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Thérèse Peters*

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Abstract

In the portfolio optimizing such as the Black Litterman the covariance matrix is essential. The usual approach is to forecast the future covariance matrices only based on equally weighted historical returns, which implies the covariance matrix is constant over time. Lately, more complex time-varying models that give a greater weight to the more recent returns, such as the multivariate GARCH models, have been developed. The aim of this thesis is to evaluate how forecasts of the Dynamic Conditional Correlation model of Engle and Sheppard (2001) performs compared to the traditional one. To evaluate the forecasts performances the unique property of the global mean-variance portfolio (GMVP) is used, namely that the most correct forecast of the covariance matrix will generate the least variance of the GMVP. Presented results show that the dynamic conditional correlation tend to out perform the covariance matrix based on historical data in the short run, while in the long run the reverse relationship holds.

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1 Introduction

In the financial industry volatility is an important conception, which measure the state of instability of the return. It is widely known that volatility varies over time and tends to cluster in periods of large volatility and periods of tranquillity. This phenomenon is called heteroscedatisity. An additional factor to consider is that the volatility has shown to be autocorrelated, which means that today's volatility depends on that of the past. Considering the fact that volatility is not directly observable the need of a good model to help estimate and forecast it is essential. One model that captures the properties mentioned above is the univariate GARCH -*General Autocorrelated Conditional Heteroscedatisity* model introduced by Bollerslev (1986). This model has shown to be successful in estimating and predicting volatility changes.

Although in the portfolio optimization model Black-Litterman, beside each individual asset's variance, the behavior between the assets is essential. A tool that quantifies all these components is the symmetric covariance matrix where the variance of each individual asset is found on the diagonal and the pair-wise covariance at the other elements. One method to estimate the covariance matrix is to extend the univariate GARCH model into a multivariate GARCH model.

The purpose of this project is to clarify problems of forecasting covariance matrices to use in the Black Litterman model, as well, to evaluate how the forecast from a multivariate GARCH model performs compared to a covariance matrix based simply on historical data. The portfolio consists of assets from several markets such as stock-, bond-, credit markets and real estate

When extending the model some difficulties appear which need to be considered;

- in order to make estimates feasible the number of parameters need to be reduced without restricting the flexibility to capture the dynamic in the conditional covariance too much.
- determine the conditions that make the covariance matrix positive definite at every point in time (as required by definition) and the conditions for the weak stationary of the process.
- the parameters should be easily interpreted.

One approach is to model the conditional covariance matrix indirectly through the conditional correlation matrix. The first model of this type was the constant conditional correlation (CCC) model of Bollerslev 1990. The conditional correlation was assumed to be constant and only the conditional variances were time varying. Thus, the assumption that the conditional correlation is constant over time is not convincing, since correlation in practice for many assets changes over time. Engle and Sheppard (2001) introduce the dynamic conditional correlation (DCC) model. This model has a two-step algorithm to estimate the parameters that makes the model relatively easy to use in practice. In the first step, the conditional variance is estimated via unvariate GARCH model for respectively asset. In the second step the parameters for the conditional correlation given the parameters from the first step are estimated. This approach makes it possible to estimate covariance with 100 assets without too cumbersome computation. Finally, the DCC-model includes conditions that make the covariance matrix positive definite at all points in time and the process covariance stationary.

This paper is organized as follows: In Section 2, the estimation of univariate GARCH models is discussed, the DCC model follows in Section 3. Forecasting the DCC model is the topic of Section 4 and in Section 5 I will show how to evaluate the performance of the DCC compared to a covariance matrix based on simply historical data. An empirical comparison of the models is presented in Section 6. Finally, Section 7 presents some conclusions.

2 Univariate GARCH

Analyzing and understanding how the univariate GARCH model works is fundamental for a study of the Dynamic Conditional Correlation multivariate GARCH model of Engle and Sheppard (2001). It is not only important because of the fact that the model is a nonlinear combination of univariate GARCH models [2], but also because the dynamic conditional correlation matrix is based on how the univariate GARCH*(1,1)* process works.

