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Geometry of meromorphic functions and intersections on moduli spaces of curves

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GEOMETRY OF MEROMORPHIC FUNCTIONS AND INTERSECTIONS ON MODULI SPACES OF CURVES

S. V. SHADRIN

Abstract. In this paper we study relations between intersection numbers on moduli spaces of curves and Hurwitz numbers. First, we prove two formulas expressing Hurwitz numbers of (generalized) polynomials via intersections on moduli spaces of curves. Then we show, how intersection numbers can be expressed via Hurwitz numbers. And then we obtain an algorithm expressing intersection numbers $\langle\tau_{n,m}\prod_{i=1}^{r-1}\tau_{0,i}^{k_i}\rangle_g$ via correlation functions of primaries.

Contents

Introduction	2
Acknowledgments	4
1. Definition of Hurwitz numbers	4
1.1. Definition of passports	4
1.2. Definition of Hurwitz numbers	4
2. Hurwitz numbers of polynomials	5
2.1. Introduction	5
2.2. Cohomological classes on $\overline{\mathcal{M}}_{0,N}$	5
2.3. Formula for Hurwitz numbers of polynomials	8
3. Admissible covers	8
3.1. Covering of $\mathcal{M}_{0,m}$	8
3.2. Boundary of $\overline{\mathcal{M}}_{0,m}$	8
3.3. Admissible covers	9
3.4. Space of admissible covers	9
4. Lemma of E. Ionel	9
4.1. Covering of space of admissible covers	9
4.2. Example	9
4.3. Lemma of E. Ionel	10
5. Proof of Theorem 1	10
5.1. Proof	10
5.2. The Poincare dual of $st_*[\widehat{H}]$	11
5.3. Independent check of Theorem 1	12
6. Hurwitz numbers of generalized polynomials	13
6.1. Two-pointed ramification cycles	13
6.2. Hurwitz numbers of generalized polynomials	14

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6.3. Genus zero case 14
7. Proof of Theorem 2 14
7.1. The first steps of the proof 14
7.2. The restriction of $\psi(x_1^1)$ to $st(\widehat{H})$ 14
7.3. Proof of the Theorem 15
8. Amusing formulas for $\langle\tau_{3g}\tau_0^2\rangle_g$ 16
8.1. Formulas 16
8.2. Check in low genera 16
9. Proofs of Theorem 3 16
9.1. First proof 17
9.2. Second proof 17
10. The conjecture of E. Witten 22
10.1. Gelfand-Dikii hierarchies 22
10.2. Intersection numbers $\langle\prod\tau_{i,j}^{k_{i,j}}\rangle_g$ 23
10.3. The conjecture 24
10.4. Boussinesq hierarchy 24
11. An algorithm to calculate $\langle\tau_{n,m}\prod_{i=1}^{r-1}\tau_{0,i}^{k_i}\rangle_g$ 25
11.1. Two-pointed ramification cycles 25
11.2. The algorithm 26
11.3. Simple examples 27
12. Proof of Theorem 4 and calculation of $\widehat{S}_1(\prod_{t=1}^{k}\eta_{m_t,b_t})$ 28
12.1. Initial values 28
12.2. First step of the algorithm 28
12.3. Equality (69) 28
12.4. Calculation of $\widehat{S}_1(\prod_{t=1}^{k}\eta_{m_t,b_t})$ 30
Appendix A. Calculation of $\langle\tau_{6,1}\rangle_3$ in the case $r=3$ 31
A.1. First step of the algorithm 31
A.2. Calculations in degree 4 32
A.3. Calculations in degree 3 34
A.4. Calculations in degree 2 35
A.5. Summary 35
References 36

Introduction

In [10] E. Ionel developed a very beautiful approach to study intersection theory of moduli space of curves. Roughly speaking, the situation is the following. Consider the space of meromorphic functions with fixed genus, degree, and ramification type. There are two mappings of this space. One mapping takes a meromorphic function to it's target curve (of genus zero) with marked critical values. Another mapping takes a meromorphic function to it's domain curve (of genus g) with marked critical points. Then one can relate intersection theories in the images of these mappings. This idea was also used in papers [22, 3] in low genera.

In this paper we just study several applications of Ionel's technique. Concretely, we express Hurwitz numbers via intersection numbers and conversely. More or less the same problem, but in much more general case, is studied by A. Okounkov and R. Pandharipande in [20, 21]. However, we do not understand if there is any relation with our work.

The paper is organized as follows.

In section 1 we define Hurwitz numbers.

In section 2 we formulate our first theorem expressing Hurwitz numbers of usual polynomials with arbitrary ramification via intersections on $\overline{\mathcal{M}}_{0,n}$.

Actually, this formula is not very intersting in any sense. There exists a combinatorial formula for the same numbers [6, 14, 27], and there is the theorem of S. Keel [12], describing cohomologies of $\overline{\mathcal{M}}_{0,n}$. Nevertheless, our formula seems to be beautiful itself, and it is a good example for our techniques.

In section 3 we recall the definition of admissible covers. It is the space which roughly can be considered as a space of meromorphic function.

In section 4 we formulate the lemma of E. Ionel, which plays the principal role in our paper.

In section 5 we prove our first theorem.

In sections 6 and 7 we formulate our second theorem. There we give an expression for Hurwitz numbers of generalized polynomials with one non-simple critical value. This means that we consider meromorphic functions defined on genus g curve with one point of total ramification, another critical value with arbitrary ramifications over it, and all other critical values are simple.

We express these numbers via integrals against so-called two-pointed ramification cycles (but not via integrals against the fundamental cycle of $\overline{\mathcal{M}}_{g,k}$). In [10] E. Ionel explains that it will be very useful to learn how to work with two-pointed ramification cycles. We hope that we make a step in this direction.

Let us note that in general the argument similar to the one we use leads to a similar formula for Hurwitz numbers of arbitrary functions with two non-simple critical values. But here some combinatorial difficulties arise, which we were able to overcome only in the polynomial case.

In sections 8 and 9 we express the intersection number $\langle\tau_{3g}\tau_0^2\rangle_g$ via Hurwitz numbers. The second proof of this formula contains all ideas necessary to calculate intersection numbers appearing in Witten's conjecture.

In section 10 we formulate the conjecture of E. Witten expressing some intersection numbers as the coefficients of the string solution of the Gelfand-Dikii hierarchy. Actually, we don't completely understand the definition of Witten's top Chern class and we replace its definition by some of its properties. We refer to [26, 11, 23] for the clarification of this question.

In sections 11 and 12 we present an algorithm for calculating all intersection numbers of the type $\langle\tau_{n,m}\prod_{i=1}^{r-1}\tau_{0,i}^{k_i}\rangle_g$ appearing in the Witten's conjecture.

Actually, we have a nice algorithm only to express such intersections via not only correlation functions of primaries, but also via some intersection numbers against two-pointed ramification cycles in genus one.

We are also able to express these additional intersection numbers via correlation functions of primaries, and we explain in this paper how to do it. But we are not satisfied with our method. The similar argument allows one to calculate any intersection number appearing in Witten's conjecture. But as one can see in subsection 12.4, such argument is too illegible and too complicated to be written down.

In the appendix we show an example of usage of our algorithm. We calculate the intersection number $\langle\tau_{6,1}\rangle_3$ in the case of Boussinesq hierarchy. As far as we know, there are no other methods to calculate this intersection number.

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1. Definition of Hurwitz numbers

1.1. **Definition of passports.** Consider a meromorphic function $f\colon C\to\mathbb{C}\mathrm{P}^1$ of degree n, defined on a smooth curve C of genus g. Let z be a point of $\mathbb{C}\mathrm{P}^1$. Then $f^{-1}(z)=a_1p_1+\dots+a_lp_l$, where $p_1,\dots,p_l$ are pairwise distinct points of C and $a_1,\dots,a_l$ are positive integers such that $\sum_{i=1}^{l}a_i=n$. Suppose that $a_1\geq a_2\geq\dots\geq a_l$. Then the tuple of numbers $(a_1,\dots,a_l)$ is called *the passport of f over z*.

For instance, the passport of f over a regular point is equal to $(1,\dots,1)$. The passport of f over a simple critical value is equal to $(2,1,\dots,1)$. If f is a polynomial of degree n, then the passport of f over infinity is equal to (n).

1.2. **Definition of Hurwitz numbers.** Two meromorphic functions, $f_1\colon C_1\to\mathbb{C}\mathrm{P}^1$ and $f_2\colon C_2\to\mathbb{C}\mathrm{P}^1$, are isomorphic if there exists a biholomorphic map $\varphi\colon C_1\to C_2$ such that $f_1=f_2\varphi$.

Consider m distinct points of $\mathbb{C}\mathrm{P}^1$. Let $A_i=(a^i_1,\dots,a^i_{l_i})$, $i=1,\dots,m$, be nonincreasing sequences of positive integers such that $\sum_{j=1}^{l_i}a^i_j=n$. Up to isomorphism there is a finite number of meromorphic functions $f\colon C\to\mathbb{C}\mathrm{P}^1$ of degree n defined on smooth curves of genus g such that the passport of f over z_i equals to A_i, $i = 1, \dots, m$, and f is unramified over $\mathbb{C}\mathrm{P}^1 \setminus \{z_1, \dots, z_m\}$.

The Hurwitz number $h(g, n|A_1, \dots, A_m)$ is the weighted count of such functions, where a function $f\colon C \to \mathbb{C}\mathrm{P}^1$ is weighted by $1/|\mathrm{aut}(f)|$.

For example, the number $h(g, 2|(2), \dots, (2))$ is equal to $1/2$.

In [2, 4, 7, 15] one can find some examples of formulas for Hurwitz numbers in special cases.

2. Hurwitz numbers of polynomials

2.1. **Introduction.** In this section we show two formulas for usual polynomials.

Let us fix $n \in \mathbb{N}$, $g = 0$, and a collection of passports $A_1, \dots, A_m$ with usual requirements (for any i we have $A_i = (a^i_1, \dots, a^i_{l_i})$, $a^i_1 \geq \dots \geq a^i_{l_i}$, $\sum_{j=1}^{l_i} a^i_j = n$).

We assume that $l_1 = 1$ and $A_1 = (n)$. Also we assume that $\sum_{i=1}^{m} \sum_{j=1}^{l_i} (a^i_j - 1) = 2n - 2$ (the Riemann-Hurwitz formula).

The following formula is proved in [6, 27, 14].

$$h(0, n|A_1, \dots, A_m) = n^{m-3} \cdot \frac{|\mathrm{aut}(l_2, \dots, l_m)|}{|\mathrm{aut}(A_2, \dots, A_m)|} \cdot \prod_{i=2}^{m} \frac{(l_i - 1)!}{|\mathrm{aut}(A_i)|}. \tag{1}$$

In this paper we prove another formula, expressing the same numbers via intersections.

2.2. Cohomological classes on $\overline{\mathcal{M}}_{0,N}$.

2.2.1. *Standart definitions.* Let N be equal to $\sum_{j=1}^{m} l_j$. We shall consider the moduli space of genus zero curves with N marked points $\overline{\mathcal{M}}_{0,N} \ni (C, x^1_1, x^2_1, \dots, x^2_{l_2}, \dots, x^m_1, \dots, x^m_{l_m})$; we mean a one-to-one correspondence between marked points and indices of a^i_j.

By $\psi(x^i_j)$ we denote the first Chern class of the line bundle, which fiber is the cotangent line at the point x^i_j. By $\pi_{p,q,k}$ we denote the projection $\overline{\mathcal{M}}_{0,N} \to \overline{\mathcal{M}}_{0,2+l_p}$ that forgets all points except for x^1_1, $x^p_1, \dots, x^p_{l_p}$, and x^q_k.

2.2.2. *Definition of Ψ_p-classes.* Let us define classes $\Psi_p(b^i_j)^{i\in\{2,\dots,m\}\setminus\{p\}}_{j=1,\dots,l_i}$ depending on $N - l_p - 1$ indices corresponding to all points x^i_j except for $x^1_1, x^p_1, \dots, x^p_{l_p}$. If $b^i_j = 0$ for all i and j, then we put $\Psi_p(b^i_j) = 0$.

Further we give the following recursive definition. Suppose we have already defined all classes with $\sum_{i=2,\dots,p-1,p+1,\dots,m} \sum_{j=1}^{l_i} b^i_j \leq s$. Consider a sequence $(b^i_j)^{i=2,\dots,p-1,p+1,\dots,m}_{j=1,\dots,l_i}$ such that $\sum_{i=2,\dots,p-1,p+1,\dots,m} \sum_{j=1}^{l_i} b^i_j = s$. Let us fix $q \in \{2, \dots, p-1, p+1, \dots, m\}$ and $k \in \{1, \dots, l_q\}$. Then we define $(\widehat{b}^i_j)^{i=2,\dots,p-1,p+1,\dots,m}_{j=1,\dots,l_i}$ in the following way. We put $\widehat{b}^q_k = \widehat{b}^q_k + 1$; for all other indices i, j we put $\widehat{b}^i_j = b^i_j$.

The formula for $\Psi_p(\widehat{b}^i_j)$ is the following one:

$$\Psi_p(\widehat{b}^i_j) = \widehat{b}^q_k \pi^*_{p,q,k}\psi(x^q_k)\Psi_p(b^i_j) - \sum_U a_U D_{U\cup\{x^q_k\}}\Psi_p((b_U)^i_j) \tag{2}$$

Here $a_U = \sum_{x^i_j\in U} b^i_j$; if $x^i_j \notin U\cup\{x^q_k\}$, then $(b_U)^i_j = b^i_j$; if $x^i_j \in U$, then $(b_U)^i_j = 0$; and $(b_U)^q_k = a_U + b^q_k$. We take the sum over all $U \subset \{x^i_j\} \setminus \{x^1_1, x^p_1, \dots, x^p_{l_p}, x^q_k\}$.

By D_V, where $V \subset \{x^i_j\}$, we denote the Poincaré dual of the cycle, defined by the divisor in $\overline{\mathcal{M}}_{0,N}$ whose generic point is represented by a two-component curve such that all points from U lie on the one component and all points from $\{x^i_j\}\setminus U$ lie on the another component.

In the foregoing formulas we shall use only classes $\Psi_p = \Psi_p(b^i_j)$, where $b^i_j = a^i_j - 1$.

2.2.3. *The Ψ_p-classes are well-defined.* We have to prove that our definition of $\Psi_p(b^i_j)$-classes is correct, i. e. it doesn't depend on the choice of the order of increasing b^i_j.

Let us fix p, $x^{q_1}_{k_1}$, $x^{q_2}_{k_2}$, and $(b^i_j)^{i\in\{2,\dots,m\}\setminus\{p\}}_{j=1,\dots,l_i}$. Let $\widehat{b}^{q_1}_{k_1} = b^{q_1}_{k_1}+1$, $\widehat{b}^{q_2}_{k_2} = b^{q_2}_{k_2}+1$, and $\widehat{b}^i_j = b^i_j$ for all other i and j.

