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THE COUNTERPART FOR HANKEL-SCHUR MULTIPLIERS OF EBERLEIN'S DECOMPOSITION

Mats Erik Andersson

Abstract. The decomposition into almost periodic and weakly null periodic part is achieved for two natural multiplier algebras. In fact, the subspaces of almost periodic multipliers turn out to be complemented subspaces in each case. Some applications are discussed.

Two sequence spaces on $\mathbb{N}$ are at the centre of attention in this text. On the one hand, the multipliers of the Hardy space, written $M(H^1)$, is the Banach algebra consisting of the sequences $b = \{b_n\}_{n=0}^{\infty}$ such that

$$\forall f \in H^1(\mathbb{T}), \quad \sum_{n=0}^{\infty} b_n \hat{f}(n) z^n \in H^1.$$

$M(H^1)$ is supplied with the natural multiplier norm.

On the other hand, M^H denotes the Banach algebra of Hankel-Schur multipliers with norm induced from V_2 in the following sense. A sequence $b \in \ell^\infty(\mathbb{N})$ is identified with a Hankel matrix Γ_b through $(\Gamma_b)_{ij} = b(i+j)$. Here

$$b \in M^H \quad \text{iff} \quad \Gamma_b \in V_2; \qquad \| b \|_{M^H} = \| \Gamma_2 \|_{V_2}.$$

The space $V_2 = M(B(\ell^2))$ of Schur multipliers consists of matrices indexed by $\mathbb{N} \times \mathbb{N}$ such that the element-wise product of matrices makes $M \in V_2$ to a bounded multiplier on $B(\ell^2)$, the bounded operators on $\ell^2(\mathbb{N})$. The norm is

$$\| M \|_{V_2} = \sup \left\{ \| M \cdot A \|_{B(\ell^2)} / \| A \|_{B(\ell^2)} \; ; A \in B(\ell^2),\ A \neq 0 \right\},$$

and using element-wise addition and multiplication, V_2 becomes a Banach algebra. Bennett [Be] is a good reference.

The subalgebra of V_2 consisting of Toeplitz matrices is isometrically isomorphic to the measure algebra $M(\mathbb{T})$, whereas the subalgebra of Hankel matrices includes $M(\mathbb{T})$ in a non-isometric manner via

$$\mu \in M(\mathbb{T}) \quad \leftrightarrow \quad \hat{\mu}\big|_{\mathbb{N}} \in M^H \quad \leftrightarrow \quad \{\hat{\mu}(i+j)\}_{i,j=0}^{\infty} \in V_2.$$

It is known that

(NCL) $$\| b \|_{M(H^1)} \le \| b \|_{M^H} \le \| b \|_{B(\mathbb{N})}$$

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whenever two of the members make sense. Here $B(\mathbb{N})$ is the Fourier-Stieltjes quotient algebra over $\mathbb{N}$ induced by $M(\mathbb{T})$.

It is also known that every member of M^H is weakly almost periodic as a member of $\ell^\infty(\mathbb{N})$ with left shift, but that $M(H^1)$ neither is included in nor includes that space of weakly almost periodic sequences. The first fact is due to Bennett, while Bożejko and Lust-Piquard have constructed the necessary counterexamples relating to $M(H^1)$.

The following text is dealing with (weak) almost periodicity and uses

$$\mathrm{AP} = \mathrm{AP}(\mathbb{Z})|_{\mathbb{N}}, \quad \mathrm{WAP} = \mathrm{WAP}(\mathbb{Z})|_{\mathbb{N}}, \quad \text{and } \mathrm{WNAP} = \mathrm{WNAP}(\mathbb{Z})|_{\mathbb{N}}$$

to denote the special closed subspaces of ℓ^∞ consisting of almost periodic, weakly almost periodic, and weakly null sequences, respectively, derived by restriction.

Recall first that $B(\mathbb{Z})$ can be topologically decomposed as

$$B(\mathbb{Z}) = M_d(\mathbb{T})\hat{}\ \oplus\ M_c(\mathbb{T})\hat{} = B_d(\mathbb{Z})\ \oplus\ B_c(\mathbb{Z}), \tag{1}$$

where the Lebesgue decomposition yields

$$x \in B(\mathbb{Z}), \quad x = x_d + x_c, \quad \| x_d \|_{B(\mathbb{Z})} \leq \| x \|_{B(\mathbb{Z})} .$$

Here $x_d \in M_d(\mathbb{T})$ is unique. According to Hewitt–Eberlein's well known theorem

$$\begin{aligned} &B_d(\mathbb{Z}) = B(\mathbb{Z}) \cap \mathrm{AP}(\mathbb{Z}) \quad \text{and} \quad \| x_d \|_{\ell^\infty} \leq \| x \|_{\ell^\infty} , \\ &B_c(\mathbb{Z}) = B(\mathbb{Z}) \cap \mathrm{WNAP}(\mathbb{Z}). \end{aligned}$$

A straightforward case of quotient norms then produces another decomposition out of the first instance above:

$$\begin{aligned} &B(\mathbb{N}) = M_d(\mathbb{T})\hat{}\big|_{\mathbb{N}} \oplus M_c(\mathbb{T})\hat{}\big|_{\mathbb{N}} = B_d(\mathbb{N})\ \oplus\ B_c(\mathbb{N}) \\ &x = x_d + x_c \quad \text{with} \quad \| x_d \|_{B(\mathbb{N})} \leq \| x \|_{B(\mathbb{N})} . \end{aligned} \tag{2}$$

Since $\mathbb{N}$ determines mean values in $\mathrm{WAP}(\mathbb{Z})$ (Glicksberg, see [W]), each $x \in \mathrm{AP}$ extends uniquely to $\tilde{x} \in \mathrm{AP}(Z)$ and thus Wells [W] gave an extension of (2) to $B(\mathbb{N})$; cf. also Porada [Po].

$$\begin{aligned} &B(\mathbb{N}) = B_d(\mathbb{N})\ \oplus\ B_c(\mathbb{N}), \quad \| x_d \|_{\ell^\infty} \leq \| x \|_{\ell^\infty} , \quad x = x_d + x_c, \\ &B_d(\mathbb{N}) = B(\mathbb{N}) \cap \mathrm{AP}, \qquad B_c(\mathbb{N}) = B(\mathbb{N}) \cap \mathrm{WNAP}. \end{aligned} \tag{3}$$

The motivation behind the present ideas was a result of Lust-Piquard:

Theorem ([LP1] or [LP2]). $M(H^1) \cap \mathrm{AP} = M_d(\mathbb{T})\hat{}\big|_{\mathbb{N}}$.

