Differential Operators on Hyperplane Arrangements

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Abstract

In this paper we show that if $\mathcal{A}$ is a hyperplane arrangement in k^n, where k is a field of characteristic zero, and if A is the coordinate ring of $\mathcal{A}$, then the ring of k-linear differential operators, $\mathcal{D}(A)$, has a nice decomposition, $\mathcal{D}(A) = \bigoplus \mathcal{D}^{(m)}(A)$, as an A-module. We also show that if $\mathcal{A}$ is generic, then $\mathcal{D}(A)$ is finitely generated as a k-algebra.

1 Introduction

Throughout this paper, we will let k be a field of characteristic zero, n a positive integer, and $R = k[x_1, \ldots, x_n]$.

Let A be a commutative k-algebra. We let $\mathcal{D}(A)$ denote the ring of k-linear differential operators on A. Then $\mathcal{D}(A)$ is an A-module in an obvious way, with $\mathrm{Der}(A)$, the derivations on A, as an A-submodule.

The differential operators, $\mathcal{D}(R)$, on R is the n-th Weyl algebra over k, i.e., the free associative algebra

$$k\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle,$$

modulo the relations

$$x_i x_j - x_j x_i = \partial_i \partial_j - \partial_j \partial_i = 0 \text{ and } \partial_i x_j - x_j \partial_i = \delta_{ij},$$

where δ_{ij} is the Kronecker δ, and where we write ∂_i in place of $\frac{\partial}{\partial x_i}$.

If I is an ideal in R and $A = R/I$, then

$$\mathcal{D}(A) \cong \mathcal{D}(I)/I\mathcal{D}(R), \tag{1}$$

where $\mathcal{D}(I)$ is the subring of $\mathcal{D}(R)$ of operators preserving I, and $I\mathcal{D}(R)$ is the ideal of operators with polynomial coefficients in I. Thus each element in $\mathcal{D}(A)$ may be written on the form

$$\sum_\alpha \overline{f_\alpha} \partial^\alpha,$$

where $f_\alpha \in R$ and $\overline{f_\alpha}$ denotes the image of f_α in A.

The ring $\mathcal{D}(I)$ is an R-submodule of $\mathcal{D}(R)$, and $\mathrm{Der}(I)$, the derivations on R preserving I, is an R-submodule of $\mathcal{D}(I)$.

An operator in $\mathcal{D}(R)$ of the form

$$\sum_{|\alpha|=m} f_\alpha \partial^\alpha \tag{2}$$

will be said to be of homogeneous order m, as will the operator in $\mathcal{D}(A)$, induced by (2).

Let $\mathcal{D}^{(m)}(I)$ and $\mathcal{D}^{(m)}(R)$ be the R-submodules of $\mathcal{D}(I)$ and $\mathcal{D}(R)$, respectively, of operators of homogeneous order m. Then the A-submodule of $\mathcal{D}(A)$ of operators of homogeneous order m will be $\mathcal{D}^{(m)}(A) \cong \mathcal{D}^{(m)}(I)/I\mathcal{D}^{(m)}(R)$.

Suppose $A = R/I$ is the coordinate ring of a hyperplane arrangement, $\mathcal{A}$, in k^n. The aim of this paper is to show that $\mathcal{D}(A)$ has a decomposition,

$$\mathcal{D}(A) = \bigoplus_{m \geq 0} \mathcal{D}^{(m)}(A),$$

as an A-module, and to find explicit, finitely many, k-algebra generators for $\mathcal{D}(A)$, if $\mathcal{A}$ is a generic hyperplane arrangement.

This will be done by first giving the corresponding results for $\mathcal{D}(I)$. From this, and (1), we immediately get our results for $\mathcal{D}(A)$.

Even more may be said about the structure of $\mathcal{D}(A)$. Namely, it follows from a result by Másson, [2, Theorem 2.4], that $\mathcal{D}(A)$ is simple if and only if $\mathcal{A}$ only consists of one hyperplane. In this case A is regular, and much is known about the structure of $\mathcal{D}(A)$.

Let us now outline the contents of this paper.

In Section 2.1 we will define $\mathcal{D}(A)$ for an arbitrary commutative k-algebra A, and state a few of its properties. Induced operators, and the isomorphism (1) are discussed in Section 2.2.

In Section 3 we show some general results on differential operators preserving ideals in R, which will be used later on. In particular we prove that

$$\mathcal{D}(\langle f_1 \cdots f_r \rangle) = \bigcap_{i=1}^{r} \mathcal{D}(\langle f_i \rangle),$$

if $f_1, \ldots, f_r \in R$ are nonconstant and pairwise relatively prime.

Hyperplane arrangements are introduced in Section 4. Our general reference for hyperplane arrangements is [4]. An arrangement of hyperplanes passing through the origin is called a central hyperplane arrangement. When we want to emphasize that all hyperplanes in an arrangement $\mathcal{A}$ need not pass through the origin, we call $\mathcal{A}$ an affine hyperplane arrangement.

A central hyperplane arrangement $\mathcal{A}$ in k^n is called generic if every intersection of less that n hyperplanes in $\mathcal{A}$ have codimension equal to the number of hyperplanes in the intersection.

Section 5 deals with differential operators on hyperplane arrangements. The first part, Section 5.1, deals with affine hyperplane arrangements. We prove that if $I \subseteq R$ defines an affine hyperplane arrangement, then

$$\mathcal{D}(I) = \bigoplus_{m \geq 0} \mathcal{D}^{(m)}(I), \tag{3}$$

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as an R-module. Hence we have

$$\mathcal{D}(A) = \bigoplus_{m \geq 0} \mathcal{D}^{(m)}(A),$$

as an A-module, if A is the coordinate ring of an affine hyperplane arrangement.

Section 5.2 deals only with central hyperplane arrangements. We introduce, for each $m \geq 0$, an operator $\varepsilon_m \in \mathcal{D}(R)$ of homogeneous order m, which preserves any homogeneous ideal in R. Then we show, in Section 5.2.1, how to associate to an ideal I defining a generic hyperplane arrangement, a subset

$$D = \{P_1 \delta_1, \ldots, P_s \delta_s\},$$

of $\mathrm{Der}(I)$, where each δ_i is a derivation with coefficients in k, and each $P_i \in R$. With this set D at hand, we have

$$\mathcal{D}^{(m)}(I) = \sum_{|\alpha|=m} (I : (I : P^\alpha)) \delta^\alpha + R \varepsilon_m, \tag{4}$$

as an R-module, for each $m \geq 0$.

The proof of this uses a method known as deletion and restriction. Given a central hyperplane arrangement $\mathcal{A}$, there is a deleted, $\mathcal{A}'$, and a restricted, $\mathcal{A}''$, hyperplane arrangement. If I, I', and I'' are the defining ideals for these hyperplane arrangements, there is (compare [4, Proposition 4.45]) an exact sequence of the form

$$0 \longrightarrow \mathrm{Der}(I') \longrightarrow \mathrm{Der}(I) \longrightarrow \mathrm{Der}(I''). \tag{5}$$

This was used in [8], where a result (Corollary 3.2) which amounts to (4) was given for the case $m = 1$, i.e., for $\mathrm{Der}(I)$, although without giving a finite set of generators for $\mathrm{Der}(I)$.

We will generalize the sequence (5) to operators of higher homogeneous order, and then follow the ideas used in [8] to prove (4)

Then we show that ε_m is not needed as a generator if the number of hyperplanes is less than or equal to n.

Combining this with (3) and (4) we reach the main result of this section:

If I defines a generic hyperplane arrangement, then

$$\mathcal{D}(I) = \bigoplus_{m \geq 0} \left(\sum_{|\alpha|=m} (I : (I : P^\alpha)) \delta^\alpha + R \varepsilon_m \right), \tag{6}$$

as an R-module, where $R \varepsilon_m \subseteq \sum_{|\alpha|=m} (I : (I : P^\alpha)) \delta^\alpha$, if the number of hyperplanes is less than or equal to n.

We end Section 5.2.1 with an example showing that (6) is not true in general if the hyperplane arrangement defined by I is not generic.

In Section 5.2.2 finally, we will use (6) to show that if I defines a generic hyperplane arrangement, then $\mathcal{D}(I)$, and hence $\mathcal{D}(A)$ if $A = R/I$, is finitely generated as a k-algebra. More precisely, for each multi-index α, the ideal $(I : (I : P^\alpha))$ is principal. For each such ideal we choose a generator, P_α, and then we show that only finitely many of the operators $P_\alpha \delta^\alpha$, together with $x_1, \ldots, x_n$ and $\varepsilon = \varepsilon_1$ are needed to generate $\mathcal{D}(I)$ as a k-algebra.

2 Differential Operators

We begin, in Section 2.1, with the definition of the ring of differential operators on a commutative k-algebra, and give some of its properties. Then, in Section 2.2, we will see what the differential operators on a quotient of a polynomial algebra looks like.

2.1 The Ring of Differential Operators

Let A be a commutative k-algebra. Every element $a \in A$ defines a k-linear endomorphism θ_a of A by $\theta_a(b) = ab$, for $b \in A$. If $\varphi \in \operatorname{End}_k(A)$, we will write $a\varphi$ and φa for $\theta_a\varphi$ and $\varphi\theta_a$, respectively. With this notation, letting A operate by left multiplication gives $\operatorname{End}_k(A)$ an A-module structure. As usual, $[\varphi, \psi]$ denotes the commutator $\varphi\psi - \psi\varphi$.

Definition 2.1. *Set*

$$\mathcal{D}_k^0(A) = \{\theta \in \operatorname{End}_k(A) \mid [\theta, a] = 0 \text{ for all } a \in A\} = \operatorname{End}_A(A) = A,$$

by identifying $a \in A$ with $\theta_a \in \operatorname{End}_k(A)$. For $m \geq 1$, define

$$\mathcal{D}_k^m(A) = \{\theta \in \operatorname{End}_k(A) \mid [\theta, a] \in \mathcal{D}_k^{m-1}(A) \text{ for all } a \in A\}.$$

The ring of differential operators *on A is now defined as*

$$\mathcal{D}_k(A) = \bigcup_{m \geq 0} \mathcal{D}_k^m(A).$$

The order *of an element $\theta \in \mathcal{D}_k(A)$ is the least m such that $\theta \in \mathcal{D}_k^m(A)$.*

This definition gives $\mathcal{D}_k(A)$ the structure of a filtered ring since, by induction,

$$\mathcal{D}_k^m(A)\mathcal{D}_k^l(A) \subseteq \mathcal{D}_k^{m+l}(A) \text{ and } \mathcal{D}_k^m(A) \subseteq \mathcal{D}_k^{m+1}(A),$$

for all m and l. Each $\mathcal{D}_k^m(A)$, and thereby also $\mathcal{D}_k(A)$, is an A-submodule of $\operatorname{End}_k(A)$.

The derivations on A are defined as

$$\operatorname{Der}_k(A) = \{\theta \in \operatorname{End}_k(A) \mid \theta(ab) = \theta(a)b + a\theta(b) \text{ for all } a, b \in A\}.$$

This too is an A-submodule of $\operatorname{End}_k(A)$.

Since there is no risk for confusion we will from now on suppress the reference to the field k in the notation.

The k-subalgebra of $\operatorname{End}(A)$ generated by A and $\operatorname{Der}(A)$ will be denoted by $\Delta(A)$. This too has the structure of a filtered ring, by letting $\Delta^m(A)$ be the A-submodule generated by all products of at most m derivations in $\operatorname{Der}(A)$.

The results in the following lemma can all be found in [3].

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Lemma 2.2.

   (i) $\mathcal{D}^1(A) = A \oplus \mathrm{Der}(A)$.

   (ii) $\Delta(A)$ *is a filtered subring of* $\mathcal{D}(A)$.

   (iii) *If A is regular, then $\mathcal{D}(A) = \Delta(A)$, and their filtrations coincide.*

   (iv) $\mathrm{Der}(R)$ *is a free R-module of rank n, generated by* $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$.

The converse of *(iii)*, that A is regular if $\mathcal{D}(A) = \Delta(A)$, in the case that A is finitely generated as a k-algebra is a well known conjecture by Nakai. It has been shown, in [6], that Nakai's conjecture holds when A is the coordinate ring of a reduced affine variety, all of whose irreducible components are nonsingular.

Part *(iii)* and *(iv)* in the lemma above shows that $\mathcal{D}(R)$ is the n-th Weyl algebra over k. For more details on the Weyl algebra, we refer to [1].

Henceforth, we will write ∂_i for $\frac{\partial}{\partial x_i}$. We will also use multi-indices, i.e.,

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \text{ and } \partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$$

if

$$\alpha = (\alpha_1, \ldots, \alpha_n) \text{ and } \beta = (\beta_1, \ldots, \beta_n).$$

The multi-index with 1 in position i and zeros elsewhere will be denoted by e_i. To further simplify notation, we will write

$$|\alpha| = \alpha_1 + \ldots + \alpha_n,$$

and

$$\alpha! = \alpha_1! \cdots \alpha_n!$$

An operator in $\mathcal{D}(R)$ of the form

$$\theta = x^\alpha \partial^\beta$$

will be called a monomial.

We will let $*$ denote the left $\mathcal{D}(A)$-module operation on A.

The following proposition appears as Proposition 1.2 in chapter one of [1].

Proposition 2.3. *Any operator $\theta \in \mathcal{D}(R)$ may be uniquely written on the form*

$$\theta = \sum c_{\alpha,\beta} x^\alpha \partial^\beta,$$

where $c_{\alpha,\beta} \in k$. That is, the monomials $\{x^\alpha \partial^\beta\}$ is a basis for $\mathcal{D}(R)$ as a vector space over k.

It is easy to see that an operator $\theta \in \mathcal{D}(R)$ is completely determined by its action on the monomials x^α, with $|\alpha| \leq$ order of θ. We formulate this as a lemma, whose proof will be omitted.

Lemma 2.4. *Let θ and ψ be operators of order m in $\mathcal{D}(R)$. If*

$$\theta * x^\alpha = \psi * x^\alpha,$$

for all α with $|\alpha| \leq m$, then $\theta = \psi$.

2.2 Induced Operators

The results in this section may all be found in [3].

Let $A = R/I$, where I is an ideal in R. For $f \in R$, we let $\overline{f}$ denote the image of f in A under the canonical projection.