If we apply formula (2) twice, first for $x^{q_1}_{k_1}$ and then for $x^{q_2}_{k_2}$, we obtain the following expression

$$\begin{aligned}
\Psi_p(\widehat{b}^i_j) = {} & \widehat{b}^{q_1}_{k_1}\widehat{b}^{q_2}_{k_2}\pi^*_{p,q_1,k_1}\psi(x^{q_1}_{k_1})\pi^*_{p,q_2,k_2}\psi(x^{q_2}_{k_2})\Psi_p(b^i_j) - \\
& \sum_U(\sum_{x^i_j\in U} b^i_j)\widehat{b}^{q_2}_{k_2} D_{U\cup\{x^{q_1}_{k_1}\}}\pi^*_{p,q_2,k_2}\psi(x^{q_2}_{k_2})\Psi_p(\dots) - \\
& \sum_U(b^{q_2}_{k_2} + \sum_{x^i_j\in U} b^i_j)\widehat{b}^{q_2}_{k_2} D_{U\cup\{x^{q_1}_{k_1},x^{q_2}_{k_2}\}}\pi^*_{p,q_2,k_2}\psi(x^{q_2}_{k_2})\Psi_p(\dots) - \\
& \sum_U(\sum_{x^i_j\in U} b^i_j)\widehat{b}^{q_1}_{k_1} D_{U\cup\{x^{q_2}_{k_2}\}}\pi^*_{p,q_1,k_1}\psi(x^{q_1}_{k_1})\Psi_p(\dots) - \\
& \sum_U(\widehat{b}^{q_1}_{k_1} + \sum_{x^i_j\in U} b^i_j)(b^{q_1}_{k_1} + \widehat{b}^{q_2}_{k_2} + \sum_{x^i_j\in U} b^i_j) D_{U\cup\{x^{q_1}_{k_1},x^{q_2}_{k_2}\}}\pi^*_{p,q_1,k_1}\psi(x^{q_1}_{k_1})\Psi_p(\dots) + \\
& \sum_U(\sum_{x^i_j\in U} b^i_j) D_{U\cup\{x^{q_2}_{k_2}\}} \sum_V(\sum_{x^i_j\in V} b^i_j) D_{V\cup\{x^{q_1}_{k_1}\}}\Psi_p(\dots) + \\
& \sum_U(\sum_{x^i_j\in U} b^i_j) D_{U\cup\{x^{q_2}_{k_2}\}} \sum_V(b^{q_2}_{k_2} + \sum_{x^i_j\in U\cup V} b^i_j) D_{U\cup V\cup\{x^{q_1}_{k_1},x^{q_2}_{k_2}\}}\Psi_p(\dots) + \\
& \sum_U(\widehat{b}^{q_1}_{k_1} + \sum_{x^i_j\in U} b^i_j) D_{U\cup\{x^{q_1}_{k_1},x^{q_2}_{k_2}\}} \sum_V(\sum_{x^i_j\in V} b^i_j) D_{U\cup V\cup\{x^{q_1}_{k_1},x^{q_2}_{k_2}\}}\Psi_p(\dots)
\end{aligned} \tag{3}$$

Here we take sums over all $U \subset \{x_j^i\} \setminus \{x_1^1, x_1^p, \dots, x_{l_p}^p, x_{k_1}^{q_1}, x_{k_2}^{q_2}\}$ and over all $V \subset \{x_j^i\} \setminus \{x_1^1, x_1^p, \dots, x_{l_p}^p, x_{k_1}^{q_1}, x_{k_2}^{q_2}\}$ such that $U \cap V = \emptyset$. We don't write down the indices in Ψ_p-classes since it is very easy to reconstruct these indices from the rest of each summand.

Now, to ensure that the Ψ_p-classes are well-defined, it is enough to prove the following lemma.

Lemma 1. *The right hand side of formula (3) is symmetric with respect to the change $x_{k_1}^{q_1} \leftrightarrow x_{k_2}^{q_2}$, $b_{k_1}^{q_1} \leftrightarrow b_{k_2}^{q_2}$.*

Proof. Obviously, this lemma can be reduced to the fact that the sum of the last two terms of the expression is symmetric. Let us prove it.

We fix the partition $U \cup V$. On the divisor $D = D_{U \cup V \cup \{x_{k_1}^{q_1}, x_{k_2}^{q_2}\}}$ the first Chern class of the cotangent line at the double point is equal to

(4)
$$\psi(*)D = D \cdot \sum_{W \ni x_\beta^\alpha} (D_{W \cup \{x_{k_2}^{q_2}\}} + D_{W \cup \{x_{k_1}^{q_1}, x_{k_2}^{q_2}\}}) = \sum_W D_{W \cup \{x_{k_1}^{q_1} x_{k_2}^{q_2}\}} D.$$

Of course, $\psi(*)$ is symmetric.

Note that

$$(5) \quad (b_{k_2}^{q_2} + \sum_{x_j^i \in U \cup V} b_j^i)(\sum_{x_j^i \in U \cup V} b_j^i)\psi(*)D =$$
$$(b_{k_2}^{q_2} + \sum_{x_j^i \in U \cup V} b_j^i) \sum_{x_\beta^\alpha \in U \cup V} b_\beta^\alpha \psi(*)D =$$
$$\sum_W (\sum_{x_j^i \in W} b_j^i)(b_{k_2}^{q_2} + \sum_{x_j^i \in U \cup V} b_j^i) D_{W \cup \{x_{k_2}^{q_2}\}} D +$$
$$\sum_W (\sum_{x_j^i \in W} b_j^i)(b_{k_2}^{q_2} + \sum_{x_j^i \in U \cup V} b_j^i) D_{W \cup \{x_{k_1}^{q_1} x_{k_2}^{q_2}\}} D$$

and

(6)
$$(b_{k_1}^{q_1} + 1)(\sum_{x_j^i \in U \cup V} b_j^i)\psi(*)D = (b_{k_1}^{q_1} + 1)(\sum_{x_j^i \in U \cup V} b_j^i) \sum_W D_{W \cup \{x_{k_1}^{q_1} x_{k_2}^{q_2}\}} D.$$

Thus we see that the sum of the last two terms of expression (3) (with fixed $U \cup V$) is equal to

$$(7) \quad \left((b_{k_1}^{q_1} + b_{k_2}^{q_2} + 1)(\sum_{x_j^i \in U \cup V} b_j^i) + (\sum_{x_j^i \in U \cup V} b_j^i)^2 \right) \psi(*) D \Psi_p(\dots) -$$
$$\sum_U (b_{k_1}^{q_1} + b_{k_2}^{q_2} + 1 + \sum_{x_j^i \in U} b_j^i)(\sum_{x_j^i \in U} b_j^i) D_{U \cup \{x_{k_1}^{q_1} x_{k_2}^{q_2}\}} D \Psi_p(\dots).$$

This expression is obviously symmetric. Hence the right hand side of formula (3) is symmetric.

□

2.3. **Formula for Hurwitz numbers of polynomials.**

Theorem 1. *If $m \geq 3$, then*

$$h(0,n|A_1,\dots,A_m) = \frac{n^{m-3}}{\prod_{j=2}^{m}|\mathrm{aut}(A_j)|}\cdot\int_{\overline{\mathcal{M}}_{0,N}}\psi(x_1^1)^{m-3}\prod_{p=2}^{m}\Psi_p \tag{8}$$

Let us recall that by Ψ_p we denote $\Psi_p(b^i_j)$, where $b^i_j = a^i_j - 1$, $i = 2,\dots,p-1,p+1,\dots,m$, $j = 1,\dots,l_i$.

Below we check this formula in some special cases independently of it's proof. Probably it is helpful to look through this section in order to see how to work with classes like $\pi^*_{p,q,i}\psi(x^q_i)$, D_U, and Ψ_p.

3. Admissible covers

3.1. **Covering of $\mathcal{M}_{0,m}$.**

Let us fix the degree n, the genus g, and a collection of passports $A_1,\dots,A_m$. Here for any $i = 1,\dots,m$ $A_i = (a^i_1,\dots,a^i_{l_i})$, $a^i_1 \geq \dots \geq a^i_{l_i}$, $\sum_{j=1}^{l_i} a^i_j = n$.

Consider all meromorphic functions $f\colon C \to \mathbb{C}\mathrm{P}^1$ of degree n defined on smooth curves of genus g such that for certain distinct points $z_1,\dots,z_m \in \mathbb{C}\mathrm{P}^1$ the passports of f over $z_1,\dots,z_m$ are equal to $A_1,\dots,A_m$ respectively and f is unramified over $\mathbb{C}\mathrm{P}^1 \setminus \{z_1,\dots,z_m\}$. Let us consider such functions up to isomorphism and up to automorphisms of $\mathbb{C}\mathrm{P}^1$ in the target. Then the space of such function is a noncompact complex manifold of dimension $m-3$. Denote this space by H.

Consider $(\mathbb{C}\mathrm{P}^1, z_1,\dots,z_m)$ as a moduli point of $\mathcal{M}_{0,m}$. Then there is a natural projection $ll\colon H \to \mathcal{M}_{0,m}$. The mapping ll is usually called the *Lyashko-Looijenga mapping*. Obviously, $ll\colon H \to \mathcal{M}_{0,m}$ is an $h(g,n|A_1,\dots,A_m)$ – sheeted unramified covering.

3.2. **Boundary of $\overline{\mathcal{M}}_{0,m}$.**

It order to obtain geometrically a moduli point of $\overline{\mathcal{M}}_{0,m} \setminus \mathcal{M}_{0,m}$ with k nodes one can choose k pairwise nonintersecting contours $c_1,\dots,c_k$ on $(\mathbb{C}\mathrm{P}^1, z_1,\dots,z_m)$ and contract each of these contours. Contours $c_1,\dots,c_k$ must contain no points $z_1,\dots,z_m$ and no points of self-intersections. The Euler characteristic of each connected component of $\mathbb{C}\mathrm{P}^1 \setminus \left(\bigcup_{i=1}^k c_i \cup \bigcup_{i=1}^m z_i\right)$ must be negative.

Let us give a more rigorous definition. A moduli point of $\overline{\mathcal{M}}_{0,m}$ is a tree of rational curves. Any two irreducible components are either disjoint or intersect transversely at a single point. Each component must contain at least three special (singular or labeled) points. For details see e. g. [16, 12].

3.3. **Admissible covers.** Our goal now is to extend the unramified covering $ll\colon H \to \mathcal{M}_{0,m}$ to a ramified covering $\overline{ll}\colon \overline{H} \to \overline{\mathcal{M}}_{0,m}$.

Suppose a moduli point on the boundary of $\overline{\mathcal{M}}_{0,m}$ is obtained from a moduli point $(\mathbb{C}\mathrm{P}^1, z_1, \dots, z_m)$ by contracting contours $c_1, \dots, c_k$. Consider a function $f \in H$, $f\colon C \to \mathbb{C}\mathrm{P}^1$ such that $ll(f) = (\mathbb{C}\mathrm{P}^1, z_1, \dots, z_m)$. All preimages of contours $c_1, \dots, c_k$ are contours on C. Let us contract each of them.

Thus we obtain a point on the boundary of $\overline{H}$. The axiomatic description of functions we obtain by such procedure gives us the definition of the space $\overline{H}$.

Consider a moduli point $(C, z_1, \dots, z_m)$ on the boundary of $\overline{\mathcal{M}}_{0,m}$. Then $\overline{ll}^{-1}(C, z_1, \dots, z_m)$ consists of holomorphic maps $f\colon C_g \to C$ of prestable curves with m labeled points to C such that over each irreducible component of C f is a n-sheeted covering, not necessary connected, with ramification only over special points (labeled or singular). It is required that ramifications over marked points are determined by their passports, and the local behavior of f at a node in the preimage is the same on both branches of C_g at this node.

Thus we defined the space $\overline{H}$ and the mapping $\overline{ll}$. For more detailed definitions we refer to [9, 8, 10].

3.4. **Space of admissible covers.** The space $\overline{H}$ can be considered as an orbifold with some glued stratas. Actually, it can be desingularized, but we need only that this space carries a fundamental class, see [1, 10].

4. Lemma of E. Ionel

4.1. **Covering of space of admissible covers.** Let us fix the degree n, the genus g, and a collection of passports $A_1, \dots, A_m$. Consider the corresponding space of admissible covers $\overline{H}$.

Let us denote by $\widehat{H}$ the space of functions from $\overline{H}$ with all labeled preimages of all points $z_1, \dots, z_m$. The natural projection $\pi\colon \widehat{H} \to \overline{H}$ is a $(\prod_{i=1}^m |\mathrm{aut}(A_i)|)$ – sheeted ramified covering.

Let us denote by $\widehat{ll}\colon \widehat{H} \to \overline{\mathcal{M}}_{0,m}$ the mapping $\overline{ll} \circ \pi$. It also is a ramified covering.

4.2. **Example.** Consider $n = 3$, $g = 0$, $A_1 = (3)$, $A_2 = A_3 = (2,1)$, and $A_4 = (1,1,1)$. Then we expect $\widehat{ll}\colon \widehat{H} \to \overline{\mathcal{M}}_{0,4}$ to be a $h(0,3|A_1, \dots, A_4) \cdot \left(\prod_{i=1}^4 |\mathrm{aut}(A_i)|\right) = 6$ – sheeted ramified covering.

Obviously, $\widehat{ll}$ is ramified only over the boundary points of $\overline{\mathcal{M}}_{0,4}$. Denote these points by $(14|23)$, $(12|34)$, and $(13|24)$ (we mean here, that $(14|23)$ corresponds to the curve, where z_1, z_4 and z_2, z_3 are labeled points on different irreducible components).

It's easy to see, that the passport of $\widehat{ll}$ over $(14|23)$ is equal to $(3,3)$ and the passports of $\widehat{ll}$ over all other boundary points are equal to

$(2,2,2)$. Thus $\widehat{ll}$ is a 6-sheeted covering of the sphere and it has 10 critical points. Hence, the covering space $\widehat{H}$ is a sphere.

4.3. **Lemma of E. Ionel.** Let the passport A_i be equal to $(a^i_1, \ldots, a^i_{l_i})$. Then a point of the space $\widehat{H}$ is a function

$$f\colon (C_g, x^1_1, \ldots, x^1_{l_1}, \ldots, x^m_1, \ldots, x^m_{l_m}) \to (C_0, z_1, \ldots, z_m). \tag{9}$$

Here $x^i_1, \ldots, x^i_{l_i}$ are labeled preimages of z_i, $f^{-1}(z_i) = \sum_{j=1}^{l_i} a^i_j x^i_j$.

By $st\colon \widehat{H} \to \overline{\mathcal{M}}_{g,N}$, $N = \sum_{i=1}^m l_i$, we denote the mapping that takes a function $\big(f\colon (C_g, x^1_1, \ldots, x^m_{l_m}) \to (C_0, z_1, \ldots, z_m)\big) \in \widehat{H}$ to the moduli point $(C_g, x^1_1, \ldots, x^1_{l_1}, \ldots, x^m_1, \ldots, x^m_{l_m}) \in \overline{\mathcal{M}}_{g,N}$.

Lemma 2. [10] *For any $i = 1, \ldots, m$ and $j = 1, \ldots, l_i$*

$$a^i_j \cdot st^*\psi(x^i_j) = (\widehat{ll})^*\psi(z_i). \tag{10}$$

Here by $\psi(x^i_j)$ (by $\psi(z_i)$) we denote the first Chern class of the line bundle, which fiber is the cotangent line at the point x^i_j (respectively, at the point z_i).

5. Proof of Theorem 1

5.1. **Proof.** Recall that we fix $n \in \mathbb{N}$, $g = 0$, and passports $A_1, \ldots, A_m$, $A_i = (a^i_1, \ldots, a^i_{l_i})$. Also we assume that $l_1 = 1$, $A_1 = (n)$, and $\sum_{i=1}^m \sum_{j=1}^{l_i} (a^i_j - 1) = 2n - 2$.

Then, if $m \geq 3$, we want to prove that

$$h(0, n|A_1, \ldots, A_m) = \frac{n^{m-3}}{\prod_{j=2}^m |\mathrm{aut}(A_j)|} \int_{\overline{\mathcal{M}}_{0,N}} \psi(x^1_1)^{m-3} \prod_{p=2}^m \Psi_p \tag{11}$$

Proof. Consider the corresponding space $\widehat{H}$. We have the following picture:

$$(C_0, x^1_1, \ldots, x^m_{l_m}) \in \overline{\mathcal{M}}_{0,N} \xleftarrow{st} \widehat{H} \xrightarrow{\widehat{ll}} \overline{\mathcal{M}}_{0,m} \ni (C_0, z_1, \ldots, z_m) \tag{12}$$

Note that $\int_{\overline{\mathcal{M}}_{0,m}} \psi(z_1)^{m-3} = 1$. Since $\widehat{ll}$ is an

$$S = h(0, n|A_1, \ldots, A_m) \cdot \left(\prod_{i=2}^m |\mathrm{aut}(A_i)| \right) \tag{13}$$

– sheeted ramified covering, it follows that $\int_{\widehat{H}} \widehat{ll}^* \psi(z_1)^{m-3} = S$.

