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How to predict crashes in financial markets with the Log-Periodic Power Law

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Abstract

Speculative bubbles seen in financial markets, show similarities in the way they evolve and grow. This particular oscillating movement can be captured by an equation called Log-Periodic Power Law. The ending crash of a speculative bubble is the climax of this Log-Periodic oscillation. The most probable time of a crash is given by a parameter in the equation. By fitting the Log-Periodic Power Law equation to a financial time series, it is possible to predict the event of a crash. With a hybrid Genetic Algorithm it is possible to estimate the parameters in the equation. Until now, the methodology of performing these predictions has been vague. The ambition is to investigate if the financial crisis of 2008, which rapidly spread through the world, could have been predicted by the Log-Periodic Power Law. Analysis of the SP500 and the DJIA showed the signs of the Log-Periodic Power Law prior to the financial crisis of 2008. Even though the analyzed indices started to decline slowly at first and the severe drops came much further, the equation could predict a turning point of the downtrend. The opposite of a speculative bubble is called an anti-bubble, moving as a speculative bubble, but with a negative slope. This log-periodic oscillation has been detected in most of the speculative bubbles that ended in a crash during the Twentieth century and also for some anti-bubbles, that have been discovered. Is it possible to predict the course of the downtrend during the financial crisis of 2008, by applying this equation? The equation has been applied to the Swedish OMXS30 index, during the current financial crisis of 2008, with the result of a predicted course of the index.

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Foreword and Acknowledgments

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Emilie Jacobsson

1 Introduction

A crash in a financial market is a sudden and dramatic decline in price of an asset or an index over a short period of time, resulting in large negative percentage drops with devastating impact on the market.

According to standard economic theory, the price of an asset in a financial market fluctuates directly proportional to the supply and demand, which is a reflection of the continuous flow of news that is interpreted by analysts and traders. A crash occurs when traders panic and place too many sell orders at the same time.

When a crash is associated with a great piece of bad information, due to external events like September 11, 2001 and the outbreak of World War One in 1914, the crash is classified as exogenous. In the absence of an exogenous triggering shock, a crash classifies as endogenous and originates from the event of a speculative bubble. A common definition of a speculative bubble is when the price of an asset or index rise significantly, and becomes overvalued, as a result of positive feedback. A crash often result in a bear market with declining prices for months or years, but this is not always the case.

Due to the dramatic nature of crashes, numbers of academics and practitioners alike have developed interest in modeling the market, prior to crashes in order to predict such events. In 1996 two independent works of Feigenheim & Freund [2] and Sornette, Johansen & Bouchaud [11], introduced a critical phenomenon that possibly could describe the scenario of endogenous crashes in financial markets. They proposed that in times of a speculative bubble, an economic index increases as a power law decorated with a log-periodic oscillation and that the ending crash is the climax of the so called Log-Periodic Power Law (LPPL) signature. The Log-Periodic Power Law is quantified as a function of time, t given by:

$$y(t) = A + B\left(t_c - t\right)^z + C\left(t_c - t\right)^z \cos\left(\omega \log\left(t_c - t\right) + \Phi\right) \tag{1}$$

where t_c denotes the most probable time of the crash, z is the exponential growth, ω is controls the amplitude of the oscillations and A, B, C and Φ are simply units and carry no structural information.

Fitting the equation to financial data, the log-periodic oscillation captures the characteristic behavior of a speculative bubble and follows the financial index to the critical time of a crash. The hallmark of the equation is the fast accelerating price of the asset or index, and when t approaches t_c the oscillations occurs more frequently with a decreasing amplitude. The most probable time of the event of a crash is when $t = t_c$, and for $t \geq t_c$, the equation transcends to complex numbers. This precursory pattern makes it possible to identify the clear signatures of near-critical behavior before the crash.

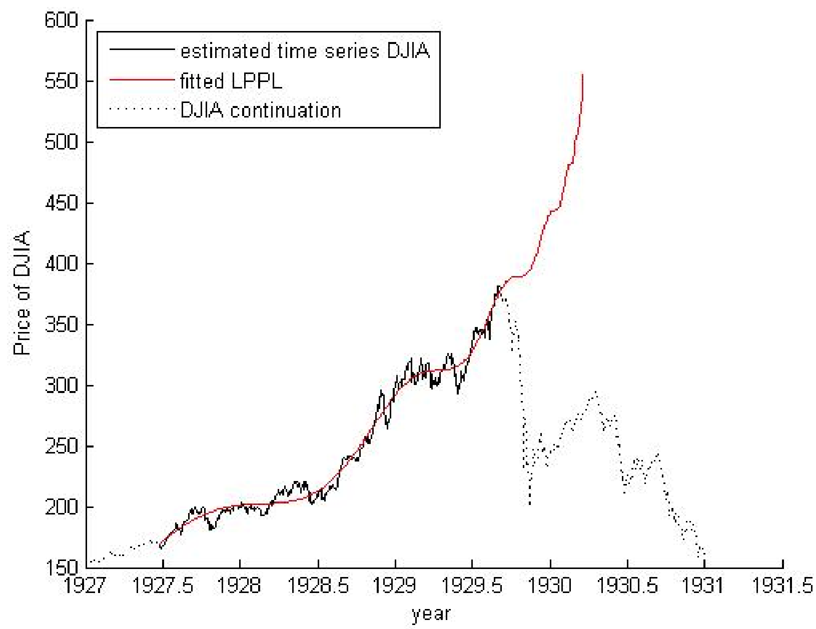

Figure 1: This is the LPPL fitted to the crash of the DJIA in October 24,1929. Parameters: A=569.988, B=-266.943, C=-14.242, tc=1930.218≈1930-03-19, Φ=-4.100, ω=7.877 and z=0.445.

In section 2, we will start by introducing a measurement of losses in a financial time series called drawdowns.This is a way to measure continuous down movements, calculated in percentage from a local maximum to the following local minimum. The largest negative drawdowns will single out abnormal movements as crashes in a financial time series, and a way to search for the log-periodic oscillation preceding a crash.

Johansen & Sornette [6] define crashes as outliers departing from a fit to a logarithmic expression of the complimentary cumulative Weibull distribution. We consider their usage of the term outlier as incorrect, when they apply a questionable fitting procedure and do not use the standard techniques of distribution fitting. Instead we define a crash in a financial market as the empirical quantile of 99.5%.

Sornette, Johansen and Ledoit [7] introduced a model of the market prior to crashes that could explain why stock markets crash, and a derivation of the Log-Periodic Power Law equation.

The hypothesis is that crashes in financial markets are a slow buildup of long-range correlations between individual traders leading to a global cooperative behavior that causes a collapse of the stock market in one critical instant. The framework of their model is roughly made up of two assumptions:

* The crash may be caused by local self- reinforcing imitations between "noise" traders.

* A crash is not a certain deterministic outcome of the bubble, and it is rational for traders to remain invested as long as they are compensated with higher rate of growth of the bubble for taking the risk of a crash.

These assumptions can be modeled, and together it gives a derivation to the LPPL-equation, which we will go through in section 3.

According to Johansen & Sornette [6], the log-periodic signatures have been detected for most of the speculative bubbles ending with crashes in the Twentieth century. The LPPL signatures can also be found for the opposite of a speculative bubble that is referred to as an anti-bubble, with a negative slope that results in a fast decline in assets price.

A number of papers have been written by Sornette & Johansen, not revealing their methodology and techniques used for crash predictions. This has been a subject of criticism and few have been able to replicate their work. In section 4, we will describe how to perform an estimation of the parameters in the LPPL equation with a method called a Hybrid Genetic Algorithm.

For the first time, we present time series of the OMXS30 index, exhibiting the features of the Log-Periodic Power Law, predicting the burst of the speculative bubble in 1998. We have also been able to fit the equation of an anti-bubble during the time of 2000-2003, predicting the turning point by a local minimum. The development of DJIA and S&P500 preceding the current financial crisis show the characteristics of the LPPL signature. The model predicted a turning point of the era of growth, but the severe crash of the fall 2008 was delayed. Analysis of the OMXS30 index shows that the current financial crisis of 2008 exhibits the pattern of log-periodicity for an anti-bubble. We wait with excitement to see when we will depart from the curve.

2 Drawdowns

We will start by introducing a measure of loss called a drawdown. It is used by Johansen & Sornette [6], as a tool to single out and define crashes in a financial markets. We will follow their methodology of calculating the drawdowns and how to fit the distribution. The indices to be examined are DJIA and OMXS30. For the DJIA, we will replicate the work of Johansen & Sornette [6], but for the first time we will fit a distribution of the OMXS30 index.

2.1 Drawdowns as a measurement of loss

The most common way to express price movements of a financial asset are returns. A return is the difference in price between times such as days, months, or year. Why do we need to introduce a new measurement of drawdowns? In case of a crash or an anti-bubble, the price of a financial index declines rapidly with large negative returns, where the largest ones are considered as extreme events on the market. The problem with returns is that their assumed independency and fixed time units do not capture the real dynamics of these extreme price movements.

To demonstrate the problem with a fixed time scale, such as for daily log-returns we use an example by Johansen & Sornette [8]. Consider a hypothetical crash of 30% over a time period of three days with successive losses of 10%. Looking at daily log-returns, this gives three independent losses of 10 % with no consideration that they occurred after one another. Consider a crash in a market, where a loss of 10% in one day, occurs about once every 4 years. A period of 4 years, with 250 trading days per year, gives the probability of 1/1000 to experience such a decline. The probability of experience three days in a row, with declines of 10%, will be eqaul to 10^{-9}, because the daily log-returns are considered to be independent. This equals to a crash of 30% that may occure once every 4 million years of trading, which is quite unrealistic.

This example is hypothetical, but shows very well how daily returns cannot accurately measure the risk of losses, since the world has experienced large drops over a few days, a number of times, in the 20th century. In times of crashes, but as well as for extreme rally days, it seems like the daily returns for a short period of time might be correlated and have a “burst of dependence”. If this is true, models based upon distributions of returns will not be valid for extreme events and there is a loss of information.

Thinking of time as a stochastic process is more accurate when we properly want to quantify the real dynamics of trading. Consider the series of 13 independent consecutive daily returns of the Dow Jones index and only consider the signs of them being positive (+), or negative (-) changes between the days: + + - - + - + - - - - - +. A drawdown, D, is defined as a persistent decrease in the value of the price at the closure for each successive trading day. A drawdown is the change in price, from a maximum to the next following minimum, calculated in percentage.

Definition: Let $\{P_{t=1,T}\}$ be a time series of a financial index price at closure.

Let $P_{max} = P_k$ be a local maximum.

Let $P_{min} = P_{k+n}$ be the next following local minimum, where $k \geq 2$ and $n \geq 1$. Here n denotes the number of days between the local maximum and the local minimum.

The following time series will define the drawdown, $P_k > P_{k+1} > ... > P_{k+n}$. The maximum, P_{max}, has to satisfy; $P_k \geq P_{k-1}$ and $P_k > P_{k+1}$. The minimum, P_{min}, has to satisfy; $P_{k+n} < P_{k+n-1}$ and $P_{k+n} \leq P_{k+n+1}$. The drawdown is calculated in percentage as;

$$D = \frac{P_{min} - P_{max}}{P_{max}} \tag{2}$$

For the 13 daily returns of the Dow Jones index, it means that the first drawdown lasted for 2 days, the second for 1 day and the third lasted for 5 days.

Hence, the measure of drawdowns captures important information of time dependence of price returns and measures a "memory" of the market. Drawdowns give a measure of what cumulative loss an investor may suffer from buying at a local maximum and selling at the next local minimum, and therefore quantify a worst-case scenario.

2.2 Distribution of drawdowns

Crashes are rare and extreme events on financial markets associated with large drawdowns. The question addressed by Johansen & Sornette [6], is whether these large drawdowns can be distinguished from the rest of the total population of drawdowns?

To be able to single out the largest negative drawdowns as extreme events; the small and intermediate drawdowns need to be characterized in a statistical way. Johansen & Sornette [6], analyzed markets globally, such as stock markets, foreign currency exchange markets, bond markets and the gold market. The vast majority of the drawdowns analyzed in the markets are well parameterized by a stretched exponential function, which is the complimentary cumulative distribution function of the Weibull distribution, that is;

$$f(x) = \begin{cases} z\left(\frac{x^{z-1}}{\lambda^z}\right) & exp\left\{-\left(\frac{x}{\lambda}\right)^z\right\}, for\ x \geq 0 \\ 0, & for\ x < 0 \end{cases} \tag{3}$$

where z is the shape parameter and λ is scale parameter. The cumulative distribution function is given by;

$$F(x) = 1 - exp\left\{-\left(\frac{x}{\lambda}\right)^z\right\} \tag{4}$$

Hence, the stretched exponential function is;

$$\overline{F}(x) = exp\left\{-\left(\frac{x}{\lambda}\right)^z\right\} \tag{5}$$

This may be viewed as a generalization of the exponential function, with one additional parameter; the stretching exponent z. For the stretched exponential, z is smaller than one and the tail is flatter than an exponential function. If z is larger than one, the function is called a super exponential. When $z = 1$, the distribution corresponds to the usual exponential function.

For approximately 98% of all the drawdowns examined by Johansen & Sornette [6], the stretched exponential function could be fitted with a remarkable good result. The rest of the 2% could not be extrapolated by the distribution and should according to them be viewed as outliers. This yields to their definition of a crash as a large negative drawdown, which is departing with a great numerical distance, from a curve of a fitted stretched exponential.

A derivation of the stretched exponential is given in Appendix A.

2.3 Price coarse-graining drawdowns

A drawdown is the cumulative loss from the last maximum to the next minimum of the price, terminated by any increase in price, no matter how small. Financial data is very noisy and by introducing a threshold ε we ignore any movement under ε in the reverse direction. Now we have a new definition of drawdown called a price coarse-grained drawdown or a ε-drawdown.

