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Abstract. Typically a hypergeometric function is a multi-valued analytic function with algebraic singularities. In this paper we give a complete description of the Newton polytope of the polynomial whose zero set naturally contains the singular locus of a nonconfluent double hypergeometric series. We show in particular that the Hadamard multiplication of such series corresponds to the Minkowski sum of the Newton polytopes of polynomials which define their singularities.

1 Introduction

One of the most important theorems in the theory of distribution of singularities of Taylor series is the classical Hadamard theorem on multiplication of singularities (see [1], § 1.4). Multidimensional versions of this result were obtained in [4],[9].

The present paper deals with singularities of nonconfluent hypergeometric functions in several variables. Such functions are defined by means of analytic continuation of hypergeometric series. There exist several related ways to define hypergeometric objects (such as functions, series, differential equations). In this paper we use the classical definition of a hypergeometric series which goes back to Horn [7]: a (formal) Laurent series is called hypergeometric if the quotient of its adjacent coefficients depends rationally on the indices of summation. Throughout the paper we identify an analytic function and its germ given by a Taylor series with a nonempty domain of convergence.

The general form of the coefficients of a (formal) hypergeometric series is given by the Ore-Sato theorem (see Section 2). Such series can be shown to satisfy a certain overdetermined system of partial differential equations with polynomial coefficients which is usually referred to as the Horn hypergeometric system (see [7],[5]). Since any differential relation for a Laurent series with a nonempty domain of convergence remains valid for its analytic continuation, it follows that the singular locus of a hypergeometric function is contained in the projection of the characteristic variety of the Horn system onto the variable space. Under some nondegeneracy conditions this projection can be shown to be an algebraic hypersurface. This algebraic hypersurface associated with a given nonconfluent hypergeometric series is the main object of study in this paper.

The singular locus of a hypergeometric function being a subset of an algebraic hypersurface, it can be naturally embedded into the zero set of the resultant of a sequence of homogeneous forms in several variables. One of important characteristics of this resultant (which is a multivariate polynomial) is its Newton polytope. The main purpose of this paper is to give a complete description of the Newton polytope of the polynomial whose zero set "naturally contains" (in the sense to be made precise in Section 2) the singularities of a given double nonconfluent hypergeometric series. We show in particular that *the Hadamard (termwise) product of double nonconfluent hypergeometric series corresponds to the Minkowski sum of the Newton polytopes of polynomials which define the singular loci of the factors (Corollary 5).*

We present examples which show that the two-dimensional situation is essentially different from the case of more variables (mainly due to the fact that in the plane any cone is simplicial). The results in the paper do not hold in the case when the dimension of the variable space exceeds two.

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2 Notations and definitions

Definition 1 A formal Laurent series

$$\sum_{s\in\mathbb{Z}^n} \varphi(s_1, ..., s_n) x_1^{s_1} ... x_n^{s_n} \tag{1}$$

is called *hypergeometric* if for any $i = 1, ..., n$ the quotient $\varphi(s + e_i)/\varphi(s)$ is a rational function in s. Throughout the paper we denote this rational function by $P_i(s)/Q_i(s+e_i)$. Here $\{e_i\}_{i=1}^n$ is the standard basis of the lattice $\mathbb{Z}^n$. By the *support* of this series we mean the subset of $\mathbb{Z}^n$ on which $\varphi(s) \neq 0$.

A *hypergeometric function* is a (multi-valued) analytic function obtained by means of analytic continuation of a hypergeometric series along all possible paths.

Theorem A (Ore, Sato [13],[5]) *The coefficients of a hypergeometric series are given by the formula*

$$\varphi(s) = t^s U(s) \prod_{i=1}^{p} \Gamma(\langle A_i, s\rangle - c_i), \tag{2}$$

where $t^s = t_1^{s_1} \dots t_n^{s_n}$, $t_i, c_i \in \mathbb{C}$, $A_i \in \mathbb{Z}^n$ *and* $U(s)$ *is a rational function.*

We will call any function of the form (2) *the Ore-Sato coefficient of a hypergeometric series.* In this paper the Ore-Sato coefficient (2) plays the role of a primary object which generates everything else: the series, the system of differential equations, the algebraic hypersurface containing the singularities of its solutions etc.

Definition 2 The Ore-Sato coefficient (2) (and the corresponding hypergeometric series (1)) is called *nonconfluent* if

$$\sum_{i=1}^{p} A_i = 0. \tag{3}$$

In this paper we only deal with nonconfluent Ore-Sato coefficients. The sum of a nonconfluent hypergeometric series cannot be an entire function. This follows for instance from the fact that the restriction of such a series to the complex line $x_2 = \dots = x_n = 0$ is a hypergeometric series in one variable with a finite radius of convergence (see [5]). A necessary and sufficient condition for a nonconfluent hypergeometric series to have a nonempty domain of convergence is given in [11].

Definition 3 *The Hadamard (termwise) product* of (formal) Laurent series $\sum_{s\in\mathbb{Z}^n} \varphi(s)x^s$ and $\sum_{s\in\mathbb{Z}^n} \psi(s)x^s$ is defined to be the series $\sum_{s\in\mathbb{Z}^n} \varphi(s)\psi(s)x^s$.

