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Normal maximal ideal in one-dimensional local rings

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Abstract

We give a criterion for the maximal ideal M of the numerical semigroup S to be normal and for 3-generated numerical semigroups we characterize those that have normal maximal ideal. We also give a criterion for the maximal ideal of Noetherian, local one-dimensional, analytically irreducible domains $(R, \mathbf{m})$ such that R and $\overline{R}$, the integral closure of R in its quotient field, have the same residue field, to be normal and we answer the question whether $\mathbf{m}$ normal implies M normal where M is the maximal ideal of $S = v(R)$. We show, in a particular case, how the property for the associated graded ring of R with respect to $\mathbf{m}$ to be Cohen-Macaulay is strictly related to the normality of $\mathbf{m}$.

MSC: 20Mxx; 13H10

1 Introduction

Let R be a local, Noetherian, one-dimensional domain; assume also that R is analytically irreducible or, equivalently, that the integral closure $\overline{R}$ of R in its quotient field is a discrete valuation ring (DVR) and a finitely generated R-module. Let K denote the quotient field of R and $\overline{R}$, let v be the discrete valuation on $K^* = K \setminus \{0\}$ associated to $\overline{R}$ and, for each subset B of K, let $v(B)$ denote the image under v of the set of nonzero elements of B.
We call $v(R) = \{v(r) \mid r \in R\}$ the *value semigroup associated to* R. It is a subsemigroup of $\mathbb{N}$ and it is well known that there is a close connection between R and $v(R)$, when R and $\overline{R}$ have the same residue field (cf. [8],[10]).

An early paper on the connection between semigroups and one-dimensional local domains is [1]. This connection has since been studied in e.g. [7] and there is an extensive study on numerical semigroups and their applications to integral domains in [2].

The key fact that allows to connect a ring to its value semigroup is that it is possible to compute the lenght $l_R(I/J)$ (where $I \supseteq J$ are fractional ideals of R) in terms of the semigroup (cf.Theorem 2.1).

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1.1 Description of the content

We now make a closer description of the content of this paper. In Section 2 we introduce the concepts of numerical semigroup S and of ideal in a numerical semigroup. Then we introduce $v(R)$, the associated value semigroup to a ring R and we recall some known results about the connection between the ring and its associated value semigroup. In Section 3, we give a criterion for the maximal ideal M of S to be normal and we use it to give a criterion for a generic ideal of S to be normal. We also answer the question whether M_n, the maximal ideal in $S_n = \langle g_1, \dots, g_n \rangle$, could be normal when M_i is not normal for some $i < n$ (where M_i denote the maximal ideal of $S_i = \langle g_1, \dots, g_i \rangle$). In Section 4 we consider the case of 3-generated semigroups and for this case we characterize the numerical semigroups that have normal maximal ideal. In Section 5 we give a criterion for the maximal ideal $\mathfrak{m}$ of R to be normal and we answer the question whether $\mathfrak{m}$ normal implies M normal where M is the maximal ideal of $S = v(R)$. In Section 6 we prove, in a particular case, that $\mathfrak{m}$ is normal if and only if $G(\mathfrak{m})$ is C-M, where $G(\mathfrak{m})$ is the associated graded ring of R with respect to $\mathfrak{m}$. In Section 7 we find a bound for $l_R(\overline{\mathfrak{m}^i}/\mathfrak{m}^i)$ for $i \gg 0$.

2 Preliminaries

Let $\mathbb{N}$ denote the natural numbers (including 0). A subsemigroup S of $(\mathbb{N}, +)$ with $0 \in S$ is called a *numerical semigroup.* Each semigroup S has a natural partial ordering $\leq_S$ where for two elements s and t in S, $s \leq_S t$ if there is an $u \in S$ such that $t = s+u$. The set $\{g_i\}$ of the minimal elements in $S \setminus \{0\}$ in this ordering is called *the minimal set of generators* for S. In fact all elements of S are linear combinations with non-negative integers coefficients of minimal elements. Note that the set $\{g_i\}$ of minimal generators is finite since for any $s \in S$, $s \neq 0$, we have $g_i \neq g_j \pmod{s}$. The same argument shows that the number of minimal generators is at most $\min\{s \in S \mid s \neq 0\}$. A numerical semigroup generated by $g_1 < g_2 < \cdots < g_n$ is called an n-generated numerical semigroup and we denote it by $\langle g_1, g_2, \dots, g_n \rangle$. Since the semigroup $\langle g_1, g_2, \dots, g_n \rangle$ is isomorphic to $\langle dg_1, dg_2, \dots, dg_n \rangle$ for any $d \in \mathbb{N} \setminus \{0\}$, we assume, in the sequel, that $\gcd(g_1, g_2, \dots, g_n) = 1$. This is easily seen to be equivalent to $|\mathbb{N} \setminus S| < \infty$. Since $|\mathbb{N} \setminus S| < \infty$, there exist in S elements s such that the set $\{s, s+1, \longrightarrow\} \subseteq S$ (where the symbol "$\longrightarrow$" means that all subsequent natural numbers belong to the set). We call the smallest of such elements s the *conductor* of S, and we denote it by $c = c(S)$.

A *relative ideal* of a semigroup S is a nonempty subset H of $\mathbb{Z}$ such that $H + S \subseteq H$ and $H + s \subseteq S$ for some $s \in S$. A relative ideal of S which is contained in S is simply called an *ideal* of S. Clearly S is an ideal of S, but $\{0\}$ is not an ideal of S. By a *proper ideal*, we mean an ideal distinct from S, i.e., an ideal not containing 0. It is straightforward to see that if H and N are relative ideals of S, then $H + N$ and $kH (= H + \cdots + H$, k summands for $k \geq 1)$ are also relative ideals of S. Sometimes it is useful to consider kH for $k = 0$; in this case we let $0H = S$. The ideal $M = \{s \in S \mid s \neq 0\}$ is called the *maximal ideal of* S. For every ideal H, we consider $\overline{H} = \{s \in S \mid s \geq \bar{h}\}$ where $\bar{h} = \min\{h \in H\}$ and we call $\overline{H}$, the *integral closure of* H *in* S. In general $iH \subseteq \overline{iH}$. We say that H is *normal* if $iH = \overline{iH}$ for every $i \geq 1$. Clearly $M = \overline{M}$.

Throughout the rest of the paper we will assume that $(R, \mathfrak{m})$ is a local, Noetherian, one-dimensional domain. We assume also that R is analytically irreducible or, equivalently, that the integral closure $\overline{R}$ of R in its quotient field is a DVR and a finite generated R-module and that R and $\overline{R}$ have the same residue field. For every such ring, $v(R)$ is a numerical semigroup and throughout the rest of the paper, we will denote it by S.