The following algorithm is used for calculations of ε-drawdowns. First we identify a local maximum in the price and the following local minimum is identified as the start of the reverse direction with the size larger than ε. The reverse direction is called a drawup and is defined as a drawdown, but the percentage is calculated from a local minimum to the following local maximum. The drawup is the cumulative gain between successive trading days. Hence, the local minimum is where the ε-drawdown, for the first time, hits a local minimum that results in a following drawup $\geq \varepsilon$. The drawups are treated the same way as the ε-drawdown, by ignoring drawdowns of a smaller size than $-\varepsilon$.

We find a threshold ε by using the volatility σ ,

$$\sigma = \sqrt{N^{-1}\sum_{i=1}^{n}(r_{i+1} - E[r])^2} \tag{6}$$

where r is the log-returns given by:

$$r_{i+1} = \log p(t_{i+1}) - \log p(t_i) \tag{7}$$

The volatility measure is the relative rate at which the price moves up and down, between days. Instead of trying to find a good fixed threshold, it is better to use the volatility, since each index fluctuates in a uniqe way.

By applying this filter of the volatility between successive days, we get more continuous drawdowns and the presence of noise will be suppressed.

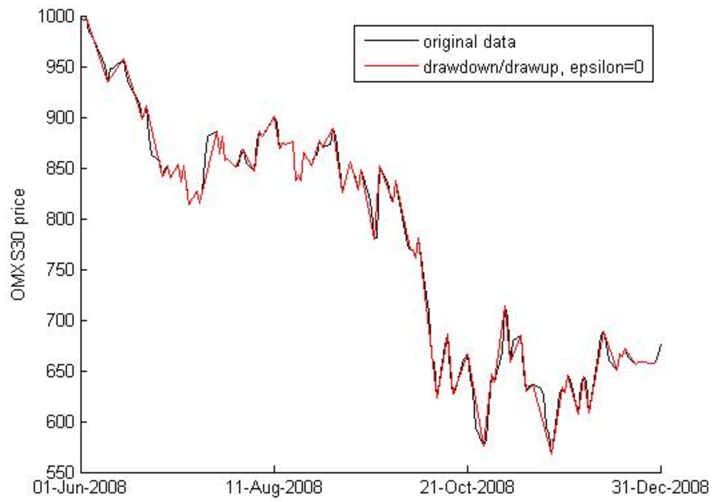

Figure 2: A demonstration of how the algorithm finds local maximums and local minimums in a financial time series. In the plot we see the OMXS30 price at closure and the drawdown algorithm when ε=0.

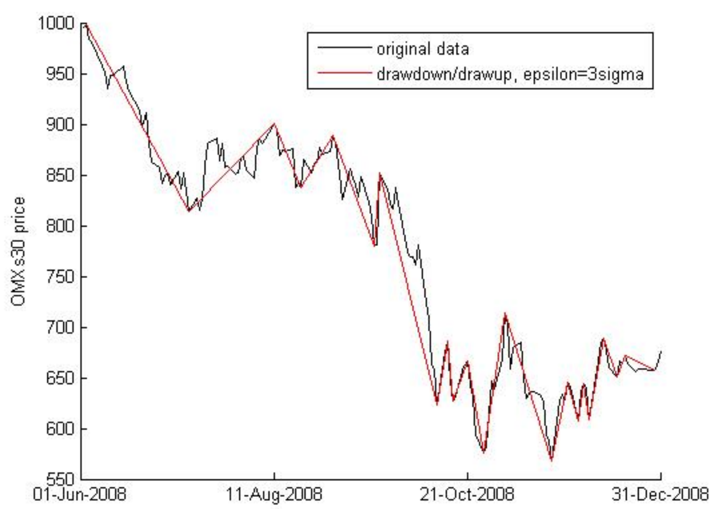

Figure 3: A demonstration of how the ε-drawdown algorithm works when $\varepsilon = 3\sigma = 0.0456$.

To prove result of the outliers, Johansen & Sornette [6] choose the following values of ε; $\sigma/4$, $\sigma/2$ and σ. They imply that with these choices of ε, some of the drawdowns are still going to depart from the distribution. This strengthens their result of that the outliers are in fact outliers. If we put ε large enough, it is possible to reduce most or all, of the outliers, hence a better fit to a distribution can be obtained. This may be viewed as the distribution of trends on financial markets.

2.4 The fitting procedure of the stretched exponential function

The statistical analysis of the drawdowns is done by applying the rank-ordering technique. The drawdowns are ordered in ascending order, $D_1 < D_2 < ... < D_N$, where D_1 is the largest negative drawdown and $n = 1, 2, \ldots, N$ is the rank number. D_n is plotted as a function of the rank number, hence the complementary cumulative distribution function can be analyzed.

By multiplying equation (5) by a constant A, we get an estimated order of rank;

$$N(D) = A \exp\left\{-b|D|^z\right\}, \text{ where } b = \lambda^{-z} \tag{8}$$

The extra parameter A is necessary in equation (8), because this parameter will adjust function to the height of the rank numbers, thus make it possible to fit the parameters of the distribution.

According to Johansen & Sornette [6], we will ensure the robustness of the fit by using the logarithmic expression of equation (8);

$$\log N(D) = \log A - b|D|^z \tag{9}$$

The fit is executed by a nonlinear regression of equation (9).

2.5 The fitted distributions

The fitting procedure is done for the Dow Jones Industrial Average from 1900-01-02 to 2008-06-17 and for the OMXS30 index from 1986-09-30 to 2009-06-17.

Table 1: Parametric fits of equation (9) for Dow Jones Industrial Average with σ=0.0115.

ε	A	b	z
0	8.9902	26.6911	0.7128
σ	8.4241	21.6291	0.7509
3σ	7.4747	16.3451	0.8220

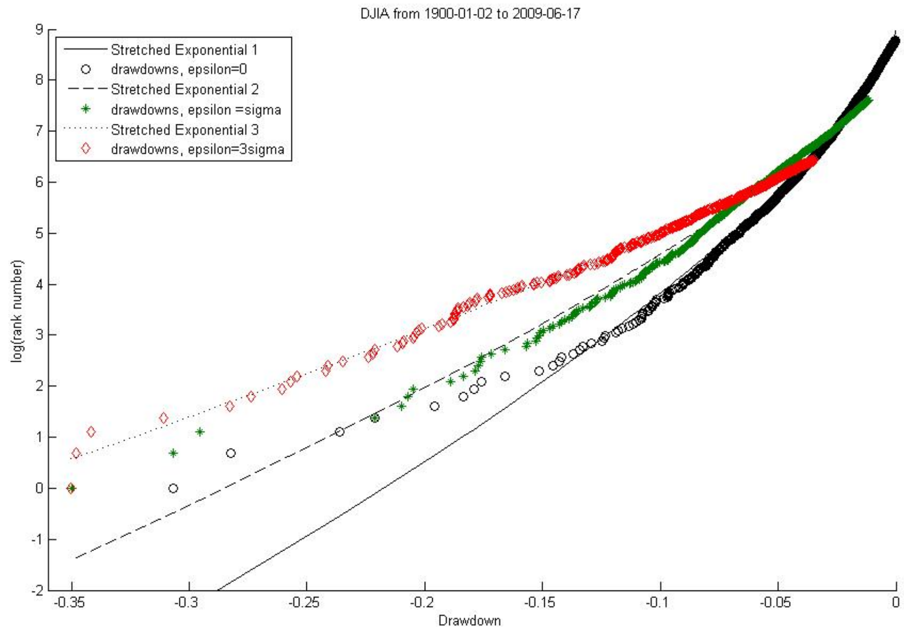

Figure 4: The stretched exponential fitted to the drawdowns of the DJIA when ε is 0, σ and 3σ. The outliers referred to by Johansen & Sornette [6] are visable over the fitted distributions in the lower left.

Table 2: The largest negative drawdowns for DJIA.

Rank	Size	Duration	Start date	Event
1	-0.3068	4	1987-10-13	Bubble
2	-0.2822	3*	1914-07-29	WW1 exogenous shock
3	-0.2822	2	1929-10-25	Bubble
4	-0.2211	8	2008-09-30	Financial crisis

Table 3: The largest ε-drawdowns when $\varepsilon = \sigma$.

Rank	Size	Duration	Start date	Event
1	-0.3499	25*	1914-07-08	WW1 exogenous shock
2	-0.3068	4	1987-10-13	Bubble
3	-0.2954	5	1929-10-22	Bubble
4	-0.2211	8	2008-09-30	Financial crisis

* The DJIA was closed from 1914-07-30 to 1914-12-12, due to the breakout of WW1.

Table 4: The largest negativeε-drawdowns, when $\varepsilon = 3\sigma$.

Rank	Size	Duration	Start date	Event
1	-0.3503	45*	1914-06-10	WW1 exogenous shock
2	-0.3480	13	1929-10-10	Bubble
3	-0.3416	11	1987-10-02	Bubble
4	-0.3109	25	1932-03-08	Great Depression

* The DJIA was closed from 1914-07-30 to 1914-12-12, due to the breakout of WW1.

Table 5: Parametric fits of the OMXS30-index when σ= 0.0152.

ε	A	b	z
0	8.9902	26.6911	0.7128
σ	8.4241	8.4241	0.7509

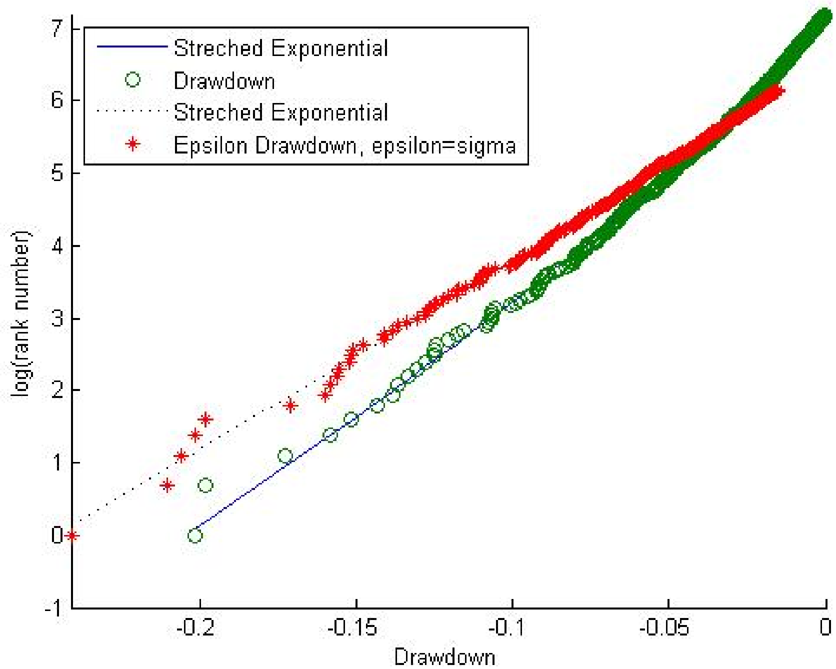

Figure 5: All the observations could be fitted to the stretched exponential without any observations departing from the curve. The largest values are to be found at the end of the tail of the distribution. We examine the ε-drawdowns when $\varepsilon = \sigma$ to test the result of that the largest drawdowns are in fact the largest ones.

Table 6: The largest negative drawdowns for OMXS30.

Rank	Size	Duration	Start date	Event
1	-0.2017	5	2008-10-03	Anti-Bubble
2	-0.1983	12	2001-08-24	Anti-Bubble*
3	-0.1729	7	1990-09-19	Swedish bank crisis, exogenous shock
4	-0.1586	5	2002-07-17	Anti-Bubble*

* The anti-bubble associated with the crash of the new economy.

Table 7: The largest negative ε-drawdowns, when $\varepsilon = \sigma$.

Rank	Size	Duration	Start date	Event
1	-0.2413	21	1990-08-30	Swedish bank crisis, exogenous shock
2	-0.2103	6	1987-10-21	Influence by the DJIA
3	-0.2062	16	1990-08-01	Swedish bank crisis, exogenous shock
4	-0.2017	5	2008-10-03	Anti-Bubble

2.6 Crashes as outliers?

The paradigm of Johansen & Sornette [6] is that the largest negative drawdowns cannot be extrapolated by the stretched exponential function and should be considered as outliers. They state that crashes are the outcome of a different population of drawdowns with another distribution.

For the OMXS30 index, the fit to the stretched exponential function was very good and there were no observations departing from the fit, with a great numerical distance. The lack of outliers does not state that the OMXS30 index has not experienced any crashes. The largest drawdown that the OMXS30 index has experienced was a severe drop of 20 %. This drawdown is consistent with the outlier of the DJIA index, with approximately the same percentage drop, and time period. Is there some way we can explain why some of the observations depart from the curve of the stretched exponential?

An outlier is an observation from a random sample that has an abnormal numerical distance from the rest of the numerical values of the data. The occurrence of an outlier may be caused by a measurement error, which is not the case here, or the outliers indicate that the distribution has a heavy tail. It may also be caused by a mixture model of two distributions.

The exponent z of the stretched exponential function is the parameter controlling the flatness of tail of the distribution, hence the fit to the large fluctuations. The fitting procedure is done by a nonlinear regression of equation (9), hence the best fit will be over the majority of small and intermediate drawdowns. Looking at figure 4, the outliers are located over the fitted stretched exponential, indicating a heavier tail of true distribution of the drawdowns.

To investigate why some of the observations cannot be extrapolated by the stretched exponential, we will take a closer look at the fitting procedure and the choice of distribution.