The Horn system of an Ore-Sato coefficient. A (formal) Laurent series $\sum_{s\in\mathbb{Z}^n} \varphi(s)x^s$ whose coefficient satisfies the relations $\varphi(s+e_i)/\varphi(s) = P_i(s)/Q_i(s+e_i)$ is a (formal) solution to the following system of partial differential equations of hypergeometric type

$$x_i P_i(\theta) y(x) = Q_i(\theta) y(x), \quad i = 1, \dots, n. \tag{4}$$

Here $\theta = (\theta_1, \dots, \theta_n)$, $\theta_i = x_i \frac{\partial}{\partial x_i}$. The system (4) will be referred to as *the Horn hypergeometric system defined by the Ore-Sato coefficient* $\varphi(s)$ (see [7] and [5]).

Singularities of solutions to the Horn system. The singular set of the hypergeometric function which is defined by means of analytic continuation of the hypergeometric series with the Ore-Sato coefficient (2) is of course heavily dependent on the set of summation. On the other hand, it was shown in [12] that for any given Ore-Sato coefficient (2) there exist finitely many ways to choose the support of the corresponding series (as long as it remains hypergeometric and has a nonempty domain of convergence). Thus one can speak of the singular locus of an Ore-Sato coefficient which we define to be the union of the singular loci of the hypergeometric series with this coefficient. Since all such series satisfy the Horn system (4), we are naturally led to the problem of describing the singularities of any solution to (4).

Let $\mathcal{D}$ denote the Weyl algebra of differential operators with polynomial coefficients in n variables, see [2]. For any differential operator $P \in \mathcal{D}$, $P = \sum_{|\alpha|\le k} c_\alpha(x)\left(\frac{\partial}{\partial x}\right)^\alpha$ its principal symbol $\sigma(P)(x,z) \in \mathbb{C}[x_1,\dots,x_n,z_1,\dots,z_n]$ is defined by

$$\sigma(P)(x,z) = \sum_{|\alpha|=k} c_\alpha(x) z^\alpha.$$

We denote by G_i the differential operator $x_i P_i(\theta) - Q_i(\theta)$ in the ith equation of the Horn system (4). Let $\mathcal{M} = \mathcal{D}/\sum_{i=1}^n \mathcal{D}G_i$ be the left $\mathcal{D}$-module associated with the system (4) and let $J \subset \mathcal{D}$ denote the left ideal generated by the differential operators $G_1,\dots,G_n$. By definition (see [2], Chapter 5, § 2) the characteristic variety $\mathrm{char}(\mathcal{M})$ of the Horn system is given by

$$\mathrm{char}(\mathcal{M}) = \{(x,z) \in \mathbb{C}^{2n} : \sigma(P)(x,z) = 0, \ \text{ for all } P \in J\}.$$

We define the set $U_\mathcal{M} \subset \mathbb{C}^n$ by

$$U_\mathcal{M} = \{x \in \mathbb{C}^n : \exists\, z \neq 0 \text{ such that } (x,z) \in \mathrm{char}(\mathcal{M})\}.$$

It follows from Proposition 8.1.3 and Theorem 8.3.1 in [8] and Theorem 7.1 in Chapter 5 of [2] that a solution to (4) can only be singular on $U_\mathcal{M}$.

Without any assumptions on the operators in the Horn system the singularities of its solutions are not necessarily algebraic. For instance, if every differential operator G_i contains the factor $(\theta_1 + ... + \theta_n)$ then any sufficiently smooth function depending on the quotients $\frac{x_1}{x_n}, ..., \frac{x_{n-1}}{x_n}$ is a solution to the system (4). To consider the case of algebraic singularities of solutions to (4) we introduce the notion of the resultant of an Ore-Sato coefficient.

The resultant of an Ore-Sato coefficient. Let $H_i(x,z)$ be the principal symbol of the differential operator G_i in the ith equation of the Horn system (4). Since the polynomials $H_1, ..., H_n$ are homogeneous in $z_1, ..., z_n$, they determine the classical resultant $R[H_1, ..., H_n]$ which is a polynomial in $x_1, ..., x_n$ (see [6], Chapter 13). We will call this resultant *the resultant of the Ore-Sato coefficient (2)* and denote it by $R[\varphi](x)$. For the convenience of future reference we formulate the following simple proposition (see [11]).

Proposition 1 *The singular locus of a hypergeometric series with the Ore-Sato coefficient $\varphi(s)$ lies in the zero set of the resultant $R[\varphi](x)$ of this coefficient.*

To prove this Proposition it suffices to notice that for $x^{(0)} \in U_{\mathcal{M}}$ the system of equations $H_1(x^{(0)}, z) = \ldots = H_n(x^{(0)}, z) = 0$ (considered as a system of algebraic equations in $z_1, \ldots, z_n$ whose coefficients depend on $x^{(0)}$) has a solution in $\mathbb{C}^n \setminus \{0\}$. This yields that the resultant of the homogeneous forms $H_1(x, z), \ldots, H_n(x, z)$ with respect to the variables $z_1, \ldots, z_n$ vanishes at $x^{(0)}$ (see [6], Chapter 13). Thus the singular locus of a solution to the Horn system (4) is contained in the zero set of the resultant $R[H_1, ..., H_n]$. Since a hypergeometric series with the Ore-Sato coefficient $\varphi(s)$ satisfies (4), its singular locus is contained in the zero set of $R[\varphi](x)$. Notice that the vanishing of this resultant at a point $x^{(0)} \in \mathbb{C}^n$ is equivalent to the condition that the sequence of the principal symbols $\{H_i(x^{(0)}, z)\}_{i=1}^n$ is not regular in the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$.