An element $\theta \in \mathrm{End}(R)$ induces a well defined element $\overline{\theta} \in \mathrm{End}(A)$ by

$$\overline{\theta}(\overline{f}) = \overline{\theta(f)}$$

if and only if $\theta(I) \subseteq I$. In this case, $\overline{\theta} = 0$ if and only if $\theta(R) \subseteq I$.

Furthermore, if $\theta \in \mathcal{D}(R)$ and $\theta * I \subseteq I$, then $\overline{\theta} \in \mathcal{D}(A)$, and the order of $\overline{\theta}$ is less than or equal to the order of θ. Let

$$\mathcal{D}(I) = \{\theta \in \mathcal{D}(R) \mid \theta * I \subseteq I\}$$

and

$$I\mathcal{D}(R) = \left\{\sum f_\alpha \partial^\alpha \in \mathcal{D}(R) \mid f_\alpha \in I \text{ for all } f_\alpha\right\}.$$

Then, if $\theta \in \mathcal{D}(R)$, we have $\theta * R \subseteq I$ if and only if $\theta \in I\mathcal{D}(R)$. It is clear that $I\mathcal{D}(R)$ is a right ideal in $\mathcal{D}(R)$, but it is also true that $\mathcal{D}(I)$ is the largest subring of $\mathcal{D}(R)$ in which $I\mathcal{D}(R)$ is a twosided ideal.

We may now formulate the following important result.

Theorem 2.5. *Let $A = R/I$. Then*

$$\mathcal{D}(A) \cong \mathcal{D}(I)/I\mathcal{D}(R),$$

and this isomorphism preserves the filtrations.

Thus any element in $\mathcal{D}(A)$ may be written as

$$\sum_\alpha \overline{f_\alpha} \partial^\alpha,$$

where $\sum f_\alpha \partial^\alpha \in \mathcal{D}(R)$, and where we mean that

$$\left(\sum_\alpha \overline{f_\alpha} \partial^\alpha\right) * \overline{g} = \sum_\alpha \overline{f_\alpha \partial^\alpha * g},$$

for all $\overline{g} \in A$.

3 General Results on Operators Preserving Ideals in R

The results in this section will be used in Section 5.

Suppose that $I = \langle f_1, \dots, f_r \rangle$ is an ideal in R, and that $\theta \in \mathcal{D}(R)$. Then θ belongs to $\mathcal{D}(I)$ if and only if $\theta * x^\alpha f_j \in I$, for all α and $j = 1, \dots, r$.

With Lemma 2.4 in mind we would expect that $\theta * x^\alpha f_j \in I$ for a finite number of the x^α should suffice to show that $\theta \in \mathcal{D}(I)$. This is indeed so, but to prove this we first need the following lemma.

Lemma 3.1. *Let $I = \langle f_1, \ldots, f_r \rangle$ be an ideal in R, and let $\theta \in \mathcal{D}(R)$. Then $\theta \in \mathcal{D}(I)$ if and only if*

(i) $[\theta, x_i] * I \subseteq I$ for $i = 1, \ldots, n$, and

(ii) $\theta * f_j \in I$ for $j = 1, \ldots, r$.

Proof. This is Proposition 2.10 *(i)* and *(ii)* in [5], from which we have taken the proof.

It is obvious that *(i)* and *(ii)* holds if $\theta \in \mathcal{D}(I)$. To prove the converse, assume $\theta \in \mathcal{D}(R)$ is such that *(i)* and *(ii)* holds. We must show that $\theta * x^\alpha f_j \in I$ for all α and for $j = 1, \ldots, r$. This is done by induction on $|\alpha|$.

The case $|\alpha| = 0$ is just assumption *(ii)* above, so assume that $\theta * x^\alpha f_j \in I$ for $|\alpha| = m$, and let $\beta = \alpha + e_i$, for some $i \in \{1, \ldots, n\}$. Then

$$\theta * x^\beta f_j = \theta * x_i x^\alpha f_j = [\theta, x_i] * x^\alpha f_j + x_i \theta * x^\alpha f_j \in I,$$

since $[\theta, x_i] * x^\alpha f_j \in I$ by *(i)*, and $x_i \theta * x^\alpha f_j \in I$ by the induction hypothesis. $\square$

Proposition 3.2. *Let $I = \langle f_1, \ldots, f_r \rangle$ be an ideal in R, and let $\theta \in \mathcal{D}(R)$ be an operator of order $m \geq 1$. Then $\theta \in \mathcal{D}(I)$ if and only if $\theta * x^\alpha f_j \in I$ for $|\alpha| \leq m - 1$ and $j = 1, \ldots, r$.*

Proof. It is clear that $\theta * x^\alpha f_j \in I$ if $\theta \in \mathcal{D}(I)$, so it remains to prove that $\theta \in \mathcal{D}(I)$ if $\theta * x^\alpha f_j \in I$ for $|\alpha| \leq m - 1$ and $j = 1, \ldots, r$. To do so, we first observe that any element in R preserves the ideal I, so we may without loss of generality assume that the order zero component of θ is zero. We will now proceed by induction on m. If $m = 1$, then $\theta \in \mathrm{Der}(R)$ and the assumption is that $\theta * f_j \in I, j = 1, \ldots, r$. Hence

$$\theta * h f_j = (\theta * h) f_j + h \theta * f_j \in I,$$

for every $h \in R$, so $\theta * I \subseteq I$.

Now assume that the proposition is true for some $m \geq 1$ and that $\theta \in \mathcal{D}(R)$ is of order $m + 1$ and such that $\theta * x^\alpha f_j \in I$ for $|\alpha| \leq m$ and $j = 1, \ldots, r$. Then $\alpha = \mathbf{0}$ shows $\theta * f_j \in I$, $j = 1, \ldots, r$. For any β with $|\beta| \leq m - 1$ we have $|\beta + e_i| \leq m$, $i = 1, \ldots, n$, so

$$I \ni \theta * x^{\beta + e_i} f_j = \theta * x_i x^\beta f_j = [\theta, x_i] * x^\beta f_j + x_i \theta * x^\beta f_j,$$

which shows that $[\theta, x_i] * x^\beta f_j \in I$ for $i = 1, \ldots, n, |\beta| \leq m - 1$, and $j = 1, \ldots, r$. But $[\theta, x_i]$ is an operator of order m, so $[\theta, x_i] * I \subseteq I$, $i = 1, \ldots, n$, by the induction hypothesis. Thus $\theta * I \subseteq I$ by lemma 3.1. $\square$

Lemma 3.3. *Let $\{I_j\}_{j \in J}$ be any family of ideals in R, and let $I = \bigcap_{j \in J} I_j$. Then*

$$\bigcap_{j \in J} \mathcal{D}(I_j) \subseteq \mathcal{D}(I).$$

Proof. Suppose $\theta \in \bigcap \mathcal{D}(I_j)$. For every $j \in J$ we have $I \subseteq I_j$, so $\theta * I \subseteq I_j$. It follows that $\theta * I \subseteq I$. $\square$

The converse of Lemma 3.3 is not true in general, as the following example shows.

Example 3.4. *Let $I_1 = \langle x \rangle$ and $I_2 = \langle x^2, xy, y^2 \rangle$ be ideals in $k[x, y]$. Then $I = I_1 \cap I_2 = \langle x^2, xy \rangle$. By Proposition 3.2, the operator $x\partial_x\partial_y - \partial_y$ preserves I, but*

$$(x\partial_x\partial_y - \partial_y) * y^2 = -2y \notin \langle x^2, xy, y^2 \rangle,$$

so it does not belong to $\mathcal{D}(I_1) \cap \mathcal{D}(I_2)$.

There are cases in which the converse of Lemma 3.3 is valid, as the next result shows. It concerns the ideal defining a union of (possibly nonreduced) hypersurfaces. It is well known for the special case of derivations, see e.g. Proposition 4.8 in [4].

Theorem 3.5. *Suppose $f_1, \dots, f_r \in R$ are nonconstant and pairwise relatively prime. Then*

$$\mathcal{D}(\langle f_1 \cdots f_r \rangle) = \bigcap_{i=1}^{r} \mathcal{D}(\langle f_i \rangle).$$

Proof. If $r = 1$ there is nothing to prove, so we may assume that $r > 1$. Since the f_i:s are pairwise relatively prime,

$$\bigcap_{i=1}^{r} \langle f_i \rangle = \langle f_1 \cdots f_r \rangle,$$

so Lemma 3.3 shows that $\bigcap \mathcal{D}(\langle f_i \rangle) \subseteq \mathcal{D}(\langle f_1 \cdots f_r \rangle)$.

To show the opposite inclusion, we observe that the polynomials

$$f = f_1 \cdots f_{r-1} \quad \text{and} \quad g = f_r$$

are also nonconstant and relatively prime, so the inclusion follows from induction on r if we can prove it for two polynomials. Therefore assume that $f, g \in R$ are nonconstant and relatively prime. We must show that

$$\mathcal{D}(\langle fg \rangle) \subseteq \mathcal{D}(\langle f \rangle) \cap \mathcal{D}(\langle g \rangle). \tag{7}$$

This will be done by induction on the order of $\theta \in \mathcal{D}(\langle fg \rangle)$. If θ is of order zero, i.e., $\theta \in R$, then θ preserves any ideal in R.

Now assume that the inclusion (7) is true for operators of some order $m \geq 0$, and let $\theta \in \mathcal{D}^{m+1}(\langle fg \rangle)$.

Then $[\theta, g] \in \mathcal{D}^m(R)$, and for every $h \in R$ we have

$$[\theta, g] * hfg = \theta * hfg^2 - g\theta * hfg \in \langle fg \rangle.$$

Thus $[\theta, g] \in \mathcal{D}^m(\langle fg \rangle)$, and it follows from the induction hypothesis that $[\theta, g] \in \mathcal{D}(\langle f \rangle)$. Thus, for any $h \in R$,

$$\langle f \rangle \ni [\theta, g] * hf = \theta * hfg - g\theta * hf.$$

Since $\theta * hfg \in \langle fg \rangle \subseteq \langle f \rangle$, this implies that $g\theta * hf \in \langle f \rangle$. But f and g are relatively prime, so we must have $\theta * hf \in \langle f \rangle$. Repeating the same argument with f and g interchanged shows that we also have $\theta \in \mathcal{D}(\langle g \rangle)$. $\qquad\square$

4 Hyperplane Arrangements

Definition 4.1. *We define a* hyperplane, H, *in k^n to be an affine subspace of dimension $(n-1)$. A* hyperplane arrangement, $\mathcal{A}$, *is a finite set of distinct hyperplanes in k^n.*

Suppose $\mathcal{A} = \bigcup H_i$. Each hyperplane $H_i \in \mathcal{A}$ is the zeroset of a polynomial p_i in R of degree one, defined up to a constant nonzero multiple. We make the convention that whenever we say that polynomials $p_1, \ldots, p_r$ define a hyperplane arrangement, we mean that they are pairwise relatively prime of degree one, so that H_i and H_j, the hyperplanes defined by p_i and p_j, are distinct if $i \neq j$.

Definition 4.2. *The product*

$$P = P(\mathcal{A}) = \prod_{H_i \in \mathcal{A}} p_i$$

is called a defining polynomial *for $\mathcal{A}$. We agree that $P = 1$ is the defining polynomial for the empty hyperplane arrangement. The ideal $I = \langle P \rangle$ in R will be called the* defining ideal *for $\mathcal{A}$.*

Unless otherwise explicitly stated, all hyperplane arrangements will be assumed to be nonempty.

Definition 4.3. *We call $\mathcal{A}$* centerless *if $\bigcap_{H \in \mathcal{A}} H = \emptyset$. If $\bigcap_{H \in \mathcal{A}} H \neq \emptyset$, we call $\mathcal{A}$* centered. *If $\mathcal{A}$ is centered, coordinates may be chosen so that each hyperplane contains the origin. In this case we call $\mathcal{A}$* central. *We agree that the empty hyperplane arrangement is central.*

From now on, we will simply use "arrangement" in place of "hyperplane arrangement". When we do not want to specify if the arrangement under consideration is centerless or not, we will use the term "affine arrangement".

The only (nonempty) central arrangement in k consists of the hyperplane $\{0\}$. Therefore, unless otherwise explicitly stated, we will always assume that $n \geq 2$.

Definition 4.4. *Let $\mathcal{A}$ be a central arrangement in k^n. The* intersection lattice, $L(\mathcal{A})$, *is the set of all intersections of elements of $\mathcal{A}$. We agree that $k^n \in L(\mathcal{A})$, considered as the intersection of the empty family. For $X \in L(\mathcal{A})$ we put $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}$ and we denote by $r(X)$ the codimension of X in k^n. We say that $\mathcal{A}$ is* generic *if for every $X \in L(\mathcal{A})$ with $r(X) < n$, we have $|\mathcal{A}_X| = r(X)$, where $|\mathcal{A}_X|$ denotes the cardinality of the set $\mathcal{A}_X$.*

Example 4.5. *For $X \in L(\mathcal{A})$, we have $r(X) = 0$ if and only if $X = k^n$, and $r(X) = 1$ if and only if $X = H \in \mathcal{A}$. Since $|\mathcal{A}_{k^n}| = |\emptyset| = 0$ and $|\mathcal{A}_H| = |\{H\}| = 1$, it follows that*

(i) *The empty arrangement is generic for any n,*

(ii) *Any central arrangement in the plane is generic.*

(iii) *In k^3, any central arrangement of less than three planes is generic. A central arrangement of three or more planes is generic if and only if any subset of three planes only intersect at the origin.*

Since being generic is only defined for central arrangements, we will use "generic arrangement" in place of "generic central arrangement".

5 Differential Operators on Hyperplane Arrangements

We will begin our discussion of differential operators on arrangements with a few results which are valid for affine arrangements. Then, in Section 5.2, we will limit our discussion to central arrangements.

5.1 Affine Arrangements

We say that an operator in $\mathcal{D}(R)$ of the form

$$\sum_{|\alpha|=m} f_\alpha \partial^\alpha$$

is of *homogeneous order m*.

From Proposition 2.3 it follows that any $\theta \in \mathcal{D}(R)$ may be uniquely written as a sum,

$$\theta = \theta_m + \ldots + \theta_0,$$

of components of homogeneous order, i.e.,

$$\theta_i = \sum_{|\alpha|=i} f_\alpha \partial^\alpha,$$

for $i = 0, \ldots, m$.