The first thing to examine is the statement of Johansen & Sornette [6]; it is "convenient and efficient" to fit the distribution by equation (9). In fact, estimating the parameters of equation (8) and (9) will result in two different distributions. To find out about the true parameters of the distribution we apply the maximum likelihood method, which is the most common way to fit a distribution to data. We will also linearize equation (5) to estimate the parameters by the linear least square method. This is done by calculating the empirical complementary cumulative distribution and use the following transformation;

$$\log(-\log(p)) = z\log(x) - z\log(\lambda) \tag{10}$$

The empirical complementary cumulative distribution is given by;

$$p_i = 1 - \frac{i-0.5}{N} \text{ for all } i = 1, ..., N \tag{11}$$

where N is the sample size.

Table 8: The table compares the parameter estimates.

method	equation	λ	z
nonlinear least square	9	0.00998	0.7128
nonlinear least square	8	0.0135	0.9262
least square	5	0.0142	0.9956
maximum likelihood	3	0.0145	0.9201

By looking at table (8), we conclude that the parameter estimates of the procedure proposed by Johansen & Sornette [6], cannot be accurate to the true distribution. Can this be the explanation of the bad parameterization resulting in observations departing from the fit? We will start by looking at the maximum likelihood estimate of the Weibull distribution to the empirical probability function.

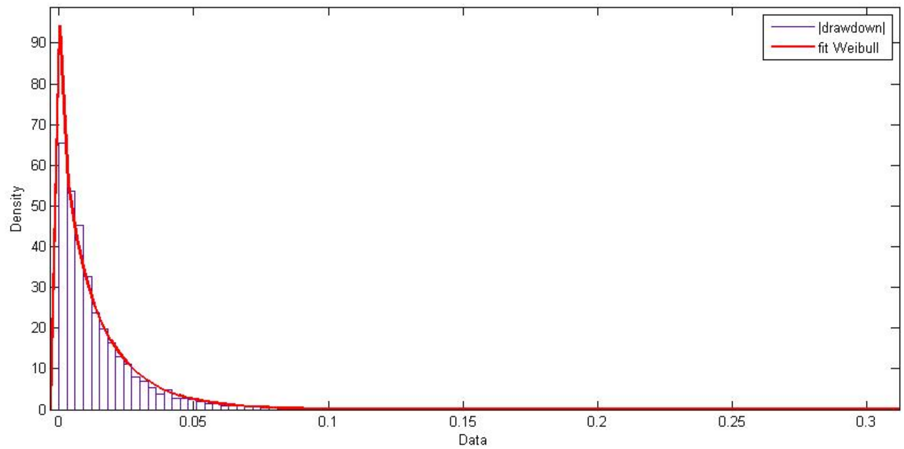

Figure 6: The maximum likelihood estimate of the Weibull PDF.

The following plot is the linearized Weibull complementary cumulative distribution function. It is easier to the human eye to determine a linear relationship rather than a curve; hence we will include the lines of the parametric fits of the maximum likelihood method and equation (9), which is denoted as "S&J fit". The parameters of equation (8) are that close to the maximum likelihood estimates that we will choose to exclude this fit in the plot.

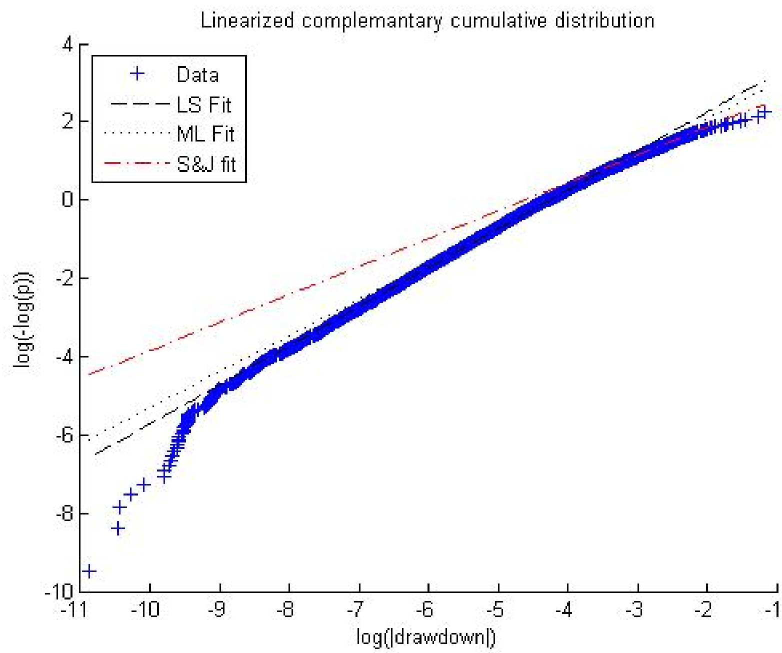

Figure 7: Linearized complementary cumulative Weibull function fitted by the least square method to the drawdowns of DJIA.

A good choice of distribution to data, will result in an approximately linear fit to the distribution. Here we see a curvature for both small drawdowns (lower left) and for larger ones (upper right). The best fit to the distribution is obviously the fit by the linear least square method, but the maximum likelihood method preformed good as well. The fitting procedure of equation (9), did as we suspected, estimate a different distribution.

The curvature of the small drawdowns can be seen as an "overshoot" of the density function when looking at figure (6). This poor fit can be found in the interval of $[0, 0.00025]$, which is the empirical quantile of approximately 1.8 %.

The curvature for the largest drawdowns can be found in the interval of $[0.035, 0.31]$, which equals the empirical quantile of 90%. This indicates once again that the true distribution of the data has a heavy tail. Looking at the distribution estimated by equation (9), we see that it fits for the largest drawdowns remarkably good. It seems like Johansen & Johansen have estimated the distribution of the tail. This is quite the opposite of their statement of crashes as outliers to the stretched exponential, fitted with this procedure.

The poor fit to the small and large drawdowns are troubling for the choice of the Weibull distribution to the data of the DJIA index. No other distribution taken into consideration has produced a better fit to this set of data. The data from the OMXS30 index could be fitted by the Weibull distribution with an excellent result.

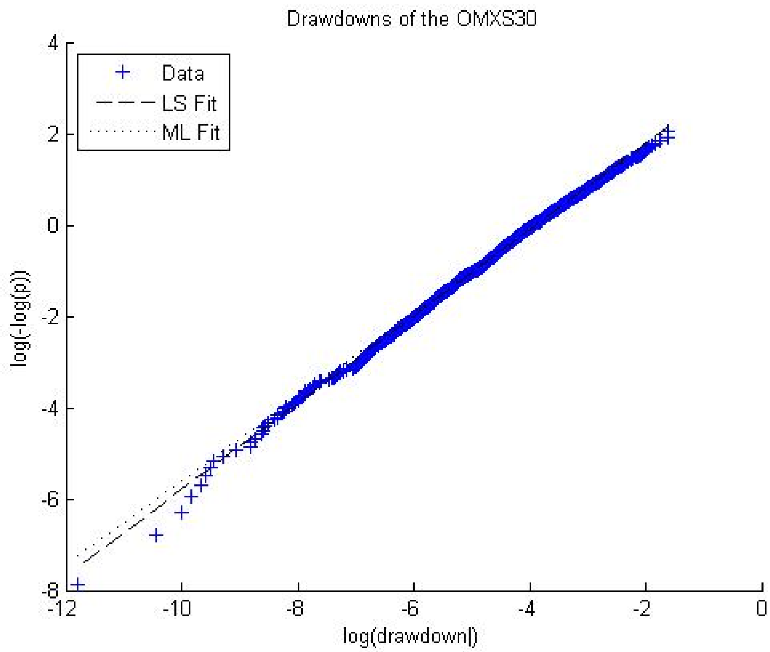

Figure 8: The Weibull fit to the drawdowns of OMXS30.

Table 9: The parametric fits of the drawdowns of OMXS30

method	equation	λ	z
maximum likelihood	3	0.0207	0.9156
least square	5	0.0206	0.9475

The discovery of the poor fit to the small drawdowns strengthens the importance of suppressing the noise and look at the ε-drawdowns. In order to examine the distribution of the ε-drawdowns, we need to transform the distribution to a three parameter Weibull. This is done by subtracting a location parameter c from all of the observations;

$$f(x) = \begin{cases} z\left(\frac{(x-c)^{z-1}}{\lambda^z}\right) & exp\left\{-\left(\frac{(x-c)}{\lambda}\right)^z\right\}, \text{for} \quad x \geq c, \text{ and } z, \lambda > 0 \\ 0, & for\ x < c \end{cases} \tag{12}$$

The parameter c is a fixed constant that equals ε. The result is;

Table 10: The estimates of the three parameter Weibull. Here $c = 3\sigma = 0.0345$.

method	equation	λ	z
least square	5	0.0466	0.9816
maximum likelihood	3	0.0460	1.0280

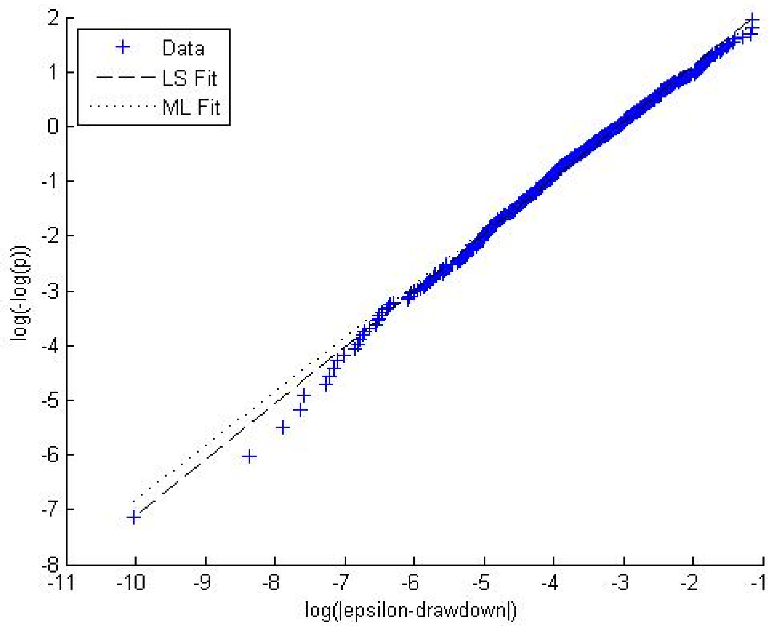

Figure 9: The complementary cumulative distribution of the ε-drawdowns of the DJIA index.

The fit to the epsilon-drawdowns is excellent, hence we have a good model for the distribution of trends.

We will make a new definition of crashes, with the benefit of not being dependent on any specific distribution. We define a crash as the 99.50% empirical quantile of the drawdowns. For the drawdowns of the DJIA is 10.50%, and for OMXS30 is 12.48%. Severe crashes are rare and do not occur very often. When a crash occurs, it is often associated with a drawdown larger than 10 %. Looking at the 99.5% quantile, all of the drawdowns are larger than 10% and the corresponding dates are historical events of financial crashes.

3 The model

The regime of a growth era can end in three ways; the regime can switch smoothly, it can end with a crash, or turning into an anti-bubble. Hence, a crash is not a deterministic outcome and has a probability of happen, or the regime can switch smoothly. The theory of this chapter is the model proposed by Sornette [10] and Johansen and *et. al.* [7], as the underlying mechanism of the log-periodic oscillation. We will start with some human characteristics that play a key role in choice of best strategy, when optimizing the profit. By modeling the best strategy in a macroscopic way and combining this with price dynamics, we can derive the equation of the LPPL signatures.

3.1 Feedback

Any price on a stock market fluctuates by the law of supply and demand. We can also describe the fluctuations of the price as a fight between order and disorder.

An agent on a stock market has three choices to consider; sell, buy or wait. During normal times on a stock market disorder rules, implying there are as many sellers as buyers and that the agents do not agree with one another. A crash happens when order rules, and too many agents want to sell at the same time. In times of a crash, the market gets organized and synchronized, which is quite the opposite of the common characterization of crashes as times at chaos. The question is, what mechanism can lead to an organized and coordinated sell off?

All traders in the world are organized into networks of family, friends, colleagues, etc. that influence each other by their opinions of whether to buy or sell. The optimal strategy for an agent on a stock market, with lack of information of how to act, is to imitate what others do, since the opinion of the majority will move the prices in that direction.

Trends on a stock market arise from feedback, that for a speculative bubble is positive and for an anti-bubble are negative. Feedback describes the situation where the output from a past event will influence the same situation in the present. Positive feedback, which moves the system in the same direction as the positive signal, is known as self-reinforcing loops and that drives a system out of equilibrium. In a stock market, positive feedback leads to speculative trends, which may dominate over fundamental and rational beliefs. A speculative bubble is a system driven out of equilibrium, and it is very vulnerable to any exogenous shock, that could trigger a crash. In real world, positive feedback is controlled by negative feedback as a limiting factor, and positive feedback do not necessarily result in a runway process.

3.1.1 Herding and imitation

Feedback are the cause of two psychological factors; herding and imitation.

Herding behavior occurs in everyday decision making, based on learning from others. The phenomenon is called information cascade and is also known as the Bandwagon effect in economics. This occurs when people observe the actions made by others and then make the same choice, independently of their own private beliefs. The main cause of this behavior is lack of information and the belief that they are better off mimicking others. Some people choose to ignore and downplay their own personal beliefs, rather than acting on them, because of the strong opinion of the crowd. As more people come to believe in something, others choose to "jump on the bandwagon", regardless of the evidence supporting the action, and the information cascade will spread with a domino-like effect.

People are rational and will always aim to maximize their utility. When people know that they are acting on limited information, they are willing to change that behavior. A single new piece of information can make a lot of people to subsequently choose a different action and in an instant turnover a previous established trend of acting. Therefore, information cascades are very fragile.

