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H_{-n}-perturbations of self-adjoint operators and Krein's resolvent formula

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$\mathcal{H}_{-n}$-perturbations of self-adjoint operators and Krein's resolvent formula.

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Abstract. Supersingular $\mathcal{H}_{-n}$ rank one perturbations of arbitrary positive self-adjoint operator A acting in the Hilbert space $\mathcal{H}$ are investigated. The operator corresponding to the formal expression

$$A_\alpha = A + \alpha\langle\varphi, \cdot\rangle\varphi, \ \alpha \in \mathbf{R}, \varphi \in \mathcal{H}_{-n}(A),$$

is determined as a regular operator with pure real spectrum acting in a certain extended Hilbert space $\mathbf{H} \supset \mathcal{H}$. The resolvent of the operator so defined is given by a certain generalization of Krein's resolvent formula. It is proven that spectral properties of the operator are described by generalized Nevanlinna functions. The results of [**23**] are extended to the case of arbitrary integer $n \geq 4$.

1. Introduction.

Finite rank singular perturbations of self-adjoint operators have been studied intensively during the recent years [**2, 3, 4, 6, 14, 15, 19, 27, 28, 36**]. In particular partial differential operators with point interactions are described in [**1, 8**], following the pioneering work by F.Berezin and L.Faddeev from 1961 [**7**]. One of the main mathematical tools to study spectral properties of these operators is well-known Krein's resolvent formula relating the resolvents of two self-adjoint extensions of one symmetric operator having finite or infinite deficiency indices [**20, 26, 33**]. Self-consistent presentation of this theory can be found in recent papers [**4, 13**].

In the current paper we continue our studies of singular rank one perturbations of a self-adjoint operator A acting in the Hilbert space $\mathcal{H}$. The perturbed operator can formally be defined by

$$A_\alpha = A + \alpha\langle\varphi, \cdot\rangle\varphi, \tag{1}$$

where $\alpha \in \mathbf{R}$ is a coupling constant and φ is the singular vector describing the interaction. To measure the singularity of the interaction one can use the scale $\mathcal{H}_s$ of Hilbert spaces[1] associated with the positive self-adjoint operator

[1]See precise definition in Section 2.

A acting in the Hilbert space $\mathcal{H}$

$$\begin{array}{ccccccccccccccccc} & & & & \mathrm{Dom}\,(A) & & & & \mathcal{H} & & & & (\mathrm{Dom}\,(A))^* & & & & \\ & & & & \| & & & & \| & & & & \| & & & & \\ \ldots \subset & \mathcal{H}_4 & \subset & \mathcal{H}_3 \subset & \mathcal{H}_2 & \subset & \mathcal{H}_1 & \subset & \mathcal{H}_0 & \subset & \mathcal{H}_{-1} & \subset & \mathcal{H}_{-2} & \subset & \mathcal{H}_{-3} \subset & \mathcal{H}_{-4} & \subset \ldots \end{array} \tag{2}$$

We say that the interaction is from the class $\mathcal{H}_{-n}$ if and only if $\varphi \in \mathcal{H}_{-n} \setminus \mathcal{H}_{-n+1}$.

The singular interactions from the classes $\mathcal{H}_{-1}$ and $\mathcal{H}_{-2}$ can be defined using operators acting in the original Hilbert space $\mathcal{H}$. The perturbation term $\alpha\langle\varphi, \cdot\rangle\varphi$ is infinitesimally form bounded with respect to the operator A If $\varphi \in \mathcal{H}_{-1}$. The perturbed operator is uniquely defined using KLMN theorem [**32**]. The perturbed operator in the case of $\mathcal{H}_{-2}$ perturbations is not defined uniquely - one dimensional family of self-adjoint operators corresponds to formal expression (1) [**18, 2, 3, 4**].

Current paper is devoted to so-called **supersingular perturbations** defined by vectors from $\mathcal{H}_{-n}, n \geq 3$. Such perturbations have been studied using certain extension of the original Hilbert space. In [**34, 35, 11, 12**] rank one supersingular perturbations were defined using self-adjoint operators acting in Pontryagin spaces. It was shown that the spectral properties of these models are described by generalized Nevanlinna functions with a finite number of negative squares [**15**]. Similar ideas were used in [**17, 29, 30, 31**] where concrete problems of mathematical physics were attacked. Different physicists and mathematicians tried to define supersingular perturbations [**9, 10, 16, 5**].

In [**22**] supersingular rank one perturbation of positive self-adjoint operator have been defined without any use of spaces with indefinite metrics. The approach was limited to the case of $\mathcal{H}_{-3}$ perturbations. In [**23**] we were able to make one step further and describe all supersingular perturbations from the class $\mathcal{H}_{-4}$. The perturbed operator has been defined in an extended Hilbert space, but no self-adjoint operator corresponds to the formal expression (1) in this case. The perturbed operator has been defined in the class of regular operators, i.e. densely defined operator having two remarkable properties:
The domain of the operator and it adjoint coincide;
The spectrum of the operator is pure real.
The formula describing the resolvent of the perturbed operator is similar to celebrated Krein's formula. The spectral properties of the operators are described by generalized Nevanlinna functions. The aim of the current paper is to generalize the ideas of [**23**] to the case of arbitrary supersingular perturbations.

We decided to remind the reader all necessary preliminary fact concerning rank one singular perturbations in Section 2. The main ideas of the current paper are described in this section. In particular it is proposed to define the perturbed operator as a restriction of a certain maximal operator. The maximal operator and the extended Hilbert space used to construct supersingular perturbations are described in Section 3. The family of regular operators corresponding to such singular perturbation is obtained in Section 4. The resolvent formula describing supersingular perturbation is calculated. The relations between this formula and Krein's resolvent formula are investigated.

2. Rank one perturbations and the extension theory.

Current paper is devoted to the construction of the operator describing rank one supersingular perturbation of a given positive self-adjoint operator A acting in a certain Hilbert space $\mathcal{H}$, given formally by (1). Recent developments in this area are described in [**2, 3, 14, 18, 21, 25, 36**]. It has been shown that if φ belongs to the original Hilbert space $\mathcal{H}$ then the perturbation $\alpha\langle\varphi,\cdot\rangle\varphi$ is a bounded symmetric operator and the perturbed operator A_α is self-adjoint on the domain of the original operator A. The resolvent of the perturbed operator is given by

$$\frac{1}{A_\alpha-\lambda}=\frac{1}{A-\lambda}-\frac{1}{\frac{1}{\alpha}+\langle\varphi,\frac{1}{A-\lambda}\varphi\rangle}\left\langle\frac{1}{A-\bar{\lambda}}\varphi,\cdot\right\rangle\frac{1}{A-\lambda}\varphi. \tag{3}$$

All spectral properties of the perturbed operator A_α are described by the Nevanlinna function $Q(\lambda)=\langle\varphi,\frac{1}{A-\lambda}\varphi\rangle$ (See e.g. [**4**]).

Consider now the scale of Hilbert spaces $\mathcal{H}_s$ associated with the positive operator A. The norm in each space $\mathcal{H}_s$ is defined by

$$\parallel U\parallel^2_{\mathcal{H}_s}=\langle U,(A+1)^sU\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in the original Hilbert space $\mathcal{H}$. In order to avoid misunderstanding only the scale of Hilbert spaces associated with the original operator A and the original Hilbert space $\mathcal{H}$ will be considered throughout the paper. All perturbations defined by vectors φ not from the original Hilbert space $\mathcal{H}$ are called singular. These perturbations are characterized by the fact that the domain of the perturbed operator does not coincide with the domain of the original one. In the case $\varphi\in\mathcal{H}_{-1}\setminus\mathcal{H}$ the perturbation is relatively form bounded with respect to the sesquilinear form of the operator A and the perturbed operator can be determined using the form perturbation technique. The resolvent of the perturbed operator is again given by (3). The main difference is that the domain of the perturbed operator does not coincide with the domain of the original operator in general, but the perturbed operator is uniquely defined and is a self-adjoint operator acting in the original Hilbert space $\mathcal{H}$ [**36, 2**]. Another way to define the perturbed operator is using the extension theory for symmetric operators. It is obvious that the perturbed and original operators coincide on the linear set of functions U satisfying the condition

$$\langle \varphi, U \rangle = 0. \tag{4}$$

Then the perturbed operator is an extension of the original operator restricted to this linear set. If $\varphi \in \mathcal{H}_{-1} \backslash \mathcal{H}$ the restricted operator is a symmetric operator with the deficiency indices $(1,1)$. Its self-adjoint extension corresponding to the formal expression (1) is uniquely defined. The resolvent of the perturbed operator can be described using Krein's formula [**20, 26**], which coincides with formula (3) in this case.