Then, using the Lemma of E. Ionel, we obtain that

$$S = n^{m-3} \int_{\widehat{H}} st^*\psi(x^1_1)^{m-3} = n^{m-3} \int_{st_*[\widehat{H}]} \psi(x^1_1)^{m-3}. \tag{14}$$

Below we shall prove that in our case the Poincare dual of $st_*[\widehat{H}]$ is equal to $\Xi = \prod_{p=2}^m \Psi_p$.

Thus we obtain our formula. $\square$

5.2. **The Poincare dual of $st_*[\widehat{H}]$.** In this subsection we prove that $st_*[\widehat{H}]$ is dual to Ξ. First we define a subvariety $V \subset \overline{\mathcal{M}}_{0,M}$; then we prove that $[V]$ is dual to Ξ; and then we prove that $[V] = st_*[\widehat{H}]$.

5.2.1. *Subvariety V.* Consider a moduli point $(C, x_1^1, \dots, x_{l_m}^m) \in \overline{\mathcal{M}}_{0,M}$. Let us fix $p \geq 2$. Consider the meromorphic 1-form ω_p with simple poles at the points $x_1^1, x_1^p, \dots, x_{l_p}^p$ with residues $-n, a_1^p, \dots, a_{l_p}^p$ respectively.

Let $C_p \subset C$ be the union of those irreducible components of C where ω_p is not identically zero. Obviously, C_p is a connected curve. Consider the collapsing map $c_p \colon C \to C_p$. Since C_p is a connected curve, it follows that the image $c_p(x_j^i)$ of any labeled point x_j^i is a nonsingular and a nonlabeled point of $(C_p, x_1^1, x_1^p, \dots, x_{l_p}^p)$.

Let $x \in C_p$ be the image of the marked points $x_{j_1}^{q_1}, \dots, x_{j_{k(x)}}^{q_{k(x)}}$ under the mapping c_p. Then we require that ω_p has exactly $\sum_{i=1}^{k(x)} (a_{j_i}^{q_i} - 1)$ zeros at x.

We define the subvariety V as follows. A moduli point $(C, x_1^1, \dots, x_{l_m}^m) \in \overline{\mathcal{M}}_{0,M}$ belongs to V iff it satisfies the last requirement for any $p = 2, \dots, m$ and for any nonsingular nonlabeled point $x \in (C_p, x_1^1, x_1^p, \dots, x_{l_p}^p)$.

5.2.2. *$[V]$ is dual to Ξ.* Let us fix $p \geq 2$ and a sequence of numbers $(b_j^i)_{j=1,\dots,l_i}^{i \in \{2,\dots,m\} \setminus \{p\}}$. By $V_p(b_j^i)$ denote the subvariety of $\overline{\mathcal{M}}_{0,M}$ such that for any $(C, x_1^1, \dots, x_{l_m}^m) \in V$ for any nonsingular nonlabeled point $x \in C_p$, $x = c_p(x_{j_1}^{q_1}) = \dots = c_p(x_{j_{k(x)}}^{q_{k(x)}})$, ω_p has at least $\sum_{i=1}^{k(x)} b_{j_i}^{q_i}$ zeros at x.

Lemma 3. *The subvariety $V_p(b_j^i)$ determines a cohomological class equal to $\Psi_p(b_j^i)$.*

Proof. Actually, if $b_j^i = 0$ for all i and j, then $V_p(b_j^i) = \overline{\mathcal{M}}_{0,M}$ and $[V_p(b_j^i)]$ is dual to $\Psi_p(0) = 1$.

Suppose that we have already proved this Lemma for all classes with $\sum_{i \in \{2,\dots,m\} \setminus \{p\}} \sum_{j=1}^{l_i} b_j^i \leq s$. Consider a sequence $(b_j^i)_{j=1,\dots,l_i}^{i \in \{2,\dots,m\} \setminus \{p\}}$ such that $\sum_{i \in \{2,\dots,m\} \setminus \{p\}} \sum_{j=1}^{l_i} b_j^i = s$. Let us fix $q \in \{2, \dots, m\} \setminus \{p\}$ and $k \in \{1, \dots, l_q\}$. Then we define $(\widehat{b}_j^i)_{j=1,\dots,l_i}^{i=2,\dots,p-1,p+1,\dots,m}$ in the following way. We put $\widehat{b}_k^q = \widehat{b}_k^q + 1$; for all other indices i, j we put $\widehat{b}_j^i = b_j^i$.

By $\mathbb{L}(x_k^q)$ denote the cotangent line bundle at the point x_k^q.

On $V_p(b_j^i)$, the restriction $\omega_p|_{x_k^q}$ determines a section of $\pi_{p,q,k}^* \mathbb{L}(x_k^q)^{\otimes \widehat{b}_k^q}$. Note that at a generic point of $V_p(b_j^i)$ the form ω_p has exactly b_k^q zeros at the point $c_p(x_k^q)$. Hence the restriction $\omega_p|_{x_k^q}$ vanishes at all curves where ω_p has more than b_k^q zeros at the point $c_p(x_k^q)$.

Let us enumerate those divisors in $V_p(b_j^i)$, where ω_p has more than b_k^q zeros at $c_p(x_k^q)$. The first one is the divisor, where $x_k^q \in C_p$ and ω_p has $\widehat{b}_k^q$ zeros at $x_k^q = c_p(x_k^q)$. The second case is when x_k^q "run away" from

C_p with some other points $x^{i_1}_{j_1},\dots,x^{i_s}_{j_s}$. The section of $\pi^*_{p,q,k}\mathbb{L}(x^q_k)^{\otimes\widehat{b}^q_k}$ has $\sum_{r=1}^s b^{i_r}_{j_r}$ zeros at such divisor.

Hence the expression defining $\Psi_p(\widehat{b}^i_j)$ is just the expression defining the same cocycle as the first divisor described above. But the first divisor is just $V_p(\widehat{b}^i_j)$. Thus we made the inductive step, which proves this Lemma. $\square$

Note that V is a transversal intersection of $V_p(a^i_j-1)$, $p=2,\dots,m$. Then an obvious corollary of Lemma 3 is the following

Lemma 4. *The subvariety V determines a cohomological class equal to Ξ.*

5.2.3. $[V]$ *is equal to* $st_*[\widehat{H}]$. Obviously, V and $st(\widehat{H})$ are the closures of $V\cap\mathcal{M}_{0,M}$ and $st(\widehat{H})\cap\mathcal{M}_{0,M}$ respectively. Let us prove that $V\cap\mathcal{M}_{0,M}=st(\widehat{H})\cap\mathcal{M}_{0,M}$.

Let $f\colon(\mathbb{C}\mathrm{P}^1,x^1_1,\dots,x^m_{l_m})\to(\mathbb{C}\mathrm{P}^1,z_1,\dots,z_m)$ be a function in $\widehat{H}$. Then $\omega_p=f^*\left(dz/(z-z_p)-dz/(z-z_1)\right)$. Since x^q_k is a critical point of f of the order a^q_k-1, if follows that ω_p has a^q_k-1 zeros at x^q_k. Thus we obtain that $V\cap\mathcal{M}_{0,M}\supset st(\widehat{H})\cap\mathcal{M}_{0,M}$.

Conversely, let $(\mathbb{C}\mathrm{P}^1,x^1_1,\dots,x^m_{l_m})\in V\cap\mathcal{M}_{0,M}$. Then there is a unique function f_p such that $f^{-1}(z_1)=n\cdot x^1_1$ and $f^{-1}(z_p)=\sum_{i=1}^{l_p}a^p_i\cdot x^p_i$. Requirements for ω_p mean that a marked point x^q_k is a critical point of f_p of order a^q_k-1. Hence $f=f_2=f_3=\dots=f_m$ is the function in $\widehat{H}$ such that $f^{-1}(z_i)=\sum_{j=1}^{l_i}a^i_jx^i_j$, $i=1,\dots,m$.

Since the map st has degree one over $V\cap\mathcal{M}_{0,M}$, it follows that $[V]=st_*[\widehat{H}]$.

5.3. **Independent check of Theorem 1.**

5.3.1. *Degenerate polynomials.* Consider the first nontrivial degenerate case. Let $A_1=(n)$, $A_2=(a^2_1,a^2_2)$, and $A_3=(2,1,\dots,1)$. Then we want to check that $h(0,n|A_1,A_2,A_3)=1/\mathrm{aut}(A_2)$.

We have

$$\begin{array}{lrcl} & \Psi_2 & = & \pi^*_{2,3,1}\psi(x^3_1),\\ (15) & \Psi_3 & = & (a^2_1-1)!(a^2_2-1)!\pi^*_{3,2,1}\psi(x^2_1)^{a^2_1-1}\pi^*_{3,2,2}\psi(x^2_2)^{a^2_2-1}-\\ & & & -((n-2)!-(a^2_1-1)!(a^2_2-1)!)D_{\{x^2_1,x^2_2\}}\pi^*_{3,2,1}\psi(x^2_1)^{n-3}.\end{array}$$

Note that Ψ_2 is dual to the divisor D whose generic point is represented by a two-component curve such that x^1_1 and x^2_1 lie on one component and x^2_2 and x^3_1 lie on the other one. The restriction of Ψ_3 to this divisor is obviously equal to $(a^2_1-1)!(a^2_2-1)!\psi(x^2_1)^{a^2_1-1}\psi(x^2_2)^{a^2_2-1}$.

The divisor D consists of several irreducible components. One can enumerate these components via subsets of $\{x^3_2,\dots,x^3_{n-1}\}$. Let D_U, $U\subset\{x^3_2,\dots,x^3_{n-1}\}$ be the divisor whose generic point is represented by a two-component curve such that x^1_1, x^2_1, and all $x^3_i\in U$ lie on one component and all other labeled points lie on the other component. So, $D = \cup_U D_U$.

It is easy to see that $\int_{D_U} \psi(x_1^2)^{a_1^2-1}\psi(x_2^2)^{a_2^2-1}$ is not vanishing iff $|U| = a_1^2 - 1$. In this case this integral is equal to 1. Hence, $\int_D \psi(x_1^2)^{a_1^2-1}\psi(x_2^2)^{a_2^2-1} = \binom{n-2}{a_1^2-1}$. It follows that $h(0,n|A_1,A_2,A_3) = 1/\mathrm{aut}(A_2)$.

5.3.2. *Generic polynomials.* The natural desire here is to check independently that out formula gives the same answer as in [15] in the case of generic polynomials. The ramification data for generic polynomials is the following: $A_1 = (n)$, $m = n$, $A_2 = \dots = A_n = (2,1,\dots,1)$. The answer given in [15] is $h(0,n|A_1,\dots,A_n) = n^{n-3}$.

Actually, we have the proof, that $h(0,n|A_1,\dots,A_n)$ counted via our formula is equal to n^{n-3}. But this proof is rather complicated and involves too many new geometric objects that we will never use in this paper again. So we shall give only the sketch of our proof, which can be deciphered to the full proof.

The first step looks like follows. One has to show that the "essential part" of Ψ_p in this case is equal to the cocycle determined by the subvariety W_p. W_p is a codimension $n-2$ subvariety whose generic point is represented by a $(n-2)$-component curve such that on one component there are points x_1^1, x_1^p, and $n-2$ singular points. All other components, except for this one, contain exactly one point from the set $\{x_1^2,\dots,x_1^{p-1},x_1^{p+1},\dots,x_1^n\}$, exactly one point from the set $\{x_2^p,\dots,x_n^p\}$, and exactly one singular point, attaching this component to the first one.

The words "essential part" mean that the integral

$$\int_{\overline{\mathcal{M}}_{0,N}} \psi(x_1^1)^{n-3}\Psi_2\dots\Psi_{p-1}(\Psi_p - [W_p])\Psi_{p+1}\dots\Psi_n \tag{16}$$

is equal to zero.

Let us remark that there are exactly $(n-2)!$ ways to split the sets $\{x_1^2,\dots,x_1^{p-1},x_1^{p+1},\dots,x_1^n\}$ and $\{x_2^p,\dots,x_n^p\}$ into pairs of points lying on the same components. Then via the standart calculations one can show that the integral part of formula (8) is equal to $(n-2)!^{n-1}$. Then we obtain that

$$h(0,n|A_1,\dots,A_n) = \frac{n^{n-3}}{\prod_{i=1}^n \mathrm{aut}(A_i)}(n-2)!^{n-1} = n^{n-3} \tag{17}$$

6. Hurwitz numbers of generalized polynomials

6.1. **Two-pointed ramification cycles.** In applications of Lemma of E. Ionel to concrete calculations one has to work with two-pointed ramification cycles. Let us define them now.

Consider a moduli space $\overline{\mathcal{M}}_{g,k}$. Let $x_1,\dots,x_k$ be marked points of curves in $\overline{\mathcal{M}}_{g,k}$. Let $b_1,\dots,b_k$ be integer numbers such that

$\sum_{i=1}^k b_i = 0$. By W denote the subvariety of $\mathcal{M}_{g,k}$ consisting of curves $(C, x_1, \dots, x_k) \in \mathcal{M}_{g,k}$ such that $\sum_{i=1}^k b_k x_k$ is the divisor of a meromorphic function.

The closure of W in $\overline{\mathcal{M}}_{g,k}$ determines a homology class $\Delta(b_1, \dots, b_k)$. This class is called the *two-pointed ramification cycle*.

6.2. **Hurwitz numbers of generalized polynomials.** By a *generalized polynomial* we mean simply the following ramification data. Let us fix integers $n > 0$ and $g \geq 0$. Let $A_1 = (n)$, $A_2 = (a_1, \dots, a_l)$, and $A_3 = \dots = A_m = (2, 1, \dots, 1)$, where $m - l - 1 = 2g$ (Riemann-Hurwitz).

Consider the space $\overline{\mathcal{M}}_{g,1+l} \ni (C, x_1^1, x_1^2, \dots, x_l^2)$. There is a two-pointed ramification cycle $\Delta(-n, a_1, \dots, a_l)$, where $-n$ corresponds to x_1^1 and a_i corresponds to x_i^2, $i = 1, \dots, l$.

Theorem 2. *If* $m \geq 3$ *then*

$$h(g, n | A_1, \dots, A_m) = \frac{n^{m-3}(m-2)!}{\operatorname{aut}(A_2)} \int_{\Delta(-n, a_1, \dots, a_l)} \psi(x_1^1)^{m-3}. \tag{18}$$

6.3. **Genus zero case.** Let us check formula (18) in the case of genus zero. We have $\Delta(-n, a_1, \dots, a_l) = [\overline{\mathcal{M}}_{0,1+l}]$. Hence

$$h(0, n | A_1, \dots, A_{l+1}) = \frac{n^{l-2}(l-1)!}{\operatorname{aut}(A_2)}. \tag{19}$$

The same answere is given by equation (1).

7. Proof of Theorem 2

7.1. **The first steps of the proof.** We start our proof in the same way as in the case of usual polynomials. Let $\overline{H}$ be the appropriated space of admissible covers. The mapping $\widehat{H} \to \overline{H}$ has degree $\operatorname{aut}(A_2) \cdot (n-2)!^{m-2}$. Then

$$\int_{\widehat{H}} \widehat{ll}^* \psi(z_1)^{m-3} = \operatorname{aut}(A_2) \cdot (n-2)!^{m-2} \cdot h(g, n | A_1, \dots, A_m). \tag{20}$$

Using the Lemma of E. Ionel, we get

$$n^{m-3} \int_{st_*[\widehat{H}]} \psi(x_1^1)^{m-3} = \operatorname{aut}(A_2) \cdot (n-2)!^{m-2} \cdot h(g, n | A_1, \dots, A_m). \tag{21}$$

7.2. **The restriction of $\psi(x_1^1)$ to $st(\widehat{H})$.** Let $\pi \colon \overline{\mathcal{M}}_{g,1+l+(m-2)(n-1)} \to \overline{\mathcal{M}}_{g,1+l}$ be the projection forgetting all marked points except for $x_1^1, x_1^2, \dots, x_l^2$.

