ISSN: 1401-5617

Order bases of subalgebras of $k[[X]]$

Vinzenco Micale
Giovanni Molica
Barbara Torrisi

Research Reports in Mathematics
Number 10, 2001

Department of Mathematics
Stockholm University

Electronic versions of this document are available at
http://www.matematik.su.se/reports/2001/10

Date of publication: May 9, 2001
2000 Mathematics Subject Classification: Primary 13F25.
Keywords: Onedimensional ring, order of power series, value semigroup.

Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.matematik.su.se
info@matematik.su.se

Order bases of subalgebras of k[[X]]

V. Micale* G. Molica† B. Torrisi‡

May 7, 2001

Abstract

Let $R = k[[f_1, \dots, f_n]]$ be a subalgebra of $k[[X]]$ such that, if we denote the integral closure of R in its quotient field by $\overline{R}$, then $\overline{R} = k[[X]]$ and $\lambda_R(\overline{R}/R) < \infty$. Every element $f = \sum_i c_i X^i \in \overline{R}^*$ has an order, that is $\min\{i \mid c_i \neq 0\}$. The set of orders of elements in R^* is a finitely generated numerical semigroup $o(R) \subseteq \mathbb{N}$.

A set of elements $g_1, g_2, \dots, g_s \in R^*$ is an order basis of R if $R = k[[g_1, \dots, g_s]]$ and $o(R)$ is minimally generated as semigroup by $o(g_1), o(g_2), \dots, o(g_s)$. We describe an algorithm to construct an order basis from the generators $f_1, f_2, \dots, f_n$ and we give some applications.

1 Introduction

Let S denote a subsemigroup in $\mathbb{N}$, that is a subset closed under addition and containing 0.

We also denote the semigroup generated by $x_1 < x_2 < \dots < x_m$ (i.e. the set $\{n_1x_1 + \dots + n_mx_m \mid n_i \in \mathbb{N}, i = 1, \dots, m\}$) by $\langle x_1, \dots, x_m\rangle$.

We call a subsemigroup $S = \langle x_1, \dots, x_m\rangle$ such that $|\mathbb{N} \setminus S| < \infty$ a *numerical semigroup*. It is easily seen that the finite complement of S to $\mathbb{N}$ is equivalent to $\gcd(x_1, \dots, x_m) = 1$.

Each numerical semigroup S has a natural partial ordering $\leq_S$ where for two elements s and r in S we have $s \leq_S r$ if there exists an $u \in S$ such that $r = s + u$. The set g_i of minimal elements in $S \setminus \{0\}$ with this ordering is called a *minimal set of generators* for S.

The set of minimal generators x_i is finite since for any $s \in S$ with $s \neq 0$, we have $x_i \not\equiv x_j \pmod{s}$ if $i \neq j$.

For everything concerning the numerical semigroups, we refer to [1] and [2].

Let $R = k[[f_1, \dots, f_n]]$ be a subalgebra of $k[[X]]$ such that, if we denote the integral closure of R in its quotient field by $\overline{R}$, then $\overline{R} = k[[X]]$ and $\lambda_R(\overline{R}/R) < \infty$.

* *email* vmicale@dipmat.unict.it
† *email* molica@dipmat.unime.it
‡ *email* bafy8@excite.it

Let $f = \sum_i c_i X^i \in \overline{R}^* = \overline{R} \setminus \{0\}$. Then we call $\min\{i \mid c_i \neq 0\}$ the *order* of f and we denote it by $o(f)$. Let us denote the set of orders of elements in R^*, i.e. $\{o(f) \mid f \in R^*\}$, by $o(R)$.

Under the hypotheses on R, we have that $o(R)$ is a numerical semigroup (cf. Proposition 2.4).

We say that a set of elements $g_1, \ldots, g_s \in R^*$ is an *order basis* of R if $R = k[[g_1, \ldots, g_s]]$ and $o(R)$ is minimally generated by $o(g_1), \ldots, o(g_s)$.

