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THE LEMPERT FUNCTION IN THE BIDISC - A PARTIAL COMPUTATION

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1. Introduction

In the preprint [C-W] Jan Wiegerinck and the author give a counterexample to a conjecture by Coman, by showing that the Lempert function and the pluricomplex Green function are different in the bidisc for the case of two poles and different weights. However, in that proof the Lempert function is not computed explicitly. Here we give a partial computation.

Let Δ be the unit disc in $\mathbb{C}$ and Ω a domain in $\mathbb{C}^n$. We fix a finite subset A of $\Omega \times \mathbb{R}^+$, and write $A = \{(w_1, \nu_1), \dots, (w_k, \nu_k)\}$. For a point $z \in \Omega$, we let F_z denote the family of analytic discs $f : \Delta \to \Omega$ such that $f(0) = z$ and there exist $\zeta_1, \dots, \zeta_k \in \Omega$ with $f(\zeta_j) = w_j$, $j = 1, \dots, k$. For each disc $f \in F_z$, we define $d(f) = \sum_{j=1}^{k} \nu_j \log |\zeta_j|$, and finally $\delta(z) = \delta(z, A) = \inf_{f \in F_z} d(f)$. We call δ the *Lempert function* of Ω with respect to A.

2. The general setup of the computation

Let Ω be the bidisc, i.e., $\Omega = \Delta \times \Delta \subset \mathbb{C}^2$. We let the poles be $(a, 0)$ and $(b, 0)$, $a \neq b$, with the weights μ and ν respectively. We may assume that $\mu \geq \nu$. The goal of this note is to compute the corresponding Lempert function, but only when $a\bar{b} < 0$ and $z = (0, \gamma)$, $\gamma \in \Delta \setminus \{0\}$.

Let $f = (f_1, f_2)$ be an analytic disc in the family F_z. Then, by definition there are points $\zeta_1, \zeta_2 \in \Delta$, such that $f_1(\zeta_1) = a, f_1(\zeta_2) = b, f_1(0) = 0$ and $f_2(\zeta_1) = f_2(\zeta_2) = 0, f_2(0) = \gamma$. We can immediately see from the Schwarz lemma that $|\zeta_1| \geq |a|$ and $|\zeta_2| \geq |b|$. We call this Condition (II).

Let T_γ be the Möbius transformation $T_\gamma(w) = (w - \gamma)/(1 - \bar{\gamma} w)$, and put $\tilde{f}_2(\zeta) = (T_\gamma \circ f_2)(\zeta)$. Then $\tilde{f}_2(\zeta_1) = \tilde{f}_2(\zeta_2) = -\gamma$, and $\tilde{f}(0) = 0$. Here we see that $\min\{|\zeta_1|, |\zeta_2|\} \geq |\gamma|$, which we call Condition (IV).

Further, let $g_1(\zeta) = f_1(\zeta)/\zeta$ and $g_2(\zeta) = \tilde{f}_2(\zeta)/\zeta$, and extend these functions analytically in the obvious way. Then, by the Schwarz lemma, $g = (g_1, g_2)$ is an analytic disc such that $g_1(\zeta_1) = a/\zeta_1, g_1(\zeta_2) = b/\zeta_2$ and $g_2(\zeta_1) = \gamma/\zeta_1, g_2(\zeta_2) = \gamma/\zeta_2$. Let us call the family of discs with these properties G_z. Obviously, for any disc $g \in G_z$, we can construct a corresponding disc $f \in F_z$. Hence the optimization in the definition of the Lempert function can be done over the set G_z instead of F_z.

From the Pick-Nevanlinna interpolation theorem (see [G]) applied to g_1, we find that a necessary and sufficient condition for such a function to exist is that the matrix

$$P_1 = \begin{pmatrix} \frac{1-\left|\frac{a}{\zeta_1}\right|^2}{1-|\zeta_1|^2} & \frac{1-\frac{a\bar{b}}{\zeta_1\bar{\zeta_2}}}{1-\zeta_1\bar{\zeta_2}} \\ \frac{1-\frac{\bar{a}b}{\bar{\zeta_1}\zeta_2}}{1-\bar{\zeta_1}\zeta_2} & \frac{1-\left|\frac{b}{\zeta_2}\right|^2}{1-|\zeta_2|^2} \end{pmatrix}$$