We also will denote by $g_1 < g_2 < \cdots < g_n$ and by $M = v(\mathfrak{m})$ respectively the generators and the maximal ideal of S.
If I is an ideal of R, we denote by $\overline{I}$ the integral closure $\{x \in R \mid x^n + r_1x^{n-1} + \cdots + r_n = 0, \text{for some } r_i \in I^i\}$. We say that I is normal if $I^j = \overline{I^j}$ for every $j \geq 1$. When I is a fractional ideal of R, then $v(I) = \{v(i) \mid i \in I\}$ is a relative ideal of the semigroup S.

With our choice of R, we have the following theorems.

Theorem 2.1. *If $I \subseteq J$ are fractional ideals of R, then $l_R(I/J) = |v(I) - v(J)|$*

Proof. Cf. [10, Proposition 1]

Corollary 2.2. *Let $I \subseteq J$ be fractional ideals of R, then $v(I) = v(J)$ if and only if $I = J$.*

For every $a \in S$, we denote in the sequel the ideal $\{r \in R \mid v(r) \geq a\}$ of R by $R(a)$, the ideal $\{r \in \overline{R} \mid v(r) \geq a\}$ of $\overline{R}$ by $\overline{R}(a)$ and the semigroup ideal $\{s \in S \mid s \geq a\}$ of S by $S(a)$.

Theorem 2.3. *Let I be a fractional ideal of R, $\overline{I}$ be the integral closure of I in R and $a = \min\{v(i) \mid i \in I\}$. Then $\overline{I} = R(a)$.*

Proof. Let $x \in I$ such that $v(x) = a$. It is known (cf. [9, Remark (a), p. 659]) that $z \in R$ is integral over the ideal xR if and only if $z/x \in \overline{R}$, i.e., if and only if $v(z) \geq v(x) = a$. Thus the integral closure of xR is $R(a)$ and $R(a)$ is integrally closed. By $xR \subseteq I \subseteq R(a)$, the claim follows.

3 Normal maximal ideal in a numerical semigroup

Let S be a numerical semigroup generated by $g_1 < g_2 < \cdots < g_n$. The following statements are easy to see:

$$\text{if } x \in S \text{ and } x > ig_n, \text{ then } x \in (i+1)M \tag{3.1}$$

$$\text{for every numerical semigroup } S \text{, we have } M = \overline{M} \tag{3.2}$$

$$\text{for any ideal } H \text{ of } S \text{, if } [a, a + g_1 - 1] \subseteq H \text{, then } [a, \infty) \subseteq H \tag{3.3}$$

From now on we denote by $\gamma = \min\{m \in \mathbb{N} \mid mg_1 \geq c(S)\}$.

Proposition 3.1. *If M is normal, then $g_2 = g_1 + 1$.*

Proof. Suppose that $g_2 = g_1 + x$ with $x > 1$. Since the second smallest element in γM is $\min\{(\gamma+1)g_1, (\gamma-1)g_1 + g_2 = \gamma g_1 + x\} > \gamma g_1 + 1$, we have that $\gamma g_1 + 1 \in \overline{\gamma M} \setminus \gamma M$, hence M is not normal.

From now on we denote by α the integer such that $(\alpha - 1)g_n < \alpha g_1$ and $\alpha g_n \geq (\alpha + 1)g_1$.

Proposition 3.2. *For every $i \leq \alpha$, $iM = \overline{iM}$.*

Proof. By definition of α, we have $ig_n < (i+1)g_1$ for every $i < \alpha$. We show that $ig_n < (i+1)g_1$ implies $(i+1)M = \overline{(i+1)M}$. We have only to show that $\overline{(i+1)M} \subseteq (i+1)M$. Let $x \in \overline{(i+1)M}$. Then $x \geq (i+1)g_1$. Since $ig_n < (i+1)g_1$, we have $x > ig_n$, hence, by (3.1), $x \in (i+1)M$. So $iM = \overline{iM}$ for every i such that $2 \leq i \leq \alpha$ and (3.2) completes the proof of the proposition.

Now we give a sufficient condition for M to be normal. We know that if $g_2 > g_1 + 1$, then M is not normal.

Proposition 3.3. *Let $g_2 = g_1 + 1$. If $\overline{iM} = iM$ for every $i \leq \gamma$, then M is normal.*

Proof. By hypothesis, $\overline{iM} = iM$ for every $i \leq \gamma$, in particular $\overline{\gamma M} = \gamma M$. So, by $\gamma g_1 \geq c$, we have $\gamma M = \overline{\gamma M} = \{\gamma g_1, \longrightarrow\}$. Thus for every $j > \gamma$, we have $jM = \{jg_1, \longrightarrow\} = \overline{jM}$. Hence M is normal.

For every $a \geq 1$, we denote by $C_a = \overline{(a+1)M} \setminus (\overline{aM} + M)$.
This is an important set for us and we use it many times in the paper.

Remark 3.4. Note that $C_1 = \overline{2M} \setminus (\overline{M} + M) = \overline{2M} \setminus 2M = \{g_i \mid g_i > 2g_1\}$ where the second equality holds by (3.2).

Lemma 3.5. *Let $x = g_{s_1} + \cdots + g_{s_t} \in C_a$ with $g_2 = g_1 + 1$. Then $s_j > 2$ for every $j = 1, \ldots, t$.*

Proof. Suppose $s_1 = 1$ or 2. Since $x \in C_a$, then $x > (a+1)g_1$, hence $x - g_{s_1} \geq ag_1$, that is $x - g_{s_1} \in \overline{aM}$ and hence $x \in \overline{aM} + M$. A contradiction to $x \in C_a$.

Lemma 3.6. *If $x = g_{s_1} + \cdots + g_{s_t} \in C_a$, then $t \leq a$.*

Proof. Suppose $t > a$. Then $x - g_{s_i} \geq ag_1$ for every i and as in the proof of Lemma 3.5, we have a contradiction to $x \in C_a$.

Remark 3.7. Note that in general $C_a \subseteq \overline{(a+1)M} \setminus (a+1)M$, since $(a+1)M \subseteq \overline{aM} + M$.

Proposition 3.8. *If $C_i = \emptyset$ for every $i \leq a-1$, then $\overline{(i+1)M} \setminus (i+1)M = \emptyset$ for every $i \leq a-1$ and $C_a = \overline{(a+1)M} \setminus (a+1)M$.*

Proof. We prove the first part of the proposition by induction on i. If $i = 1$, then $\overline{2M} \setminus 2M = \emptyset$ follows by $C_1 = \emptyset$ and by Remark 3.4.
Suppose now that for every $j < i \leq a-1$, $\overline{(j+1)M} \setminus (j+1)M = \emptyset$ (i.e. $\overline{(j+1)M} = (j+1)M$ for every $j < i \leq a-1$) and we prove that $\overline{(i+1)M} \setminus (i+1)M = \emptyset$. In fact $\emptyset = C_i = \overline{(i+1)M} \setminus (\overline{iM} + M) = \overline{(i+1)M} \setminus (iM + M) = \overline{(i+1)M} \setminus (i+1)M$, where the third equality holds by the inductive hypothesis.
Now we prove the second part of the proposition. Since $\overline{(i+1)M} \setminus (i+1)M = \emptyset$ for every $i \leq a-1$, in particular $\overline{aM} = aM$. Hence $C_a = \overline{(a+1)M} \setminus (\overline{aM}+M) = \overline{(a+1)M} \setminus (aM + M) = \overline{(a+1)M} \setminus (a+1)M$.