In this paper we describe an algorithm to construct an order basis from the generators $f_1, \ldots, f_n$ and we will give some applications.

2 Preliminaries

Throughout the rest of the paper S will denote a numerical semigroup. Since $|\mathbb{N} \setminus S| < \infty$, then there exist elements $a \in \mathbb{N}$ such that if $i \geq a$, then $i \in S$. We denote $\min\{a \mid i \in S \text{ if } i \geq a\}$ by $c = c(S)$ and we call it the *conductor* of S.

Throughout the rest of the paper we will consider rings R that are subalgebras of $k[[X]]$ and such that, if we denote the integral closure of R in its quotient field by $\overline{R}$, then $\overline{R} = k[[X]]$ and $\lambda_R(\overline{R}/R) < \infty$.

Proposition 2.1. *Let f_1, f_2 be elements of $\overline{R}^*$ and let $a = \min\{o(f_1), o(f_2)\}$. Then:*

(i) $a \leq o(f_1 + f_2)$.

(ii) *If $o(f_1) \neq o(f_2)$ then $a = o(f_1 + f_2)$.*

(iii) $o(f_1 f_2) = o(f_1) + o(f_2)$.

Proof. This follows easily from the definition of order.

Proposition 2.2. *[4, Lemma 3, p.486] Let R_1 and R_2 be rings of our type such that $R_1 \subseteq R_2$ and $o(R_1) = o(R_2)$. Then $R_1 = R_2$.*

Proposition 2.3. *[4, Proposition 1, p.488] Let R be a ring of our type. Then $\lambda_R(\overline{R}/R) = |\mathbb{N} \setminus o(R)|$.*

Proposition 2.4. *Let R be a ring of our type. Then $o(R)$ is a numerical semigroup.*

Proof. Since R is a ring and by iii) of Proposition 2.1, we have that $o(R)$ is a subsemigroup of $\mathbb{N}$. By $\lambda_R(\overline{R}/R) < \infty$ and Proposition 2.3, we have the proof.

Proposition 2.5. *Let R be a ring of our type. Then R contains every element $f \in k[X]]$ of order $o(f) \geq c(o(R))$.*

Proof. Use Proposition 2.2 with $R_1 = R$ and $R_2 = R + X^{c(o(R))} k[[X]]$.

3 The main Theorem

In this section we describe an algorithm to construct an order basis of a ring R of our type.

For each finite set of power series $\{h_i\}$ we can make an ordered list using the condition $o(h_1) \leq o(h_2) \leq \cdots \leq o(h_r)$. We call this "ordering the elements" and we denote such an ordered list by $[h_1, h_2, \ldots, h_r]$.

By "monic" power series f we mean that the coefficient of the lowest term of f is equal to 1.

Theorem 3.1. *Let $R = k[[f_1, f_2, \ldots, f_n]]$ be a subalgebra of $k[[X]]$ such that, if we denote the integral closure of R in its quotient field by $\overline{R}$, then $\overline{R} = k[[X]]$ and $\lambda_R(\overline{R}/R) < \infty$. Then it is possible, after a finite number of steps, to construct an order basis of R from the generators $f_1, f_2, \ldots, f_n$.*

Proof. We will inductively construct "monic" elements $g_1, g_2, \ldots, g_i \in R$ and an ordered list of "monic" elements $[g_1, g_2, \ldots, g_i, h_1, \ldots, h_s]$ with $R = k[[g_1, g_2, \ldots, g_i, h_1, \ldots, h_s]]$.

We start the algorithm supposing, without loss of generality, that f_i's are "monic" and $o(f_1) \leq o(f_2) \leq \cdots \leq o(f_n)$. We start with the ordered list $[f_1, f_2, \ldots, f_n]$. We denote this ordered list by $\mathcal{F}_{1,0}$ and the set $\{g_1 = f_1\}$ by $\mathcal{G}_1$. We note that, by $R = k[[f_1, \ldots, f_n]]$ and by Proposition 2.1, $o(g_1)$ is minimal in $o(R) \setminus \{0\}$. If $o(g_1) = 1$, then $R = k[[X]]$ and the algorithm terminates.