3.2 Microscopic modeling

Consider a network of agents and denote each one of them with an integer $i = 1, ..., I$. Let $N(i)$ be the set of agents that are directly connected with agent i. According to the concepts of herding and imitation, the agents in $N(i)$ will influence each other. Say we isolate one single trader i, and the network $N(i)$ consisting of $j = 1, \dots, n$ individuals surrounding that trader.

For every unit of time, each member in the network will be in one of two possible states; $selling = -1$ or $buying = +1$, which could represent being pessimistic or positive.

The sum of all agents on the market $\sum_{i=1}^{N} s_i$ will move the price proportionally, where s denotes the two states selling of bying. The best choice when the sum is negative is to sell and buy when the sum is positive. If the sum is zero, there are as many buyers as sellers, and the price should not change. Since this sum is unknown to the single trader, the best strategy that will maximize the return, is polling the known neighborhood and hopefully this sample will be a representation of the whole market. The trader has to form a priori distribution of the probability for each trader selling P_- and buying P_+. The simplest case correspond to a market with no drift, $P_- = P_+$

The general opinion in a network of a single trader is given by $\sum_{j=1}^{n} s_j$. The optimal choice is given by:

$$s_i(t+1) = sign\left(K\sum_{j=1}^{n} s_j(t) + \sigma\varepsilon_i\right) \tag{13}$$

where ε is a $N(0,1)$-distributed error compensating that the network did not represent the whole market correctly. K is a positive constant measuring the strength of imitation. The variable K is inversely proportional to the market depth, which gives the relative impact of a given unbalance of the aggregated buy and sells orders, depending on the size of the market; the larger market, the smaller impact of a given unbalance. σ governs the tendency toward idiosyncratic behavior, and together with the constant K, it determines the outcome of the battle between order and disorder on the market.

3.3 Macroscopic modeling

The variable K measures the strength of imitation and there exists a variable K_c that determines the property of the system of imitation. When $K < K_c$, disorder rules on the market, and the agents are are not agreeing with one another. As K gets closer to K_c, order starts appearing where agents who agree with each other form large groups, and the behavior of imitation is widely spread. At this point, the system is extremely unstable and sensitive to any small global permutation that may result in large collective behavior triggering a rally or a crash. It should be stressed that K_c is not the value for the crash to happen, it could happen anyway at any level of K, but the probability of the crash is not very likely for small values of K. In Natural Science, this is known as a critical phenomenon. The hallmark of a critical phenomenon is a power law, where the susceptibility goes to infinity. The susceptibility is given by:

$$\chi \approx A\,(K_c - K)^{-\gamma} \tag{14}$$

where A is a positive constant and γ is called the critical exponent.

Sornette & Johansen [7] model this imitation process by defining a crash hazard rate evolving over time. The critical point is now t_c, which is the most probable time of the crash to take place, and the death of a speculative bubble.

$$h(t) = B\,(t_c - t)^{-\alpha} \tag{15}$$

where t denotes the time and B is a positive constant. The exponent is given by $\alpha = (\xi - 1)^{-1}$, where ξ is the number of traders in one typical network. The typical trader must be connected to at least one other trader, hence $2 < \xi < \infty$. This result is an exponent that will always be smaller than one and the price will not continue to infinity.

There exists a probability:

$$1 - \int_{t_0}^{t_c} h(t)dt > 0 \tag{16}$$

that the crash will not take place and the speculative bubble will land smoothly. Hence, we see that the hazard rate behaves in the similar way as the approximated susceptibility.

3.4 Price dynamics

For simplicity of the model, Johansen & Sornette [7] choose to ignore interest rate, dividends, risk aversion, information asymmetry, and the market clearing condition.

The rational expectation model says that in an efficient market every agent takes all pieces of available information, looking rational into the future trying to maximize their well being. The aggregated predictions of the future price will be the expected value conditioned of the information revealed up to time t, which will be the price of the asset at time t. This gives the famous martingale hypothesis of rational expectations:

$$E_t[p(t')] = p(t), \text{ for all } t' > t \tag{17}$$

In a maket without noise, we get the solution of the martingale equation $p(t) = p(t_0) = 0$, where t_0 denotes the initial time. The rational expectation model can be interpreted as price in excess of the fundamental value and for a positive value of p(t) that constitutes a speculative bubble.

Assume for simplicity that when the crash takes place, the price drops by a fixed percentage $\kappa \in (0, 1)$. The dynamics of the price of an asset before the crash is given by:

$$dp = \mu(t)p(t)dt - \kappa p(t)dj \tag{18}$$

where j denotes a jump process taking the value 0 before the crash and 1 afterwards. The drift coefficient $\mu(t)$ is chosen so that the martingale condition is satisfied:

$$E_t[dp] = \mu(t)p(t)dt - \kappa p(t)h(t)dt = 0 \tag{19}$$

This yields:

$$\mu(t) = \kappa h(t) \tag{20}$$

The martingale condition states that investors must be compensated by a high return of the assets price to keep invested, when the probability of a crash rises. Hence, when the price rises, the risk of a crash increases, therefore is no such thing as a free lunch. Plugging equation (20) into equation (18) gives:

$$dp = \kappa h(t)p(t)dt - \kappa h(t)dj \tag{21}$$

When j=0 before the crash, we get an ordinary differential equation

$$p'(t) = \kappa h(t)p(t)$$

$$\frac{p'(t)}{p(t)} = \kappa h(t) \tag{22}$$

Integrating equation (22) on both sides gives the corresponding solution before the crash:

$$\log\left[\frac{p(t)}{p(t_0)}\right] = \kappa \int_{t_0}^{t} h(t')dt'$$

$$\log(p(t)) = \log(p(t_0)) + \kappa \int_{t_0}^{t} h(t')dt' \tag{23}$$

where

$$\int_{t_0}^{t} h(t')dt' = B \int_{t_0}^{t} (t_c - t')^{-\alpha} dt = -B \left[\frac{(t_c - t')^{-(\alpha+1)}}{\alpha + 1}\right]_{t'=t}^{t'=t} =$$

$$= -B \left(\frac{2t_c - t - t_0}{\alpha + 1}\right) = \{\text{let } t_0 = t_c \text{ and } \beta = \alpha + 1 \in (0,1)\} = -\frac{B}{\beta}(t_c - t)^{\beta} \tag{24}$$

Plug equation (24) into equation (23):

$$\log(p(t)) = \log(p(t_o)) - \frac{\kappa B}{\beta}(t_c - t)^{\beta} \tag{25}$$

We have now reached the final step of deriving the equation of the Log-Periodic formula. Recall equation (14) of the susceptibility of the system $\chi \approx A\,(K_c - K)^{-\gamma}$ that we imitated by the hazard rate. We will make the assumption of the susceptibility of the system to be a bit more realistic, by letting the exponent of the power law be a complex number. When the exponent is a complex number, one can model a system with hierarchical structures in a network. In the real world, this is a more accurate way of describing the market, since different agents hold different amounts of shares and their decision of buying and selling has different impact on the market. The most powerful agents on the market, who are on the top of the hierarchy holding large amounts of shares, are pension funds etc. A more general version of the equation describing the susceptibility is sums of terms like, $\chi \approx A\,(K_c - K)^{-\gamma}$with complex exponents.

$$\chi \approx Re\left[A_0\,(K_c - K)^{-\gamma} + A_1\,(K_c - K)^{-\gamma + i\omega} + ...\right] \approx$$

$$\approx A_0'\,(K_c - K)^{-\gamma} + A_1'\,(K_c - K)^{-\gamma}\, cos(\omega \log(K_c - K) + \psi)... \tag{26}$$

where A_0', A_1', ω and ψ are real numbers, and Re[] denotes the real part of a complex number. These accelerating oscillations are called log-periodic, and their frequency are $\lambda = \omega/2\pi$.

Following the same step, we can back up the hazard rate:

$$h(t) \approx Re\left[B_0\,(t_c - t)^{-\gamma} + B_1\,(t_c - t)^{-\gamma + i\omega} + ...\right] \approx$$

$$\approx B_0'\,(t_c - t)^{-\gamma} + B_1'\,(t_c - t)^{-\gamma}\, cos(\omega \log(t_c - t) + \psi) \tag{27}$$

Plugging equation (27) into equation (25), we get the evolution of the price before the crash:

$$\log(p(t)) = \log(p(t_o)) - \frac{\kappa}{\beta}\Big(B_0\,(t_c-t)^{\beta} + B_1'\,(t_c-t)^{-\gamma}\,cos(\omega\log(t_c-t)+\psi)\Big) \tag{28}$$

Simplifying this gives:

$$y = A + B\,(t_c-t)^{z} + C\,(t_c-t)^{z}\,cos\,(\omega\log\,(t_c-t)+\Phi)$$

and we now have the same expression as equation (1).

3.5 The equation

Following the recommendations of Gazola, Fernandez, Pizzinga, Riera [3], the qualification of the parameters in the equation (1) will be:

$y(t)$ should be the logarithm of the price if we follow the derived formula. Indeed, the LPPL-equation may be fitted to any choice of presentation of the financial data such as price at closure or normated prices.

A has to be larger than zero and equals the predicted price of the asset at the time of the crash $p(t_c)$. For a bullish speculative bubble and for a bearish anti-bubble B has to be smaller than zero. If B is larger than zero in case of an anti-bubble, this is referred to as a bullish anti-bubble and is discussed in Zhou & Sornette [13]. $C \neq 0$ to ensure the log-periodic behavior.

The parameter t_c is the most probable time of the crash, if it actually occurs.

z will lie between $[0,1]$ and the empirical findings of J&S [6] for speculative bubbles are $z = 0.33 \pm 0.18$.

ω controls the amplitude in the oscillations and will lie between $[5,15]$. The empirical findings of J&S [6] for speculative bubbles are $\omega = 6.36 \pm 1.56$.

ϕ has not been considered elsewere, but for Gazola *el. al.* [3] it is assumed to be $[0, 2\pi]$.

A, B, C and Φ are simply units and carry no structural information.

Equation (1) exemplifies two characteristics of a bubble with a preceding endogenous crash:

* A growth that is faster than an exponential growth, which is captured by the power law $B\,(t_c-t)^{z}$.

* An accelerating oscillation decorating the power law quantified by the term $cos\,(\omega\log\,(t_c-t)+\Phi)$.

4 The fitting process

Fitting a set of financial data to a complex equation, like equation (1), involves a number of considerations to secure that the best possible fit is being obtained.

4.1 Number of free parameters

In order to reduce the number of free parameters in the fit, the three linear variables have been "slaved" into the fitting procedure and calculated from the obtained values of the nonlinear parameters, suggested by Sornette & Johansen [7]. This is done by rewriting equation (1) as:

$$y(t) \approx A + Bf(t) + Cg(t) \tag{29}$$

where

$$f(t) = \begin{cases} (t_c - t)^z & \text{for a speculative bubble} \\ (t - t_c)^z & \text{for an anti-bubble} \end{cases} \tag{30}$$

$$g(t) = \begin{cases} cos\left(\omega \log\left(t_c - t\right) + \Phi\right) & \text{for a speculative bubble} \\ cos\left(\omega \log\left(t - t_c\right) + \Phi\right) & \text{for an anti-bubble} \end{cases} \tag{31}$$

For each choice of the nonlinear parameters, we obtain the best values of the linear parameters by using ordinary least square (OLS) method:

$$\begin{pmatrix} \sum_i^N y(t_i) \\ \sum_i^N y(t_i)f(t_i) \\ \sum_i^N y(t_i)g(t_i) \end{pmatrix} = \begin{pmatrix} N & \sum_i^N f(t_i) & \sum_i^N g(t_i) \\ \sum_i^N f(t_i) & \sum_i^N f(t_i)^2 & \sum_i^N f(t_i)g(t_i) \\ \sum_i^N g(t_i) & \sum_i^N f(t_i)g(t_i) & \sum_i^N g(t_i)^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} \tag{32}$$

We write this system of equations in a compact representation:

$$\mathbf{X}'\mathbf{y} = (\mathbf{X}^{'}\mathbf{X})\mathbf{b} \tag{33}$$

where $X = \begin{pmatrix} 1 & f(t_1) & g(t_1) \\ \vdots & \vdots & \vdots \\ 1 & f(t_n) & g(t_n) \end{pmatrix}$ and $b = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

The general solution of this eqaution is:

$$\hat{\mathbf{b}} = (\mathbf{X}^{'}\mathbf{X})^{-\mathbf{1}}\mathbf{X}'\mathbf{y} \tag{34}$$

This leaves four physical parameters controlling the fit.

4.2 The optimization problem

Fitting a function to data is a minimization of the residual sum of squares, SSE, which is the objective function given by:

$$\underset{\theta}{min} F(\theta) = \sum_{i}^{N} \left(y_\theta(t_i) - \hat{y_\theta}(t_i) \right)^2, \text{where } \theta = (t_c, \phi, \omega, z) \tag{35}$$

The function of the residual sum of squares, $F(\theta)$ is not a well behaved convex function and consists of multiple local minimums with fairly similar values. The consideration is to find the global minimum, which is the local minimum with the smallest value.

The unconstrained global optimization problem, A. Georgiva & I. Jordanov [4], is defined as to find a point $P^* \in \Pi$ from a non empty, compact feasible interval domain $\Pi \subset \mathbf{R^n}$ that minimizes the objective function F:

$$F^* = F(P^*) \leq F(P), \text{ for all } P \in \Pi \tag{36}$$

where $F^* : \mathbf{R}^n \to \mathbf{R}$
Working in a parameter space of four nonlinear parameters, one cannot get a true picture of the characteristics of the objective function. To demonstrate the problems with optimizing the objective function, we use data from the speculative bubble of the DJIA that ended with a crash in October 1929. We let two parameters vary and keep the rest of the nonlinear parameters fixed. The parameters are varying over the given intervals, but now we let ϕ vary over $[-2\pi, 2\pi]$. The linear parameters are estimated by equation (34).