The case $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$ has to be treated using the extension theory for symmetric operators, since the perturbation is not form bounded with respect to the original operator. The restricted symmetric operator can be defined in a way similar to $\mathcal{H}_{-1}$-case. But the perturbed operator is not uniquely defined anymore. One can only conclude that the perturbed operator is equal to one of the self-adjoint extensions of the restricted operator. All such operators can be parametrized by one real parameter $\gamma \in \mathbf{R} \cup \{\infty\}$ as follows

$$\frac{1}{A^\gamma - \lambda} = \frac{1}{A-\lambda} - \frac{1}{\gamma + \langle \varphi, \frac{1+\lambda}{A-\lambda}\frac{1}{A+1}\varphi \rangle} \left\langle \frac{1}{A-\bar{\lambda}}\varphi, \cdot \right\rangle \frac{1}{A-\lambda}\varphi. \tag{5}$$

The relation between the real parameter γ describing the self-adjoint extensions of the restricted operator and the additive real parameter α appearing in formula (1) cannot be established without additional assumptions like homogeneity of the original operator and the perturbation vector.[2] The Nevanlinna function $Q(\lambda) = \langle \varphi, \frac{1+\lambda}{A-\lambda}\frac{1}{A+1}\varphi \rangle$ can be considered as a regularization of the resolvent $\langle \varphi, \frac{1}{A-\lambda}\varphi \rangle$ which is not defined in the case of $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$

$$Q(\lambda) = \langle \varphi, \frac{1+\lambda}{A-\lambda}\frac{1}{A+1}\varphi \rangle \overset{\text{formally}}{=} \langle \varphi, \frac{1}{A-\lambda}\varphi \rangle - \langle \varphi, \frac{1}{A+1}\varphi \rangle. \tag{6}$$

Observe that the two scalar products appearing in the right hand side of the last formula are not defined for $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, but their difference is in contrast well defined. The Nevanlinna function $\langle \varphi, \frac{1+A\lambda}{A-\lambda}\frac{1}{A^2+1}\varphi \rangle$ just coincides with Krein's Q-function appearing in the formula for the difference between the resolvents of two different self-adjoint extensions of one symmetric operator with the deficiency indices $(1,1)$ [**20, 26**].

The next step is to consider $\varphi \in \mathcal{H}_{-3}$. The restriction defined by (4) is defined only if one considers the original operator A as an operator acting in the Hilbert space $\mathcal{H}_1$. Then the domain of the unperturbed operator A coincides with the space $\mathcal{H}_3$ and the restriction (4) determines a symmetric operator. From another hand formula (3) is valid only if one considers the

[2] This approach has been developed in [**2, 3**].

extended Hilbert space containing vectors $\frac{1}{A-\lambda}\varphi \in \mathcal{H}_{-1}$. It appears that such extension is in fact one-dimensional, since for arbitrary $\lambda, \mu \notin \mathbf{R}_+$ the following inclusion is valid

$$\frac{1}{A-\lambda}\varphi - \frac{1}{A-\mu}\varphi = (\varphi - \mu)\frac{1}{(A-\lambda)(A-\mu)}\varphi \in \mathcal{H}_1.$$

Hence it is enough to include the one dimensional subspace generated by the vector $\frac{1}{A+1}\varphi$ only. Hence the perturbed operator can be defined in the Hilbert space $\mathbf{H}_{-3} = \mathcal{H}_1 \oplus \mathbf{C}$ equipped with the natural embedding ρ_{-3}

$$\begin{array}{ll} \rho_{-3}: & \mathbf{H}_{-3} \to \mathcal{H}_{-1} \\ & \mathbf{U} = (U, u_1) \mapsto U + u_1 \frac{1}{A+1}\varphi. \end{array} \tag{7}$$

The perturbed operator corresponding to the formal expression (1) has been constructed in [**22**] by first defining certain maximal operator acting in $\mathbf{H}$ and then restricting it to a self-adjoint operator. The maximal operator is similar to the adjoint operator appearing in the restriction-extension procedure used to construct $\mathcal{H}_{-2}$-perturbations. The set of self-adjoint restrictions of the maximal operator are described by one real parameter. Therefore formula (1) does not determine the perturbed operator uniquely, but a one parameter family of operators like in the case of $\mathcal{H}_{-2}$-perturbations. The resolvent of the perturbed operator restricted to the original Hilbert space is given by the formula

$$\rho \frac{1}{\mathbf{A}_\theta - \lambda}|_{\mathcal{H}_1} = \frac{1}{A-\lambda} - \frac{1}{(\lambda+1)\cot\theta + \langle \varphi, \frac{1}{A-\lambda}\frac{(\lambda+1)^2}{(A+1)^2}\varphi\rangle - 1} \left\langle \frac{1}{A-\lambda}\varphi, \cdot \right\rangle \frac{1}{A-\lambda}\varphi, \tag{8}$$

where $\theta \in [0, \pi)$ is the real number parametrizing the restrictions. The similarity between formulas (3) and (8) is obvious. The function

$$\begin{array}{rcl} Q(\lambda) & = & \langle \varphi, \dfrac{1}{A-\lambda}\dfrac{(\lambda+1)^2}{(A+1)^2}\varphi\rangle \\ & \overset{\text{formally}}{=} & \langle \varphi, \dfrac{1}{A-\lambda}\varphi\rangle - \langle \varphi, \dfrac{1}{A+1}\varphi\rangle - (\lambda+1)\langle \varphi, \dfrac{1}{(A+1)^2}\varphi\rangle \end{array} \tag{9}$$

is a double regularization of the resolvent function. This function describes the spectral properties of the self-adjoint perturbed operator.

Supersingular perturbation from the class $\mathcal{H}_{-4}$ have been studied in [**23**]. Our original aim was simply to generalize the ideas developed in [**22**] to the case of more singular perturbations. The main difference with the case $\varphi \in \mathcal{H}_{-3}$ is that the original operator A should be considered as an operator acting in the Hilbert space $\mathcal{H}_2$ from the scale of Hilbert spaces. Moreover this Hilbert space should be extended to include not only the vector

$$g_1 = \frac{1}{A+1}\varphi \in \mathcal{H}_{-2}$$

but the vector

$$g_2 = \frac{1}{(A+1)^2}\varphi \in \mathcal{H}$$

as well. Hence one has to consider the Hilbert space

$$\mathbf{H}_{-4} = \mathcal{H}_2 \oplus \mathbf{C}^2 \tag{10}$$

equipped with the standard imbedding

$$\begin{array}{rl} \rho_{-4}: & \mathbf{H}_{-4} \to \mathcal{H}_{-2} \\ & \mathbf{U} = (U, u_2, u_1) \mapsto U + u_2 \frac{1}{(A+1)^2} + u_1 \frac{1}{A+1}\varphi. \end{array} \tag{11}$$

The maximal operator can be defined in the way similar to $\mathcal{H}_{-3}$-perturbations. The main difference is that any symmetric restriction of the maximal operator is not self-adjoint. Hence no self-adjoint operator corresponds to formal expression (1). Instead one can consider the restrictions of the maximal operator that are regular operators.[3] All such restrictions are parametrized by one real parameter in the way similar to $\mathcal{H}_{-3}$ perturbations. The real and imaginary parts of these operators were calculated explicitly. The resolvent of the perturbed operator is also calculated and it is shown that the spectrum of the perturbed regular operator is pure real. The resolvent restricted to the original Hilbert space is given by the formula similar to Krein's formula (5). All spectral properties of the perturbed operator are described by the Nevanlinna function Q given by

$$\begin{aligned} Q_{-4}(\lambda) \quad &= \quad \langle \varphi, \frac{1}{A-\lambda}\frac{(\lambda+1)^4}{(A+1)^4}\varphi\rangle \\ &\overset{\text{formally}}{=} \quad \langle \varphi, \frac{1}{A-\lambda}\varphi\rangle - \langle \varphi, \frac{1}{A+1}\varphi\rangle - (\lambda+1)\langle \varphi, \frac{1}{(A+1)^2}\varphi\rangle \qquad (12) \\ &\quad -(\lambda+1)^2\langle \varphi, \frac{1}{(A+1)^3}\varphi\rangle - (\lambda+1)^3\langle \varphi, \frac{1}{(A+1)^4}\varphi\rangle \end{aligned}$$

The aim of the current paper is to generalize the ideas of [**23**] to the case of arbitrary supersingular perturbations from the class $\mathcal{H}_{-n}$, $n \geq 4$.