It follows that

$$M(H^1) \cap \mathrm{AP} = M^H \cap \mathrm{AP} = B(\mathbb{N}) \cap \mathrm{AP} = B_d(\mathbb{N}),$$

where as introduced above $B_d(\mathbb{N}) = M_d(\mathbb{T})\hat{}\big|_{\mathbb{N}}$.

The aim now is to establish decompositions of M^H and $M(H^1)$ reminiscent of (1), say. Toward this end Eberlein's classical theory of ergodicity, [E1], presents the right means to accomplish this. A preparatory property must first be secured in order to apply the mentioned framework.

Proposition 1. *The norm closed unit balls of* $B(\mathbb{N})$, M^H, *and* $M(H^1)$ *are closed in the topology of pointwise convergence in* ℓ^∞.

Proof. Consider the case of $B(\mathbb{N})$ and a net c_α in the unit ball with $c(k) = \lim_\alpha c_\alpha(k)$ for all $k \geq 0$. To any positive ε there are measures μ_α with $\hat{\mu}_\alpha|_{\mathbb{N}} = c_\alpha$ and $\| \mu_\alpha \|_{M(\mathbb{T})} \leq 1 + \varepsilon$. By weak-$*$ compactness for measures, there is a subnet β with

$$\mu_\beta \xrightarrow{*} \mu \quad \text{and} \quad \hat{\mu}(k) = \lim_\beta \hat{\mu}_\beta(k) = c(k), \text{ all } k \geq 0.$$

Thus $\hat{\mu} = c$ in $B(\mathbb{N})$ and by the semicontinuity of the norm

$$\| c \|_{B(\mathbb{N})} \leq \| \hat{\mu} \|_{B(\mathbb{Z})} = \| \mu \|_{M(\mathbb{T})} \leq \underline{\lim}_{\beta} \| \mu_\beta \|_{M(\mathbb{T})} \leq 1 + \varepsilon.$$

It follows that the weak limit c belongs to the unit ball of $B(\mathbb{N})$.

Consider next $c_\alpha \in M(H^1)$, $\| c_\alpha \|_{M(H^1)} \leq 1$, and $c_\alpha \to c$ weakly in ℓ^∞. For each polynomial $f(z) = \sum_0^N a_n z^n$ the multiplier acts by $[cf](z) = \sum_{n=0}^N c(n) a_n z^n$. Thus

$$\begin{aligned} \| cf \|_{H^1} &= \int_0^{2\pi} \left| \sum_{n=0}^N c(n) a_n e^{in\theta} \right| \frac{d\theta}{2\pi} \leq \underline{\lim}_{\alpha} \| c_\alpha f \|_{H^1} \\ &\leq \underline{\lim}_{\alpha} \| c_\alpha \|_{M(H^1)} \| f \|_{H^1} \leq \| f \|_{H^1} . \end{aligned}$$

Due to the density of polynomials in H^1, it follows that $c \in M(H^1)$ and additionally $\| c \|_{M(H^1)} \leq 1$.

Finally, the case of M^H; take c_α analogous to the above. For any $A \in B(\ell^2)$ and finitely supported x, $y \in \ell^2$, the Hankel matrix Γ_c acts according to

$$\langle (\Gamma_c \cdot A)x, y \rangle = \sum_{k,l=0}^N x_l c(k+l) a_{kl} \overline{y_k}$$

for some finite N only depending on the support of x and y. Thus

$$\begin{aligned} |\langle (\Gamma_c \cdot A)x, y \rangle| &\leq \underline{\lim}_{\alpha} \left| \textstyle\sum_{k,l=0}^N x_l c_\alpha(k+l) a_{kl} \overline{y_k} \right| \\ &= \underline{\lim}_{\alpha} |\langle (\Gamma_{c_\alpha} \cdot A)x, y \rangle| \leq \underline{\lim}_{\alpha} \| \Gamma_{c_\alpha} \|_{V_2} \| A \|_{B(\ell^2)} \| x \|_{\ell^2} \| y \|_{\ell^2} . \end{aligned}$$

From this follow $\mathcal{M}c \in V_2$ and $\| \mathcal{M}c \|_{V_2} \leq 1$, which is the actual membership in the unit ball of M^H for the weak limit sequence c. This completes the proof.

Returning to Eberlein's theory, it is convenient to remind oneself of the essential notions and definitions, here tailored to the present needs.

Take $E = \ell^\infty$, the shift $Tx(n) = x(n+1)$, $T^0 = \text{id}$, and $G = \{T^n ; n \geq 0\}$. This is Eberlein's Example 1! Then G is an abelian, bounded semigroup of norm one operators on ℓ^∞ possessing as almost invariant means the operators $U_n = (n+1)^{-1} \sum_0^n T^j$. Thus G is ergodic with the natural von Neumann mean

$$\mathfrak{M}x = \lim_{n\to\infty} U_n x(0) = \lim_{n\to\infty} (n+1)^{-1} \sum_{j=0}^n x(j).$$

The orbit of $x \in \ell^\infty$ is the set $\mathcal{O}(x)$ consisting of the finite convex combination of different $T^n x$.

Definition. (1) $x \in \ell^\infty$ is ergodic if $\{U_n x\}_{n=0}^\infty$ has a weak accumulation point.
(2) $x \in \ell^\infty$ is (weakly) almost periodic if $\{T^n x\}_{n=0}^\infty$ is (weakly) relatively compact.