Since the singularities of an analytic function propagate along analytic hypersurfaces, it follows from Proposition 1 that if the resultant of the coefficient of a hypergeometric series is not identically zero, the singular locus of the series equals the union of some of the irreducible components of the zero set of this resultant. Thus a hypergeometric series has algebraic singularities provided that the resultant of its coefficient is a nonzero polynomial. Throughout the paper we assume that the resultants of all Ore-Sato coefficients we are dealing with are not equal to zero identically.

Normalization of resultants. The resultant $R(f_1, ..., f_n)$ of n homogeneous forms in n variables is a homogeneous polynomial in the coefficients of each form f_i of degree $\prod_{j \neq i} \deg f_j$ (see Proposition 1.1 in Chapter 13 of [6]). Throughout the paper we use the following normalization of resultants:

$$R(\alpha_1 z_1^{k_1}, ..., \alpha_n z_n^{k_n}) = \prod_{i=1}^{n} \alpha_i^{\prod_{j \neq i} k_j}$$

for $n > 1$ and $R(\alpha_1 z_1^{k_1}) = \alpha_1^{k_1}$ for $n = 1$. This normalization agrees with the multiplicative property of resultants: if $f_1 = f_1^{'} f_1^{''}$ is a product of two homogeneous forms then $R(f_1, ..., f_n) = R(f_1^{'}, f_2, ..., f_n) R(f_1^{''}, f_2, ..., f_n)$, see Proposition 1.3 in Chapter 13 of [6].

Remark 1 Given a $n \times n$ matrix (w_{ij}) with integer entries and an Ore-Sato coefficient $\varphi(s)$, one can consider the quotients $\varphi(s + w_i)/\varphi(s)$, where w_i denotes the ith row of (w_{ij}). By the definition of an Ore-Sato coefficient these quotients are rational functions. Denoting them by $P_i^{(w)}(s)/Q_i^{(w)}(s + w_i)$, one can consider the system of partial differential equations

$$x^{w_i} P_i^{(w)}(\theta) y(x) = Q_i^{(w)}(\theta) y(x), \ \ i = 1, ..., n, \tag{5}$$

to which the hypergeometric series (1) is a solution. Arguing as in Proposition 1 we conclude that the singularities of (1) are contained in the zero set of the resultant of the principal symbols of the operators in (5). Thus there is a certain ambiguity in choosing a polynomial (the resultant of an Ore-Sato coefficient) whose zero set "naturally contains" the singularities of a hypergeometric series with this coefficient. However, using the multiplicative property of resultants one can conclude that the resultant of an Ore-Sato coefficient is well-defined in the following sense: if $|\det(w_{ij})| = 1$ then the resultant of the principal symbols of the operators in the Horn system (4) differs from the resultant of the principal symbols of the operators in (5) only by a monomial factor. For arbitrary (w_{ij}) the resultant of the Ore-Sato coefficient $\varphi(s)$ divides the resultant of the principal symbols of the operators in (5). Thus it is sufficient in this context to consider the system (4).

The principal symbol of an Ore-Sato coefficient. The set of vectors $\{A_1, ..., A_p\}$ will be called *the principal symbol of the Ore-Sato coefficient (2).* Since the set of functions of the form (2) satisfying the condition (3) is closed under multiplication, we have a semigroup structure on the set of principal symbols of Ore-Sato coefficients, the union being the semigroup operation. Notice that the union of the principal symbols of Ore-Sato coefficients corresponds to the Hadamard product of hypergeometric series.

The polygon of a nonconfluent Ore-Sato coefficient in two variables. Using, if necessary, the Gauss multiplication formula for the Γ-function, we may without loss of generality assume that for any $i = 1, ..., p$ the nonzero components of the vector A_i are relatively prime. Let l_i denote the generator of the sublattice $\{s \in \mathbb{Z}^2 : \langle A_i, s\rangle = 0\}$ and let m_i be the number of elements in the set $\{A_1, ..., A_p\}$ which coincide with A_i. The nonconfluency condition (3) implies that there exists a uniquely determined integer convex polygon whose sides are translations of the vectors $m_i l_i$, the vectors $A_1, ..., A_p$ being the outer normals to its sides. (The number of sides of this polygon coincides with the number of different elements in the set of vectors $\{A_1, ..., A_p\}$.) We call this polygon *the polygon of the Ore-Sato coefficient (2)* and denote it by $\mathcal{P}(\varphi)$. For instance, the polygon of the Ore-Sato coefficient $\Gamma(as_1 + bs_2)\Gamma^a(-s_1)\Gamma^b(-s_2)$ is the triangle $\{(s_1, s_2) \in \mathbb{R}^2 : s_i \geq 0,\ s_1/b + s_2/a \leq 1\}$. The principal symbol of this Ore-Sato coefficient is the set of vectors $\{(a, b), \underbrace{(-1, 0), ..., (-1, 0)}_{a \text{ times}}, \underbrace{(0, -1), ..., (0, -1)}_{b \text{ times}}\}$.

For a Taylor polynomial $p(x) = \sum_{|\alpha|\leq k} c_\alpha x^\alpha$ we denote by $h(p)(x)$ its homogeneous component of the highest possible degree, i.e., $h(p)(x) = \sum_{|\alpha|=k} c_\alpha x^\alpha$.

The Newton polytope of a Laurent polynomial f is denoted by $\mathcal{N}(f)$, the set of its vertices by $\mathrm{vert}(\mathcal{N}(f))$. We say that a polytope $\triangle_1$ is a Minkowski summand of another polytope $\triangle_2$ if there exists a polytope $\triangle_3$ such that $\triangle_2 = \triangle_1 + \triangle_3$.