is positive definite. Let Condition (I) be the statement that $\det(P_1) \geq 0$. Then the Pick-Nevanlinna condition is equivalent to (I) and (II) combined. Then we apply the same theorem to g_2. Now we find that the matrix

$$P_2 = \begin{pmatrix} \frac{1-\left|\frac{\gamma}{\zeta_1}\right|^2}{1-|\zeta_1|^2} & \frac{1-\frac{|\gamma|^2}{\zeta_1\bar{\zeta_2}}}{1-\zeta_1\bar{\zeta_2}} \\ \frac{1-\frac{|\gamma|^2}{\bar{\zeta_1}\zeta_2}}{1-\bar{\zeta_1}\zeta_2} & \frac{1-\left|\frac{\gamma}{\zeta_2}\right|^2}{1-|\zeta_2|^2} \end{pmatrix}$$

must be positive definite. This is equivalent to Condition (IV) combined with $\det(P_2) \geq 0$, which we call Condition (III).

Summing up we have four conditions:

$$\text{(I)} \quad \det(P_1) \geq 0,$$
$$\text{(II)} \quad |\zeta_1| \geq |a| \wedge |\zeta_2| \geq |b|,$$
$$\text{(III)} \quad \det(P_2) \geq 0,$$
$$\text{(IV)} \quad |\zeta_1|, |\zeta_2| \geq |\gamma|.$$

Let S be the set $(x, y) \in \mathbb{R}^2$ such that there exist $\zeta_1, \zeta_2 \in \Delta$, $\zeta_1 \neq \zeta_2$, with $|\zeta_1| = x, |\zeta_2| = y$ and such that conditions (I)-(IV) hold. We now want to minimize the function $L(x, y) := \mu \log x + \nu \log y$ over S. Clearly, the minimum must be attained at the boundary of S, i.e., at a point where one of the conditions (I)-(IV) is satisfied with equality.

The conditions (III) and (IV) are invariant under rotations in γ. Hence we may assume that $\gamma \in (0, 1)$.

3. Conditions (III) and (IV)

We will now take a closer look at Conditions (III) and (IV). Obviously, they only depend on the modulus of ζ_1, ζ_2 and the real part of $\zeta_1\bar{\zeta_2}$. Hence it is natural to write $x = |\zeta_1|, y = |\zeta_2|, \operatorname{Re} \zeta_1\bar{\zeta_2} = xyt$, where $-1 \leq t \leq 1$. The conditions now take the form

$$\frac{1 - \frac{\gamma^2}{x^2}}{1 - x^2} \cdot \frac{1 - \frac{\gamma^2}{y^2}}{1 - y^2} - \frac{1 + \frac{\gamma^4}{x^2y^2} - \frac{2\gamma^2 t}{xy}}{1 + x^2y^2 - 2xyt} \geq 0,$$
$$x, y \geq \gamma.$$

All denominators in the fractions that occur here are positive. After multiplying the first inequality by the three denominators and by x^2y^2, and after cancelling some positive factors, we can reduce Condition (III) to

$$(xy - \gamma)(x^2 + y^2 - 2xyt) \geq 0.$$

Now, the second bracket in the last expression is non-negative, and is strictly positive unless $x = y$ and $t = 1$. However, this would correspond to $\zeta_1 = \zeta_2$, which is not allowed. We conclude that the third condition can be replaced by $xy \geq \gamma$. This in turn implies the fourth condition. Therefore we can replace Conditions (III) and (IV) by the single condition

$$\text{(H)} \quad xy \geq \gamma,$$

(H for hyperbola).

4. Conditions (I) and (II)

Our assumption that $a\bar{b}$ is real and negative implies that Conditions (I) and (II) only depend on the modulus of ζ_1, ζ_2 and the real part of $\zeta_1\bar{\zeta}_2$, as was the case for the other conditions. We keep the notation x, y, t, and also set $c = |a|, d = |b|$. It follows that $a\bar{b} = -cd$.

