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THE RING OF ARITHMETICAL FUNCTIONS WITH UNITARY CONVOLUTION: GENERAL TRUNCATIONS.

JAN SNELLMAN

ABSTRACT. Let $(\mathcal{A}, +, \oplus)$ denote the ring of arithmetical functions with unitary convolution, and let $V \subseteq \mathbb{N}^+$ have the property that for every $v \in V$, all unitary divisors of v lie in V. If in addition V is finite, then $\mathcal{A}_V$ is an artinian monomial quotient of a polynomial ring in finitely many indeterminates, and isomorphic to the "Artinified" Stanley-Reisner ring $\mathbb{C}[\overline{\Delta(V)}]$ of a certain simplicial complex $\Delta(V)$. We describe some ring-theoretical and homological properties of $\mathcal{A}_V$.

1. INTRODUCTION

The ring of arithmetical functions with Dirichlet convolution

$$f * g(n) = \sum_{d|n} f(d)g(n/d)$$

is well-studied and well understood. From a ring-theoretical point of view, the most important fact is the theorem by Cashwell-Everett [5] which states that this ring is a unique factorization domain (it is isomorphic to the power series ring over $\mathbb{C}$ on countably many variables). It is also interesting that this ring can be given a natural norm, with respect to which it is a normed, valued ring [11, 10].

The ring of arithmetical functions with unitary convolution

$$f * g(n) = \sum_{\substack{d|n \\ \gcd(d,n/d)=1}} f(d)g(n/d)$$

is, by contrast, not even a domain. However, the same norm as before makes it into a normed (not valued) ring [14] and it is at least présimplifiable, atomic, and has bounded length on factorizations of a given element.

In [13] certain truncations of the ring of arithmetical functions with Dirichlet convolution was studied. These rings were defined as follows: fix a positive integer n, and consider all arithmetical functions supported on $[n] = \{1, 2, \ldots, n\}$. If we modify the multiplication slightly, this becomes a zero-dimensional algebra, which is the monomial quotient of a polynomial ring (on finitely many indeterminates). Furthermore, the defining ideals are *strongly stable*, so the homological properties of these truncated algebras are easily determined, using the Eliahou-Kervaire resolution [6]. A way of stating that these truncated algebras "approximate" the original algebra

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is to note that they form an inverse system (of discretely topologized Artinian $\mathbb{C}$-algebras) whose inverse limit is precisely the ring of arithmetical functions with Dirichlet convolution (and with the topology induced by the above mentioned norm).

Of course, the same truncation can be performed on the ring of arithmetical functions with unitary convolution. In this article, we'll be somewhat more general, and consider truncations to any subset $V \subseteq \mathbb{N}^+$ that is closed w.r.t taking unitary divisors. Of course, the case when V is finite is the most interesting one. Here, the truncated algebra is again a monomial quotient of a polynomial ring, but the defining ideals need not be strongly stable. They are, however, of the form $I + (x_1^2, \ldots, x_v^2)$ where I is square-free, so they can be regarded as *Artinified Stanley-Reisner rings* on certain simplicial complexes. As a matter of fact, we show (Theorem 5.16) that the truncations comprise *all* Artinified Stanley-Reisner rings.

In a forthcoming article, we'll study the particular truncations $V = [n]$, exactly as in [13]. The defining ideals are now multi-stable rather than stable, so the minimal free resolutions are more involved. However, the associated simplicial complexes, as defined above, are *shellable*, which makes it possible to give some information about the Poincaré-Betti series.

2. The ring of arithmetical functions with unitary convolution

We denote the i'th prime number by p_i, and the set of all prime numbers by $\mathbb{P}$. The set of prime powers is denoted by $\mathbb{PP}$. For a positive integer n, we put $[n] = \{1, \ldots, n\}$.

2.1. Unitary multiplication, unitary convolution, the norm of an arithmetical function.

The unitary multiplication for positive integers is defined by

$$d \oplus m = \begin{cases} dm & \text{if } \gcd(d,m) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

With this operation, $(\mathbb{N}^+, \oplus)$ becomes a monoid-with-zero (see Definition 2.2) . We write $d||n$ (or sometimes $d \leq_\oplus n$) when d is a unitary divisor of n, i.e. when $n = d \oplus m$ for some m. Then $(\mathbb{N}^+, \leq_\oplus)$ is a partially ordered set, and a well-order, but not a lattice, since for instance $\{2, 4\}$ have no upper bound.

The algebra $(\mathcal{A}, +, \oplus)$ of arithmetical functions with unitary convolution is defined as the set of all functions

$$\mathbb{N}^+ \to \mathbb{C},$$

with the structure of $\mathbb{C}$-vector space given by component-wise addition and multiplication of scalars, and with multiplication defined by the *unitary convolution*

$$(f \oplus g)(n) = \sum_{d||n} f(d)g(n/d) = \sum_{d \oplus m = n} f(d)g(m). \tag{2}$$

$(\mathcal{A}, +, \oplus)$ is a non-Noetherian, quasi-local ring, where the non-units (i.e. the elements in the unique maximal ideal) consists of those f with $f(1) = 0$.

It is natural to endow the algebra $\mathcal{A}$ with the non-archimedean norm

$$|f| = \frac{1}{\operatorname{ord}(f)}, \qquad \operatorname{ord}(f) = \min \operatorname{supp}(f), \tag{3}$$

where for $f \in \mathcal{A}$, $\operatorname{supp}(f) = \{\, n \in \mathbb{N}^+ | f(n) \neq 0 \,\}$. If we give $\mathbb{C}$ the trivial norm

$$|c| = \begin{cases} 1 & c \neq 0 \\ 0 & c = 0 \end{cases}$$

then $\mathcal{A}$ becomes a normed vector space over C, and $(\mathcal{A}, +, \oplus)$ becomes a normed (not valued) algebra [14]. In detail: by a *normed C-vector space* we mean a $\mathbb{C}$-vector space A equipped with a norm, such that for $f \in A$, $c \in \mathbb{C}$ we have that

$$\begin{aligned} |f + g| &\leq \max(|f|, |g|) \\ |cf| &= |c||f| \end{aligned} \tag{4}$$

A $\mathbb{C}$-algebra A is normed if its underlying vector space is normed, and if for $f, g \in A$,

$$|fg| \leq |f||g| \tag{5}$$

For $n \in \mathbb{N}^+$ we denote by e_n the function which is 1 on n and zero otherwise; we'll use the convention that $e_0 = 0$ denotes the zero function. The function e_1 is the multiplicative identity in $\mathcal{A}$.

Every f in $\mathcal{A}$ can be written uniquely as a convergent sum

$$f = \sum_{n \in \mathbb{N}^+} f(n) e_n, \tag{6}$$

thus the e_n's form a Schauder basis, and the set

$$\{\, e_{p^r} | r \in \mathbb{N},\, p \text{ prime } \}$$

is a minimal generating set in the sense that it generates a dense sub-algebra of $\mathcal{A}$, and no proper subset has this property.

Lemma 2.1. *For $a, b \in \mathbb{N}^+$,*

$$e_a \oplus e_b = e_{a \oplus b} = \begin{cases} e_{ab} & \textit{if } \gcd(a, b) = 1 \\ 0 & \textit{otherwise} \end{cases}$$

2.2. **The ring of arithmetical functions as a power series ring on a monoid-with-zero.**

Definition 2.2. A monoid-with-zero $(M, 0, 1, \cdot)$ is a set M, a zero element $0 \notin M$, a multiplicative unit $1 \in M$, and a commutative, associative operation

$$(M \cup \{0\}) \times (M \cup \{0\}) \to M \cup \{0\}$$

such that for all $m \in M$,

$$\begin{aligned} 0 \cdot m &= 0 \\ 1 \cdot m &= m \end{aligned} \tag{7}$$

It is said to be cancellative as a monoid-with-zero if

$$\forall a, b, c \in M : \; a \cdot b = a \cdot c \neq 0 \implies b = c. \tag{8}$$

A homomorphism

$$f : (M, 0, 1, \cdot) \to (M', 0', 1', \cdot') \tag{9}$$

of monoids-with-zero is a mapping $M \cup \{0\} \to M' \cup \{0'\}$ such that $f(0) = 0'$, $f(1) = 1'$ and $f(x \cdot y) = f(x) \cdot' f(y)$.

A subset $S \subseteq M$ is a sub-monoid-with-zero if

$$x, y \in S \implies x \cdot y \in S \cup \{0\},$$

and a monoid ideal-with-zero if

$$x \in S,\ y \in M \qquad \implies \qquad x \cdot y \in S \cup \{0\}.$$

Remark 2.3. Dropping the demand for a unit 1, we get a *semigroup-with-zero.*

Lemma 2.4. $(\mathbb{N}^+, \oplus)$ *is a monoid-with-zero which is cancellative as a monoid-with-zero.*

Proof. We let define $n \oplus 0 = 0$ for all $n \in \mathbb{N}^+$. Then (7) hold, with $1 \in \mathbb{N}^+$ as the multiplicative unit. If $\gcd(a, b) = \gcd(a, c) = 1$ and $ab = ac$ then $b = c$, so $(\mathbb{N}^+, \oplus)$ is cancellative. □

We recall some of the definitions made in [14]. Let

$$Y = \{\, y_{i,j} | 1 \le i, j < \infty \,\}$$

be an $\mathbb{N}^+ \times \mathbb{N}^+$-indexed set of indeterminates, let Y^* denote the free abelian monoid on Y, and let

$$Y^* \supseteq \mathcal{M} = \{1\} \cup \{\, y_{i_1,j_1} \cdots y_{i_r,j_r} | i_1 < \cdots < i_r \,\} \tag{10}$$

denote the subset of *separated monomials.* We regard $\mathcal{M}$ as a monoid-with-zero, the multiplication given by

$$a \oplus b = \begin{cases} ab & \text{if } ab \in \mathcal{M} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Note that $(\mathcal{M}, 0, 1, \oplus)$ is not a sub-monoid-with-zero of $(Y^*, 0, 1, \cdot))$, but it is an epimorphic image, under the map

$$Y \ni m \mapsto \begin{cases} m & \text{if } m \in \mathcal{M} \\ 0 & \text{if } m \not\in \mathcal{M} \end{cases}$$

Denote by Φ the map between $\mathcal{M}$ and $\mathbb{N}^+$ determined by

$$\Phi\left(\prod_{i=1}^{r} y_{a_i,b_i}\right) = \prod_{i=1}^{r} p_{a_i}^{b_i} \tag{12}$$

Lemma 2.5. Φ *induces an isomorphism of monoids-with-zero.*

Proof. We extend Φ by putting $\Phi(0) = 0$. If $a, b \in \mathcal{M}$ then ab is separated if and only if $\Phi(a)$ and $\Phi(b)$ are relatively prime. Hence, Φ is a homomorphism. It follows from the fundamental theorem of arithmetic (unique factorization of positive integers) that Φ is a bijection. □

The following definition is similar to the more general construction of Ribenboim [8, 9].