As a rule, a hypergeometric series contains terms with negative powers of the variables and hence has singularities on the union of the coordinate hyperplanes $x_1 \ldots x_n = 0$. Those are trivial singularities of a hypergeometric series with generic parameters. To avoid dealing with them we consider all mappings as being defined in the torus $(\mathbb{C}^*)^n$. We will identify resultants of Ore-Sato coefficients which differ only by a monomial factor as well as polytopes which differ by a translation. The *essential part* of a Taylor polynomial $p(x)$ is defined to be the quotient $p(x)/x^a$ where x^a is the monomial of the maximal possible degree which divides $p(x)$.

3 The main result

In this section we restrict our attention to the case of two variables. Examples 3 and 4 show that the results in this section are not true in higher dimensions.

The main result in the paper is the following theorem.

Theorem 2 *Suppose that the resultant of a nonconfluent Ore-Sato coefficient in two variables is not identically equal to zero. Then its Newton polytope coincides with the polygon of this coefficient.*

A convex polygon is uniquely determined by the lengths of its sides and the outer normals to them. Let φ be a nonconfluent Ore-Sato coefficient. We will show that for any side of the polygon $\mathcal{P}(\varphi)$ there exists a side of $\mathcal{N}(R[\varphi])$ with the same length and the same outer normal and that the polygon $\mathcal{N}(R[\varphi])$ does not have any other sides. This will imply the conclusion of Theorem 2.

To prove Theorem 2 we need some intermediate results. Our first observation is the following lemma which holds for arbitrary (and not only hypergeometric) power series.

Lemma 3 *Let $y_1(x)$ be the analytic function defined by means of analytic continuation of the power series $\sum_{s\in\mathbb{N}_0^n} \varphi(s)x^s$ with a nonempty domain of convergence. Then for any rational function $U(s)$ which has neither zeros nor poles in $\mathbb{N}_0^n$ the function $y_2(x)$ defined by means of analytic continuation of the series $\sum_{s\in\mathbb{N}_0^n} U(s)\varphi(s)x^s$ has the same singularities as $y_1(x)$.*

Proof. The domain of convergence of the series $\sum_{s\in\mathbb{N}_0^n} U(s)\varphi(s)x^s$ is nonempty and contains the origin. Let $U(s) = V(s)/W(s)$ where V and W are polynomials. Consider the analytic function $y_3(x)$ defined by means of analytic continuation of the series $\sum_{s\in\mathbb{N}_0^n}(\varphi(s)/W(s))x^s$. The domain of convergence of this series is also nonempty and contains the origin. The function $y_3(x)$ satisfies the relation $W(\theta)y_3(x) = y_1(x)$. Let us define the set

$$E = \{x \in \mathbb{C}^n : \exists z \neq 0 \text{ such that } h(W)(x_1z_1, ..., x_nz_n) = 0\},$$

and let $\operatorname{sing}(y(x))$ denote the singular locus of the (multi-valued) analytic function $y(x)$. Proposition 8.1.3 and Theorem 8.3.1 in [8] and Theorem 7.1 in Chapter 5 of [2] yield that $\operatorname{sing}(y_3(x)) \subset E \cup \operatorname{sing}(y_1(x))$. Since the function $y_3(x)$ is holomorphic at the origin and for any $z \in \mathbb{C}^n \setminus \{0\}$ the polynomial $h(W)(x_1z_1, ..., x_nz_n)$ is homogeneous in $x_1, ..., x_n$ (and hence its zero set hits the origin), it follows that $y_3(x)$ cannot have singularities on E. This yields the inclusion $\operatorname{sing}(y_3(x)) \subset \operatorname{sing}(y_1(x))$. Using the equality $y_2(x) = V(\theta)y_3(x)$, which implies that $\operatorname{sing}(y_2(x)) \subset \operatorname{sing}(y_3(x))$, we conclude that $\operatorname{sing}(y_2(x)) \subset \operatorname{sing}(y_1(x))$. Since the function $U(s)$ was assumed to have neither zeros nor poles in $\mathbb{N}_0^n$, its inverse $(U(s))^{-1}$ has the same property. We can therefore repeat the above argument with y_1 and y_2 interchanged and conclude that $\operatorname{sing}(y_1(x)) \subset \operatorname{sing}(y_2(x))$ and hence $\operatorname{sing}(y_1(x)) = \operatorname{sing}(y_2(x))$. The proof is complete.

Making the change of variables $\xi_i = t_ix_i$ we may without loss of generality assume that the exponential factor t^s in (2) equals 1.

Proposition 5 in [11] states that for a nonconfluent hypergeometric series to have a nonempty domain of convergence its support must be contained in a translation of a strongly convex cone (i.e. a cone which does not contain linear subspaces). Since any such cone can be mapped into the positive octant by a suitable linear mapping, we can without loss of generality restrict our attention to the case of Taylor series. It follows by Proposition 5 in [11] and Lemma 3 that the singular locus of a hypergeometric Taylor series only depends on the principal symbol of its coefficient. Thus we may without loss of generality assume that $U(s) \equiv 1$ in (2).

