ISSN: 1401-5617

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Research Reports in Mathematics
Number 9, 2001

Department of Mathematics
Stockholm University

Electronic versions of this document are available at
http://www.matematik.su.se/reports/2001/9

Date of publication: May 8, 2001
2000 Mathematics Subject Classification: Primary 35J60, Secondary 35Q55, 58E05.
Keywords: Semilinear Schrödinger equation, linking, asymptotically periodic.

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An asymptotically periodic Schrödinger equation with indefinite linear part

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Abstract

We consider the Schrödinger equation $-\Delta u + V(x)u = f(x,u)$, where V is periodic and f asymptotically periodic in the x-variables, 0 is in a spectral gap of $-\Delta + V$ and f is either asymptotically linear or superlinear as $|u| \to \infty$. We show that this equation has a solution $u \in H^1(\mathbf{R}^N)$, $u \neq 0$.

1 Introduction and statement of the main results

In this paper we shall be concerned with the existence of nontrivial solutions of the semilinear Schrödinger equation

$$(1) \qquad -\Delta u + V(x)u = f(x,u), \qquad u \in H^1(\mathbf{R}^N),$$

where $V(x)$ is continuous and periodic in x_j for $j = 1, \dots, N$, 0 is in a gap of the spectrum of the operator $-\Delta + V$ and f is asymptoticaly periodic in x_j and either asymptotically linear or superlinear (but subcritical) as $|u| \to \infty$. The above problem with nonlinearities periodic in the space variables has been considered by numerous authors, see e.g. [3, 4, 7, 8, 9, 13, 16, 17] and the references there. For asymptotically periodic nonlinearities there are considerably fewer results; here we mention [1] and [11]. In these two papers solutions of (1) were found as critical points of a functional which has the mountain pass geometry. In [1] two problems were considered. In the first one 0 was in a spectral gap of $-\Delta + V$ and $f(x,u) = W(x)|u|^{p-2}u$, where $W > 0$ and $2 < p < 2^* := 2N/(N-2)$. In the second problem f was more general but $V(x)$ was a positive constant (so the spectrum $\sigma(-\Delta + V) \subset (0, +\infty)$). Under these conditions the dual functional associated with the first and the natural (Lagrangian) functional associated with the second problem have mountain pass geometry. This fact was used in [1] in order to obtain the ground states and

2000 *Mathematics Subject Classification: 35J60, 35Q55, 58E05.*
Key words and phrases: Semilinear Schrödinger equation, linking, asymptotically periodic.
The first author was supported in part by NSFC and Academy of Finland.
The second author was supported in part by the Swedish Natural Science Research Council.

subsequently the multibump solutions of (1). In [11] $V(x)$ was a positive constant and $f(x,u)$ was asymptotically linear. Using a version of the mountain pass theorem and comparing with appropriate solutions of a periodic problem associated with (1) a nontrivial solution was found.

In this paper we assume that 0 is in a spectral gap of $-\Delta+V$. The associated functional has then a linking structure and it seems that under our hypotheses the problem cannot be reformulated in terms of a functional having the mountain pass geometry.

Recall that the spectrum of $-\Delta+V$ in $L^2(\mathbf{R}^N)$ is bounded below (but not above) and is the union of disjoint closed intervals (see e.g. p. 161 and Theorem 4.5.9 in [10]). Portions of the real axis contained between such intervals are called spectral gaps.

Let (μ_{-1},μ_1) be the spectral gap containing 0 and denote $\mu_0 := \min\{-\mu_{-1},\mu_1\}$. Suppose $V\in C(\mathbf{R}^N,\mathbf{R})$, $f\in C(\mathbf{R}^N\times\mathbf{R},\mathbf{R})$ and let $F(x,u) := \int_0^u f(x,s)\,ds$. We introduce the following hypotheses:

($\mathbf{H_1}$) V is 1-periodic in x_j for $j=1,\dots,N$ and 0 is in a spectral gap of $(-\Delta+V)$;

($\mathbf{H_2}$) $f(x,u)/u\to 0$ uniformly in x as $u\to 0$;

($\mathbf{H_3}$) $f(x,u)=V_\infty(x)u+f_\infty(x,u)$, where V_∞ is 1-periodic in x_j for $j=1,\dots,N$, $f_\infty(x,u)/u\to 0$ uniformly in x as $|u|\to\infty$ and $V_\infty(x)\ge\mu$ for all x and some $\mu>\mu_1$;

($\mathbf{H_4}$) $\frac{1}{2}uf(x,u)\ge F(x,u)\ge 0$ for all x,u;

($\mathbf{H_5}$) There exists $\delta\in(0,\mu_0)$ such that if $f(x,u)/u\ge\mu_0-\delta$, then $\frac{1}{2}uf(x,u)-F(x,u)\ge\delta$.

Note that it follows from the continuity of f that V_∞ and f_∞ are necessarily continuous. We shall also need some conditions on the asymptotic behaviour of f as $|x|\to\infty$. Let $g\in C(\mathbf{R}^N\times\mathbf{R},\mathbf{R})$, $g_u\in C(\mathbf{R}^N\times\mathbf{R},\mathbf{R})$, $G(x,u) := \int_0^u g(x,s)\,ds$ and suppose

($\mathbf{H_6}$) g is 1-periodic in x_j for $j=1,\dots,N$ and $0<ug(x,u)<u^2g_u(x,u)\le V_\infty(x)u^2$ whenever $u\ne 0$;

($\mathbf{H_7}$) $F(x,u)>G(x,u)$ whenever $u\ne 0$ and $|f(x,u)-g(x,u)|\le a(x)|u|$, where $a\in L^\infty(\mathbf{R}^N)$ and $a(x)\to 0$ as $|x|\to\infty$.

This last condition can be made slightly weaker, see Remark 3.9. We note that if f,g respectively satisfy (H_2)-(H_5) and (H_6), (H_7), then g must necessarily satisfy (H_2)-(H_5). Indeed, using periodicity, (H_2) and (H_3) are easy to verify for g. Furthermore, it follows from (H_6) that $G\ge 0$ and $u\mapsto g(x,u)/u$ is strictly increasing for $u>0$ and strictly decreasing for $u<0$. Hence also

$$\frac{1}{2}ug(x,u)-G(x,u)=\int_0^u\left(\frac{g(x,u)}{u}-\frac{g(x,s)}{s}\right)s\,ds \tag{2}$$

strictly increases with $|u|$ and (H_4), (H_5) follow. Obviously, (H_2) implies that (1) has the trivial solution $u=0$.