Lemma 5.1. *Suppose $p \in R$ is a polynomial of degree one, and that $\theta \in \mathcal{D}(R)$. Then*

$$\theta \in \mathcal{D}(\langle p \rangle) \text{ if and only if } \theta_i \in \mathcal{D}(\langle p \rangle), \text{ for } i = 0, \ldots, m,$$

where

$$\theta = \theta_m + \ldots + \theta_0,$$

is the decomposition of θ into components of homogeneous order.

Proof. It is trivial that $\theta \in \mathcal{D}(\langle p \rangle)$ if the same thing holds for each θ_i. To prove the other implication, we first observe that the homogeneous component θ_0 of θ belongs to R, so it preserves any ideal. Thus $\theta_m + \ldots + \theta_1 \in \mathcal{D}(\langle p \rangle)$. Then

$$(\theta_m + \ldots + \theta_1) * p \in \langle p \rangle,$$

but $\deg p = 1$, so

$$(\theta_m + \ldots + \theta_1) * p = \theta_1 * p \in \langle p \rangle.$$

By Proposition 3.2, this means that $\theta_1 \in \mathcal{D}(\langle p \rangle)$. This in turn also means that $\theta_m + \ldots + \theta_2 \in \mathcal{D}(\langle p \rangle)$. In particular, we have

$$(\theta_m + \ldots + \theta_2) * x_j p \in \langle p \rangle,$$

for $j = 1, \ldots, n$. Now, $\deg x_j p = 2$, so

$$(\theta_m + \ldots + \theta_2) * x_j p = \theta_2 * x_j p \in \langle p \rangle,$$

and again Proposition 3.2 shows that $\theta_2 \in \mathcal{D}(\langle p \rangle)$. Continuing like this we see that each $\theta_i \in \mathcal{D}(\langle p \rangle)$, which concludes the proof. $\square$

Notation. If I is an ideal in R we will write $\mathcal{D}^{(m)}(R)$ and $\mathcal{D}^{(m)}(I)$ respectively to denote the operators in $\mathcal{D}(R)$ and $\mathcal{D}(I)$ of homogeneous order m. If $A = R/I$, we say that an operator in $\mathcal{D}(A)$ is of homogeneous order m if it is induced by an element in $\mathcal{D}^{(m)}(I)$. Thus, writing $\mathcal{D}^{(m)}(A)$ for the operators in $\mathcal{D}(A)$ of homogeneous order m, we have $\mathcal{D}^{(m)}(A) \cong \mathcal{D}^{(m)}(I)/I\mathcal{D}^{(m)}(R)$.

From Lemma 2.2 *(i)* we know that $\mathcal{D}^1(R) = R \oplus \mathrm{Der}(R)$, which implies that

$$\mathcal{D}^1(I) = R \oplus \mathrm{Der}(I) = \mathcal{D}^{(0)}(I) \oplus \mathcal{D}^{(1)}(I),$$

for any ideal I in R. For $m > 1$ though, we generally just have a strict inclusion

$$\bigoplus_{l \leq m} \mathcal{D}^{(l)}(I) \subsetneq \mathcal{D}^m(I).$$

Example 5.2. *Let $I = \langle x^r \rangle \subseteq k[x]$. Then we have, e.g.,*

$$x\partial^2 - (r-1)\partial \in \mathcal{D}(\langle x^r \rangle),$$

but neither $x\partial^2$ nor ∂ belong to $\mathcal{D}(\langle x^r \rangle)$ if $r > 1$, so

$$\bigoplus_{l \leq 2} \mathcal{D}^{(l)}(I) \subsetneq \mathcal{D}^2(I),$$

in this case.

This is what makes the following proposition important.

Proposition 5.3. *If $I = \langle p_1 \cdots p_r \rangle$ defines an affine arrangement, then*

$$\mathcal{D}(I) = \bigoplus_{m \geq 0} \mathcal{D}^{(m)}(I),$$

as an R-module.

Proof. We need to show that $\mathcal{D}(I) \subseteq \bigoplus \mathcal{D}^{(m)}(I)$. To do so, assume that $\theta = \theta_m + \ldots + \theta_0 \in \mathcal{D}(I)$, where θ_i is the component of θ of homogeneous order i. We must show that each θ_i also belongs to $\mathcal{D}(I)$.

By Theorem 3.5, $\theta \in \mathcal{D}(\langle p_j \rangle)$, for $j = 1, \ldots, r$. Thus, by Lemma 5.1, it follows that $\theta_i \in \mathcal{D}(\langle p_j \rangle)$, for each $i = 0, \ldots, m$, and $j = 1, \ldots, r$. Hence, by Theorem 3.5 again, $\theta_i \in \mathcal{D}(I)$, for each i. $\square$

Corollary 5.4. *If A is the coordinate ring of an affine arrangement, then*

$$\mathcal{D}(A) = \bigoplus_{m \geq 0} \mathcal{D}^{(m)}(A),$$

as an A-module.

In view of Proposition 5.3, to find R-module generators for $\mathcal{D}(I)$, where I defines an affine arrangement, we may proceed by finding such generators for $\mathcal{D}^{(m)}(I)$, one m at a time.

It will be convenient to study the case of a single hyperplane in some detail, before turning to more general arrangements.

Lemma 5.5. *Let $p \in R$ be a polynomial of degree one. Then $\mathcal{D}^{(m)}(\langle p \rangle)$ is generated, as an R-module, by the homogeneous components of highest order of products of m derivations in $\mathrm{Der}(\langle p \rangle)$, together with $\{p\partial^{\alpha} \mid |\alpha| = m\}$.*

Proof. In this case $A = R/\langle p \rangle$ is regular. Therefore, by Lemma 2.2 *(iii)*, $\mathcal{D}(A)$ is generated, as a k-algebra, by A and $\mathrm{Der}(A)$, i.e. $\mathcal{D}^m(A)$ is generated by $(\mathrm{Der}(A))^j$, $j \leq m$, as an A-module.

By Theorem 2.5 the isomorphism $\mathcal{D}(A) \cong \mathcal{D}(\langle p \rangle)/\langle p \rangle \mathcal{D}(R)$ preserves the order filtration on $\mathcal{D}(A)$, so

$$\mathrm{Der}(A) \cong \mathrm{Der}(\langle p \rangle)/\langle p \rangle \mathrm{Der}(R).$$

Hence $\mathcal{D}^m(\langle p \rangle)$ is generated by $(\mathrm{Der}(\langle p \rangle))^j + \langle p \rangle \mathcal{D}^m(R)$, $j \leq m$, as R-module. Thus it follows from Proposition 5.3 that $\mathcal{D}^{(m)}(\langle p \rangle)$ is generated by the homogeneous components of highest order of products of m derivations in $\mathrm{Der}(\langle p \rangle)$, together with $\{p\partial^{\alpha} \mid |\alpha| = m\}$. $\qquad\square$

Let us now describe the R-module $\mathrm{Der}(\langle p \rangle)$. In the following general lemma we do not require that the polynomial f is of degree one.

Lemma 5.6. *Let f be any polynomial in R such that there is an element $\delta \in \mathrm{Der}(R)$ with $\delta * f = af$, for some $a \in k \setminus \{0\}$. Let $\mathcal{N}$ be the R-module of derivations annihilating f. Then*

$$\mathrm{Der}(\langle f \rangle) = R\delta + \mathcal{N}.$$

Proof. If $\theta \in \mathrm{Der}(R)$, we know from Proposition 3.2 that $\theta \in \mathrm{Der}(\langle f \rangle)$ if and only if $\theta * f \in \langle f \rangle$. Since this holds for δ it follows immediately that $R\delta + \mathcal{N} \subseteq \mathrm{Der}(\langle f \rangle)$. To show the opposite inclusion, assume $\theta \in \mathrm{Der}(\langle f \rangle)$, say $\theta * f = gf$, $g \in R$. Then

$$\theta * f = gf = \frac{g}{a}\delta * f,$$

so

$$\left(\theta - \frac{g}{a}\delta\right) \in \mathcal{N}.$$

This shows that $\theta \in R\delta + \mathcal{N}$. $\qquad\square$

Suppose $\delta = \sum g_i \partial_i$ is such that $\delta * f = af$, as in the lemma. Then

$$f = \frac{1}{a}\delta * f = \frac{1}{a}\sum g_i \partial_i * f \in \langle \partial_1 * f, \dots, \partial_n * f \rangle.$$

Conversely, if $f \in \langle \partial_1 * f, \dots, \partial_n * f \rangle$, say $f = \sum g_i \partial_i * f$, then

$$af = a\sum g_i \partial_i * f = \delta * f,$$

with $\delta = a\sum g_i \partial_i$. Hence a derivation as in the lemma exists if and only if $f \in \langle \partial_1 * f, \dots, \partial_n * f \rangle$. This is obviously the case if $\langle \partial_1 * f, \dots, \partial_n * f \rangle = R$, which holds for any f of degree one, i.e., for any polynomial defining a hyperplane.

Recall that a polynomial $f \in R$ is called quasihomogeneous if there are integers $t_1, \dots, t_n \geq 0$, not all zero, such that f is homogeneous with respect to $\deg x^{\alpha} = t_1\alpha_1 + \dots + t_n\alpha_n$. We have $f \in \langle \partial_1 * f, \dots, \partial_n * f \rangle$ for any polynomial f which, possibly after a translation, is quasihomogeneous.

When p is a polynomial of degree one, the module $\mathcal{N}$ in Lemma 5.6 is particularly nice.

Definition 5.7. *Let V be the k-vector space*

$$V = \sum_{i=1}^{n} k\partial_i,$$

and, if H is a hyperplane through the origin in k^n, define $V_H \subseteq V$ to be the $(n-1)$-dimensional subspace

$$V_H = \left\{ \sum_{i=1}^{n} b_i\partial_i \mid (b_1, \dots, b_n) \in H \right\}.$$

If p is any polynomial of degree one, then $p - p(0)$ is a nonzero linear form, which is equal to p in case p is already homogeneous, so $p - p(0)$ defines a hyperplane through the origin.

Lemma 5.8. *Let p be a polynomial of degree one, and let $\mathcal{N}$ be the module of derivations annihilating p. Let H be the hyperplane through the origin defined by $p - p(0)$. Then*
$$\mathcal{N} = RV_H.$$

Proof. Suppose $p = a_0 + \sum a_i x_i$. A vector $(b_1, \dots, b_n) \in k^n$ lies in H if and only if $\sum a_i b_i = 0$, i.e., if and only if $(b_1, \dots, b_n)$ is a syzygy of $(a_1, \dots, a_n)$. In fact, the vectors in H generate the whole R-module of syzygies of $(a_1, \dots, a_n)$.

On the other hand, a derivation $\delta = \sum f_i\partial_i \in \mathrm{Der}(R)$ lies in $\mathcal{N}$ if and only if $0 = \delta * p = \sum f_i a_i$, i.e., if and only if $(f_1, \dots, f_n)$ is a syzygy of $(a_1, \dots, a_n)$. Hence $\mathcal{N} = RV_H$. $\qquad\square$

Thus, let p be a polynomial of degree one, and H the hyperplane defined by $p - p(0)$. If we choose a basis $\{\delta_1, \dots, \delta_{n-1}\}$ for the vector space V_H in Definition 5.7, we may extend this to a basis $\{\delta_1, \dots, \delta_n\}$ for V.

Since δ_n does not lie in the span of $\delta_1, \dots, \delta_{n-1}$ it cannot annihilate p. Whence $p\delta_n \in \mathrm{Der}(R)$ meets the requirements of Lemma 5.6, and it follows that $\mathrm{Der}(\langle p \rangle)$ is generated by $\{\delta_1, \dots, \delta_{n-1}, p\delta_n\}$.

We may now improve on Lemma 5.5:

Proposition 5.9. *Suppose $p \in R$ is a polynomial of degree one. Let H be the hyperplane through the origin defined by the polynomial $p - p(0)$, and let $M = \{\delta_1, \dots, \delta_n\}$ be a basis for V such that $\{\delta_1, \dots, \delta_{n-1}\}$ is a basis for the subspace V_H from Definition 5.7. Then*

$$\mathcal{D}^{(m)}(\langle p \rangle) = \sum_{\substack{|\alpha|=m \\ \alpha_n=0}} R\delta^\alpha + \sum_{\substack{|\alpha|=m \\ \alpha_n>0}} Rp\delta^\alpha,$$

as an R-module.

Proof. We know that $\mathrm{Der}(\langle p \rangle)$ is generated by $\{\delta_1, \dots, \delta_{n-1}, p\delta_n\}$. Thus, by Lemma 5.5, $\mathcal{D}^{(m)}(\langle p \rangle)$ is generated by the homogeneous components of highest order of products of m of these derivations, together with $\{p\partial^\beta \mid |\beta| = m\}$.

The homogeneous component of highest order of $(p\delta_n)^l$ is $p^l\delta_n^l$. Since the derivations $\delta_1, \dots, \delta_n$ belong to V they all commute with each other and since

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$\delta_i * p = 0$ for $i = 1, \ldots, n-1$, it follows that $\delta_1, \ldots, \delta_{n-1}$ commute with $p\delta_n$. Thus the homogeneous component of highest order of a product of the derivations $\delta_1, \ldots, \delta_{n-1}, p\delta_n$ is of the form $p^{\alpha_n}\delta^\alpha$. Any such element is clearly a multiple of one of the proposed generators.

Since $\{\delta_1, \ldots, \delta_n\}$ is a basis for V it follows that any ∂^β with $|\beta| = m$ may be expressed as a k-linear combination of $\{\delta^\alpha \mid |\alpha| = m\}$, so any element of the form $p\partial^\beta$, $|\beta| = m$, is an R-linear combination of the proposed generators. $\quad\square$

We end this section with a few examples where we use the tools we have developed so far to find the differential operators on some affine arrangements.

Example 5.10. *Suppose that $I = \langle p_1 \cdots p_r \rangle$ defines an affine arrangement of parallel hyperplanes. Then there is a nonzero linear form p such that*

$$p_i = p + a_i, \; i = 1, \ldots, r,$$

where $a_i \in k$, and $a_i \neq a_j$ if $i \neq j$.