It must be remarked that in all three cases the accumulation point is unique and T-invariant, thus constant. The corresponding spaces $\mathfrak{A} \subseteq \mathfrak{W} \subseteq \Gamma$ are closed T-invariant subspaces of ℓ^∞. The proof of [E1], Thm 12.1 goes through verbatim and proves $\mathfrak{W}$ to be a B^*-algebra. Another important closed subspace of $\mathfrak{W}$ to mention is $\mathfrak{W}_0 = \{x \in \mathfrak{W}\,;\, \mathfrak{M}|x| = 0\}$. It is known that

$$\mathfrak{A} = \mathrm{AP} \oplus c_0 \quad \text{and} \quad \mathfrak{W} = \mathrm{WAP},\ \mathfrak{W}_0 = \mathrm{WNAP}$$

with equal norms. This direct sum is observed in [dLG2], section 6. The unitary, irreducible representations of $\mathbb{N}$ are in one-to-one correspondence with $\mathbb{T}$ via the character $\rho_\alpha(n) = \alpha^n$. Save for the restriction to $\mathbb{N}$, these are identical with the same kind of representations of $\mathbb{Z}$. Thus one concludes $\mathfrak{W}_p = \mathrm{AP}$, where $\mathfrak{W}_p$ is the closure in ℓ^∞ of the unitary subspaces in the sense of [dLG1], section 4. Then the same paper in Theorem 5.7 establishes for the present case a decomposition:

Lemma 2. $\mathfrak{W} = AP \oplus \mathfrak{W}_0$.

Observe in passing that $c_0 \subseteq \mathfrak{A}$ is contained in $\mathfrak{W}_0$. It is the semicharacters of $\mathbb{N}$, standing in bijective correspondence to the closed unit disk, that accounts [dLG2] for the decomposition $\mathfrak{A} = AP \oplus c_0$.

A peculiar convolution-like product is necessary for the following development.

Definition. Let $x \in \mathfrak{W}$ and $y \in \mathrm{WAP}(\mathbb{Z})$. Their quasi-convolution $z = x * y$ is for $t \in \mathbb{Z}$ defined by the expression

$$z(t) = \mathfrak{M}[x(\cdot)y(t-\cdot)] = \lim_{n\to\infty} (n+1)^{-1} \sum_{j=0}^{n} x(j)y(t-j).$$

That the above von Neumann mean exists follows on grounds that $y(t-\cdot)\big|_{\mathbb{N}} \in \mathfrak{W}$ for all $t \in \mathbb{Z}$, and that $\mathfrak{W}$ is a B^*-algebra.

Lemma 3. *$z = x * y$ is a member of* $\mathrm{AP}(\mathbb{Z})$.

Proof. (Imitating Eberlein [E1], Thm 15.1.) Let (u_n) be a sequence in $\mathbb{N}$. Then $z(t+u_n) = \mathfrak{M}[x(\cdot+u_n)y(t-\cdot)]$. From $x \in \mathfrak{W}$ follows the existence of a subsequence (u_n') as well as $\tilde{x} \in \mathfrak{W}$ such that $x(\cdot + u_n') \to \tilde{x}$ weakly.

Next the Cauchy-Schwarz inequality produces

$$\begin{aligned}
\big|z(t+u_n') - \mathfrak{M}[\tilde{x}(\cdot)y(t-\cdot)]\big|^2 &= \big|\mathfrak{M}[\{x(\cdot+u_n') - \tilde{x}(\cdot)\}\, y(t-\cdot)]\big|^2 \\
&\le \mathfrak{M}[|\tilde{x}(\cdot) - x(\cdot+u_n')|^2]\ \mathfrak{M}[|y(t-\cdot)|^2] \\
&\le \mathfrak{M}[|\tilde{x}(\cdot) - x(\cdot+u_n')|^2]\ \ \|\, y\,\|_{\ell^\infty}^2 .
\end{aligned}$$

The last expression tends to zero with increasing n, due to $|\tilde{x}(\cdot) - x(\cdot+u_n')|^2 \to 0$ weakly, so the uniform convergence in t of $z(t+u_n') \to \mathfrak{M}[\tilde{x}(\cdot)y(t-\cdot)]$ as $n \to \infty$ is a consequence. Thus $z \in \mathrm{AP}(\mathbb{Z})$.

With the quasi-convolution we have the possibility of imitating Eberlein's "abstract summability theory" as presented in [E2], section two. The original proofs were there performed on an lca group. However, the quasi-convolution easily allows the proofs of [E2] to be performed on $\mathbb{N}$ instead of $\mathbb{Z}$, almost verbatim.

Lemma 4. *Let $x \in \mathrm{WAP}$ and let $y \in \mathrm{AP}(\mathbb{Z})$ be such that $y(t) \geq 0$, $y(t) = y(-t)$, for all $t \in \mathbb{Z}$, and $\mathfrak{M}y = 1$. Then $x * y\big|_{\mathbb{N}}$ is a member of $\overline{\mathcal{O}(x)}$, the closed hull of the orbit of x in ℓ^∞.*

Proof (Essentially [E2], Lemma 1). Recall first that $\{U_n\}_0^\infty$ is an equicontinuous collection of operators. By almost periodicity $\{y_t = y(\cdot - t)\}_{t\in\mathbb{Z}}$ is relatively compact in ℓ^∞, whence the same holds for $\{xy_t = x(\cdot)y_t(\cdot - t)\}_{t\in\mathbb{Z}}$. It follows that

$$(*) \qquad x * y(t) = \mathfrak{M}[x(\cdot)y(\cdot - t)] = \lim_n U_n[xy_t]$$

is uniform in t. Observe however, that the right-hand side of $(*)$ deals with sequences in ℓ^∞, whereas the left-hand side of $(*)$ has to be interpreted for fixed t as the constant sequence $\{x * y(t)\}_{s=0}^\infty$. By definition of U_n the above uniform convergence expresses the relation

$$\lim_{n\to\infty} \sup_{s\geq 0} \sup_{t\in\mathbb{Z}} \left| x * y(t) - \sum\nolimits_{j=0}^{n} (n+1)^{-1} x(j+s) y(j+s-t) \right| = 0.$$

This insures to each $\varepsilon > 0$ the existence of $\nu = \nu(\varepsilon)$ such that $n \geq \nu$ implies simultaneously (take $s = t$)

$$\sup_{t\geq 0} \left| x * y(t) - \sum\nolimits_{j=0}^{n} (n+1)^{-1} x(j+t) y(j) \right| < \varepsilon$$

and

$$\left|1 - b_n^{-1}\right| < \varepsilon / \|x\|\,\|y\|, \quad b_n = \sum\nolimits_{j=0}^{n} (n+1)^{-1} y(j).$$

The last property is due to $\mathfrak{M}y = 1$, granted the exclusion of the trivial case $x = 0$.