[3]We call a densely defined operator B **regular** if its domain coincides with the domain of the adjoint operator.

3. The extended Hilbert space and the maximal operator.

The operator corresponding to the formal expression (1) will be constructed as a restriction of a certain maximal operator acting in a certain extended Hilbert space. The extended Hilbert space and the maximal operator are described in the current section. To avoid not essential discussion we limit our consideration to the case where $\varphi \in \mathcal{H}_{-n} \setminus \mathcal{H}_{-n+1},\ n \geq 4$.

Following the ideas expressed in Section 2, consider the Hilbert space $\mathbf{H} \equiv \mathbf{H}_{-n} = \mathbf{C}^{(n-2)} \oplus \mathcal{H}_{n-2}$ equipped with the scalar product

$$\begin{aligned} \ll \mathbf{U}, \mathbf{V} \gg &= \bar{u}_1 v_1 + \ldots + \bar{u}_{n-3} v_{n-3} + \bar{u}_{n-2} v_{n-2} + \langle U, V \rangle_{\mathcal{H}_{n-2}} \\ &= \langle \vec{u}, \vec{v} \rangle_{\mathbf{C}^{n-2}} + \langle U, (1+A)^{(n-2)} V \rangle, \end{aligned} \tag{13}$$

where we used the following vector notation

$$\vec{u} = (u_1, u_2, \ldots, u_{n-2}).$$

Different scalar products can be defined in the vectors space $\mathbf{H}$. The simplest case is considered in the current paper in order to avoid not necessary complications. The general case will be studied in one of the following publications.

The space $\mathbf{H}$ can be embedded into the space $\mathcal{H}_{-n+2}$ as follows

$$\begin{aligned} \rho \mathbf{U} &= u_1 g_1 + u_2 g_2 + \ldots + u_{n-2} g_{n-2} + U \\ &= \sum_{k=1}^{n-2} u_k g_k + U, \end{aligned} \tag{14}$$

where the vectors $g_k, k = 1, 2, \ldots, n-2$ are defined by

$$g_0 = \varphi, \quad g_k = \frac{1}{A+1} g_{k-1} = \frac{1}{(A+1)^k} \varphi,\ k = 1, 2, \ldots, n-2. \tag{15}$$

Note that the embedding operator ρ depends on the order n.

The operator A can be considered as an operator acting in the scale of Hilbert spaces. Let us remind that the spaces $\mathcal{H}_s \subset \mathcal{H} \subset \mathcal{H}_{-s}$ form Gelfand triplet for any $s = 1, 2, \ldots$: the original space $\mathcal{H}$ is Hilbert and $\mathcal{H}_s^* = \mathcal{H}_{-s}$ with respect to the original scalar product in $\mathcal{H}$.

Consider arbitrary Gelfand triplet $\mathcal{K} \subset \mathcal{H} \subset \mathcal{K}^*$. Let B be a densely defined operator in the space $\mathcal{K}$ then the **triplet adjoint** operator $B^\dagger$ acting in $\mathcal{K}^*$ is defined on the domain

$$\mathrm{Dom}\,(B^\dagger) = \{ f \in \mathcal{K}^* : g \in \mathrm{Dom}\,(B) \Rightarrow |\langle Bg, f \rangle| \leq C_f \parallel g \parallel_{\mathcal{H}_s} \}$$

by the following equality

$$\langle Bg, f \rangle = \langle g, B^\dagger f \rangle.$$

Note that the scalar product appearing in the last definition is the scalar product in the original Hilbert space $\mathcal{H}$. The triplet adjoint operator coincides with the standard adjoint operator in the case $\mathcal{K} = \mathcal{H} = \mathcal{K}^*$. Otherwise the triplet adjoint operator is different from the adjoint operator B^* - operator adjoint to B considered as an operator in the Hilbert space $\mathcal{H} \supset \mathcal{K}$.

Really consider the restriction $A_{\mathcal{H}_{n-2}}$ of the operator A to the Hilbert space $\mathcal{H}_{n-2}$. This operator is a self-adjoint operator in this Hilbert space with the domain $\mathrm{Dom}\,(A_{\mathcal{H}_{n-2}}) = \mathcal{H}_n$. The triplet adjoint operator $A^\dagger_{\mathcal{H}_{n-2}}$ coincides with the extension of the operator A to the Hilbert space $\mathcal{H}_{-n+2}$ (the space adjoint to $\mathcal{H}_{n-2}$ with respect to the original scalar product). The domain of the triplet adjoint operator coincides with the space $\mathcal{H}_{-n+4}$

$$\mathrm{Dom}\,(A^\dagger_{\mathcal{H}_{n-2}}) = \mathcal{H}_{-n+4}.$$

Summing up we conclude that the triplet adjoint operator to $A_{\mathcal{H}_{n-2}}$ with respect to the triplet $\mathcal{H}_{n-2} \subset \mathcal{H} \subset \mathcal{H}_{-n+2}$ coincides with the operator $A_{\mathcal{H}_{-n+2}}$.

We define the minimal operator $A_{\min}$ corresponding to the formal expression (1) as the restriction of the operator $A_{\mathcal{H}_{n-2}}$ to the domain of function orthogonal to φ

$$\mathrm{Dom}\,(A_{\min}) = \{\psi \in \mathcal{H}_n : \langle \varphi, \psi \rangle = 0\}. \tag{16}$$

Then the maximal operator $A_{\max}$ coincides with the triplet adjoint operator to $A_{\min}$ with respect to the triplet $\mathcal{H}_{n-2} \subset \mathcal{H} \subset \mathcal{H}_{-n+2}$

$$A_{\max} = A^\dagger_{\min}. \tag{17}$$

Lemma 1. *The maximal operator $A_{\max}$ is defined on the domain*

$$\mathrm{Dom}\,(A_{\max}) = \left\{ f = \tilde{f} + f_1 g_1 \in \mathcal{H}_{-n+2}, \tilde{f} \in \mathcal{H}_{-n+4}, f_1 \in \mathbf{C} \right\}. \tag{18}$$

by the following formula

$$A^\dagger(\tilde{f} + f_1 g_1) = A\tilde{f} - f_1 g_1. \tag{19}$$

Remark. In the case $n = 2$ the minimal operator $A_{\min}$ is a symmetric operator in the original Hilbert space having the domain $\mathrm{Dom}\,(A_{\min}) = \{\psi \in \mathrm{Dom}\,(A) = \mathcal{H}_2(A) : \langle \varphi, \psi \rangle = 0\}$. Then the maximal operator $A_{\max}$ coincides with the usual adjoint operator to $A_{\min}$ with the domain given by (18). The action of the adjoint operator is given by (19).

Proof. The domain of the triplet adjoint operator $A^\dagger_{\min}$ consists of all elements $f \in \mathcal{H}_{-n+2}$ such that the sesquilinear form $\langle (A+1)\psi, f \rangle = \langle \psi, (A+1)f \rangle$ can be estimated as follows

$$|\langle \psi, (A+1)f \rangle| \leq C_f \parallel \psi \parallel_{\mathcal{H}_{n-2}}$$

for all $\psi \in \mathrm{Dom}\,(A_{\min})$, since the operator $A_{\min}$ is a restriction of the operator A. The last estimate holds for all $\psi \in \mathcal{H}_n$, $\langle \psi, \varphi \rangle = 0$ if and only if

$$(A+1)f = \hat{f} + f_1 \varphi,$$

where $\hat{f} \in \mathcal{H}_{-n+2}$, $f_1 \in \mathbf{C}$. It follows that the function f possesses representation (18).