Theorem 1.1 *Suppose that the hypotheses (H_1)-(H_7) are satisfied. Then (1) has a nontrivial solution.*

The function f in the above theorem was assumed to be asymptotically linear. If f is superlinear as $|u| \to \infty$, the same result remains true under appropriate hypotheses. More precisely, we replace (H_3)-(H_7) with

($\mathbf{H_3'}$) $|f(x,u)| \le c_0(1+|u|^{p-1})$ for some $c_0 > 0$ and $p \in (2, 2^*)$ if $N \ge 3$, $p > 2$ if $N = 1$ or 2;

($\mathbf{H_4'}$) There is $\gamma > 2$ such that $0 < \gamma F(x,u) \le uf(x,u)$ and $0 < \gamma G(x,u) \le ug(x,u)$ whenever $u \ne 0$;

($\mathbf{H_5'}$) g is 1-periodic in x_j for $j = 1, \dots, N$ and $0 < ug(x,u) < u^2 g_u(x,u)$ whenever $u \ne 0$;

($\mathbf{H_6'}$) $F(x,u) > G(x,u)$ whenever $u \ne 0$ and $|f(x,u) - g(x,u)| \le a(x)(|u| + |u|^{p-1})$, where p is as in (H_3'), $a \in L^\infty(\mathbf{R}^N)$ and $a(x) \to 0$ as $|x| \to \infty$.

Theorem 1.2 *Suppose that the hypotheses (H_1), (H_2) and (H_3')-(H_6') are satisfied. Then (1) has a nontrivial solution.*

We remark that instead of (H_3') and (H_4') it is possible to assume superlinearity conditions in the spirit of [8]; however, the proof would be technically more difficult without bringing any new ideas (it would involve an adaptation of the arguments in [8] and Sections 3, 4 of the present paper).

A rather general example of a function f satisfying the assumptions of Theorem 1.1 is $f(x,u) = \alpha(u)u + a(x)\beta(u)u$, where $a > 0$, $a(x) \to 0$ as $|x| \to \infty$, $\alpha(0) = \beta(0) = 0$, $\alpha(u) + a(x)\beta(u) \to \mu > \mu_1$ as $|u| \to \infty$, $\alpha'(u)u > 0$ and $\beta'(u)u > 0$ as $u \ne 0$, and $(\alpha(u)u)' \le \mu$. Here we have $V_\infty(x) = \mu$ and $g(x,u) = \alpha(u)u$; note that (H_4) and (H_5) are satisfied because $u \mapsto f(x,u)/u$ is strictly increasing as $u > 0$ and strictly decreasing as $u < 0$ (cf. (2)). In Theorem 1.2 one can take e.g. $f(x,u) = |u|^{p-2}u + a(x)h(u)$, where $2 < p < 2^*$, $a > 0$, $a(x) \to 0$ as $|x| \to \infty$, and h satisfies (H_2), (H_3'), (H_4'). Note in particular that $H(x,u) := \int_0^u h(x,s)\,ds$ need not be convex in u, hence the methods of [1] are not applicable.

Our main topological tool will be a linking theorem which we discuss in the next section. In Section 3 we prove the main theorems except for a few lemmas whose proofs we defer to Section 4. The proofs of both results are very similar; therefore we mainly concentrate on the more difficult Theorem 1.1 and only make a few comments on Theorem 1.2.

Notation $B(a,r)$ is an open ball of radius r and center a, $\|\cdot\|_p$ is the usual norm in $L^p(\mathbf{R}^N)$, $\rightharpoonup$ denotes weak convergence and $\Phi_a := \{u \in E : \Phi(u) \ge a\}$, $\Phi^b := \{u \in E : \Phi(u) \le b\}$, $\Phi_a^b := \Phi_a \cap \Phi^b$.

Acknowledgement The second author would like to thank A.A. Pankov for sending him the unpublished notes [12].

2 Linking

Let E be a real Hilbert space and let $\Phi \in C^1(E, \mathbf{R})$. Recall that $(u_n) \subset E$ is called a *Palais-Smale sequence* at the level c ($(PS)_c$-sequence for short) if $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ as $n \to \infty$. If $\Phi(u_n) \to c$ and $(1 + \|u_n\|)\Phi'(u_n) \to 0$, then (u_n) will be called a *Cerami sequence* at the level c ($(C)_c$-sequence for short), cf. [5] and [2].

In this section we extend a linking theorem of [9] (see also [3, 18]).

First we recall some terminology from [9]. Let E^- be a closed subspace of a separable Hilbert space E and let $E^+ := (E^-)^\perp$. On E we define a new norm

$$\|u\|_\tau := \max\left\{\|u^+\|, \sum_{k=1}^{\infty} \frac{1}{2^k}|\langle u^-, e_k\rangle|\right\},$$

where (e_k) is a total orthonormal sequence in E^- and $u = u^+ + u^-$, $u^\pm \in E^\pm$. The topology induced by $\|\cdot\|_\tau$ will be called the *τ-topology.* A homotopy $h = I - g : A \times [0,1] \to E$ ($A \subset E$) is said to be *admissible* if:
(i) h is τ-continuous, i.e. $h(u_n, s_n) \xrightarrow{\tau} h(u, s)$ whenever $u_n \xrightarrow{\tau} u$ and $s_n \to s$,
(ii) g is τ-locally finite-dimensional, i.e. for each $(u, s) \in A \times [0,1]$ there is a neighbourhood U of (u, s) in the product topology of (E, τ) and $[0,1]$ such that $g(U \cap (A \times [0,1]))$ is contained in a finite-dimensional subspace of E.
Admissible maps are defined in a similar way. Recall further that admissible maps and homotopies are continuous in the strong topology, and on bounded sets (E, τ) coincides with the product topology of E^-_{weak} and E^+_{strong} (this last topology was used in [3] also on unbounded sets).