Now the inequality

$$\begin{aligned} \Big| x * y(t) - &\sum\nolimits_{j=0}^{n} (n+1)^{-1} b_n^{-1} y(j) x(j+t) \Big| \\ &\leq \Big| x * y(t) - \sum\nolimits_{j=0}^{n} (n+1)^{-1} y(j) x(j+t) \Big| \\ &\qquad + \left|1 - b_n^{-1}\right| \Big| \sum\nolimits_{j=0}^{n} (n+1)^{-1} y(j) x(j+t) \Big| \\ &< \varepsilon + \frac{\varepsilon}{\|x\|\,\|y\|} \cdot \|x\|\,\|y\| = 2\varepsilon \end{aligned}$$

holds for all $t \geq 0$ and $n \geq \nu$. Rewrite this as

$$\Big\| x * y|_{\mathbb{N}} - \sum\nolimits_{j=0}^{n} (n+1)^{-1} b_n^{-1} y(j) x(\cdot + j) \Big\|_{\ell^\infty} < 2\varepsilon.$$

On grounds that $\sum_{j=0}^{n} (n+1)^{-1} b_n^{-1} y(j) = 1$, clearly $\sum_{j=0}^{n} (n+1)^{-1} b_n^{-1} y(j) x(\cdot + j)$ becomes a member of $\mathcal{O}(x)$. The work above lets us conclude $x * y|_{\mathbb{N}} \in \overline{\mathcal{O}(x)}$, which is exactly the claim.

A combination of the two lemmata shows

Corollary 5. *For x and y as in Lemma 4, the membership $x * y|_{\mathbb{N}} \in \mathrm{AP} \cap \overline{\mathcal{O}(x)}$ obtains.*

It is clear from the definition of the quasi-convolution that $\|x * y\| \leq \|x\| \, \|y\|$ in $\ell^\infty(\mathbb{Z})$. In particular, $V_y : \mathfrak{W} \to \mathrm{AP}(\mathbb{Z}),\ x \mapsto x * y$ is norm-to-norm-continuous for each fixed $y \in \mathrm{WAP}(\mathbb{Z})$.

Consideration of the almost periodic compactification of $\mathbb{Z}$ allows by standard procedures the selection of a bounded approximative unit $\{y_\alpha\}$ in $\mathrm{AP}(\mathbb{Z})$ with the properties $y_\alpha \geq 0$, $y_\alpha(t) = y_\alpha(-t)$, and $\mathfrak{M} y_\alpha = 1$. These are clearly chosen to fit with Corollary 5. Write for convenience $V_\alpha = V_{y_\alpha}$. The mechanisms employed by Eberlein [E2], Lemma 2, are perfectly applicable to prove the following result, whose proof will not be repeated.

Lemma 6. *The collection of operators defined on $\mathfrak{W}$ by $T_y x = x - V_y x\big|_{\mathbb{N}}$ as y ranges over $\mathrm{AP}(\mathbb{Z})$ constitutes a convex abelian semigroup admitting the family of almost invariant integrals T_α introduced by $T_\alpha x = x - V_\alpha x\big|_{\mathbb{N}}$.*

Consider next any fixed $x \in \mathfrak{W}$. Since $\overline{\mathcal{O}(x)}$ is weakly compact, the collection $\{x * y_\alpha\big|_{\mathbb{N}}\}$ is relatively weakly compact, so possesses a weak accumulation point $x_a \in \overline{\mathcal{O}(x)}$. Then $x - x * y_\alpha\big|_{\mathbb{N}}$ has a weak accumulation point $x_0 = x - x_a$. The ergodic theorem of Eberlein thus implies the norm-convergence (in ℓ^∞) and T_y-invariance expressed in

$$\lim_\alpha \left(x - x * y_\alpha\big|_{\mathbb{N}}\right) = \lim_\alpha T_\alpha x = x_0, \qquad V_y x_0 = x_0, \quad \text{all } y \in \mathrm{AP}(\mathbb{Z}).$$

In particular, x_0 and x_a are unique. The latter property above expresses $x_0 * y\big|_{\mathbb{N}} = 0$ for all $y \in \mathrm{AP}(\mathbb{Z})$. Furthermore,

$$x_a = x - \lim_\alpha T_\alpha x = \lim_\alpha x * y_\alpha\big|_{\mathbb{N}} \in \overline{\mathrm{AP}}^{\ell^\infty} = \mathrm{AP}.$$

This means that $x_a = \overline{\mathcal{O}(x)} \cap \mathrm{AP}$.