Lemma 5. $\pi^* \psi(x_1^1)|_{st(\widehat{H})} = \psi(x_1^1)|_{st(\widehat{H})}$.

Proof. Recall that $\pi^*\psi(x_1^1) = \psi(x_1^1) - [D_\pi]$. Here $[D_\pi]$ is the cocycle determined by the divisor D_π in $\overline{\mathcal{M}}_{g,1+l+(m-2)(n-1)}$. The generic point of D_π is represented by a two-component curve such that one component has genus zero and contains x_1^1, and the other component has genus g and contains $x_1^2, \dots, x_l^2$.

Let us prove that this divisor does not intersect $st(\widehat{H})$. Assume the converse, i. e., we assume that there exists a function in $\widehat{H}$ such that its domain belongs to D_π.

Consider the image of such function. It is a stable curve of genus zero with marked point $z_1, \dots, z_m$. It follows from the definition of D_π that z_1 and z_2 lie on different irreducible components of the target curve.

Consider the irreducible component C_1 of the target curve containing z_1. This component also contains the point z_* and at least one more special point. (the point z_* separates C_1 from the component containing z_2)

If follows from the definition of admissible covers that the point z_* is a point of total ramification. Since the preimage of C_1 has genus zero, it follows that any point of this component (except for z_1 and z_*) has exactly n simple preimages.

Note that all marked points in the target curve are critical values of the function. After being cut at z_*, the target curve splits in two halves, and the total preimage of the half containing z_1 has genus 0. It follows that the third special point on C_1 has a ramification in the preimage.

This contradiction proves that D_π doesn't intersect $st(\widehat{H})$. Hence $\pi^*\psi(x_1^1)|_{st(\widehat{H})} = \psi(x_1^1)|_{st(\widehat{H})}$. □

7.3. **Proof of the Theorem.**

Proof. We have

(22)
$$n^{m-3}\int_{st_*[\widehat{H}]} \psi(x_1^1)^{m-3} = \mathrm{aut}(A_2)\cdot(n-2)!^{m-2}\cdot h(g,n|A_1,\dots,A_m).$$

Using Lemma 5 we get

$$\int_{st_*[\widehat{H}]} \psi(x_1^1)^{m-3} = \int_{\pi_* st_*[\widehat{H}]} \psi(x_1^1)^{m-3}. \tag{23}$$

Since the mapping $\pi \circ st|_{\widehat{H}}$ has degree $(m-2)!\cdot(n-2)!^{(m-2)}$, it follows that

$$\pi_* st_*[\widehat{H}] = (m-2)!\cdot(n-2)!^{(m-2)}\cdot\Delta(-n,a_1,\dots,a_l). \tag{24}$$

Hence

$$h(g,n|A_1,\dots,A_m) = \frac{n^{m-3}\cdot(m-2)!}{\mathrm{aut}(A_2)}\cdot\int_{\Delta(-n,a_1,\dots,a_l)} \psi(x_1^1)^{m-3}. \tag{25}$$

$\square$

8. Amusing formulas for $\langle\tau_{3g}\tau_0^2\rangle_g$

Consider the moduli space of curves $\overline{\mathcal{M}}_{g,n} \ni (C, x_1, \dots, x_n)$. By $\langle\tau_{k_1}\dots\tau_{k_n}\rangle_g$ denote $\int_{\overline{\mathcal{M}}_{g,n}} \psi(x_1)^{k_1}\dots\psi(x_n)^{k_n}$. It is known from [25, 13] that $\langle\tau_{3g}\tau_0^2\rangle_g = 1/(24^g g!)$.

In this section we express $\langle\tau_{3g}\tau_0^2\rangle_g$ via Hurwitz numbers of generalized polynomials.

8.1. **Formulas.** By $H(g;n)$ we denote $h(g,n|A_1,\dots,A_m)$, where $A_1 = (n)$, $A_2 = \dots = A_m = (2,1,\dots,1)$, $m = 2g+n$.

Theorem 3. *For any $l \geq 0$,*

$$\langle\tau_{3g}\tau_0^2\rangle_g = \sum_{i=0}^{g}(-1)^i\binom{g}{i}\frac{(g+l+1-i)!H(g;g+l+1-i)}{g!(3g+l-i)!(g+l+1-i)^{3g+l-1-i}}. \tag{26}$$

Thus we have an infinite number (for any $l \geq 0$) of formulas for $\langle\tau_{3g}\tau_0^2\rangle_g$.

8.2. **Check in low genera.**

8.2.1. *Genus zero.* In genus zero we have the following:

$$\langle\tau_0^3\rangle_0 = \frac{(l+1)!H(0;l+1)}{l!(l+1)^{l-1}}. \tag{27}$$

Since $H(0;l+1) = (l+1)^{l-2}$, if follows that

$$\langle\tau_0^3\rangle_0 = \frac{(l+1)!(l+1)^{l-2}}{l!(l+1)^{l-1}} = 1. \tag{28}$$

8.2.2. *Genus one.* In genus one we have the following:

$$\langle\tau_3\tau_0^2\rangle_1 = \frac{(l+2)!H(1;l+2)}{(l+3)!(l+2)^{l+2}} - \frac{(l+1)!H(1;l+1)}{(l+2)!(l+1)^{l+1}}. \tag{29}$$

From [24] it follows that

$$H(1;l) = \frac{(l+1)!l^l(l-1)}{l!\cdot 24}. \tag{30}$$

Combining these formulas we obtain that $\langle\tau_3\tau_0^2\rangle_1 = 1/24$.

9. Proofs of Theorem 3

We shall give two proofs of Theorem 3. The first one is purely combinatorial. It is based on the formula of T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein [4]. The second proof is purely geometric. This proof is based on our Theorem 2.

Historically the second proof was the first one. It is based on ideas, which will be very useful in the rest of the paper. The second proof we obtained trying to check our formula in genera 2 and 3.

9.1. **First proof.**

Proof. Let us recall the formula for $H(g;k)$ from [4].

$$H(g;k) = \frac{(2g+k-1)! \cdot k^k}{k!} \int_{\overline{\mathcal{M}}_{g,1}} \frac{\sum_{i=0}^{g}(-1)^i \lambda_i}{1-k\psi(x_1)}. \tag{31}$$

Here x_1 is the unique marked point on curves in $\overline{\mathcal{M}}_{g,1}$, and $\lambda_i = c_i(\mathbb{E})$, where $\mathbb{E}$ is the rank g vector bundle over $\overline{\mathcal{M}}_{g,1}$ with the fiber over (C, x_1) equal to $H^0(C, \omega_C)$.

Putting this formula for $H(g;k)$ in our formula (26) we obtain the following:

(32)

$$\langle \tau_{3g}\tau_0^2 \rangle_g = \frac{1}{g!} \sum_{i=0}^{g} \sum_{j=0}^{g} (-1)^{i+j} \binom{g}{j} (g+l+1-j)^{g-i} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_i \psi(x_1)^{3g-2-i}.$$

It follows from the combinatorial Lemma below that the right hand side of this formula is equal to $\int_{\overline{\mathcal{M}}_{g,1}} \psi(x_1)^{3g-2}$. The equality $\langle \tau_{3g}\tau_0^2 \rangle_g = \langle \tau_{3g-2} \rangle_g$ follows from the string equation [25]:

$$\langle \tau_0 \prod_{i=1}^{m} \tau_{k_i} \rangle = \sum_{i=1^m} \langle \tau_{k_1} \dots \tau_{k_{i-1}} \tau_{k_i - 1} \tau_{k_{i+1}} \dots \tau_{k_m} \rangle. \tag{33}$$

□

Lemma 6.

$$\sum_{i=0}^{g} (-1)^i \binom{g}{i} (g+1-i)^k = \left\{ \begin{array}{ll} 0, & k < g \\ g!, & k = g \end{array} \right. . \tag{34}$$

Proof. By $f(g,k)$ denote the left hand side of the Lemma statement. Note that $f(0,0) = 1$, and $f(g,0) = 0$ if $g > 0$. Since

(35)

$$f(g+1, k+1) = (g+1) \cdot \left(\binom{k+1}{0} f(g,k) + \dots + \binom{k+1}{k} f(g,0) \right),$$

the statement of the Lemma follows. □

9.2. **Second proof.** First we shall express the intersection number $\langle \tau_{3g}\tau_0^2 \rangle_g = \langle \tau_{3g-2} \rangle_g$ via integrals over two-pointed ramification cycles. Then using our Theorem 2 we shall obtain formula (26).

9.2.1. *Two-pointed ramification cycles.* Let us fix $g \geq 0$. By Δ_k denote the two-pointed ramification cycle $\Delta(-k, 1, \dots, 1)$ in the moduli space $\overline{\mathcal{M}}_{g,1+k} \ni (C, y, t_1, \dots, t_k)$.

Consider the intersection number $\langle \tau_{3g}\tau_0^2 \rangle_g$. Below we prove the following Lemma.

Lemma 7. *For any $l \geq 0$*

$$\langle \tau_{3g}\tau_0^2\rangle_g = \sum_{i=0}^{g} \frac{(-1)^g}{g!}\binom{g}{i}\int_{\Delta_{g+l-i+1}} \psi(y_1)^{3g+l-i-1}. \tag{36}$$

9.2.2. *Proof of Theorem 3.*

Proof. From Theorem 2 we know that

$$\int_{\Delta_{g+l-i+1}} \psi(y_1)^{3g+l-i-1} = \frac{(g+l-i+1)!H(g,g+l-i+1)}{(g+l-i+1)^{3g+l-i-1}(3g+l-i)!}. \tag{37}$$

If we combine this with Lemma 7 we get

$$\langle \tau_{3g}\tau_0^2\rangle_g = \sum_{i=0}^{g} \frac{(-1)^g}{g!}\binom{g}{i}\frac{(g+l-i+1)!H(g,g+l-i+1)}{(g+l-i+1)^{3g+l-i-1}(3g+l-i)!}. \tag{38}$$

□

9.2.3. *Proof of Lemma 7.* Consider the moduli space $\overline{\mathcal{M}}_{g,2+g+l} \ni (C, y, t_1, \dots, t_{g+l+1})$. By $V(k_1|i_1,\dots,i_{k_2})$ denote the subvariety of $\mathcal{M}_{g,2+g+l}$ consisting of curves $(C, y_1, t_1, \dots, t_{g+l+1})$ such that there exists a meromorphic function of degree k with pole of multiplicity k at y and simple zeros at $t_{i_1}, \dots, t_{i_{k_2}}$.

Let $E(k_1|i_1,\dots,i_{k_2})$ be the cycle in homologies of $\overline{\mathcal{M}}_{g,1+g+l+1}$ determined by the closure of $V(k_1|i_1,\dots,i_{k_2})$. By $S(k_1,k_2)$ denote

$$\int_{E(k_1|i_1,\dots,i_{k_2})} \psi(y)^{3g+l-1+k_1-k_2}. \tag{39}$$

Obviously, the last number depends only on k_1 and k_2, but not on the choice of $i_1,\dots,i_{k_2}$.

Below, we always suppose that $k_1 \leq k_2 + g$, and $k_2 \geq 1$.

The proof of Lemma 7 is based on the following Lemma.

Lemma 8. *If $k_1 > k_2$, then*

$$S(k_1,k_2) = \frac{1}{k_2-k_1}\left(S(k_1,k_2+1) - S(k_1-1,k_2)\right). \tag{40}$$

The dependence of $S(k_1,k_2)$ on l is explained by the following Lemma.

Lemma 9. *The number $S(k_1,k_2)$ doesn't depend on the choice of l satisfying $g+l+1 \geq k_2$.*

Both this Lemmas we prove in the next subsection.

Proof of Lemma 7. Note that $[\overline{\mathcal{M}}_{0,2+g+l}] = E(g{+}l{+}1|1,2,\dots,l{+}1)$ (on the generic curve $(C, y, t_1, \dots, t_{l+1})$ there exists a meromorphic function of degree $g+l+1$ such that y is the pole of multiplicity $g+l+1$ and

$t_1, \dots, t_{l+1}$ are simple zeros; for the proof of such statements see [17], section 7). Then

$$(41) \qquad \langle \tau_{3g}\tau_0^2 \rangle_g = \int_{\overline{\mathcal{M}}_{g,g+l+2}} \psi(y)^{4g+l-1} = S(g+l+1|l+1).$$

Lemma 8 implies that

$$(42) \quad S(g+l+1|l+1) = \frac{1}{g} S(g+l+1|l+2) - \frac{1}{g} S(g+l|l+1) =$$
$$= \frac{S(g+l+1|l+3)}{g(g-1)} - \frac{2 \cdot S(g+l|l+2)}{g(g-1)} + \frac{S(g+l-1|l+1)}{g(g-1)} =$$
$$= \dots = \sum_{i=0}^{g} \frac{(-1)^i}{g!} \binom{g}{i} S(g+l-i+1|g+l-i+1).$$

Lemma 9 implies that

$$(43) \qquad S(g+l-i+1|g+l-i+1) = \int_{\Delta_{g+l-i+1}} \psi(y)^{3g+l-i-1}.$$

Combining (42) and (43) we obtain the statement of Lemma 7. □

9.2.4. *Proofs of Lemmas 8 and 9.*

Proof of Lemma 9. Let $E = E(k_1|1, \dots, k_2)$ be the cycle in the homology of $\overline{\mathcal{M}}_{g,2+g+l}$. Let $\widehat{E} = E(k_1|1, \dots, k_2)$ be the cycle in the homology of $\overline{\mathcal{M}}_{g,2+g+\widehat{l}}$, where $\widehat{l} = l+1$.

By $\pi \colon \overline{\mathcal{M}}_{g,2+g+\widehat{l}} \to \overline{\mathcal{M}}_{g,2+g+l}$ denote the projection forgetting the labeled point $t_{g+\widehat{l}+1}$. Note that $\psi(y)$ in the cohomology of $\overline{\mathcal{M}}_{g,2+g+\widehat{l}}$ is equal to $\pi^*\psi(y) + D$, where D is the class dual to the divisor whose generic point is represented by a two-component curve such that one component has genus zero and contains points y and $t_{g+\widehat{l}+1}$, and the other component has genus g and contains all labeled points except for y and $t_{g+\widehat{l}+1}$.

It's easy to see that $(\pi^*\psi(y)+D)^K = \pi^*\psi(y)^K + D \cdot \pi^*\psi(y)^{K-1}$. Also note that $\pi_*(\widehat{E} \cdot D) = E$.

Since (from dimensional conditions)

$$(44) \qquad \int_{\widehat{E}} \pi^*\psi(y)^{3g+\widehat{l}-1+k_1-k_2} = 0,$$

it follows that

(45)
$$\int_{\widehat{E}} \psi(y)^{3g+\widehat{l}-1+k_1-k_2} = \int_{\widehat{E}} D \cdot \pi^*\psi(y)^{3g+l-1+k_1-k_2} = \int_{E} \psi(y)^{3g+l-1+k_1-k_2}.$$

This proves the Lemma. □

Proof of Lemma 8. Using Lemma 9 we can consider $S(k_1, k_2)$ as

$$\int_{E(k_1|1,\dots,k_2)} \psi(y)^{2g+k_1-1}, \tag{46}$$

where $g + l = k_2$ (l can be negative).