The conditions now take the form

$$\frac{1-\frac{c^2}{x^2}}{1-x^2} \cdot \frac{1-\frac{d^2}{y^2}}{1-y^2} - \frac{1+\frac{c^2d^2}{x^2y^2}+\frac{2cdt}{xy}}{1+x^2y^2-2xyt} \geq 0,$$

$$x \geq c \wedge y \geq d.$$

Eventually we will minimize L under these conditions. The variable t does not occur in the function to be minimized, but is still important; in the minimization we are only allowed to choose x and y such that there exists t, $-1 \leq t \leq 1$, with Condition (I) satisfied.

After similar manipulations as in the previous section we can write Condition (I) as $Q_1 - Q_2 t \geq 0$, where

$$Q_1 = (x^2-c^2)(y^2-d^2)(1+x^2y^2) - (1-x^2)(1-y^2)(x^2y^2+c^2d^2),$$

and

$$Q_2 = 2xy[(x^2-c^2)(y^2-d^2) + cd(1-x^2)(1-y^2)].$$

Using Condition (II) we see that $Q_2 > 0$, hence we may write Condition (I) as $t \leq Q_1/Q_2$. Recall that t takes values in the interval $[-1,1]$. Hence, we can always find a suitable t as long as Q_1/Q_2 is at least -1. Therefore Condition (I) is equivalent to $Q_1+Q_2 \geq 0$. It turns out that Q_1+Q_2 can be factorized into a product of two cubic polynomials in x and y. Condition (I) takes the form

$$(dx+cdx-cy+cdy-x^2y+dx^2y-xy^2-cxy^2)\times$$
$$\times(-dx+cdx+cy+cdy-x^2y-dx^2y-xy^2+cxy^2) \geq 0.$$

Let us call the two factors C_1 and C_2 respectively.

Next, we need to take a closer look at these two cubics $C_i = 0$, $i = 1, 2$. We are interested in their behaviour in the rectangle $c \leq x \leq 1, d \leq y \leq 1$. We claim the following.

1. The curve $C_1 = 0$ passes through the points (c,d) and $(1,d)$, and it is a graph over the x-axis in the interval $[c,1]$.
2. The curve $C_2 = 0$ passes through the points (c,d) and $(c,1)$, and it is a graph over the y-axis in the interval $[d,1]$.
3. The only point of intersection between the two curves in the rectangle is (c,d).
4. The part of the rectangle where the Condition (I) holds is situated above $C_1 = 0$ and to the right of $C_2 = 0$. There we have $C_1, C_2 < 0$.

It is easy to check all these statements. For instance, to show that $C_2 = 0$ is a graph over the y-axes we compute

$$\frac{\partial C_2}{\partial x} = -(y^2+d)d(1-c) - 2xy(1+d) < 0.$$

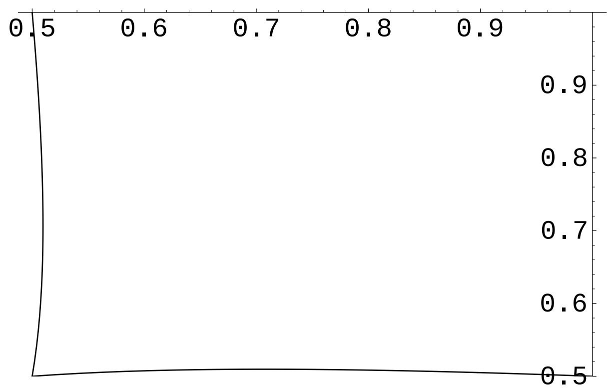

FIGURE 1. C_1 and C_2 $(c = d = 1/2)$.

We arrive at the following reformulation of Conditions (I) and (II) :

$$\text{(C)} \qquad C_1 < 0, \quad C_2 < 0,$$

(C for cubic).

It will turn out that only C_2 and not C_1 will play a role in the minimization. The reason is that $\mu \geq \nu$, hence it makes more sense to take a small x than a small y, and $C_2 < 0$ is an obstruction to that. On the other hand $C_1 < 0$ for all the points that interests us. See the graph of the region (Fig. 1).

5. THE MINIMIZATION

Having described the admissible region, we turn to the minimization itself. There are three cases.

Case 1. $\gamma \leq cd$.