Definition 2.6. Let $(M, \cdot)$ be a commutative monoid (or let $(M, 0, 1, \cdot)$ be a monoid-with-zero) such that for any $m \in M$, the equation

$$x \cdot y = m$$

have only finitely many solutions $(x, y) \in M \times M$. Then the generalized power series ring on M with coefficients in $\mathbb{C}$, denoted $\mathbb{C}[[M]]$, is the set of all maps $f : M \to \mathbb{C}$, with the obvious structure as a C-vector space, and with multiplication given by the convolution

$$f * g(m) = \sum_{x \cdot y = m} f(x)g(y).$$

We give $\mathbb{C}[[M]]$ the topology of point-wise convergence, where $\mathbb{C}$ is discretely topologized, so if $(f_v)_{v=1}^{\infty}$ is a sequence in $\mathbb{C}[[M]]$ then

$$f_v \to f \in \mathbb{C}[[M]] \iff \forall m \in M : \exists V(m) : \forall v > V(m) : \ f_v(m) = f(m) \tag{13}$$

As an example, $\mathbb{C}[[Y^*]]$ is the "large power series ring" in the sense of Bourbaki [4] on the set of indeterminates Y.

The following easy result was proved in [14].

Theorem 2.7. *Let*

$$\mathfrak{D} = \{ f \in \mathbb{C}[[Y^*]] \big| \operatorname{supp}(f) \cap \mathcal{M} = \emptyset \} \tag{14}$$

Then $\mathfrak{D}$ is a closed ideal in $\mathbb{C}[[Y^*]]$, and the ideal minimally generated by the set

$$\left\{ y_{i,a} y_{i,b} \big| i, a, b \in \mathbb{N}^+, \ a \leq b \right\} \tag{15}$$

has $\mathfrak{D}$ as its closure. Furthermore,

$$\mathcal{A} \simeq \mathbb{C}[[(\mathbb{N}^+, \oplus)]] \simeq \mathbb{C}[[\mathcal{M}]] \simeq \frac{\mathbb{C}[[Y^*]]}{\mathfrak{D}} \tag{16}$$

The isomorphisms in (16) are homeomorphisms, so the norm-topology on $\mathcal{A}$ coincides with the topology of point-wise convergence.

Definition 2.8. We let $\mathbb{C}[M] \subseteq \mathbb{C}[[M]]$ denote the dense sub-algebra of finitely supported maps $M \to \mathbb{C}$.

We denote by $\mathcal{A}^f$ the sub-algebra of finitely supported maps $\mathbb{N}^+ \to \mathbb{C}$. It follows that

$$\mathcal{A}^f \simeq \mathbb{C}[(\mathbb{N}^+, \oplus)] \simeq \mathbb{C}[\mathcal{M}] \simeq \frac{\mathbb{C}[Y^*]}{(\{ y_{i,a} y_{i,b} \big| i, a, b \in \mathbb{N}^+, \ a \leq b \})} \tag{17}$$

3. The truncations $\mathcal{A}_V$ — definition and basic properties

In what follows, V will, unless otherwise stated, denote a subset of $\mathbb{N}^+$.

Definition 3.1. Define

$$\begin{aligned} \mathcal{M}_V &= \Phi^{-1}(V) \\ \mathcal{A}_V &= \{ f \in \mathcal{A} | \operatorname{supp}(f) \subseteq V \} \\ \mathcal{A}_V^f &= \left\{ f \in \mathcal{A}^f \middle| \operatorname{supp}(f) \subseteq V \right\} \end{aligned} \tag{18}$$

With component-wise addition, and the modified multiplication

$$(f \oplus_V g)(k) = \begin{cases} (f \oplus g)(k) & k \in V \\ 0 & k \notin V \end{cases} \tag{19}$$

$\mathcal{A}_V$ becomes a commutative algebra over $\mathbb{C}$. The modified multiplication

$$a \oplus_V b = \begin{cases} a \oplus b & a \oplus b \in V \\ 0 & a \oplus b \notin V \end{cases} \tag{20}$$

makes $(V, 0, \oplus_V)$ into a semigroup-with-zero. We regard $\mathcal{M}_V$ as a semigroup-with-zero isomorphic to $(V, 0, \oplus_V)$.

In what follows, we will often (by abuse of notation) use $\oplus$ for $\oplus_V$.

We define

$$a \leq_{\oplus_V} c \iff \exists c : a \oplus_V b = c \tag{21}$$

$\mathcal{A}_V$ and $\mathcal{A}_V^f$ are unital, and $\mathcal{M}_V$ is a monoid-with-zero, if and only if $1 \in V$.

Lemma 3.2. *$\mathcal{A}_V$ is a closed subspace and a sub vector space of $\mathcal{A}$. If V is finite, then $\mathcal{A}_V$ has the discrete topology, and $\mathcal{A}_V = \mathcal{A}_V^f$. If V is infinite, then $\mathcal{A}_V^f$ is dense in $\mathcal{A}_V$.*

Proof. Suppose that $f_v \to f$, $f_v \in \mathcal{A}_V$, $f \in \mathcal{A} \setminus \mathcal{A}_V$. Let

$$j = \min\left(\operatorname{supp}(f) \setminus V)\right).$$

Then $|f_v - f| \geq 1/j$, a contradiction. □

Lemma 3.3.

$$\begin{aligned} \mathcal{A}_V &\simeq \mathbb{C}[[(V, \oplus_V)]] \simeq \mathbb{C}[[\mathcal{M}_V]] \\ \mathcal{A}_V^f &\simeq \mathbb{C}[(V, \oplus_V)] \simeq \mathbb{C}[\mathcal{M}_V] \end{aligned} \tag{22}$$

where the isomorphisms are also homeomorphisms.

Definition 3.4. Let $\mathcal{U}$ denote the set of non-trivial sub-monoids of the monoid-with-zero $(\mathbb{N}^+, \oplus)$. Thus $W \in \mathcal{U}$ if and only if $1 \in W$ and $a, b \in W$ implies that $a \oplus b \in W \cup \{0\}$.

Lemma 3.5. *Let $1 \in W \subseteq \mathbb{N}^+$. Then $\mathcal{A}_W$ is a closed sub-algebra of $\mathcal{A}$ if and only if $W \in \mathcal{U}$.*

Proof. From Lemma 3.2 we have that $\mathcal{A}_W$ is a closed subspace of $\mathcal{A}$, and a vector subspace. So $\mathcal{A}_W$ is a closed sub-algebra if and only if $(\mathcal{A}_W, \oplus)$ is a sub-monoid of the multiplicative monoid-with-zero of $\mathcal{A}$.

Suppose first that $W \in \mathcal{U}$, and that $f, g \in \mathcal{A}_W$. To prove that

$$\operatorname{supp}(f \oplus g) \subseteq W,$$

take $n \notin W$. Since W is a sub-monoid of $(\mathbb{N}, \oplus)$), n can not be written as $n = a \oplus b$ with $a, b \in W$. Thus

$$(f \oplus g)(n) = \sum_{a \oplus b = n} f(a) g(b) = 0.$$

On the other hand, if $W \notin \mathcal{U}$, then there are $a, b \in W$ with $\gcd(a, b) = 1$ such that $a \oplus b \notin W$. It follows that

$$e_a \oplus e_b = e_{a \oplus b} \notin \mathcal{A}_W.$$

□

Definition 3.6. For $1 \in V \subseteq W \subseteq \mathbb{N}^+$ we define the *truncation map*

$$\rho_{W,V} : \mathcal{A}_W \to \mathcal{A}_V$$
$$\rho_{W,V}(f)(k) = \begin{cases} f(k) & k \in V \\ 0 & k \notin V \end{cases} \tag{23}$$

We also define

$$\rho_V : \mathcal{A} \to \mathcal{A}_V$$
$$\rho_V(f)(k) = \begin{cases} f(k) & k \in V \\ 0 & k \neq V \end{cases} \tag{24}$$

Lemma 3.7. *The truncation maps ρ_V and $\rho_{W,V}$ are surjective, $\mathbb{C}$-linear and continuous. We have that*

$$\ker \rho_V = \mathfrak{S}_V = \{\, f \in \mathcal{A} | f(k) = 0 \text{ for all } k \in V \,\} = \mathcal{A}_{V^c} \tag{25}$$

$\mathfrak{S}_V$ is a closed subset of $\mathcal{A}$.

Proof. It is obvious that the maps are surjective and C-linear. Let us prove continuity for $\rho_{W,V}$. If $f \in \mathcal{A}_W \setminus \{0\}$, then

$$|\rho_{W,V}(f)| = \begin{cases} 0 & \operatorname{supp}(f) \cap V = \emptyset \\ \frac{1}{\min\{\, k \in V | f(k) \neq 0 \,\}} & \text{otherwise} \end{cases}$$
$$\leq |f| = \frac{1}{\min\{\, k \in W | f(k) \neq 0 \,\}}$$

It follows that $\rho_{W,V}$ is continuous. The case of ρ_V is similar. Since $\mathcal{A}$ is Hausdorff, so is $\mathcal{A}_W$, and thus $\ker \rho_V = \rho_V^{-1}(\{0\})$ is the inverse image (under a continuous map) of a closed set, and consequently closed. □

Definition 3.8. Let $\mathcal{O}$ denote the collection of non-empty order ideals of $(\mathbb{N}^+, \geq_\oplus)$, and let $\mathcal{O}^f$ denote the collection of finite, non-empty order ideals of $(\mathbb{N}^+, \geq_\oplus)$. Thus $V \in \mathcal{O}$ if and only if

$$n \in V,\ d \in \mathbb{N}^+,\ d || n \implies d \in V. \tag{26}$$

Lemma 3.9. *If $V \in \mathcal{O}$ then $1 \in V$, and V^c is either empty or infinite.*

Proof. If $V = \mathbb{N}^+$ then both assertions hold. If $V \neq \mathbb{N}^+$ then there is some $n \in \mathbb{N}^+ \setminus V$. Hence 1, which is a unitary divisor of n, is in V. Since V is an order ideal, it follows that whenever $\gcd(n, m) = 1$, then $nm = n \oplus m \notin V$. Thus the complement of V is infinite. □

Theorem 3.10. *Let $1 \in V \subseteq \mathbb{N}^+$. The following are equivalent:*

(i) $V \in \mathcal{O}$,
(ii) *V^c is a monoid ideal,*
(iii) *$\mathfrak{S}_V$ is a closed ideal of $\mathcal{A}$, and $\mathcal{A}_V = \frac{\mathcal{A}}{\mathfrak{S}_V}$.*
(iv) *The truncation map ρ_V is an algebra homomorphism.*

Proof. Suppose that $V \in \mathcal{O}$. If $x \in V^c$, $y \in \mathbb{N}^+$ then either $x \oplus y = 0$ or $x \oplus y \in V^c$, because otherwise $x \oplus y \in V \implies x \in V$, a contradiction. So V^c is a monoid ideal.

Let $f, g \in \mathcal{A}$, $n \in V$. Then

$$\rho_V(f \oplus g)(n) = f \oplus h(n) = \sum_{d||n} f(d)h(n/d)$$

whereas

$$\left[\rho_V(f) \oplus_V \rho_V(h)\right](n) = \sum_{\substack{d||n \\ d \in V \\ n/d \in V}} f(d)h(n/d) = \sum_{d||n} f(d)h(n/d)$$

If $m \not\in V$ then

$$\rho_V(f \oplus g)(m) = \left[\rho_V(f) \oplus_V \rho_V(g)\right](m) = 0,$$

so we have shown that ρ_V is a homomorphism, hence the kernel $\mathfrak{S}_V$ is an ideal, and $\mathcal{A}_V = \frac{\mathcal{A}}{\mathfrak{S}_V}$.