Suppose now that representation (18) holds. Then the sesquilinear form can be written as follows

$$\begin{aligned} \langle (A+1)\psi, f \rangle &= \langle (A+1)\psi, \tilde{f} \rangle + \langle (A+1)\psi, \frac{1}{A+1}\varphi \rangle \\ &= \langle \psi, (A+1)\tilde{f} \rangle + 0. \end{aligned}$$

It follows that

$$(A+1)^\dagger (\tilde{f} + f_1 g_1) = (A+1)\tilde{f}$$

and hence (19) holds. The lemma is proven. □

The operator $A_{\max}$ will be used to define the maximal operator acting in the extended Hilbert space $\mathbf{H}$.

Definition 2. The **maximal operator $\mathbf{A_{max}}$** acting in the Hilbert space $\mathbf{H} \subset \mathcal{H}_{-n+2}$ is the restriction of the operator $A_{\max}$ to the Hilbert space $\mathbf{H}$ defined by the following equality

$$A_{\max} \rho = \rho \mathbf{A_{max}} \tag{20}$$

The following lemma describes in details the maximal operator $\mathbf{A_{max}}$.

Lemma 3. *The maximal operator $\mathbf{A_{max}}$ determined by Definition 2 is defined on the domain*

$$\begin{aligned} \mathrm{Dom}(\mathbf{A_{max}}) \;=\; & \{\mathbf{U} = (u_1, u_2, \dots, u_{n-2}, U_r + u_{n-1} g_{n-1}), \\ & u_1, u_2, \dots, u_{n-2}, u_{n-1} \in \mathbf{C}, U_r \in \mathcal{H}_n\} \end{aligned} \tag{21}$$

by the formula

$$\mathbf{A_{max}} \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_{n-2} \\ U_r + u_{n-1} g_{n-1} \end{pmatrix} = \begin{pmatrix} u_2 - u_1 \\ u_3 - u_2 \\ \dots \\ u_{n-1} - u_{n-2} \\ AU_r - u_{n-1} g_{n-1} \end{pmatrix}. \tag{22}$$

Proof. Consider any vector $\mathbf{U}$ from the domain of the operator $\mathbf{A_{max}}$ and let us denote its image by $\mathbf{W} = (w_1, w_2, \dots, w_{n-2}, W)$. Then equality (20) can be written as follows

$$\begin{aligned} & w_1 g_1 + w_2 g_2 + \ldots + w_{n-2} g_{n-2} + W \\ =& (u_2 - u_1) g_1 + (u_3 - u_2) g_2 + \ldots + (u_{n-2} - u_{n-3}) g_1 + AU\varphi. \end{aligned} \tag{23}$$

We conclude that

$$\begin{array}{ccc} w_1 & = & u_2 - u_1; \\ w_2 & = & u_3 - u_2; \\ \ldots & & \\ w_{n-3} & = & u_{n-2} - u_{n-3}; \\ W + w_{n-2} g_{n-2} & = & AU - u_{n-2} g_{n-2}. \end{array} \tag{24}$$

The last equality can be written as

$$W + U + v_{n-2} g_{n-2} + u_{n-2} g_{n-2} = (A+1) U$$

and therefore

$$U = \frac{1}{A+1}(W+U) + (w_{n-2} + u_{n-2}) \frac{1}{A+1} g_{n-2}.$$

It follows that the element U possesses the following representation

$$U = U_r + u_{n-1} g_{n-1},$$

where $U_r \in \mathcal{H}_n$, $u_{n-1} \in \mathbf{C}$. Then equality (23) can be written as

$$\begin{aligned} & w_1 g_1 + w_2 g_2 + \ldots + w_{n-2} g_{n-2} + W \\ = & (u_2 - u_1) g_1 + (u_3 - u_2) g_2 + \ldots + (u_{n-1} - u_{n-2}) g_{n-2} + AU_r - u_{n-1} g_{n-1}, \end{aligned}$$

and one can deduce that formula (22) holds. The Lemma is proven. $\square$

The spectrum of the operator $\mathbf{A_{max}}$ covers the whole complex plane. Really consider any complex number λ. Then the element

$$\mathbf{U} = \begin{pmatrix} 1 \\ (1+\lambda) \\ \ldots \\ (1+\lambda)^{n-3} \\ \frac{(1+\lambda)^{n-1}}{A-\lambda} g_{n-1} + (1+\lambda)^{n-2} g_{n-1} \end{pmatrix} = \begin{pmatrix} 1 \\ (1+\lambda) \\ \ldots \\ (1+\lambda)^{n-3} \\ (1+\lambda)^{n-2} \frac{A+1}{A-\lambda} g_{n-1} \end{pmatrix}$$

solves the equation $\mathbf{A_{max}U} = \lambda \mathbf{U}$. Note that the last formula reads as follows in the special case $\lambda = -1$

$$\mathbf{A_{max}} \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix} = - \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix}.$$

Let us calculate the adjoint operator $\mathcal{A}_{\mathbf{min}} = \mathbf{A}^*_{\mathbf{max}}$. Note that the operator $\mathbf{A_{min}}$ is different from the minimal operator $A_{\min}$ considered earlier. In fact the operator $A_{\min}$ is a restriction of the operator $\mathbf{A_{min}}$.

LEMMA 4. *The operator* $\mathbf{A_{min}}$*, adjoint to* $\mathbf{A_{max}}$ *in* $\mathbf{H}$*, is defined on the domain*

$$\begin{array}{rcl} \mathrm{Dom}\,(\mathbf{A_{min}}) & = & \{\mathbf{U} = (u_1, u_2, \dots, u_{n-2}, U_r); u_1, u_2, \dots, u_{n-2} \in \mathbf{C}, \\ & & U_r \in \mathcal{H}_n,\, u_{n-2} = \langle \varphi, U_r \rangle\} \end{array} \tag{25}$$

by the formula

$$\mathbf{A_{min}} \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_{n-2} \\ U_r \end{pmatrix} = \begin{pmatrix} -u_1 \\ u_1 - u_2 \\ \dots \\ u_{n-3} - u_{n-2} \\ AU_r \end{pmatrix}. \tag{26}$$

PROOF. Consider arbitrary elements $\mathbf{U} \in \mathrm{Dom}\,(\mathbf{A_{max}})$ and $\mathbf{V} = (v_1, v_2, \dots, v_{n-2}, V) \in \mathbf{H}$. The sesquilinear form of the operator $\mathbf{A_{max}}$ is

$$\begin{array}{rcl} \ll (\mathbf{A_{max}} + 1)\mathbf{U}, \mathbf{V} \gg & = & \bar{u}_2 v_1 + \bar{u}_3 v_2 + \dots + \bar{u}_{n-1} v_{n-2} \\ & & + \langle (A+1)U_r, (1+A)^{n-2} V \rangle \\ & = & \bar{u}_2 v_1 + \bar{u}_3 v_2 + \dots + \bar{u}_{n-2} v_{n-3} \\ & & + \bar{u}_{n-1} \left\{ v_{n-2} - \langle g_{n-1}, (1+A)^{n-1} V \rangle \right\} \\ & & + \langle U_r + u_{n-1} g_{n-1}, (1+A)^{n-1} V \rangle. \end{array} \tag{27}$$

Consider first the subset of elements $\mathbf{U} \in \mathrm{Dom}\,(\mathbf{A})$ with

$$u_k = 0, k = 1, 2, \dots, n-1.$$

Then the last term in (27) is a bounded functional with respect to $\mathbf{U} \in \mathbf{H}$ if and only if $V = V_r \in \mathcal{H}_n$. Consider next arbitrary $\mathbf{U} \in \mathrm{Dom}\,(\mathbf{A})$. Since the functional $\mathbf{U} \mapsto u_{n-1}$ is not bounded in the norm of $\mathbf{H}$, the last formula determines bounded linear functional if and only if the expression in $\{\quad\}$ vanishes, i.e.

$$v_{n-2} = \langle g_{n-1}, (1+A)^{n-1} V_r \rangle \equiv \langle \varphi, V_r \rangle. \tag{28}$$

Hence the domain of $\mathbf{A_{min}}$ is formed by the elements possessing representation

$$\mathbf{V} = (v_1, v_2, \dots, v_{n-2}, V_r), \quad V_r \in \mathcal{H}_n, v_k \in \mathbf{C}$$

and satisfying (28). Taking into account these relations formula for the adjoint operator can be written as follows

$$\begin{aligned} \ll \mathbf{A_{max}} \mathbf{U}, \mathbf{V} \gg &= \ll \mathbf{U}, \mathbf{A_{min}} \mathbf{V} \gg \\ &= \bar{u}_1 v_1 + \bar{u}_2 (v_1 - v_2) + \ldots + \bar{u}_{n-2} (v_{n-3} - v_{n-2}) \\ &\quad + \langle U_r + u_{n-1} g_{n-1}, (1+A)^{n-2} A V_r \rangle. \end{aligned}$$