Consider the following ramification data: $A_1 = (k_1)$, $A_2 = \cdots = A_{2g+k_1} = (2, 1, \dots, 1)$, $A_{2g+k_1+1} = A_{2g+k_1+2} = (1, \dots, 1)$. Consider the space $\widehat{H}$ built using this ramification data. The map st takes a point of $\widehat{H}$ to a curve

$$\begin{aligned}(C, x_1^1, x_1^2, \dots, x_{k_1-1}^2, \dots, x_1^{2g+k_1}, \dots, x_{k_1-1}^{2g+k_1}, x_1^{2g+k_1+1}, \dots,\\ x_{k_1}^{2g+k_1+1}, x_1^{2g+k_1+2}, \dots, x_{k_1}^{2g+k_1+2}) \in \overline{\mathcal{M}}_{g,1+2k_1+(2g+k_1-1)(k_1-1)}.\end{aligned} \tag{47}$$

Consider the projection $\pi\colon \overline{\mathcal{M}}_{g,1+2k_1+(2g+k_1-1)(k_1-1)} \to \overline{\mathcal{M}}_{g,2+k_2}$, which takes the labeled point x_1^1 to y; $x_1^{2g+k_1+1}, \dots, x_{k_2}^{2g+k_1+1}$ to $t_1, \dots, t_{k_2}$; $x_1^{2g+k_1+2}$ to t_{k_2+1}; and forgets all other labeled points. Note that $\pi_* st_*[\widehat{H}] = K \cdot E(k_1|1, \dots, k_2)$, where

$$K = (k_1 - 1)!(k_1 - k_2)!(2g + k_1 - 1)!(k_1 - 2)^{(2g+k_1-1)}. \tag{48}$$

The Lemma of E. Ionel implies that

$$\psi(y) \cdot E(k_1|1, \dots, k_2) = \frac{1}{K}\pi_* \left(\frac{1}{k_1} st_*(\widehat{ll}^* \psi(z_1) \cdot [\widehat{H}]) - D \cdot st_*[\widehat{H}]\right), \tag{49}$$

where D is dual to the divisor whose generic point is represented by a two-component curve such that one component has genus g and contains the preimages of the points $t_1, \dots, t_{k_2+1}$, and the other component has genus zero and contains the point x_1^1. It is the standart expression for $\psi(x_1^1)$ via $\pi^*\psi(y)$.

One can consider $D \cdot st_*[\widehat{H}]$ as the class determined by $D \cap st(\widehat{H})$. We are interested only in the irreducible components of $D \cap st(\widehat{H})$ whose image under the map π has codimension one.

Let us describe the generic point of such component. It is a three-component curve. One component has genus zero and contains x_1^1, the whole tuple of preimages of one critical value, say $x_1^2, \dots, x_{k_1-1}^2$, and two nodes, $*_1$ and $*_2$. There exists a meromorphic function of degree k_1 whose divisor is $-k_1 x_1^1 + (k_1 - 1) *_1 + *_2$, and $x_1^2, \dots, x_{k_1-1}^2$ are the preimages of its simple critical value. The second component has genus zero and is attached to the first one at the point $*_2$. It contains exactly one point from each of the sets $\{x_2^3, \dots, x_{k_1-1}^3\}$, $\dots$, $\{x_2^{2g+k_1}, \dots, x_{k_1-1}^{2g+k_1}\}$, $\{x_{k_2+1}^{2g+k_1+1}, \dots, x_{k_1}^{2g+k_1+1}\}$, $\{x_2^{2g+k_1+2}, \dots, x_{k_1}^{2g+k_1+2}\}$. The last component has genus g, is attached to the first component at the point $*_1$ and contains all other points. There is a function of degree $k_1 - 1$ such that $*_1$ is the point of total ramification, $x_1^3, \dots, x_1^{2g+k_1}$ are simple critical points, and all points x^i_j with fixed superscript are the preimages of a single point in the image.

From this description it is obvious that

$$\pi_*(D\cdot st_*[\widehat{H}]) = K\cdot E(k_1-1|1,\dots,k_2). \tag{50}$$

Now we shall describe $\pi_* st_*(\widehat{ll}^*\psi(z_1))$. Note that $\psi(z_1)$ is dual to the divisor whose generic point is represented by a two-component curve such that z_1 lies on one component, and z_{2g+k_1+1} and z_{2g+k_1+2} lie on the other component. Let us describe only those components of the preimage of this divisor under the map $\widehat{ll}$ whose π_*st_*-image does not vanish $\psi(y)^{2g+k_1-1}$.

There are two possible cases. The first one looks like follows. Consider the divisor in $\overline{\mathcal{M}}_{0,2g+k_1+2}$ whose generic point is represented by a two-component curve such that z_1 and exactly one point from the set $\{z_2,\dots,z_{2g+k_1}\}$ lie on one component, and all other points (including z_{2g+k_1+1} and z_{2g+k_1+2}) lie on the other component. Then we obtain just the same picture as in the case of $D\cdot st_*[\widehat{H}]$, but with coefficient k_1-1, since $\widehat{ll}$ is ramified along this divisor with multiplicity k_1-1.

The next case is as follows. Consider the divisor in $\overline{\mathcal{M}}_{0,2g+k_1+2}$ whose generic point is represented by a two-component curve such that z_{2g+k_1+1} and z_{2g+k_1+2} lie on one component and all other marked points lie on the other component. The map $\widehat{ll}$ is not ramified over this divisor. There are $k_2+(k_1-k_2)$ possible cases in the preimage of this divisor under the map $\widehat{ll}$.

The first k_2 cases mean that $x_1^{2g+k_1+2}$ lies on the same genus zero irreducible component of the domain curve as one of the marked points $x_1^{2g+k_1+1},\dots,x_{k_2}^{2g+k_1+1}$. Under the map π_*st_* this gives us $k_2\cdot K\cdot E(k_1|1,\dots,k_2)$ in the space $\overline{\mathcal{M}}_{g,2+l'+g}$, where $l'=l-1$.

The other k_1-k_2 cases mean that $x_1^{2g+k_1+2}$ lie on the same genus zero irreducible component of the domain curve, as one of the marked points $x_{k_2+1}^{2g+k_1+1},\dots,x_{k_1}^{2g+k_1+1}$. Under the map π_*st_* this gives us $K\cdot E(k_1|1,\dots,k_2,k_2+1)$ in the space $\overline{\mathcal{M}}_{g,2+l+g}$ (it is easy to check the coefficient by direct calculations).

Thus we obtain that

$$\begin{aligned}\int_{E(k_1|1,\dots,k_2)}\psi(y)^{2g+k_1-1} &= \frac{k_1-1}{k_1}\int_{E(k_1-1|1,\dots,k_2)}\psi(y)^{2g+k_1-2}+\\ &+\frac{k_2}{k_1}\int^*_{E(k_1|1,\dots,k_2)}\psi(y)^{2g+k_1-2}+\frac{1}{k_1}\int_{E(k_1|1,\dots,k_2,k_2+1)}\psi(y)^{2g+k_1-2}-\\ &-\int_{E(k_1-1|1,\dots,k_2,k_2)}\psi(y)^{2g+k_1-2}.\end{aligned}\tag{51}$$

Here the sign $\int^*$ means that we calculate the intersection number in the (co)homology of $\overline{\mathcal{M}}_{g,2+l'+g}$.

Combining the last equality and Lemma 9 we obtain that $S(k_1, k_2) = (S(k_1, k_2 + 1) - S(k_1 - 1, k_2))/(k_1 - k_2)$. $\square$

10. The conjecture of E. Witten

In this section we recall the conjecture of E. Witten [26]. Unfortunately, we give some definitions very briefly. More details one can find in [26] and [11].

10.1. **Gelfand-Dikii hierarchies.** Let us fix r. In this section we define the string solution of the r-Gelfand-Dikii (or r-KdV) hierarchy.

Consider the differential operator

$$Q = D^r - \sum_{i=0}^{r-2} u_i(x) D^i, \tag{52}$$

where

$$D = \frac{i}{\sqrt{r}} \frac{\partial}{\partial x}. \tag{53}$$

There is a pseudo-differential operator $Q^{1/r} = D + \sum_{i>0} w_i D^{-i}$. By $Q^{n+m/r}$ denote $(Q^{1/r})^{nr+m}$. Note that $[Q_+^{n+m/r}, Q]$ is a differential operator of order at most $r-2$.

The Gelfand-Dikii equations read

$$i \frac{\partial Q}{\partial t_{n,m}} = [Q_+^{n+m/r}, Q] \cdot \frac{c_{n,m}}{\sqrt{r}}, \tag{54}$$

where

$$c_{n,m} = \frac{(-1)^n r^{n+1}}{(m+1)(r+m+1)\dots(nr+m+1)}. \tag{55}$$

The string solution of the Gelfand-Dikii hierarchy is the formal series F in variables $t_{i,j}$, $i = 0, 1, 2, \dots$, $j = 0, \dots, r-1$, such that

$$\frac{\partial F}{\partial t_{0,0}} = \frac{1}{2} \sum_{i,j=0}^{r-2} \delta_{i+j,r-2} t_{0,i} t_{0,j} + \sum_{n=1}^{\infty} \sum_{m=0}^{r-2} t_{n+1,m} \frac{\partial F}{\partial t_{n,m}} \tag{56}$$

and

$$\frac{\partial^2 F}{\partial t_{0,0} \partial t_{n,m}} = -c_{n,m} \mathrm{res}(Q^{n+\frac{m+1}{r}}), \tag{57}$$

where Q satisfies the Gelfand-Dikii equations and $t_{0,0}$ is identified with x.

One can prove that F is uniquely determined by this equations, up to an additive constant. Below we shall discuss an effective way to calculate the coefficients of F in the case of $r = 3$ (Boussinesq hierarchy).

10.2. **Intersection numbers $\langle \prod \tau_{i,j}^{k_{i,j}} \rangle_g$.** Our definitions in this section can seem to be not entirely clear. But our goal now is only to give a general idea of Witten's definitions. As usual, we refer to [26, 11, 23] for details.

10.2.1. *A covering of $\mathcal{M}_{g,s}$.* Consider the moduli space $\overline{\mathcal{M}}_{g,s} \ni (C, x_1, \ldots, x_s)$. Fix an integer $r \geq 2$. Label each marked point x_i by an integer m_i, $0 \leq m_i \leq r-1$.

By K denote the canonical line bundle of C. Consider the line bundle $S = K \otimes (\otimes_{i=1}^s \mathcal{O}(x_i)^{-m_i})$ over C. If $2g-2-\sum_{i=1}^s m_i$ is divisible by r, then there are r^{2g} isomorphism classes of line bundles $\mathcal{T}$ such that $\mathcal{T}^{\otimes r} \cong S$.

The choice of an isomorphism class of $\mathcal{T}$ determines a cover $\mathcal{M}'_{g,s}$ of $\mathcal{M}_{g,s}$. To extend it to a covering of $\overline{\mathcal{M}}_{g,s}$ we have to discuss the behavior of $\mathcal{T}$ near a double point.

10.2.2. *Behavior near a double point.* Let C be a singular curve with one double point. By $\pi\colon C_0 \to C$ denote its normalization. The preimage of the double point consists of two points, say x' and x''. There are r possible cases of behavior of $\mathcal{T}$ near the double point.

The first $r-1$ cases are the following ones: $\mathcal{T} \cong \pi_* \mathcal{T}'$, where $\mathcal{T}'$ is a locally free sheaf on C_0 with a natural isomorphism $\mathcal{T}'^{\otimes r} \cong K \otimes (\otimes_{i=1}^s \mathcal{O}(x_i)^{-m_i}) \otimes \mathcal{O}(x')^{-m} \otimes \mathcal{O}(x'')^{-(r-2-m)}$, $m = 0, \ldots, r-2$.

The last case is when $\mathcal{T}$ is defined by the following exact sequence: $0 \to \mathcal{T}' \to \mathcal{T} \to \mathcal{O} \to$. Here $\mathcal{T}' = \pi_* \mathcal{T}''$, where $\mathcal{T}''^{\otimes r} \cong K \otimes (\otimes_{i=1}^s \mathcal{O}(x_i)^{-m_i}) \otimes \mathcal{O}(x')^{-(r-1)} \otimes \mathcal{O}(x'')^{-(r-1)}$. The map $\mathcal{T} \to \mathcal{O}$ is the residue map taking a section of $\mathcal{T}$ to the coefficient of $(dx/x)^{1/r}$.

10.2.3. *The top Chern class.* If $H^0(C, \mathcal{T})$ vanishes everywhere, then one can consider the vector bundle $\mathcal{V}$ over $\overline{\mathcal{M}}'_{g,s}$ whose fiber is the dual space to $H^1(C, \mathcal{T})$. What we need is the top Chern class $c_D(\mathcal{V})$ of this bundle; here

$$D = \frac{(g-1)(r-2)}{r} + \frac{1}{r} \sum_{i=1}^{s} m_i \tag{58}$$

In the case when $H^0(C, \mathcal{T})$ is not identically zero, there is another definition of the corresponding cohomology class. We don't want to recall it. Let us denote this class by the same notation, $c_D(\mathcal{V})$.

We shall use only the following properties of this class. First, consider a component of the boundary consisting of curves with one double point, where $\mathcal{T}$ is defined by one of the first $r-1$ possible cases. If this component of the boundary consists of two-component curves, then $c_D(\mathcal{V}) = c_{D_1}(\mathcal{V}_1) \cdot c_{D_2}(\mathcal{V}_2)$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are the corresponding spaces on the components, $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$. If this component of the boundary consists of one-component self-intersecting curves, then $c_D(\mathcal{V}) = c_D(\mathcal{V}')$, where $c_D(\mathcal{V}')$ is defined on the normalization of these curves. The next case is when we restrict $c_D(\mathcal{V})$ to a component of the boundary with the last possible behavior of $\mathcal{T}$ at a double point. Then we require that $c_D(\mathcal{V})$ vanishes.

It will be easy to see that such properties uniquely determine the desired intersection numbers. It is not obvious that these properties define the Witten's top Chern class. But in any mathematical paper reformulating the conjecture of Witten [11, 23], it is done in the similar way.

10.2.4. *The Mumford-Morita-Miller intersection numbers.* Let us label each marked point x_i by an integer $n_i \geq 0$. By

$$\langle \tau_{n_1,m_1} \dots \tau_{n_s,m_s} \rangle_g \tag{59}$$

denote the intersection number

$$\frac{1}{r^g} \int_{\overline{\mathcal{M}}'_{g,s}} \prod_{i=1}^{s} \psi(x_i)^{n_i} \cdot c_D(\mathcal{V}). \tag{60}$$

Of course, this number is not zero only if

$$3g - 3 + s = \sum_{i=1}^{s} n_i + D. \tag{61}$$

10.3. **The conjecture.** Consider the formal series F in variables $t_{n,m}$, $n = 0, 1, 2, \dots;\ m = 0, \dots, r-1$;

$$F(t_{0,0}, t_{0,1}, \dots) = \sum_{d_{n,m}} \langle \prod_{n,m} \tau_{n,m}^{d_{n,m}} \rangle \prod_{n,m} \frac{t_{n,m}^{d_{n,m}}}{d_{n,m}!}. \tag{62}$$

The conjecture is that this F is the string solution of the r-Gelfand-Dikii hierarchy.

10.4. **Boussinesq hierarchy.** We need to be able to calculate the coefficients of F starting from the Gelfand-Dikii hierarchy. The general methods, like [18], are very complicated. We show how to do this in the case $r = 3$ (Boussinesq hierarchy).

Let us denote $\partial^k F / \partial t_{i_1,j_1} \dots \partial t_{i_k,j_k}$ by $\langle\langle \tau_{i_1,j_1} \dots \tau_{i_k,j_k} \rangle\rangle$. Note that $\operatorname{res}(Q^{1/3}) = -\langle\langle \tau_{0,0}^2 \rangle\rangle / 3$ and $\operatorname{res}(Q^{2/3}) = -2 \langle\langle \tau_{0,0} \tau_{0,1} \rangle\rangle / 3$.