Let H be the hyperplane through the origin defined by p, and let $\mathcal{N}_i$ be the R-module of derivations annihilating p_i, for $i = 1, \ldots, r$. Then, by Lemma 5.8,

$$\mathcal{N}_1 = \ldots = \mathcal{N}_r = RV_H,$$

so letting $M = \{\delta_1, \ldots, \delta_n\}$ be a basis for V such that $\{\delta_1, \ldots, \delta_{n-1}\}$ is a basis for V_H, it follows from Proposition 5.9 that

$$\mathcal{D}^{(m)}(\langle p_i \rangle) = \sum_{\substack{|\alpha|=m \\ \alpha_n=0}} R\delta^\alpha + \sum_{\substack{|\alpha|=m \\ \alpha_n>0}} Rp_i\delta^\alpha,$$

for $i = 1, \ldots, r$.

The set $\{\delta^\alpha \mid |\alpha| = m\}$ is a basis, and the p_i:s are pairwise relatively prime. Thus, by Theorem 3.5,

$$\mathcal{D}^{(m)}(I) = \bigcap_{i=1}^r \mathcal{D}^{(m)}(\langle p_i \rangle) = \sum_{\substack{|\alpha|=m \\ \alpha_n=0}} R\delta^\alpha + \sum_{\substack{|\alpha|=m \\ \alpha_n>0}} I\delta^\alpha.$$

Let $A = R/I$. Then Proposition 5.3 and Theorem 2.5 give that

$$\mathcal{D}(A) = A[\delta_1, \ldots, \delta_{n-1}],$$

where the δ_i:s commute with each other, but not with elements in the coefficient ring A. In this case, $\mathcal{D}(A)$ is an Ore extension of A.

Remark. After a linear change of coordinates we could have assumed that $p = x_1$, so that $P = p_1 \cdots p_r$ would be a polynomial of degree r in one variable, with r distinct zeros in k. Therefore we could have reached the result in Example 5.10 by other, well known, methods. However, this example illustrates how our results, in particular Theorem 3.5 and Proposition 5.3, may be used.

Example 5.11. *Suppose that $\mathcal{A}$ is an affine arrangement of three lines in the plane, defined by $I = \langle p_1 p_2 p_3 \rangle \subseteq k[x, y] = R$, and let $A = R/I$. There are four possibilities:*

(i) *Three parallel lines.*

   This case follows from Example 5.10.

(ii) *Two parallel lines intersecting a third.*

   Suppose that the lines defined by p_1 and p_2 are parallel to each other, but not to the line defined by p_3.

   Let $M = \{\delta_1, \delta_2\}$ be a basis for $V = k\partial_x + k\partial_y$ such that

   $$\delta_1 * p_1 = \delta_1 * p_2 = \delta_2 * p_3 = 0,$$

   but $\delta_1 * p_3, \delta_2 * p_1$, and $\delta_2 * p_2$ are all nonzero. Then

   $$\mathcal{D}^{(m)}(\langle p_1 p_2 \rangle) = R\,\delta_1^m + \sum_{j=0}^{m-1} R\,p_1 p_2 \delta_1^j \delta_2^{m-j},$$

   by Example 5.10, and

   $$\mathcal{D}^{(m)}(\langle p_3 \rangle) = \sum_{j=1}^{m} R\,p_3 \delta_1^j \delta_2^{m-j} + R\,\delta_2^m,$$

   by Proposition 5.9. As in Example 5.10, the set $\{\delta_1^j \delta_2^{m-j} \mid j = 0, \ldots, m\}$ is a basis, and $p_1 p_2$ is relatively prime to p_3, so from Theorem 3.5 it follows that

   $$\mathcal{D}^{(m)}(I) = \mathcal{D}^{(m)}(\langle p_1 p_2 \rangle) \cap \mathcal{D}^{(m)}(\langle p_3 \rangle)$$

   $$= R\,p_3 \delta_1^m + R\,p_1 p_2 \delta_2^m + \sum_{j=1}^{m-1} I \delta_1^j \delta_2^{m-j},$$

   for $m \geq 1$. Hence, again by Proposition 5.3 and Theorem 2.5,

   $$\mathcal{D}(A) = A + \bigoplus_{m \geq 1} \left(A\overline{p_3}\,\delta_1^m + A\overline{p_1 p_2}\,\delta_2^m \right).$$

(iii) *Three nonparallel lines with no common point of intersection.*

   After a translation we may assume that two of the lines intersect at the origin. Then, after a linear change of coordinates, we may assume that the three lines are defined by the ideal $I = \langle xy(x - y + 1) \rangle$.

   With $M = \{\partial_x, \partial_y\}$ it follows from Proposition 5.9, that

   $$\mathcal{D}^{(m)}(\langle x \rangle) = \sum_{j=1}^{m} Rx\partial_x^j \partial_y^{m-j} + R\partial_y^m,$$

   and

   $$\mathcal{D}^{(m)}(\langle y \rangle) = R\partial_x^m + \sum_{j=0}^{m-1} Ry\partial_x^j \partial_y^{m-j},$$

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so, by Theorem 3.5, we have

$$\mathcal{D}^{(m)}(\langle xy \rangle) = R\,x\partial_x^m + \sum_{j=1}^{m-1} R\,xy\partial_x^j\partial_y^{m-j} + R\,y\partial_y^m.$$

From Proposition 5.9 it follows, with $M = \{\partial_x + \partial_y, \partial_x\}$, that

$$\mathcal{D}^{(m)}(\langle x - y + 1 \rangle) = R\,(\partial_x + \partial_y)^m + \sum_{j=1}^{m} R\,(x - y + 1)\partial_x^j(\partial_x + \partial_y)^{m-j}$$

$$= R\,(\partial_x + \partial_y)^m + \sum_{j=1}^{m} R\,(x - y + 1)\partial_x^j\partial_y^{m-j}.$$

Applying Theorem 3.5 again gives

$$\begin{aligned}
\mathcal{D}^{(m)}(I) =&\mathcal{D}^{(m)}(\langle xy \rangle) \cap \mathcal{D}^{(m)}(\langle x - y + 1 \rangle)\\
=&R\,x(x - y + 1)\partial_x^m + R\,xy(\partial_x + \partial_y)^m + R\,y(x - y + 1)\partial_y^m\\
&+ \sum_{j=1}^{m-1} I\partial_x^j\partial_y^{m-j},
\end{aligned}$$

for $m \geq 1$. Thus

$$\begin{aligned}
\mathcal{D}(A) =&A\\
&+ \bigoplus_{m \geq 1}\left(A\overline{x(x - y + 1)}\partial_x^m + A\overline{xy}(\partial_x + \partial_y)^m + A\overline{y(x - y + 1)}\partial_y^m\right).
\end{aligned}$$

(iv) Three lines intersecting in one point.

After a translation we may assume that the common point of intersection is the origin. This case will follow from Corollary 5.31 in Section 5.2.1.

5.2 Central Arrangements

Let us now concentrate on central arrangements. In this case, if $\mathcal{A}$ is defined by $p_1, \ldots, p_r$, each p_i is a nonzero linear form, and the defining polynomial $P(\mathcal{A})$ is a homogeneous polynomial whose degree is the cardinality of $\mathcal{A}$.

There is a derivation preserving any homogeneous ideal, namely the Euler derivation,

$$\varepsilon = \sum_{i=1}^{n} x_i\partial_i.$$

This derivation is easily seen to have the property

$$\varepsilon * f = rf,$$

if f is homogeneous of degree r. In particular, ε preserves any ideal defining a central arrangement. By Lemma 5.5, we need to know what the component of highest order of a power of the Euler derivation looks like.

Let $h = \sum_{i=1}^{n} x_i y_i \in k[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Mapping y_i to ∂_i gives ε. For $m \in \mathbb{N}$ we have $h^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} x^\alpha y^\alpha$, and again mapping y_i to ∂_i gives an operator

$$\varepsilon_m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} x^\alpha \partial^\alpha.$$

We see that $\varepsilon_0 = 1$ and $\varepsilon_1 = \varepsilon$, but for $m \neq 0, 1$ we have of course that $\varepsilon_m \neq \varepsilon^m$. However

Lemma 5.12.

  (i) $\varepsilon_{m+1} = \varepsilon_m(\varepsilon - m)$.

  (ii) $\varepsilon_m = \varepsilon(\varepsilon - 1) \cdots (\varepsilon - m + 1)$, if $m > 0$.

  (iii) ε_m is the homogeneous component of highest order of ε^m.

  (iv) If $f \in R$ is homogeneous of degree r, then

$$\varepsilon_m * f = \begin{cases} \frac{r!}{(r-m)!} f & \text{if } m \leq r, \\ 0 & \text{if } m > r. \end{cases}$$

Proof. It is easily seen that $(i) \Rightarrow (ii) \Rightarrow (iii)$. Condition (iv) is obvious for $m = 0$, and for $m > 0$ this too follows from (ii), because

$$\varepsilon_m * f = \varepsilon(\varepsilon - 1) \cdots (\varepsilon - m + 1) * f = r(r-1) \cdots (r - m + 1)f,$$

since $\varepsilon * f = rf$.

It remains to show (i). This is clear if $m = 0$. If $m > 0$ we get

$$\varepsilon_m \cdot \varepsilon = \sum_{|\alpha|=m} \sum_{i=1}^{n} \frac{m!}{\alpha!} x^\alpha \partial^\alpha x_i \partial_i = \sum_{|\alpha|=m} \sum_{i=1}^{n} \frac{m!}{\alpha!} x^\alpha \left(x_i \partial^\alpha + \alpha_i \partial^{\alpha - e_i} \right) \partial_i$$

$$= \sum_{|\alpha|=m} \sum_{i=1}^{n} \frac{m!}{\alpha!} x^{\alpha + e_i} \partial^{\alpha + e_i} + \sum_{|\alpha|=m} \sum_{i=1}^{n} \frac{m!}{\alpha!} \alpha_i x^\alpha \partial^\alpha$$

$$= \sum_{|\beta|=m+1} \sum_{i=1}^{n} \beta_i \frac{m!}{\beta!} x^\beta \partial^\beta + \sum_{|\alpha|=m} \frac{m!}{\alpha!} \left(\sum_{i=1}^{n} \alpha_i \right) x^\alpha \partial^\alpha$$

$$= \sum_{|\beta|=m+1} (m+1) \frac{m!}{\beta!} x^\beta \partial^\beta + m \sum_{|\alpha|=m} \frac{m!}{\alpha!} x^\alpha \partial^\alpha$$

$$= \sum_{|\beta|=m+1} \frac{(m+1)!}{\beta!} x^\beta \partial^\beta + m\varepsilon_m = \varepsilon_{m+1} + m\varepsilon_m.$$

$\square$

5.2.1 Module Generators

In this section we will find generators for $\mathcal{D}(I)$ as a module over R, when I defines a generic arrangement. By Theorem 2.5, this will give A-module generators for $\mathcal{D}(A)$, if $A = R/I$ is the coordinate ring of the arrangement.

First suppose that $r \geq 1$ and that $p_1, \ldots, p_r$ define a central arrangement. Let H_i be the hyperplane defined by p_i, for $i = 1, \ldots, r$, and let $V_i = V_{H_i} \subseteq V$ be the subspace generating the module of derivations annihilating p_i.

If $\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, r\}$, then an element $\delta \in V_{i_1} \cap \ldots \cap V_{i_s}$ annihilates $p_{i_1} \cdots p_{i_s}$. Thus, if $\{i_{s+1}, \ldots, i_r\} = \{1, \ldots, r\} \setminus \{i_1, \ldots, i_s\}$, then the derivation $p_{i_{s+1}} \cdots p_{i_r} \delta$ preserves the ideal $\langle p_1 \cdots p_r \rangle$. We will show that if the arrangement defined by $\langle p_1 \cdots p_r \rangle$ is generic, then $\mathcal{D}^{(m)}(\langle p_1 \cdots p_r \rangle)$ is generated by ε_m together with elements of the form $p_{i_{s+1}} \cdots p_{i_r} \theta$, where θ is a product of m derivations in $V_{i_1} \cap \ldots \cap V_{i_s}$.

To do so we should have a basis for each intersection of V_i:s, including the whole of V, considered as the intersection of the empty family. Therefore, assume that $p_1, \ldots, p_r$ define a generic arrangement. Since each V_i is isomorphic to H_i, and since $\mathcal{A} = \bigcup H_i$ is generic, we know that

$$\dim(V_{i_1} \cap \ldots \cap V_{i_s}) = \dim(H_{i_1} \cap \ldots \cap H_{i_s}) = \begin{cases} n - s & \text{if } s < n, \\ 0 & \text{if } s \geq n. \end{cases}$$

We will now show how to find a subset M of V containing a basis for each intersection of V_i:s.

Lemma 5.13. *Let $p_1, \ldots, p_r$ be linear forms defining a generic arrangement. Choose a subset M of V as follows.*

> *If $r < n$: choose a basis $\{\delta_{r+1}, \ldots, \delta_n\}$ for $\bigcap_{i=1}^r V_i$. Then, for each $i = 1, \ldots, r$, choose an element δ_i so that $\{\delta_i, \delta_{r+1}, \ldots, \delta_n\}$ is a basis for $\bigcap_{j \neq i} V_j$, and let $M = \{\delta_1, \ldots, \delta_n\}$.*

> *If $r \geq n$: choose a basis element for each intersection of $n - 1$ of the V_i:s, and let $M = \{\delta_1, \ldots, \delta_t\}$ be the set of all of these.*

A set M chosen in this way contains a basis for each intersection of V_i:s, including the whole of V.

Proof. The case $r > n$ follows from the case $r = n$. Any intersection $V_{i_1} \cap \ldots \cap V_{i_s}$ with $s \geq n$ is zero, and for an intersection $V_{i_1} \cap \ldots \cap V_{i_s}$ with $s \leq n - 1$, choose any $V_{i_{s+1}}, \ldots, V_{i_n}$ to give n V_i:s. The subset of n elements of M which are bases for the intersections of $n - 1$ of these V_i:s contains a basis for $V_{i_1} \cap \ldots \cap V_{i_s}$ by the case $r = n$.

Now suppose that $r \leq n$. In this case $M = \{\delta_1, \ldots, \delta_n\}$, and an intersection

$$V_{i_1} \cap \ldots \cap V_{i_s},$$

with $1 \leq s \leq r$, has dimension $n - s$ and contains $\delta_{i_{s+1}}, \ldots, \delta_{i_r}, \delta_{r+1}, \ldots, \delta_n$, so these form a basis if they are linearly independent. Hence we are done if we can show that M is a basis for V, i.e., that $\delta_1, \ldots, \delta_n$ are linearly independent.