For $z_0 \in E^+ \setminus \{0\}$ and $R > r > 0$, let

$$M := \{u = u^- + tz_0 : \|u\| \le R,\ t \ge 0\}, \quad N := \{u \in E^+ : \|u\| = r\}, \tag{3}$$

and

$$\Gamma := \{h \in C(M \times [0,1], E) : h \text{ is admissible, } h(u,0) = u \text{ and } \Phi(h(u,s)) \le \max\{\Phi(u), -1\} \text{ for all } s \in [0,1]\}.$$

Theorem 2.1 *Let $E = E^+ \oplus E^-$ be a separable Hilbert space with E^- orthogonal to E^+. Suppose*
(i) $\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \psi(u)$, where $\psi \in C^1(E, \mathbf{R})$ is bounded below, weakly sequentially lower semicontinuous and ψ' is weakly sequentially continuous;
(ii) There exist $z_0 \in E^+ \setminus \{0\}$, $\beta > 0$ and $R > r > 0$ such that $\Phi|_N \ge \beta$ and $\Phi|_{\partial M} \le 0$.
Then there exists a $(C)_c$-sequence for Φ, where

$$c := \inf_{h \in \Gamma} \sup_{u \in M} \Phi(h(u,1)).$$

Moreover, $c \ge \beta$.

This extends Theorem 3.4 in [9], where the conclusion was that there is a $(PS)_c$-sequence for Φ and some $c \ge \beta$.

Proof Since the argument is similar to that in Section 3 of [9] and Chapter 6 of [18] (see also [16]), we omit some details.

First we show that $c \ge \beta$. Let $H : M \times [0,1] \to \mathbf{R}z_0 \oplus E^-$ be given by

$$H(u,s) = (\|h(u,s)^+\| - r)\frac{z_0}{\|z_0\|} + h(u,s)^-,$$

where $h \in \Gamma$ and $h(u,s) = h(u,s)^+ + h(u,s)^- \in E^+ \oplus E^-$. Then H is an admissible homotopy and $H(u,s) = 0$ if and only if $h(u,s) \in N$. Since $\Phi|_{\partial M} \le 0$ and $\Phi(u) \ge \Phi(h(u,s)) \ge \beta$ whenever $h(u,s) \in N$, $0 \notin H(\partial M \times [0,1])$. Therefore the degree of [9] is well-defined and

$$\deg(H(\cdot,1), M, 0) = \deg(H(\cdot,0), M, 0) = 1.$$

Consequently, $H(\bar{u},1)=0$ for some $\bar{u}$; hence $h(\bar{u},1)\in N$ and $\Phi(h(\bar{u},1))\geq\beta$. So $c\geq\beta$.

Suppose now that there is no $(C)_c$-sequence for Φ. Then there exists $\varepsilon>0$ such that $(1+\|u\|)\|\Phi'(u)\|\geq\varepsilon$ whenever $|\Phi(u)-c|\leq\varepsilon$. We shall need a special vector field V which we obtain by slightly modifying the construction in [9, pp. 454-455] or [18, Lemmas 6.7-6.8]. For $u\in\Phi_{c-\varepsilon}^{c+\varepsilon}$, let

$$w(u):=\frac{2\Phi'(u)}{\|\Phi'(u)\|^2}.$$

Recall that the set Φ_a is τ-closed for each a and if $(u_n)\subset\Phi_a$, then $u_n\xrightarrow{\tau}u$ if and only if $u_n^+\to u^+$ and $u_n^-\rightharpoonup u^-$ [9, Remark 2.1]. Using this and weak sequential continuity of Φ' we see that the function $v\mapsto\langle\Phi'(v),w(u)\rangle$ is τ-continuous on Φ_a for any a; in particular, it is τ-continuous on Φ_{-1}. Moreover, the set $\{v\in E:\|v\|\leq b\}$ is τ-closed for each b (indeed, if (v_n) is a bounded sequence, then $v_n\rightharpoonup v$ whenever $v_n\xrightarrow{\tau}v$). Hence there exists a τ-open neighbourhood U_u of u in E such that

$$\langle\Phi'(v),w(u)\rangle>1\quad\text{ for all } v\in\Phi_{-1}\cap U_u \tag{4}$$

and

$$\|u\|<2\|v\|\quad\text{ for all } v\in U_u. \tag{5}$$

Let $U_0:=\Phi^{-1}(-\infty,c-\varepsilon)$; then U_0 is τ-open (since $\Phi_{c-\varepsilon}$ is τ-closed). The family $(U_u)_{u\in\Phi_{c-\varepsilon}^{c+\varepsilon}}\cup U_0$ is a τ-open covering of $\Phi^{c+\varepsilon}$ and there exists a τ-locally finite τ-open refinement $(N_j)_{j\in J}$ with a corresponding τ-Lipschitz continuous partition of unity $(\lambda_j)_{j\in J}$. If $N_j\subset U_0$, we set $w_j=0$, otherwise we choose u_j such that $N_j\subset U_{u_j}$ and set $w_j:=w(u_j)$. Let $N:=\bigcup_{j\in J}N_j$,

$$V(u):=\sum_{j\in J}\lambda_j(u)w_j,\quad u\in N\supset\Phi^{c+\varepsilon},$$

and consider the initial value problem

$$\frac{d\eta}{ds}=-V(\eta),\quad \eta(u,0)=u\in\Phi^{c+\varepsilon}.$$

By [9, 18], V is τ-locally and locally Lipschitz continuous. Hence for each u as above there exists a unique solution $\eta(u,\cdot)$. By (5), if $w_j\neq0$ and $u\in N_j$, then

$$\|w_j\|=\|w(u_j)\|=\frac{2}{\|\Phi'(u_j)\|}\leq\frac{2(1+\|u_j\|)}{\varepsilon}\leq\frac{2(1+2\|u\|)}{\varepsilon}.$$

So $\|V(u)\|\leq 2(1+2\|u\|)/\varepsilon$ and $\eta(u,s)$ exists for all $s\geq0$. Since $w_j=0$ if $N_j\subset U_0$, it follows from (4) that $\langle\Phi'(u),V(u)\rangle\geq0$ whenever $u\in\Phi_{-1}^{c+\varepsilon}$ and $\langle\Phi'(u),V(u)\rangle>1$ for $u\in\Phi_{c-\varepsilon}^{c+\varepsilon}$. Hence $\Phi(\eta(u,s))\leq\max\{\Phi(u),-1\}$ and if $\Phi(\eta(u,s))\geq c-\varepsilon$, then

$$\Phi(\eta(u,s))-\Phi(u)=\int_0^s\frac{d}{dt}\Phi(\eta(u,t))\,dt=-\int_0^s\langle\Phi'(\eta(u,t)),V(\eta(u,t))\rangle\,dt<-s.$$

Therefore $\Phi(\eta(u,2\varepsilon))\leq c-\varepsilon$. Moreover, according to Proposition 2.2 in [9] or Lemma 6.8 in [18], $\eta:\Phi^{c+\varepsilon}\times[0,2\varepsilon]\to E$ is an admissible homotopy.