If $V \not\in \mathcal{O}$ then there exist $n \in V$, $a, b \in \mathbb{N}^+$, $a \not\in V$, such that $a \oplus b = n$. Thus V^c is not a monoid ideal, and

$$\rho_V(e_a \oplus e_b) = \rho_V(e_n) = e_n \neq 0 = \rho_V(e_a) \oplus \rho_V(e_b).$$

So ρ_V is not an algebra homomorphism. Furthermore, $e_a \in \mathfrak{S}_V$, $e_b \in \mathcal{A}$, but

$$e_a \oplus e_b = e_n \not\in \mathfrak{S}_V,$$

so $\mathfrak{S}_V$ is not an ideal. □

Corollary 3.11. *Let* $1 \in V \subseteq \mathbb{N}^+$. *The following are equivalent:*

(i) $\mathcal{A}_V$ *is a continuous algebra retract of* $\mathcal{A}$.
(ii) $V \in \mathcal{O} \cap \mathcal{U}$.
(iii) V *is a sub-monoid of* $(\mathbb{N}^+, \oplus)$ *generated by a some subset of* $\mathbb{PP}$.

Proof. Since a continuous homomorphism from $\mathcal{A}$ to $\mathcal{A}_V$ is determined by its values on the Schauder basis $\{ e_n | n \in \mathbb{N}^+ \}$, $\mathcal{A}_V$ is an algebra retract of $\mathcal{A}$ if an only if it is a sub-algebra and the restriction map $\rho_V : \mathcal{A} \to \mathcal{A}_V$ is an algebra homomorphism. This yields the equivalence between (i) and (ii).

If V is a non-empty order ideal in $(\mathbb{N}^+, \geq_\oplus)$, then it contains along with any $n \in V$ all of its prime power unitary divisors, hence if it is also a monoid ideal in $(\mathbb{N}^+, \oplus)$, it must contain the sub-monoid generated by those prime powers. Conversely, a sub-monoid of $(\mathbb{N}^+, \oplus)$ generated by a some subset of $\mathbb{PP}$ is an order ideal. So (ii) and (iii) are equivalent. □

Definition 3.12. Let $\tilde{\mathbb{N}}$ denote the set of positive, square-free integers; in particular, $1 \in \tilde{\mathbb{N}}$.

Lemma 3.13. *Let* $n \in \mathbb{N}^+$. *Then the following hold:*

(i) $[n] \in \mathcal{O}^f$.
(ii) $\{ q \in \mathbb{N}^+ | q \,||\, n \} \in \mathcal{O}^f$.
(iii) $\tilde{\mathbb{N}} \in \mathcal{O} \cap \mathcal{U}$.
(iv) $[n] \cap \tilde{\mathbb{N}} \in \mathcal{O}^f$.
(v) $\left\{ q \in \tilde{\mathbb{N}} \middle| q \,||\, n \right\} \in \mathcal{O}^f$.

Proof. (i): If $w\,||n$ then $w \leq n$ so $w \in [n]$.

(ii): In any locally finite poset with a minimal element, the principal order ideal on an element is a finite order ideal.

(iii): The divisors of a square-free integer are square-free, so in particular the unitary divisors are square-free. If a, b are square-free then $a \oplus b$ is either square-free (if $\gcd(a, b) = 1$) or zero.

(iv), (v): Intersections of order ideals are order ideals. □

Lemma 3.14. *Let $1 \in V \subseteq W \subseteq \mathbb{N}^+$, $V \in \mathcal{O}$. Then $\rho_{W,V}$ is an algebra epimorphism.*

Proof. Let $f, g \in \mathcal{A}_W$. If $n \in W \setminus V$ then

$$0 = \rho_{W,V}(f \oplus_W g)(n) = \big(\rho_{W,V}(f) \oplus_V \rho_{W,V}(g)\big)(n) = 0,$$

whereas if $n \in V$ then

$$\begin{aligned}\rho_{W,V}(f \oplus_W g)(n) &= (f \oplus_W g)(n) \\ &= \sum_{\substack{d||n \\ d\in W}} f(d)g(n/d) \\ &= \sum_{\substack{d||n \\ d\in V}} f(d)g(n/d)\end{aligned}$$

where the last equality follows since V is an order ideal in W, and consequently any unitary divisor of n is in V, so $d, n/d \in V$. Using this observation, we have

$$\begin{aligned}\big(\rho_{W,V}(f) \oplus_V \rho_{W,V}(g)\big)(n) &= \sum_{\substack{d||n \\ d\in V}} \rho_{W,V}(f)(d)\rho_{W,V}(g)(n/d) \\ &= \sum_{\substack{d||n \\ d\in V}} f(d)g(n/d) \\ &= \rho_{W,V}(f \oplus_W g)(n)\end{aligned}$$

□

Corollary 3.15. *If $1 \in V \subseteq W \subseteq \mathbb{N}^+$, $V \in \mathcal{O}$, then $\mathcal{A}_V$ is a cyclic $\mathcal{A}_W$-module.*

The following theorem gives a motivation for our studies of the truncations $\mathcal{A}_V$: they approximate $\mathcal{A}$ in the natural sense.

Theorem 3.16. *Let $\mathcal{FIN}$ denote the set of finite subsets of $\mathbb{N}^+$.*

(A) The set of all $\mathcal{A}_V$, $V \in \mathcal{FIN}$, and truncation maps

$$\rho_{W,V} : \mathcal{A}_W \to \mathcal{A}_V, \qquad V, W \in \mathcal{FIN},\, V \subseteq W$$

forms an inverse system of normed vector spaces over $\mathbb{C}$, and

$$\varprojlim_{V\in\mathcal{FIN}} \mathcal{A}_V \simeq \mathcal{A} \tag{27}$$

as normed vector spaces over $\mathbb{C}$.

(B) $\mathcal{O}^f$ *is cofinal in* $\mathcal{FIN}$*, and*

$$\varprojlim_{V \in \mathcal{O}^f} \mathcal{A}_V \simeq \mathcal{A} \tag{28}$$

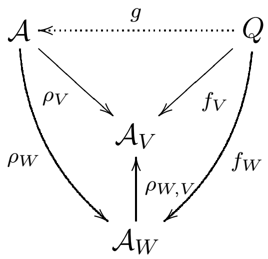
as normed vector spaces over $\mathbb{C}$.

(C) The set of all $\mathcal{A}_V$, $V \in \mathcal{O}^f$*, together with all truncation maps*

$$\rho_{W,V} : \mathcal{A}_W \to \mathcal{A}_V, \qquad V \subseteq W,\ V, W \in \mathcal{O}^f,$$

forms an inverse system of normed Artinian $\mathbb{C}$*-algebras, and* (28) *is an isomorphism of normed* $\mathbb{C}$*-algebras.*

Proof. (A) Let Q be a normed $\mathbb{C}$-vector space, and consider the diagram



(29)

where $W \supseteq V$ and f_W, f_V are given continuous homomorphisms. If the diagram without the dotted line commutes, the whole diagram commutes when we define

$$g(x)(k) = f_{V'}(x)(k), \qquad V' \ni k \tag{30}$$

Clearly, g is $\mathbb{C}$-linear; we claim that it is also continuous. To prove this, let $x_n \to 0$ in Q. Then $f_V(x_n) \to 0$ in every $\mathcal{A}_V$, that is, $|f_V(x_n)| \to 0$. Fix a $k \in \mathbb{N}^+$, and suppose that $k \in V$. Since $|f_V(x_n)| \to 0$, there is an N such that $|f_V(x_n)| < 1/k$ whenever $n > N$. Hence for $n > N$

$$|g(x_n)| = |f_V(x_n)| < 1/k.$$

(B): If $V \subseteq \mathbb{N}^+$ is finite, so is the set obtained by adding all unitary divisors of elements in V. This proves that $\mathcal{O}^f$ is cofinal in $\mathcal{FIN}$.

(C) Let Q be a normed algebra, and consider once again the diagram (29), where now the f_V's are continuous algebra homomorphisms. We must show that now g is also a ring homomorphism. So let $x, y \in Q$, fix $k \in \mathbb{N}^+$, and take $V \in \mathcal{O}^f$ with $k \in V$. Then

$$\begin{aligned} g(xy)(k) &= f_V(xy)(k) \\ &= \big(f_V(x) \oplus_V f_V(y)\big)(k) \\ &= \sum_{d||k} f_V(x)(d) f_V(y)(k/d) \\ &= \sum_{d||k} g(x)(d) g(y)(k/d) \\ &= \big(g(x) \oplus_V g(y)\big)(k), \end{aligned}$$

where we have used that f_V is a ring homomorphism, and that $k \in V \in \mathcal{O}^f$ implies that any unitary divisor of k lies in V.

Hence, $g(xy) = g(x) \oplus g(y)$. ∎

With minor modifications, the preceding proof works for the following generalization:

Theorem 3.17. *Let* $U \subseteq \mathbb{N}^+$*.*

$$\varprojlim_{\substack{V \subseteq U \\ V \in \mathcal{FIN}}} \mathcal{A}_V \simeq \mathcal{A}_U \tag{31}$$

as normed vector spaces over $\mathbb{C}$*, and*

$$\varprojlim_{\substack{V \in \mathcal{O}^f \\ V \subseteq U}} \mathcal{A}_V \simeq \mathcal{A}_U \tag{32}$$

as normed algebras over $\mathbb{C}$*.*

Lemma 3.18. *Let* $V \in \mathcal{O}$*.*

(i) The multiplicative unit of $\mathcal{A}_V$ *is* $\rho_V(e_1)$*.*
(ii) $\mathcal{A}_V$ *is quasi-local:* $f \in \mathcal{A}_V$ *is a unit if and only if* $f(1) \neq 0$*, and the non-units form the unique maximal ideal.*
(iii) If V *is finite, then all non-units are nilpotent.*
(iv) $\mathcal{A}_V$ *is Artinian if and only if* V *is finite.*
(v) $\mathcal{A}_V$ *is Noetherian if and only if* V *is finite.*

Proof. We'll prove (ii) and (v), the rest is trivial.

(ii): If $f(1) \neq 0$, then let f' denote any "lift" of f to $\mathcal{A}$, i.e. $\rho_V(f') = f$. Then there is a $g' \in \mathcal{A}$ such that $f'g' = e_1$, hence

$$\rho_V(f'g') = f\rho_V(g') = \rho_V(e_1).$$

Thus f is a unit, hence regular.

Conversely, it is easy to see that if $f(0) = 0$ then f is not a unit. If V is finite, let $n = \max V$, and let $g \in \mathcal{A}$ be such that $\rho_V(g) = f$. We claim that if $a, b \in \mathcal{A}$ are (non-zero) non-units, then

$$\operatorname{ord}(a \oplus b) \geq \operatorname{ord}(a)\operatorname{ord}(b) > \max\{\operatorname{ord}(a), \operatorname{ord}(b)\}.$$

Thus we conclude that $\operatorname{ord}(g^{n+1}) \geq n + 1$, which implies that

$$\rho_n(g^{n+1}) = f^{n+1} = 0.$$

To prove the claim, let $i = \operatorname{ord}(a)$, $j = \operatorname{ord}(b)$. Then for $k < ij$,

$$a \oplus b(k) = \sum_{\ell || k} a(\ell)b(k/\ell) = 0,$$

since it is impossible that $\ell \geq i$ and $k/\ell \geq j$.