The volatility of an assets return refers to the standard deviation of the changes in value during a specific time horizon. In the long run returns tend to move towards a mean value (mean reverting). The changes in value that appear during this time are both positive and negative (asymmetric), mostly close to the mean value but some changes obtain extreme values (leptokurtic). As mentioned in the introduction, the volatility of today's returns is conditional on the past volatility and tends to cluster in volatility. [3]

Suppose that the stochastic process $\{r_t\}_t^T$ describes the return during a specific time horizon where r_t is the return observed at time t. Consider, for instance, the following model for the return:

$$r_t = \mu_t + \eta_t \tag{2.1}$$

where $\mu_t = E(r_t \mid \psi_{t-1})$ denotes the conditional expectations of the return series, η_t the conditional errors and $\psi_{t-1} = \sigma(\{r_s : s \le t-1\})$ represent the information set (sigma field) generated by the values of the return until time $t-1$.[17] This model take in the characteristics of the return stated above.

Assume that the conditional errors are the conditional standard deviations of the returns $h_t^{1/2} = Var(r_t \mid \psi_{t-1})^{1/2}$ times the independent and identically normally distributed zero mean unit variance stochastic variable z_t. [17] Also note, h_t and z_t are supposed to be independent of for all t.

$$\eta_t = \sqrt{h_t} z_t \sim N(0, h_t) \text{ given } \psi_{t-1} \tag{2.2}$$

Finally, suppose that the conditional expectation $\mu_t = 0$, which implies that

$$r_t = \sqrt{h_t} z_t \text{ and } r_t | \psi_t \sim N(0, h_t) \tag{2.3}$$

In practice if $\mu_t \neq 0$, the returns can be either ARMA filtered or demeaning.[12] However, in this case when $\mu_t = 0$ the variance of the return coincides with the variance of the errors and their conditional expectation is zero, therefore the error process is an innovation process. [10]

Often in financial models, conditioning is stated as regressions of a variable's present values of a variable on the same variable's past values, as well in the GARCH*(p,q)* model of Bollerslev (1986). This discrete process is given by equation (2.1) and

$$h_t = \omega + \sum_{i=1}^{q} \delta_i \eta_{t-i}^2 + \sum_{i=1}^{p} \gamma_i h_{t-i} \tag{2.4}$$

$$p \geq 0, \quad q > 0$$

$$\omega \geq 0 \quad \delta_i \geq 0 \text{ for } i = 1,...,q \quad \gamma_i \geq 0 \text{ for } i = 1,..., p$$

In words the GARCH*(p,q)* consists of three terms

- ω - the weighted long run variance
- $\sum_{i=1}^{q} \delta_i \eta_{t-i}^2$ -the moving average term, which is the sum of the q previous lags of squared-innovations times the assigned weight δ_i for each lagged square innovation. $i = 1,...,q$

- $\sum_{i=1}^{p} \gamma_i h_{t-i}$ - the autoregressive term, which is the sum of the p previous lagged variances times the assigned weight γ_i for each lagged variance $i = 1,..., p$

Note that the innovations η_t in the moving average term is squared and the GARCH*(p,q)* does not include the asymmetry of the errors, which is a drawback. The EGARCH model of Nelson (1991) and the GJR-GARCH model of Glosten, Jagannathan and Runkle (1993) are two examples of extended GARCH models that accommodate the asymmetry of the returns.[24] These two models would not be further analyzed in these thesis.

Since the variance is non-negative by definition, the process $\{h_t\}_{t=0}^{\infty}$ needs to be non-negative valued. Exact constraints for the GARCH*(p,q)* process are complicated and can be found in Nelson and Cao (1992).

2.1 GARCH (1,1)

The simplest and very popular GARCH model is the GARCH*(1,1)* which is given by equation (2.1) and

$$h_t = \omega + \delta \, \eta_{t-1}^2 + \gamma \, h_{t-1} \tag{2.5}$$

$$\text{where } \omega \geq 0, \ \delta \geq 0, \ \gamma \geq 0$$

The three terms can be interpreted as for the GARCH*(p,q)* but with only one lag each for the squared innovation and variance respectively. [4]

Successively backward substituting h_t to time $t - J$ yields the alternative expression of the GARCH*(1,1)*

$$h_t = \omega \, (1 + \gamma + \gamma^2 + ... + \gamma^{J-1}) + \delta \sum_{k=1}^{J} \gamma^{k-1} \eta_{t-k}^2 + \gamma^J h_{t-J} =$$

$$= \omega \frac{1-\gamma^{J}}{1-\gamma} + \delta \sum_{k=1}^{J} \gamma^{k-1} \eta_{t-k}^{2} + \gamma^{J} h_{t-J} \tag{2.6}$$

If J approaches infinity then

$$\lim_{J \to \infty} h_t = \frac{\omega}{1-\gamma} + \delta \sum_{k=1}^{\infty} \gamma^{j-1} \eta_{t-k}^{2} \quad \text{since } 0 < \gamma < 1 \tag{2.7}$$

This implies that current volatility is an exponentially weighted moving average of past squared innovations. Although, there are crucial differences between the GARCH*(1,1)* and EMWA (exponential weighted moving average) model, in the GARCH case the parameters need to be estimated. [7] and mean reversion has been incorporated in the model.