To do so, assume $\sum a_i \delta_i = 0$. We have numbered the δ_i:s so that

$$\delta_i * p_j = \begin{cases} b_j \in k \setminus \{0\} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus

$$0 = \left(\sum a_i \delta_i \right) * p_j = a_j b_j,$$

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for each $j = 1, \ldots, r$, so $a_i = 0$ for $i = 1, \ldots, r$. This finishes the proof for the case $r = n$. If $r < n$ this shows that $\sum_{i=r+1}^{n} a_i \delta_i = 0$, but $\{\delta_{r+1}, \ldots, \delta_n\}$ is a basis, so we must have $a_i = 0$ for $i = r+1, \ldots, n$, too. $\qquad\square$

Remark. When $r = 1$, this gives a set M as in Proposition 5.9, although with a different numbering of the basis elements. As in Proposition 5.9, the set M in Lemma 5.13 is not unique. However, we will use M to define certain R-submodules of $\mathcal{D}(R)$, in Definition 5.15, and as we shall see in Lemma 5.17, given $p_1, \ldots, p_r$, any choice of basis elements will give the same module, as long as they are chosen as in the lemma. Therefore we will feel justified to refer to *the* set M from Lemma 5.13.

To find bases for intersections of V_i:s is easy. Because of the isomorphism between V_H and H, finding a basis for $V_{i_1} \cap \ldots \cap V_{i_s}$ amounts to the same thing as finding the solutions of the system

$$p_{i_1} = \ldots = p_{i_s} = 0.$$

Let us now define a new subset of $\mathrm{Der}(R)$, associated to a generic arrangement. Actually, this is a subset of $\mathrm{Der}(I)$, as we shall see in Proposition 5.18.

Definition 5.14. *Suppose $p_1, \ldots, p_r$ are linear forms defining a generic arrangement. We define a subset D of $\mathrm{Der}(R)$ as*

$$D = \{P_1 \delta_1, \ldots, P_s \delta_s\},$$

where $\{\delta_1, \ldots, \delta_s\}$ is the set M from Lemma 5.13, and where each P_i is the product of those p_j:s which are not *annihilated by δ_i. If all p_j:s are annihilated by δ_i, we put $P_i = 1$.*

Remark. The set D depends on the set M from Lemma 5.13, so it is not uniquely defined. However, to choose some other basis element γ_i instead of δ_i in M, will not change the coefficient P_i, i.e., we will get $P_i \gamma_i$ instead of $P_i \delta_i$ in D, since δ_i and γ_i annihilate the same p_j:s. Just as for the set M, we will refer to *the* set D from Definition 5.14.

For any ideals I, J in R, we define their *ideal quotient* as

$$(I : J) = \{f \in R \mid fJ \subseteq I\}.$$

If $J = \langle f \rangle$ we will simply write $(I : f)$ for $(I : J)$.

We may now make the following definition.

Definition 5.15. *Suppose $I = \langle p_1 \cdots p_r \rangle$ defines a generic arrangement, and that*

$$D = \{P_1 \delta_1, \ldots, P_s \delta_s\}$$

is the set associated to I according to Definition 5.14. Then we define an R-module $\Theta^m(I)$ by

$$\Theta^m(I) = \sum_{|\alpha|=m} (I : (I : P^\alpha)) \delta^\alpha + R\varepsilon_m,$$

for each $m \in \mathbb{N}$.

Observe that, in the definition above, the multi-indices α belong to $\mathbb{N}^s$, where $s > n$ if $r > n$.

We also observe that $(I : (I : P^\alpha))$ is a principal ideal in R. More precisely, if $P^\alpha = 1$, then $(I : (I : P^\alpha)) = R$, and if

$$P^\alpha = p_{i_1}^{a_1} \cdots p_{i_t}^{a_t},$$

with all $a_j \neq 0$, then

$$(I : (I : P^\alpha)) = \langle p_{i_1} \cdots p_{i_t} \rangle.$$

This follows from unique factorization, since the p_i:s are pairwise relatively prime. The generator $p_{i_1} \cdots p_{i_t}$ of $(I : (I : P^\alpha))$ is the product of those p_j:s such that some δ_i with $\alpha_i \neq 0$ does not annihilate p_j.

For ease of notation, let us make the following

Definition 5.16. *Suppose $I = \langle p_1 \cdots p_r \rangle$ defines a generic arrangement and that $\{\delta_1, \dots, \delta_s\}$ is the set M associated to I according to Lemma 5.13.*

For each $\alpha \in \mathbb{N}^s$, define P_α as the product of those p_j:s such that some δ_i with $\alpha_i \neq 0$ does not annihilate p_j. If every δ_i with $\alpha_i \neq 0$ annihilate all p_j:s, we define $P_\alpha = 1$.

Thus, for each $\alpha \in \mathbb{N}^s$, we have

$$RP_\alpha = (I : (I : P^\alpha)),$$

so

$$\Theta^m(I) = \sum_{|\alpha|=m} RP_\alpha \delta^\alpha + R\varepsilon_m.$$

Finally we observe that, although the set $\{P_\alpha \delta^\alpha \mid |\alpha| = m\} \cup \{\varepsilon_m\}$ of generators for $\Theta^m(I)$ is finite, it is in general not minimal. A precise statement for the case $m = 1$ may be found in [7].

All results from Lemma 5.17 to Lemma 5.27, inclusive, are obvious for $m = 0$, so in the proofs of all of these results, we may without loss of generality assume that $m \geq 1$.

Lemma 5.17. *If $I = \langle p_1 \cdots p_r \rangle$ defines a generic arrangement, then different choices of basis elements in M in Lemma 5.13 give the same R-module $\Theta^m(I)$, for each m.*

Proof. This is obvious for $r \geq n$, since the elements in M are bases for one-dimensional spaces in this case, so that they are unique up to nonzero constants. Therefore we may assume that $r < n$. If

$$M_\delta = \{\delta_1, \dots, \delta_n\} \text{ and } M_\gamma = \{\gamma_1, \dots, \gamma_n\}$$

are chosen as in Lemma 5.13, the sets D from Definition 5.14 will be

$$D_\delta = \{P_1 \delta_1, \dots, P_n \delta_n\} = \{p_1 \delta_1, \dots, p_r \delta_r, \delta_{r+1}, \dots, \delta_n\}$$

and

$$D_\gamma = \{P_1\gamma_1, \dots, P_n\gamma_n\} = \{p_1\gamma_1, \dots, p_r\gamma_r, \gamma_{r+1}, \dots, \gamma_n\}.$$

Let $\Theta_\delta^m(I)$ and $\Theta_\gamma^m(I)$ be the modules given by D_δ and D_γ according to Definition 5.15. By symmetry, to see that $\Theta_\delta^m(I) = \Theta_\gamma^m(I)$, it is enough to show that $\Theta_\delta^m(I) \subseteq \Theta_\gamma^m(I)$.

Since

$$\{\delta_{r+1}, \dots, \delta_n\} \text{ and } \{\gamma_{r+1}, \dots, \gamma_n\}$$

and, for $i = 1, \dots, r$,

$$\{\delta_i, \delta_{r+1}, \dots, \delta_n\} \text{ and } \{\gamma_i, \gamma_{r+1}, \dots, \gamma_n\},$$

respectively, are bases for the same vectorspaces, there are $a_i, a_{ij} \in k$ such that

$$\delta_i = a_i\gamma_i + \sum_{j=r+1}^{n} a_{ij}\gamma_j,$$

for $i = 1, \dots, r$, and

$$\delta_i = \sum_{j=r+1}^{n} a_{ij}\gamma_j,$$

for $i = r+1, \dots, n$. Thus, if $|\alpha| = m$,

$$\delta^\alpha = a\gamma^\alpha + \sum_{|\beta|=m} a_\beta\gamma^\beta,$$

where $a, a_\beta \in k$, and $a_\beta \neq 0$ only if γ^β is a product of $\gamma_{r+1}, \dots, \gamma_n$, and the γ_i:s with $\alpha_i \neq 0$, i.e., such that P_α is a multiple, $f_\beta P_\beta$, of P_β, where f_β is 1, or a product of p_j:s. Hence

$$P_\alpha\delta^\alpha = aP_\alpha\gamma^\alpha + \sum_{|\beta|=m} a_\beta P_\alpha\gamma^\beta$$

$$= aP_\alpha\gamma^\alpha + \sum_{|\beta|=m} a_\beta f_\beta P_\beta\gamma^\beta \in \Theta_\gamma^m(I),$$

for all α with $|\alpha| = m$. $\qquad\square$

We will show that $\mathcal{D}^{(m)}(I) = \Theta^m(I)$. To begin with we have

Proposition 5.18. *Suppose that $I = \langle p_1 \cdots p_r \rangle$ defines a generic arrangement. Then*
$$\Theta^m(I) \subseteq \mathcal{D}^{(m)}(I),$$
as R-modules.

Proof. From Lemma 5.12, *(iv)*, it follows that $R\varepsilon_m \subseteq \mathcal{D}^{(m)}(I)$. To see that

$$\sum_{|\alpha|=m} RP_\alpha \delta^\alpha \subseteq \mathcal{D}^{(m)}(I), \tag{8}$$

we observe that, if $f \in R$ and θ is a derivation annihilating f, we have $\theta^l * gf = (\theta^l * g)f$, for any $g \in R$. Thus δ^α preserves the ideal generated by the product of those p_j:s that all δ_i with $\alpha_i \neq 0$ annihilate. It follows that $P_\alpha \delta^\alpha$ preserves $\langle p_1 \cdots p_r \rangle$, i.e., (8) holds. $\qquad\square$

To show that $\mathcal{D}^{(m)}(I) \subseteq \Theta^m(I)$, we will use a method for doing induction on the cardinality of an arrangement $\mathcal{A}$, known as *deletion and restriction*. Let us therefore make the following definition.

Definition 5.19. *Let $\mathcal{A}$ be a central arrangement and let $H \in \mathcal{A}$ be a fixed hyperplane. Let $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ and let $\mathcal{A}'' = \{K \cap H \mid K \in \mathcal{A}'\}$. Then $\mathcal{A}'$ is called a* deleted *arrangement and $\mathcal{A}''$ is called a* restricted *arrangement. We call H the* distinguished *hyperplane.*

Observe that if $\mathcal{A}$ is a generic arrangement, so are $\mathcal{A}'$ and $\mathcal{A}''$. For a fixed hyperplane $H \in \mathcal{A}$, we may assume, after a linear change of coordinates, that H is defined by the equation $x_1 = 0$.

Therefore, suppose $\mathcal{A}$ is defined by the ideal $I = \langle x_1 p_1 \cdots p_r \rangle$, and let the distinguished hyperplane be the hyperplane defined by x_1. Let $I' = \langle p_1 \cdots p_r \rangle$, and let $I'' = \langle q_1 \cdots q_r \rangle \subseteq k[x_2, \ldots, x_n]$, where

$$q_i(x_2, \ldots, x_n) = p_i(0, x_2, \ldots, x_n), \, i = 1, \ldots, r.$$

Then I' is the defining ideal for $\mathcal{A}'$ and $\sqrt{I''}$, the radical of I'', is the defining ideal for $\mathcal{A}''$. If $\mathcal{A}$ is generic and $n > 2$, I'' is radical, so it is the defining ideal for $\mathcal{A}''$.

Since $I'' \subseteq k[x_2, \ldots, x_n]$, we will consider $\mathcal{D}^{(m)}(I'')$ and, when applicable, $\Theta^m(I'')$ as $k[x_2, \ldots, x_n]$-submodules of $\mathcal{D}(k[x_2, \ldots, x_n])$.

Let $j : \mathcal{D}^{(m)}(I') \longrightarrow \mathcal{D}^{(m)}(I)$ be left multiplication by x_1, and define π by

$$\pi : \sum f_\beta(x_1, \ldots, x_n)\partial^\beta \longmapsto \sum f_\beta(0, x_2, \ldots, x_n)\partial^\beta.$$

It is clear that if $\theta \in \mathcal{D}^{(m)}(R)$, then $\pi(\theta)$ is an operator of homogeneous order m. In [4] it was shown that the sequence

$$0 \longrightarrow \mathrm{Der}(I') \overset{j}{\longrightarrow} \mathrm{Der}(I) \overset{\pi}{\longrightarrow} \mathrm{Der}(\sqrt{I''}), \tag{9}$$

is exact. We will show that the corresponding sequence for homogeneous operators of higher order is also exact. However, in our proof that $\mathcal{D}^{(m)}(I) = \Theta^m(I)$, we will need a slightly different sequence. This is because when $n = 2$, $\mathcal{A}''$ is the arrangement $\{0\} \subseteq k$, but $I'' = \langle x_2^r \rangle$, so I'' is *not* the defining ideal for $\mathcal{A}''$ if $r > 1$.

Lemma 5.20. *Suppose $r \geq 1$, and that $I = \langle x_1 p_1 \cdots p_r \rangle$ defines a central arrangement. Then*

$$\pi\left(\mathcal{D}^{(m)}(I)\right) \subseteq \mathcal{D}^{(m)}(I''),$$

and the sequence

$$0 \xrightarrow{} \mathcal{D}^{(m)}(I') \xrightarrow{j} \mathcal{D}^{(m)}(I) \xrightarrow{\pi} \mathcal{D}^{(m)}(I''), \qquad (10)$$

of k-vector spaces, is exact.

Proof. Suppose $\theta \in \mathcal{D}^{(m)}(I)$. Then θ belongs to both $\mathcal{D}^{(m)}(\langle x_1 \rangle)$ and $\mathcal{D}^{(m)}(I')$, by Theorem 3.5.

We know from Proposition 5.9 that $\mathcal{D}^{(m)}(\langle x_1 \rangle)$ is generated as an R-module by $\{\partial^\beta \mid |\beta| = m \text{ and } \beta_1 = 0\}$ together with $\{x_1 \partial^\beta \mid |\beta| = m \text{ and } \beta_1 > 0\}$. Thus any monomial in θ containing ∂_1 must also contain x_1. This shows that we may write

$$\theta = x_1 \theta_1 + \theta_2,$$

where $\theta_2 = \pi(\theta) \in \mathcal{D}^{(m)}(k[x_2, \dots, x_n])$.