Example 3.9. Consider the numerical semigroup $S = \langle 13, 14, 19 \rangle$. It is easy to check that $\overline{iM} \setminus iM = \emptyset$ for every $i \neq 4, 5$, $\overline{4M} \setminus 4M = \{57\}$ and $\overline{5M} \setminus 5M = \{76\}$. By Remark 3.7, we have $C_i = \emptyset$ for every $i \neq 3, 4$. By Proposition 3.8, $C_3 = \{57\}$. By Remark 3.7 and since $76 = 57 + 19$, where $57 \in \overline{4M}$, we have $C_4 = \emptyset$.

Remark 3.10. By definitions of γ and C_i, it is straightforward to prove that $C_i = \emptyset$ for every $i \geq \gamma$. If $g_2 = g_1 + 1$, then, by (3.3), we have $\overline{iM} \setminus iM = \emptyset$ for every $i \gg 0$ (e.g. $i \geq g_1 - 1$). However if $g_2 > g_1 + 1$, then, by the proof of Proposition 3.1, $\overline{iM} \setminus iM \neq \emptyset$ for every $i \geq \gamma$.

Now we give a criterion for the maximal ideal M to be normal.

Theorem 3.11. *The following statements are equivalent:*

(i) M is normal.

(ii) $\overline{(a+1)M} = \overline{aM} + M$ for every $a \geq 0$.

Proof. (i) $\Rightarrow$ (ii): This follows by the definition of normality of M.
(ii) $\Rightarrow$ (i): We want to prove that $\overline{iM} = iM$ for every $i \geq 1$. Since $C_a = \emptyset$ for every a, then by (3.2) and by Proposition 3.8, we have the proof.

We recall that we always assume $g_1 < \cdots < g_n$. If M_i denote the maximal ideal of $S_i = \langle g_1, \ldots, g_i \rangle$, it is natural to ask whether M_n could be normal when M_i is not normal for some $i < n$. The following theorem answers this question.

Theorem 3.12. *If M_i is not normal, then M_n is not normal.*

Proof. Since M_i is not normal, then, by the definition of normal maximal ideal and by Proposition 3.8, there exist an integer a and an element $x = g_{s_1} + \cdots + g_{s_t}$ such that $x \in \overline{(a+1)M_i} \setminus (a+1)M_i = \overline{(a+1)M_i} \setminus \overline{aM_i} + M_i$.
Now we prove that $x \in \overline{(a+1)M_n} \setminus (a+1)M_n$, hence M_n is not normal.
Clearly $x \in \overline{(a+1)M_i} \subseteq \overline{(a+1)M_n}$. Since $x \in \overline{(a+1)M_i} \setminus \overline{aM_i} + M_i$, we

have $x - g_{s_j} < ag_1$ for every $j = 1, \dots, t$. If $x \in (a+1)M_n$, then x is a sum of at least $a+1$ generators with at least one generator greater than g_i since $x \notin (a+1)M_i$. But this is impossible since, by $g_{s_j} \leq g_i$ for every $j = 1, \dots, t$, we have $x - g_{i+1} < x - g_{s_j} < ag_1$.

We can use what studied till now about the normality of maximal ideals for the study of the normality of generic ideals of a numerical semigroup.

Let H be an ideal of S. We recall that $\overline{H} = \{s \in S \mid s \geq \bar{h}\}$, where $\bar{h} = \min\{h \in H\}$, is the integral closure of H in S. Thus H is integrally closed if and only if $H = S(\bar{h})$.

By definition of ideal in a numerical semigroup, we have that $H \cup \{0\}$ is a numerical semigroup. We denote it by S_H and we denote its maximal ideal by M_H.

Remark 3.13. Let H be an integrally closed ideal of S, then $H = [\bar{h}, \infty) \cap S = [\bar{h}, \infty) \cap S_H = M_H$.

Proposition 3.14. *Let H be an integrally closed ideal of S. Then $\overline{iH} = \overline{iM_H}$ for every $i \geq 1$.*

Proof. By $H = M_H$, we have $\overline{iH} = \{s \in S \mid s \geq i\bar{h}\} = \{s \in S_H \mid s \geq i\bar{h}\} = \overline{iM_H}$.

Hence, by Remark 3.13 and Proposition 3.14, we have that the study of the normality of any integrally closed ideal H of S, is related to the study of the normality of the maximal ideal of a new numerical semigroup S_H. This allows us to translate the results of the first part of this section. In particular we have

$$H \text{ is normal if and only if } \overline{(a+1)H} = \overline{aH} + H \text{ for every } a \geq 0. \tag{3.4}$$

Remark 3.15. We remark that there is no connection between the generators of S and the generators of S_H. We know only that the number of generator of S are less or equal of those of S_H, $S_H \subseteq S$ and, if H is integrally closed, $c(S) = c(S_H)$.

Example 3.16. Let $S = \langle 7, 8, 9, 12, 13\rangle$ and let $H = S(9)$, $K = S(8)$ two integrally closed ideals of S.
For H, we have $S_H = \langle 9, 12, 13, 14, 15, 16, 17, 19, 20\rangle$. Hence H is not normal by Proposition 3.1 applied to S_H.

For K, we have $S_K = \langle 8, 9, 12, 13, 14, 15, 19\rangle$. Since $19 \in C_1 = \overline{2M_H} \setminus 2M_H$, we have K is not normal by Remark 3.4 applied to S_K .

Example 3.17. Let $S = \langle 10, 12, 15, 16, 17\rangle$ and let $H = S(15)$, $K = S(12)$ two integrally closed ideals of S.

For H we have $S_H = \langle 15, 16, 17, 20, 22, 24, 25, 26, 27, 28, 29\rangle$. Since $\gamma = 2$ and H is integrally closed, we have H is normal by Proposition 3.3 applied to S_H.

Instead, as in the example 3.16, K is not normal by Proposition 3.1 applied to S_K.

4 The 3-generated case

In this section we consider only 3-generated numerical semigroups and we determine which of them that have normal maximal ideal M. We know, by Propositions 3.1 and 3.11 and Remark 3.4, that if $g_2 > g_1 + 1$ or $g_3 > 2g_1$, then M is not normal. Hence throughout the rest of this section we assume $g_2 = g_1 + 1$ and $g_3 < 2g_1$. We recall that, by Lemmas 3.5 and 3.6, C_a is empty or contains elements only of the form tg_3 with $t \leq a$.