(v): If V is finite, then $\mathcal{A}_V$ is Artinian, hence Noetherian. If V is infinite, then since $V \in \mathcal{O}$ we must have that $V \cap \mathbb{PP}$ is infinite, as well. The ideal generated by

$$\{\, e_q | q \in V \cap \mathbb{PP} \,\}$$

is not finitely generated. □

Theorem 3.19. *Let* $V \in \mathcal{O}$*. The set*

$$M_V = \{\, e_k | k \notin V, \text{ but } d \in V \text{ for all proper unitary divisors } d \text{ of } k \,\} \tag{33}$$

form a minimal generating set of an ideal I_V *whose closure is* $\mathfrak{S}_V$*.*

Proof. Recall that $\{ e_n | n \in \mathbb{N}^+ \}$ is a Schauder base for $\mathcal{A}$. Thus if

$$\mathcal{A} \ni f = \sum_{n=1}^{\infty} c_n e_n, \qquad f \in \mathfrak{S}_V$$

then

$$f = \sum_{n \not\in V} c_n e_n \tag{34}$$

It follows that

$$\{ e_n | n \not\in V \}$$

forms a generating set of an ideal I_V s.t. $I_V \subseteq \mathfrak{S}_V = \overline{I_V}$. Clearly, M_V is a minimal generating set of I_V. □

Lemma 3.20. *Let $V \in \mathcal{O}$, $V \neq \mathbb{N}^+$. Then I_V is not finitely generated, and $I_V \subsetneq \mathfrak{S}_V$.*

Proof. From the proof of Lemma 3.9 we have that there exists $n \in V^c$, $q_1, q_2, q_3, \dots \in \mathbb{P}$ such that $(n, q_i) = 1$ and $q_i, nq_i \in V^c$. We can furthermore assume that all unitary divisors of n belong to V. Then e_{nq_i} is a minimal generator of I_V. The sums of these e_{nq_i}'s is an element in $\mathfrak{S}_V$ but not (since the sum is infinite) in I_V. □

Theorem 3.21. *Let $V \in \mathcal{O}$. Then there is a smallest $W \in \mathcal{O} \cap \mathcal{U}$ containing V, and a subset $P \subseteq \mathbb{PP}$ such that W is the sub-monoid of $(\mathbb{N}^+, \oplus)$ generated by P. Define*

$$\begin{aligned} Y \supseteq Y(V) &= \Phi^{-1}(\bar{V}) \cap Y \\ &= \left\{ y_{i,j} \middle| \exists v \in V : p_i^j \,||v \right\} \\ &= \left\{ y_{i,j} \middle| p_i^j \in V \right\} \end{aligned} \tag{35}$$

and denote (by abuse of notation) by $\mathbb{C}[[Y(V)]]$ the generalized power series ring on the free abelian sub-monoid of Y^* which $Y(V)$ generates. Similarly, denote by $\mathbb{C}[Y(V)]$ the polynomial ring on the set of variables $Y(V)$. Then $\mathcal{A}_V$ is a cyclic $\mathbb{C}[[Y(V)]]$-module, as well as a cyclic $\mathcal{A}_W$-module; $\mathcal{A}_V^f$ is a cyclic $\mathbb{C}[Y(V)]$-module, as well as a cyclic $\mathcal{A}_W^f$-module;

Proof. Let $P = V \cap \mathbb{PP}$, and let W be the sub-monoid it generates. Then $W \in \mathcal{O}$, proving the first assertion. By Corollary 3.15 we have that $\mathcal{A}_V$ is a cyclic module over $\mathcal{A}_W$, which in turn is a cyclic module over $\mathcal{A}$, which in turn is a cyclic module over $\mathbb{C}[[Y^*]]$. So $\mathcal{A}_V$ is a cyclic $\mathbb{C}[[Y^*]]$-module. Since every variable not in $Y(P)$ will act trivially on $\mathcal{A}_V$, the latter is in fact a cyclic $\mathbb{C}[[Y(P)]]$-module. □

4. Hilbert and Poincaré-Betti series

Definition 4.1. We denote by $\mathbb{N}^\omega$ the abelian monoid of all finitely supported functions $\mathbb{N}^+ \to \mathbb{N}$, with component-wise addition. A $\mathbb{C}$-vector space R is ω-multi-graded it is graded over $\mathbb{N}^\omega$. The grading is *locally finite* if

$R_{\boldsymbol{\alpha}}$ is a finite dimensional for all $\boldsymbol{\alpha} \in \mathbb{N}^\omega$. For such an R, we define the $\mathbb{N}^\omega$-graded Hilbert series of R as

$$R(\boldsymbol{u}) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^\omega} \dim_{\mathbb{C}} R_{\boldsymbol{\alpha}} \boldsymbol{u}^{\boldsymbol{\alpha}} \in \mathbb{Z}[[\boldsymbol{u}]] \tag{36}$$

If $c : \mathbb{N}^\omega \to \mathbb{N}^s$ is a monoid homomorphism, then an ω-multi-graded R is s-multi-graded by

$$R_{\boldsymbol{\gamma}} := \bigoplus_{\substack{\boldsymbol{\alpha} \in \mathbb{N}^\omega \\ c(\boldsymbol{\alpha}) = \boldsymbol{\gamma}}} R_{\boldsymbol{\alpha}} \tag{37}$$

We say that the s-multi-grading is obtained by *collapsing* the $\mathbb{N}^\omega$-grading. c is called locally finite if $c^{-1}(\boldsymbol{\gamma})$ is finite for all $\boldsymbol{\gamma} \in \mathbb{N}^s$. Hence, if the $\mathbb{N}^\omega$-grading is locally finite then the collapsed grading is locally finite if and only if c is locally finite. Then, we define

$$\begin{aligned} R(t_1, \ldots, t_s) &= R(c(u_1), c(u_2), \ldots) \\ &= \sum_{\boldsymbol{\gamma} \in \mathbb{N}^s} \dim_{\mathbb{C}} R_{\boldsymbol{\gamma}} \boldsymbol{t}^{\boldsymbol{\gamma}} \in \mathbb{Z}[[t_1, \ldots, t_s]] \end{aligned} \tag{38}$$

Definition 4.2. Define a bijection

$$\begin{aligned} &\nu : Y \to \mathbb{N}^+ \\ \nu(y_{i',j'}) > \nu(y_{i,j}) \iff &\quad \Phi(y_{i',j'}) > \Phi(y_{i,j}) \end{aligned} \tag{39}$$

Then ν extends to a monoid isomorphism $Y^* \to \mathbb{N}^\omega$, such that

$$y_{i,j}^k \mapsto (0, \ldots, 0, k, 0, \ldots) \in \mathbb{N}^\omega,$$

with k in the $\nu(y_{i,j})$'th position. Hence, we may regard a $\mathbb{N}^\omega$-graded algebra as Y^*-graded. Furthermore, we'll call an Y^*-graded algebra, which is concentrated in the degrees $\mathcal{M} \subseteq Y^*$, $\mathcal{M}$-graded; a $\mathbb{N}^\omega$-graded algebra concentrated in the corresponding multi-degrees of $\mathbb{N}^\omega$ will also be called $\mathcal{M}$-graded.

The enumeration of Y given by (39) is illustrated in Table 1, as well as the induced enumeration for the variables corresponding to square-free elements.

We see that $\mathbb{C}[Y^*]$ is Y^*-graded, hence $\mathcal{A}^f = \frac{\mathbb{C}[Y^*]}{(y_{i,j}y_{i,k})}$ is also Y^*-graded; in fact, it is $\mathcal{M}$-graded. We have that

$$\begin{aligned} \mathbb{C}[Y^*](\boldsymbol{u}) &= \sum_{m \in Y^*} m = \prod_{i,j=1}^{\infty} \frac{1}{1 - u_{i,j}} \in \mathbb{Z}[[u_{i,j}]] \\ \mathcal{A}^f(\boldsymbol{u}) &= \prod_{i=1}^{\infty} \left(1 + \sum_{j=1}^{\infty} u_{i,j} \right) \\ &\quad 1 + u_{1,1} + u_{2,1} + u_{1,2} + u_{3,1} + u_{1,1}u_{2,1} + \cdots \in \mathbb{Z}[[u_{i,j} \,|\, i, j \in \mathbb{N}^+]] \\ \mathcal{A}^f_{\tilde{\mathbb{N}}}(\boldsymbol{u}) &= \prod_{i=1}^{\infty} (1 + u_{i,1}) \\ &= 1 + u_{1,1} + u_{2,1} + u_{3,1} + u_{1,1}u_{2,1} + \cdots \in \mathbb{Z}[[u_{1,j} \,|j \in \mathbb{N}^+]] \end{aligned} \tag{40}$$

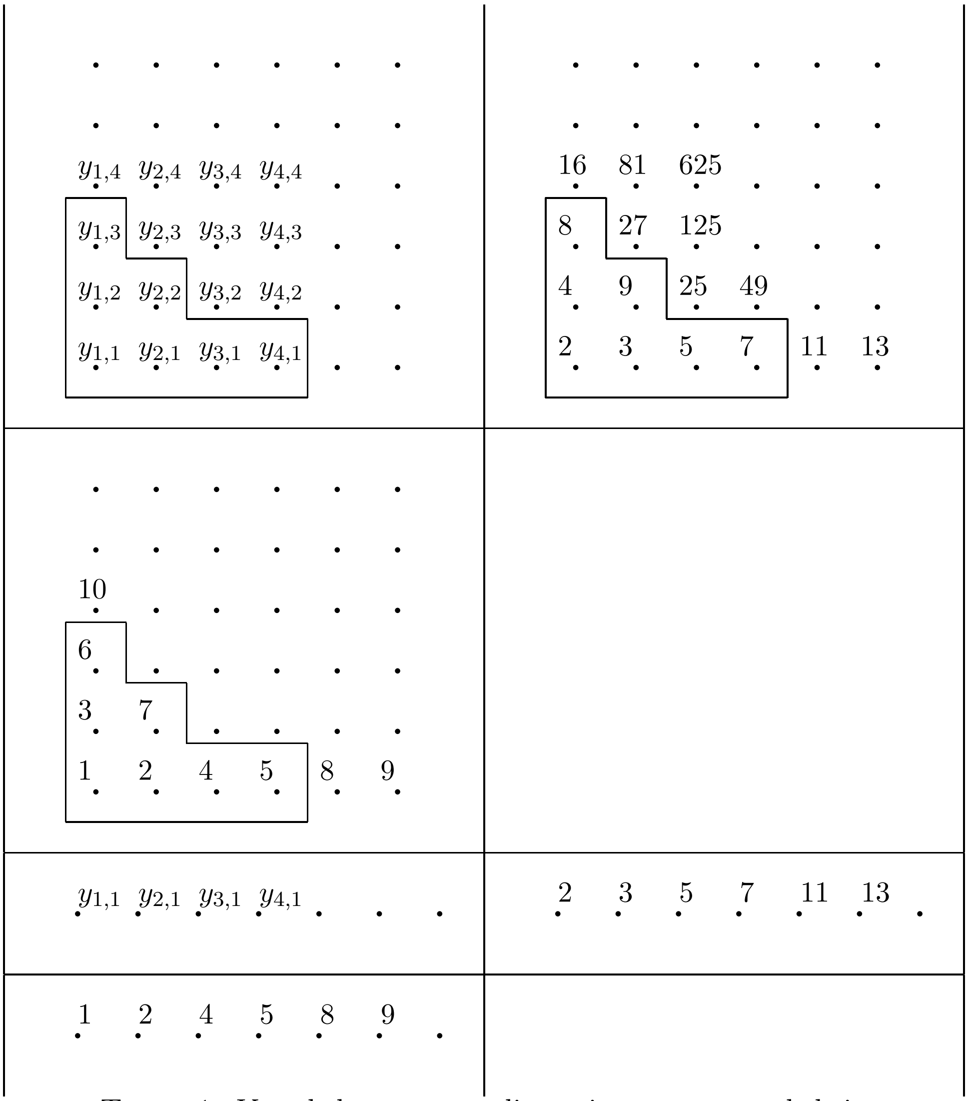


TABLE 1. Y and the corresponding prime powers, and their enumeration. $Y([10])$ is enclosed. The square-free part of Y and the corresponding prime powers, and their enumeration.