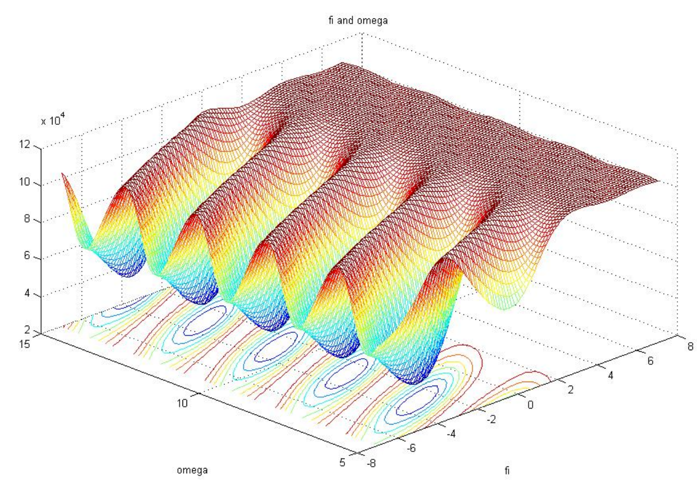

Figure 10: The objective function when Φ and ω are varying.

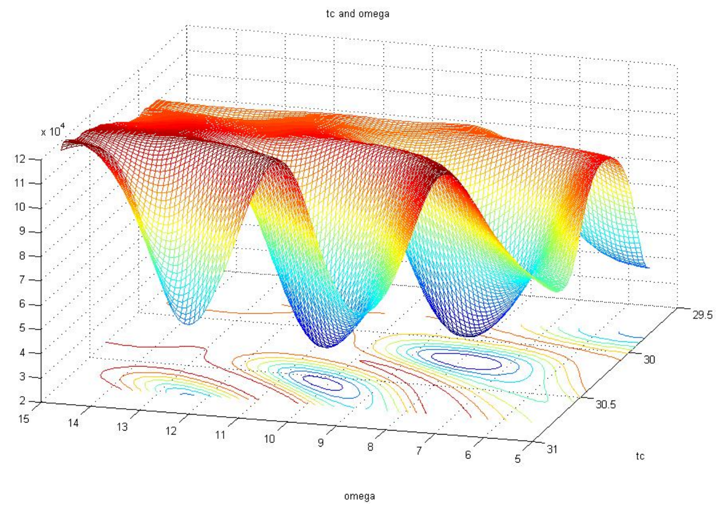

Figure 11: The objective function when ω and t_c are varying.

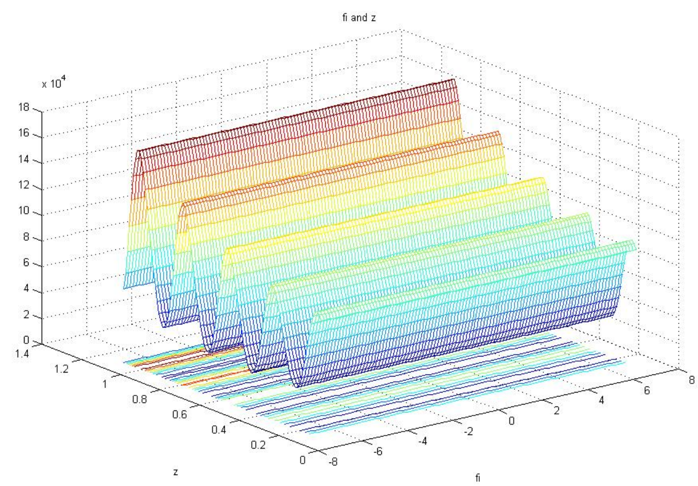

Figure 12: The objective function when Φ and z are varying.

With these plots it becomes clear when trying to optimize the objective function with methods like the downhill simplex method or the Quasi-Newton method, we risk to get trapped in local minimum, rather than finding the global minimum. These methods are a very efficient, but they can only find the local minimum in the basin of attraction in which the initial starting point is set at. How can we set the initial starting points in the right basin of attraction when we do not know where the global minimum is at? There are many ways to solve a global optimization problem such as Simulated Annealing, Taboo Search and Genetic Algorithm etc. The Genetic Algorithm was used here.

4.3 Genetic Algorithm

The Genetic Algorithm, GA, is an evolutionary algorithm inspired by Darwin's "survival of the fittest idea". The theory of the GA was developed by John Holland, known as the father of GA by his pioneer work in 1975.

The GA is a computer simulation that mimics the natural selection in biological systems governed by a selection mechanism, a crossover (breeding) mechanism, a mutation mechanism, and a culling mechanism. There are many ways to apply the mechanisms of the GA, here we follow the guidelines of Gulsten, Smith & Tate [5].

Algorithm 1 A pseudo code for the Genetic Algorithm

1. Randomly generate an initial solution set (population) and calculate the value of the objective function.
2. Repeat {

   Select Parents for crossover.

   Generate offspring.

   Mutate some of the members of the original population.

   Merge mutants and offspring to into the population.

   Cull some members of the population to maintain constant population size.

   }
3. Continue until desired termination criterion is met.

The main benefits of the GA are that no special information about the solution surface is needed, such as gradient or curvature. The objective function do not need to be smooth or continuous. The GA has been proven to be robust in finding the parameter values in the objective function.

4.3.1 Generating the initial population

Each member of the population is a vector of our coefficients (t_c, ϕ, ω, z). The initial population consists of members with coefficients drawn at random, from a uniform distribution with a pre-specified range. The population has 50 members and the fitness value, the residual sum of squares, is calculated for each member.

4.3.2 Breeding mechanism

The 25 superior individuals with the best value of fitness are selected from the population into the breeding mechanism. An offspring is produced by randomly drawing two parents, without replacement, and taking the arithmetic mean of them.

Table 11: An example of the breeding mechanism.

	t_c	ϕ	ω	z
parent A	30.02	-3.99	7.83	0.42
parent B	30.25	-4.15	8.20	0.46
offspring	30.135	-4.07	8.015	0.44

This is repeated 25 times where each pair of parents is drawn randomly with replacement, so one parent can produce offspring with another parent.

4.3.3 Mutation mechanism

Mutations in nature occur when offspring are born with a variation within the gene pool, which results in beneficial or less favorable characteristics. Beneficial mutations play a key role in evolution when it increases the fitness of the species.

In the GA, mutation perturbs solutions so that new regions of the search space can be explored. This is an important mechanism to prevent premature convergence in a local minimum. The mutation process is done by determine the range of maximum and minimum for each coefficient in the current population. By multiplying the range for each coefficient, with factor $\pm k$, we obtain the perturbation variable ε that is uniformly distributed over the interval [-k*(coefficient range), k*(coefficient range)]. The factor used here was $k = 2$. The range of the perbutation variable adapts to the extreme values found in the current popultation.

25 individuals are drawn randomly, without replacement, from the population of 50. For each selected individual, we add an exclusive vector of random perturbation values for every coefficient.

Because of the mutation mechanism, the parameters are free to drift away from the pre-set range of the parameter solutions, which will compensate if one took an inaccurate guess of the initial intervals in the solution space.

4.3.4 Culling mechanism

After breeding and mutation, we merge the newly generated individuals into the population, so that 100 solutions are present (50 old, 25 offspring, and 25 mutations). All of the 100 solutions are ranked according to their fitness value in ascending order, and the 50 best solutions are culled, and live on into the next generation. The rest is deleted.

4.3.5 A demonstration of the Genetic Algorithm

Recall figure (11), when we let tc and ω vary. We will perform an optimization of the objective function $\underset{\theta}{min} F(\theta) = \sum_i^N \left(y_\theta(t_i) - \hat{y_\theta}(t_i)\right)^2$, where $\theta = (t_c, \omega)$, using the Genetic Algorithm. The rest of the parameters are fixed. Here we have used the population size to 40 and the elite- and mutation number is 10. The algorithm is iterated 10 times after the initial population is created. The following pictures shows how the algorithm finds the global minimum.

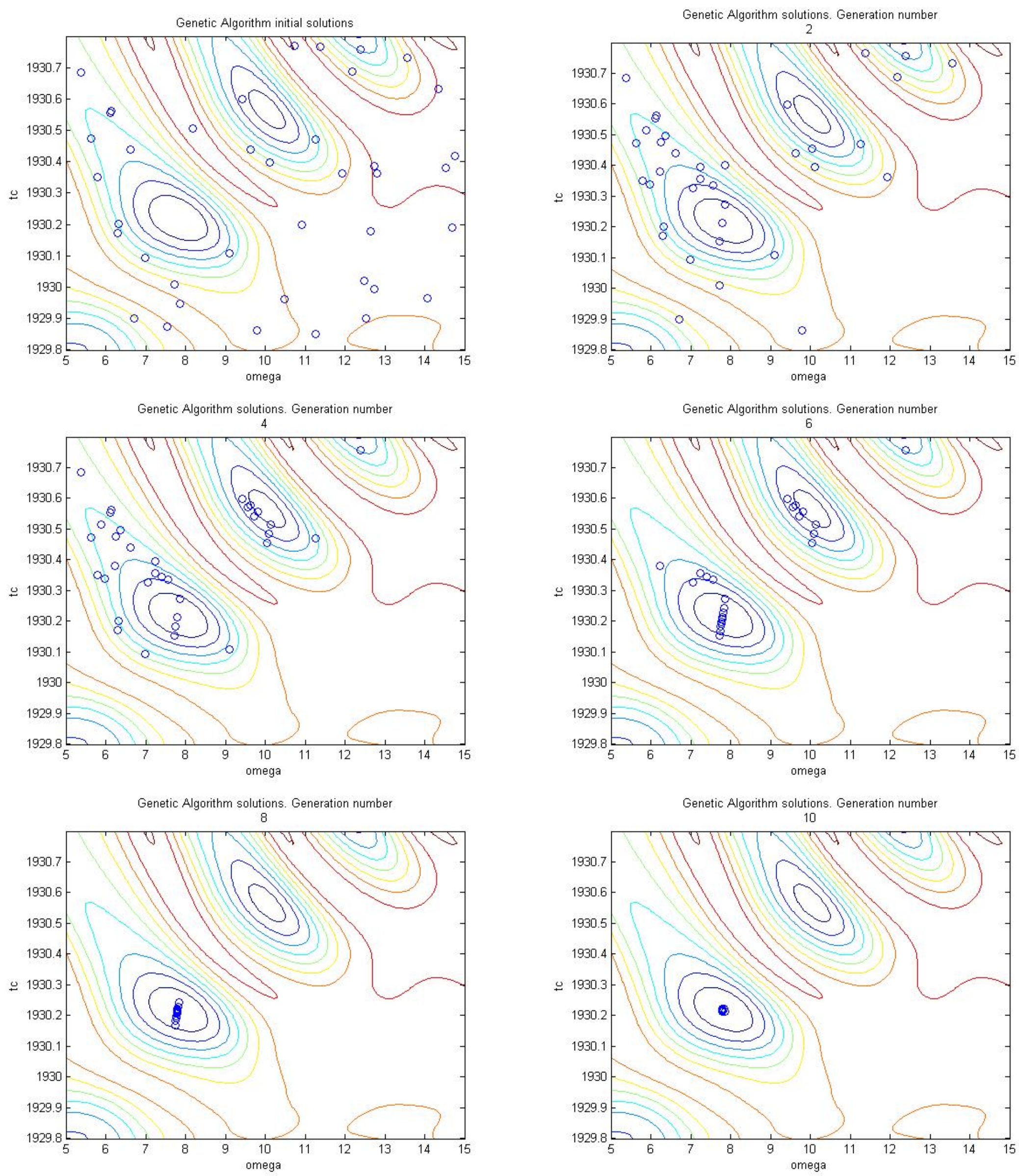

Figure 13: A demonstration of the Genetic Algorithm, when optimizing for t_c and ω. The algorithm is iterated 10 times and the generations in the pictures are from the upper left and down; the initial randomly drawn population and then generation number 2,4,6,8 and 10.

4.3.6 Hybrid Genetic Algorithm

To improve and guarantee the best result of the parameter estimation we apply a Nelder-Mead Simplex method, also known as the downhill simplex method, with the estimated parameters from the GA as the initial values. We use the same objective function as for the Genetic Algorithm, when estimating the nonlinear paramaters with the downhill simplex method.

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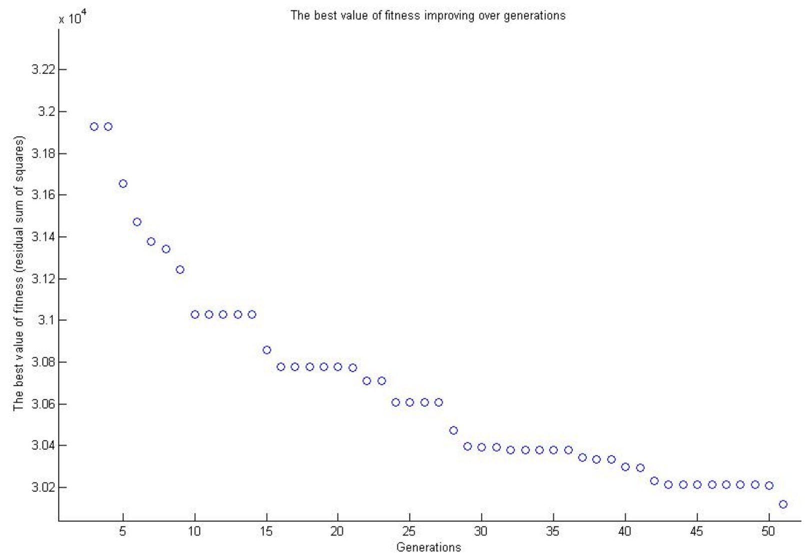

Figure 14: A demonstration how the GA improves the best value of fitness for the objective function over generations. Generation number 51 is the Nelder-Mead Simplex method improving the result with the estimated parameters from the GA as initial values. The LPPL-equation is fitted to the speculative bubble seen in DJIA that ended in a crash 1929.