Next, the membership $x_0 \in \mathfrak{W}_0$ is claimed. In fact, Lemma 2 admits a decomposition $x_0 = x_1 + x_2$, with $\tilde{x}_1 \in \mathrm{AP}(\mathbb{Z})$, $x_1 = \tilde{x}_1|_{\mathbb{N}}$, and $x_2 \in \mathfrak{W}_0$. However, it is clear by definition that this property of x_2 forces $x_2 * y = 0$ for any $y \in \mathrm{WAP}(Z)$. Thus

$$x_1 * y = x_0 * y = 0, \quad \text{for all } y \in \mathrm{AP}(Z).$$

This holds in particular for any character $y(t) = e^{it\theta}$, $\theta \in [0, 2\pi[$, whence Eberlein's version [E1], Thm 15.2, of Parseval's equation uncovers $x_1 = 0$. This is so since the invariant means of $z \in \mathrm{AP}(\mathbb{Z})$ and $z|_{\mathbb{N}} \in \mathrm{AP}$ over $\mathbb{Z}$ and $\mathbb{N}$, respectively, coincide. Now the membership $x_0 = x_2 \in \mathfrak{W}_0$ follows. In condensed form:

Theorem 7. $\mathfrak{W} = \mathrm{AP} \oplus \mathfrak{W}_0, \quad x = x_a + x_0, \quad \|x_a\| \leq \|x\|, \quad x_a \in \overline{\mathcal{O}(x)} \cap \mathrm{AP}.$

This result is only a refinement of Lemma 2 in the way it nominally gives a description how to calculate the almost periodic part x_a, a procedure whose for now only relevant property is expressed in the membership $x_a \in \overline{\mathcal{O}(x)} \cap \mathrm{AP}$, for given $x \in \mathfrak{W}$. It can be applied to the present multiplier spaces as follows.

Theorem 8. *The following direct sum decompositions hold as stated.*

$$B(\mathbb{N}) = B_d(\mathbb{N}) \oplus B_c(\mathbb{N}) \quad \text{in } \ell^\infty \text{ and } B(\mathbb{N}). \tag{4}$$

$$M^H = B_d(\mathbb{N}) \oplus \mathfrak{W}_0 \cap M^H \quad \text{in } \ell^\infty \text{ and } M^H. \tag{5}$$

$$M(H^1) \cap \mathfrak{W} = B_d(\mathbb{N}) \oplus \mathfrak{W}_0 \cap M(H^1) \quad \text{in } \ell^\infty \text{ and } M(H^1). \tag{6}$$

Here the decomposition $x = x_a + x_0$ has in each case $x_a \in B_d(\mathbb{N})$ and $\| x_a \|_X \leq \| x \|_X$, where $X = \ell^\infty$, $B(\mathbb{N})$, M^H, or $M(H^1)$. In fact, $B_d(\mathbb{N})$ is a complemented subspace in each of $B(\mathbb{N})$, M^H, and $M(H^1)$ via a norm one projection.

Remark. In the case of ℓ^∞-norm, the direct sum is to be interpreted in the algebraic sense, since no one of the spaces is closed in uniform norm. Observe also that $M(H^1) \cap \mathfrak{W}$ is a closed subspace of $M(H^1)$.

Proof. Let X denote any of $B(\mathbb{N})$, M^H, or $M(H^1) \cap \mathfrak{W}$. They are all subsets of $\mathfrak{W}$. If $x \in X$, Proposition 1 says $\overline{\mathcal{O}(x)} \subseteq X$. The previous theorem produces the unique decomposition $x = x_a + x_0$ with $\| x_a \|_\infty \leq \| x \|_\infty$, $x_a \in \overline{\mathcal{O}(x)} \cap \mathrm{AP}$, so the convex combinations involved in $\mathcal{O}(x)$ then contribute with $\| x_a \|_X \leq \| x \|_X$. The theorem of Lust-Piquard provides the membership

$$x_a \in X \cap \mathrm{AP} = B_d(\mathbb{N}) = M_d(\mathbb{T})\hat{}\,\big|_{\mathbb{N}}.$$

Except for the last sentence of the statement, the proof is complete in the light of $x_0 = x - x_a \in \mathfrak{W}_0 \cap X$ and the singular observation $\mathfrak{W}_0 \cap B(\mathbb{N}) = B_c(\mathbb{N})$.

The proof of Theorem 7 shows $Lx = \lim_\alpha x * y_\alpha\big|_{\mathbb{N}}$ to give a well-defined linear mapping $L\colon \mathfrak{W} \to \mathrm{AP} \cap \overline{\mathcal{O}(x)}$. The first part of the present proof adds the fact that L is of norm one when restricted to $B(\mathbb{N})$, M^H, or $M(H^1) \cap \mathfrak{W}$, and it then has values in $\mathrm{AP} = B_d(\mathbb{N})$. Should $x \in \mathrm{AP}$, then $x = Lx + x_0$ by Theorem 7, where $x_0 \in \mathfrak{W}_0$. But then $x_0 = x - Lx \in \mathrm{AP} \cap \mathfrak{W}_0 = \{0\}$, so $x = Lx$. This shows that L is a projection with image AP. The proof is complete.

Remark. The statement (4) was included to stress the similarities with the new results, in spite of it already having been verified earlier in the text.

Example 9. Consider the Paley multiplier $\gamma(n) = 1$ if $n = 3^k$ and 0 otherwise. It is well known that γ is a member of $M(H^1)$ as well as M^H. For the latter, Bożejko's paper [Bo] is suitable. Clearly γ is an idempotent and is a member of $\mathfrak{W}_0$. This fact relates to Corollary 11.

It is also long known that $\gamma \in \overline{B(\mathbb{N})}$, completion in ℓ^∞, which is best achieved using Riesz products; one explicit construction can be found in [DR], Thm. 3.2. It must be remarked that this procedure generates a sequence $a_n \in B(\mathbb{N})$, such that $\| \gamma - a_n \|_{\ell^\infty} \to 0$, and $\| a_n \|_{B(\mathbb{N})} \to \infty$. In fact, this last property cannot be avoided of the following reason.

Were there a sequence $\{b_n\}$ in $B(\mathbb{N})$ such that $\| \gamma - b_n \|_{\ell^\infty} \to 0$, with a bound $\sup_n \| b_n \|_{B(\mathbb{N})} < \infty$, then Proposition 1 would force $\gamma \in B(\mathbb{N})$. However, by Rudin's generalisation of F. and M. Riesz' theorem with lacunary components, it is already known that γ cannot be a member of $B(\mathbb{N})$.