Choose $h\in\Gamma$ such that $h(M\times\{1\})\subset\Phi^{c+\varepsilon}$ and let

$$\tilde{h}(u,s):=\begin{cases} h(u,2s), & 0\leq s\leq\frac12\\ \eta(h(u,1),4\varepsilon s-2\varepsilon), & \frac12\leq s\leq1.\end{cases}$$

Then $\tilde{h}\in\Gamma$ and $\tilde{h}(M\times\{1\})\subset\Phi^{c-\varepsilon}$, contradicting the definition of c. □

3 Proof of Theorems 1.1 and 1.2

In this section we assume that the hypotheses (H_1)-(H_7) are satisfied except in the proof of Theorem 1.2 where we assume (H_1), (H_2) and (H_3')-(H_6'). As is well-known (see e.g. [9, 14, 18]), the functional

$$\Phi(u) := \frac{1}{2}\int_{\mathbf{R}^N}(|\nabla u|^2 + V(x)u^2)\,dx - \int_{\mathbf{R}^N} F(x,u)\,dx$$

is of class C^1 in the Sobolev space $E := H^1(\mathbf{R}^N)$ and critical points of Φ are solutions of (1). Since 0 is in a spectral gap of $-\Delta + V$, $E = E^+ \oplus E^-$, where E^+ and E^- correspond to the positive and the negative part of the spectrum of $-\Delta + V$ in E. Moreover, $E^\pm$ are infinite-dimensional. Let $u = u^+ + u^- \in E^+ \oplus E^-$. Since the quadratic part of Φ is positive definite on E^+ and negative definite on E^- [14, Sections 8 and 9], we may introduce an equivalent inner product $\langle .,. \rangle$ and an equivalent norm $\|\,.\,\| = \langle .,. \rangle^{1/2}$ on E such that

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \psi(u),$$

where

$$\psi(u) := \int_{\mathbf{R}^N} F(x,u)\,dx.$$

Obviously, $\psi \geq 0$ and it is easy to see that ψ is weakly lower semicontinuous and ψ' is weakly continuous [9, 18]. Hence (i) of Theorem 2.1 is satisfied. A well-known argument using (H_2) shows that if r is small enough, then $\Phi|_N \geq \beta$ for some $\beta > 0$.

Lemma 3.1 *If $z_0 \in E^+ \setminus \{0\}$ is such that*

$$\tau^2\|z_0\|^2 - \|v^-\|^2 - \int_{\mathbf{R}^N} V_\infty(x)(\tau z_0 + v^-)^2\,dx < 0 \tag{6}$$

for all $\tau > 0$, $v^- \in E^-$, and M is as in (3), then $\Phi|_{\partial M} \leq 0$ provided R is large enough.

This result is essentially contained in [16] but for the reader's convenience we reprove it in the next section.

Let

$$\tilde{\Phi}(u) := \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \tilde{\psi}(u),$$

where

$$\tilde{\psi}(u) := \int_{\mathbf{R}^N} G(x,u)\,dx.$$

Critical points of $\tilde{\Phi}$ are solutions of (1) with $f(x,u)$ replaced by $g(x,u)$. In [16] it was shown, in fact under weaker hypotheses, that $\tilde{\Phi}$ has a nontrivial critical point. We shall also see from (7) below that $\tilde{\Phi}(u_0) > 0$ if $\tilde{\Phi}'(u_0) = 0$ and $u_0 \neq 0$. In order to prove that also Φ has a nontrivial critical point, we shall make a comparison between $(C)_c$-sequences of Φ and $\tilde{\Phi}$, and for this purpose we need some additional information on $\tilde{\Phi}$. The next lemma will be crucial for what follows.

Lemma 3.2 *If $u_0 \neq 0$ is a critical point of $\tilde{\Phi}$, then $\tilde{\Phi}''(u_0)$ is negative definite on $\mathbf{R}u_0 \oplus E^-$. More generally, if $\tilde{E} = \mathbf{R}u_0 \oplus \tilde{E}^-$, where $\tilde{E}^- \subset E^-$ and $u_0 \neq 0$ is a critical point of $\tilde{\Phi}|_{\tilde{E}}$, then $\tilde{\Phi}''(u_0)$ is negative definite on $\tilde{E}$.*

The proof will be given in the next section.

Corollary 3.3 *If $u_0 \neq 0$ is a critical point of $\tilde{\Phi}$, then*

$$\tau^2\|u_0^+\|^2 - \|v^-\|^2 - \int_{\mathbf{R}^N} V_\infty(x)(\tau u_0^+ + v^-)^2\,dx < 0$$

for all $\tau > 0$ and $v^- \in E^-$. So in particular, if we choose $z_0 = u_0^+$ in the definition of M, then the conclusion of Lemma 3.1 holds for Φ.

Proof Since $\mathbf{R}u_0 \oplus E^- \equiv \mathbf{R}u_0^+ \oplus E^-$, it follows from Lemma 3.2 that $\langle \tilde{\Phi}''(u_0)(\tau u_0^+ + v^-), \tau u_0^+ + v^-\rangle < 0$. Therefore, using (H_6), we obtain

$$\tau^2\|u_0^+\|^2 - \|v^-\|^2 < \int_{\mathbf{R}^N} g_u(x,u_0)(\tau u_0^+ + v^-)^2\,dx \le \int_{\mathbf{R}^N} V_\infty(x)(\tau u_0^+ + v^-)^2\,dx.$$

Note that $\tilde{\Phi}(u_0) > 0$ implies $u_0^+ \neq 0$, so we can choose $z_0 = u_0^+$ in the definition of M. □

Lemma 3.1 and Corollary 3.3 imply that Φ satisfies (ii) of Theorem 2.1. Hence there exists a $(C)_c$-sequence (u_n) for Φ. By modifying an argument in [16] we shall prove the following result in the next section:

Lemma 3.4 *If (u_n) is a $(C)_c$-sequence for Φ (or $\tilde{\Phi}$), then (u_n) is bounded.*

Remark 3.5 The fact that $(C)_c$-sequences are bounded for asymptotically linear equations (1) in $\mathbf{R}^N$ (in the mountain pass case) has been observed in [11] and [15].