4.4 Fitting the LPPL equation

Fitting the LPPL equation to historical data of a speculative bubble, one typically chooses the window of estimation with the start at the point where the LPPL signatures first seem to appear, and the peak of the price as the end. Including the data of the crash will be less favorable to the fit, and one risk is to hit the point of t_c, with the result of complex numbers that will destroy the fit. When fitting the LPPL equation for the purpose of prediction, the start will be at the point where the signatures start and then use all available data.

Fitting the LPPL equation to an anti-bubble, the fitting procedure is preceded in a different manner. For a speculative bubble, the end of the curve is at the point of t_c, recall $(t_c - t)$. The parameter t_c for an anti-bubble is the start of the LPPL signatures, hence $(t_c - t)$.

The consideration is where to set the start of the window of estimation for an anti-bubble. If we knew the time t_c, then the obvious choice would be a t_{start} next to t_c, which is the most optimal choice when it allows us to use the longest possible time span. Not knowing the critical time precisely, we have to ensure that $t_c < t_{start}$ to avoid complex numbers and ensure an accurate fit. This can result in a bit of work testing for different times of t_{start}, by trial and error.

If we remove the constraint $t_c < t_{start}$, by taking the absolute value of $|t_c - t|$, it is possible to scan for different t_{start} and attain a flexibility in the search space of t_c.

When optimizing with this new procedure, we use the peak as the start of the window of estimation, or at the point where the LPPL signature first seem to appear. The end is at the bottom of the anti-bubble for historical data, or the data available when fitting a current anti-bubble.

The fits produced by this new procedure are robust and stable, but face some minor problems. Adding an absolute value operator into the equation, it is assumed that an anti-bubble is always associated with a symmetric bubble with the same parameters, and the same value of t_c. This is not true, but we choose to neglect this when t_c with this new procedure often is very close to t_{start}, according to Sornette & Zhou [12]. A value t_c much larger than t_{start} will compromise the fit, as the GA is trying to optimize the symmetric bubble. If this happens, choose a larger value of t_{start}.

Another consideration about the optimization of the objective function, equation (35) is the presence of multiple local minimum with approximately the same value of fitness.

Table 12: Multiple local minimums with approximately the same fitness value . Here we optimize the parameters of the LPPL equation to the speculative bubble of DJIA, prior to the crash in October 1929. We optimize the equation over the standard intervals, but let ϕ take values in the interval $[-2\pi, 2\pi]$.

R^2	A	B	C	t_c	ϕ	ω	z
0.98307160189274	590.2883	-285.8670	-14.2892	30.2247	2.0879	7.9276	0.4204
0.98307160189287	590.2909	-285.8691	14.2892	30.2247	-1.0537	7.9276	0.4203
0.98307160189315	590.2842	-285.8628	14.2892	30.2247	5.2295	7.9276	0.4204
0.98307160189311	590.2804	-285.8596	-14.2892	30.2247	-4.1952	7.9276	0.4204

5 Results

This chapter presents the results of fitting the LPPL-equation to different financial time series. We follow the guidelines of Johansen & Sornette [6] when searching for the LPPL signature in financial time series. We look at the largest drawdowns and look at the time series preceding them. The ambition is to find the LPPL signature in the time series before and within the current financial crisis of 2008. If it is possible to fit the LPPL-equation to time series preceding the financial crisis, could we have been able to predict the crash? Is it possible to make a prediction of the down trend in the financial indexes resulting from the current financial crisis of 2008?

5.1 OMXS30

OMX Stockholm 30 is a value weighted index consisting of the 30 most traded stocks of the Stockholm stock exchange.

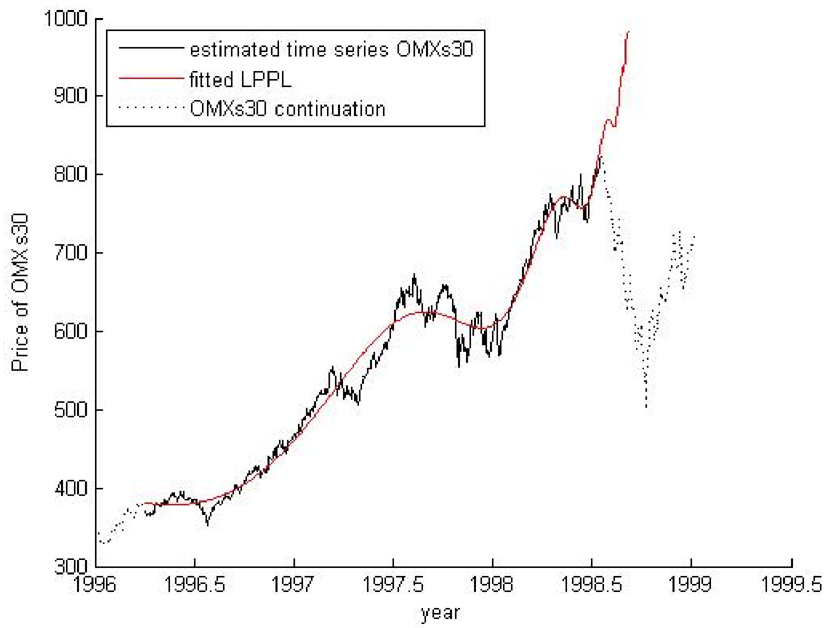

Figure 15: The speculative bubble of OMXS30 ending with a crash. The ε-drawdown for this period is the seventh largest negative with a start in 1998-08-18, a duration of 12 days and a drop of -16.02%.
Parameters: A=1077.799, B=-489.258, C=48.907, t_c=1998.686, Φ=5.556, ω= 5.518 and z=0.3424. The goodness of fit: R^2=0.9794

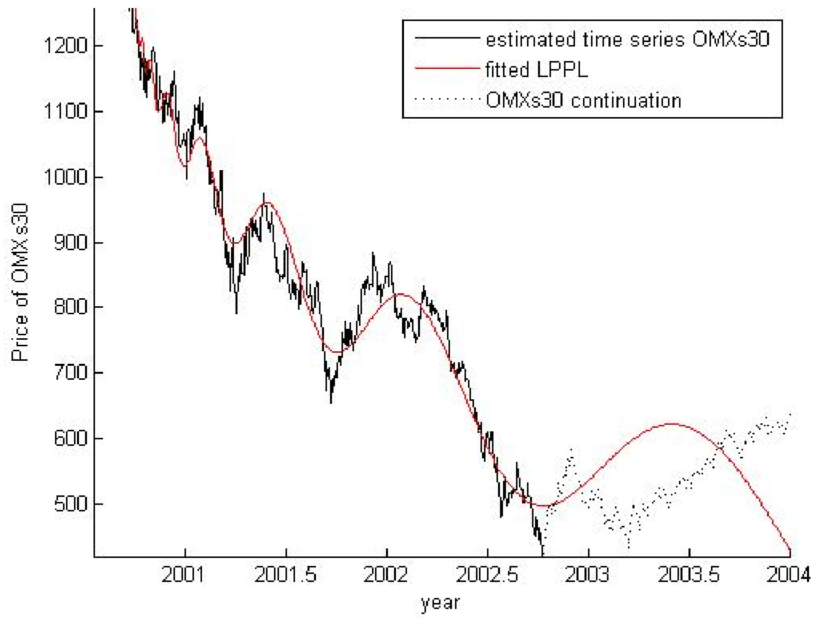

Figure 16: The LPPL-equation fitted to OMXS30. The crash of the new economy resulted in an anti-bubble that spread world wide affecting the OMXS30. The bottom of the anti-bubble was predicted by a local minimum of the LPPL equation.
Parameters: A=1299.98, B=-493.427, C=78.805, t_c=2000.74885, Φ=3.478, ω=8.982 and 0.4934.
The goodness of fit: R^2=0.961

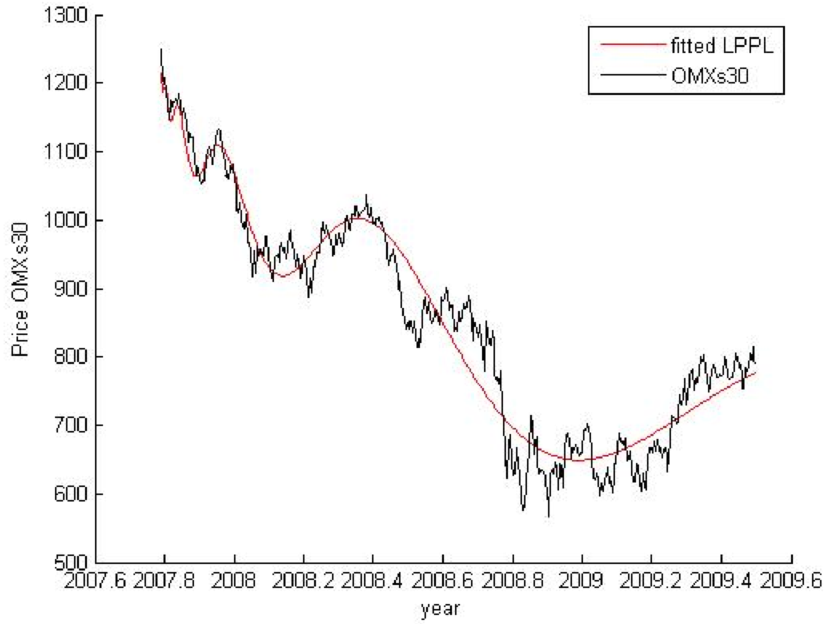

Figure 17: OMXS30 anti-bubble resulting from the worldwide spread financial crisis that started in the fall of 2008. Parameters: A=1235.168, B=-426.01, 120.264, t_c=2007.7886, Φ=2.6511, ω=5.1048 and z=0.4997.
The goodness of fit: R^2=0.9404

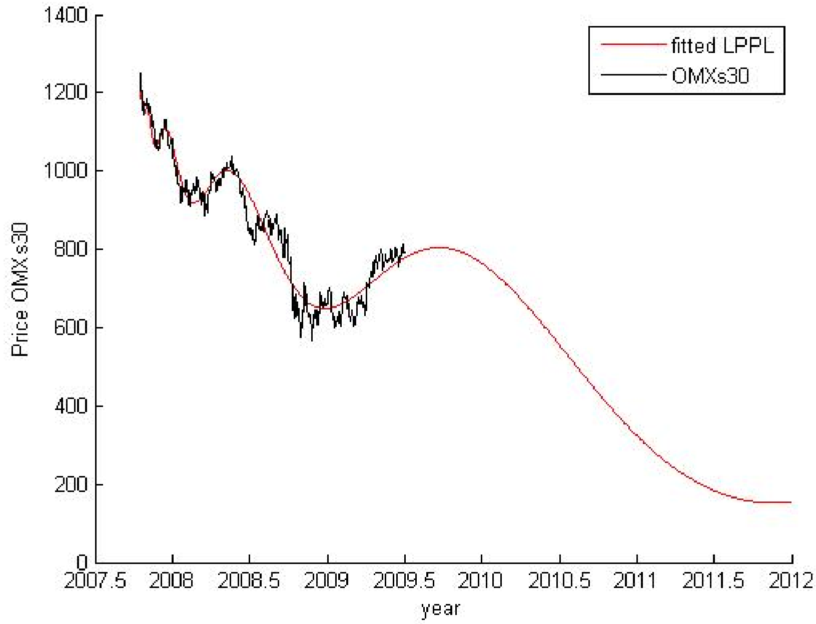

Figure 18: The LPPL fitted to OMXS30 for the anti-bubble resulting from the financial crisis of 2008. The curve is plotted over a longer time span with the next local minimum in the end of 2011.

The LPPL equation could absolutely not be fitted to the time span prior to the crash of the New Economy, nor the financial crisis of 2008. The growth of the periods resembles linear trends.

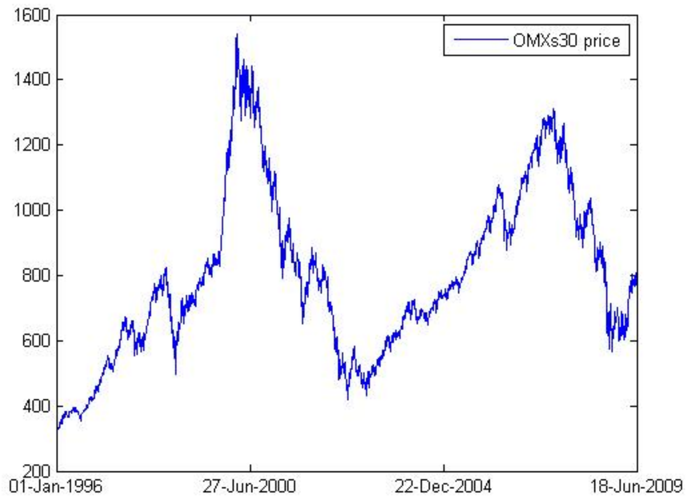

Figure 19: The price development of OMXS30.

5.2 DJIA

The Dow Jones Industrial average was founded in 1896 and represented the price average of the 12 most important industrial stocks in the United States. Today the DJIA has little to do with the traditional heavy industry and is a weighted price average of the 30 largest and most widely held public companies.