Since $\theta \in \mathcal{D}^{(m)}(I')$ too, it follows in particular that for any x^γ with $\gamma_1 = 0$ we have $\theta * x^\gamma p_1 \cdots p_r = f p_1 \cdots p_r$, for some $f \in R$. Each p_i is of the form $p_i = a_i x_1 + q_i$, for some $a_i \in k$, so

$$p_1 \cdots p_r = (a_1 x_1 + q_1) \cdots (a_r x_1 + q_r) = x_1 g + q_1 \cdots q_r,$$

for some $g \in R$. Thus

$$\begin{aligned}
\theta * x^\gamma p_1 \cdots p_r &= (x_1 \theta_1 + \theta_2) * x^\gamma p_1 \cdots p_r \\
&= x_1 \theta_1 * x^\gamma p_1 \cdots p_r + \theta_2 * x^\gamma (x_1 g + q_1 \cdots q_r) \\
&= x_1 \theta_1 * x^\gamma p_1 \cdots p_r + x_1 \theta_2 * x^\gamma g + \theta_2 * x^\gamma q_1 \cdots q_r,
\end{aligned}$$

and

$$f p_1 \cdots p_r = x_1 f g + f q_1 \cdots q_r.$$

We may write $f = x_1 f_1 + f_2$, where no monomial in f_2 is divisible by x_1, so

$$f p_1 \cdots p_r = x_1 f g + x_1 f_1 q_1 \cdots q_r + f_2 q_1 \cdots q_r.$$

Hence

$$x_1 (f g + f_1 q_1 \cdots q_r - \theta_1 * x^\gamma p_1 \cdots p_r - \theta_2 * x^\gamma g) = \theta_2 * x^\gamma q_1 \cdots q_r - f_2 q_1 \cdots q_r,$$

where the right hand side is divisible by x_1 only if it is zero, so

$$\theta_2 * x^\gamma q_1 \cdots q_r = f_2 q_1 \cdots q_r.$$

We conclude that $\pi(\theta) \in \mathcal{D}^{(m)}(I'')$ whenever $\theta \in \mathcal{D}^{(m)}(I)$.

To see that the sequence (10) is exact, we begin by concluding that if $x_1 \theta = 0$, then $\theta = 0$, so j is an injection. It is clear that $\operatorname{Im} j \subseteq \ker \pi$. To show the reverse inclusion, take $\theta \in \ker \pi$. Then, by the above, $\theta = x_1 \theta_1$, and $\theta \in \mathcal{D}^{(m)}(I')$. Hence, for any $f \in R$,

$$x_1 \theta_1 * f p_1 \cdots p_r = g p_1 \cdots p_r,$$

for some $g \in R$. Since x_1 does not divide $p_1 \cdots p_r$, it must divide g. Thus

$$\theta_1 * f p_1 \cdots p_r = h p_1 \cdots p_r,$$

where $x_1 h = g$. It follows that $\theta_1 \in \mathcal{D}^{(m)}(I')$, so $\theta \in x_1 \mathcal{D}^{(m)}(I') = \operatorname{Im} j$. $\square$

Again, the defining ideal for $\mathcal{A}''$ is $\sqrt{I''}$, which may differ from I''. However, as a corollary to the preceding lemma we have

Corollary 5.21. *If $r \geq 1$, and $I = \langle x_1 p_1 \cdots p_r \rangle$ defines a central arrangement, then, also,*

$$\pi\left(\mathcal{D}^{(m)}(I)\right) \subseteq \mathcal{D}^{(m)}(\sqrt{I''}),$$

so the generalization

$$0 \longrightarrow \mathcal{D}^{(m)}(I') \xrightarrow{\ j\ } \mathcal{D}^{(m)}(I) \xrightarrow{\ \pi\ } \mathcal{D}^{(m)}(\sqrt{I''}),$$

of the sequence (9), is exact for each $m \geq 0$.

Proof. It is enough to show that $\pi\left(\mathcal{D}^{(m)}(I)\right) \subseteq \mathcal{D}^{(m)}(\sqrt{I''})$. The only way in which I'' may differ from $\sqrt{I''}$ is that π maps different p_i:s onto (nonconstant multiples of) the same q_j. Thus, after a renumbering, we may assume that $q_1, \ldots, q_t$, $t \leq r$, are pairwise relatively prime, and that $I'' = \langle q_1^{l_1} \cdots q_t^{l_t} \rangle$, where $\pi(p_i) = q_i$, for $i = 1, \ldots, t$, each $l_j \geq 1$, and $l_1 + \ldots + l_t = r$. Write $J = \langle x_1 p_1 \cdots p_t \rangle$ and $J'' = \langle q_1 \cdots q_t \rangle = \sqrt{I''}$. Then

$$\mathcal{D}^{(m)}(I) = \mathcal{D}^{(m)}(J) \cap \mathcal{D}^{(m)}(\langle p_{t+1} \cdots p_r \rangle) \subseteq \mathcal{D}^{(m)}(J),$$

by Theorem 3.5, and $\pi\left(\mathcal{D}^{(m)}(J)\right) \subseteq \mathcal{D}^{(m)}(J'')$, by Lemma 5.20. $\qquad\square$

If the arrangement defined by I is not generic, π need not be onto.

Example 5.22. *Let $R = k[x, y, z]$ and let $\mathcal{A}$ be the arrangement in k^3 defined by $I = \langle xyz(x+y)(x+z) \rangle \subseteq R$. Then $\mathcal{A}$ is not generic, by Example 4.5 (iii). Here $I'' = \langle y^2 z^2 \rangle$, so $\sqrt{I''} = \langle yz \rangle$. Let $S = k[y, z]$. Using Proposition 3.2, we see that*

$$\operatorname{Der}(I'') = \operatorname{Der}(\sqrt{I''}) = Sy\partial_y + Sz\partial_z.$$

On the other hand, by [4, Theorem 4.23], $\operatorname{Der}(I)$ is generated by ε, $y(x+y)\partial_y$, and $z(x+z)\partial_z$, as an R-module, so

$$\pi(\operatorname{Der}(I)) = S(y\partial_y + z\partial_z) + Sy^2\partial_y + Sz^2\partial_z \subsetneq \operatorname{Der}(I'') = \operatorname{Der}(\sqrt{I''}).$$

Thus π is not onto if $m = 1$.

The fact that $\pi : \mathcal{D}^{(m)}(I) \longrightarrow \mathcal{D}^{(m)}(I'')$ *is* onto whenever the arrangement defined by I is generic, is the content of Corollary 5.26.

We will now see what $\mathcal{D}^{(m)}(I'')$ is when $n = 2$.

Lemma 5.23. *If $r \geq 1$, and $I = \langle x_1 p_1 \cdots p_r \rangle$ defines a central arrangement in k^2, then $I'' = \langle x_2^r \rangle$ and*

$$\mathcal{D}^{(m)}(I'') = k[x_2] x_2^{\min\{r,m\}} \partial_2^m.$$

Proof. Any element in $\mathcal{D}^{(m)}(k[x_2])$ is of the form $f\partial_2^m$, where $f \in k[x_2]$. By Proposition 3.2, $f\partial_2^m$ preserves $\langle x_2^r \rangle$ if and only if

$$\langle x_2^r \rangle \ni f\partial_2^m * x_2^{r+j} = \begin{cases} \frac{(r+j)!}{(r+j-m)!} f x_2^{r+j-m} & \text{if } m \leq r+j, \\ 0 & \text{if } m > r+j, \end{cases}$$

for every $j \leq m - 1$. From this, the lemma follows easily. $\qquad\square$

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Lemma 5.24. *If $r \geq 1$, and $I = \langle x_1 p_1 \cdots p_r \rangle$ defines a generic arrangement, then the restriction of π to $\Theta^m(I)$ is a surjection onto $\Theta^m(I'')$ when $n > 2$, and a surjection onto $\mathcal{D}^{(m)}(I'')$ when $n = 2$. In both cases the restriction of j to $\Theta^m(I')$ is an injection into $\Theta^m(I)$.*

Thus, when $n > 2$, the sequence

$$0 \longrightarrow \Theta^m(I') \xrightarrow{\ j\ } \Theta^m(I) \xrightarrow{\ \pi\ } \Theta^m(I'') \longrightarrow 0, \qquad (11)$$

is exact, except possibly at $\Theta^m(I)$, and when $n = 2$, the same thing holds for the sequence

$$0 \longrightarrow \Theta^m(I') \xrightarrow{\ j\ } \Theta^m(I) \xrightarrow{\ \pi\ } \mathcal{D}^{(m)}(I'') \longrightarrow 0.$$

As in Lemma 5.20, these are sequences of k-vector spaces.

Proof. Let $M = \{\delta_1, \ldots, \delta_s\}$ be the set associated to $x_1, p_1, \ldots, p_r$ according to Lemma 5.13. Then M contains a basis for each intersection of V_i:s; in particular for each intersection of V_i:s corresponding to $p_1, \ldots, p_r$. Thus the set M' associated to $p_1, \ldots, p_r$ according to Lemma 5.13 may be taken as a subset of M. From this it follows that

$$\Theta^m(I') = \sum R P'_\alpha \delta^\alpha + R \varepsilon_m,$$

where the sum ranges over those α such that $\alpha_i \neq 0$ only if δ_i belongs to M', and the factors in P'_α are chosen among $p_1, \ldots, p_r$. It follows that

$$j(\Theta^m(I')) = x_1 \Theta^m(I') \subseteq \Theta^m(I).$$

Now assume $n > 2$. Number the V_i:s so that V_0 corresponds to x_1, and V_i corresponds to p_i, for $i = 1, \ldots, r$. Note that since the derivations in V_0 annihilate x_1, they all have ∂_1-coefficient equal to zero. Let $W = \sum_{j=2}^n k \partial_j$, and let W_i be the subspace of W generating the module of derivations annihilating q_i, $i = 1, \ldots, r$. Then

$$W_{i_1} \cap \ldots \cap W_{i_t} = V_0 \cap V_{i_1} \cap \ldots \cap V_{i_t} \subseteq W,$$

by the isomorphism between V_H and H. Let M'' be the subset of M consisting of the δ_i:s with ∂_1-coefficient equal to zero. From the above, it follows that M'' is the set associated to $q_1, \ldots, q_r$ according to Lemma 5.13. Thus, letting $Q_\alpha(x_2, \ldots, x_n) = P_\alpha(0, x_2, \ldots, x_n)$, we see that the generators for $\Theta^m(I'')$ as $k[x_2, \ldots, x_n]$-module are

$$\{Q_\alpha \delta^\alpha \mid |\alpha| = m\} \cup \{\epsilon_m\},$$

where the δ_i:s with $\alpha_i \neq 0$ are chosen from M'', and $\epsilon_m = \pi(\varepsilon_m)$ is the operator corresponding to ε_m in $\mathcal{D}(k[x_2, \ldots, x_n])$. Thus any element in $\Theta^m(I'')$ is the image, under π, of some operator in $\Theta^m(I)$.

On the other hand, for any R-module generator $P_\alpha \delta^\alpha$ of $\Theta^m(I)$, in which some δ_i with $\alpha_i \neq 0$ does not belong to M'', x_1 must be a factor in P_α, so $\pi(x^\beta P_\alpha \delta^\alpha) = 0$, for any monomial $x^\beta \in R$. Whence π maps $\Theta^m(I)$ onto $\Theta^m(I'')$.

When $n = 2$, the set M from Lemma 5.13 will consist of $r+1$ derivations, each one annihilating precisely one of $x_1, p_1, \ldots, p_r$. Thus all R-module generators

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of $\Theta^m(I)$ of the form $P_\alpha \delta^\alpha$, except for $p_1 \cdots p_r \partial_2^m$, will have x_1 as a factor in P_α. Hence the image of $\Theta^m(I)$ under π is generated, as a $k[x_2]$-module, by

$$\pi(\varepsilon_m) = x_2^m \partial_2^m$$

and

$$\pi(p_1 \cdots p_r \partial_2^m) = a x_2^r \partial_2^m,$$

where a is some nonzero constant. This means that

$$\pi(\Theta^m(I)) = k[x_2] x_2^{\min\{r,m\}} \partial_2^m,$$

which, by Lemma 5.23, finishes the proof. $\qquad\square$

We can now show

Proposition 5.25. *Suppose that $I = \langle p_1 \cdots p_r \rangle$ defines a generic arrangement. Then*
$$\mathcal{D}^{(m)}(I) = \Theta^m(I),$$

as R-modules.

Proof. By Proposition 5.18 it is enough to show that

$$\mathcal{D}^{(m)}(I) \subseteq \Theta^m(I), \tag{12}$$

as k-vector spaces. To do so we proceed by double induction, on the number $r \geq 1$ of hyperplanes, and on the dimension $n \geq 2$ of the ambient space of the arrangement.

The case $r = 1$ follows from Proposition 5.9. The case $n = 2$ will be dealt with in a while.

Thus suppose that $r \geq 1$ and $n \geq 2$, and that the inclusion (12) is true for r and n, and for r and $n+1$, respectively. Suppose that $p_1, \ldots, p_{r+1}$ are linear forms in $k[x_1, \ldots, x_{n+1}]$ defining a generic arrangement. We may assume, after a linear change of coordinates, that $p_{r+1} = x_1$. Thus, letting $I = \langle x_1 p_1 \cdots p_r \rangle$, it follows from Lemmas 5.20 and 5.24 that the rows in the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Theta^m(I') & \overset{j}{\longrightarrow} & \Theta^m(I) & \overset{\pi}{\longrightarrow} & \Theta^m(I'') & \longrightarrow & 0 \\
 & & \ \downarrow{\scriptstyle i_1} & & \ \downarrow{\scriptstyle i} & & \ \downarrow{\scriptstyle i_2} & & \\
0 & \longrightarrow & \mathcal{D}^{(m)}(I') & \underset{j}{\longrightarrow} & \mathcal{D}^{(m)}(I) & \underset{\pi}{\longrightarrow} & \mathcal{D}^{(m)}(I''), & &
\end{array}
$$

of k-vector spaces are exact, except possibly at $\Theta^m(I)$. Here $i, i_1,$ and i_2 are inclusions, by Proposition 5.18.