Since $Q = D^3 + 3(\operatorname{res}(Q^{1/3}))D + (3/2)(\operatorname{res}(Q^{2/3}) + D \operatorname{res}(Q^{1/3}))$, it follows that $Q = D^3 + \gamma_1 D + \gamma_2$, where $\gamma_1 = -\langle\langle \tau_{0,0}^2 \rangle\rangle$ and $\gamma_2 = -\langle\langle \tau_{0,0}\tau_{0,1} \rangle\rangle - i \langle\langle \tau_{0,0}^3 \rangle\rangle / (2\sqrt{3})$.

Let us consider the pseudo-differential operator $Q^{n-1+m/r}$. Since $[Q, Q^{n-1+m/r}] = 0$, it follows that $[Q, Q_-^{n-1+m/r}]$ is a differential operator. Let $Q_-^{n-1+m/r} = \alpha_1 D^{-1} + \alpha_2 D^{-2} + \dots$. Then the coefficients at D^{-1} and D^{-2} of the operator $[Q, Q_-^{n-1+m/r}]$ are equal to

$$D^3 \alpha_1 + 3D^2 \alpha_2 + 3D\alpha_3 + D(\alpha_1 \gamma_1) = 0 \tag{63}$$

and
(64)
$$D^3\alpha_2 + 3D^2\alpha_3 + 3D\alpha_4 + \gamma_1 D\alpha_2 - \alpha_1 D^2\gamma_1 + 2\alpha_2 D\gamma_1 + \alpha_1 D\gamma_2 = 0$$
respectively.

Note that $\mathrm{res}(Q^{n+m/3}) = \alpha_4 - \alpha_1 D\gamma_1 + \alpha_2\gamma_1 + \alpha_1\gamma_2$. From the Gelfand-Dikii equations it follows that

$$(65)\quad \frac{i\sqrt{3}}{c_{n-1,m}}\frac{\partial\gamma_1}{\partial t_{n-1,m}} = 3D\alpha_1,$$
$$\frac{i\sqrt{3}}{c_{n-1,m}}\frac{\partial\gamma_2}{\partial t_{n-1,m}} = 3D^2\alpha_1 + 3D\alpha_2,$$
$$\frac{i\sqrt{3}}{c_{n,m}}\frac{\partial\gamma_1}{\partial t_{n,m}} = 3D\,\mathrm{res}(Q^{n+m/3}).$$

Using equations (63) and (64) we can express $D\alpha_4$ in terms of α_1 and α_2. Using the first two equations in (65) and the string equation we can express α_1 and α_2 in terms of derivatives of F. If we replace all α_i and γ_i in the third equation in (65) with their expressions we obtain the following:

$$(66)\quad (3n+m+1)\langle\langle\tau_{n,m}\tau_{0,0}^2\rangle\rangle = \langle\langle\tau_{n-1,m}\tau_{0,1}\rangle\rangle\langle\langle\tau_{0,0}^3\rangle\rangle + 2\langle\langle\tau_{n-1,m}\tau_{0,0}\rangle\rangle\langle\langle\tau_{0,1}\tau_{0,0}^2\rangle\rangle + 2\langle\langle\tau_{n-1,m}\tau_{0,1}\tau_{0,0}\rangle\rangle\langle\langle\tau_{0,0}^2\rangle\rangle + \frac{2}{3}\langle\langle\tau_{n-1,m}\tau_{0,1}\tau_{0,0}^3\rangle\rangle + 3\langle\langle\tau_{n-1,m}\tau_{0,0}^2\rangle\rangle\langle\langle\tau_{0,1}\tau_{0,0}\rangle\rangle.$$

This equation allows us to calculate the coefficients of F in the examples below. The similar equation is used in [25] in the case of $r=2$. It is easy to see that using the same argument one can obtain the similar equation for any Gelfand-Dikii hierarchy.

11. An algorithm to calculate $\langle\tau_{n,m}\prod_{i=1}^{r-1}\tau_{0,i}^{k_i}\rangle_g$

Let us fix $r \geq 2$.

11.1. Two-pointed ramification cycles. By $V_{g,m}^*(\prod_{i=1}^s \eta_{q_i,a_i})$ denote the subvariety of $\mathcal{M}'_{g,1+s}$ consisting of curves $(C, x_1 \dots, x_{1+s}, \mathcal{T})$ such that $-(\sum_{i=1}^s a_i)x_1 + \sum_{i=1}^s a_i x_{1+i}$ is the divisor of a meromorphic function. The covering $\mathcal{M}'_{g,1+s} \to \mathcal{M}_{g,1+s}$ is defined here by $m_1 = m$, $m_2 = q_1, \dots, m_{1+s} = q_s$. All a_i are supposed to be positive integers and also we require $0 \leq m, q_1, \dots, q_s \leq r-1$.

Let us denote by $V_{g,m}(\prod_{i=1}^s \eta_{q_i,a_i})$ the closure of $V_{g,m}^*(\prod_{i=1}^s \eta_{q_i,a_i})$.

Let us denote by $S_{g,m}^n(\prod_{i=1}^s \eta_{q_i,a_i})$ the intersection number

$$(67)\qquad \frac{1}{r^g}\int_{V_{g,m}(\prod_{i=1}^s \eta_{q_i,a_i})} \psi(x_1)^n \cdot c_D(\mathcal{V}).$$

Note that $S^n_{g,m}(\prod_{i=1}^s \eta_{m_i,a_i})$ is defined iff $g \geq 0$; $0 \leq m, q_1, \ldots, q_s \leq r-1$; $n \geq 0$; $s \geq 1$; $a_1, \ldots, a_s \geq 1$. Moreover, for convenience we put $S^n_{0,m}(\eta_{m_1,1}) = 0$; and if $g < 0$, then we also put $S^n_{g,m}(\prod_{i=1}^s \eta_{m_i,a_i}) = 0$.

Another definition we need is the following one. Consider the moduli space $\overline{\mathcal{M}}'_{1,k} \ni (C, x_1, \ldots, x_k, \mathcal{T})$ determined by some labels $m_1, \ldots, m_k$. Let $b_1, \ldots, b_k$ be non-zero integers such that $\sum_{t=1}^k b_t = 0$. By $W(b_1, \ldots, b_k)$ denote the closure of the subvariety consisting of smooth curves $(C, x_1, \ldots, x_k, \mathcal{T})$ such that there exists a meromorphic function whose divisor is equal to $\sum_{t=1}^k b_t x_t$.

By $\widehat{S}_1(\prod_{t=1}^k \eta_{m_t,b_t})$ denote $(1/r)\int_{W(b_1,\ldots,b_k)} c_D(\mathcal{V})$. We shall discuss later how to calculate $\widehat{S}_1(\prod_{t=1}^k \eta_{m_t,b_t})$.

11.2. **The algorithm.** The algorithm is based on the following statements.

Theorem 4. *If $s \geq 1$, then*

$$\langle \tau_{n,m} \prod_{i=1}^{s} \tau_{0,m_i} \rangle_g = \sum_{j=0}^{g} \frac{(-1)^j}{g!} \binom{g}{j} S^{n-j}_{g,m} (\prod_{i=1}^{s} \eta_{m_i,1} \cdot \eta_{0,1}^{g-j}). \tag{68}$$

If $n \geq 1$, then

$$\begin{aligned} &S^n_{g,m}(\prod_{i=1}^{s} \eta_{m_i,a_i}) = \\ &\sum_{I \subset \{1,\ldots,s\}} \sum_{j=1}^{a(I)} \sum_{B(j,a(I))} \Bigg(\frac{|I| + j - 2}{(\sum_{r=1}^s a_r) \cdot (2g+s-1)} \cdot \frac{\prod_{r=1}^j b_r}{\mathrm{aut}(b_1, \ldots, b_j)} \cdot \\ &\sum_{u_1,\ldots,u_j=0}^{r-2} S^{n-1}_{g+1-j,m}(\prod_{i \notin I} \eta_{m_i,a_i} \prod_{t=1}^{j} \eta_{u_t,b_t}) \cdot \langle \prod_{t=1}^{j} \tau_{0,r-2-u_t} \prod_{i \in I} \tau_{0,m_i} \rangle_0 + \\ &\quad + \frac{|I| + j}{(\sum_{r=1}^s a_r) \cdot (2g+s-1)} \cdot \frac{\prod_{r=1}^j b_r}{\mathrm{aut}(b_1, \ldots, b_j)} \cdot \\ &\sum_{u_1,\ldots,u_j=0}^{r-2} S^{n-1}_{g-j,m}(\prod_{i \notin I} \eta_{m_i,a_i} \prod_{t=1}^{j} \eta_{u_t,b_t}) \cdot \widehat{S}_1(\prod_{i \notin I} \eta_{m_i,-a_i} \prod_{t=1}^{j} \eta_{r-2-u_t,b_t}) \Bigg). \end{aligned} \tag{69}$$

Here the first sum is taken over all subsets I of $\{1, \ldots, s\}$. Then, $a(I)$ is defined to be $\sum_{k \in I} a_k$. The third sum is taken over all possible partitions $B = (b_1, \ldots, b_j)$ of $a(I)$ of length j; $\sum_{k=1}^j b_k = a(I)$, $b_1 \geq \cdots \geq b_j$, and all b_k are positive integers.

The initial values look like follows:

$$(70)\quad S^0_{g,m}(\prod_{i=1}^{s}\eta_{m_i,a_i})=0,\quad \textit{if}\ g>1;$$

$$S^0_{1,m}(\prod_{i=1}^{s}\eta_{m_i,a_i})=\widehat{S}_1(\eta_{m,-\sum_{i=1}^s a_i}\cdot\prod_{i=1}^{s}\eta_{m_i,a_i});$$

$$S^n_{0,m}(\prod_{i=1}^{s}\eta_{m_i,a_i})=\langle\tau_{n,m}\prod_{i=1}^{s}\tau_{0,m_i}\rangle_0.$$

Applying Eq. (68) and then several times Eq. (69) we express the intersection number $\langle\tau_{n,m}\prod_{i=1}^{r-1}\tau_{0,i}^{k_i}\rangle_g$ via the intersection numbers $\langle\prod_{i=1}^{r-1}\tau_{0,i}^{p_i}\rangle_0$ and $\widehat{S}_1(\prod_{t=1}^{k}\eta_{m_t,b_t})$. Later we explain how to express the intersection numbers $\widehat{S}_1(\prod_{t=1}^{k}\eta_{m_t,b_t})$ via $\langle\prod_{i=1}^{r-1}\tau_{0,i}^{p_i}\rangle_0$.

11.3. **Simple examples.**

11.3.1. $\langle\tau_{1,0}\rangle_1$. Let us consider the case $r=4$. From the topological recursion relation (see [5]) we know that $\langle\tau_{1,0}\rangle_1=1/8$. Let us prove this independently.

Note that $\langle\tau_{1,0}\rangle_1=\langle\tau_{2,0}\tau_{0,0}\rangle_1$. Equation (68) implies that $\langle\tau_{2,0}\tau_{0,0}\rangle_1=S^2_{1,0}(\eta^2_{0,1})$.

We know from [26] that in the case $r=4$, $\langle\tau^2_{0,0}\tau_{0,2}\rangle_0=\langle\tau_{0,0}\tau^2_{0,1}\rangle_0=1$, $\langle\tau^2_{0,1}\tau^2_{0,2}\rangle_0=1/4$, $\langle\tau^5_{0,2}\rangle_0=1/8$, and all other $\langle\prod_{i=1}^{r-1}\tau_{0,i}^{p_i}\rangle_0$ are equal to zero. Another fact we need here is that $\widehat{S}_1(\prod_{t=1}^{k}\eta_{m_t,b_t})$ is equal to zero if one m_i is equal to zero.

Then, from (69) and formulas for initial values, it follows that

$$(71)\quad S^2_{1,0}(\eta^2_{0,1})=\frac{1}{3}S^1_{1,0}(\eta_{0,2})\langle\tau^2_{0,0}\tau_{0,2}\rangle_0;$$

$$S^1_{1,0}(\eta_{0,2})=\frac{1}{4}S^0_{0,0}(\eta_{0,1}\eta_{2,1})\langle\tau^2_{0,0}\tau_{0,2}\rangle_0+\frac{1}{8}S^0_{0,0}(\eta^2_{1,1})\langle\tau_{0,0}\tau^2_{0,1}\rangle_0;$$

$$S^0_{0,0}(\eta_{0,1}\eta_{2,1})=\langle\tau^2_{0,0}\tau_{0,2}\rangle_0;$$

$$S^0_{0,0}(\eta^2_{1,1})=\langle\tau_{0,0}\tau^2_{0,1}\rangle_0.$$

Thus we obtain that $\langle\tau_{1,0}\rangle_1=1/8$, as it has to be.

11.3.2. $\langle\tau_{1,1}\tau^3_{0,1}\tau_{0,0}\rangle_0$. Let us calculate $\langle\tau_{1,1}\tau^3_{0,1}\tau_{0,0}\rangle_0$ in the case $r=3$. It follows from the string equation that $\langle\tau_{1,1}\tau^3_{0,1}\tau_{0,0}\rangle_0=\langle\tau^4_{0,1}\rangle_0=1/3$, but we want to calculate this using our algorithm.

Recall that in the case $r=3$, $\langle\tau^2_{0,0}\tau_{0,1}\rangle_0=1$, $\langle\tau^4_{0,1}\rangle_0=1/3$, and all other $\langle\prod_{i=1}^{r-1}\tau_{0,i}^{p_i}\rangle_0$ are equal to zero.

From our algorithm we have the following

$$\langle\tau_{1,1}\tau_{0,1}^3\tau_{0,0}\rangle_0 = S_{0,1}^1(\eta_{1,1}^3\eta_{0,1}) \tag{72}$$
$$S_{0,1}^1(\eta_{1,1}^3\eta_{0,1}) = \frac{1}{2}S_{0,1}^0(\eta_{1,1}^2\eta_{1,2})\langle\tau_{0,0}^2\tau_{0,1}\rangle_0 + \frac{1}{2}S_{0,1}^0(\eta_{0,3}\eta_{0,1})\langle\tau_{0,1}^4\rangle_0$$
$$S_{0,1}^0(\eta_{1,1}^2\eta_{1,2}) = \langle\tau_{0,1}^4\rangle_0$$
$$S_{0,1}^0(\eta_{0,3}\eta_{0,1}) = \langle\tau_{0,0}^2\tau_{0,1}\rangle_0$$

Thus we obtain that $\langle\tau_{1,1}\tau_{0,1}^3\tau_{0,0}\rangle_0 = 1/3$.

12. Proof of Theorem 4 and calculation of $\widehat{S}_1(\prod_{t=1}^k \eta_{m_t,b_t})$

12.1. **Initial values.**

Proof of the third statement of Theorem 4. Consider the intersection number $S = S_{g,m}^0(\prod_{i=1}^s \eta_{m_i,a_i}) = 0$, where $g > 0$. We want to prove that $S = 0$. Recall that S is defined to be

$$S = \frac{1}{r^g}\int_{V_{g,m}(\prod_{i=1}^s \eta_{q_i,a_i})} c_D(\mathcal{V}) \tag{73}$$

Note that $\dim V_{g,m}(\prod_{i=1}^s \eta_{q_i,a_i}) = 2g+s-2$ and $D < g+s$. If $g \geq 2$, then $2g+s-2 \geq g+s$, and we obtain $S=0$.

Since $V_{1,m}(\prod_{i=1}^s \eta_{q_i,a_i}) = W(-\sum_{i=1}^s a_i, a_1, \dots, a_s)$, where $m_1 = m$, $m_2 = q_1, \dots, m_{s+1} = q_s$, if follows that $S_{1,m}^0(\prod_{i=1}^s \eta_{m_i,a_i}) = \widehat{S}_1(\eta_{m,-\sum_{i=1}^s a_i} \cdot \prod_{i=1}^s \eta_{m_i,a_i})$.