We recall also that for us α is the unique integer such that $(\alpha - 1)g_3 < \alpha g_1$ and $\alpha g_3 \geq (\alpha+1)g_1$, i.e. the unique integer such that $(\alpha+1)g_1 \leq \alpha g_3 < \alpha g_1+g_3$.

Lemma 4.1. *$C_a = \emptyset$ for every $a < \alpha$ and $C_\alpha = \overline{(\alpha + 1)M} \setminus (\alpha + 1)M$.*

Proof. By Proposition 3.2 and by definition of C_a, we have $C_a = \emptyset$ for every $a < \alpha$. Hence, by Proposition 3.8, we have the second part of the proof.

Lemma 4.2. *C_a is empty or of the form $\{tg_3\}$ where $t \leq a$ is the unique integer which satisfies $(a + 1)g_1 < tg_3 < ag_1 + g_3$. In particular C_α is empty or of the form $\{\alpha g_3\}$.*

Proof. Let $tg_3 \in C_a$. By Lemma 3.6 and by definition of C_a, we have $t \leq a$ and $(a + 1)g_1 < tg_3$. Since $tg_3 \notin \overline{aM} + M$, then $(t - 1)g_3 \in S \setminus \overline{aM}$, hence $(t - 1)g_3 < ag_1$, i.e. $tg_3 < ag_1 + g_3$.
The second part of the theorem follows immediately by definion of α.

Remark 4.3. We note that could happen that $(a + 1)g_1 < tg_3 < ag_1 + g_3$, but $C_a = \emptyset$.

Lemma 4.4. *$C_\alpha = \emptyset$ if and only if $\alpha g_3 \leq (\alpha + 1)g_2$.*

Proof. Suppose $\alpha g_3 > (\alpha + 1)g_2$. We want to prove that $\alpha g_3 \in \overline{(\alpha + 1)M} \setminus (\alpha + 1)M$, hence, by Lemma 4.1, $C_\alpha \neq \emptyset$. Suppose $\alpha g_3 \in (\alpha + 1)M$. So $\alpha g_3 = xg_1 + yg_2 + zg_3$ with $x + y + z \geq \alpha + 1$. By $(\alpha - z)g_3 \geq (\alpha - z + 1)g_1$ and by the definition of α, we have $z = 0$. Furthermore, by $\alpha g_3 > (\alpha + 1)g_2$, we have that αg_3 is a sum of at least $(\alpha + 2)$ generators g_1 and g_2, hence, by $\alpha g_3 \geq (\alpha+2)g_1$ and $(\alpha-1)g_3 < \alpha g_1$ (by definition of α), we have a contradiction to $g_3 < 2g_1$.
Suppose now $\alpha g_3 \leq (\alpha + 1)g_2$. Since $g_2 = g_1 + 1$, we have that every element between $(\alpha+1)g_1$ and $(\alpha+1)g_2$ is a sum of $\alpha+1$ elements g_1 or g_2. In particular $\alpha g_3 \in (\alpha + 1)M$. Hence, by the second part of the Lemma 4.2 and By Lemma 4.1, we have $C_\alpha = \emptyset$.

Lemma 4.5. *If $C_\alpha = \emptyset$, then $C_a = \emptyset$ for every a.*

Proof. Let $t \leq a$ the number which satisfies $(a + 1)g_1 < tg_3 < ag_1 + g_3$. By $C_\alpha = \emptyset$ and by Lemma 4.4, we have $\alpha g_3 \leq (\alpha + 1)g_2$. Moreover, by definition of α, we have $(\alpha + 1)g_1 \leq \alpha g_3$ and $\alpha \leq t$ (since $(t+1)g_1 \leq (a+1)g_1 < tg_3$). We prove that $tg_3 \in \overline{aM} + M$, then $C_a = \emptyset$ follows by Lemma 4.2.
Consider first the case $\alpha g_3 = (\alpha+1)g_2$. Since $(a+1)g_1 < tg_3$ and $g_2 = g_1+1$, we

have $ag_1+g_2 \leq tg_3$. Thus $ag_1 \leq tg_3-g_2 = (t-\alpha)g_3+\alpha g_3-g_2 = (t-\alpha)g_3+\alpha g_2$, that is $tg_3 - g_2 \in \overline{aM}$. Hence $tg_3 \in \overline{aM} + M$.
Suppose now $\alpha g_3 < (\alpha+1)g_2$. By $g_2 = g_1+1$, we have $\alpha g_3 = g_1+s$ with $s \in S$. Thus $ag_1 < tg_3-g_1 = (t-\alpha)g_3+\alpha g_3-g_1 = (t-\alpha)g_3+s$, that is $tg_3-g_1 \in \overline{aM}$. Hence $tg_3 \in \overline{aM} + M$.

Theorem 4.6. *Let $g_2 = g_1+1$ and $g_3 < 2g_1$. Then M is normal if and only if $\alpha g_3 \leq (\alpha+1)g_2$.*

Proof. By Theorem 3.11 and definition of C_a, we have that M is normal if and only if $C_a = \emptyset$ for every a. Hence, by Lemmas 4.5 and 4.4, we have the proof.

Example 4.7. Let $S = \langle 10, 11, 10+x\rangle$. We want to study for which values of x, M is normal. By Proposition 3.11 and Remark 3.4, we only consider values of x for which $10+x < 20$, hence $2 \leq x \leq 9$. By definition of α and Theorem 4.6, M is normal if and only if there exist integers α satisfying the following system of inqualities

$$\begin{cases} (\alpha-1)(10+x) < 10\alpha \\ \alpha(10+x) \geq 10(\alpha+1) \\ \alpha(10+x) \leq 11(\alpha+1) \end{cases} \iff \begin{cases} x\alpha < 10+x \\ x\alpha \geq 10 \\ x\alpha \leq \alpha+11 \end{cases}$$

It easy to check that the system has solution only for $x = 2, 3, \dots, 6$. Hence M is normal if and only if $S = \langle 10, 11, 10+x\rangle$ with $x = 2, 3, \dots, 6$.

Example 4.8. Let us consider the numerical semigroup $\langle 100, 101, 117\rangle$. Since $5 \cdot 117 = 585 < 6 \cdot 100 = 600$ and $6 \cdot 117 = 702 > 7 \cdot 100 = 700$, we have $\alpha = 6$. Since $702 \leq 7 \cdot 101 = 707$ and by Theorem 4.6, M is normal. Now let us consider the numerical semigroup $\langle 100, 101, 118\rangle$. Since $5 \cdot 118 = 590 < 6 \cdot 100 = 600$ and $6 \cdot 188 = 708 > 7 \cdot 100 = 700$, we have $\alpha = 6$. Since $708 > 7 \cdot 101 = 707$ and by Theorem 4.6, M is not normal. We note that we can find semigroups for which M is normal and $g_3 > 118$. In fact using the same argument as above one can easily check that $\langle 100, 101, 150\rangle$, has normal maximal ideal M.