Remark 3.6 As we have already mentioned, it was shown in [16] that $\tilde{\Phi}$ has a nontrivial critical point. This result also follows by our arguments here. Indeed, since $V_\infty(x) \ge \mu > \mu_1$, we can find $z_0 \in E^+ \setminus \{0\}$ such that the quadratic form corresponding to $-\Delta + V(x) - \mu$ is negative definite on $\mathbf{R}z_0 \oplus E^-$. Therefore

$$\tau^2\|z_0^+\|^2 - \|v^-\|^2 - \int_{\mathbf{R}^N} V_\infty(x)(\tau z_0^+ + v^-)^2\,dx \le \tau^2\|z_0^+\|^2 - \|v^-\|^2 - \mu\|\tau z_0 + v^-\|_2^2 < 0.$$

Hence we can apply Lemma 3.1 to see that $\tilde{\Phi}$ satisfies the assumptions of Theorem 2.1. Consequently, there exists a $(C)_c$-sequence (where $c > 0$) for $\tilde{\Phi}$, and this sequence is bounded according to Lemma 3.4. Now the usual concentration-compactness argument (see e.g. [9, 16, 18]) implies that $\tilde{\Phi}'(u_0) = 0$ for some $u_0 \neq 0$. We note that this is the only place where the assumption $V_\infty(x) \ge \mu > \mu_1$ is used explicitly.

Proposition 3.7 *There exists u_0 such that $\tilde{\Phi}(u_0) = \min\{\tilde{\Phi}(u) : u \neq 0,\ \tilde{\Phi}'(u) = 0\}$. Moreover, $\tilde{\Phi}(u_0) > 0$.*

Proof Let (u_n) be a sequence such that $\tilde{\Phi}'(u_n) = 0$ and $\tilde{\Phi}(u_n) \to \inf\{\tilde{\Phi}(u) : u \neq 0,\ \tilde{\Phi}'(u) = 0\}$. Then, using (2), we obtain

$$\tilde{\Phi}(u_n) = \tilde{\Phi}(u_n) - \frac{1}{2}\langle \tilde{\Phi}'(u_n), u_n\rangle = \int_{\mathbf{R}^N}\left(\frac{1}{2}u_n g(x,u_n) - G(x,u_n)\right)dx > 0. \tag{7}$$

Hence $(\tilde{\Phi}(u_n))$ is bounded and therefore (u_n) is a $(C)_c$-sequence which is bounded according to Lemma 3.4. Moreover, since $\tilde{\Phi}'(u) = u^+ - u^- + o(\|u\|)$ as $u \to 0$, (u_n) is bounded away from 0. It follows therefore from Lemma 2.25 in [7] or Lemma 1.7 in [9] that

$$\liminf_{n\to\infty} \int_{B(y_n,r)} u_n^2 \, dx \geq \eta$$

for some $r, \eta > 0$ and $(y_n) \subset \mathbf{R}^N$ (in [7, 9] the function g was superlinear but the argument remains the same for asymptotically linear g). Choosing a larger r we may assume $(y_n) \subset \mathbf{Z}^N$. Since V and g are 1-periodic in the x-variables, $\tilde{\Phi}'(\tilde{u}_n) = 0$ and $\tilde{\Phi}(\tilde{u}_n) = \tilde{\Phi}(u_n)$, where $\tilde{u}_n(x) := u_n(x + y_n)$. Since $\int_{B(y_n,r)} u_n^2 \, dx = \int_{B(0,r)} \tilde{u}_n^2 \, dx$, $\tilde{u}_n \rightharpoonup u_0 \neq 0$ after passing to a subsequence. Then $\tilde{\Phi}'(u_0) = 0$ and we see as in (7) that $\tilde{\Phi}(u_0) > 0$. □

In what follows we assume that u_0 has been chosen as in the above proposition and that $z_0 = u_0^+$ in the definition (3) of M. Then $\tilde{\Phi}$ does not admit any $(C)_c$-sequence with $0 < c < \tilde{\Phi}(u_0)$. Indeed, if (u_n) is such a sequence, then it is bounded according to Lemma 3.4 and it follows as in [7] or [9, Proposition 4.2] that $\tilde{\Phi}'(v) = 0$ and $\tilde{\Phi}(v) \leq c < \tilde{\Phi}(u_0)$ for some $v \neq 0$. If

$$\tilde{c} := \inf_{h\in\Gamma} \sup_{u\in M} \tilde{\Phi}(h(u,1)),$$

then there exists a $(C)_{\tilde{c}}$-sequence for $\tilde{\Phi}$ and hence $\tilde{c} \geq \tilde{\Phi}(u_0)$.

Proposition 3.8 $\sup_M \tilde{\Phi}(u) = \tilde{\Phi}(u_0)$. *So in particular,* $\tilde{c} = \tilde{\Phi}(u_0)$.

Proof We have $M \subset \mathbf{R}u_0^+ \oplus E^- \equiv \mathbf{R}u_0 \oplus E^-$. Let $u = tu_0^+ + u^- \in M$. Then $\tilde{\Phi}(u) = \frac{1}{2}t^2\|u_0^+\|^2 - \frac{1}{2}\|u^-\|^2 - \tilde{\psi}(u)$. Since the mapping $u \mapsto \frac{1}{2}\|u^-\|^2 + \tilde{\psi}(u)$ is weakly lower semicontinuous, $\tilde{\Phi}$ is weakly upper semicontinuous on M. Moreover, $\tilde{\Phi}$ is bounded on M and $\tilde{\Phi}|_{\partial M} \leq 0$; therefore $\tilde{\Phi}$ takes its maximum on M at some v_0 and $\tilde{\Phi}(v_0) > 0$. We shall show that u_0 is the only critical point of $\tilde{\Phi}|_{M\setminus\{0\}}$. It will then follow that $v_0 = u_0$ and $\sup_M \tilde{\Phi} = \tilde{\Phi}(u_0)$.