With the direct sum decompositions of Theorem 7 at hand, it is easy to get more information than expressed through $\|x_a\| \leq \|x\|$, simply by following the lead of Glicksberg and Wik [GW].

Proposition 10. *Let $x \in B(\mathbb{N})$, M^H, or $M(H^1) \cap \mathrm{WAP}$. Decompose $x = x_a + x_0$ as above. Then $x_a(\mathbb{N}) \subseteq x(\mathbb{N})^-$ meaning that the closure of $\{x(n)\,;\, n \geq 0\}$ in $\mathbb{C}$ contains the set $\{x_a(n)\,;\, n \geq 0\}$. More generally, $x_a(n_0 + k\mathbb{N}) \subseteq x(n_0 + k\mathbb{N})^-$ for each positive integer k and each $n_0 \geq 0$.*

Proof (Inessential variation of [GW].). Fix a positive integer k and take $\varepsilon > 0$ as well as $n_0 \in \mathbb{N}$ arbitrary. In the light of $x_0 \in \mathfrak{W}_0$ and hence also the same property for any of its compressions $n \mapsto x_0(n_0 + kn)$, it is possible to choose inductively an increasing sequence of integers $\{n_m\}_{m=1}^{\infty}$ such that

$$\bigl|x_0\bigl(n_0 + k[n_m - n_j]\bigr)\bigr| < \varepsilon \quad \text{all} \quad j < m.$$

The component x_a can be represented as

$$x_a(t) = \sum_{r=1}^{\infty} a_r e^{it\theta_r}, \qquad \sum |a_r| < \infty,$$

by Theorem 8. Thus there is an N with $\sum_{r=N+1}^{\infty} |a_r| < \varepsilon$. For the indices $1 \leq r \leq N$ one passes to a subsequence $\{n'_m\}$ of $\{n_m\}$ with the added property that

$$e^{ikn'_m\theta_r} \quad \text{converges as} \quad m \to \infty \quad \text{for each} \quad 1 \leq r \leq N.$$

Then $\bigl|1 - \exp\bigl(ik[n'_j - n'_m]\theta_r\bigr)\bigr| \to 0$ as $m, j \to \infty$. It follows that for $m > j$ sufficiently large

$$\bigl|x_a(n_0) - x\bigl(n_0 + k[n'_m - n'_j]\bigr)\bigr| < 4\varepsilon,$$

which contains the claimed property.

Corollary 11. *Any idempotent in $B(\mathbb{Z})$, $B(\mathbb{N})$, M^H or $M(H^1) \cap \mathrm{WAP}$ can be written as $x = a + b_1 - b_2$. Here $a = \hat{\nu}|_{\mathbb{N}}$ for a measure $\nu \in M_d(\mathbb{T})$ of finite support in $\pi\mathbb{Q} \pmod{2\pi}$, $\hat{\nu}$ being the characteristic function of a union of arithmetic progressions. Furthermore, b_1 and b_2 are idempotents in the same space as x is, with the added property $b_1, b_2 \in \mathfrak{W}_0$.*

Proof. By proposition 3.10 (or [GW] for $x \in B(\mathbb{Z})$) the idempotent x decomposes as $x = a + b$ with $a \in B_d(\mathbb{N})$, $b \in \mathfrak{W}_0$, and $\{a(n)\,;\, n \geq 0\} \subseteq \{0, 1\}$. This range forces a to be periodic and therefore the underlying measure ν with $a = \hat{\nu}|_{\mathbb{N}}$ must have finite support in $\pi\mathbb{Q} \pmod{2\pi}$. The identification of the arithmetic progressions is clear once the periodicity is used to transfer the analysis to a finite cyclic group.

From $b = x - a$ it is obvious that b belongs to the same space as x does, and that additionally its range $\{b(n)\,;\, n \geq 0\}$ is contained in $\{-1, 0, 1\}$. Thus $b_1 = (b + b^2)/2$ and $b_2 = (b^2 - b)/2$ are idempotents in whatever space x belongs to and additionally $b_j \in \mathfrak{W}_0$. Clearly $b = b_1 - b_2$.

Remark. The proof gives the norm estimates $\|b\| \leq 2\|x\|$ and $\|b_j\| \leq \|x\|\bigl(1 + 2\|x\|\bigr)$. These are weaker than the corresponding results in full proofs of Cohen's theorem.

Example 12. Let $E \subseteq \mathbb{N}$ be a Paley set, i.e., a finite union of lacunary sets, then $\chi_E \in \mathfrak{W}_0$ and by Paley's inequality $\chi_E \in M^H \cap M(H^1)$. Thus χ_E is a non-trivial idempotent which is not generated by a measure, the latter since $\mathbb{Z}_+ \cup E$ is a Riesz set, which in turn forces $B(E) \subseteq c_0$ for the Fourier-Stieltjes restriction algebra. Already Hartman [Ha] expressed this observation.

More generally, write $E = \{n(k)\}_{k=0}^{\infty}$ and introduce the mapping $\rho : \ell^\infty \to M^H \subset M(H^1) \cap \mathfrak{W}_0$, $x \mapsto y_\rho$ by $y_\rho(j) = x_{n(k)}$ in case $j = n(k)$ and 0 otherwise. Letting Q_ρ denote the image of ρ, Bożejko's theorem [Bo] proves the unit ball of ℓ^∞ to have image inside a ball in M^H. Clearly $\rho : \ell^\infty \to Q_\rho$ becomes an isomorphism between two infinite dimensional B^*-algebras. Thus M^H and $M(H^1) \cap \mathfrak{W}_0$ do contain infinite dimensional B^*-algebras. This must be compared to the next well-known result and borne in mind when reading Theorem 13 below.

The preceding paragraph includes in particular the fact that the norm of $\chi_{E\cap[0,n]}$ is uniformly bounded in M^H as well as in $M(H^1) \cap \mathrm{WAP}$. This is relevant for Theorem 15.