or, regarding instead these algebras as $\mathbb{N}^\omega$-graded,

$$
\begin{aligned}
\mathbb{C}[Y^*](\boldsymbol{t}) &= \sum_{\boldsymbol{\alpha} \in \mathbb{N}^\omega} \boldsymbol{t}^{\boldsymbol{\alpha}} = \prod_{i=1}^{\infty} \frac{1}{1-t_i} \in \mathbb{Z}[[t_1, t_2, t_3, \dots]] \\
\mathcal{A}^f(\boldsymbol{t}) &= \sum_{i=1}^{\infty} \boldsymbol{t}^{\mathrm{multideg}(e_i)} \\
&= 1 + t_1 + t_2 + t_3 + t_4 + t_1 t_2 + \cdots \in \mathbb{Z}[[t_1, t_2, t_3, t_4, t_5, \dots]] \\
\mathcal{A}^f_{\tilde{\mathbb{N}}}(\boldsymbol{t}) &= \sum_{i \in \tilde{\mathbb{N}}} \boldsymbol{t}^{\mathrm{multideg}(e_i)} \\
&= 1 + t_1 + t_2 + t_4 + t_1 t_2 + \cdots \in \mathbb{Z}[[t_1, t_2, t_4, t_5, t_8, t_9, \dots]]
\end{aligned} \tag{41}
$$

Note that the collapsing $\mathcal{M} \to \mathbb{N}$ obtained by giving each $y_{i,j}$ degree 1 is *not* locally finite, so we can not define the ordinary $\mathbb{N}$-graded Hilbert series of $\mathcal{A}^f$. There are however other ways of collapsing the grading that work:

Example 4.3. Define a locally finite $\mathbb{N}^2$-grading on $\mathbb{C}[Y^*]$ by giving $y_{i,j}$ bi-grade $(1, p_i^j)$. Then

$$\begin{aligned}\mathcal{A}^f(u_1,u_2) &= \prod_{i=1}^{\infty}\left(1+\sum_{j=1}^{\infty} u_1 u_2^{p_i^j}\right)\\ &= 1+\sum_{i,j=1}^{\infty} d(i,j) u_1 u_2^j\\ &= 1+\sum_{j=1}^{\infty} u_2^j \left(\sum_{i=1} d(i,j) u_1^i\right)\\ &= 1+u_1u_2^2+u_1u_2^3+u_1u_2^4+(u_1+u_1^2)u_2^5+(u_1+2u_1^2)u_2^7+\dots\end{aligned} \tag{42}$$

where $d(i,j)$ is the number of ways the positive integer j can be written as a sum of i pairwise co-prime prime powers. Collapsing this grading further, so that $y_{i,j}$ gets degree p_i^j, we have that

$$\begin{aligned}\mathcal{A}^f(u) &= \prod_{i=1}^{\infty}\left(1+\sum_{j=1}^{\infty} u^{p_i^j}\right)\\ &= \sum_{i=1}^{\infty} g(i) u^i\\ &= 1+u^2+u^3+u^4+2u^5+3u^7+\dots\end{aligned} \tag{43}$$

where $g(i)$ is the number of ways the positive integer i can be written as a finite sum of pairwise co-prime prime powers.

Example 4.4. If R denotes the ring of finitely supported arithmetical functions *with Dirichlet convolution*, then

$$R \simeq \mathbb{C}[x_1, x_2, x_3, \dots],$$

by letting e_n, for $n = p_{i_1}^{a_1} \cdots p_{i_r}^{a_r}$, correspond to the monomial $x_{i_1}^{a_1} \cdots x_{i_r}^{a_r}$. We have that R is $\mathbb{N}^\omega$-graded, with Hilbert series

$$\begin{aligned}R(\boldsymbol{u}) &= \sum_{\boldsymbol{\alpha} \in \mathbb{N}^\omega} \boldsymbol{u}^{\boldsymbol{\alpha}}\\ &= \prod_{i=1}^{\infty} \frac{1}{1-u_i}\end{aligned}$$

Giving e_n the same bi-grade as in the previous example means giving x_i bi-grade $(1, p_i)$. Then

$$\begin{aligned} R(u_1, u_2) &= \prod_{i=1}^{\infty} \frac{1}{1 - u_1 u_2^{p_i}} \\ &= \sum_{i,j} c_{i,j} u_1^i u_2^j \end{aligned}$$

where $c_{i,j}$ is the number of ways of writing j as the sum of i primes. Thus, by Vinogradov's three-primes theorem, $c_{i,3} > 0$ for odd $i \gg 0$, and if Goldbach's conjecture is true, then $c_{i,3} > 0$ for $i > 5$, and $c_{i,2} > 0$ for odd $i > 4$.

Comparing this to (42), one can ask if there is an j such that $d_{i,j} > 0$ for all sufficiently large i, and what the minimal such j might be. We have checked that for any integer $6 < i < 485$, either $i \in \mathbb{PP}$ and $d_{i,1} > 0$, or $d_{i,2} > 0$, thus one might conjecture that any integer > 6 is either a prime power or can be written as the sum of two relatively prime prime powers.

Definition 4.5. For an $\mathbb{N}^r$-graded $\mathbb{C}$-algebra R and a $\mathbb{N}^r$-graded R-module M, we regard $\mathbb{C}$ as a cyclic R-module, and define the $\mathbb{N}^r$-graded *Betti numbers* as

$$\beta_{i,\boldsymbol{a}}(R, M) = \dim_{\mathbb{C}} \operatorname{Tor}_R^i(M, \mathbb{C})_{\boldsymbol{a}} \tag{44}$$

and the *Poincaré-Betti series* by

$$P_R^M(t, \boldsymbol{u}) = \sum_{i=0}^{\infty} \sum_{\boldsymbol{a} \in \mathbb{N}^r} \beta_{i,\boldsymbol{a}}(R, M) t^i \boldsymbol{u}^{\boldsymbol{a}} \tag{45}$$

We use the convention $P_R(t, \boldsymbol{u}) = P_R^{\mathbb{C}}(t, \boldsymbol{u})$.

If R and M are instead locally finite and $\mathbb{N}^\omega$-graded, then each $\operatorname{Tor}_R^i(M, \mathbb{C})$ will be locally finite and $\mathbb{N}^\omega$-graded, so we can define the $\mathbb{N}^\omega$-graded, or Y^*-graded, Poincaré-Betti series. That means that

$$P_R^M(t, \boldsymbol{u}) \in \mathbb{Z}[[t]][[u_{i,j} \,|\, i, j \in \mathbb{N}^+]].$$

We say that R is *ω-Koszul* [1] if for every term $t^i \boldsymbol{u}^{\boldsymbol{\alpha}}$ occurring in $P_R^M(t, \boldsymbol{u})$, $|\boldsymbol{u}| = i$.

Lemma 4.6. *Let $W \in \mathcal{O}$, and let*

$$V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots \subseteq W, \quad \cup_{n=1}^{\infty} V_n = W$$

be an ascending family in $\mathcal{O}^f$. Then

$$\lim_{n \to \infty} \mathcal{A}_{V_n}(\boldsymbol{u}) = \mathcal{A}_W^f(\boldsymbol{u}) \tag{46}$$

$$\lim_{n \to \infty} P_{\mathbb{C}[Y(V_n)]}^{\mathcal{A}_{V_n}}(t, \boldsymbol{u}) = P_{\mathcal{A}_{\mathbb{C}[Y(W)]}}^{\mathcal{A}_W^f}(t, \boldsymbol{u}) \tag{47}$$

$$\lim_{n \to \infty} P_{\mathcal{A}_V}(t, \boldsymbol{u}) = P_{\mathcal{A}_W^f}(t, \boldsymbol{u}) \tag{48}$$

where the topology on $\mathbb{Z}[[u_{i,j} \,|\, i, j \in \mathbb{N}^+]]$ and on $\mathbb{Z}[[t]][[u_{i,j} \,|\, i, j \in \mathbb{N}^+]]$ is that of point-wise convergence.

---

[1]Recall that a commutative $\mathbb{N}$-graded $\mathbb{C}$-algebra R is *Koszul* if $\operatorname{Tor}_R^i(\mathbb{C}, \mathbb{C})_j = 0$ if $i \neq j$. If R is Koszul, then $P_R(t, u) = R(-tu)^{-1}$. An important result by Fröberg[7] is that if $R = \frac{\mathbb{C}[x_1, \ldots, x_n]}{(m_1, \ldots, m_r)}$, with m_i monomials (not of degree 1) then R is Koszul if and only if all m_i are quadratic.

Definition 4.7. Let $\boldsymbol{x} = (x_1, x_2, x_3, \dots)$. For $\sigma \subseteq \mathbb{N}^+$ we put

$$\boldsymbol{x}_\sigma = \prod_{i \in \sigma} x_i, \tag{49}$$

and if $j \in \sigma$, we put

$$\boldsymbol{x}_{\sigma,j} = x_j \boldsymbol{x}_\sigma. \tag{50}$$

Lemma 4.8. *Let $R = \mathbb{C}[x_1, \dots, x_n]$, and let $\mathbf{m} = (x_1, \dots, x_n)$ denote the graded maximal ideal. Put $M = R/\mathbf{m}^2$. Then*

$$M(u_1, \dots, u_n) = 1 + u_1 + \dots + u_n \tag{51}$$

M is Koszul, so

$$P_M(t, u_1, \dots, u_n) = \frac{1}{M(-tu_1, \dots, -tu_n)} = \frac{1}{1 - tu_1 - \dots - tu_n} \tag{52}$$

Furthermore,

$$P_R^M(t, u_1, \dots, u_n) = 1 + \sum_{\substack{\sigma \subseteq \{1,\dots,n\} \\ 1 \le |\sigma| \le n \\ j \in \sigma}} t^{|\sigma|} \boldsymbol{u}_{\sigma,j} + \sum_{\substack{\sigma \subseteq \{1,\dots,n\} \\ 2 \le |\sigma| \le n}} (|\sigma| - 1) t^{|\sigma|-1} \boldsymbol{u}_\sigma \tag{53}$$

Proof. As a $\mathbb{C}$-vector space, M is spanned by $1, x_1, \dots, x_n$, so (51) follows. Since the defining ideal of M is a quadratic monomial ideal, a theorem by Fröberg [7] shows that M is Koszul. Hence (52) follows.

For the result on the relative Poincaré-Betti series, we'll use a result by Bayer [3], which yields that the coefficient of the $t^i \boldsymbol{u}$-term is equal to the rank of the torsion-free part of the $(i-2)$'th reduced homology of the complex

$$\Delta_{\boldsymbol{u}} = \left\{ F \subset [n] \middle| \boldsymbol{x}^{\boldsymbol{u}-F} \in \mathbf{m} \right\} \tag{54}$$

Here, F is identified with its characteristic vector, which is the square-free vector in $\mathbb{N}^n$ with a 1 in position i if $i \in F$, and zero otherwise.