Suppose now that the elements $g_1, g_2, \ldots, g_i$, with $g_j = X^{o(g_j)} + \sum_{l \geq 1} a_{j_l} X^{j_l}$, are constructed and let us denote the ordered list of "monic" elements $[g_1, g_2, \ldots, g_i, h_1, \ldots, h_s]$ such that $R = k[[g_1, g_2, \ldots, g_i, h_1, \ldots, h_s]]$ by $\mathcal{F}_{i,0}$. We also denote the set $\{g_1, g_2, \ldots, g_i\}$ by $\mathcal{G}_i$ and the semigroup $\langle o(g_1), o(g_2), \ldots, o(g_i)\rangle$ by $\langle \mathcal{G}_i \rangle$.

We look for an element $g_{i+1} \in R$ with $o(g_{i+1})$ minimal in $o(R) \setminus \langle o(g_1), o(g_2), \ldots, o(g_i)\rangle$. We will see that either we will find it or the algorithm terminates.

We note that the only way to find g_{i+1} is from the h_i's and from expressions $\sum_{j_1,\ldots,j_i} g_1^{i_1} g_2^{i_2} \cdots g_i^{j_i}$ where the lowest terms cancel. To calculate all possible cancellations, we determine the kernel of the map

$$\phi_i : k[Y_1, Y_2, \ldots, Y_i] \to k[X]$$

$$Y_j \to X^{o(g_j)}.$$

Let $\{T_a(Y_1, Y_2 \ldots, Y_i)\}$ be the set of binomials which generate $\ker \phi_i$ and let $\{T_a(g_1, g_2 \ldots, g_i)\}$ be the set of the corresponding elements in R. Without loss of generality we can suppose that the $T_a(g_1, g_2 \ldots, g_i)$'s are "monic". We adjoin to $\mathcal{F}_{i,0}$ all $T_a(g_1, g_2 \ldots, g_i) \neq 0$ with $o(g_i) < o(T_a(g_1, g_2 \ldots, g_i))$ and, if $\langle \mathcal{G}_i \rangle$ is a numerical semigroup, with $o(T_a(g_1, g_2 \ldots, g_i)) < c(\langle \mathcal{G}_i \rangle)$. In fact if $o(T_a(g_1, g_2 \ldots, g_i)) \leq o(g_i)$, then $T_a(Y_1, Y_2 \ldots, Y_i) \in k[Y_1, Y_2, \ldots, Y_{i-1}]$ and hence it has alredy been considered in an earlier kernel. We note that if, in the second step, $\gcd(o(g_1), o(g_2)) = 1$, we can avoid to compute the kernel of the map ϕ_2. In fact it is known (cf. [2]) that $c(\langle \mathcal{G}_2 \rangle) = o(g_1) \cdot o(g_2) - o(g_1) - o(g_2) + 1$, hence either $T(g_1, g_2) = 0$ or $c(\langle \mathcal{G}_2 \rangle) < o(g_1) \cdot o(g_2) < o(T(g_1, g_2))$, where $T(g_1, g_2))$ is the corresponding element of the generator of $\ker \phi_2$.

By ordering the elements of the new set obtained as the union of $\mathcal{F}_{i,0}$ and $\{T_a(g_1, g_2 \ldots, g_i) \neq 0\}$ with $o(g_i) < o(T_a(g_1, g_2 \ldots, g_i))$ (and, if $\langle \mathcal{G}_i \rangle$ is a numerical semigroup, with $o(T_a(g_1, g_2 \ldots, g_i)) < c(\langle \mathcal{G}_i \rangle))$, we get the new ordered list of "monic" elements $\mathcal{F}_{i,1} = [g_1, g_2, \ldots, g_i, h_{i,1}^{(1)}, \ldots, h_{i,t_1}^{(1)}]$.