It follows that the action of the minimal operator is given by (26). The Lemma is proven. □

The operator $\mathbf{A_{min}}$ is an extension of the operator $A_{\rm min}$. Really the domain of $A_{\rm min}$ defined by (16) belongs to Dom $(\mathbf{A_{min}})$ and moreover $A_{\rm min} = \mathbf{A_{min}}|_{{\rm Dom}(A_{\rm min})}$.[4]

The domain of the minimal operator $\mathbf{A_{min}}$ is contained in the domain of the maximal operator $\mathbf{A_{max}}$, but the minimal operator does not coincide with the restriction of the maximal operator to the domain of the minimal one. Therefore no restriction of the operator $\mathbf{A_{max}}$ is self-adjoint like in the case of $\mathcal{H}_{-3}$-perturbations [**23**]. Therefore no self-adjoint can be associated with the formal operator (1). One can proceed now along two possible lines:
1. Construct non self-adjoint operator corresponding to (1).
2. Consider the maximal common symmetric restriction of the operators $\mathbf{A_{min}}$ and $\mathbf{A_{max}}$ and describe all its self-adjoint extensions.

We decided to follow the first possibility, since the resolvent of the operator obtained in this way is given by formula (45) similar to Krein's formula for generalized resolvents. The second approach is described in [**24**].

Let us have another look at the extension problem for $\mathbf{A_{min}}$. Since the operator $\mathbf{A_{max}}$ is closed, it coincides with the operator adjoint to $\mathbf{A_{min}}$. Hence all possible self-adjoint extensions of the operator $\mathbf{A_{min}}$ are described by Lagrangian planes of the symplectic boundary form of $\mathbf{A_{max}}$.

LEMMA 5. *The boundary form of the maximal operator* $\mathbf{A}$ *is given by*

$$\ll \mathbf{A_{max}} \mathbf{U}, \mathbf{V} \gg - \ll \mathbf{U}, \mathbf{A_{max}} \mathbf{V} \gg$$

[4] The operators $\mathbf{A_{min}}$ and $A_{\rm min}$ coincide only in the cases $n = 1, 2$.

$$= \left\langle \begin{pmatrix} 0 & 1 & \dots & 0 & 0 & 0 \\ -1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & -1 & 0 & 1 \\ 0 & 0 & \dots & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_{n-2} \\ u_{n-3} \\ \langle \varphi, U_r \rangle \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_{n-2} \\ v_{n-3} \\ \langle \varphi, V_r \rangle \end{pmatrix} \right\rangle \tag{29}$$

PROOF. The following straightforward calculations prove the Lemma

$$\ll \mathbf{A_{max} U}, \mathbf{V} \gg - \ll \mathbf{U}, \mathbf{A_{max} V} \gg$$

$$= \ll \begin{pmatrix} u_2 - u_1 \\ u_3 - u_2 \\ \dots \\ u_{n-1} - u_{n-2} \\ AU_r - u_{n-1} g_{n-1} \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_{n-2} \\ V_r + v_{n-1} g_{n-1} \end{pmatrix} \gg$$

$$- \ll \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_{n-2} \\ U_r + u_{n-1} g_{n-1} \end{pmatrix}, \begin{pmatrix} v_2 - v_1 \\ v_3 - v_2 \\ \dots \\ v_{n-1} - v_{n-2} \\ AV_r - v_{n-1} g_{n-1} \end{pmatrix} \gg$$

$$= +\bar{u}_2 v_1 + \bar{u}_3 v_2 + \dots + \bar{u}_{n-1} v_{n-2} - \bar{u}_1 v_2 - \bar{u}_2 v_3 - \dots - \bar{u}_{n-2} v_{n-1}$$

$$+ \langle U_r, \varphi \rangle v_{n-1} - \bar{u}_{n-1} \langle \varphi, V_r \rangle.$$

We have used that $\varphi = (1+A)^{n-1} g_{n-1}$ in these calculations. The Lemma is proven. □

The matrix describing the boundary form

$$\mathbf{B} \equiv \begin{pmatrix} 0 & -1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & -1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & -1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

is symplectic and has rank n for even n and $n-1$ for odd n. Hence any symmetric restriction of the operator $\mathbf{A_{max}}$ is described by at least $[\frac{n}{2}]$ boundary conditions.[5] Such restriction cannot be self-adjoint, since the kernel of

[5] $[\cdot]$ denotes the integer part here.

the adjoint operator for nonreal λ is one dimensional. We have proven another one time that no restriction of the operator $\mathbf{A_{max}}$ is self-adjoint in the Hilbert space $\mathbf{H}$ and no self-adjoint operator corresponds to formal expression (1) in the case $\varphi \in \mathcal{H}_{-n} \setminus \mathcal{H}_{-n+1},\ n \geq 4$. To define a non self-adjoint operator corresponding to this formal expression the class of regular operators will be introduced in the following section.

If $n = 3$, then the rank of the matrix $\mathbf{B}$ is 2 and all Lagrangian planes of the boundary form are described by one condition. Thus the restrictions of $\mathbf{A_{max}}$ to the corresponding subspaces are self-adjoint operators [**22**].

4. Regular operators and supersingular perturbation of self-adjoint operators.

The operator corresponding to the formal expression (1) is a certain restriction of the maximal operator. In this section we are going to study the set of regular restrictions of the maximal operator.

Definition 6. Densely defined operator B is called **regular** if its domain coincides with the domain of the adjoint operator.

The set of regular operator contains all self-adjoint operators. The class of self-adjoint operators can be characterized by one additional restriction: the operator B is self-adjoint if it is regular and symmetric. Obviously the set of regular operators extends the set of self-adjoint operators enormously.

All regular restrictions of the maximal operator are characterized by the following theorem.

Theorem 7. *Let* $\mathbf{A}$ *be a regular restriction of the maximal operator* $\mathbf{A_{max}}$*. Then there exists a three dimensional nonzero real vector* $(a, b, c) \in \mathbf{R}^3$ *orthogonal to the vector* $(1, 0, 1)$ *such that the operator* $\mathbf{A}$ *coincides with the restriction of the maximal operator to the domain*

$$a\langle \varphi, U_r \rangle + b u_{n-1} + c u_{n-2} = 0. \tag{30}$$

Proof. The domain $\mathrm{Dom}\,(\mathbf{A_{max}})$ of the maximal operator contains the domain $\mathrm{Dom}\,(\mathbf{A_{min}})$ of its adjoint. Therefore every regular restriction of $\mathbf{A_{max}}$ is an extension of $\mathbf{A_{min}}$. This picture is similar to the extension theory of symmetric operators. The domain $\mathrm{Dom}\,(\mathbf{A_{max}})$ consists of all elements $\mathbf{U} \in \mathbf{H}$ possessing the representation

$$\mathbf{U} = (u_1, u_2, \dots, u_{n-2}, U_r + u_{n-1} g_{n-1}),$$

where $U_r \in \mathcal{H}_n,\ u_k \in \mathbf{C}, k = 1, 2, \dots, n-1$. The domain $\mathrm{Dom}\,(\mathbf{A_{min}})$ of the adjoint operator is a subdomain of $\mathrm{Dom}\,(\mathbf{A_{max}})$ characterized by the boundary conditions

$$\begin{cases} u_{n-1} = 0, \\ u_{n-2} = \langle \varphi, U_r \rangle. \end{cases}$$

Thus the dimension of the quotient space Dom $(\mathbf{A_{max}})$/Dom $(\mathbf{A_{min}})$ is equal to 2. Any linear subset D of Dom $(\mathbf{A_{max}})$ which does not coincide with Dom $(\mathbf{A_{min}})$ and Dom $(\mathbf{A_{max}})$ is described by the boundary conditions of the form

$$a\langle \varphi, U_r \rangle + b u_{n-1} + c u_{n-2} = 0, \tag{31}$$

where (a, b, c) is a three dimensional nonzero complex vector. (If all constants a, b, and c are equal to zero, then the boundary condition (31) is satisfied by all functions from Dom $(\mathbf{A_{max}})$ and the linear subset coincides with the domain of maximal operator.)