Lemma 4 *Let $y(x) = \sum_{s\in\mathbb{Z}^n}\varphi(s)x^s$ be a solution to the Horn system (4). For any $u \in \mathbb{Z}^n$ there exist nonzero polynomials ρ_u, τ_u of equal degrees such that $y(x)$ satisfies the equation*

$$x^u\rho_u(\theta)y(x) = \tau_u(\theta)y(x).$$

To prove this Lemma it suffices to notice that by the definition of a hypergeometric series the quotient $\varphi(s+u)/\varphi(s)$ is a rational function for any $u \in \mathbb{Z}^n$. We denote this function by $\rho_u(s)/\tau_u(s+u)$. Since $y(x)$ is nonconfluent, it follows that $\deg \rho_u = \deg \tau_u$. A straightforward computation shows that the function $y(x)$ is annihilated by the differential operator $x^u \rho_u(\theta) - \tau_u(\theta)$.

Observe that the homogeneous parts of the highest degree of the polynomials ρ_u, τ_u are given by

$$h(\rho_u)(s) = \prod_{i:\langle A_i,u\rangle>0} \langle A_i, s\rangle^{\langle A_i,u\rangle}, \qquad h(\tau_u)(s) = \prod_{i:\langle A_i,u\rangle<0} \langle A_i, s\rangle^{-\langle A_i,u\rangle}. \tag{6}$$

If $A_k = -A_l$ for some $k, l \in \{1, ..., p\}$ and $A_{k1}, A_{k2} \neq 0$ then it follows from (6) that the polynomials $h(P_i)(s), h(Q_i)(s)$, $i = 1, 2$ are divisible by $\langle A_k, s\rangle$. This yields that the principal symbols of the operators in the Horn system defined by the Ore-Sato coefficient (2) vanish along the line $A_{k1}x_1z_1 + A_{k2}x_2z_2 = 0$ and hence the resultant of (2) is identically zero. We do not consider this degenerate case.

Suppose now that only one of the integers A_{k1}, A_{k2} is different from zero. Let $A_{k1} \neq 0$, then using, if necessary, the Gauss multiplication formula for the Γ-function, we may assume that $A_k = e_1$ or $A_k = -e_1$. Suppose that $A_k = e_1$ (the case $A_k = -e_1$ can be treated similarly). Using the multiplicative property of resultants (see Proposition 1.3 in Chapter 13 of [6]), we conclude that $R[\varphi](x) = (x_1x_2)^{\deg P_2}(x_2 - 1)\tilde{R}(x)$, where $\tilde{R}(x)$ is the resultant of an Ore-Sato coefficient with the principal symbol $\{A_1, ..., [k], ..., [l], ..., A_p\}$ (here $[k]$ is the sign of omission). Thus the Newton polytope $\mathcal{N}(R[\varphi])$ is given by the Minkowski sum of the segment $\mathcal{N}(x_2 - 1)$ and the Newton polytope of the resultant of another nonconfluent Ore-Sato coefficient whose principal symbol contains fewer vectors. By the construction of the polygon of a nonconfluent Ore-Sato coefficient it is also equal to the Minkowski sum of $\mathcal{N}(x_2 - 1)$ and the polygon of an Ore-Sato coefficient with the principal symbol $\{A_1, ..., [k], ..., [l], ..., A_p\}$. Similar arguments can be used in the case when $A_{k2} \neq 0$. It is therefore sufficient to prove Theorem 2 in the case when the principal symbol of the Ore-Sato coefficient in question does not contain opposite vectors.

With each vertex $v \in \mathcal{P}(\varphi)$ of the polygon of an Ore-Sato coefficient $\varphi(s)$ we associate the cone

$$C_v = \{s \in \mathbb{R}^n : t(s - v) + v \in \mathcal{P}(\varphi) \text{ for some } t > 0\}.$$

Let $\mu^{(v)}, \nu^{(v)} \in \mathbb{Z}^2$ be the generators of the cone C_v.

We can now complete the proof of Theorem 2.

PROOF OF THEOREM 2. Let $\varphi(s)$ be a nonconfluent Ore-Sato coefficient in two variables. Fix $i \in \{1, \ldots, p\}$ and let v be a vertex of the polygon $\mathcal{P}(\varphi)$ such that A_i is normal to one of the sides of $\mathcal{P}(\varphi)$ which meet at v. We may without loss of generality assume that $\mu^{(v)} = (-A_{i2}, A_{i1})$. Choose a vector $u \in \mathbb{Z}^2$ such that the cone generated by $\mu^{(v)}$ and u contains C_v and $|\det(\mu^{(v)}, u)| = 1$. Such a choice of the vector u is possible since by the assumption the nonzero components of A_k are relatively prime for any k and hence so are the components of $\mu^{(v)}$. Consider the system of partial differential equations

$$\begin{cases} x^{\mu^{(v)}} \rho_{\mu^{(v)}}(\theta) y(x) & = \tau_{\mu^{(v)}}(\theta) y(x), \\ x^u \rho_u(\theta) y(x) & = \tau_u(\theta) y(x) \end{cases} \tag{7}$$

with the polynomials $\rho_{\mu^{(v)}}, \rho_u, \tau_{\mu^{(v)}}, \tau_u$ defined as in Lemma 4. By Lemma 4 the hypergeometric series with the coefficient $\varphi(s)$ satisfies the system (7). By Remark 1 the essential part of the resultant of the Ore-Sato coefficient $\varphi(s)$ coincides with the essential part of the resultant of the principal symbols of the operators in (7). Let $\xi = x^{\mu^{(v)}}$, $\eta = x^u$ and consider the restriction of the essential part of the latter resultant to $\eta = 0$. The multiplicative property of resultants (see Proposition 1.3 in Chapter 13 of [6]) together with (6) lead to the equality

$$R(\xi h(\rho_{\mu^{(v)}})(z) - h(\tau_{\mu^{(v)}})(z), h(\tau_u)(z)) =$$

$$\prod_{i:\langle A_i, u\rangle < 0} \left(R(\xi h(\rho_{\mu^{(v)}})(z) - h(\tau_{\mu^{(v)}})(z), \langle A_i, z\rangle) \right)^{-\langle A_i, u\rangle}. \tag{8}$$