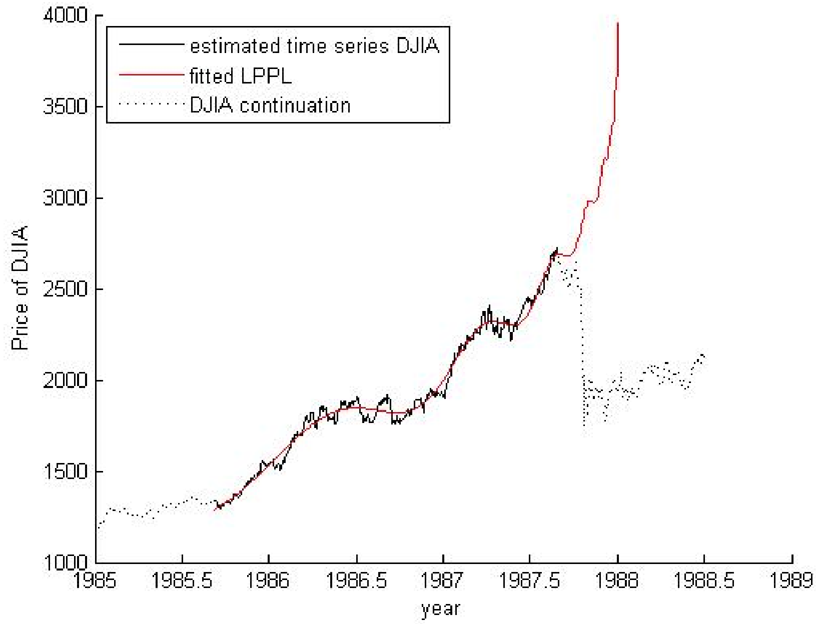

Figure 20: The speculative bubble of DJIA that ended with a crash in 1987-10-13. DJIA fell during the Black Monday of October 19, 1987 with -22.61 %. The drawdown for this crash is -30.68% and is the largest drawdown the DJIA has experienced.
Parameters: A=4047.71, B=-1997.52, C=101.831, t_c=1988.00, Φ=2.018, ω= 8.471 and z=0.327.
The goodness of fit: R^2=0.9872

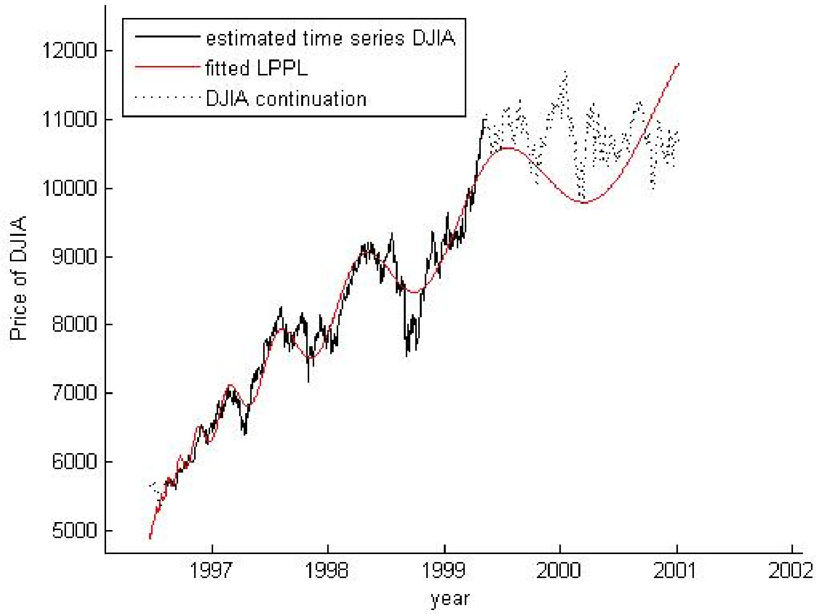

Figure 21: A bullish anti-bubble of the New Economy.
Parameters: A=4868.05, B=2510.17, C=-359.17, t_c=1996.4656, Φ=1.909, ω=12.617 and z=0.6227.
The goodness of fit: R^2=0.9528.

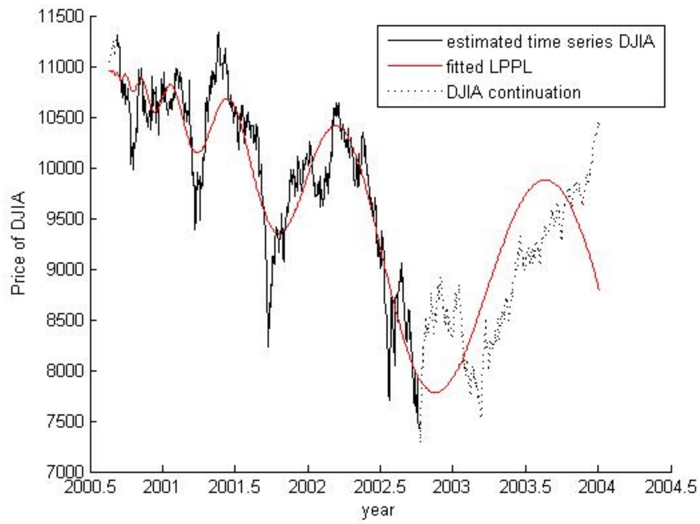

Figure 22: The anti-bubble of DJIA after the crash of the New Economy in 2000.
Parameters: A=10957.03, B=-863.55, C=-522.31, t_c=2000.63, Φ=5.0673, ω= 9.611 and z=1.043.
The goodness of fit: R^2=0.8415

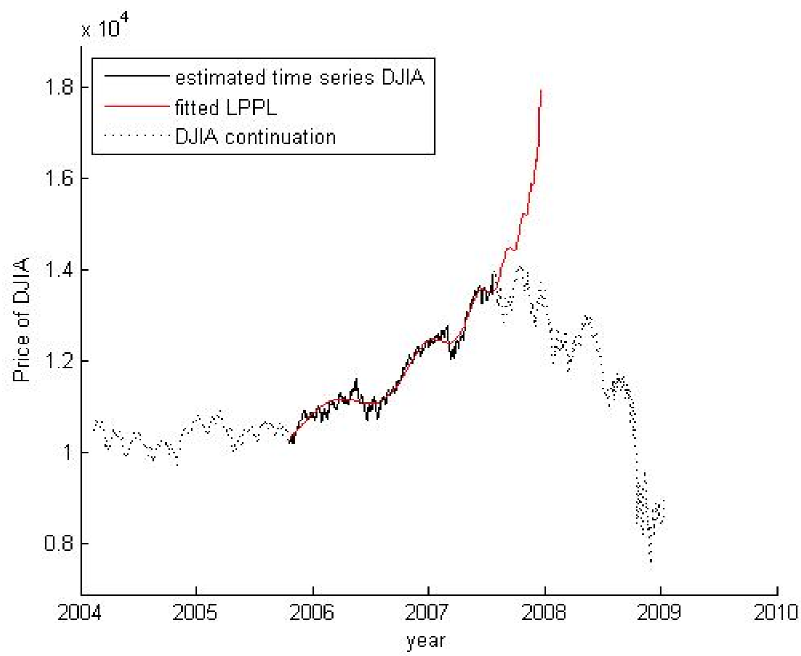

Figure 23: The LPPL signatures found before the financial crisis in 2008.
Parameters: A=19148.19, B=-7024.61, C=-283.499, t_c=2007.967, Φ=3.245, ω= 10.205 and z=0.2910.
The goodness of fit: R^2=0.9737

The bubble of the New Economy, also known as the "dot-com" bubble, is one of the most famous speculative bubbles in the world. Failing to fit the LPPL-equation for a speculative bubble for some time series prior to the crash, the equation for an anti-bubble could be fitted. This is known as a bullish anti-bubble with the parameter t_c indicating the start of the anti-bubble. The bullish anti-bubble has a positive slope. Like the bearish anti-bubble, there is no possibility to make a prediction at the end of the anti-bubble. The appearance of the bullish anti-bubble in one of the most famous bubbles is an interesting feature.

DJIA did a fast recovery after the bottom of the anti-bubble of 2000. Between January 2004 and October 2005, the DJIA hits a plateau with slow development and the LPPL signatures were found in the time span of 2005-10-06 to 2007-07-13. An endogenous bubble is consistent with an immediate crash with large drawdowns. Indeed, the time period of 2008-2009 did contain some of the largest drawdowns of DJIA, but they were delayed. The fit of the peak is not that perfect either. One can argue whether this is a true endogenous bubble or an exogenous shock. The time span of 2008-2009 can absolutely not be fitted to the LPPL equation as an anti-bubble.

5.3 S&P500

Standard & Poor 500 is a value weighted index published in 1957 of the prices of the largest 500 American large-cap stocks. It is one of the most widely followed indexes in the world and is considered to reflect the wealth of the US economy. The data used for the curve fits are S&P500 Total Return, which reflects the effects of dividends reinvested in the market.

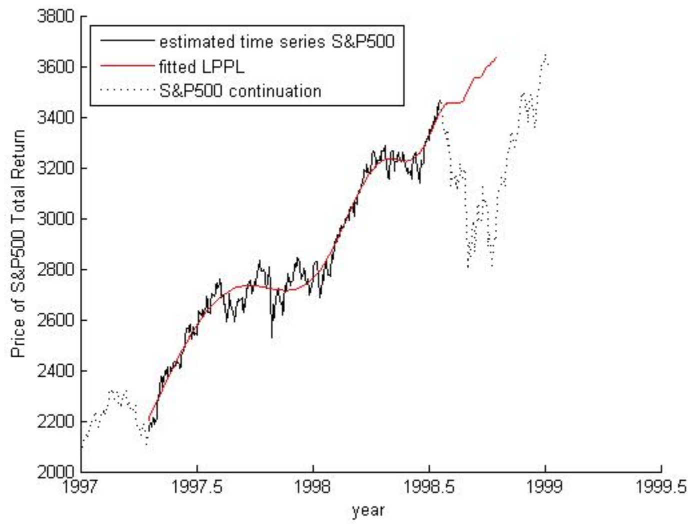

Figure 24: The speculative bubble in S&P500 ending in a crash with a drawdown of -13.21 % in 1998-08-24.
Parameters: A=3636.24, B=-950.388, C=146.00, t_c=1998.7903, Φ=5.010, ω=7.4305 and z=0.9561 .
The goodness of fit: R^2=0.9768

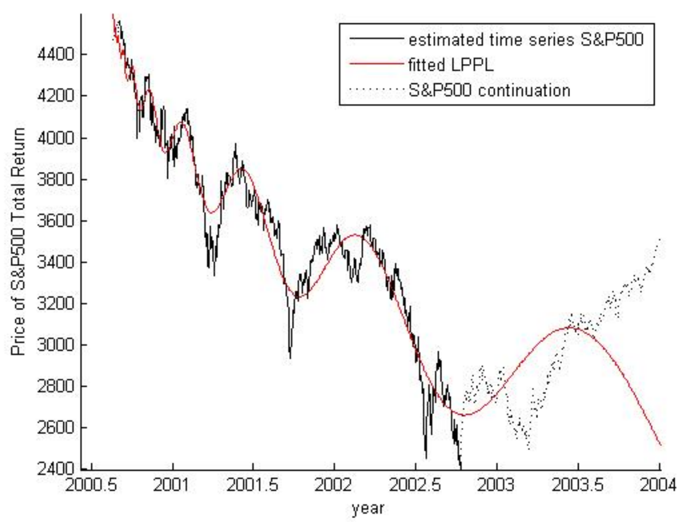

Figure 25: The anti-bubble resulting from the crash of the new economy in 2000.
Parameters: A=4630.459, B=-1092.43, C=-213.488, t_c=2000.6318, Φ=5.21089, ω=9.918 and z=0.540.
The goodness of fit: R^2=0.9429

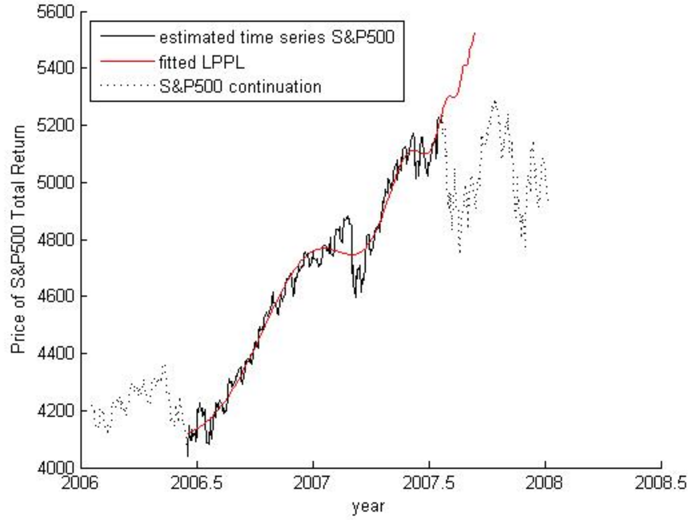

Figure 26: The LPPL fit to the time series of S&P500 prior the financial crisis of 2008.
Parameters: A=5543.743, B=-1124.797, C=137.258, t_c=2007.6976, Φ=2.236, ω=6.935 and z=0.6458. The goodness of fit: R^2=0.9793

As for DJIA this fit can be questioned if it is really a speculative bubble or an exogenous shock. The LPPL-equation could not be fitted to the time series of the financial crisis of 2008.

6 Discussion

Looking at drawdowns rather than daily returns is a good alternative to examine the fluctuations of the price in a financial market. The beauty of fitting a distribution to data is the possibility to calculate probabilities of future events, based on the distribution of historical data. To be able to fit a distribution to these movements results in a good tool for i.e. risk management. Instead of trying to prove that crashes are outliers as Johansen & Sornette [6], we would be better off trying to fit these observations to a distribution.