Thus every operator $\mathbf{A}$ which is a regular restriction of $\mathbf{A_{max}}$ is characterized by the boundary conditions (31). It remains to study which boundary conditions lead to regular operators.

The sesquilinear form of the operator $\mathbf{A_{max}}|_{\mathbf{D}}$ is given again by formula (27), where now $\mathbf{U} \in$ D. Consider vectors $\mathbf{U}$ with $u_{n-2} = u_{n-1} = \langle \varphi, U_r \rangle = 0$. Then the scalar product

$$\langle U_r, (A+1)^{n-1} V \rangle$$

generates a bounded linear functional with respect to the vector $(0, 0, \dots, 0, U_r) \in \mathbf{H}$ and the standard norm in $\mathbf{H}$ if and only if the following representation holds[6]

$$(A+1)^{n-1} V = c\varphi + \tilde{f},$$

where $c \in \mathbf{C},\ \tilde{f} \in \mathcal{H}_{-n+2}$. This implies that

$$V = c g_{n-1} + \frac{1}{(A+1)^{n-1}} \tilde{f},$$

and it follows that the vector V possesses the representation

$$V = V_r + v_{n-1} g_{n-1},$$

where $V_r \in \mathcal{H}_n,\ \ v_{n-1} \in \mathbf{C}$. Then the sesquilinear form is given by

$$\begin{array}{rl} & \ll (\mathbf{A}+1)\mathbf{U}, \mathbf{V} \gg \\ = & \langle U_r + u_{n-1} g_{n-1}, (A+1)^{n-1} V_r \rangle - \bar{u}_{n-1} \langle \varphi, V_r \rangle + v_{n-1} \langle U_r, \varphi \rangle \\ & + \bar{u}_{n-1} v_{n-2} + \bar{u}_{n-2} v_{n-3} + \dots + \bar{u}_3 v_2 + \bar{u}_2 v_1. \end{array}$$

The domain of the operator adjoint to $\mathbf{A_{max}}|_{\mathbf{D}}$ is characterized by the condition that the last formula determines bounded linear functional with respect to $\mathbf{U} \in \mathbf{H}$. We are going consider all possible values of the parameters a, b and

[6] Remember that U_r is orthogonal to φ.

c and study the question whether the domain of the adjoint operator coincides with the domain of the restricted operator or not. Let us call **admissible** the three dimensional real vectors leading to regular restrictions of the maximal operator. The following three cases cover all possible values of the the parameters.

1. The general case:
$a \neq 0$**,** b **and** c **arbitrary.**
The boundary condition can be presented in the form

$$\langle \varphi, U_r \rangle = -\frac{b}{a} u_{n-1} - \frac{c}{a} u_{n-2}$$

and the sesquilinear form of the operator is given by

$$\begin{aligned} & \ll (\mathbf{A}+1)\mathbf{U}, \mathbf{V} \gg \\ = \; & \langle U_r + u_{n-1} g_{n-1}, (A+1)^{n-1} V_r \rangle + \bar{u}_{n-1} \left(-\langle \varphi, V_r \rangle - \frac{\bar{b}}{\bar{a}} v_{n-1} + v_{n-2} \right) \\ & + \bar{u}_{n-2} \left(v_{n-3} - \frac{\bar{c}}{\bar{a}} v_{n-1} \right) + \bar{u}_{n-3} v_{n-4} + \ldots + \bar{u}_3 v_2 + \bar{u}_2 v_1. \end{aligned}$$

The last expression determines a bounded linear functional if and only if the following relation holds

$$\bar{a} \langle \varphi, V_r \rangle + \bar{b} v_{n-1} - \bar{a} v_{n-2} = 0.$$

This condition coincides with (31) if and only if

$$a = -c$$

and the complex numbers a and b have the same phase. Hence without loss of generality the constants a, b, c can be chosen real and such that $c = -a$, i.e. $\langle (a,b,c), (1,0,1) \rangle = 0$. Any vector $(a,b,c) \in \mathbf{R}^3$, $(a,b,c) \perp (1,0,1)$, $a \neq 0$ is admissible.

2. The first special case:
$a = 0$, $b \neq 0$**,** c **arbitrary.**
The boundary condition takes the form

$$u_{n-1} = -\frac{c}{b} u_{n-2}. \tag{32}$$

Hence the sesquilinear form of the operator is given by

$$\begin{aligned} & \ll (\mathbf{A}+1)\mathbf{U}, \mathbf{V} \gg \\ = & \langle U_r + u_{n-1} g_{n-1}, (A+1)^{n-1} V_r \rangle + \bar{u}_{n-2} \left[\frac{\bar{c}}{\bar{b}} \left(\langle \varphi, V_r \rangle - v_{n-2} \right) + v_{n-3} \right] \\ & + \langle U_r, \varphi \rangle v_{n-1} + \bar{u}_{n-3} v_{n-4} + \ldots + \bar{u}_3 v_2 + \bar{u}_2 v_1. \end{aligned}$$

This form defines bounded linear functional with respect to $\mathbf{U} \in \mathbf{H}$ if and only if

$$v_{n-1} = 0,$$

since $\langle U_r, \varphi \rangle$ is not a bounded linear functional. The last condition coincides with (32) only if $c = 0$. Then condition (32) reads as follows

$$b u_{n-1} = 0.$$

Without loss of generality the constant b can be chosen real. Every vector $(0, b, 0)$ is orthogonal to $(1, 0, 1)$. Any vector $(0, b, 0) \in \mathbf{R}^3$ is admissible.

3. The second special case:
$a = 0,\ b = 0,\ c \neq 0.$
The boundary condition is

$$u_{n-2} = 0. \tag{33}$$

The sesquilinear form

$$\begin{aligned} \ll (\mathbf{A}+1)\mathbf{U}, \mathbf{V} \gg \; = \; & \langle U_r + u_{n-1} g_{n-1}, (A+1)^{n-1} V_r \rangle + \bar{u}_{n-1} \left(v_{n-2} - \langle \varphi, V_r \rangle \right) \\ & + v_{n-1} \langle U_r, \varphi \rangle + \bar{u}_{n-3} v_{n-4} + \ldots + \bar{u}_3 v_2 + \bar{u}_2 v_1. \end{aligned},$$

determines bounded linear functional if and only if the following conditions are satisfied

$$v_{n-1} = 0 \ \text{ and } \ v_{n-2} = \langle \varphi, V_r \rangle.$$

These conditions never coincide with the condition $u_{n-2} = 0$. Hence the boundary condition (33) never determine a regular restriction of the operator $\mathbf{A}$. No vector $(0, 0, c) \in \mathbf{R}^3$ is admissible.

Summing up our studies we conclude that the set of admissible vectors coincides with the set of three dimensional real nonzero vectors orthogonal to the vector $(1, 0, 1)$. The Theorem is proven. $\square$

The last theorem states that all regular restrictions of the operator $\mathbf{A_{max}}$ are described by real three dimensional vectors (a, b, c) subject to the orthogonality condition $(a, b, c) \perp (1, 0, 1)$. The length of the vector (a, b, c) plays no rôle and therefore all boundary conditions can be parametrized by one real parameter - "angle" $\theta \in [0, \pi)$ as follows:

$$\sin\theta\langle\varphi, U_r\rangle + \cos\theta u_{n-1} - \sin\theta u_{n-2} = 0. \tag{34}$$

The following definition will be used.

DEFINITION 8. The operator $\mathbf{A}_\theta$ is the restriction of the maximal operator $\mathbf{A}_{\mathbf{max}}$ to the set of functions satisfying boundary conditions (34).

The domain of the operator $\mathbf{A}_\theta$ is formed by the functions from Dom $(\mathbf{A}_{\mathbf{max}})$ (given by (21)) subject to the boundary conditions (34). The action of the operator $\mathbf{A}_\theta$ is given by (22).

Thus the regular operator corresponding to the formal expression (1) is not defined uniquely. Like in the case of $\mathcal{H}_{-2}$ and $\mathcal{H}_{-3}$-perturbations one parameter family of operators has been constructed.