Then we have

$$\text{either } o(h_{i,1}^{(1)}) \notin \langle \mathcal{G}_i \rangle \text{ or } o(h_{i,1}^{(1)}) \in \langle \mathcal{G}_i \rangle.$$

Suppose $o(h_{i,1}^{(1)}) \in \langle \mathcal{G}_i \rangle$, i.e. $o(h_{i,1}^{(1)}) = \sum_{j=1}^{i} m_j o(g_j)$ with $m_j \in \mathbb{N}$. Hence $o(h_{i,1}^{(1)}) < o(h_{i,1}^{(1)} - \prod_{j=1}^{i} g_j^{m_j})$. Let us denote $h_{i,1}^{(1)} - \mu \cdot \prod_{j=1}^{i} g_j^{m_j}$ by $\bar{h}_{i,1}^{(1)}$ and without loss of generality we can suppose $\bar{h}_{i,1}^{(1)}$ "monic". Clearly $R = k[[g_1, g_2, \ldots, g_i, \bar{h}_{i,1}^{(1)}, h_{i,2}^{(1)}, \ldots, h_{i,t_1}^{(1)}]]$.

Consider now the ordered list $[g_1, g_2, \ldots, g_i, h_{i,1}^{(2)}, \ldots, h_{i,t_2}^{(2)}]$ obtained by ordering the elements of $\{g_1, g_2, \ldots, g_i, \bar{h}_{i,1}^{(1)}, h_{i,2}^{(1)}, \ldots, h_{i,t_1}^{(1)}\}$ and denote it by $\mathcal{F}_{i,2}$. As above we have

$$\text{either } o(h_{i,1}^{(2)}) \notin \langle \mathcal{G}_i \rangle \text{ or } o(h_{i,1}^{(2)}) \in \langle \mathcal{G}_i \rangle.$$

By Proposition 2.3 and by $o(h_{i,1}^{(1)}) < o(h_{i,1}^{(2)})$, we will find, after a finite number of steps, an ordered list of "monic" elements $\mathcal{F}_{i,s} = [g_1, g_2, \ldots, g_i, h_{i,1}^{(s)}, \ldots, h_{i,t_s}^{(s)}]$ with either $o(h_{i,1}^{(s)}) \notin \langle \mathcal{G}_i \rangle$ or, if $\langle \mathcal{G}_i \rangle$ is a numerical semigroup, $c(\langle \mathcal{G}_i \rangle) \leq o(h_{i,1}^{(s)})$.

If we are in the second case, then, by Proposition 2.2, the algorithm terminates. In fact $k[[g_1, g_2, \ldots, g_i]] \subseteq R$ and, by definition of $c(\langle o(g_1), o(g_2), \ldots, o(g_i) \rangle)$, we have that for every $f \in R^*$, $o(f)$ can be written as a sum of $o(g_1), o(g_2), \ldots, o(g_i)$, i.e. $o(k[[g_1, g_2, \ldots, g_i]]) = o(R)$. Then $\{g_1, g_2, \ldots, g_i\}$ is an order basis.

Consider hence the first case. Let us denote $h_{i,1}^{(s)}$ by g_{i+1}, the set $\mathcal{G}_i \cup \{g_{i+1}\}$ by $\mathcal{G}_{i+1}$ and the ordered list of "monic" elements $[g_1, g_2, \ldots, g_i, h_{i,1}^{(s)}, \ldots, h_{i,t_s}^{(s)}] = [g_1, g_2, \ldots, g_{i+1}, h_{i+1,1}^{(1)}, \ldots, h_{i,t_s-1}^{(1)}]$ by $\mathcal{F}_{i+1,0}$.

We note that, using the same argument as above, after a finite number of steps the algorithm terminates since, by Proposition 2.4 $o(R)$ is a numerical semigroup and these are finitely generated.