In this case Condition (C) implies Condition (H); the hyperbola does not enter into the region defined by the two cubics (not even into the rectangle). We can always let $x = c, y = d$, and this gives us the minimum $\mu \log c + \nu \log d$. Hence the Lempert function is simply this constant.

Case 2. $c \leq \gamma$.

Here, we first minimize L over the hyperbola $xy = \gamma$ that is contained in the rectangle. It is easy to check that the gradient of L and the gradient of $xy - \gamma$ are non-parallel for all points on the hyperbola. The minimum is hence attained at an endpoint. A trivial calculation shows that the minimum is $\mu \log \gamma$, attained at the point $(\gamma, 1)$ (recall that $\mu \geq \nu$). If this point is allowed by Condition (C) we are done. This is easily seen to be the case. Thus the Lempert function becomes $\mu \log \gamma$.

Geometrically this means that the hyperbola is to the right of $C_2 = 0$, and they never intersect. We remark that the hyperbola might cut the other cubic, but that doesn't matter here. See Fig. 2.

Case 3. $cd < \gamma < c$.

This is the interesting case. It is natural to modify the idea in the previous case, i.e., to minimize L over the part of the hyperbola. This would give $(\mu - \nu) \log c + \nu \log \gamma$ in the point $(c, \gamma/c)$. However, it is easily seen that this

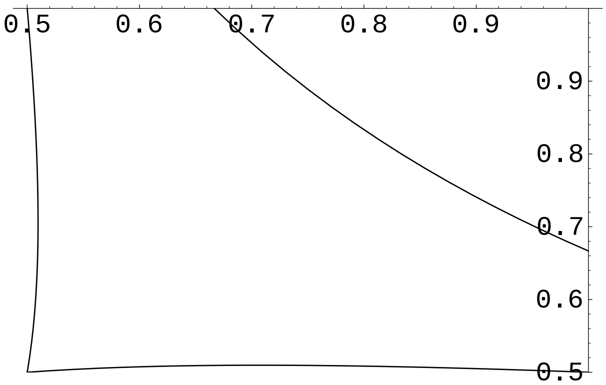

FIGURE 2. $\gamma = 2/3, c = d = 1/2$.

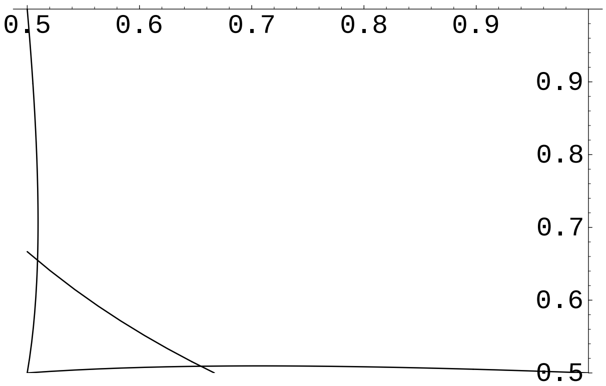

FIGURE 3. $\gamma = 1/3, c = d = 1/2$.

point violates Condition (C). We are led to look for the minimum in a point of intersection between the hyperbola and the cubic $C_2 = 0$. See Figure 3.

We now claim:

1. The gradient of L and the gradient of C_2 are non-parallel for all points on the cubic $C_2 = 0$.
2. There is exactly one point of intersection between the hyperbola and the cubic.

We omit the easy but tedious proof of the first statement. For the second, let $y = \gamma/x$ in the equation of the cubic and solve for x. The result is

$$x = \sqrt{\gamma}\sqrt{\frac{c(1+d) - \gamma(1-c)}{\gamma(1+d) + d(1-c)}}.$$

We conclude that the minimum must be attained either in this intersection point or possibly in the point where the cubic leaves the rectangle, i.e. in the point $(c, 1)$. We compute first the value of the function $\mu \log x + \nu \log y$ in the intersection point:

$$\begin{aligned}\mu\log x+\nu\log y &= (\mu-\nu)\log x+\nu\log\gamma\\ &= \frac{\mu-\nu}{2}[\log\gamma+\log\frac{c(1+d)-\gamma(1-c)}{\gamma(1+d)+d(1-c)}]+\nu\log\gamma\\ &= \frac{\mu+\nu}{2}\log\gamma+\frac{\mu-\nu}{2}\log\frac{c(1+d)-\gamma(1-c)}{\gamma(1+d)+d(1-c)}.\end{aligned}$$

The value in the other point is easily seen to be larger, hence we have found the Lempert function in this case.