Let us calculate the operator adjoint to $\mathbf{A}_\theta$. The domain of this operator coincides with the domain Dom $(\mathbf{A}_\theta)$. The sesquilinear form of the operator $\mathbf{A}_\theta$ can be presented by the following expression using the fact, that the functions from the domains of the operators $\mathbf{A}_\theta$ and $\mathbf{A}_\theta^*$ satisfy (34)

$$\begin{aligned} \ll (\mathbf{A}_\theta + 1)\mathbf{U}, \mathbf{V} \gg \; &= \; \langle U_r + u_{n-1}g_{n-1}, (A+1)^{n-1}V_r\rangle + \bar{u}_{n-2}\left(v_{n-1} + v_{n-3}\right) \\ &\quad + \sum_{k=1}^{n-4} \bar{u}_{k+1}v_k, \end{aligned}$$

and it follows that

$$(\mathbf{A}_\theta^* + 1)\begin{pmatrix} v_1 \\ v_2 \\ \cdots \\ v_{n-3} \\ v_{n-2} \\ V_r + v_{n-1}g_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ v_1 \\ \cdots \\ v_{n-4} \\ v_{n-1} + v_{n-3} \\ (A+1)V_r \end{pmatrix}.$$

Hence the action of the operator $\mathbf{A}_\theta^*$ is given by

$$\mathbf{A}_\theta^* \begin{pmatrix} v_1 \\ v_2 \\ \cdots \\ v_{n-3} \\ v_{n-2} \\ V_r + v_{n-1}g_{n-1} \end{pmatrix} = \begin{pmatrix} -v_1 \\ v_1 - v_2 \\ \cdots \\ v_{n-4} - v_{n-3} \\ v_{n-1} + v_{n-3} - v_{n-2} \\ AV_r - v_{n-1}g_{n-1} \end{pmatrix}. \tag{35}$$

The real and imaginary parts of the operator $\mathbf{A}_\theta$ are given by

$$\mathbf{A}_\theta = \Re\mathbf{A}_\theta + i\Im\mathbf{A}_\theta;$$

$$(\Re \mathbf{A}_\theta) \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_{n-3} \\ u_{n-2} \\ U_r + u_{n-1} g_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}u_2 - u_1 \\ \frac{1}{2}(u_3 + u_1) - u_2 \\ \dots \\ \frac{1}{2}(u_{n-2} + u_{n-4}) - u_{n-3} \\ u_{n-1} + \frac{1}{2}u_{n-3} - u_{n-2} \\ AU_r - u_{n-1} g_{n-1} \end{pmatrix};$$

$$\Im \mathbf{A}_\theta = \frac{1}{2} \begin{pmatrix} 0 & -i & 0 & \dots & 0 & 0 & 0 \\ i & 0 & -i & \dots & 0 & 0 & 0 \\ 0 & i & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & -i & 0 \\ 0 & 0 & 0 & \dots & i & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}. \tag{36}$$

The real part of the operator $\mathbf{A}_\theta$ is a self-adjoint operator on the domain $\mathrm{Dom}\,(\mathbf{A}_\theta)$. The imaginary part of $\mathbf{A}_\theta$ is a bounded self-adjoint operator, which does not depend on the parameter θ.

Let us study the operator $\mathbf{A}_0$ in more details. This operator is equal to the orthogonal sum of two operators acting in the spaces $\mathbf{C}^{n-2}$ and $\mathcal{H}_{n-2}$. Really the domain of the operator $\mathbf{A}_0$ can be decomposed as follows

$$\mathrm{Dom}\,(\mathbf{A}_0) = \mathbf{C}^{n-2} \oplus \mathcal{H}_n \subset \mathbf{C}^{n-2} \oplus \mathcal{H}_{n-2} \equiv \mathbf{H}.$$

The two operators appearing in the corresponding decomposition of the operator $\mathbf{A}_0$

$$\mathbf{A}_0 = \mathbf{T} \oplus A,$$

are the operator in $\mathbf{C}^{n-2}$ given by the upper triangular matrix

$$\mathbf{T} = \begin{pmatrix} -1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 \end{pmatrix}$$

and the operator A in $\mathcal{H}_{n-2}$ with the domain $\mathcal{H}_n$. The resolvent of the operator $\mathbf{A}_0$ for arbitrary nonreal λ can easily be calculated

$$\frac{1}{\mathbf{A}_0-\lambda} = \begin{pmatrix} \frac{-1}{1+\lambda} & \frac{-1}{(1+\lambda)^2} & \frac{-1}{(1+\lambda)^3} & \dots & \frac{-1}{(1+\lambda)^{n-2}} \\ 0 & \frac{-1}{(1+\lambda)} & \frac{-1}{(1+\lambda)^2} & \dots & \frac{-1}{(1+\lambda)^{n-3}} \\ 0 & 0 & \frac{-1}{(1+\lambda)} & \dots & \frac{-1}{(1+\lambda)^{n-4}} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{-1}{1+\lambda} \end{pmatrix} \oplus \frac{1}{A-\lambda}. \tag{37}$$

To prove that the spectrum of the operator $\mathbf{A}_\theta$ is real we calculate its resolvent for arbitrary nonreal value of λ.

Theorem 9. *The resolvent of the operator $\mathbf{A}_\theta$ for all nonreal λ is given by the $(n-1)\times(n-1)$ bounded matrix operator*

$$\frac{1}{\mathbf{A}_\theta-\lambda} = \frac{1}{\mathbf{A}_0-\lambda}$$
$$-\frac{\sin\theta}{D(\lambda)} \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{1}{(1+\lambda)^{n-1}} & \frac{1}{(1+\lambda)^{n-2}}\langle \frac{1}{A-\overline{\lambda}}\varphi, \cdot\rangle \\ 0 & 0 & \dots & 0 & \frac{1}{(1+\lambda)^{n-2}} & \frac{1}{(1+\lambda)^{n-3}}\langle \frac{1}{A-\overline{\lambda}}\varphi, \cdot\rangle \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \frac{1}{(1+\lambda)^2} & \frac{1}{1+\lambda}\langle \frac{1}{A-\overline{\lambda}}\varphi, \cdot\rangle \\ 0 & 0 & \dots & 0 & \frac{1}{1+\lambda}\frac{1}{A-\lambda}g_{n-2} & \left(\frac{1}{A-\lambda}g_{n-2}\right)\langle \frac{1}{A-\overline{\lambda}}\varphi, \cdot\rangle \end{pmatrix} \tag{38}$$

where the function $D(\lambda,\theta)$ is the following Nevanlinna function

$$D(\lambda,\theta) = \left(\langle\varphi, \frac{1+\lambda}{A-\lambda}g_{n-1}\rangle - \frac{1}{1+\lambda}\right)\sin\theta + \cos\theta. \tag{39}$$

PROOF. Consider arbitrary $\mathbf{F} = (f_1, f_2, \dots , f_{n-2}, F) \in \mathbf{H}$. Then the resolvent equation

$$(\mathbf{A} - \lambda) \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_{n-2} \\ U_r + u_{n-1} g_{n-1} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \dots \\ f_{n-2} \\ F \end{pmatrix} \tag{40}$$

together with the boundary condition (34) imply that

$$\begin{pmatrix} -(1+\lambda) & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -(1+\lambda) & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -(1+\lambda) & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -(1+\lambda) & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -\langle \varphi, \dfrac{1+\lambda}{A-\lambda} g_{n-1} \rangle & 1 \\ 0 & 0 & 0 & \dots & -\sin\theta & \cos\theta & \sin\theta \end{pmatrix}$$

$$\times \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \dots \\ u_{n-2} \\ u_{n-1} \\ \langle \varphi, U_r \rangle \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \dots \\ f_{n-2} \\ \langle \varphi, \frac{1}{A-\lambda} F \rangle \\ 0 \end{pmatrix}. \tag{41}$$

To derive the last equation we used the following transformation of the last equation in the system (40)

$$(A - \lambda) U_r - (1+\lambda) u_{n-1} g_{n-1} = F \tag{42}$$

$$\Rightarrow \langle \varphi, U_r \rangle - (1+\lambda) u_{n-1} \langle \varphi, \frac{1}{A-\lambda} g_{n-1} \rangle = \langle \varphi, \frac{1}{A-\lambda} F \rangle.$$

The determinant of the matrix appearing in the last equation is equal to $(-1)^{n-1}(1+\lambda)^{n-2} D(\lambda, \theta)$ and it vanishes for nonreal λ only if $D(\lambda,\theta) = 0$. The imaginary part of the function $\langle \varphi, \dfrac{1+\lambda}{A-\lambda} \dfrac{1}{(A+1)^{n-1}} \varphi \rangle - \dfrac{1}{1+\lambda}$ is given by

$$\Im \langle \varphi, \tfrac{1+\lambda}{A-\lambda} \tfrac{1}{(A+1)^{n-1}} \varphi \rangle - \tfrac{1}{1+\lambda} = y \left(\langle \varphi, \tfrac{(A+1)^2}{(A-x)^2+y^2} \tfrac{1}{(A+1)^n} \varphi \rangle + \tfrac{1}{(1+x)^2+y^2} \right),$$

where $\lambda = x + iy$, $x, y \in \mathbf{R}$. The imaginary part cannot vanish for nonreal values of λ if $\theta \neq 0$. In the case $\theta = 0$ the function $D(\lambda, 0) \equiv 1$ is constant.