Proposition 3.2. *Let R be a ring of our type. Then R has a unique order basis $\{h_1, h_2, \ldots, h_s\}$ where $h_i \in k[X]$, and $h_i = X^{o(h_i)} + \sum_{j \geq 1} b_{i_j} X^{i_j}$ where $b_{i_j} \neq 0$ implies $i_j \notin o(R)$.*

Proof. Let $\{g_1, g_2, \ldots, g_s\}$ be a "monic" order basis of R. Then $R = k[[g_1, g_2, \ldots, g_s]]$, $o(R) = \langle o(g_1), o(g_2), \ldots, o(g_s) \rangle$ and $g_i = X^{o(g_i)} + \sum_{j \geq 1} a_{i_j} X^{i_j} \in k[[X]]$.

For every g_i let us consider the set $\{i_j \mid i_j \in o(R), i_j < c(o(R))\}$. Suppose that this set is non empty and let $f \in R^*$ such that $o(f) = i_{j_0} = \min\{i_j \mid$

$i_j \in o(R), i_j < c(o(R))\}$. Then there exists $\lambda \in k^*$ such that $\overline{g}_i = g_i - \lambda f = X^{o(g_i)} + \sum_{j\geq 1} b_{\overline{i_j}} X^{\overline{i_j}}$ with $i_{j_0} < \overline{i_j}$ for every j such that $\overline{i_j} \in o(R)$.

Continuing in this way, we have that we can represent the order basis with elements $k_i = X^{o(g_i)} + \sum_{j\geq 1} b_{i_j} X^{i_j}$ with $\{i_j \mid i_j \in o(R), i_j < c(o(R))\} = \emptyset$, that is $k_i = X^{o(g_i)} + b_{i_1} X^{i_1} + \cdots + b_{i_r} X^{i_r} + \sum_{j>r} b_{i_j} X^{i_j}$ where $i_j \notin o(R)$ for every $j = 1, 2, \ldots, r$ and where $c(o(R)) \leq i_j$ for every $j > r$.

Since $o(\sum_{j>r} b_{i_j} X^{i_j}) \geq c(o(R))$, we have $\sum_{j>r} b_{i_j} X^{i_j} \in R$ by Proposition 2.5. Hence $h_i = X^{o(g_i)} + \sum_{j\geq 1}^{r} b_{i_j} X^{i_j} \in R$. Since $\langle o(h_1), \ldots, o(h_s)\rangle = o(R)$, we have $k[[h_1, ..., h_s]] = R$ by Proposition 2.2.

Now we prove that R has a unique representation $k[[h_1, h_2, \ldots, h_s]]$ as above. In fact if there exists another representation, say $R = k[[h'_1, h'_2, \ldots, h'_s]]$, where $h'_i = X^{o(h_i)} + \sum c_{i_j} X^{i_j}$ where $c_{i_j} \neq 0$ implies $i_j \notin o(R)$, we would have $h_i - h'_i = \sum_{i_j} (b_{i_j} - c_{i_j}) X^{i_j} \in R$ contradicting $i_j \notin o(R)$.

We call the unique order basis, as in the Proposition 3.2, *reduced*.

4 Examples and applications

In this section we use the algorithm given in the proof of Theorem 3.1 in some examples and applications.

Example 4.1. Let $R = k[[X^4 + X^5, X^6, X^{15} + X^{16}]]$, with k a field of characteristic zero. Then R is a one-dimensional ring and we will show that $\lambda_R(k[[X]]/R) < \infty$. Let us denote $X^4 + X^5$ by f_1, X^6 by f_2 and $X^{15} + X^{16}$ by f_3.

Then $\mathcal{F}_{1,0} = [X^4 + X^5, X^6, X^{15} + X^{16}]$ and $\mathcal{G}_1 = \{g_1 = X^4 + X^5\}$.

Let us consider the semigroup generated by $o(g_1)$, that is $\langle \mathcal{G}_1\rangle = \langle o(g_1)\rangle = \{0, 4, 8, 12, \ldots\} = 4\mathbb{N}$. Since $o(f_2) = 6 \notin \langle o(g_1)\rangle$, then we denote X^6 by g_2 and we consider the set $\mathcal{G}_2 = \mathcal{G}_1 \cup \{g_2\} = \{X^4 + X^5, X^6\}$.