If $\langle A_i, u\rangle < 0$ and $\langle A_i, \mu^{(v)}\rangle > 0$ then $R(\xi h(\rho_{\mu^{(v)}})(z) - h(\tau_{\mu^{(v)}})(z), \langle A_i, z\rangle)$ is a constant, since by (6) the factor $\langle A_i, z\rangle$ is present in the form $h(\rho_{\mu^{(v)}})(z)$. Therefore the resultant (8) is equal (up to a constant factor) to the resultant

$$\prod_{i:\begin{cases}\langle A_i, u\rangle < 0, \\ \langle A_i, \mu^{(v)}\rangle \le 0\end{cases}} \left(R(\xi h(\rho_{\mu^{(v)}})(z) - h(\tau_{\mu^{(v)}})(z), \langle A_i, z\rangle) \right)^{-\langle A_i, u\rangle}. \tag{9}$$

As we have remarked earlier, we may without loss of generality assume that the principal symbol of the Ore-Sato coefficient $\varphi(s)$ does not contain opposite vectors. Thus by the choice of u the only vector in the set $\{A_1, \ldots, A_p\}$ which satisfies the conditions $\langle A_i, u\rangle < 0$, $\langle A_i, \mu^{(v)}\rangle \le 0$ is A_i. The condition $|\det(\mu^{(v)}, u)| = 1$ implies that $-\langle A_i, u\rangle = 1$. Hence the degree of the polynomial (9) in ξ equals the number of elements in the set $\{A_1, \dots, A_p\}$ which coincide with A_i, i.e., the multiplicity m_i. This yields that the side of the Newton polytope of the resultant of the principal symbols in (7) with the outer normal A_i is congruent to $m_i l_i$ (here l_i is the generator of the sublattice $\{s \in \mathbb{Z}^2 : \langle A_i, s\rangle = 0\}$). By Remark 1 this polytope coincides with the Newton polytope of the resultant of the Ore-Sato coefficient $\varphi(s)$. Thus for any side of the polygon $\mathcal{P}(\varphi)$ there exists a side of the polygon $\mathcal{N}(R[\varphi])$ with the same length and the same outer normal.

It remains to show that the polygon $\mathcal{N}(R[\varphi])$ does not have any other sides. To do this we fix $v \in \mathrm{vert}(\mathcal{P}(\varphi))$ and consider the system of equations

$$\begin{cases} x^{\mu^{(v)}} \rho_{\mu^{(v)}}(\theta) y(x) &= \tau_{\mu^{(v)}}(\theta) y(x), \\ x^{\nu^{(v)}} \rho_{\nu^{(v)}}(\theta) y(x) &= \tau_{\nu^{(v)}}(\theta) y(x). \end{cases} \tag{10}$$

Here $\mu^{(v)}, \nu^{(v)}$ are the generators of the cone C_v. By Lemma 4 the series (1) satisfies the system (10). Let $\xi_1 = x^{\mu^{(v)}}, \xi_2 = x^{\nu^{(v)}}$ and consider the essential part $\tilde{R}(\xi_1, \xi_2)$ of the resultant of the principal symbols of the operators in (10). It follows from (6) that the restriction of $\tilde{R}(\xi_1, \xi_2)$ to any of the lines $\xi_1 = 0$, $\xi_2 = 0$ is a nonconstant polynomial in the remaining variable with a nonzero constant term. Since by Remark 1 the Newton polytope of the resultant of $\varphi(s)$ is a Minkowski summand of the Newton polytope of $\tilde{R}(x^{\mu^{(v)}}, x^{\nu^{(v)}})$, it follows that for any $v \in \mathrm{vert}(\mathcal{P}(\varphi))$ the normals to the vectors $\mu^{(v)}, \nu^{(v)}$ coincide with the normals to some adjacent sides of the polytope $\mathcal{N}(R[\varphi])$. This shows that no extra sides can appear in the polygon $\mathcal{N}(R[\varphi])$ and completes the proof of Theorem 2. □

Remark 2 Examples 3 and 4 show that the conclusion of Theorem 2 is not valid in the case of more than two variables. The main difference between the two-dimensional situation and the case of more than two variables, which makes the geometric argument in the proof of Theorem 2 fail, is the fact that in higher dimensions not every cone is simplicial.

By the construction the polygon of the product of two nonconfluent Ore-Sato coefficients is given by the Minkowski sum of the polygons of the factors. Using Theorem 2 we arrive at the following corollary.