We conclude that the linear system (41) has unique solution for all nonreal λ. It follows that the spectrum of the operator $\mathbf{A}_\theta$ is real.

To calculate the resolvent exactly consider the system of equations for $u_{n-2}, u_{n-1}, \langle \varphi, U_r \rangle$

$$\begin{pmatrix} -(1+\lambda) & 1 & 0 \\ 0 & -\langle \varphi, \dfrac{1+\lambda}{A-\lambda} g_{n-1} \rangle & 1 \\ -\sin\theta & \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} u_{n-2} \\ u_{n-1} \\ \langle \varphi, U_r \end{pmatrix} = \begin{pmatrix} f_{n-2} \\ \langle \varphi, \frac{1}{A-\lambda} F \rangle \\ 0 \end{pmatrix} \tag{43}$$

The solution to this linear system reads as follows

$$\begin{aligned} u_{n-2} &= -\frac{(\sin\theta \langle \varphi, \frac{1+\lambda}{A-\lambda} g_{n-1} \rangle + \cos\theta) f_{n-2} + \sin\theta \langle \varphi, \frac{1}{A-\lambda} F \rangle}{(1+\lambda) D(\lambda,\theta)}; \\ u_{n-1} &= -\frac{\big((1+\lambda)\langle \varphi, \frac{1}{A-\lambda} F \rangle + f_{n-2}\big) \sin\theta}{(1+\lambda) D(\lambda,\theta)}; \\ \langle \varphi, U_r \rangle &= -\frac{\sin\theta \langle \varphi, \frac{1+\lambda}{A-\lambda} g_{n-1} \rangle f_{n-2} + (\sin\theta - (1+\lambda)\cos\theta) \langle \varphi, \frac{1}{A-\lambda} F \rangle}{(1+\lambda) D(\lambda,\theta)}. \end{aligned} \tag{44}$$

Then all other components of the vector $\vec{u}$ can be calculated from the recursive relations

$$u_l = \frac{1}{1+\lambda} u_{l+1} - \frac{1}{1+\lambda} f_l, \quad l = 1, 2, \dots, n-3,$$

which coincide with the first $n-3$ equations of the system (41). The following formula holds

$$u_l = \frac{1}{(1+\lambda)^{n-2-l}} u_{n-2} - \sum_{m=l}^{n-3} \frac{1}{(1+\lambda)^{m+1-l}} f_m.$$

The component U can be calculated from (42)

$$\begin{aligned} U &= U_r + u_{n-1} g_{n-1} \\ &= \frac{1}{A-\lambda} F + (1+\lambda) u_{n-1} \frac{1}{A-\lambda} g_{n-1} + u_{n-1} g_{n-1} \\ &= \frac{1}{A-\lambda} F + u_{n-1} \frac{1}{A-\lambda} g_{n-2}. \end{aligned}$$

This completes the calculation of the resolvent of the operator $\mathbf{A}_\theta$ given by formula (38) for all nonreal λ. The Theorem is proven. □

The theorem implies that the spectrum of the operator $\mathbf{A}_\theta$ is real. Consider the restriction of the resolvent to the subspace $\mathcal{H}_{n-2} \subset \mathbf{H}$ combined with the embedding ρ

$$\rho \frac{1}{\mathbf{A}_\theta - \lambda}|_{\mathcal{H}_{n-2}} = \frac{1}{A-\lambda}$$
$$- \frac{1}{(\lambda+1)^{n-2}\left\{\cot\theta + \langle \varphi, \frac{1}{A-\lambda}\frac{1+\lambda}{(A+1)^{n-1}}\varphi\rangle - \frac{1}{1+\lambda}\right\}} \left\langle \frac{1}{A-\bar{\lambda}}\varphi, \cdot \right\rangle \frac{1}{A-\lambda}\varphi, \tag{45}$$

The last formula is analogous to Krein's formula connecting the resolvents of two self-adjoint extensions of one symmetric operator and is very similar to formula (8) describing the restricted resolvent of the self-adjoint operator corresponding to the singular $\mathcal{H}_{-3}$-perturbations. The main difference between this formula and well-known Krein's formula is that conventional Krein's formula describes the resolvent of self-adjoint operator, while the formula obtained comes from a certain non self-adjoint operator if the perturbation is singular enough $\varphi \in \mathcal{H}_{-n},\ n \geq 4$.

The last formula can be called **Krein's formula** for supersingular interactions. The spectral properties of the operator are described by the generalized Nevanlinna function

$$Q(\lambda) = (\lambda+1)^{n-2}\left\{\cot\theta + \langle \varphi, \frac{1+\lambda}{A-\lambda}\frac{1}{(A+1)^{n-1}}\varphi\rangle - \frac{1}{1+\lambda}\right\}. \tag{46}$$

The zeroes of this function determine the singularities of the resolvent. The function $\cot\theta + \langle \varphi, \frac{1+\lambda}{A-\lambda}\frac{1}{(A+1)^{n-1}}\varphi\rangle - \frac{1}{1+\lambda}$ is a standard Nevanlinna function tending to $-\infty$ and $+\infty$ when $\lambda \to -\infty$ and $\lambda \to -1^-$ respectively. Therefore the function has at least one zero in the interval $(-\infty, -1)$. Another one zero can be situated in the interval $(-1, 0)$ depending on the behavior of the function $\langle \varphi, \frac{1+\lambda}{A-\lambda}\frac{1}{(A+1)^{n-1}}\varphi\rangle$ at the origin and the coupling parameter θ. Let $\lambda_0 < 0,\ \ \lambda_0 \neq -1$ be a zero of the function $Q(\lambda)$. Then the vector

$$\begin{pmatrix} \frac{1}{(1+\lambda_0)^{n-2}} \\ \frac{1}{(1+\lambda_0)^{n-3}} \\ \cdots \\ \frac{1}{1+\lambda_0} \\ \frac{1}{A-\lambda_0}\frac{1}{(A+1)^{n-2}}\varphi \end{pmatrix} \tag{47}$$

is an eigenvector of the operator $\mathbf{A}_\theta$ corresponding to the eigenvalue λ_0. The point $\lambda = -1$ is an eigenvalue of the operator $\mathbf{A}_\theta$ with the eigenvector

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The function $Q_n(\lambda) = \left\langle \varphi, \frac{1}{A-\lambda}\frac{(\lambda+1)^{n-1}}{(A+1)^{n-1}}\varphi \right\rangle$ is an $n-1$-times regularized resolvent function

$$Q(\lambda) \stackrel{\text{formally}}{=} \langle \varphi, \frac{1}{A-\lambda}\varphi\rangle - \langle \varphi, \frac{1}{A+1}\varphi\rangle - \langle \varphi, \frac{\lambda+1}{(A+1)^2}\varphi\rangle - \ldots - \langle \varphi, \frac{(1+\lambda)^{n-2}}{(A+1)^{n-1}}\varphi\rangle.$$

5. Conclusions.

Rank one singular perturbations of self-adjoint determined by arbitrary vectors from the class $\mathcal{H}_{-n}$ have been defined in this article. It has been shown that such operators can be defined in the class of non self-adjoint operators acting in a certain extended Hilbert space. The final operator obtained is nevertheless close to a self-adjoint one - the imaginary part of the operator is a bounded operator. It has been proven that the spectrum of the perturbed operator is pure real. It remains to study in more details the spectral properties of the operator obtained. It is not clear whether the operator constructed is similar to a certain self-adjoint one. These questions will be considered in one of the future publications. The approach developed in this paper has to be generalized in order to include perturbations of not finite rank following the main ideas of [**25**].

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