If $\boldsymbol{u}$ is square-free, $|\boldsymbol{u}| = k$, then

$$F \in \Delta_{\boldsymbol{u}} \iff \#(\boldsymbol{u} - F) \ge 2 \iff \#F \le |\boldsymbol{u}| - 2 = k - 2,$$

so $\Delta_{\boldsymbol{u}}$ is the $(k-3)$-skeleton of a $(k-1)$-simplex, hence the reduced homology is concentrated in degree $k-3$, where it is $\mathbb{Z}^{k-1}$.

If $\boldsymbol{x}^{\boldsymbol{u}} = x_a^2 \boldsymbol{x}^{\boldsymbol{v}}$, where $\boldsymbol{v}$ is square-free, and does not contain a, and if $|\boldsymbol{u}| = k$, (so $|v| = k-2$), then

$$F \in \Delta_{\boldsymbol{u}} \iff F \subseteq \boldsymbol{v} \quad \text{or} \quad a \in F \text{ and } \#F \le k - 3,$$

thus $\Delta_{\boldsymbol{u}}$ is the disjoint union of a $(k-3)$-simplex, which has no reduced homology, and a $(k-3)$-sphere, which has homology concentrated in degree $(k-3)$, where it is $\mathbb{Z}$.

If $x_a^2 \,| \boldsymbol{x}^{\boldsymbol{u}}$ for two different a's, so that e.g. $\boldsymbol{x}^{\boldsymbol{u}} = x_1^2 x_2^2 \boldsymbol{x}^{\boldsymbol{v}}$, then

$$\forall F \subset [n] : \quad \boldsymbol{x}^{\boldsymbol{u}-F} \in \mathbf{m},$$

thus

$$\Delta_{\boldsymbol{u}} = 2^{[n]}$$

is acyclic.

Summarizing we have that

$$\dim_{\mathbb{C}} \operatorname{Tor}^R_{i,\boldsymbol{u}}(M,\mathbb{C}) = \begin{cases} i & \boldsymbol{u} \text{ square-free of total degree } i+1 \\ 1 & u_a = 2,\ u_j \le 1 \text{ for } j \neq a \text{ and } |\boldsymbol{u}| = i+1 \\ 0 & \text{otherwise} \end{cases} \tag{55}$$

From this, (53) follows. □

Theorem 4.9. *$\mathcal{A}^f$ is ω-Koszul, and*

$$P_{\mathcal{A}^f} = \frac{1}{\mathcal{A}^f(-t\boldsymbol{u})} = \prod_{i=1}^{\infty} \frac{1}{1 - t\sum_{j=1}^{\infty} u_{i,j}} \tag{56}$$

Define

$$A(t,\boldsymbol{x}) = 1 + \sum_{\substack{\sigma \subseteq \mathbb{N}^+ \\ 1 \le |\sigma| \\ j \in \sigma}} t^{|\sigma|}\boldsymbol{x}_{\sigma,j} + \sum_{\substack{\sigma \subseteq \mathbb{N}^+ \\ 2 \le |\sigma|}} (|\sigma|-1)t^{|\sigma|-1}\boldsymbol{x}_\sigma \tag{57}$$

For each positive integer i, put

$$\boldsymbol{u}_i = (u_{i,1}, u_{i,2}, u_{i,3}, \dots).$$

Put

$$\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \dots) = (u_{1,1}, u_{1,2}, \dots; u_{2,1}, u_{2,2}, \dots; u_{3,1}, u_{3,2}, \dots; \dots)$$

Then

$$P^{\mathcal{A}^f}_{\mathbb{C}[Y^*]}(t,\boldsymbol{u}) = \prod_{i=1}^{\infty} A(t,\boldsymbol{u}_i) \tag{58}$$

Similarly: $\mathcal{A}^f_{\tilde{\mathbb{N}}}$ is ω-Koszul, and

$$P_{\mathcal{A}^f_{\tilde{\mathbb{N}}}} = \frac{1}{\mathcal{A}^f_{\tilde{\mathbb{N}}}(-t\boldsymbol{u})} = \prod_{i=1}^{\infty} \frac{1}{1 - tu_{i,1}} \tag{59}$$

$$P^{\mathcal{A}^f_{\tilde{\mathbb{N}}}}_{\mathbb{C}[y_{1,1},y_{2,1},y_{3,1},\dots]}(t,\boldsymbol{u}) = A(t,\boldsymbol{u}_1) \tag{60}$$

Proof. We apply Theorem 4.6, putting $W = \mathbb{N}^+$, $V_n =$ the sub-monoid of $(\mathbb{N}^+, \oplus)$ generated by $\left\{ p_i^j \middle| 1 \le i,j \le n \right\}$. Then

$$\begin{aligned} \mathcal{A}^f_{V_n} &= \frac{\mathbb{C}[\{ y_{i,j} | 1 \le i,j \le n \}]}{(\{ y_{i,j}y_{i,j'} | 1 \le i,j,j' \le n \})} \\ &= \frac{\mathbb{C}[\{ y_{1,j} | 1 \le j \le n \}]}{(\{ y_{1,j}y_{1,j'} | 1 \le j,j' \le n \})} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \frac{\mathbb{C}[\{ y_{n,j} | 1 \le j \le n \}]}{(\{ y_{n,j}y_{n,j'} | 1 \le j,j' \le n \})} \\ &= \left(\frac{\mathbb{C}[x_1,\dots,x_n]}{\mathbf{m}^2} \right)^{\otimes n} \end{aligned} \tag{61}$$

By Lemma 4.8, $\frac{\mathbb{C}[x_1,\dots,x_n]}{\mathbf{m}^2}$ is Koszul and has absolute Poincaré-Betti series

$$\frac{1}{1 - tu_1 - \cdots - tu_n},$$

hence $\mathcal{A}^f_{V_n}$ is Koszul and

$$P_{\mathcal{A}^f_{V_n}}(t, u_{1,1}, \ldots, u_{n,n}) = \prod_{i=1}^{n} \frac{1}{1 - tu_{i,1} - \cdots - tu_{i,n}}.$$

It follows that

$$\begin{aligned} P_{\mathcal{A}^f}(t, \boldsymbol{u}) &= \lim_{n\to\infty} P_{\mathcal{A}^f_{V_n}}(t, u_{1,1}, \ldots, u_{n,n}) \\ &= \lim_{n\to\infty} \prod_{i=1}^{n} \frac{1}{1 - tu_{i,1} - \cdots - tu_{i,n}} \\ &= \prod_{i=1}^{\infty} \frac{1}{1 - t\sum_{j=1}^{\infty} u_{i,j}}. \end{aligned}$$

The assertion about the relative Poincaré-Betti series is proved in a similar way. The square-free case is analogous. □

Example 4.10. We continue our study of Example 4.3. Let V_n be as above, so that

$$\mathbb{C}[Y(V_n)] = \mathbb{C}[x_1, \ldots, x_n]^{\otimes n},$$

and $\mathcal{A}^f_{V_n}$ is as in (61). The induced bi-grading on the i'th copy of $C[x_1, \ldots, x_n]$ gives x_j bi-grade $(1, p_i^j)$, hence

$$\mathcal{A}^f_{V_n}(u_1, u_2) = \prod_{i=1}^{n} \left(1 + \sum_{j=1}^{n} u_1 u_2^{p_i^j} \right).$$

We have that

$$\begin{aligned} &\mathcal{A}^f_{V_1}(u_1, u_2) = 1 + u_2^2 u_1 \\ &\mathcal{A}^f_{V_2}(u_1, u_2) = 1 + u_1u_2^2 + u_1u_2^3 + u_1u_2^4 + u_1^2u_2^5 + u_1^2u_2^7 + u_1u_2^9 + u_1^2u_2^{11} + u_1^2u_2^{13} \\ &\mathcal{A}^f_{V_3}(u_1, u_2) = 1 + u_1u_2^2 + u_1u_2^3 + u_1u_2^4 + (u_1 + u_1^2)u_2^5 + \ldots \end{aligned}$$

and $\mathcal{A}^f_{V_n}(u_1, u_2)$ converges to (42) as $n \to \infty$.

Now collapse this grading to an $\mathbb{N}$-grading. Since $\mathcal{A}_{V_n}$ is Koszul, we have that

$$P_{\mathcal{A}_{V_n}}(t, u) = \frac{1}{\mathcal{A}_{V_n}(-tu)}$$

which converges to

$$\begin{aligned} \frac{1}{\mathcal{A}^f(-tu)} &= \frac{1}{1 - t^2u^2 - t^3u^3 - t^4u^4 - 2t^5u^5 - 3t^7u^7 - \ldots} \\ &= 1 + u^2t^2 + u^3t^3 + 2u^4t^4 + \ldots \end{aligned}$$

We have that

$$P_{\mathcal{A}_{V_1}}(t, u) = \frac{1}{1 - t^2u^2} = 1 + t^2u^2 + t^4u^4 + t^6u^6 + t^8u^8 + \ldots$$

$$\begin{aligned} P_{\mathcal{A}_{V_2}}(u) &= \frac{1}{1 - t^2u^2 - t^3u^3 - t^4u^4 - t^5u^5 - t^7u^7 - t^9u^9 - t^{11}u^{11}} t^{13}u^{13} \\ &= 1 + u^2t^2 + u^3t^3 + 2u^4t^4 + 3u^5t^5 + 4u^6t^6 + 8u^7t^7 + 10u^8t^8 + \ldots \end{aligned}$$

Finally,

$$\begin{aligned}P_{\mathcal{A}^f}(t,u) &= \prod_{i=1}^{\infty} A(t,u^{p_i})\\ &= \prod_{i=1}^{\infty}\left(1+\sum_{\substack{\sigma\subseteq\mathbb{N}^+\\ 1\le|\sigma|\\ j\in\sigma}} t^{|\sigma|}u^{p_i^j+\sum_{v\in\sigma}p_i^v}+\sum_{\substack{\sigma\subseteq\mathbb{N}^+\\ 2\le|\sigma|}}(|\sigma|-1)t^{|\sigma|-1}u^{\sum_{v\in\sigma}p_i^v}\right)\end{aligned}$$

The reader is cordially invited to simplify this expression, and to compute its first few terms.

5. The case of a finite V

5.1. **Simple properties.** We now assume that $V \in \mathcal{O}^f$. To avoid trivial special cases, we assume that V contains at least two elements. Then $\mathcal{A}_V = \mathcal{A}_V^f$ is a local Artin ring, where the maximal ideal is spanned (as a vector space) by $\{\, e_j | j \in V \setminus \{1\}\,\}$. It is easy to see that the maximal ideal is nilpotent, thus elements of $\mathcal{A}_V$ are either units or nilpotent.