Now we give, for each $k \geq 3$, an example of a numerical semigroup for which $iM = \overline{iM}$ for every $i < k$, but $kM \neq \overline{kM}$.

Example 4.9. Let $S = \langle k^2-3, k^2-2, k^2+k-1\rangle$. Since

$$(k-1)(k^2+k-1) \geq k(k^2-3) \iff 1 \geq -3k$$

and

$$(k-2)(k^2+k-1) < (k-1)(k^2-3) \iff 2 < 3,$$

we have $\alpha = k-1$. Hence by

$$(k-1)(k^2+k-1) > k(k^2-2) \iff 1 > 0$$

and by Lemmas 4.1 and 4.4 and by definition of C_a, the semigroup satisfies the condition above.

Let us denote the number of 3-generated numerical semigroups with normal maximal ideal and $g_1 = m$ by $N(m)$, the number of numerical semigroups with $g_1 = m$ and $g_2 = m+1$ by $W(m)$ and the number of 3-generated numerical semigroups $\langle m, m+1, m+x\rangle$ with $x < m$ by $B(m)$.

Remark 4.10. Since $2 \le x < m$, then $B(m) = m-2$. Consider $S = \langle m, m+1\rangle$. Clearly $W(m)$ is equal to the number integers y such that $y > m+1$ and $y \notin \langle m, m+1\rangle$ (so that we have the numerical semigroup $\langle m, m+1, y\rangle$). It is well known (cf. e.g. [6]) that any 2-generated numerical semigroup is symmetric i.e. has just as many elements as non-elements below the conductor and that $c(S) = m(m+1) - 2m$. Hence $W(m) = \frac{c(S)}{2} - (m-1) = \frac{(m-2)(m-1)}{2}$.

By Proposition 3.11 and Remark 3.4, we know that if $\langle m, m+1, m+x\rangle$ has normal maximal ideal, then $x < m$. Hence $N(m)$ is very small compared to $W(m)$ for large m. The following theorem shows that $N(m)$ is very small compared also to $B(m)$ for large m.

Theorem 4.11. *Let $N(m)$ and $B(m)$ be as above. Then $\lim_{m\to\infty} \frac{N(m)}{B(m)} = 0$.*

Proof. By Theorem 4.6, the values of x such that $\langle m, m+1, m+x\rangle$ has normal maximal ideal satisfy the following system of inequalities

$$\begin{cases} (\alpha-1)(m+x) < \alpha m - 1 \\ \alpha(m+x) \ge (\alpha+1)m \\ \alpha(m+x) \le (\alpha+1)m + (\alpha+1) \end{cases} \iff \begin{cases} (\alpha-1)x < m-1 \\ \alpha x \ge m \\ \alpha x \le m + \alpha + 1 \end{cases}$$

that is $\frac{m}{\alpha} \le x \le \min\{\frac{m-1}{\alpha-1}, \frac{m}{\alpha} + 1 + \frac{1}{\alpha}\}$.
We note that $\min\{\frac{m-1}{\alpha-1}, \frac{m}{\alpha} + 1 + \frac{1}{\alpha}\} - \frac{m}{\alpha} \le (\frac{m}{\alpha} + 1 + \frac{1}{\alpha}) - \frac{m}{\alpha} = 1 + \frac{1}{\alpha} < 2$, since by $x < m$, we have $\alpha \ge 2$. Hence for every α there are at most two integers in the interval $[\frac{m}{\alpha}, \min\{\frac{m-1}{\alpha-1}, \frac{m}{\alpha} + 1 + \frac{1}{\alpha}\}]$.
Suppose $\alpha - 1 \ge \sqrt{m-1}$, then $\frac{m-1}{\alpha-1} \le \sqrt{m-1}$. Thus for every such α, we have $2 \le x \le \sqrt{m-1}$. Hence there are at most $\sqrt{m-1} - 1$ such values of x such that $\langle m, m+1, m+x\rangle$ has normal maximal ideal. Suppose now $\alpha - 1 < \sqrt{m-1}$. Thus $2 \le \alpha < 1 + \sqrt{m-1}$. Since for every α there are at most two integers in the interval $[\frac{m}{\alpha}, \min\{\frac{m-1}{\alpha-1}, \frac{m}{\alpha} + 1 + \frac{1}{\alpha}\}]$, then if $2 \le \alpha < 1 + \sqrt{m-1}$, there exist at most $2(1 + \sqrt{m-1})$ such values of x such that $\langle m, m+1, m+x\rangle$ has normal maximal ideal.
Hence $N(m) \le (\sqrt{m-1} - 1) + (2 + 2\sqrt{m-1}) = 3\sqrt{m-1} + 1$. By $\frac{N(m)}{B(m)} \le \frac{3\sqrt{m-1}+1}{m-2}$, we have the proof.

5 Normality of the maximal ideal in a ring

We recall that for us R is a local, Noetherian, one-dimensional domain such that the integral closure $\overline{R}$ of R in its quotient field is a DVR and a finitely generated R-module and such that R and $\overline{R}$ have the same residue field. We recall also that $v(R) = S = \langle g_1, \dots, g_n\rangle$ with $g_1 < g_2 < \cdots < g_n$.

In this section we study the connection between the normality of $\mathfrak{m}$ and the normality of $v(\mathfrak{m}) = M$.

We note that in general $iv(\mathfrak{m}) \subseteq v(\mathfrak{m}^i)$ and the inclusion could be strict.

Example 5.1. Let $R = K[[t^4+t^5, t^6, t^{11}]]$ with K a field of characteristic different from 3 and let $S = \langle 4, 6, 11, 13\rangle$. One can easily check that $v(R) = S$ and, by $(t^4+t^5)^3 - (t^6)^2 = 3t^{13} + 3t^{14} + t^{15}$, we have $13 \in v(\mathfrak{m}^2)$. However $13 \notin 2v(\mathfrak{m})$.

Proposition 5.2. *For every $i \geq 1$ we have $v(\overline{\mathfrak{m}^i}) = \overline{iv(\mathfrak{m})}$.*

Proof. Since, by Theorem 2.3, $\overline{\mathfrak{m}^i} = R(ig_1)$, then $v(\overline{\mathfrak{m}^i}) = S(ig_1) = \overline{iv(\mathfrak{m})}$.

The following theorem is an analogue for rings to Theorem 3.11. We recall that $\overline{\mathfrak{m}} = \mathfrak{m}$ is always true.