To reduce the number of parameters from three to two in the GARCH*(1,1)* and thereby make the computation easier "variance targeting" of Engle and Mezrich (1996) can be used. To illustrate how this works, denote the unconditional variance $\overline{h}$ and rewrite equation (2.5) as the deviations from the unconditional variance

$$h_t - \overline{h} = \omega - \overline{h} + \delta\ (\eta_{t-1}^{2} - \overline{h}) + \gamma\ (h_{t-1} - \overline{h}) + \gamma\overline{h} + \delta\overline{h} \tag{2.8}$$

After rearranging equation (2.8)

$$\begin{aligned} h_t &= \omega - (1-\delta\ -\gamma\ )\overline{h} + (1-\delta\ -\gamma\ )\overline{h} + \delta\ \eta_{t-1}^{2} + \gamma\ h_{t-1} = \\ &= \omega - (1-\delta\ -\gamma\ )\overline{h} + (1-\delta\ -\gamma\ )\overline{h} + \delta\ \eta_{t-1}^{2} + \gamma\ h_{t-1} \end{aligned} \tag{2.9}$$

If $\omega = (1-\delta\ -\gamma\ )\overline{h}$ then equation (2.5) can be written as following

$$h_t = (1-\delta\ -\gamma\ )\overline{h} + \delta\ \eta_{t-1}^{2} + \gamma\ h_{t-1} \tag{2.10}$$

The model is not only easier to compute but it also implies that the unconditional variance $\overline{h} = \omega / (1-\delta\ -\gamma\ )$. This simply works under the assumption that $\gamma + \delta < 1$ and it only makes sense if the weights $\omega > 0$, $\delta > 0$ and $\gamma > 0$ [7]

2.1.1 The Inequality Constrains of the GARCH*(1,1)*

As mentioned before, the conditional variance h_t must by definition remain non-negative with probability one. To ensure this for the GARCH*(1,1)* process the following conditions are sufficient $\omega \geq 0$, $\delta \geq 0$ and $\gamma \geq 0$.

Another feature that keeps the conditional variance non-negative is if the process behaves similar in any epoch that might be observed, in other words, if the process is stationary. [9] Thus, when discussing the stationarity of a process we consider the unconditional moments of the series [15].

The GARCH*(1,1)*-process is weakly[1] stationary with the unconditional expected value and covariance

$$E[r_t] = 0$$
$$Cov[r_t, r_{t-s}] = \omega/(1-\delta-\gamma) \tag{2.11}$$

if and only if $\gamma + \delta < 1$.[17].

Recall, when using variance targeting in the GARCH*(1,1)* the unconditional variance $\bar{h} = \omega/(1-\delta-\gamma)$ which coincides with the unconditional variance when the process is weakly stationary. [11]

As a short summary, the inequalities constraints to regard in the GARCH*(1,1)* when using variance targeting is $\omega > 0$, $\delta > 0$, $\gamma > 0$ and $\gamma + \delta < 1$. The process is considered to be covariance stationary.

2.2.2 Estimate GARCH *(1,1)*

The question arises whether all of the constraints are necessary when estimating the parameters with the log likelihood function. Both yes and no! Analyzing the quasi

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[1] The process $\{r_t\}_{t=1}^{T}$ is weakly stationary (covariance stationary) if neither the mean nor the autocovariances depend on the time *t*. That is if $E(r_t) = \mu$ for all *t* and $E(r_t - \mu)(r_{t-j} - \mu) = \gamma_j$ for all t and j [14]

log-likelihood function it is obvious that it will not generate negative conditional variances in sample since the natural logarithmic function will explode to minus infinity when the conditional variances approaches zero. Another reason is that the logarithmic function is also ill defined for values less then zero. However, out-of-sample it can be negative from the estimated parameters without any restriction. [21]

The assumption that the conditional return is normally distributed with zero mean and variance h_t gives the following (quasi) log-likelihood function of the GARCH*(1,1)*

$$\log L(\gamma,\delta) = -\frac{1}{2}\sum_{t=1}^{T}\left(\log(2\pi) + \log(h_t) + \frac{\eta_t^2}{h_t}\right) =$$

$$= -\frac{1}{2}\left(T\log(2\pi) + \sum_{t=1}^{T}\left(\log(h_t) + \frac{r_t^2}{h_t}\right)\right) \tag{2.12}$$

The non-linear log-likelihood function needs to be maximized numerically with respect to its inequality constraints. For instance, the matlab command *fmincon* in the optimizing toolbox can be used.