Suppose that $\tilde{\Phi}$ has another critical point on $M \setminus \{0\}$ and let $\tilde{E} = \mathbf{R}u_0^+ \oplus \tilde{E}^-$ be a finite-dimensional space such that $\tilde{E}^- \subset E^-$ and $\tilde{E}$ contains u_0 and this second critical point. Let $\tilde{M} := (M \setminus B(0,\varepsilon)) \cap \tilde{E}$ and $\tilde{J} := \tilde{\Phi}|_{\tilde{E}}$. Since $ug(x,u) - 2G(x,u) > 0$ for $u \neq 0$ (cf. (2)), we have

$$\langle \tilde{J}'(u), u\rangle = \langle \tilde{\Phi}'(u), u\rangle < 2\tilde{\Phi}(u) \leq 0 \quad \text{whenever } u \in \partial M \setminus \{0\}. \tag{8}$$

Moreover, $\tilde{J}'(u) = tu_0^+ - u^- + o(\|u\|)$ as $u \to 0$; hence $\tilde{J}'(u) \neq 0$ whenever $\|u\| = \varepsilon$ and it follows that $\tilde{J}$ has no critical points on $\partial\tilde{M}$ provided $\varepsilon > 0$ is small enough. Consider now the mapping $-\tilde{J}' : \tilde{M} \to \tilde{E}$. We want to show that the Brouwer degree $\deg(-\tilde{J}', \tilde{M}, 0) = 1$ and each $u \in \tilde{M}$ with $-\tilde{J}'(u) = 0$ is isolated with local degree 1. It will then follow from the additivity property of the degree that no second critical point can exist in $\tilde{M}$.

Consider the homotopy $H : \tilde{M} \times [0,1] \to \tilde{E}$ given by

$$H(u,s) := (1-s)(-tu_0^+ + u^- + \tilde{\psi}(u)) + s((t-1)u_0^+ + u^-) = -(1-s)\tilde{J}'(u) + s(u - u_0^+).$$

If $\|u\| = R$, then $\langle \tilde{J}'(u), u\rangle < 0$ (cf. (8)) and since $\|u_0^+\| < R$, $t\|u_0^+\| \leq R$, we obtain

$$\langle H(u,s), u\rangle = (1-s)\langle -\tilde{J}'(u), u\rangle + s(\|u\|^2 - t\|u_0^+\|^2) > 0. \tag{9}$$

Similarly, (9) is satisfied for $u \in \tilde{E}^- \setminus \{0\}$. If $\|u\| = \varepsilon$ and $t \geq 0$, it is easy to see that $H(u,s) = (1-s)(-tu_0^+) + s(t-1)u_0^+ + u^- + o(\|u\|) \neq 0$ whenever ε is small enough. Hence H is an admissible homotopy. Since $H(u,0) = -\tilde{J}'(u)$, $H(u,1) = u - u_0^+$ and $u_0^+ \in \tilde{M}$,

$$\deg(-\tilde{J}', \tilde{M}, 0) = \deg(I, \tilde{M}, u_0^+) = 1$$

(I denotes the identity mapping).

By Lemma 3.2, if $v_0 \in \tilde{M}$ is a critical point of $\tilde{J}'$, then $-\tilde{J}''(v_0)$ is positive definite. Hence v_0 is an isolated zero of $-\tilde{J}'$ and the local degree at v_0, $\deg(-\tilde{J}', v_0, 0) = 1$. Therefore $v_0 = u_0$. □

Proof of Theorem 1.1 Since $F(x,u) > G(x,u)$ for $u \neq 0$ (cf. (H_7)) and since $\sup_M \Phi(u) = \Phi(v_0)$ for some v_0,

$$0 < c = \inf_{h\in\Gamma} \sup_{u\in M} \Phi(h(u)) \leq \sup_{u\in M} \Phi(u) = \Phi(v_0) < \tilde{\Phi}(v_0) \leq \sup_{u\in M} \tilde{\Phi}(u) = \tilde{\Phi}(u_0) = \tilde{c}.$$

We have asserted earlier in this section that there exists a $(C)_c$-sequence (u_n) for Φ, and by Lemma 3.4, this sequence is bounded. Passing to a subsequence we may assume $u_n \rightharpoonup u$. If $u \neq 0$, then $\Phi'(u) = 0$ by the weak continuity of Φ' and the theorem is proved. So assume $u = 0$. Since $a(x)$ in (H_7) tends to 0 as $|x| \to \infty$, it is easy to see that the mapping $\tilde{\psi}' - \psi'$ is compact. Hence $\tilde{\psi}'(u_n) - \psi'(u_n) \to 0$ and it follows that

$$\tilde{\Phi}(u_n) = \Phi(u_n) + \psi(u_n) - \tilde{\psi}(u_n) \to c \quad \text{and} \quad \tilde{\Phi}'(u_n) = \Phi'(u_n) + \psi'(u_n) - \tilde{\psi}'(u_n) \to 0.$$

So (u_n) is a $(C)_c$ (in fact a $(PS)_c$)-sequence for $\tilde{\Phi}$ and therefore $\tilde{\Phi}$ has a critical point $\tilde{u} \neq 0$ with $0 < \tilde{\Phi}(\tilde{u}) \leq c < \tilde{c}$ (see the argument following the proof of Proposition 3.7). But according to Propositions 3.7 and 3.8, $\tilde{c}$ is the least positive critical level for $\tilde{\Phi}$, so in particular $\tilde{\Phi}(\tilde{u}) \geq \tilde{c}$, a contradiction. □

Remark 3.9 The second part of hypothesis (H_7) can be slightly weakened. We have used it in order to assure compactness of $\tilde{\psi}' - \psi'$. Weaker sufficient conditions for this may be found e.g. in [6, Theorem A.3.1].