Corollary 5 *Let $\mathcal{C}$ denote the semigroup of the principal symbols of nonconfluent Ore-Sato coefficients in two variables with the operation m and let $\mathcal{P}$ be the semigroup of the convex polygons in $\mathbb{R}^2$, the operation M being the Minkowski sum. Denote by $\mathcal{N}$ the mapping which assigns to a polynomial its Newton* *polytope and by R the mapping which maps an Ore-Sato coefficient to its resultant. Suppose that none of the Ore-Sato coefficients we are dealing with has zero resultant. Then the following diagram is commutative.*

$$\begin{tikzcd}
 & \mathcal{C} \times \mathcal{C} \arrow[dl, "R\times R"'] \arrow[dr, "m"] & \\
\mathbb{C}[x] \times \mathbb{C}[x] \arrow[d, "\mathcal{N}\times\mathcal{N}"'] & & \mathcal{C} \arrow[d, "R"] \\
\mathcal{P} \times \mathcal{P} \arrow[dr, "M"'] & & \mathbb{C}[x] \arrow[dl, "\mathcal{N}"] \\
 & \mathcal{P} &
\end{tikzcd}$$

In other words, multiplication in the semigroup of the principal symbols of Ore-Sato coefficients in two variables corresponds to the Minkowski sum of the Newton polytopes of their resultants.

Combining Corollary 5 with Proposition 1 and taking into account the fact that multiplication in the semigroup of the principal symbols of the Ore-Sato coefficients corresponds to the Hadamard product of hypergeometric series, we arrive at the following result.

Corollary 6 *The Newton polytope of the polynomial defining the singular locus of the Hadamard product of nonconfluent double hypergeometric series is a Minkowski summand in the sum of the Newton polytopes of the resultants of the Ore-Sato coefficients of the factors.*

Suppose that a nonconfluent Ore-Sato coefficient φ defines a regular holonomic Horn system (for definitions see Chapter 5 of [3]). Then by the Andronikof-Kashiwara theorem (see Theorem 8.11.8 in [3]) the zero set of the resultant $R[\varphi](x)$ coincides with the singular locus of the general solution to this Horn system, or, equivalently, with the singular locus of a generic hypergeometric series with the coefficient φ. Thus in the case when nonconfluent Ore-Sato coefficients φ_1, φ_2, $\varphi_1\varphi_2$ define regular holonomic Horn systems we have equality in Corollary 6: the singular locus of the Hadamard product of the series with the coefficients φ_1, φ_2 can be defined as the zero set of a polynomial whose Newton polytope is given by the Minkowski sum of the Newton polytopes of some polynomials whose zero sets are the singular loci of the respective factors.

4 Examples

Example 1 Consider the Ore-Sato coefficient

$$\varphi(s) = \Gamma(2s_1 - s_2)\Gamma(s_1 + 2s_2)\Gamma^3(-s_1)\Gamma(s_2)\Gamma^2(-s_2). \tag{11}$$

It is the product of the nonconfluent Ore-Sato coefficients

$$\varphi_1(s) = \Gamma(2s_1 - s_2)\Gamma^2(-s_1)\Gamma(s_2) \ \text{ and } \ \varphi_2(s) = \Gamma(s_1 + 2s_2)\Gamma(-s_1)\Gamma^2(-s_2).$$

By Corollary 5 the Newton polytope of the resultant of $\varphi(s)$ is given by the Minkowski sum of the Newton polytopes of the resultants of $\varphi_1(s)$ and $\varphi_2(s)$. These polytopes are shown in Figure 1. Notice that the resultants of the Ore-Sato coefficients in question are given by

$$\begin{aligned} R[\varphi] &= (x_1x_2)^9(4x_1 - 1)(3125x_1^2x_2^2 - 1000x_1x_2^2 + 64x_2^3 - \\ &\qquad\qquad 50x_1x_2 + 48x_2^2 - 4x_1 + 12x_2 + 1), \\ R[\varphi_1] &= (x_1x_2)^2(4x_1x_2^2 - x_2^2 - 2x_2 - 1), \\ R[\varphi_2] &= (x_1x_2)^2(x_1^2 - 2x_1 - 4x_2 + 1). \end{aligned}$$

Example 2 The Ore-Sato coefficient

$$\varphi(s) = (-1)^{s_1+\ldots+s_n}\Gamma(s_1 + \ldots + s_n + 1)\Gamma(-s_1)\ldots\Gamma(-s_n) \tag{12}$$

defines the Horn system

$$x_i(\theta_1 + \ldots + \theta_n + 1)y(x) = \theta_i y(x), \ \ i = 1, \ldots, n.$$

The principal symbols of the differential operators in this system of equations are the linear forms $\sum_{j=1}^n (x_j - \delta_{ij})x_iz_j$, $i = 1, \ldots, n$. The resultant of these forms with respect to the variables $z_1, \ldots, z_n$ is given by the determinant

$$\begin{vmatrix} x_1^2 - x_1 & x_1x_2 & \ldots & x_1x_n \\ x_1x_2 & x_2^2 - x_2 & \ldots & x_2x_n \\ \ldots & \ldots & \ldots & \ldots \\ x_1x_n & x_2x_n & \ldots & x_n^2 - x_n \end{vmatrix} = (-1)^n x_1\ldots x_n(1 - x_1 - \ldots - x_n).$$

Thus the Newton polytope of the resultant of the Ore-Sato coefficient (12) is (a translation of) the standard simplex in $\mathbb{R}^n$.

Example 3 Consider the nonconfluent Ore-Sato coefficient

$$\varphi(s) = (-1)^{s_1+s_2}\Gamma(s_1+s_3)\Gamma(s_2+s_3)\Gamma(-s_1)\Gamma(-s_2)\Gamma^2(-s_3) \tag{13}$$

and the corresponding Horn system

$$\begin{cases} x_1(\theta_1+\theta_3)y(x) & = \theta_1 y(x), \\ x_2(\theta_2+\theta_3)y(x) & = \theta_2 y(x), \\ x_3(\theta_1+\theta_3)(\theta_2+\theta_3)y(x) & = \theta_3^2 y(x). \end{cases} \tag{14}$$

The resultant of the principal symbols of the differential operators in this system of equations is given by the polynomial

$$R[\varphi](x) = (x_1x_2x_3)^2(x_1-1)(x_2-1)(x_1x_2-x_1-x_2-x_3+1).$$

The Newton polytope of this polynomial is displayed in Figure 2.