It is necessary to compute the variance at time t, h_t, recursively and therefore a startup value for h_0 and η_0^2 greater than zero has to be selected. To make sure that the variance process $\{h_t\}_{t=0}^{\infty}$ is non-negative with probability one given earlier mentioned constraints h_0 and η_0 is chosen as $h_0 = \eta_0^2 = \overline{h}$. In practice the unconditional variance is estimated as [21]

$$\hat{\overline{h}} = \frac{1}{T}\sum_{t=1}^{T}\eta_t^2 \tag{2.13}$$

3 The Dynamic Conditional Correlation model

To extend the assumptions of the return in section 2 to the multivariate case, suppose that we have n assets in a portfolio and the return vector is the colon vector

$r_t = (r_{1t}, r_{2t}, ..., r_{nt})'$. Furthermore, assume that the conditional returns are normally distributed with zero mean and conditional covariance matrix $H_t = E\left[r_t r_t' \middle| \psi_{t-1}\right]$. This implies that

$$r_t = H_t^{1/2} z_t \text{ and } r_t \middle| \psi_{t-1} \sim N(0, H_t) \tag{3.1}$$

where $z_t = (z_{1t}, z_{2t}, ..., z_{nt})' \sim N(0, I_n)$ and I_n the identity matrix of order n. $H_t^{1/2}$ may be obtained by Cholesky factorization of H_t

In the DCC-model, the covariance matrix is decomposed into

$$H_t \equiv D_t R_t D_t \tag{3.2}$$

What do the matrices D_t and R_t represent? D_t is a diagonal matrix of time varying standard variation from univariate GARCH -processes

$$D_t = \begin{bmatrix} \sqrt{h_{1t}} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{h_{2t}} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{h_{3t}} & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \sqrt{h_{nt}} \end{bmatrix} \tag{3.3}$$

The specification of elements in the D_t matrix is not only restricted to the GARCH*(p,q)* described in section 2 but to any GARCH process with normally distributed errors which meet the requirements for suitable stationary and non-negative conditions. The number of lags for each assets and series do not need to be the same either.

However, R_t is the conditional correlation matrix of the standardized disturbances ε_t

$$R_t = \begin{bmatrix} 1 & q_{12,t} & q_{13,t} & \cdots & q_{1n,t} \\ q_{21,t} & 1 & q_{23,t} & \cdots & q_{2n,t} \\ q_{31,t} & q_{32,t} & 1 & & q_{3n,t} \\ \vdots & \vdots & & \ddots & \vdots \\ q_{n1,t} & q_{n2,t} & q_{n3,t} & \cdots & 1 \end{bmatrix} \tag{3.4}$$

$$\varepsilon_t = D_t^{-1} r_t \sim N(0, R_t) \tag{3.5}$$

Thus, the conditional correlation is the conditional covariance between the standardized disturbances.

Before analyzing R_t further, recall that H_t has to be positive definite by the definition of the covariance matrix. Since H_t is a quadratic form based on R_t it follows from basics in linear algebra that R_t has to be positive definite to ensure that H_t is positive definite. Furthermore, by the definition of the conditional correlation matrix all the elements have to equal or less than one. To guarantee that both these requirements are met R_t is decomposed into

$$R_t = Q_t^{*-1} Q_t Q_t^{*-1} \tag{3.6}$$

where Q_t is a positive definite matrix defining the structure of the dynamics and Q_t^{*-1} rescales the elements in Q_t to ensure $\left|q_{ij}\right| \leq 1$. In other words Q_t^{*-1} is simply the inverted diagonal matrix with the square root of the diagonal elements of Q_t

$$Q_t^{*-1} = \begin{bmatrix} 1/\sqrt{q_{11t}} & 0 & 0 & \cdots & 0 \\ 0 & 1/\sqrt{q_{11t}} & 0 & \cdots & 0 \\ 0 & 0 & 1/\sqrt{q_{11t}} & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1/\sqrt{q_{11t}} \end{bmatrix} \tag{3.7}$$

Suppose that the Q_t has the following dynamics

$$Q_t = (1 - \alpha - \beta)\overline{Q} + \alpha \varepsilon_{t-1} \varepsilon'_{t-1} + \beta Q_{t-1} \tag{3.8}$$

where $\overline{Q}$ is the unconditional covariance of these standardized disturbances

$$\overline{Q} = Cov(\varepsilon_t \varepsilon'_t) = E\left[\varepsilon_t \varepsilon'_t\right] \tag{3.9}$$

and α and β are scalars.