Proof of Theorem 1.2 The proof is very much the same as that of Theorem 1.1 except that a few details become simpler. In particular, the conclusion of Lemma 3.1 remains valid for any $z_0 \in E^+ \setminus \{0\}$ (see [9, Lemma 1.4], [17, Lemma 4.2] or [18, Lemma 6.14]) and Lemma 3.4 holds true in a stronger form: any $(PS)_c$-sequence is bounded [9, 17, 18]. It is therefore not necessary (though possible) to use Theorem 2.1 - it suffices to invoke the linking theorem of [9]. □

4 Proofs of lemmas

Proof of Lemma 3.1 Let $u = tz_0 + u^-$, where $u^- \in E^-$. Since $F \geq 0$ and hence $\psi \geq 0$, $\Phi(u^-) = -\frac{1}{2}\|u^-\|^2 - \psi(u) \leq 0$. So it suffices to prove that $\Phi(u) \leq 0$ whenever $u \in \partial M$, $\|u\| = R$ and R is large enough. Arguing by contradiction, we find a sequence (u_n) such that $u_n = t_n z_0 + u_n^-$, $\|u_n\| \to \infty$ and

$$\frac{\Phi(u_n)}{\|u_n\|^2} = \frac{1}{2}\tau_n^2\|z_0\|^2 - \frac{1}{2}\|v_n^-\|^2 - \int_{\mathbf{R}^N} \frac{F(x,u_n)}{\|u_n\|^2}\,dx \geq 0, \tag{10}$$

where $\tau_n := t_n/\|u_n\|$ and $v_n^- := u_n^-/\|u_n\|$. Clearly, $\tau_n\|z_0\| \geq \|v_n^-\|$ and since $\tau_n^2\|z_0\|^2 + \|v_n^-\|^2 = 1$, $\tau_n \to \tau > 0$ and $v_n^- \rightharpoonup v^-$ in E after passing to a subsequence. Let $v := \tau z_0 + v^-$ and $v_n := \tau_n z_0 + v_n^-$. By (6), there exists a bounded domain $\Omega \subset \mathbf{R}^N$ such that

$$\tau^2\|z_0\|^2 - \|v^-\|^2 - \int_\Omega V_\infty(x)(\tau z_0 + v^-)^2\,dx < 0. \tag{11}$$

As $F(x,u) = \frac{1}{2}V_\infty(x)u^2 + F_\infty(x,u)$ according to (H_3) and $F \geq 0$, it follows from (10) that

$$\begin{aligned} 0 &\leq \frac{1}{2}\tau_n^2\|z_0\|^2 - \frac{1}{2}\|v_n^-\|^2 - \int_\Omega \frac{F(x,u_n)}{\|u_n\|^2}\,dx \\ &= \frac{1}{2}\tau_n^2\|z_0\|^2 - \frac{1}{2}\|v_n^-\|^2 - \frac{1}{2}\int_\Omega V_\infty(x)v_n^2\,dx - \int_\Omega \frac{F_\infty(x,u_n)}{\|u_n\|^2}\,dx. \end{aligned}$$

Clearly, $|F_\infty(x,u)| \leq c_0u^2$ for some c_0 and $F_\infty(x,u)/u^2 \to 0$ as $|u| \to \infty$. Since $v_n \rightharpoonup v$ in E, then $v_n \to v$ in $L^2(\Omega)$ and it is easy to see from the Lebesgue dominated convergence theorem that

$$\int_\Omega \frac{F_\infty(x,u_n)}{\|u_n\|^2}\,dx = \int_\Omega \frac{F_\infty(x,u_n)}{u_n^2}v_n^2\,dx \to 0.$$

Hence

$$0 \leq \frac{1}{2}\tau^2\|z_0\|^2 - \frac{1}{2}\|v^-\|^2 - \frac{1}{2}\int_\Omega V_\infty(x)v^2\,dx,$$

a contradiction to (11). □

Proof of Lemma 3.2 It suffices to prove the second statement. As in the preceding section, we set $\tilde{J} = \tilde{\Phi}|_{\tilde{E}}$. Suppose $u_0 \neq 0$ is a critical point of $\tilde{J}$. For $u \in \tilde{E}$ we write $u = tu_0 + v$, where $v \in \tilde{E}^-$ (note that u_0 and v need not be orthogonal). The computation below is inspired by (and very similar to) an argument in [12]. Since $\tilde{J}'(u_0) = 0$,

$$\begin{aligned} &\langle \tilde{J}''(u_0)(tu_0+v), tu_0+v\rangle = \langle \tilde{J}''(u_0)(tu_0+v), tu_0+v\rangle - \langle t\tilde{J}'(u_0), tu_0+2v\rangle \\ &= t^2\|u_0^+\|^2 - t^2\|u_0^-\|^2 - 2t\langle u_0^-, v\rangle - \|v\|^2 - \int_{\mathbf{R}^N} g_u(x,u_0)(tu_0+v)^2\,dx \\ &\quad - t^2\|u_0^+\|^2 + t^2\|u_0^-\|^2 + 2t\langle u_0^-, v\rangle + t\int_{\mathbf{R}^N} g(x,u_0)(tu_0+2v)\,dx \\ &= -\|v\|^2 - \int_{\mathbf{R}^N}(g_u(x,u_0)u_0^2 - g(x,u_0)u_0)t^2\,dx - \int_{\mathbf{R}^N} 2(g_u(x,u_0)u_0 - g(x,u_0))tv\,dx \\ &\quad - \int_{\mathbf{R}^N} g_u(x,u_0)v^2\,dx. \end{aligned}$$

For a fixed x and $u_0 = u_0(x)$ the sum of the integrands above is a quadratic form in t and v. Using (H_6) we obtain

$$(g_u(x,u_0)u_0 - g(x,u_0))^2 - (g_u(x,u_0)u_0^2 - g(x,u_0)u_0)g_u(x,u_0) = g(x,u_0)^2 - u_0g(x,u_0)g_u(x,u_0) < 0$$

for each x such that $u_0(x) \neq 0$. It is now easy to see that $\tilde{J}''(u_0)$ is negative definite on $\tilde{E}$. □

Let (v_n) be a bounded sequence in E. Then (v_n) is either

(i) *Vanishing:* For each $r > 0$, $\lim\limits_{n\to\infty} \sup\limits_{y\in\mathbf{R}^N} \int_{B(y,r)} v_n^2\,dx = 0$, or

(ii) *Non-vanishing:* There exist $r, \eta > 0$ and a sequence $(y_n) \subset \mathbf{Z}^N$ such that

$$\limsup_{n\to\infty} \int_{B(y_n,r)} v_n^2\, dx \geq \eta.$$

Note that if (i) does not hold, then (ii) is satisfied with $(y_n) \subset \mathbf{R}^N$ but as we already have observed, we may assume $y_n \in \mathbf{Z}^N$ by choosing a larger r if necessary.

Proof of Lemma 3.4 We shall modify an argument in [16] which in turn goes back to [8]. Assume $\|u_n\| \to \infty$ and let $v_n := u_n/\|u_n\|$. We shall obtain a contradiction by showing that (v_n) is neither vanishing nor nonvanishing.