The equality $S_{0,m}^n(\prod_{i=1}^s \eta_{m_i,a_i}) = \langle\tau_{n,m}\prod_{i=1}^s \tau_{0,m_i}\rangle_0$ we obtain just from the fact that in the case of genus zero $V_{0,m}(\prod_{i=1}^s \eta_{q_i,a_i})$ is equal to the appropriate space $\overline{\mathcal{M}}'_{0,1+s}$. □

12.2. **First step of the algorithm.**

Proof of equality (68). The first statement of Theorem 4 is proved by the very same argument as Lemma 7. The only difference is the following one. We have to formulate and prove an analogue of Lemma 8. Proving this analogue and using the Lemma of E. Ionel we represent the ψ-class as a sum of divisors. Then we choose only those divisors, where $\psi(y)^k \cdot c_D(\mathcal{V})$ is not zero. Here we have to use one more additional argument: $\langle\tau_{0,0}\cdot\prod_{i=1}^l \tau_{0,m_i}\rangle_0 \neq 0$, iff $l=2$ and $m_1+m_2 = r-2$. Then this intersection number is equal to 1. All other steps of the proof are just the same. □

12.3. **Equality (69).**

Proof of equality (69). This is also proved in the same way as Lemma 7. Let us use the Lemma of E. Ionel. In the target moduli space of the $\widehat{ll}$ mapping we express the ψ-class as the divisor whose generic point is represented by a two-component curve such that the point corresponding to x_1 lies on the first component and the point corresponding to $x_2, \dots, x_s$ with a fixed critical value lie on the other component.

Then we express $\psi(x_1)$ as a sum of some divisors in $\widehat{H}$. The mapping $\sigma \circ st$ (σ is the projection forgetting all marked point except for $x_1, \dots, x_{1+s}$) takes each divisor to a subvariety of $\pi(V_{g,m}(\prod_{i=1}^s \eta_{q_i,a_i}))$ (π is the projection $\overline{\mathcal{M}}'_{g,1+s} \to \overline{\mathcal{M}}_{g,1+s}$). We need only subvarieties of codimension one. This condition means that all critical points of the corresponding functions are lying exactly on two components of a curve in $st(\widehat{H})$. In other words, this means that a curve in $\sigma(st(\widehat{H}))$ consists of two components.

Consider such irreducible divisor in $\pi(V_{g,m}(\prod_{i=1}^s \eta_{q_i,a_i})) = \sigma(st(\widehat{H}))$. Two components of a curve representing a generic point of this divisor can intersect at j points. One component contains points x_{i+1}, $i \in I \subset \{1, \dots, s\}$, and j points of intersection. The other component contains points x_1 and x_{i+1}, $i \notin I$, and also j points of intersection. The first component determines a two-pointed ramification cycle, where the divisor is $\sum_{i \in I} a_i x_{1+i} - \sum_{t=1}^j b_t *_t$ (by $*_t$ denote the points of intersection). The other component also determines a two-pointed ramification cycle, where the divisor is $-(\sum_{t=1}^s a_t)x_1 + \sum_{i \notin I} a_i x_{1+i} + \sum_{t=1}^j b_t *_t$.

Let the first component have genus g_1 and the second one have genus g_2. We have $g_1 + g_2 + j - 1 = g$. We must consider the preimage of this divisor under the mapping π and then to integrate against it the class $\psi(x_1)^{n-1} \cdot c_D(\mathcal{V})$. Note that when $c_D(\mathcal{V})$ doesn't vanish, it factorizes to $c_{D_1}(\mathcal{V}_1) \cdot c_{D_2}(\mathcal{V}_2)$. Then we have to integrate $c_{D_1}(\mathcal{V}_1)$ over the two-pointed ramification cycle determined by the first component, and $\psi(x_1)^{n-1} \cdot c_{D_2}(\mathcal{V}_2)$ over the two-pointed ramification cycle determined by the second component.

In the first case we see that dimensional conditions (as in Sec. 12.1) imply that the integral does not vanish iff $g = 1$ or $g = 0$. Thus we obtain the second multipliers in the formula. The integral corresponding to the second component obviously gives us the first multipliers in the formula.

Now we have only to explain the coefficients in the formula. The coefficient $1/r^g$ appearing in the definition of $S^n_{g,m}(\prod_{i=1}^s \eta_{m_i,a_i})$ behaves properly since π is a ramified covering with correspoding multiplicities. Then, $\prod_{r=1}^j b_r$ is the multiplicity of $\widehat{ll}$ along the corresponding divisor in $\widehat{H}$; the coefficient $\mathrm{aut}(b_1, \dots, b_j)$ appears since we have to mark the points of intersection of two components; $(\sum_{r=1}^s a_r)$ comes from the Lemma of E. Ionel; and $(|I| + j - 2)/(2g + s - 1)$ in the case of genus 0 (or $(|I| + j)/(2g + s - 1)$ in the case of genus 1) is the fraction of multiplicities of $\sigma \circ st$ over the divisor and over the initial subvariety $\pi(V_{g,m}(\prod_{i=1}^s \eta_{q_i,a_i}))$.

Thus we obtain the required formula. $\square$

12.4. **Calculation of $\widehat{S}_1(\prod_{t=1}^k \eta_{m_t,b_t})$.** In this subsection we sketch an algorithm to calculate $\widehat{S}_1(\prod_{t=1}^k \eta_{m_t,b_t})$. Actually, this algorithm is a very complicated and a very inefficient one. We are not satisfied with it in any sense, and sketched it here only for the completeness of the exposition.

12.4.1. *Notation.* First we would like to introduce another notation for $\widehat{S}_1(\prod_{t=1}^k \eta_{m_t,b_t})$.

Let us consider the moduli space

$$\overline{\mathcal{M}}_{1,k+l+n} \ni (C, x_1, \dots, x_k, y_1, \dots, y_l, z_1, \dots, z_n). \tag{74}$$

Let $a_1, \dots, a_k$ and $b_1, \dots, b_l$ be positive integers such that $\sum_{i=1}^k a_i = \sum_{i=1}^l b_i$. Consider the subvariety $V \subset \mathcal{M}_{1,k+l+n}$ consisting of curves such that $\sum_{i=1}^k a_i x_i - \sum_{i=1}^l b_l y_l$ is the divisor of a meromorphic function.

Let us fix $0 \le m_1, \dots, m_l, m'_1, \dots, m'_l, m''_1, \dots, m''_n \le r-1$. Consider the corresponding covering $\pi\colon \overline{\mathcal{M}}'_{1,k+l+n} \to \overline{\mathcal{M}}_{1,k+l+n}$. By $V(\,{}^{a_1}_{m_1} \cdots {}^{a_k}_{m_k} \,|\, {}^{b_1}_{m'_1} \cdots {}^{b_l}_{m'_l} | m''_1, \dots, m''_n)$ denote the closure of the preimage of V under the mapping π.

By $R(\,{}^{a_1}_{m_1} \cdots {}^{a_k}_{m_k} \,|\, {}^{b_1}_{m'_1} \cdots {}^{b_l}_{m'_l} | m''_1, \dots, m''_n)$ denote

$$\frac{1}{r} \int_{V(\,{}^{a_1}_{m_1} \cdots {}^{a_k}_{m_k} \,|\, {}^{b_1}_{m'_1} \cdots {}^{b_l}_{m'_l} | m''_1, \dots, m''_n)} c_D(\mathcal{V}). \tag{75}$$

Obviously, $R(\,{}^{a_1}_{m_1} \cdots {}^{a_k}_{m_k} \,|\, {}^{b_1}_{m'_1} \cdots {}^{b_l}_{m'_l} | \emptyset) = \widehat{S}_1(\prod_{i=1}^k \eta_{m_i,a_i} \prod_{i=1}^k \eta_{m'_i,-b_i})$.

12.4.2. *The simplest cases.* First, note that $R(\emptyset|\emptyset|m''_1, \dots, m''_n) = 0$ (this follows from the dimensional conditions). Then $R(\,{}^{1}_{m_1} \,|\, {}^{1}_{m'_1} | m''_1, \dots, m''_n) = 0$.

Let us show how to caculate $R(\,{}^{2}_{m_1} \,|\, {}^{2}_{m'_1} | m''_1, \dots, m''_n)$. If $n = 0$, then we can consider the intersection number $\langle \tau_{1,m_1} \tau_{0,m'_1} \rangle_1$. We can calculate it using the topological recursion relation. Trying to calculate it using our usual technique, we express it via $R(\,{}^{2}_{m_1} \,|\, {}^{2}_{m'_1} | \emptyset)$ with a non-zero coefficient. Then we can calculate $R(\,{}^{2}_{m_1} \,|\, {}^{2}_{m'_1} | \emptyset)$.

Suppose we managed to calculate all intersection numbers $R(\,{}^{2}_{m_1} \,|\, {}^{2}_{m'_1} | m''_1, \dots, m''_n)$ with $n < n_0$. Let us calculate it for $n = n_0$. Consider the intersection number $\langle \tau_{1,m_1} \tau_{0,m'_1} \prod_{i=1}^{n_0} \tau_{0,m''_i} \rangle_1$. We try to calculate it using our usual technique. Therefore, we express it as a linear combination of $R(\,{}^{2}_{m_1} \,|\, {}^{2}_{m'_1} | m''_1, \dots, m''_n)$ with $n < n_0$ and $R(\,{}^{2}_{m_1} \,|\, {}^{2}_{m'_1} | m''_1, \dots, m''_{n_0})$ with a non-zero coefficient. Then we can calculate $R(\,{}^{2}_{m_1} \,|\, {}^{2}_{m'_1} | m''_1, \dots, m''_{n_0})$.

The next step looks like follows. Suppose we managed to calculate all intersection numbers $R(\,{}^{c}_{m_1} \,|\, {}^{c}_{m'_1} | m''_1, \dots, m''_n)$ with $c < c_0$. Let us do the same for $c = c_0$. Consider the intersection number

$\langle\tau_{1,m_1}\tau_{0,m'_1}\prod_{i=1}^n \tau_{0,m''_i}\rangle_1$. Applying admissible covers of degree c_0 (totally ramified at x_1 and with multiplicity c_0-1 at y_1) we express it as a linear combination of already calculated intersection numbers and $R(\begin{smallmatrix} c_0 \\ m_1 \end{smallmatrix}|\begin{smallmatrix} c_0 \\ m'_1 \end{smallmatrix}|m''_1,\dots,m''_n)$ with a non-zero coefficient.

12.4.3. *The general inductive step.* Suppose we managed to calculate all intersection numbers $R = R(\begin{smallmatrix} a_1 \\ m_1 \end{smallmatrix}\dots\begin{smallmatrix} a_k \\ m_k \end{smallmatrix}|\begin{smallmatrix} b_1 \\ m'_1 \end{smallmatrix}\dots\begin{smallmatrix} b_l \\ m'_l \end{smallmatrix}|m''_1,\dots,m''_n)$ with $\sum_{i=1}^k a_i < s$. For $\sum_{i=1}^k a_i = s$ suppose we managed to calculate R with $k+l<k_0$. For $k+l=k_0$ suppose we managed to calculate R with, say, $b_l < (b_l)_0$.

We shall consider admissible covers f such that $f^{-1}(z_1) = a_1x_1 + \dots + a_kx_k$ and $f^{-1}(z_2) = b_1y_1 + \dots + ((b_l)_0 - 1)x_l + 1\cdot *$ (here $*$ is an additional point). Applying the Lemma of Ionel and our usual argument we see that $\langle\tau_{1,m_1}\prod_{i=1}^k \tau_{0,m_i}\prod_{i=1}^l \tau_{0,m'_i}\prod_{i=1}^n \tau_{0,m''_i}\rangle$ is expressed as a linear combination of already calculated intersection numbers and $R(\begin{smallmatrix} a_1 \\ m_1 \end{smallmatrix}\dots\begin{smallmatrix} a_k \\ m_k \end{smallmatrix}|\begin{smallmatrix} b_1 \\ m'_1 \end{smallmatrix}\dots\begin{smallmatrix} (b_l)_0 \\ m'_l \end{smallmatrix}|m''_1,\dots,m''_n)$ with a non-zero coefficient.

12.4.4. *Examples.* Consider the case $r=4$. It's easy to see that $\widehat{S}_1(\prod_{t=1}^k \eta_{m_t,b_t}) \neq 0$, iff $k=2$, $m_1=m_2=2$, and $b_1=-b_2$. Let us calculate $\widehat{S}_1(\eta_{2,2}\eta_{2,-2})$ and $\widehat{S}_1(\eta_{2,3}\eta_{2,-3})$.

Note that

$$\frac{1}{3\cdot 2^5} = \langle\tau_{1,2}\tau_{0,2}\rangle_1 = S^1_{1,2}(\eta_{2,1}\eta_{0,1}) = \frac{1}{3}\widehat{S}_1(\eta_{2,2}\eta_{2,-2}). \tag{76}$$

Therefore, $\widehat{S}_1(\eta_{2,2}\eta_{2,-2}) = 1/2^5$.

Then

$$\begin{aligned} \langle\tau_{1,2}\tau_{0,2}\rangle_1 &= S^1_{1,2}(\eta_{2,2}\eta_{0,1}) - \widehat{S}_1(\eta_{2,2}\eta_{2,-2}); \\ S^1_{1,2}(\eta_{2,2}\eta_{0,1}) &= \frac{1}{3}\widehat{S}_1(\eta_{2,3}\eta_{2,-3}) + \frac{4}{9}\widehat{S}_1(\eta_{2,2}\eta_{2,-2}). \end{aligned} \tag{77}$$

Therefore, $\widehat{S}_1(\eta_{2,3}\eta_{2,-3}) = 1/12$.

Appendix A. Calculation of $\langle\tau_{6,1}\rangle_3$ in the case $r=3$

In this section we calculate the intersection number $\langle\tau_{6,1}\rangle_3$ in the case $r=3$, using our algorithm (Theorem 4). Then we calculate the corresponding coefficient of the string solution of the Boussinesq hierarchy using the relation explained in Sec. 10.4. The results will appear to be the same.

A.1. **First step of the algorithm.** The first step is the following:

$$\langle\tau_{6,1}\rangle_3 = \langle\tau_{7,1}\tau_{0,0}\rangle_3 = \frac{1}{6}S^7_{3,1}(\eta^4_{0,1}) - \frac{1}{2}S^6_{3,1}(\eta^3_{0,1}) + \frac{1}{2}S^5_{3,1}(\eta^2_{0,1}). \tag{78}$$

In the next three subsections we calculate separately the summands of this expression. Let us recall once again that in the case of $r=3$,

$\langle\tau_{0,0}^2\tau_{0,1}\rangle_0 = 1$, $\langle\tau_{0,1}^4\rangle_0 = 1/3$, and all other $\langle\prod_{i=1}^{r-1}\tau_{0,i}^{p_i}\rangle_0$ are equal to zero.

For convenience, we shall denote $\langle\tau_{0,0}^2\tau_{0,1}\rangle_0$ by $\langle 1\rangle$ and $\langle\tau_{0,1}^4\rangle_0$ by $\langle 1/3\rangle$.