5.2. **Presentations.**

Theorem 5.1. *Let $V \in \mathcal{O}^f$, $|V| = r$. $\mathcal{A}_V$ is a cyclic module over $\mathbb{C}[Y(V)]$. Furthermore, if we define the following ideals of $\mathbb{C}[Y(V)]$,*

$$\begin{aligned}A_V &= \mathbb{C}[Y(V)]\left\{\, y_{i,j}^2 \middle| y_{i,j}\in Y(V)\,\right\}\\ B_V &= \mathbb{C}[Y(V)]\left\{\, y_{i,j}y_{i,j'} \middle| y_{i,j}, y_{i,j'}\in Y(V),\ j<j'\,\right\}\\ C_V &= \mathbb{C}[Y(V)]\left\{\prod_{\ell=1}^{r} y_{i_\ell,j_\ell}\,\middle|\,\prod_{\ell=1}^{r} p_{i_\ell}^{j_\ell}\notin V,\ \forall 1\le v\le r: \prod_{\substack{1\le\ell\le r\\ \ell\neq j}} p_{i_\ell}^{j_\ell}\in V\right\}\end{aligned} \tag{62}$$

then the indicated generating sets are in fact the unique multi-graded minimal generating sets, and

$$\mathcal{A}_V = \frac{C[Y(V)]}{A_V + B_V + C_V} \tag{63}$$

If we let W denote the monoid ideal that V generates, then

$$\mathcal{A}_W = \frac{C[Y(V)]}{A_V + B_V}, \tag{64}$$

and $\mathcal{A}_V$ is a cyclic $\mathcal{A}_W$-module, since

$$\mathcal{A}_V = \frac{\mathcal{A}_W}{C_V}.$$

We put

$$\mathbb{C}[\overline{Y(V)}] = \frac{\mathbb{C}[Y(V)]}{A_V} \simeq \left(\frac{\mathbb{C}[x]}{(x^2)}\right)^{\otimes r},$$

an r-fold tensor power of the ring of dual numbers. Then $\mathcal{A}_V$ is also a cyclic $\mathbb{C}[\overline{Y(V)}]$-module, since $\mathcal{A}_V = \frac{\mathbb{C}[\overline{Y(V)}]}{B_V + C_V}$.

Proof. We have that $W \in \mathcal{O}^f \cap \mathcal{U}^f$, and $Y(V) = Y(W)$, so by Theorem 3.21, $\mathcal{A}_V$ and $\mathcal{A}_W$ are cyclic $\mathbb{C}[Y(V)]$-modules. Clearly, $\mathcal{A}_W$ has a $\mathbb{C}$-basis consisting of

$$\{\, e_k | k \in W \,\} = \{\, e_{i_1} \oplus e_{i_2} \oplus \cdots \oplus e_{i_r} | i_1, \ldots, i_r \in V \cap \mathbb{PP} \,\}$$

which in $C[Y(W)]$ corresponds to separated monomials in the variables in $Y(W)$. From this, (64) follows, since $A(V)+B(V)$ is precisely what wee need to divide out with in order to have only separated monomials. Similarly, $\mathcal{A}_V$ has a $\mathbb{C}$-basis consisting of

$$\{\, e_k | k \in V \,\} = \{\, e_{i_1} \oplus e_{i_2} \oplus \cdots \oplus e_{i_r} | i_1, \ldots, i_r \in V \cap \mathbb{PP},\ i_1 \oplus \cdots \oplus i_r \in V \,\}$$

so the defining ideal of $\mathcal{A}_V$ in $\mathbb{C}[Y(V)]$ contains, in addition, those separated monomials (in $y_{i,j}$-variables) which do not correspond to an element in V; from this observation, and Theorem 3.19, (63) follows.

The remaining assertions follow trivially. □

Example 5.2. Let

$$V = [10] = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

Then

$$Y([10]) = \{y_{1,1},\ y_{1,2},\ y_{1,3},\ y_{2,1},\ y_{2,2},\ y_{3,1},\ y_{4,1}\},$$

and

$$A_{[10]} = (y_{1,1}y_{1,1},\ y_{1,2}y_{1,2},\ y_{1,3}y_{1,3},\ y_{2,1}y_{2,1},\ y_{2,2}y_{2,2},\ y_{3,1}y_{3,1},\ y_{4,1}y_{4,1}),$$

$$B_{[10]} = ({y_{1,1}}^2,\ {y_{1,1}}^2,\ {y_{1,2}}^2,\ {y_{2,1}}^2),$$

and finally

$$C_{[10]} = (y_{1,1}y_{2,2},\ y_{1,1}y_{4,1},\ y_{2,2}y_{3,1},\ y_{1,2}y_{2,1},\ y_{1,3}y_{2,1},\ y_{2,1}y_{3,1},\ y_{2,1}y_{4,1},$$
$$y_{1,2}y_{2,2},\ y_{1,2}y_{3,1},\ y_{1,2}y_{4,1},\ y_{1,3}y_{3,1},\ y_{1,3}y_{4,1},\ y_{2,2}y_{4,1},\ y_{1,3}y_{2,2},\ y_{3,1}y_{4,1})$$

We have that

$$\mathcal{A}_{[10]} \simeq \frac{C[y_{1,1},\ y_{1,2},\ y_{1,3},\ y_{2,1},\ y_{2,2},\ y_{3,1},\ y_{4,1}]}{A_{[10]} + B_{[10]} + C_{[10]}}$$

has a $\mathbb{C}$-basis consisting of (the images of) the following separated monomials:

$$1,\ y_{1,1},\ y_{1,2},\ y_{1,3},\ y_{2,1},\ y_{1,1}y_{2,1}\,y_{2,2},\ y_{3,1},\ y_{4,1},\ y_{1,1}y_{3,1}.$$

5.3. **The associated simplicial complex.** For terminology and general results regarding simplicial complexes and Stanley-Reisner rings, see [15].

Definition 5.3. Let S be a finite set. A simplicial complex on S is a subset $\Delta \subseteq 2^S$ of the power-set of S, such that

(i) $\{s\} \in \Delta$ for all $s \in S$,
(ii) $\tau \subseteq \sigma \in \Delta \implies \tau \in \Delta$.

If $U \subseteq S$, we denote by Δ_U the simplicial complex on U given by $\Delta_U = \Delta \cap 2^U$.

If the reduced homology $\widetilde{H}^i(\Delta, \mathbb{C})$ is non-zero for $i = j$, but vanishes for $i > j$ we say that Δ has homological degree i.

Definition 5.4. Let $S = \{s_1, \ldots, s_n\}$ be a finite set, and let $S_1, \ldots, S_r$ be a partition of S, i.e. $S = \sqcup_{i=1}^{r} S_i$, $S_i \cap S_j = \emptyset$ if $i \neq j$. A partitioned simplicial complex Δ (corresponding to the partition $S = \sqcup_{i=1}^{r} S_i$) on S is an order ideal in the sub-poset

$$\overline{2^S} = \{\, \sigma \subseteq S | \forall 1 \leq i \leq r : \#(\sigma \cap S_i) \leq 1 \,\} \subseteq 2^S \tag{65}$$

such that $\{s\} \in \Delta$ for every $s \in S$.

We define the following ideals in $\mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, \ldots, x_n]$:

$$\begin{aligned} A_S &= \mathbb{C}[\boldsymbol{x}] \left\{\, x_a^2 \middle| 1 \leq a \leq n \,\right\} \\ B_S &= \mathbb{C}[\boldsymbol{x}] \left\{\, x_a x_b \middle| \exists i : s_a, s_b \in X_i \,\right\} \\ C_\Delta &= \mathbb{C}[\boldsymbol{x}] \left\{\, x_{a_1} \cdots x_{a_v} \middle| 1 \leq a_1 < a_2 < \cdots < a_v \leq n, \{a_1, \ldots, a_v\} \in \overline{2^X} \setminus \Delta \,\right\} \end{aligned} \tag{66}$$

We define the Stanley-Reisner algebra of Δ as

$$\mathbb{C}[\Delta] = \frac{\mathbb{C}[x_1, \ldots, x_n]}{B_S + C_\Delta} \tag{67}$$

and the Artinified Stanley-Reisner ring as

$$\mathbb{C}[\overline{\Delta}] = \frac{\mathbb{C}[x_1, \ldots, x_n]}{A_S + B_S + C_\Delta} \tag{68}$$

Note that a partitioned simplicial complex is also a simplicial complex in the ordinary sense, that is, an order ideal in 2^X, and that $\mathbb{C}[\Delta]$ coincides with the usual definition; the only difference is that we have partitioned the Stanley-Reisner ideal $I_\Delta = B_X + C_\Delta$. The Artinified Stanley-Reisner ring

$$\mathbb{C}[\overline{\Delta}] \simeq \frac{\mathbb{C}[\boldsymbol{x}]}{A_X + I_\Delta}$$

is identical to the one introduced by Sköldberg [12]. The only difference is that we may choose to regard it as a cyclic $\frac{\mathbb{C}[\boldsymbol{x}]}{A_X + B_X}$-module as well as a cyclic $\mathbb{C}[\boldsymbol{x}]$-module.

Example 5.5.

$$\Delta([10]) = \{\emptyset, \{2\}, \{3\}, \{4\}, \{5\}, \{2, 3\}, \{7\}, \{8\}, \{9\}, \{2, 5\}\}$$

looks like Figure 1.

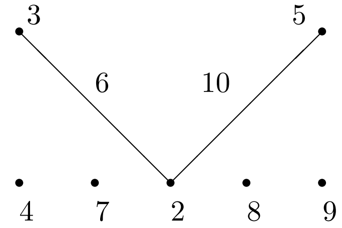


FIGURE 1. $\Delta([10])$

For subsequent use, we state the following theorem:

Theorem 5.6. *Let Δ be a simplicial complex, and $\mathbb{C}[\overline{\Delta}]$ its Artinified Stanley-Reisner ring. Let $\mathbb{C}[\overline{\boldsymbol{x}}]$ denote the (smallest) polynomial ring of which $\mathbb{C}[\overline{\Delta}]$ is an epimorphic image. Then the multi-graded Betti numbers*

$$\beta_{i,\boldsymbol{\alpha}} = \dim_{\mathbb{C}} \operatorname{Tor}_{\boldsymbol{\alpha}}^{\mathbb{C}[\overline{\boldsymbol{x}}]}(\mathbb{C}[\overline{\Delta}], \mathbb{C})$$

are given by

$$\beta_{i,\boldsymbol{\alpha}} = \dim_{\mathbb{C}} \widetilde{H}^{|\boldsymbol{\alpha}|-i-1}(\Delta_U; \mathbb{C}), \qquad \textit{where } U = \operatorname{supp}(\boldsymbol{\alpha}) \tag{69}$$

Proof. This is the formula for the Betti numbers of the corresponding indicator algebra $\mathbb{C}\{\Delta\}$ [1, 2]. This skew-commutative algebra has the same Betti numbers as $\mathbb{C}[\overline{\Delta}]$ [12]. □

Theorem 5.7. *For $V \in \mathcal{O}^f$, let $P = V \cap \mathbb{PP}$, $Q = V \cap \mathbb{P}$. Number the elements in Q as $q_1, \ldots, q_r$ such that $q_1 < q_2 < \cdots < q_r$. Partition $P = P_1 \cup P_2 \cdots \cup P_r$, $P_i = \left\{ q_i^j \,\middle|\, q_i^j \in P \right\}$. Define a partitioned simplicial complex $\Delta(V)$ on P by*

$$\{q_{i_1}^{a_1}, \ldots, q_{i_s}^{a_s}\} \in \Delta(V) \qquad \Longleftrightarrow \qquad q_{i_1}^{a_1} \cdots q_{i_s}^{a_s} \in V \tag{70}$$

Then

$$\mathbb{C}[\overline{\Delta(V)}] \simeq \mathcal{A}_V \tag{71}$$

as graded $\mathbb{C}$-algebras, and

$$(\Delta(V), \subseteq) \simeq (V, \leq_{\oplus V}) \tag{72}$$

as posets.