By the induction hypothesis, i_1 and i_2 are actually identity maps. Thus assume that $\theta \in \mathcal{D}^{(m)}(I)$. Then $\pi(\theta) \in \mathcal{D}^{(m)}(I'') = \pi(\Theta^m(I))$. Hence there exists $\psi \in \Theta^m(I) \subseteq \mathcal{D}^{(m)}(I)$ such that $\pi(\psi) = \pi(\theta)$. Then

$$\theta - \psi \in \ker \pi = \operatorname{Im} j = x_1 \mathcal{D}^{(m)}(I') = x_1 \Theta^m(I') \subseteq \Theta^m(I).$$

This implies that $\theta \in \Theta^m(I)$, which shows that $\mathcal{D}^{(m)}(I) \subseteq \Theta^m(I)$.

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To show the proposition for $n = 2$, we use induction on the number of hyperplanes. Proceeding as above we get the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Theta^m(I') & \xrightarrow{\ j\ } & \Theta^m(I) & \xrightarrow{\ \pi\ } & \mathcal{D}^{(m)}(I'') & \longrightarrow & 0 \\
& & \ \downarrow{\scriptstyle i_1} & & \ \downarrow{\scriptstyle i} & & \ \| & & \\
0 & \longrightarrow & \mathcal{D}^{(m)}(I') & \xrightarrow{\ j\ } & \mathcal{D}^{(m)}(I) & \xrightarrow{\ \pi\ } & \mathcal{D}^{(m)}(I''), & &
\end{array}
$$

where i_1, by the induction hypothesis, is the identity map. The same diagram chase as above concludes the proof. □

Corollary 5.26. *If $r \geq 1$, and $I = \langle x_1 p_1 \cdots p_r \rangle$ defines a generic arrangement, then the sequence of k-vector spaces*

$$
0 \longrightarrow \mathcal{D}^{(m)}(I') \xrightarrow{\ j\ } \mathcal{D}^{(m)}(I) \xrightarrow{\ \pi\ } \mathcal{D}^{(m)}(I'') \longrightarrow 0
$$

is exact for each $m \geq 0$.

In Proposition 5.9 we did not need ε_m as a generator for $\mathcal{D}^{(m)}(I)$. This remains true for all $r \leq n$.

Lemma 5.27. *If $r \leq n$ and $p_1, \ldots, p_r$ define a generic arrangement, then*

$$
\varepsilon_m \in \sum_{|\alpha| = m} (I : (I : P^\alpha)) \delta^\alpha.
$$

Proof. It is enough to show how ε may be expressed as a linear combination of $P_i \delta_i$, since the homogeneous component of highest order of a power of such an expression must be a linear combination of the $P_\alpha \delta^\alpha$, $|\alpha| = m$.

If $r = n$, not only is $M = \{\delta_1, \ldots, \delta_n\}$ from Lemma 5.13 a basis for V, but $\{p_1, \ldots, p_n\}$ is a basis for $\sum k x_i$. Thus

$$
x_l = \sum_j a_{lj} p_j,
$$

for some $a_{lj} \in k$, and by the proposition above, $\mathrm{Der}(\langle p_1 \cdots p_n \rangle)$ is generated by $p_1 \delta_1, \ldots, p_n \delta_n$ together with ε.

Let

$$
\theta = \sum_i \frac{p_i \delta_i}{\delta_i * p_i} \in \mathrm{Der}(\langle p_1 \cdots p_n \rangle).
$$

Then

$$
\theta * x_l = \left(\sum_i \frac{p_i \delta_i}{\delta_i * p_i} \right) * \left(\sum_j a_{lj} p_j \right) = \sum_j a_{lj} \frac{p_j \delta_j * p_j}{\delta_j * p_j} = \sum_j a_{lj} p_j = x_l,
$$

for $l = 1, \ldots, n$, so $\theta = \varepsilon$, by Lemma 2.4.

If $r < n$ there are linear forms $p_{r+1}, \ldots, p_n \in R$ such that δ_i annihilates all p_j:s, except p_i, for $i = r + 1, \ldots, n$, and such that $\{p_1, \ldots, p_n\}$ is a basis for $\sum k x_i$. Then the same result as above holds. □

We are now ready to formulate the main result of this section.

Theorem 5.28. *Suppose $I = \langle p_1 \cdots p_r \rangle$ defines a generic arrangement. Let*

$$D = \{P_1 \delta_1, \dots, P_s \delta_s\}$$

be the set associated to I according to Definition 5.14. Then

$$\mathcal{D}(I) = \bigoplus_{m \geq 0} \left(\sum_{|\alpha|=m} (I : (I : P^\alpha)) \delta^\alpha + R\varepsilon_m \right),$$

as an R-module, where $R\varepsilon_m \subseteq \sum_{|\alpha|=m}(I : (I : P^\alpha))\delta^\alpha$, if $r \leq n$.

Proof. This follows immediately from Proposition 5.3, Proposition 5.25, and from Lemma 5.27, together. $\qquad\square$

Let us write $\overline{\varepsilon_m}$ for $\sum_{|\alpha|=m} \frac{m!}{\alpha!}\overline{x^\alpha}\partial^\alpha$, the operator in $\mathcal{D}(R/I)$ induced by ε_m, as in Section 2.2. Then, by Theorem 2.5 we get

Corollary 5.29. *Suppose A is the coordinate ring of a generic arrangement, and let*

$$D = \{P_1 \delta_1, \dots, P_s \delta_s\}$$

be the set associated to the defining ideal $I = \langle p_1 \cdots p_r \rangle$, according to Definition 5.14. Then

$$\mathcal{D}(A) = \bigoplus_{m \geq 0} \left(\sum_{|\alpha|=m} \overline{(I : (I : P^\alpha))} \delta^\alpha + A\overline{\varepsilon_m} \right),$$

as an A-module, where $A\overline{\varepsilon_m} \subseteq \sum_{|\alpha|=m} \overline{(I : (I : P^\alpha))}\delta^\alpha$, if $r \leq n$.

Remark. It is clear that $\overline{(I : (I : P^\alpha))} \neq 0$ if and only if $P^\alpha \notin I$. This is the case if and only if there is some p_i with $\delta_j * p_i = 0$ for all j with $\alpha_j \neq 0$, i.e., if and only if $\delta^\alpha \in V_i^{|\alpha|}$, for some i.

The case $n = 2$ is special, since by Example 4.5 *(ii)*, any central arrangement in the plane is generic. We therefore state this case explicitly in the following corollary to Theorem 5.28.

Corollary 5.30. *Suppose that $I = \langle p_1 \cdots p_r \rangle \subseteq k[x,y] = R$ defines a central arrangement. Let*

$$P_i = p_1 \cdots \widehat{p_i} \cdots p_r,$$

with $P_1 = 1$ if $r = 1$, and

$$\delta_i = \begin{cases} \partial_y & \text{if } p_i = ax, a \in k \setminus \{0\}, \\ \partial_x + a_i \partial_y & \text{if } p_i = a(y - a_i x), a \in k \setminus \{0\}, \end{cases}$$

for $i = 1, \dots, r$. Then

$$\mathcal{D}(I) = R + \bigoplus_{m \geq 1} (RP_1 \delta_1^m + \dots + RP_r \delta_r^m + R\varepsilon_m) + I\mathcal{D}(R),$$

as an R-module.

Proof. If $r > 1$, the set D associated to I according to Definition 5.14 will be

$$D = \{P_1\delta_1, \dots, P_r\delta_r\},$$

with P_i and δ_i as above. If $r = 1$, let P_1 and δ_1 be as above. Let $P_2 = p_1$ and let δ_2 be any derivation in $k\partial_x + k\partial_y$, which is not a multiple of δ_1. Then

$$D = \{P_1\delta_1, P_2\delta_2\}.$$

Finally, for a product, δ^α, of distinct δ_i:s, we have $P_\alpha = p_1 \cdots p_r$. $\qquad\square$

Corollary 5.31. *Suppose that A is the coordinate ring of a central arrangement in the plane, defined by $I = \langle p_1 \cdots p_r \rangle \subseteq k[x,y]$. Then, with notation as in Corollary 5.30,*

$$\mathcal{D}(A) = A + \bigoplus_{m \geq 1} \left(A\overline{P_1}\delta_1^m + \dots + A\overline{P_r}\delta_r^m + A\overline{\varepsilon_m} \right),$$

as an A-module.

Observe that, if k is algebraically closed, the result in Corollaries 5.30 and 5.31 are true for any principal ideal which is homogeneous and radical.

We have only defined $\Theta^m(I)$ for the case when I defines a generic arrangement. It is of course possible to do the same for any ideal defining a central arrangement:

Suppose $I = \langle p_1 \cdots p_r \rangle$ defines a central arrangement.

Let $M = \{\delta_1, \dots, \delta_s\}$ be a subset of V containing a basis for each intersection of V_i:s.

Let $D = \{P_1\delta_1, \dots, P_s\delta_s\}$, where P_i is the product of the p_j:s which are not annihilated by δ_i.

Define $\Theta^m(I) = \sum_{|\alpha|=m} (I : (I : P^\alpha))\delta^\alpha + R\varepsilon_m$.

Then, as in Proposition 5.18, and with the same proof, we have

$$\Theta^m(I) \subseteq \mathcal{D}^{(m)}(I).$$

We will end this section with an example showing that the opposite inclusion need not hold if the arrangement defined by I is not generic. Hence Theorem 5.28 is not true for nongeneric arrangements.

Example 5.32. *Let $I = \langle xy(x - y) \rangle \subseteq k[x,y,z] = R$. Let $\mathcal{A}$ be the central arrangement of three planes in k^3, defined by I. Then $\mathcal{A}$ is nongeneric, by Example 4.5 (iii).*

We have that ε, ∂_z, and $z\partial_z^2$ all preserve I, so

$$(x\partial_x + y\partial_y)\partial_z = \varepsilon\partial_z - z\partial_z^2 \in \mathcal{D}^{(2)}(I).$$

To find $\Theta^2(I)$, let $x = p_1$, $y = p_2$, and $x - y = p_3$. Then

$$V_1 \cap V_2 \cap V_3 = V_1 \cap V_2 = V_1 \cap V_3 = V_2 \cap V_3 = k\partial_z.$$

Extending this to bases for the V_i:s, we may choose

$$V_1 = k\partial_y + k\partial_z,$$
$$V_2 = k\partial_x + k\partial_z,$$
$$V_3 = k(\partial_x + \partial_y) + k\partial_z.$$

Thus $M = \{\partial_x, \partial_x + \partial_y, \partial_y, \partial_z\}$ has the property of containing a basis for each intersection of V_i:s, as well as for V, so

$$D = \{x(x-y)\partial_x, xy(\partial_x + \partial_y), y(x-y)\partial_y, \partial_z\}.$$

Hence, after removing unnecessary generators,

$$\begin{aligned}\Theta^2(I) =&Rx(x-y)\partial_x^2 + Rxy(\partial_x + \partial_y)^2 + Ry(x-y)\partial_y^2 \\ &+ Rx(x-y)\partial_x\partial_z + Rxy(\partial_x + \partial_y)\partial_z + Ry(x-y)\partial_y\partial_z \\ &+ R\partial_z^2 + R\varepsilon_2.\end{aligned}$$

All the polynomial coefficients in these generators, except for ∂_z^2, have degree two, but the polynomial coefficients in the element $(x\partial_x+y\partial_y)\partial_z$ of $\mathcal{D}^{(2)}(I)$, only have degree one, so it cannot possibly belong to $\Theta^2(I)$. Whence

$$\Theta^2(I) \subsetneq \mathcal{D}^{(2)}(I),$$

in this case.

5.2.2 Algebra Generators

As pointed out earlier, Nakai's conjecture holds for the coordinate ring of an affine arrangement. Thus, in particular, if A is the coordinate ring of a generic arrangement of more than one hyperplane, then $\mathcal{D}(A) \neq \Delta(A)$. We will use Theorem 5.28 to find finitely many k-algebra generators for $\mathcal{D}(I)$. Then, by Theorem 2.5, it follows that, although not generated by A and derivations, $\mathcal{D}(A)$ is still finitely generated as a k-algebra.

Lemma 5.33. *Let $p_1, \ldots, p_t$, $t \geq 1$, be polynomials in R of degree one and let $\delta \in \mathrm{Der}(R)$ be a derivation with coefficients in k. Then*

$$\sum_{i=0}^{t}(-1)^i \binom{t}{i}\delta^{2t+1-i}p_1\cdots p_t\delta^i = (\delta * p_1)\cdots(\delta * p_t)t!\delta^{t+1}.$$

Proof. We will prove this by induction on t. If $t = 1$, then

$$\sum_{i=0}^{1}(-1)^i \binom{1}{i}\delta^{3-i}p_1\delta^i = \delta^3 p_1 - \delta^2 p_1\delta = \delta^2(\delta p_1 - p_1\delta)$$

$$= \delta^2(\delta * p_1) = (\delta * p_1)\delta^2.$$

Now assume that the lemma is true for some $t \geq 1$. Agreeing that $\binom{t}{t+1} =$

$\binom{t}{-1} = 0$, we get

$$\sum_{i=0}^{t+1}(-1)^i\binom{t+1}{i}\delta^{2t+3-i}p_1\cdots p_{t+1}\delta^i$$

$$= \sum_{i=0}^{t+1}(-1)^i\left(\binom{t}{i}+\binom{t}{i-1}\right)\delta^{2t+3-i}p_1\cdots p_{t+1}\delta^i$$

$$= \sum_{i=0}^{t}(-1)^i\binom{t}{i}\delta^{2t+3-i}p_1\cdots p_{t+1}\delta^i - \sum_{i=0}^{t}(-1)^i\binom{t}{i}\delta^{2t+2-i}p_1\cdots p_{t+1}\delta^{1+i}$$

$$= \sum_{i=0}^{t}(-1)^i\binom{t}{i}\delta^{2t+2-i}\left(p_1\cdots p_{t+1}\delta + \sum_{j=1}^{t+1}(\delta*p_j)p_1\cdots \widehat{p_j}\cdots p_{t+1}\right)\delta^i$$

$$- \sum_{i=0}^{t}(-1)^i\binom{t}{i}\delta^{2t+2-i}p_1\cdots p_{t+1}\delta^{1+i}$$

$$= \sum_{i=0}^{t}(-1)^i\binom{t}{i}\delta^{2t+2-i}\sum_{j=1}^{t+1}(\delta*p_j)p_1\cdots\widehat{p_j}\cdots p_{t+1}\delta^i$$

$$= \sum_{j=1}^{t+1}(\delta*p_j)\delta\left(\sum_{i=0}^{t}(-1)^i\binom{t}{i}\delta^{2t+1-i}p_1\cdots\widehat{p_j}\cdots p_{t+1}\delta^i\right)$$