Let us compute the generator of the kernel of the map

$$\phi : k[Y_1, Y_2] \to k[X]$$

$$Y_1 \to X^4, Y_2 \to X^6.$$

The kernel is generated by $Y_1^3 - Y_2^2$ and the corresponding element in R of the generator is $T(g_1, g_2) = g_1^3 - g_2^2 = (X^4 + X^5)^3 - (X^6)^2 = 3X^{13} + 3X^{14} + X^{15}$. Since $o(g_2) = 6 < 13 = o(T(g_1, g_2))$, we add $X^{13} + X^{14} + \frac{1}{3}X^{15}$ to the list $\mathcal{F}_{1,0}$. By ordering the elements of the set $\mathcal{F}_{1,0} \cup \{X^{13} + X^{14} + \frac{1}{3}X^{15}\}$ we have the ordered list $\mathcal{F}_{2,0} = [X^4 + X^5, X^6, X^{13} + X^{14} + \frac{1}{3}X^{15}, X^{15} + X^{16}]$. Since $o(X^{13} + X^{14} + \frac{1}{3}X^{15}) = 13 \notin \langle \mathcal{G}_2\rangle = \langle 4, 6\rangle$, then we denote $X^{13} + X^{14} + \frac{1}{3}X^{15}$ by g_3 and we consider the set $\mathcal{G}_3 = \mathcal{G}_2 \cup \{g_3\} = \{X^4 + X^5, X^6, X^{13} + X^{14} + \frac{1}{3}X^{15}\}$.

Now we have a numerical semigroup $\langle \mathcal{G}_3\rangle = \langle 4, 6, 13\rangle$ with $c(\langle \mathcal{G}_3\rangle) = 16$. Let us compute a minimal set of generators of the map

$$\phi : k[Y_1, Y_2, Y_3] \to k[X]$$

$$Y_1 \to X^4, Y_2 \to X^6, Y_3 \to X^{13}.$$

The kernel is generated by $Y_1^3 - Y_2^2, Y_1^2Y_2^3 - Y_3^2$ and the corresponding elements in R are $T(g_1, g_2)$ and $T(g_1, g_2, g_3) = X^{27} - \frac{2}{3}X^{28} - \frac{2}{3}X^{29} - \frac{1}{9}X^{30}$. Since $o(T(g_1, g_2)) = 13 = o(g_3)$ and $c(\langle \mathcal{G}_3 \rangle) = 16 < 27 = o(T(g_1, g_2, g_3))$, we do not add $T(g_1, g_2)$ or $T(g_1, g_2, g_3)$ to the list $\mathcal{F}_{2,0}$

Since $o(X^{15} + X^{16}) = 15 \notin \langle \mathcal{G}_3 \rangle$, then we denote $X^{15} + X^{16}$ by g_4 and we consider the set $\mathcal{G}_4 = \mathcal{G}_3 \cup \{g_4\} = \{g_1, g_2, g_3, g_4\}$.

We note that the numerical semigroup $\langle \mathcal{G}_4 \rangle = \langle 4, 6, 13, 15 \rangle$ has conductor equal to 12. Since the order of the corresponding elements of the generators of the kernel of the map

$$\phi : k[Y_1, Y_2, Y_3, Y_4] \to k[X]$$
$$Y_1 \to X^4, Y_2 \to X^6, Y_3 \to X^{13}, Y_4 \to X^{15},$$

is greater than 12, then for every $f \in R^*$, we have that $o(f) = \sum_{i=1}^4 n_i o(g_i)$, $n_i \in \mathbb{N}$. Hence, by Proposition 2.2, the algorithm terminates since $k[[g_1, \ldots, g_4]] \subseteq R$ and $o(k[[g_1, \ldots, g_4]]) = o(R)$. Furthermore by Proposition 3.2, we have that $\{X^4 + X^5, X^6, X^{13}, X^{15}\}$ is the reduced order basis of R.