Proof. Since (70) gives a bijection between the basis vectors of $\mathbb{C}[\overline{\Delta(V)}]$ and those of $\mathcal{A}_V$, we need only check that the multiplication is the same. Let $\sigma, \tau \in \Delta(V)$, and define

$$\sigma(i) = \begin{cases} 1 & q_i \in \sigma \\ 0 & q_i \notin \sigma \end{cases}$$

$$x_\sigma = \prod_{i=i}^{r} x_i^{\sigma(i)}$$

$$e_\sigma = e_m \qquad \text{where } m = \prod_{j \in \sigma} j$$

and similarly for τ. Then

$$e_\sigma e_\tau = \begin{cases} e_{\sigma \cup \tau} & \text{if } \sigma \cap \tau = \emptyset,\ \sigma \cup \tau \in \Delta(V) \\ 0 & \text{otherwise} \end{cases}$$

Furthermore $x_\sigma, x_\tau \in \mathbb{C}[\overline{\Delta(V)}]$, and

$$x_\sigma x_\tau = \begin{cases} x_{\sigma \cup \tau} & \text{if } \sigma \cap \tau = \emptyset,\ \sigma \cup \tau \in \Delta(V) \\ 0 & \text{otherwise} \end{cases}$$

This shows that the multiplications are the same.

The bijection (70) is easily seen to be the desired poset isomorphism (72). □

5.4. **The socle.** There is a short exact sequence of complex vector spaces and linear maps

$$\begin{aligned} 0 \to K_1(V) \to \mathcal{A}_V \otimes \mathcal{A}_V &\to \mathcal{A}_V \to 0 \\ f \otimes g &\mapsto f \oplus g \end{aligned} \tag{73}$$

which restrict to

$$0 \to K_2(V) \to \mathcal{A}_V^+ \otimes \mathcal{A}_V^+ \to \mathcal{A}_V^+ \tag{74}$$

We denote by pr the projection $\mathcal{A}_V^+ \otimes \mathcal{A}_V^+ \to \mathcal{A}_V^+$ to the first factor.

We call elements in $K_1(V)$ of the form $e_a \otimes e_b$, $e_a \oplus e_b = 0$, *monomial multiplicative syzygies* and those of the form $e_a \otimes e_b - e_c \otimes e_d$ *binomial multiplicative sysygies.*

Lemma 5.8. *$K_1(V)$ is spanned (as a $\mathbb{C}$-vector space) by monomial and binomial multiplicative syzygies; consequently, so is $K_2(V)$. Let $n = |V| = \dim_{\mathbb{C}} \mathcal{A}_V$, then $\dim_{\mathbb{C}} K_1(V) = n^2 - n$, $\dim_{\mathbb{C}} K_2(V) = (n-1)^2 - (n-1)$.*

Proof. Since the map (73) is multi-homogeneous, it is enough to study it in one multi-degree

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r),$$

where

$$k = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \in V \oplus V.$$

If

$$f = \sum_{\substack{a,b \in V \\ ab=k}} c_{a,b} e_a \otimes e_b$$

is an element of $K_1(V)$ of multi-degree $\boldsymbol{\alpha}$, then we can write $f = f_1 + f_2$ with

$$f_1 = \sum_{\substack{a,b \in V \\ ab=k \\ a \oplus_V b = 0}} c_{a,b} e_a \otimes e_b, \qquad f_2 = \sum_{\substack{a,b \in V \\ ab=k \\ a \oplus_V b = k}} c_{a,b} e_a \otimes e_b.$$

Then f_1 is a linear combination of monomial syzygies, so it is enough to show that f_2 is a linear combination of binomial syzygies. By construction,

$$\sum_{\substack{a,b \\ ab=k \\ a \oplus_V b = k}} c_{a,b} = 0.$$

If we order the pairs (a, b) with $a, b \in V$, $a \oplus_V b = k$ linearly, this becomes

$$\sum_{i=1}^{L} c_i = 0,$$

or

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_L \end{bmatrix} = 0$$

It is an elementary fact that the solution set is spanned by

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$$

Thus f_2 can be written as a linear combination of binomial syzygies.

Since dimension of vector spaces is additive function over short exact sequences, the assertions on the dimension of K_1 and K_2 follows from the fact that $\dim_{\mathbb{C}} \mathcal{A}_V = n$, $\dim_C \mathcal{A}_V \otimes \mathcal{A}_V = n^2$, $\dim_{\mathbb{C}} \mathcal{A}_V^+ = n-1$, $\dim_C \mathcal{A}_V^+ \otimes \mathcal{A}_V^+ = (n-1)^2$. □

Definition 5.9. The *socle* of $\mathcal{A}_V$ consists of all elements in the maximal ideal that annihilates the maximal ideal. We denote it by $\mathrm{Socle}(\mathcal{A}_V)$.

Lemma 5.10. *The socle of $\mathcal{A}_V$ is generated (as a $\mathbb{C}$-vector space) by*

$$\begin{aligned}\{\, e_k,\ k \in V \setminus \{1\} | km \notin V \ \textit{if}\ m \in V \setminus \{1\} \ \textit{and}\ \gcd(k,m) = 1 \,\} &= \\ = \left\{\, e_k \middle| e_k \otimes e_m \in K_2(V) \ \textit{for all}\ m \in \mathcal{A}_V^+ \,\right\} & \quad (75)\end{aligned}$$

In fact, the above set is a basis.

Proof. $\mathcal{A}_V$ and its maximal ideal are multi-graded: $\mathcal{A}_V$ has the $\mathbb{C}$-basis

$$\{\, e_k | k \in V \,\},$$

and

$$\{\, e_k | k \in V \setminus \{1\} \,\}$$

is a basis for the maximal ideal. Hence, the set

$$\begin{aligned}\{\, e_k | k \in V \setminus \{1\},\ e_k \oplus_V e_m = 0 \text{ for } m \in V \setminus \{1\} \,\} &= \\ = \{\, e_k,\ k \in V \setminus \{1\} | km \notin V \text{ if } m \in V \setminus \{1\} \text{ and } \gcd(k,m) = 1 \,\}\end{aligned}$$

is a basis for the socle. This is precisely the set of e_k, $k \in V \setminus \{1\}$ such that $e_k \otimes e_m$ is a multiplicative syzygy for all $e_m \in \mathcal{A}_V^+$. □

Lemma 5.11. *Let $k \in V$. The following are equivalent:*

(i) $e_k \in \mathrm{Socle}(\mathcal{A}_V)$.
(ii) k is a maximal element in $(V, \leq_{\oplus_V})$.
(iii) k corresponds to a facet in $\Delta(V)$.
(iv) $\mathrm{pr}^{-1}(e_k) \subseteq K_2$.

Proof. If k is maximal, then $k \oplus_V m = 0$ for all $m \in V \setminus \{1\}$, hence $e_k \oplus_V e_m = 0$ for all $m \in V \setminus \{1\}$, hence $e_k \in \mathrm{Socle}(\mathcal{A}_v)$. If k is not maximal, so that $k <_{\oplus_V} m$, then $m = k \oplus_v c$ for some $c \in V \setminus \{1\}$, hence $e_m = e_k \oplus_v e_c$, hence e_k does not annihilate all elements in the maximal ideal, hence $e_k \notin \mathrm{Socle}(\mathcal{A}_v)$.

By (72), maximal elements in $(V, \leq_{\oplus_V})$ correspond to facets in $\Delta(V)$. □

Since $\mathcal{A}_V$ is Artin, it is Cohen-Macaulay. It is Gorenstein if and only if the socle is one-dimensional [15, Theorem 12.4], hence

Lemma 5.12. *If $k = \max V$ then $e_k \in \mathrm{Socle}(\mathcal{A}_V)$. The following are equivalent:*

(i) *$\mathcal{A}_V$ is Gorenstein,*
(ii) $\mathrm{Socle}(\mathcal{A}_V) = \mathbb{C}e_k$,
(iii) *For every $j \in V \setminus \{k, 1\}$ there exists at least one $i = i(j) \in V \setminus \{k, 1\}$ such that $\gcd(i, j) = 1$, $ij \in V$.*
(iv) *V is a principal order ideal.*
(v) *$\Delta(V)$ is a simplex.*

Proof. If $k = \max V$ then it is a maximal element in $(V, \geq_{\oplus V})$, hence $e_k \in \mathrm{Socle}(\mathcal{A}_V)$ by the previous Lemma. In fact, the only $V \in \mathcal{O}^f$ which have only one maximal element are the principal order ideals. □

5.5. **The Koszul property.**

Theorem 5.13. *Let $V \in \mathcal{O}^f$ contain at least two elements, and let $Q = V \cap \mathbb{PP}$. Let W denote the sub-monoid of $(\mathbb{N}^+, \oplus)$ generated by Q. The following are equivalent:*

(i) *$\mathcal{A}_V$ is Koszul,*
(ii) *C_V is 0 or generated in degree 2.*
(iii) *If $w \in W \setminus V$ then*

$$\exists q_1, q_2 \in W: \quad \gcd(q_1, q_2) = 1, \quad q_1 q_2 \,||w, \quad q_1 q_2 \in W \setminus V.$$

(iv) *For any $U \subseteq Q$ with more than two elements,*

$$\tilde{H}^{|U|-2}(\Delta(V)_U, \mathbb{C}) = 0.$$

Proof. By (63) and the fact that A_V and B_V are quadratic, Fröberg's result [7] that a monomial algebra is Koszul if and only if it is quadratic gives the equivalence between (i) and (ii). (iii) says that any e_w in the defining ideal of $\mathcal{A}_V$ as a cyclic $\mathcal{A}_W$-module is divisible by some $e_{q_1 q_2}$, i.e. by some quadratic element. This is equivalent to all minimal generators being quadratic.

Finally, let $\beta_{1,j}$ denote the number of minimal generators of $A_V + B_V + C_V$ of degree j. Since $A_v + B_V$ are generated in degree 2, it follows from Theorem 5.6 that for $j > 2$,

$$\beta_{1,j} = \sum_{\substack{U \subseteq Q \\ |U| = j}} \dim_{\mathbb{C}} \tilde{H}^{|U|-2}(\Delta(V)_U, \mathbb{C}).$$

We want $\beta_{1,j} = 0$ for $j > 2$, so (iv) follows. □

Corollary 5.14. *If $\mathcal{A}_V$ is Koszul then for $d > 1$, $\Delta(V)$ can not contain a "punctured d-simplex" i.e. a subset of the form $2^U \setminus U$ with $|U| = d + 1$.*

Proof. If it does, $\Delta(V)_U \simeq \mathbb{S}^{d-1}$ so $\tilde{H}^{d-1}(\Delta(V)_U, \mathbb{C}) \neq 0$. □

Corollary 5.15. *Let $Q \subseteq \mathbb{PP}$ be finite and non-empty, and let V be the monoid ideal in $(\mathbb{N}^+, \oplus)$ generated by Q. Then $V \in \mathcal{O}^f$ and $\mathcal{A}_V$ is Koszul.*

5.6. **Universality.**

Theorem 5.16. *Let* Γ *be a finite simplicial complex. Then there exists infinitely many* $V \in \mathcal{O}$ *such that* $\Delta(V) \simeq \Gamma$*.*

Proof. Without loss of generality we may assume that Γ is a simplicial complex on the set $\{1, \ldots, n\}$. Let $q_1, \ldots, q_n$ be any set of distinct primes, and define

$$V = \left\{ \prod_{i \in \sigma} q_i \middle| \sigma \in \Gamma \right\}.$$

Then $\Delta(V) \simeq \Gamma$. □

From this result we conclude that it isn't feasible to study the (homological) properties of all truncations $\mathcal{A}_V$, for V finite. In a sequel to this article, we'll restrict ourselves to the special cases $V = [n]$ and $V = [n] \cap \tilde{\mathbb{N}}$.

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