Proposition [DR, p. 30]**.** *Let B be a commutative B^*-algebra such that B is algebraically $*$-isomorphic to a subalgebra of the Fourier-Stieltjes algebra $M(G)\hat{}$ for some group G. Then B is finite dimensional.*

Bearing Example 12 in mind the next result is the best to hope for.

Theorem 13. *Let B be a commutative B^*-algebra such that B is algebraically $*$-isomorphic to a subalgebra of M^H or $M(H^1) \cap \mathfrak{W}_0$. Then this counterpart of B is contained in $Q_a + J$, where $Q_a \subseteq B_d(\mathbb{N})$ is a finite dimensional, closed subalgebra, while $J \subseteq M^H \cap \mathfrak{W}_0$ or $J \subseteq M(H^1) \cap \mathfrak{W}_0$ is a closed ideal, possibly of infinite dimension*

Proof. Recall first that each $x \mapsto x(n)$ gives a regular maximal ideal for each $n \geq 0$. Thus the radicals of M^H and $M(H^1)$ are trivial and both multiplier spaces are thus semisimple.

Let T be the indicated algebraic $*$-isomorphism. The semisimplicity insures that T in each of the two cases is continuous, and it is topological while the sup-norm derived from B is minimal on the image. Let Q denote the image of B under T, which is closed and therefore $T : B \to Q$ is a B^*-isomorphism. The norm used on Q is the one induced by M^H or $M(H^1)$ according to the case of interest.

Denote next the projection $\tau : M^H \to B_d(\mathbb{N})$ and $\tau : M(H^1) \cap \mathrm{WAP} \to B_d(\mathbb{N})$, respectively, as provided by Theorem 8. Then τ is of norm one in either case. Put $Q_a = \tau(Q)$, so $Q_a \subseteq B_d(\mathbb{N})$. The kernel $J = \ker \tau|_Q$ is a self-adjoint, closed ideal, it being contained in $\mathfrak{W}_0$. Clearly $Q \subseteq Q_a + J$. Next, Q_a turns out to be an algebra of the obvious reason: x_a, $y_a \in Q_a$ provide elements in Q with $x = x_a + x_0$, $y = y_a + y_0$. Then

$$xy = x_a y_a + (x_a y_0 + x_0 y_a + x_0 y_0) \in B_d(\mathbb{N}) + J,$$

since J is an ideal and $x_a y_a$ corresponds to the discrete measure $\mu * \nu$, given that $x_a = \hat{\mu}$, $y_a = \hat{\nu}$ restricted to $\mathbb{N}$.

The composition $\tau \circ T : B \to Q_a$ is an algebraic $*$-homomorphism. Then the natural quotient map $B/\ker(\tau \circ T) \to Q_a$ is an algebraic $*$-isomorphism too. Since $\ker(\tau \circ T)$ is a closed, self-adjoint ideal of the B^*-algebra B, it is well known that the quotient becomes a B^*-algebra, see for example [Ri, p. 249]. The result from [DR] cited above shows Q_a and thus the quotient algebra to be finite dimensional. On grounds that the range of T is contained in $Q_a + J$, the theorem has now been fully established.

As a further application of Theorem 8, let us recall Helson's proof of Steinhaus' conjecture.

Proposition [He]. *Let the symmetric partial Fourier sums of a measure $\mu \in M(\mathbb{T})$ be uniformly bounded in L^1-norm. Then $\mu \in M_0(\mathbb{T})$, the ideal of all measures having Fourier coefficients in $c_0(\mathbb{Z})$.*

The natural condition to try in M^H would be $\sup_n \|x\chi_{[0,n]}\| < \infty$. In the light of the last paragraph from Example 12 it is possible to have $x \notin c_0$, so the Helson-Steinhaus statement has to be changed somewhat and clearly suggests to try $\mathfrak{W}_0$. For reasons of comparison and completeness, a natural proof for the result in $B(\mathbb{N})$ is included first. This achieves the same thing as Helson did, but with nominally weaker assumptions.

Proposition 14. *Suppose $x \in B(\mathbb{N})$ has $\sup_n \|x\chi_{[0,n]}\|_{B(\mathbb{N})}$ finite. Then $x \in c_0$.*

Remark. In particular, any measure $\mu \in M(\mathbb{T}) = x$ achieving $\hat{\mu}|_{\mathbb{N}}$ is by necessity a continuous measure. Examples by M. Weiss or Y. Katznelson are easily seen to produce cases where such μ has a non-trivial singular component.

Proof. Consider any increasing integer sequence n_k such that $x(n_k)$ converges; the existence of such indices inside any prescribed subsequence is granted by $x \in \ell^\infty$. It suffices for the proof to demonstrate $\lim_{k\to\infty} x(n_k) = 0$. Interpret in the following text the relation $\nu \sim y$ as $\hat{\nu}|_{\mathbb{N}} = y|_{\mathbb{N}}$.

Choose a measure $\mu \in M(\mathbb{T})$ such that $\mu \sim x$ and let A be a finite number strictly larger than $\|\mu\|$ and $\sup_n \|x\chi_{[0,n]}\|$. Passing to a subsequence $\{n'_k\}$ there are measures ν, λ, and μ_k in $M(\mathbb{T})$, all of whose variation norms do not exceed A and such that

$$\mu_k \sim x\chi_{[0,n'_k]}, \qquad \nu = \underset{n\to\infty}{*\text{-lim}}\, e^{-in'_k\theta}\, d\mu_k, \qquad \lambda = \underset{n\to\infty}{*\text{-lim}}\, e^{-in'_k\theta}\, d\mu.$$

This involving weak-$*$ limits in $M(\mathbb{T})$. Helson's translation lemma [DR, p. 64] now forces λ to be a singular measure.

On the other hand, each $e^{-in'_k\theta}\, d\mu_k$ has Fourier spectrum in $\mathbb{Z}_- \cup \{0\}$ and is hence absolutely continuous. Thus the same is true of their weak-$*$ limit ν. Clearly $\lambda - \nu$ has Fourier spectrum in $\mathbb{Z}_+$ by construction and is therefore absolutely continuous. The same property is thus present for $\lambda = \nu + (\lambda - \nu)$. Hence $\lambda \in M_c(\mathbb{T}) \cap M_s(\mathbb{T}) = \{0\}$, from which

$$\lim_{k\to\infty} x(n_k) = \lim_{k\to\infty} \hat{\mu}(n'_k) = \hat{\lambda}(0) = 0$$

follows. Thus $x \in c_0$, the desired conclusion.