Johansen & Sornette [6] used an unusual fitting procedure to estimate the parameters of the stretched exponential. By a linearization of the complementary cumulative Weibull distribution, we concluded that the parameter estimates from this fitting procedure did not result in a good fit to the data. In fact, the logarithmic transformation used in equation (9) is not a good transformation to fit the parameters of the distribution. A good transformation of a function will result in robust estimates of the parameters, and this transformation fails.

We could fit the Weibull distribution with a very good result to the drawdowns of the OMXS30 index. The drawdowns of DJIA could not be fitted as well as for the OMXS30 index. The Weibull distribution could not extrapolate the small and large drawdowns, resulting in a curvature. The bad fit to the Weibull distribution could indicate some sort of mixture model, or a heavy tail of the true distribution. We stress that this does not prove the point of Johansen & Sornette [6] that crashes are outliers. The curvature starts at the drawdown size of 3.5% and that is not considered as a crash in a stock market.

The curvature may also be a result of noise. By suppressing noise in financial time series, it is possible to obtain more continuous drawdowns with two main benefits; we prove that the largest drawdowns are in fact the largest ones, and we can obtain a better fit to a distribution.

Very few guidelines about the ε-drawdown algorithm used by Johansen & Sornette [6] were given and their results could not be replicated. Looking at their plots in of the ε-drawdowns it becomes clear that they do not treat drawups as drawdowns, when they include drawdowns smaller than ε. With their treatment of the ε-drawdowns, they faced some problems; *"Those very few drawdowns (drawups) initiated with this algorithm which end up as drawups are discarded"*, Johansen & Sornette [6]. Not a single observation had to be discarded with our algorithm because of positive signs for the ε-drawdowns and our interpretation of the problem must be slightly better. Not treating the ε-drawups like ε-drawdowns may result in an incorrect distribution of trends.

A concern is how to choose a good value of ε. Johansen & Sornette [6] suggest that the volatility is the factor that should be used to suppress the noise in financial time series. The volatility is calculated as the standard deviation of the logarithmic returns between successive days, and the drawdowns are calculated as the percentage drop from a local maximum to the following local minimum. Even though we have replicated their work, we believe that comparing one measurement with another is fundamentally wrong. A better value of epsilon would probably be the volatility of the daily returns calculated in precentage, eventhough the difference between these two measurements is quite small. By letting the value of ε be large enough, it is possible to fit the ε-drawdowns to a three-parameter Weibull distribution with an excellent result.

A common definition of a speculative bubble is when the price of an index or asset rise significantly over a short period of time and becomes overvalued, ending with a crash. Above this definition, there are many models and definitions of a speculative bubble. Some even argue if there is such thing as a speculative bubble. Johansen & Sornette have developed their definition of an endogenous speculative bubble exhibiting the LPPL oscillation. We cannot argue that these LPPL-signatures do not exist, but the main question is why they exist. Is it in fact a result of a feedback loop driven by imitation and herding, leading to a collapse in one critical instant? Can the explanation simply be that the LPPL signature is a result of an accidental feature of the stochastic process that drives the price on a financial market?

The most famous speculative bubble was the "dot-com" bubble in 1998-2000. If we study the development of the ending years of the "dot-com" bubble in OMXS30, DJIA and S&P500, there are approximately linear trends and no signs of the LPPL-signatures.

The LPPL equation could be fitted to the DJIA and S&P500 prior to the current financial crisis of 2008, but are the fits just accidental? We know that the financial crisis of 2008 originated from a speculative bubble in the credit market. This speculative bubble inflated when the credit market were suppressing risks and aiming for larger profits. In the United States the credit market bubble was a result of a house market bubble, which peaked approximately in 2005-2006. We know that the price development in a stock market is the result the information interpreted by analyst and traders. Therefore the speculative bubble of the housing and credit market ought to affect the American stock market as well.

The definition made by Johansen & Sornette [6], is that a speculative bubble, exhibiting the LPPL signature, should have its crash at some time $t \leq t_c$. If the crash does not occur, the ending of the speculative bubble is to be considered as a soft landing. An exogenous crash can be preceded by any development of the price. Can the financial crisis be considered as a burst of an endogenous bubble?

The concern is about the development of the indices of the S&P500 and the DJIA prior to the financial crisis of 2008. The LPPL equation could be fitted to both of the indices, but not to the highest peak. Using the second highest peak in the window of estimation, the model predicted the turning point at the highest peak. At first, the indices began to decline slowly, which indicates a slow landing of the speculative bubble. Looking at the time series after the turning point, the severe crashes of the indices looks like the definition of an exogenous crash. We do not consider the financial crisis as an exogenous event, and we believe the model simply failed to predict the crash, that was delayed. Nevertheless, the model accurately predicted the turning point at the maximum peak.

How well does this model perform when predicting crashes? There are two major problems of the model:

* When looking at the plots . of the estimated LPPL curve, it becomes clear that the parameter t_c overshoots the actual time of the crash. Sornette *et. al.* [7], defend this by the rational expectation model and stress that the probability of a crash increases as the time approaches t_c and that t_c is the most probable time of a crash.

* Another concern is how much time prior to the peak of a speculative bubble that is needed to obtain an accurate estimation of the parameter t_c .

According to Sornette *et. al.* [7], it is possible to lock in the parameter t_c up to a year prior to the time of crash. They used data from a speculative bubble in S&P500 over the time period of January 1980 to Sepeptember 1987 that could be fitted by a LPPL curve containing 6 local maximum. They took the end date of 1985.5 and added 0.16 years consecutively to the time of the crash. The number of oscillations varied between the different fits, meaning the appearance of the curve varied, but the parameter t_c were quite robust. It is possible to make this statement by choosing a long speculative bubble with many oscillations. Detecting a LPPL signature in financial data requires at least two local maximum or two local minimum. Many of the speculative bubbles only contain between 2-4 local maximum or local minimum and do not stretch over such a long period of time. Removing a couple of month could destroy the parametric fits of the curve if that time period contains a local maximum or minimum. These considerations are troubling for the LPPL equation as a tool for predictions.

We preformed a fit of the LPPL equation to the time series of OMXS30 during the current financial crisis of 2008. For a speculative bubble the parameter t_c is the most probable time of the crash, but for an anti-bubble this is the start of the curve, hence we do not have the same tool of prediction. The next local minimum in the end of 2011 could imply a turning point, but the time and at a price level is quite unrealistic. The start of the LPPL signatures for a speculative bubble can appear at any point of an index and continue to the parameter time of t_c. For an anti-bubble the LPPL signatures of an index can stop at any point of the curve. To our knowledge, Sornette and his co-workers have not been able to predict an end of an anti-bubble.

7 Conclusions

We reject the definition of crashes as outliers to the stretched exponential, used by Johansen & Sornette [6], with the following motivation:

* The lack of outliers does not state that an index has not experienced any crashes.
* The curvature for the drawdowns of DJIA starts at 3.5%, which is not considered as crashes in a stock market.
* The fitting procedure proposed by Johansen & Sornette does not estimate the true parameters of the stretched exponential.

Instead, we define a crash as an observation in the empirical quantile of 99.5%. The empirical quantile for the drawdowns of DJIA is 10.50%, and for OMXS30 is 12.48%. Crashes are often associated with drawdowns larger than 10% and all of the observations in the quantile are consistent with historical crashes. The benefit of this definition is that it does not depend on any specific distribution, only the empirical distribution.

By choosing a good value of ε and reduce the presence of noise, we could fit a three-parameter Weibull distribution, to the DJIA index. This was done with an excellent result.

We preformed fits of the LPPL equation for OMXS30 and discovered the LPPL signatures between 1985-1988, 1996-1999, 2000-2003 and 2007-2009. The

LPPL signatures seen in the DJIA for the time periods of 1927-1929, 1985-1988 and 2000-2003 have been discovered by Sornette and coworkers. To our knowledge the fits of 1996-2000 and 2006-2007 has not been considered else were. We could fit the LPPL equation to the S&P500 during 1997-1999 and 2006-2008, that to our knowledge has not been considered else were. Sornette and coworkers found the LPPL signatures of the time period of 2000-2003. The main purpose of this project is to search for the LPPL signatures prior to the current financial crisis of 2008 and look for the characteristics of an anti-bubble during the financial crisis. The conclusion is that it can be found in time series of DJIA and S&P500, predicting a turning point of the indices. The LPPL-signature of an anti-bubble can be found for the OMXS30 during the current financial crisis of 2008. The next local minimum of the predicted curve is quite unrealistic and only time will reveal when we will depart from the curve.

7.1 Further Studies

We are aware of that the parameter t_c overshoots the actual time of the crash and we wish to make an empirical study to establish if there is a dependence of how far off the estimation of the parameter t_c will be depending on the parameters ω and z. The parameter ω controls the amplitude of the oscillations and further studies about the typical size off this parameter and how it affects the number of oscillations in the fit.

It would be interesting to study more about predictions on anti-bubbles. Prediction of the crash for a speculative bubble is the estimation of the parameter t_c, but so far there is no way to make a prediction of where a financial index will depart from the LPPL curve. One theory is that the Power Law can be viewed as a "center of gravity" and the further the Log-Periodic oscillations depart from the Power Law the higher the probability for the index to depart from the curve. This could explain why the next following minimum is considered to be the turning point, but this is not always the case. The empirical studies of speculative bubbles and anit-bubbles requires a large amount of work and this is left for further studies.

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9 Appendix

9.1 Appendix A

The parameter z is the stretching parameter of the Weibull distribution. When $z = 1$, the distribution recovers to an exponential function. When searching for a proof of the stretched exponential, the null hypothesis will be an exponential function.

Let X_i be independent identically distributed random variables of losses between successive days defined over $]-\infty, 0]$. Let $D_N = X_1 + X_2 + ... + X_N$, where N is a random variable independent of X_i.

D is the cumulative loss from local maximum to the following local minimum, regarding that we have at least one loss before the fist gain. N can be written as N= 1 + M, where M is a geometric random variable, hence N has the distribution of a First Success. A First success distribution $Fs(p_+)$, has the probability function $p(n) = p_-^{n-1} p_+$, where p_+ is the probability for the market to go up and p_- is the probability for that the market go down, $p_+ = 1 - p_-$.

We use the law of total probability to calculate the probability of experience a drawdown D of a given magnitude: Let f(x) be the probability density function for the random variable for a loss X. We start by applying the moment generating function for one random variable of a loss X, with the argument k.

$$\Psi_X(k) = E[e^{kX}] = \int_{-\infty}^{0} e^{kx} f(x) dx \tag{37}$$

For a given N=n, D is the sum of X_i, i =1,. . . ,n, such that;

$$\Psi_D(k) = \sum_{n=1}^{\infty} E[e^D | N = n] P(N = n) = \sum_{n=1}^{\infty} E[e^D] p_-^{n-1} p_+ =$$

$$= p_+ \sum_{n=1}^{\infty} E[e^{k(X_1+...+X_n)}] p_-^{n-1} = p_+ \sum_{n=1}^{\infty} E[e^{kX_1} ... e^{kX_n}] p_-^{n-1} =$$

$$= \left\{ \text{Geometric series: } \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ and } \sum_{n=1}^{\infty} \frac{1}{1-x} - 1 = \frac{x}{1-x} \right\} =$$

$$= \frac{p_+}{p_-} \sum_{n=1}^{\infty} (\Psi_X(k) p_-)^n =$$

$$= \left\{ \text{let } P(k) = p_- \Psi_X(k) = p_- \int_{-\infty}^{0} e^{kx} f(x) dx \text{ so that } \hat{P}(0) = p_- \right\} =$$

$$= \frac{p_+}{p_-}\left(\frac{p_-\Psi_X(k)}{1-p_-\Psi_X(k)}\right) = \frac{p_+}{p_-}\left(\frac{\hat{P}(k)}{1-\hat{P}(k)}\right) = \frac{1}{1-\frac{1}{p_+}\left(\frac{\hat{P}(k)-p_-}{\hat{P}(k)}\right)} =$$

$$= \left\{\text{use that } \hat{P}(0) = p_-\right\} = \frac{1}{1-\frac{1}{p_+}\left(\frac{\hat{P}(k)-\hat{P}(0)}{\hat{P}(k)}\right)} \tag{38}$$

If the distribution of P(D) does not declay more slowely than an exponential we can according to Sornette & Johansen [8] expand $\frac{\hat{P}(k)-\hat{P}(0)}{\hat{P}(k)}$ for small k (corresponding to large |D|) as:

$$\frac{\hat{P}(k)-\hat{P}(0)}{\hat{P}(k)} \approx \frac{k\hat{P}'(0)}{\hat{P}(0)} + \mathcal{O}(k^2) \tag{39}$$

which yieldes to:

$$\frac{k\hat{P}'(0)}{\hat{P}(0)} = \frac{p_-\Psi_X(k)}{p_-} = \int_{-\infty}^{0} xf(x)dx = E[X] \tag{40}$$

We define λ as the expected return of of D:

$$\lambda = E[D_N] = E[\sum_{j=1}^{N} X_j|N=n] = E\left[NE[X]\right] = E[N]E[X] = \frac{1}{p_+}E[X] \tag{41}$$

This yields:

$$P(k) = \frac{1}{1-\frac{1}{p_+}\left(\frac{\hat{P}(k)-\hat{P}(0)}{\hat{P}(k)}\right)} = \frac{1}{1+\frac{k}{p_+}E[X]} = \frac{1}{1+k\lambda} \tag{42}$$

Equation (40) is the moment generating function of an exponential function with mean λ, hence;

$$f(d) = \frac{e^{-\frac{|d|}{\lambda}}}{\lambda} \tag{43}$$

The typical size of λ for approximately symmetric distributions of daily losses, where $p_+ = p_- = 1/2$, is about 2 times the average daily drop.