A.2. **Calculations in degree** 4**.** Using Eq. (69) we get

$$S^7_{3,1}(\eta^4_{0,1}) = \frac{1}{3}S^6_{3,1}(\eta^2_{0,1}\eta_{0,2})\langle 1\rangle. \tag{79}$$

Then

$$\begin{aligned}
S^6_{3,1}(\eta^2_{0,1}\eta_{0,2}) &= \frac{3}{16}S^5_{3,1}(\eta_{0,1}\eta_{0,3})\langle 1\rangle + \\
&\quad + \frac{1}{16}S^5_{3,1}(\eta^2_{0,2})\langle 1\rangle + \frac{1}{32}S^5_{2,1}(\eta^3_{0,1}\eta_{1,1})\langle 1\rangle; \\
S^5_{3,1}(\eta_{0,1}\eta_{0,3}) &= \frac{1}{7}S^4_{3,1}(\eta_{0,4})\langle 1\rangle + \\
&\quad \frac{1}{14}S^4_{2,1}(\eta^2_{0,1}\eta_{1,2})\langle 1\rangle + \frac{1}{14}S^4_{2,1}(\eta_{0,1}\eta_{0,2}\eta_{1,1})\langle 1\rangle; \\
S^5_{3,1}(\eta^2_{0,2}) &= \frac{1}{7}S^4_{3,1}(\eta_{0,4})\langle 1\rangle + \frac{1}{14}S^4_{2,1}(\eta_{0,1}\eta_{0,2}\eta_{1,1})\langle 1\rangle; \\
S^5_{2,1}(\eta^3_{0,1}\eta_{1,1}) &= \frac{3}{14}S^4_{2,1}(\eta^2_{0,1}\eta_{1,2})\langle 1\rangle + \frac{3}{14}S^4_{2,1}(\eta_{0,1}\eta_{0,2}\eta_{1,1})\langle 1\rangle.
\end{aligned} \tag{80}$$

Therefore,

$$S^7_{3,1}(\eta^4_{0,1}) = \frac{1}{84}S^4_{3,1}(\eta_{0,4}) + \frac{3}{448}S^4_{2,1}(\eta^2_{0,1}\eta_{1,2}) + \frac{11}{1344}S^4_{2,1}(\eta_{0,1}\eta_{0,2}\eta_{1,1}). \tag{81}$$

Then

$$\begin{aligned}
S^4_{3,1}(\eta_{0,4}) &= \frac{1}{8}S^3_{2,1}(\eta_{0,1}\eta_{1,3})\langle 1\rangle + \\
&\quad \frac{1}{8}S^3_{2,1}(\eta_{0,3}\eta_{1,1})\langle 1\rangle + \frac{1}{6}S^3_{2,1}(\eta_{0,2}\eta_{1,2})\langle 1\rangle; \\
S^4_{2,1}(\eta^2_{0,1}\eta_{1,2}) &= \frac{1}{4}S^3_{2,1}(\eta_{0,1}\eta_{1,3})\langle 1\rangle + \\
&\quad \frac{1}{12}S^3_{2,1}(\eta_{0,2}\eta_{1,2})\langle 1\rangle + \frac{1}{48}S^3_{1,1}(\eta^2_{0,1}\eta^2_{1,1})\langle 1\rangle; \\
S^4_{2,1}(\eta_{0,1}\eta_{0,2}\eta_{1,1}) &= \frac{1}{8}S^3_{2,1}(\eta_{0,1}\eta_{1,3})\langle 1\rangle + \frac{1}{12}S^3_{2,1}(\eta_{0,2}\eta_{1,2})\langle 1\rangle + \\
&\quad \frac{1}{8}S^3_{2,1}(\eta_{0,3}\eta_{1,1})\langle 1\rangle + \frac{1}{24}S^3_{1,1}(\eta^2_{0,1}\eta^2_{1,1})\langle 1\rangle.
\end{aligned} \tag{82}$$

Therefore,

$$\begin{aligned}
S^7_{3,1}(\eta^4_{0,1}) &= \frac{15}{2^9\cdot 7}S^3_{2,1}(\eta_{0,1}\eta_{1,3}) + \frac{13}{2^6\cdot 3^2\cdot 7}S^3_{2,1}(\eta_{0,2}\eta_{1,2}) + \\
&\quad \frac{9}{2^9\cdot 7}S^3_{2,1}(\eta_{0,3}\eta_{1,1}) + \frac{31}{2^{10}\cdot 3^2\cdot 7}S^3_{1,1}(\eta^2_{0,1}\eta^2_{1,1}).
\end{aligned} \tag{83}$$

Note that

$$(84)\quad S^3_{2,1}(\eta_{0,1}\eta_{1,3}) = \frac{1}{5}S^2_{2,1}(\eta_{1,4})\langle 1\rangle + \frac{1}{10}S^2_{1,1}(\eta_{0,1}\eta_{1,1}\eta_{1,2})\langle 1\rangle + \frac{1}{60}S^2_{0,1}(\eta^4_{0,0})\langle\frac{1}{3}\rangle;$$

$$S^3_{2,1}(\eta_{0,2}\eta_{1,2}) = \frac{1}{5}S^2_{2,1}(\eta_{1,4})\langle 1\rangle + \frac{1}{20}S^2_{1,1}(\eta_{0,1}\eta_{1,1}\eta_{1,2})\langle 1\rangle + \frac{1}{40}S^2_{1,1}(\eta_{0,2}\eta^2_{1,1})\langle 1\rangle;$$

$$S^3_{2,1}(\eta_{0,3}\eta_{1,1}) = \frac{1}{5}S^2_{2,1}(\eta_{1,4})\langle 1\rangle + \frac{1}{10}S^2_{1,1}(\eta_{0,1}\eta_{1,1}\eta_{1,2})\langle 1\rangle + \frac{1}{10}S^2_{1,1}(\eta_{0,2}\eta^2_{1,1})\langle 1\rangle;$$

$$S^3_{1,1}(\eta^2_{0,1}\eta^2_{1,1}) = \frac{2}{5}S^2_{1,1}(\eta_{0,1}\eta_{1,1}\eta_{1,2})\langle 1\rangle + \frac{1}{10}S^2_{1,1}(\eta_{0,2}\eta^2_{1,1})\langle 1\rangle + \frac{1}{20}S^2_{0,1}(\eta^4_{0,0})\langle\frac{1}{3}\rangle.$$

Then

$$(85)\quad S^2_{2,1}(\eta_{1,4}) = \frac{3}{16}S^1_{1,1}(\eta_{1,3}\eta_{1,1})\langle 1\rangle + \frac{1}{8}S^1_{1,1}(\eta^2_{1,2})\langle 1\rangle + \frac{1}{8}S^1_{0,1}(\eta^2_{0,1}\eta_{0,2})\langle\frac{1}{3}\rangle;$$

$$S^2_{1,1}(\eta_{0,1}\eta_{1,1}\eta_{1,2}) = \frac{3}{16}S^1_{1,1}(\eta_{1,3}\eta_{1,1})\langle 1\rangle + \frac{1}{8}S^1_{1,1}(\eta^2_{1,2})\langle 1\rangle + \frac{1}{4}S^1_{0,1}(\eta^2_{0,1}\eta_{0,2})\langle\frac{1}{3}\rangle + \frac{1}{32}S^1_{0,1}(\eta_{0,1}\eta^3_{1,1})\langle 1\rangle;$$

$$S^2_{1,1}(\eta_{0,2}\eta^2_{1,1}) = \frac{3}{8}S^1_{1,1}(\eta_{1,3}\eta_{1,1})\langle 1\rangle + \frac{1}{16}S^1_{0,1}(\eta_{0,1}\eta^3_{1,1})\langle 1\rangle + \frac{1}{16}S^1_{0,1}(\eta^2_{0,1}\eta_{0,2})\langle\frac{1}{3}\rangle.$$

Since

$$(86)\quad S^1_{1,1}(\eta_{1,3}\eta_{1,1}) = \frac{1}{6}S^0_{0,1}(\eta^2_{1,1}\eta_{1,2})\langle 1\rangle + \frac{1}{2}S^0_{0,1}(\eta_{0,1}\eta_{0,3})\langle\frac{1}{3}\rangle + \frac{1}{3}S^0_{0,1}(\eta^2_{0,2})\langle\frac{1}{3}\rangle = \frac{1}{3};$$

$$S^1_{1,1}(\eta^2_{1,2}) = \frac{1}{12}S^0_{0,1}(\eta^2_{1,1}\eta_{1,2})\langle 1\rangle + \frac{1}{2}S^0_{0,1}(\eta_{0,1}\eta_{0,3})\langle\frac{1}{3}\rangle + \frac{1}{3}S^0_{0,1}(\eta^2_{0,2})\langle\frac{1}{3}\rangle = \frac{11}{2^2\cdot 3^2},$$

it follows that

$$(87)\quad S^2_{2,1}(\eta_{1,4}) = \frac{41}{2^5\cdot 3^2};$$
$$S^2_{1,1}(\eta_{0,1}\eta_{1,1}\eta_{1,2}) = \frac{7}{2^2\cdot 3^2};$$
$$S^2_{1,1}(\eta_{0,2}\eta^2_{1,1}) = \frac{1}{2\cdot 3};$$

and, therefore,

$$(88)\quad S^3_{2,1}(\eta_{0,1}\eta_{1,3}) = \frac{77}{2^5\cdot 3^2\cdot 5}$$
$$S^3_{2,1}(\eta_{0,2}\eta_{1,2}) = \frac{61}{2^5\cdot 3^2\cdot 5}$$
$$S^3_{2,1}(\eta_{0,3}\eta_{1,1}) = \frac{31}{2^5\cdot 3\cdot 5}$$
$$S^3_{1,1}(\eta^2_{0,1}\eta^2_{1,1}) = \frac{1}{3^2}.$$

Thus, we have

$$(89)\qquad S^7_{3,1}(\eta^4_{0,1}) = \frac{209}{2^7\cdot 3^4\cdot 5\cdot 7}.$$

A.3. **Calculations in degree 3.** We have

$$(90)\quad S^6_{3,1}(\eta^3_{0,1}) = \frac{1}{4}S^5_{3,1}(\eta_{0,1}\eta_{0,2})\langle 1\rangle;$$
$$S^5_{3,1}(\eta_{0,1}\eta_{0,2}) = \frac{1}{7}S^4_{3,1}(\eta_{0,3})\langle 1\rangle + \frac{1}{21}S^4_{2,1}(\eta^2_{0,1}\eta_{1,1})\langle 1\rangle;$$
$$S^4_{3,1}(\eta_{0,3}) = \frac{1}{9}S^3_{2,1}(\eta_{0,1}\eta_{1,2})\langle 1\rangle + \frac{1}{9}S^3_{2,1}(\eta_{0,2}\eta_{1,1})\langle 1\rangle;$$
$$S^4_{2,1}(\eta^2_{0,1}\eta_{1,1}) = \frac{2}{9}S^3_{2,1}(\eta_{0,1}\eta_{1,2})\langle 1\rangle + \frac{1}{9}S^3_{2,1}(\eta_{0,2}\eta_{1,1})\langle 1\rangle;$$
$$S^3_{2,1}(\eta_{0,1}\eta_{1,2}) = \frac{1}{5}S^2_{2,1}(\eta_{1,3})\langle 1\rangle + \frac{1}{30}S^2_{1,1}(\eta_{0,1}\eta^2_{1,1})\langle 1\rangle;$$
$$S^3_{2,1}(\eta_{0,2}\eta_{1,1}) = \frac{1}{5}S^2_{2,1}(\eta_{1,3})\langle 1\rangle + \frac{1}{15}S^2_{1,1}(\eta_{0,1}\eta^2_{1,1})\langle 1\rangle;$$
$$S^2_{2,1}(\eta_{1,3}) = \frac{1}{6}S^1_{1,1}(\eta_{1,1}\eta_{1,2})\langle 1\rangle + \frac{1}{36}S^1_{0,1}(\eta^3_{0,1})\langle \frac{1}{3}\rangle;$$
$$S^2_{1,1}(\eta_{0,1}\eta^2_{1,1}) = \frac{1}{3}S^1_{1,1}(\eta_{1,1}\eta_{1,2})\langle 1\rangle + \frac{1}{12}S^1_{0,1}(\eta^3_{0,1})\langle \frac{1}{3}\rangle;$$
$$S^1_{1,1}(\eta_{1,1}\eta_{1,2}) = \frac{4}{9}S^0_{0,1}(\eta_{0,1}\eta_{0,2})\langle \frac{1}{3}\rangle + \frac{1}{18}S^0_{0,1}(\eta^3_{1,1})\langle 1\rangle = \frac{1}{2\cdot 3}.$$

Thus, we have

$$(91)\quad S^2_{1,1}(\eta_{0,1}\eta^2_{1,1}) = \frac{1}{2^2\cdot 3};$$
$$S^2_{2,1}(\eta_{1,3}) = \frac{1}{3^3};$$
$$S^3_{2,1}(\eta_{0,2}\eta_{1,1}) = \frac{7}{2^2\cdot 3^3\cdot 5};$$
$$S^3_{2,1}(\eta_{0,1}\eta_{1,2}) = \frac{11}{2^3\cdot 3^3\cdot 5};$$
$$S^4_{2,1}(\eta^2_{0,1}\eta_{1,1}) = \frac{1}{2\cdot 3^3\cdot 5};$$
$$S^4_{3,1}(\eta_{0,3}) = \frac{5}{2^3\cdot 3^5};$$
$$S^5_{3,1}(\eta_{0,1}\eta_{0,2}) = \frac{37}{2^3\cdot 3^5\cdot 5\cdot 7}$$

and therefore,

$$(92)\qquad S^6_{3,1}(\eta^3_{0,1}) = \frac{37}{2^5\cdot 3^5\cdot 5\cdot 7}.$$

A.4. **Calculations in degree** 2. We have

$$(93)\quad S^5_{3,1}(\eta^2_{0,1}) = \frac{1}{7}S^4_{3,1}(\eta_{0,2})\langle 1\rangle = \frac{1}{7}\cdot\frac{1}{12}S^3_{2,1}(\eta_{0,1}\eta_{1,1})\langle 1\rangle =$$
$$\frac{1}{2^2\cdot 3\cdot 7}\cdot\frac{1}{5}S^2_{2,1}(\eta_{1,2})\langle 1\rangle = \frac{1}{2^2\cdot 3\cdot 5\cdot 7}\cdot\frac{1}{16}S^1_{1,1}(\eta^2_{1,1})\langle 1\rangle =$$
$$\frac{1}{2^6\cdot 3\cdot 5\cdot 7}\cdot\frac{1}{6}S^0_{0,1}(\eta^2_{0,1})\langle\frac{1}{3}\rangle = \frac{1}{2^7\cdot 3^3\cdot 5\cdot 7}.$$

A.5. **Summary.** We have

$$(94)\quad \langle\tau_{6,1}\rangle_3 = \frac{209}{2^8\cdot 3^5\cdot 5\cdot 7} - \frac{37}{2^6\cdot 3^5\cdot 5\cdot 7} + \frac{1}{2^8\cdot 3^3\cdot 5\cdot 7} = \frac{1}{2^7\cdot 3^5}.$$

Let us calculate the corresponding coefficient of the string solution of the Boussinesq hierarchy. It follows from (66) that

$$(95)\quad 26\langle\tau_{8,1}\tau^2_{0,0}\rangle_3 = 2\langle\tau_{7,1}\tau_{0,0}\rangle_3\langle\tau_{0,1}\tau^2_{0,0}\rangle_0 + \frac{2}{3}\langle\tau_{7,1}\tau_{0,1}\tau^3_{0,0}\rangle_2;$$
$$20\langle\tau_{6,1}\tau_{0,1}\tau^2_{0,0}\rangle_2 = 4\langle\tau_{5,1}\tau_{0,1}\tau_{0,0}\rangle_2\langle\tau_{0,1}\tau^2_{0,0}\rangle_0 + \frac{2}{3}\langle\tau_{5,1}\tau^2_{0,1}\tau^3_{0,0}\rangle_1;$$
$$14\langle\tau_{4,1}\tau^2_{0,1}\tau^2_{0,0}\rangle_1 = 6\langle\tau_{3,1}\tau^2_{0,1}\tau_{0,0}\rangle_1\langle\tau_{0,1}\tau^2_{0,0}\rangle_0 + \frac{2}{3}\langle\tau_{4,1}\tau^3_{0,1}\tau^3_{0,0}\rangle_0.$$

Therefore, $\langle\tau_{6,1}\rangle_3 = 1/(2^7\cdot 3^5)$. Thus we have checked Witten's conjecture for $\langle\tau_{6,1}\rangle_3$ in the case of Boussinesq hierarchy.

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