Suppose first that (v_n) is nonvanishing and let (y_n) be as in (ii) above. Since $\Phi'(u_n)/\|u_n\| \to 0$,

$$\frac{1}{\|u_n\|}\langle \Phi'(u_n), \varphi_n\rangle = \langle v_n^+ - v_n^-, \varphi_n\rangle - \int_{\mathbf{R}^N} V_\infty(x) v_n \varphi_n\, dx - \int_{\mathbf{R}^N} \frac{f_\infty(x,u_n)}{u_n} v_n\varphi_n\, dx \to 0, \tag{12}$$

where $\varphi_n(x) := \varphi(x-y_n)$ and $\varphi \in C_0^\infty(\mathbf{R}^N)$. Let $\tilde{v}_n(x) := v_n(x+y_n)$ and $\tilde{u}_n(x) := u_n(x+y_n)$. Then $\tilde{v}_n \rightharpoonup \tilde{v}$ in E and $\tilde{v}_n \to \tilde{v}$ in $L^2_{loc}(\mathbf{R}^N)$ after taking a subsequence. Since $|f_\infty(x,u)| \leq c_0|u|$ for some c_0 and $f_\infty(x,u)/u \to 0$ as $|u| \to \infty$, it is easy to see by the Lebesgue dominated convergence theorem that

$$\int_{\mathbf{R}^N} \frac{f_\infty(x,u_n)}{u_n} v_n\varphi_n\, dx = \int_{\mathbf{R}^N} \frac{f_\infty(x+y_n,\tilde{u}_n)}{\tilde{u}_n} \tilde{v}_n\varphi\, dx \to 0.$$

Moreover,

$$\limsup_{n\to\infty} \int_{B(0,r)} \tilde{v}_n^2\, dx = \limsup_{n\to\infty} \int_{B(y_n,r)} v_n^2\, dx \geq \eta.$$

Hence $\tilde{v} \neq 0$ and it follows from (12) that

$$\langle \tilde{v}^+ - \tilde{v}^-, \varphi\rangle - \int_{\mathbf{R}^N} V_\infty(x)\tilde{v}\varphi\, dx = 0 \quad \text{for all } \varphi \in C_0^\infty(\mathbf{R}^N),$$

i.e.,

$$-\Delta\tilde{v} + (V(x) - V_\infty(x))\tilde{v} = 0 \quad \text{in } \mathbf{R}^N.$$

However, since $V - V_\infty$ is periodic, the spectrum of $-\Delta + V - V_\infty$ is absolutely continuous and therefore has no eigenvalues. This rules out the possibility of (v_n) being nonvanishing.

Suppose now (v_n) is vanishing. As in (12), we obtain

$$\frac{1}{\|u_n\|}\langle \Phi'(u_n), v_n^+\rangle = \|v_n^+\|^2 - \int_{\mathbf{R}^N} \frac{f(x,u_n)}{u_n} v_n v_n^+\, dx \to 0$$

and

$$\frac{1}{\|u_n\|}\langle \Phi'(u_n), v_n^-\rangle = -\|v_n^-\|^2 - \int_{\mathbf{R}^N} \frac{f(x,u_n)}{u_n} v_n v_n^-\, dx \to 0.$$

Since $\|v_n\| = 1$, it follows that

$$\int_{\mathbf{R}^N} \frac{f(x,u_n)}{u_n} v_n(v_n^+ - v_n^-)\, dx \to 1.$$

Recall from Section 1 that (μ_{-1}, μ_1) is the spectral gap of $-\Delta + V$ containing 0 and $\mu_0 := \min\{-\mu_{-1}, \mu_1\}$. Hence

$$\|v^+\|^2 \equiv \int_{\mathbf{R}^N} (|\nabla v^+|^2 + V(x)(v^+)^2)\, dx \geq \mu_1 \|v^+\|_2^2\,;$$

similarly, $\|v^-\|^2 \geq -\mu_{-1}\|v^-\|_2^2$, and therefore

$$\|v\|^2 \geq \mu_0 \|v\|_2^2 \tag{13}$$

(cf. [14, Section 8]). Let $\Omega_n := \{x \in \mathbf{R}^N : f(x,u_n)/u_n \leq \mu_0 - \delta\}$. Then, using Hölder's inequality, (13) and the fact that v_n^+, v_n^- are orthogonal in $L^2(\mathbf{R}^N)$, we obtain

$$\int_{\Omega_n} \frac{f(x,u_n)}{u_n} v_n(v_n^+ - v_n^-)\,dx \leq (\mu_0 - \delta)\int_{\Omega_n} |v_n|\,|v_n^+ - v_n^-|\,dx \leq (\mu_0-\delta)\|v_n\|_2^2 \leq \frac{\mu_0 - \delta}{\mu_0} < 1.$$

Hence

$$\liminf_{n\to\infty} \int_{\mathbf{R}^N\setminus\Omega_n} \frac{f(x,u_n)}{u_n} v_n(v_n^+ - v_n^-)\,dx > 0$$

and since $|f(x,u)| \leq c_0|u|$, it follows from Hölder's inequality that

$$\int_{\mathbf{R}^N\setminus\Omega_n} \frac{f(x,u_n)}{u_n} v_n(v_n^+ - v_n^-)\,dx \leq c_0 \operatorname{meas}(\mathbf{R}^N \setminus \Omega_n)^{(p-2)/p}\|v_n\|_p^{2/p},$$

where $2 < p < 2^*$. Since (v_n) is vanishing, $v_n \to 0$ in $L^p(\mathbf{R}^N)$ according to P.L. Lions' lemma (see e.g. Lemma 1.21 in [18]). Thus $\operatorname{meas}(\mathbf{R}^N \setminus \Omega_n) \to \infty$ and it follows from (H_4), (H_5) that

$$\int_{\mathbf{R}^N}\left(\frac{1}{2}u_n f(x,u_n) - F(x,u_n)\right)dx \geq \int_{\mathbf{R}^N\setminus\Omega_n}\left(\frac{1}{2}u_n f(x,u_n) - F(x,u_n)\right)dx \geq \int_{\mathbf{R}^N\setminus\Omega_n}\delta\,dx \to \infty.$$

However, since (u_n) is a $(C)_c$-sequence, $\langle \Phi'(u_n), u_n\rangle \to 0$ and therefore

$$\int_{\mathbf{R}^N}\left(\frac{1}{2}u_n f(x,u_n) - F(x,u_n)\right)dx = \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle \to c,$$

a contradiction. □

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