Theorem 15. *Let $x \in M^H$ or $M(H^1) \cap \mathrm{WAP}$ be such that $\sup_n \|x\chi_{[0,n]}\|$ is finite. Then $x \in M^H \cap \mathfrak{W}_0$ or $x \in M(H^1) \cap \mathfrak{W}_0$, respectively.*

Proof. Take for definiteness $x \in M(H^1) \cap \mathrm{WAP}$, the other case being mutatis mutandis identical.

Write $x = x_a|_{\mathbb{N}} + x_0$ for $x_a \in B_d(\mathbb{Z})$ and $x_0 \in M(H^1) \cap \mathfrak{W}_0$ in accordance with Theorem 8. By almost periodicity there are integers $n_k \to \infty$ such that $\lim_{k\to\infty} x_a(\cdot + n_k)$ exists in ℓ^∞. This limit is in fact x_a itself.

The membership $x \in \mathrm{WAP}$ contains the weak relative compactness of $\{x(\cdot + n_k)\}$. Thinning to a subsequence allows the weak convergence on $\mathbb{N}$ and by passage to a still further subsequence, the existence of

$$\rho(m) = \lim_{k\to\infty} x(m + n_k)$$

can be granted for all $m \in \mathbb{Z}$, which clearly includes $\| \rho \|_{\ell^\infty(\mathbb{Z})} \leq \| x \|_{\ell^\infty(\mathbb{N})}$.

An argument due to Lust-Piquard [LP2] will yield $\rho \in B(\mathbb{Z})$. In fact, consider any polynomial f. Then

$$\begin{aligned}\Big\| \sum \hat{f}(m)\rho(m)e^{im\theta} \Big\|_{L^1} &= \lim_{k\to\infty} \Big\| \sum \hat{f}(m)x(m+n_k)e^{im\theta} \Big\|_{L^1} \\ &= \lim_{k\to\infty} \Big\| \sum \hat{f}(m)x(m+n_k)e^{i(m+n_k')\theta} \Big\|_{L^1} \\ &\leq \sup_k \|x(\cdot + n_k)\|_{M(H^1)} \Big\| \sum \hat{f}(m)e^{im\theta} \Big\|_{L^1} \\ &\leq \| x \|_{M(H^1)} \; \| f \|_{L^1} .\end{aligned}$$

The density of polynomials in $L^1(\mathbb{T})$ now implies $\rho \in M(L^1) = B(\mathbb{Z})$, $\| \rho \|_{B(\mathbb{Z})} \leq \| x \|_{M(H^1)}$.

Putting $A = \sup_k \|x\chi_{[0,n]}\|_{M(H^1)}$ a similar calculation for polynomial f gives

$$\begin{aligned}\Big\| \sum\nolimits_{m\leq 0} \hat{f}(m)\rho(m)e^{im\theta} \Big\|_{L^1} &= \lim_{k\to\infty} \Big\| \sum\nolimits_{m\leq 0} \hat{f}(m)x(m+n_k)e^{i(m+n_k)\theta} \Big\|_{L^1} \\ &\leq \sup_k \|x(\cdot + n_k)\chi_{[0,n_k]}\|_{M(H^1)} \big\| \sum \hat{f}(m)e^{im\theta} \big\|_{L^1} \\ &\leq A \, \|f\|_{L^1},\end{aligned}$$

from which $\rho\chi_{]-\infty,0]} \in B(\mathbb{Z})$ follows. Hence also $\rho\chi_{[1,\infty[} = \rho - \rho\chi_{]-\infty,0]}$ is a member of $B(\mathbb{Z})$. By the theorem of F. and M. Riesz one concludes that $\rho\chi_{]-\infty,0]}$ and $\rho\chi_{[1,\infty]}$ are in $L^1(\mathbb{T})\hat{}$ and thus $\rho \in L^1(\mathbb{T})\hat{} \subseteq c_0$.

Pointwise for $m \geq 0$ we have an identity

$$\lim_{k\to\infty} \big[\rho(m) - x_0(m + n_k)\big] = \lim_{k\to\infty} x_a(m + n_k) = x_a(m)$$

and still $x_a \in \mathrm{AP}(\mathbb{Z}) \cap M(H^1) = B_d(\mathbb{Z})$. Now all $\rho - x_0(\cdot + n_k)$ are members of $\mathfrak{W}_0 \cap \mathcal{C} \subseteq \ell^\infty$, where $\mathcal{C}$ is a suitable closed ball of $M(H^1)$. This intersection $\mathfrak{W}_0 \cap \mathcal{C}$ is convex and norm-closed in ℓ^∞ by Proposition 1, hence also weakly closed in ℓ^∞, so the above identity says

$$x_a = \operatorname*{weak-lim}_{k\to\infty} [\rho - x_0(\cdot + n_k)] \quad \text{in } \ell^\infty,$$

bearing in mind the uniqueness of weak limits and the existence of the right-hand side by the choice of n_k. Thus one concludes

$$x_a|_{\mathbb{N}} \in B_d(\mathbb{N}) \cap \mathfrak{W}_0 \cap M(H^1) = \{0\}$$

and so

$$x = x_0 \in \mathfrak{W}_0 \cap M(H^1),$$

which is the claimed analogue of Helson–Steinhaus.

Remark. It is clear from Proposition 1 that the membership $x \in B(\mathbb{N})$ or M^H in Prop. 14 or Thm. 15, respectively, need not be a priori given. By the same token $x \in M(H^1) \cap \mathrm{WAP}$ can be relaxed to $x \in \mathrm{WAP}$ without loss. Whether this needs to be presupposed or not is still unclear.

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