6. Conclusion and an example

We have proved the following formula.

Proposition 6.1. *Let $a\neq b$ be two points in the unit disc such that $a\bar{b}<0$, $\mu\geq\nu>0$ and $0<|\gamma|<1$. Then the Lempert function for the bidisc with poles $(a,0),(b,0)$ and weights μ,ν is given by*

$$\delta((0,\gamma))=\begin{cases}\mu\log c+\nu\log d, & 0<|\gamma|\leq cd\\ \frac{\mu+\nu}{2}\log|\gamma|+\frac{\mu-\nu}{2}\log\frac{c(1+d)-|\gamma|(1-c)}{|\gamma|(1+d)+d(1-c)}, & cd<|\gamma|<c\\ \mu\log|\gamma|, & c\leq|\gamma|\end{cases}$$

where $c=|a|$ and $d=|b|$.

The Lempert function is not plurisubharmonic, since it does not equal the Green function (the latter equals $(\mu-\nu)\log c+\nu\log|\gamma|$ in the middle interval). However we can also see this directly from the formula. Recall that a subharmonic function in the unit disc that only depends on the modulus of z must be an increasing convex function of $\log|z|$. Replacing $|\gamma|$ by e^u and differentiating twice with respect to u, we obtain $(\mu-\nu)R(u)$, where R is negative in the critical interval. This shows that the Lempert function is subharmonic on the disc $\{0\}\times\Delta$ only if $\mu=\nu$, in which case we know that it equals the Green function in the whole bidisc.

In [C-W], the case $\mu=2,\nu=1$ is used in the given counterexample. The pluricomplex Green function is in this case

$$g((0,\gamma))=\begin{cases}2\log c+\log d, & 0<\gamma\leq cd\\ \log\gamma+\log c & cd<\gamma<c\\ 2\log\gamma, & c\leq\gamma\end{cases}$$

However, we have just seen that the Lempert function is

$$\delta((0,\gamma))=\begin{cases}2\log c+\log d, & 0<\gamma\leq cd\\ \frac{3}{2}\log|\gamma|+\frac{1}{2}\log\frac{c(1+d)-|\gamma|(1-c)}{|\gamma|(1+d)+d(1-c)}, & 1/4<|\gamma|<1/2\\ 2\log\gamma, & c\leq\gamma\end{cases}$$

We conclude this note with a graph for the case $a=-b=1/2$, $\mu=2,\nu=1$ (Fig. 4). It shows the Lempert function and the Green function in the critical interval $1/4\leq\gamma\leq1/2$.

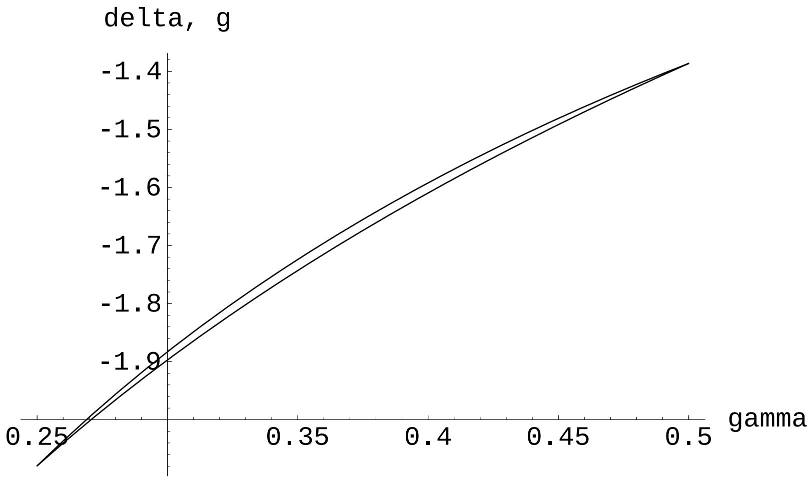

FIGURE 4. The Lempert function and the Green function.

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