A basis in the space of holomorphic solutions to the Horn system (14) is given by the functions 1, $\log\frac{x_3}{(x_1-1)(x_2-1)}$. Indeed, the general solution to the first equation in the system (14) is given by $f(x_2, \frac{x_3}{x_1-1})$, where f is an arbitrary differentiable function. Since the second equation in (14) has the same form as the first one, it follows that the general solution to the subsystem of (14) which consists of the first two equations, is given by the function $F\left(\frac{x_3}{(x_1-1)(x_2-1)}\right)$, where F is an arbitrary differentiable function. Substituting it into the third equation we arrive at the ordinary differential equation $tF^{''}(t) + F^{'}(t) = 0$ whose general solution is $C_1 + C_2\log t$. The fact that the space of holomorphic solutions to (14) has dimension 2 at a generic point follows from Theorem 8 in [12].

This example shows that, unlike the case of two variables, there is in general no one-to-one correspondence between the vectors in the principal symbol of an Ore-Sato coefficient and the outer normals to the facets of the Newton polytope of its resultant. Indeed, the vector $(0,0,1)$ is not in the principal symbol of the Ore-Sato coefficient (13). However, the Newton polytope of its resultant has a facet with the outer normal $(0,0,1)$ (see Figure 2).

Example 4 Consider the quartic equation

$$y^4 + x_1y^3 + x_2y^2 + x_3y - 1 = 0. \tag{15}$$

It is shown in [10] (see also [14]) that its solution $y(x)$ satisfies the system of partial differential equations

$$
\left\{\begin{array}{ll}
-256\frac{\partial^4 y(x)}{\partial x_1^4} & = \left((\mathcal{H}-1)\prod_{j=0}^{2}(\mathcal{G}+4j+1)\right) y(x), \\
256\frac{\partial^4 y(x)}{\partial x_2^4} & = \left(\prod_{j=0}^{1}(\mathcal{G}+4j+1)\prod_{j=0}^{1}(\mathcal{H}+4j-1)\right) y(x), \\
-256\frac{\partial^4 y(x)}{\partial x_3^4} & = \left((\mathcal{G}+1)\prod_{j=0}^{2}(\mathcal{H}+4j-1)\right) y(x),
\end{array}\right. \tag{16}
$$

where

$$\mathcal{G} = 3\theta_1 + 2\theta_2 + \theta_3, \quad \mathcal{H} = \theta_1 + 2\theta_2 + 3\theta_3.$$

The discriminant of the equation (15) is given by the polynomial

$$x_1^2x_2^2x_3^2 - 4x_1^3x_3^3 + 4x_1^2x_2^3 - 4x_2^3x_3^2 - 18x_1^3x_2x_3 + 18x_1x_2x_3^3 - 27x_1^4 - 16x_2^4 - 27x_3^4 +$$

$$80x_1x_2^2x_3 + 6x_1^2x_3^2 + 144x_1^2x_2 - 144x_2x_3^2 - 192x_1x_3 - 128x_2^2 - 256. \tag{17}$$

Its Newton polytope is displayed in Figure 3.

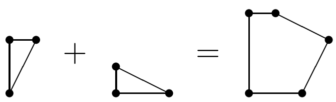

Fig. 1 The Newton polytope of the resultant of the Ore-Sato coefficient (11)

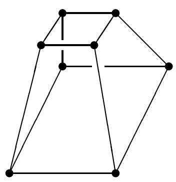

Fig. 2 The Newton polytope of the resultant of the Ore-Sato coefficient (13)

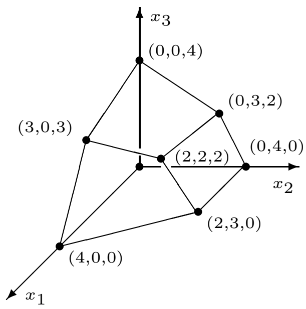

Fig. 3 The Newton polytope of the discriminant of the equation (15)

The resultant of the principal symbols of the differential operators in the system (16) is not equal to zero identically. Indeed, it has a nonzero constant term. Since the solution $y(x)$ to the equation (15) satisfies the system (16) and has branching on the zero set of the polynomial (17), it follows that the polytope in Figure 3 is a Minkowski summand of the Newton polytope of the resultant of the principal symbols of the operators in (16). In particular, the latter polytope has a facet with the outer normal $(1, 2, 1)$. Using the identity

$$x_i^4 \frac{\partial^4}{\partial x_i^4} = \theta_i(\theta_i - 1)(\theta_i - 2)(\theta_i - 3)$$

and making the monomial change of variables $\xi_i = x_i^4$ in (16) (which contracts the Newton polytope of the resultant of the principal symbols of the operators in the system (16) by the factor of 4) we arrive at a Horn system defined by a nonconfluent Ore-Sato coefficient with the principal symbol

$$\{(3,2,1),(1,2,3),(-4,0,0),(0,-4,0),(0,0,-4)\}.$$

The Newton polytope of the resultant of this coefficient has a facet with the outer normal $(1,2,1)$ although this vector is not present in its principal symbol. This shows that the conclusion of Theorem 2 is in general not valid in the case of more than two variables.

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