Remark 4.2. It is known (cf. [1, Section II.1]) that there exist relations between algebraic characters and invariants of the semigroup $o(R)$ and the ring R. Hence, in the Example 4.1, by $o(R) = \langle 4, 6, 13, 15 \rangle = \{0, 4, 6, 8, 10, 12, \longrightarrow\}$, we have that $\lambda_R(\overline{R}/R) = |[0, c(o(R))-1] \cap (\mathbb{N} \setminus o(R))| = 7$, $\lambda_R(R/(R : \overline{R})) = |[0, c(o(R))-1] \cap o(R)| = 5$, $t(R) \leq t(o(R)) = 3$, where $t(R)$ is the type of the ring R.

Example 4.3. Let $R = k[[X^4, X^6 + X^7, X^{13} + a_{14}X^{14} + a_{15}X^{15} + \ldots]]$ with k a field. Using the same argument as in the Example 4.1, we find that

if char $k \neq 2$, we have that if $a_{15} - a_{14} + 1/2 = 0$, then $\{X^4, X^6 + X^7, X^{13}\}$ is the reduced order basis of the R. Furthermore, since $\langle 4, 6, 13 \rangle$ is a symmetric numerical semigroup (cf. [2, Lemma 1]), then, by [3, Theorem], R is Gorenstein. Finally $\lambda_R(\overline{R}/R) = 8$. Otherwise if $a_{15} - a_{14} + 1/2 \neq 0$, then $\{X^4, X^6 + X^7, X^{13}, X^{15}\}$ is the reduced order basis of R with R a non Gorenstein ring. Furthermore $\lambda_R(\overline{R}/R) = 7$.

Otherwise, if char $k = 2$, then the reduced order basis of R is $\{X^4, X^6 + X^7, X^{13}, X^{15}\}$ and R is not a Gorenstein ring. Finally, $\lambda_R(\overline{R}/R) = 7$

Example 4.4. Let $R = k[[X^4 + \sum_{i \geq 5} a_i X^i, X^5 + \sum_{i \geq 6} b_i X^i, X^6 + \sum_{i \geq 7} c_i X^i]]$, with k a field. Using the same argument as in the Example 4.1, we have that $\{X^4 + l_1X^7, X^5 + l_2X^7, X^6 + l_3X^7\}$ is the reduced order basis of the Gorenstein ring R for some l_1, l_2, l_3. Furthermore, we have $\lambda_R(\overline{R}/R) = 4$.

Example 4.5. Let $R = k[[X^8, X^{12} + X^{14} + X^{15}]]$, with k a field of characteristic zero. Using the same argument as in the Example 4.1, we have that

$$\{X^8, X^{12} + X^{14} + X^{15}, X^{26} + X^{27} + X^{29} - \frac{1}{2}X^{31}, X^{53} + \frac{1}{2}X^{55} - \frac{1}{2}X^{57} - \frac{1}{8}X^{63} +$$

$$\frac{25}{8}X^{67} - \frac{95}{32}X^{71} - \frac{15}{16}X^{75} - \frac{135}{32}X^{83}\}$$

is the reduced order basis of the Gorenstein ring R. Furthermore, we have $\lambda_R(\overline{R}/R) = 42$.

Acknowledgements

The authors wish to thank R. Fröberg and C. Gottlieb for their helpful comments and for a careful reading of the manuscript.

References

[1] V. Barucci-D. E. Dobbs-M. Fontana, *Properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains*, Mem. Amer. Math. Soc. vol 125, **598** (1997).

[2] R. Fröberg-C. Gottlieb-R. Häggkvist, *On numerical semigroups*, Semigroup forum **35** (1987), 63-83.

[3] E. Kunz, *The value-semigroup of a one-dimensional Gorenstein ring*, Proc. Amer. Math. Soc. **25** (1970), 748-751.

[4] T. Matsuoka, *On the degree of singularity of one-dimensional analytically irreducible noetherian local rings*, J. Math. Kyoto Univ., **11-3** (1971), 485-494.