113 – 119.

[4] F. Noo, J. M. Wagner: Image reconstruction in 2D SPECT with 180° acquisition, *Inverse Problems* **17** (2001), 1357 – 1371.

[5] R. G. Novikov: Une formule d'inversion pour la transformation d'un rayonnement X atténué, *C. R. Acad. Sci. Paris Ser. I Math.* **332** (2001), 1059 – 1063.

[6] O. Tretiak, C. Metz: The exponential Radon transform, *SIAM J. Appl. Math.* **39** (1980), 341 – 354.