Proposition 5.3. *The maximal ideal $\mathfrak{m}$ is normal if and only if $\overline{\mathfrak{m}^{i+1}} = \overline{\mathfrak{m}^i}\mathfrak{m}$ for every $i \geq 1$.*

Proof. Clearly $\mathfrak{m}$ normal implies $\overline{\mathfrak{m}^{i+1}} = \overline{\mathfrak{m}^i}\mathfrak{m}$ for every i.
Suppose now $\overline{\mathfrak{m}^{i+1}} = \overline{\mathfrak{m}^i}\mathfrak{m}$ for every $i \geq 1$ and we want to prove that $\overline{\mathfrak{m}^i} = \mathfrak{m}^i$. We prove it by induction on i. We know that $\overline{\mathfrak{m}} = \mathfrak{m}$. Suppose $\overline{\mathfrak{m}^i} = \mathfrak{m}^i$ for every $i \leq n$ and we prove that $\overline{\mathfrak{m}^{n+1}} = \mathfrak{m}^{n+1}$. In fact $\overline{\mathfrak{m}^{n+1}} = \overline{\mathfrak{m}^n}\mathfrak{m} = \mathfrak{m}^n\mathfrak{m} = \mathfrak{m}^{n+1}$.

Theorem 5.4. *If $v(\mathfrak{m})$ is normal, then $\mathfrak{m}$ is normal.*

Proof. By $\overline{iv(\mathfrak{m})} + v(\mathfrak{m}) \subseteq v(\overline{\mathfrak{m}^i}\mathfrak{m})$ and by Proposition 5.2, we have that $v(\overline{\mathfrak{m}^{i+1}}) \setminus v(\overline{\mathfrak{m}^i}\mathfrak{m}) \subseteq \overline{(i+1)v(\mathfrak{m})} \setminus (\overline{iv(\mathfrak{m})} + v(\mathfrak{m}))$.
Hence if $v(\mathfrak{m})$ is normal, then $\mathfrak{m}$ normal follows by Corollary 2.2 and Proposition 5.3.

It is natural to ask whether $\mathfrak{m}$ normal implies $v(\mathfrak{m})$ normal.
By $iv(\mathfrak{m}) \subseteq v(\mathfrak{m}^i)$ and Proposition 5.2, for every $i \geq 1$, the following lemma follows immediately.

Lemma 5.5. *For every $i \geq 1$, $v(\overline{\mathfrak{m}^i}) \setminus v(\mathfrak{m}^i) \subseteq \overline{iv(\mathfrak{m})} \setminus iv(\mathfrak{m})$.*

Example 5.6. Consider the ring $R = K[[t^{12}+t^{18}, t^{13}+t^{18}, t^{15}+t^{20}, t^{22}]]$ with K a field of characteristic different from 2 and $S = \langle 12, 13, 15, 22\rangle$. One can easily check that $v(R) = S$. We have $\overline{iv(\mathfrak{m})} = iv(\mathfrak{m})$ for every $i \neq 3$ and $\overline{3v(\mathfrak{m})} \setminus 3v(\mathfrak{m}) = \{44\}$, hence $v(\mathfrak{m})$ is not normal. By Lemma 5.5 and Corollary 2.2, $\mathfrak{m}^i = \overline{\mathfrak{m}^i}$ for every $i \neq 3$. Since $(t^{13}+t^{18})^3 - (t^{12}+t^{18})^2(t^{15}+t^{20}) = 2t^{44} - 2t^{45} - 2t^{50} - t^{51} + t^{54} - t^{56} \in \mathfrak{m}^3$ and by Lemma 5.5, we also have $\mathfrak{m}^3 = \overline{\mathfrak{m}^3}$. Hence $\mathfrak{m}$ is normal.

We denote $\min\{(i+1)g_1, (i-1)g_1 + g_2\}$ by μ_i.

Lemma 5.7. *For every $i \geq 1$, the two smallest values in $v(\mathfrak{m}^i)$ are ig_1 and μ_i.*

Proof. Let $\mathfrak{m} = (r_1, \ldots, r_n)$ with $v(r_j) = g_j$ for every $j = 1, 2, \ldots, n$ (this is not in general a minimal set of generators of $\mathfrak{m}$). We know that every element $x \in \mathfrak{m}^i$ is of the type $x = b_1 r_1^i + b_2 r_1^{i-1} r_2 + y$ with $v(y) > \mu_i$. By properties of valuation, we also know that if b_1 is not unit, then $v(b_1 r_1^i) \geq (i+1)g_1 \geq \mu_i$ and if b_2 is unit then $v(b_2 r_1^{i-1} r_2) = (i-1)g_1 + g_2 \geq \mu_i$. Hence if b_1 is a unit, then $v(x) = ig_1$, otherwise $v(x) \geq \mu_i$. Since r_1^{i+1} and $r_1^{i-1}r_2$ are in $\mathfrak{m}^i$, we have the proof.

The following proposition is an analogue for rings to Proposition 3.1.

Proposition 5.8. *If $\mathfrak{m}$ is normal, then $g_2 = g_1 + 1$.*

Proof. Suppose $g_2 > g_1 + 1$. By the proof of Proposition 3.1, we have that if i is an integer such that $i \geq \gamma$, then $ig_1 + 1 \in \overline{iv(\mathfrak{m})} \setminus iv(\mathfrak{m})$. Hence by Lemma 5.7, $ig_1 + 1 \in v(\overline{\mathfrak{m}^i}) \setminus v(\mathfrak{m}^i)$ that is $\overline{\mathfrak{m}^i} \neq \mathfrak{m}^i$.

6 The Cohen-Macaulay property of $G(\mathfrak{m})$ in a particular case

Let R be a ring as in the Section 5. Throughout the rest of this section we assume that $g_2 = g_1 + 1$.

It is known (cf. [4, Corollary 17]) that for a ring R of our type, principal minimal reductions always exist and that if $x \in \mathfrak{m}$ then xR is a minimal reduction of $\mathfrak{m}$ if and only if x is of minimal value in $\mathfrak{m}$. For everything concerning reduction of an ideal, we refer to [11].

We denote by $G(\mathfrak{m})$ the *associated graded ring* of R with respect to $\mathfrak{m}$, that is $G(\mathfrak{m}) = \oplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. The question whether, for a local ring, $G(\mathfrak{m})$ is Cohen-Macaulay is an important one. In fact it is important and often difficult to compute the Hilbert function of a local ring, however, if the associated graded ring is Cohen-Macaulay, then the computation of the Hilbert function can be reduced to the computation of the Hilbert function of an Artinian local ring.

An element $\bar{0} \neq \bar{z} = z + \mathfrak{m}^{s+1} \in G(\mathfrak{m})$ is a zero divisor in $G(\mathfrak{m})$ if and only if there exists an element $\bar{0} \neq \bar{y} = y + \mathfrak{m}^{r+1} \in G(\mathfrak{m})$ such that $\bar{z} \cdot \bar{y} = 0$ i.e. $z \cdot y \in \mathfrak{m}^{s+r+1}$.