(by the induction hypothesis)

$$= \sum_{j=1}^{t+1}(\delta*p_j)\delta(\delta*p_1)\cdots\widehat{(\delta*p_j)}\cdots(\delta*p_{t+1})t!\delta^{t+1}$$

$$= \sum_{j=1}^{t+1}(\delta*p_1)\cdots(\delta*p_{t+1})t!\delta^{t+2}$$

$$= (\delta*p_1)\cdots(\delta*p_{t+1})(t+1)!\delta^{t+2}.$$

$\square$

Lemma 5.34. *Let $p_1,\ldots,p_t$, $t\geq 1$, be polynomials in R of degree one and let $\delta\in\operatorname{Der}(R)$ be a derivation with coefficients in k such that $\delta*p_j\neq 0$, for $j=1,\ldots,t$. Then, for any $m>0$, we may write $p_1\cdots p_t\delta^m$ as a k-linear combination of products of $p_1\cdots p_t\delta^l$ with $1\leq l\leq 2t+1$.*

Proof. If $m\leq 2t+1$ there is nothing to prove, so assume that $m>2t+1$. Then, by Lemma 5.33, we have

$$\sum_{i=0}^{t}(-1)^i\binom{t}{i}p_1\cdots p_t\delta^{m-1-i}p_1\cdots p_t\delta^{t+1+i}$$

$$= p_1\cdots p_t\delta^{m-2t-2}\left(\sum_{i=0}^{t}(-1)^i\binom{t}{i}\delta^{2t+1-i}p_1\cdots p_t\delta^i\right)\delta^{t+1}$$

$$= p_1\cdots p_t\delta^{m-2t-2}(\delta*p_1)\cdots(\delta*p_t)t!\delta^{t+1}\delta^{t+1}$$

$$= (\delta*p_1)\cdots(\delta*p_t)t!p_1\cdots p_t\delta^m,$$

i.e.,

$$p_1 \cdots p_t \delta^m = \sum_{i=0}^{t} a_i p_1 \cdots p_t \delta^{m-1-i} p_1 \cdots p_t \delta^{t+1+i}, \tag{13}$$

where $a_i \in k \setminus \{0\}$, for $i = 0, \ldots, t$. We observe that in each term in the sum (13) we have

$$1 \le m - 1 - i < m \quad \text{and} \quad 1 \le t + 1 + i \le 2t + 1.$$

Hence the lemma follows, by induction on m. $\qquad\square$

Example 5.11 (continued from page 16). *Finding generators for $\mathcal{D}(A)$ as a k-algebra, where A is the coordinate ring of three distinct lines in the plane, we have, using the same notation as before,*

(i) *Three parallel lines.*

    By Example 5.10, $\mathcal{D}(A)$ is generated by A and a derivation δ_1, with coefficients in k, which annihilates all of $p_1, p_2,$ and p_3.

In cases (ii), (iii), and (iv) it is clear that, as in the proof of Corollary 5.30, any product of two generators of the form $\overline{p}\delta_i$, with different i:s, equals zero. Thus, using Lemma 5.34, we get:

(ii) *Two parallel lines intersecting a third.*

    In this case $\mathcal{D}(A)$ is generated by A and

$$\{\overline{p_3}\delta_1^m \mid m = 1, 2, 3\} \cup \{\overline{p_1 p_2}\delta_2^m \mid m = 1, 2, 3, 4, 5\}.$$

(iii) *Three nonparallel lines with no common point of intersection.*

    Here $\mathcal{D}(A)$ is generated by

$$\{\overline{x(x - y + 1)}\partial_x^m, \overline{xy}(\partial_x + \partial_y)^m, \overline{y(x - y + 1)}\partial_y^m \mid m = 1, 2, 3, 4, 5\},$$

    together with A.

(iv) *Three lines intersecting in one point.*

    This case follows from Proposition 5.38.

Theorem 5.35. *Suppose that $I = \langle p_1 \cdots p_r \rangle$ defines a generic arrangement. Then $\mathcal{D}(I)$ is finitely generated as a k-algebra. More precisely:*

    If $r \le n$, then $\mathcal{D}(I)$ is generated by

$$\{x_1, \ldots, x_n, \delta_{r+1}, \ldots, \delta_n\} \cup \{p_i \delta_i, p_i \delta_i^2, p_i \delta_i^3 \mid i = 1, \ldots, r\},$$

    where $\{p_1 \delta_1, \ldots, p_r \delta_r, \delta_{r+1}, \ldots, \delta_n\}$ is the set D from Definition 5.14.

    If $r > n$, then $\mathcal{D}(I)$ is generated by

$$\{x_1, \ldots, x_n, \varepsilon\} \cup \{P_\alpha \delta^\alpha \mid \alpha_i \le 2(r - n) + 3 \text{ for } i = 1, \ldots, s\},$$

    with P_α as in Definition 5.16 for each $\alpha \in \mathbb{N}^s$.

Proof. First suppose that $r \leq n$. Then every R-module generator of $\mathcal{D}(I)$ of order ≥ 1 is of the form $P_\alpha \delta^\alpha$, as defined in 5.16. However, the elements in D commute with each other, so it is enough to show that for $i = 1, \ldots, r$ we may write $p_i \delta_i^m$ as a sum of products of operators of lower order, if m is large enough. It follows from Lemma 5.34 that this is possible if $m > 3$.

Hence $\{x_1, \ldots, x_n, \delta_{r+1}, \ldots, \delta_n\} \cup \{p_i \delta_i, p_i \delta_i^2, p_i \delta_i^3 \mid i = 1, \ldots, r\}$ generate $\mathcal{D}(I)$ as a k-algebra.

Now suppose that $r > n$. The R-module generators ε_m are k-linear combinations of powers of ε, so it remains to show that each R-module generator of the form $P_\alpha \delta^\alpha$ may be expressed using lower order operators, if $|\alpha|$ is large enough. In fact, we will show that any operator of the form $P_\alpha \delta^\alpha$ may be written as a k-linear combination of products of operators of the form $P_\beta \delta^\beta$ with $\beta_i \leq 2(r-n) + 3$, for $i = 1, \ldots, s$, and such that $\beta_i = 0$ if $\alpha_i = 0$.

Therefore assume that $P_\alpha \delta^\alpha$ is an R-module generator of $\mathcal{D}(I)$ with $\alpha_i > 2(r-n) + 3$ for some i.

The derivations $\delta_1, \ldots, \delta_s$ all commute with each other, so any product $\delta^\alpha = \delta_1^{\alpha_1} \cdots \delta_s^{\alpha_s}$ may be rearranged as $\delta^\alpha = \delta_i^{\alpha_i} \delta_1^{\alpha_1} \cdots \widehat{\delta_i^{\alpha_i}} \cdots \delta_s^{\alpha_s} = \delta_i^{\alpha_i} \delta^{\alpha - \alpha_i e_i}$.

Furthermore, $P_\alpha = P' P_i$, where P' is the product of those p_j:s in P_α which are annihilated by δ_i, or $P' = 1$ if no p_j in P_α is annihilated by δ_i.

Since the arrangement defined by $p_1, \ldots, p_r$ is generic, each δ_i annihilates precisely $n-1$ of the p_j:s. Thus, for every $i = 1, \ldots, s$, we have $\deg P_i = r - n + 1$. Lemma 5.34 shows that we may write $P_\alpha \delta^\alpha = P' P_i \delta_i^{\alpha_i} \delta^{\alpha - \alpha_i e_i}$ as a sum in which each term is of the form

$$a P' P_i \delta_i^{m_1} \cdots P_i \delta_i^{m_l} \delta^{\alpha - \alpha_i e_i},$$

where $a \in k \setminus \{0\}$ and $1 \leq m_j \leq 2(r-n) + 3$, for $j = 1, \ldots, l$.

Since δ_i annihilates all factors in P', it annihilates P'. Whence P' and δ_i commute, so

$$
\begin{aligned}
a P' P_i \delta_i^{m_1} \cdots P_i \delta_i^{m_l} \delta^{\alpha - \alpha_i e_i} &= a P_i \delta_i^{m_1} \cdots P_i \delta_i^{m_{l-1}} P' P_i \delta_i^{m_l} \delta^{\alpha - \alpha_i e_i} \\
&= a P_i \delta_i^{m_1} \cdots P_i \delta_i^{m_{l-1}} P_\alpha \delta_1^{\alpha_1} \cdots \delta_i^{m_l} \cdots \delta_s^{\alpha_s} \\
&= a P_i \delta_i^{m_1} \cdots P_i \delta_i^{m_{l-1}} P_\nu \delta^\nu,
\end{aligned}
$$

where $\nu = (\alpha_1, \ldots, \alpha_{i-1}, m_l, \alpha_{i+1}, \ldots, \alpha_s)$, and $P_\nu = P_\alpha$, since $m_l \geq 1$.

We may now do the same thing with $P_\nu \delta^\nu$, if some $\nu_j > 2(r-n) + 3$, and continuing like this, we will eventually end up with a k-linear combination of products of operators of the form $P_\beta \delta^\beta$ with $\beta_i \leq 2(r-n) + 3$, for each $i = 1, \ldots, s$, and such that $\beta_i \neq 0$ only if $\alpha_i \neq 0$.

Hence $\{x_1, \ldots, x_n, \varepsilon\} \cup \{P_\alpha \delta^\alpha \mid \alpha_i \leq 2(r-n) + 3 \text{ for } i = 1, \ldots, s\}$ generate $\mathcal{D}(I)$ as a k-algebra. $\qquad \square$

By applying Theorem 2.5 we get

Corollary 5.36. *Let A be the coordinate ring of a generic arrangement, defined by the ideal $\langle p_1 \cdots p_r \rangle \subseteq R$. Then $\mathcal{D}(A)$ is finitely generated as a k-algebra.*

With notation as in Theorem 5.35:

If $r \leq n$, then $\mathcal{D}(A)$ is generated by

$$\{\overline{x_1}, \ldots, \overline{x_n}, \delta_{r+1}, \ldots, \delta_n\} \cup \{\overline{p_i} \delta_i, \overline{p_i} \delta_i^2, \overline{p_i} \delta_i^3 \mid i = 1, \ldots, r\}.$$

If $r > n$, then $\mathcal{D}(A)$ is generated by

$$\{\overline{x_1}, \ldots, \overline{x_n}, \overline{\varepsilon}\} \cup \{\overline{P_\alpha} \delta^\alpha \mid \alpha_i \leq 2(r - n) + 3 \text{ for } i = 1, \ldots, s\},$$

where $\overline{\varepsilon} = \sum \overline{x_i} \partial_i$.

Remark. As we remarked after Corollary 5.29, $\overline{P_\alpha} \neq 0$ if and only if $\delta^\alpha \in V_i^{|\alpha|}$, for some i.

Again, the case $n = 2$ takes a special position. Before stating our next proposition we will need a definition to facilitate its proof.

Definition 5.37. *For a monomial $x^\alpha \partial^\beta \in \mathcal{D}(R)$, we define its* total degree, *as* $\text{totdeg}(x^\alpha \partial^\beta) = |\alpha| - |\beta|$. *A nonzero operator θ is of* homogeneous total degree t, *if all monomials occuring in θ have total degree t. In this case we write $t = \text{totdeg}(\theta)$.*

It is clear that if θ and ψ are operators of homogeneous total degree, then so is $\theta\psi$, and

$$\text{totdeg}(\theta\psi) = \text{totdeg}(\psi\theta) = \text{totdeg}(\theta) + \text{totdeg}(\psi).$$

Proposition 5.38. *Suppose $p_1, \ldots, p_r \subseteq k[x, y]$ define a central arrangement. Let P_i and δ_i be as in Corollary 5.30, and let A be the coordinate ring of the arrangement. Then $\mathcal{D}(A)$ is generated, as a k-algebra, by*

$$\{\overline{y}, \delta_1\} \quad or \quad \{\overline{x}, \overline{y}\} \cup \{\overline{p_1} \delta_1^m, \overline{p_2} \delta_2^m \mid m = 1, 2, 3\},$$

if r is one or two, respectively, and by

$$\{\overline{x}, \overline{y}, \overline{\varepsilon}\} \cup \{\overline{P_i} \delta_i^m \mid i = 1, \ldots, r \text{ and } m = 1, \ldots, 2r - 1\},$$

if $r \geq 3$. Thus $\mathcal{D}(A)$ is generated by operators of order $\leq 2r - 1$. Moreover, $\mathcal{D}(A)$ cannot be generated by operators of lower order than this.

Proof. The first part of the proposition follows from Corollary 5.36 together with the subsequent remark.

To show the second part, let I be the defining ideal for the arrangement under consideration. By Corollary 5.30, any product of k-algebra generators of $\mathcal{D}(I)$ is a k-linear combination of products of x, y, ε, and operators of the form $P_i \delta_i^m$ for some m, together with operators with polynomial coefficients in I. All operators of the form $P_i \delta_i^m$, as well as ε, have homogeneous total degree. Since $\text{totdeg}(x) = \text{totdeg}(y) = 1$ and $\text{totdeg}(\varepsilon) = 0$, if θ is any product of k-algebra generators for $\mathcal{D}(I)$, not inducing the zero operator in $\mathcal{D}(A)$, it must be a sum of operators ψ of homogeneous total degree, such that

$$\text{totdeg}(\psi) \geq \text{totdeg}(P_i \delta_i^{m_1} \cdots P_i \delta_i^{m_t}) = t(r - 1) - \sum_{l=1}^{t} m_l,$$

for some i. Assuming that $m_l < 2r - 1$ for each l we get

$$\text{totdeg}(P_i \delta_i^{m_1} \cdots P_i \delta_i^{m_t}) = t(r - 1) - \sum_{l=1}^{t} m_l \geq t(r - 1) - 2t(r - 1)$$

$$= t(1 - r) > -r = \text{totdeg}(P_j \delta_j^{2r-1}),$$

for each j. Hence no $\overline{P_j} \delta_j^{2r-1}$ can be expressed using lower order operators in $\mathcal{D}(A)$. $\square$

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