Remark 6.1. Let $\mathfrak{m} = (x_1, x_2, \ldots, x_r)$ the maximal ideal of R. By the hypotheses above on R, we can assume $v(x_1) = g_1$ and $v(x_2) = g_1 + 1$. Hence $\mathfrak{m}^h = \overline{\mathfrak{m}^h} = \overline{R}(hg_1)$ for every $h \gg 0$ (for example $h \geq g_1 - 1$).

Throughout the rest of the section we denote by $x_1 R$ a principal minimal reduction of $\mathfrak{m}$. It is known that $G(\mathfrak{m})$ is C-M if and only if $\overline{x_1}$ is a non-zero divisor in $G(\mathfrak{m})$ (cf. [5, Remark 3.1]).

Lemma 6.2. *If $G(\mathfrak{m})$ is C-M, then for every $i \geq 1$ and for every $w \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$ we have $v(w) < (i+1)g_1$.*

Proof. By Remark 6.1, we have only to consider $i < h$ and we use decreasing induction on i. For every $w \in \mathfrak{m}^{h-1} \setminus \mathfrak{m}^h = \mathfrak{m}^{h-1} \setminus \overline{\mathfrak{m}^h}$ clearly we have $v(w) < hg_1$. Now we prove that if for every $w \in \mathfrak{m}^{h-a} \setminus \mathfrak{m}^{h-a+1}$ we have $v(w) < (h-a+1)g_1$, then for every $f \in \mathfrak{m}^{h-a-1} \setminus \mathfrak{m}^{h-a}$ we have $v(f) < (h-a)g_1$. Since $G(\mathfrak{m})$ is C-M it follows that for every $f \in \mathfrak{m}^{h-a-1} \setminus \mathfrak{m}^{h-a}$ we have $x_1 f \in \mathfrak{m}^{h-a} \setminus \mathfrak{m}^{h-a+1}$. By inductive hypothesis we have that $v(x_1 f) = g_1 + v(f) < (h-a+1)g_1$ which is $v(f) < (h-a)g_1$.

Now we are ready to characterize when $G(\mathfrak{m})$ is C-M for a ring R under the hypotheses at the beginning of this section.

Theorem 6.3. *Let R be a ring as above, then $G(\mathfrak{m})$ is C-M if and only if $\mathfrak{m}$ is normal.*

Proof. If $\mathfrak{m}$ is normal, then $G(\mathfrak{m})$ is C-M (cf. [3, Proposition 2.1]).
Suppose now $G(\mathfrak{m})$ is C-M and that $\mathfrak{m}$ is not normal. Let $x \in \overline{\mathfrak{m}^\beta} \setminus \mathfrak{m}^\beta$ and let r the integer such that $x \in \mathfrak{m}^r \setminus \mathfrak{m}^{r+1}$ (hence $r < \beta$). By Lemma 6.2 we have $v(x) < (r+1)g_1$ and, by $r < \beta$ and $x \in \overline{\mathfrak{m}^\beta} \setminus \mathfrak{m}^\beta$, we have $(r+1)g_1 \leq \beta g_1 < v(x)$. A contradiction.

Let $\mathfrak{m} = (x_1, \ldots, x_r)$, $r \leq n$ be the maximal ideal of R, with $v(x_1) = g_1$ and $v(x_2) = g_1 + 1$. We will study the connection between the property that $G(\mathfrak{m})$ is C-M and $G(\mathfrak{m}_T)$ is C-M, where $\mathfrak{m}_T = (t^{g_1}, t^{g_2}, \ldots, t^{g_n})$ is the maximal ideal of the semigroup ring $T = K[[S]] = K[[t^{g_1}, t^{g_2}, \ldots, t^{g_n}]]$.

Proposition 6.4. *If $G(\mathfrak{m}_T)$ is C-M, then $G(\mathfrak{m})$ is C-M.*

Proof. Suppose $G(\mathfrak{m}_T)$ is C-M. By Theorem 6.3, we have $\mathfrak{m}_T$ normal, hence $\mathfrak{m}$ normal by Corollary 5.4. Again by Theorem 6.3, $G(\mathfrak{m})$ is C-M.

It is natural to ask whether the converse to Proposition 6.4 holds.

Remark 6.5. If $G(\mathfrak{m})$ is C-M, then in general is not true that $G(\mathfrak{m}_T)$ is C-M. In fact consider $\mathfrak{m} = (t^{12}+t^{18}, t^{13}+t^{18}, t^{15}+t^{20}, t^{22})$ and $\mathfrak{m}_T = (t^{12}, t^{13}, t^{15}, t^{22})$ in the Example 5.6. Since $\mathfrak{m}$ is normal, then $G(\mathfrak{m})$ is C-M. However, by Theorem 6.3, $G(\mathfrak{m}_T)$ is not C-M since $\mathfrak{m}_T$ is not normal.

7 A bound for $l_R(\overline{\mathfrak{m}^i}/\mathfrak{m}^i)$

Let R be a ring as in Section 5, with maximal ideal $\mathfrak{m} = (x_1, x_2, \ldots, x_p)$, where $v(x_1) = g_1$ and $v(x_1) < v(x_2) < \cdots < v(x_p)$. Let us denote by $r = r(\mathfrak{m}) = \min\{n \mid \mathfrak{m}^{n+1} = z\mathfrak{m}^n \text{ for some } z \in \mathfrak{m}\}$ the *reduction number* of $\mathfrak{m}$.

We let $\mu(R) = l_R(\mathfrak{m}^i/\mathfrak{m}^{i+1})$, with $i \geq r$ and we call $\mu(R)$ the multiplicity of R.

As in Section 6, we assume that $x_1 R$ is a principal minimal reduction of $\mathfrak{m}$. By definition of r and Theorem 2.1, the following statements are easy to see:

$$\text{for every } i \geq r,\ \mathfrak{m}^{i+1} = x_1 \mathfrak{m}^i \quad (7.1)$$

$$\text{for every } i \geq r,\ \mu(R) = v(x_1) = g_1 \quad (7.2)$$

By [10, Lemma 2], we have that $(R:\overline{R})=\{x\in\overline{R}\mid v(x)\geq c(S)\}$. Hence $\min\{a\in\mathbb{N}\mid x_1^a\in(R:\overline{R})\}=\gamma$, where γ is the integer introduced in the Section 3. From now on we denote by $\beta=\max\{\gamma,r\}$ and by $q=l_R(\overline{\mathfrak{m}^\beta}/\mathfrak{m}^\beta)$.

Proposition 7.1. *For every $i\geq\beta$, $l_R(\overline{\mathfrak{m}^i}/\mathfrak{m}^i)=q$.*

Proof. Let $i\geq\beta$. By $\beta\geq\gamma$, we have $\overline{\mathfrak{m}^\beta}=\overline{R}(\beta g_1)=R(\beta g_1)$. Since x_1R is a principal minimal reduction and $i\geq r$, we have $\overline{\mathfrak{m}^i}=x_1^{i-\beta}\overline{\mathfrak{m}^\beta}$ and $\mathfrak{m}^i=x_1^{i-\beta}\mathfrak{m}^\beta$. Hence $l_R(\overline{\mathfrak{m}^i}/\mathfrak{m}^i)=l_R(x_1^{i-\beta}\overline{\mathfrak{m}^\beta}/x_1^{i-\beta}\mathfrak{m}^\beta)=l_R(\overline{\mathfrak{m}^\beta}/\mathfrak{m}^\beta)=q$.

Theorem 7.2. *For every $i\geq\beta$, $0\leq l_R(\overline{\mathfrak{m}^i}/\mathfrak{m}^i)\leq l_R(\overline{R}/R)-\mu(R)+1$.*

Proof. Let us consider the second inequality and let $T=k[[t^{g_1},t^{g_2},\ldots,t^{g_n}]]$ be the semigroup ring associated to S with maximal ideal $\mathfrak{m}_T=(t^{g_1},t^{g_2},\ldots,t^{g_n})$. We first prove that the inequality holds for T. Let $x\in\overline{\mathfrak{m}_T^i}\setminus\mathfrak{m}_T^i$ with $i\geq 0$. Then $x=t^{ig_1}t^y$ with $y>0$ and $t^{g_1}t^y\notin\mathfrak{m}$ (if not $x=t^{(i-1)g_1}t^{g_1}t^y\in\mathfrak{m}_T^i$). Clearly the map that associates the element $ig_1+y\in v(\overline{\mathfrak{m}_T^i})\setminus v(\mathfrak{m}_T^i)$ to $g_1+y\in v(\overline{T}(g_1))\setminus v(\mathfrak{m}_T)$ is injective, hence $l_T(\overline{\mathfrak{m}_T^i}/\mathfrak{m}_T^i)=|v(\overline{\mathfrak{m}_T^i})\setminus v(\mathfrak{m}_T^i)|\leq|v(\overline{T}(g_1))\setminus v(\mathfrak{m}_T)|=l_T(\overline{T}/T)-g_1+1$.

Noting that $v(\mathfrak{m}_T^i)=iv(\mathfrak{m})$ and $v(\overline{\mathfrak{m}_T^i})=\overline{iv(\mathfrak{m})}$, by Theorem 2.1, Lemma 5.5 and 7.2, we have

$$l_R(\overline{\mathfrak{m}^i}/\mathfrak{m}^i)\leq l_T(\overline{\mathfrak{m}_T^i}/\mathfrak{m}_T^i)\leq l_T(\overline{T}/T)-g_1+1=l_R(\overline{R}/R)-\mu(R)+1.$$

Now we show that the bounds are the best possible.

Proposition 7.3. *We have $l_R(\overline{\mathfrak{m}^i}/\mathfrak{m}^i)=0$ for every $i\geq\beta$ if and only if $v(x_2)=g_1+1$.*

Proof. Let $l_R(\overline{\mathfrak{m}^i}/\mathfrak{m}^i)=0$. Suppose $v(r_2)>g_1+1$. By the proof of Proposition 5.8, for every $i\gg 0$ we have $\overline{\mathfrak{m}^i}\neq\mathfrak{m}^i$. A contradiction to $l_R(\overline{\mathfrak{m}^i}/\mathfrak{m}^i)=0$ for every $i\geq\beta$.
Vice versa, let $v(r_2)=g_1+1$. Then, as in the Remark 6.1, we have $\mathfrak{m}^h=\overline{\mathfrak{m}^h}$ for $h\gg 0$, hence, by Proposition 7.1, $\mathfrak{m}^\beta=\overline{\mathfrak{m}^\beta}$. Using again Proposition 7.1, we have $l_R(\overline{\mathfrak{m}^i}/\mathfrak{m}^i)=0$ for every $i\geq\beta$.

Now we give a class of rings for which $l_R(\overline{\mathfrak{m}^i}/\mathfrak{m}^i)=l_R(\overline{R}/R)-\mu(R)+1$ for every $i\geq\beta$.

Example 7.4. Let $R=k[[t^2,t^{2a+1}]]$ with $a\geq 1$.
Thus $\mathfrak{m}=\{t^2,t^4,\ldots,t^{2a},t^{2a+1},\ldots\}$, hence $l_R(\overline{R}/R)-v(t^2)+1=|\mathbb{N}\setminus S|-1=a-1$.

Since $\mathfrak{m}^i=\{t^{2i},\ldots,t^{2(a+(i-1))},t^{2(a+(i-1))+1},\ldots\}$ for every $i\geq 1$, thus for every $i\geq 1$, $l_R(\overline{\mathfrak{m}^i}/\mathfrak{m}^i)=|v(\overline{\mathfrak{m}^i})\setminus v(\mathfrak{m}^i)|=|\{2i,\longrightarrow\}\setminus\{2i,\ldots,2(a+i-1),\longrightarrow\}|=|\{2i,\longrightarrow\}\setminus\{2i,\ldots,2i+2(a-1),\longrightarrow\}|=a-1$.

Remark 7.5. The rings $R = k[[t^2, t^{2a+1}]]$ with $a \geq 1$, are not the only rings for which $l_R(\overline{\mathfrak{m}^i}/\mathfrak{m}^i) = l_R(\overline{R}/R) - \mu(R) + 1$ for every $i \geq \beta$. For example every ring $R = k[[t^{g_1}, t^{g_2}, \dots, t^{g_n}]] = K[[S]]$, with $g_2 < c(S) \leq 2g_1$ and $3g_1 = 2g_2$, satisfies the equality above.

Example 7.6. Now we give a class of rings $R = k[[t^{g_1}, t^{g_2}, \dots, t^{g_n}]]$ for which $l_R(\overline{\mathfrak{m}^i}/\mathfrak{m}^i)$, for every $i \geq \beta$, is equal to a fixed $x \geq 1$.
Let $R = k[[t^{2x+1}, t^{3x+2}, t^{3x+3}, \dots, t^{4x+1}]]$. So $\mathfrak{m} = \{t^{2x+1}, t^{3x+2}, t^{3x+3}, \dots, t^{4x+2}, t^{5x+3}, t^{5x+4}, \dots\}$. In fact for every $1 \leq y \leq x$, we have $6x + 3 + y = (3x + 2) + (3x+1+y)$. Thus $\mathfrak{m}^i = \{t^{i(2x+1)}, t^{i(2x+1)+x+1}, t^{i(2x+1)+x+2}, \dots\}$ for every $i \geq 2$, hence $l_R(\overline{\mathfrak{m}^i}/\mathfrak{m}^i) = |\{i(2x+1), \longrightarrow\} \setminus \{i(2x+1), i(2x+1)+x+1, \longrightarrow\}| = x$ for every $i \geq \beta \geq 2$.

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