At first sight the proposed structure of the dynamic might seem complicated and taken from out of the blue, but considering the resemblance between equation (3.8) with (2.5) it is obvious that the structure is similar to the GARCH*(1,1)*-process with "variance targeting".[25] Actually, the dynamic structure defined above is the simplest multivariate GARCH called Scalar GARCH. One fact to emphasize is that this structure implies that all correlations obey the same structure, which can be regarded as a drawback of the model. [11]

The structure can be extended to the general the DCC*(P,Q)*.

$$Q_t = (1 - \sum_{i=1}^{P} \alpha_i - \sum_{j=1}^{Q} \beta_j)\overline{Q} + \sum_{i=1}^{P} \alpha_i \varepsilon_{t-i} \varepsilon'_{t-i} + \sum_{j=1}^{Q} \beta_j Q_{t-j} \tag{3.10}$$

In this thesis only the DCC*(1,1)* model will be studied. For more information regarding the DCC*(P,Q)* see Engle and Sheppard (2002).

3.1 Constraints of the DCC(1,1) model

If the covariance matrix is not positive definite then it is impossible to invert the covariance matrix which is essential in the Black Litterman as well in portfolio optimizing in general.

To guarantee a positive definite H_t for all t simple conditions on the parameters are imposed. First, the conditions for the univariate GARCH model in section 2.1.1 have to be satisfied. Similar conditions on the dynamic correlations are required, namely [13]

$$\alpha \geq 0 \text{ and } \beta \geq 0 \tag{3.11}$$

$$\alpha + \beta < 1 \tag{3.12}$$

$$\text{and finally } Q_0 \text{ has to be positive definite} \tag{3.13}$$

3.2 Estimation of the DCC(1,1) model

In order to estimate the parameters of H_t, that is to say $\theta = (\phi, \varphi)$, the following log-likelihood function ℓ can be used when the errors are assumed to be multivariate normally distributed:

$$\ell(\theta) = -\frac{1}{2}\sum_{t=1}^{T}\left(n\log(2\pi) + \log(|H_t|) + r_t^{'}H_t^{-1}r_t\right)$$

$$= -\frac{1}{2}\sum_{t=1}^{T}\left(n\log(2\pi) + \log(|D_tR_tD_t|) + r_t^{'}D_t^{-1}R_t^{-1}D_t^{-1}r_t\right) =$$

$$= -\frac{1}{2}\sum_{t=1}^{T}\left(n\log(2\pi) + 2\log(|D_t|) + \log(|R_t|) + \varepsilon_t^{'}R_t^{-1}\varepsilon_t\right) \qquad (3.14)$$

Conveniently, the parameters in the DCC*(1,1)* model can be divided in to two groups $\phi = (\omega_1, \delta_1, \gamma_1, ..., \omega_n, \delta_n, \gamma_n)$ and $\varphi = (\alpha, \beta)$ and estimated via two following steps:

3.2.1 Step one

The R_t matrix in the log-likelihood function (3.14) is replaced with the identity matrix I_n, which gives the following log-likelihood function:

$$\ell_1(\phi|r_t) = -\frac{1}{2}\sum_{t=1}^{T}\left(n\log(2\pi) + 2\log(|D_t|) + \log(|I_n|) + r_t^{'}D_t^{-1}I_nD_t^{-1}r_t\right) =$$

$$= -\frac{1}{2}\sum_{t=1}^{T}\left(n\log(2\pi) + 2\log(|D_t|) + r_t^{'}D_t^{-1}D_t^{-1}r_t\right) =$$

$$= -\frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{n}\left(\log(2\pi) + \log(h_{it}) + \frac{r_{it}^2}{h_{it}}\right) =$$

$$= -\frac{1}{2}\sum_{i=1}^{n}\left(T\log(2\pi) + \sum_{t=1}^{T}\left(\log(h_{it}) + \frac{r_{it}^2}{h_{it}}\right)\right) \qquad (3.15)$$

Comparing equation (3.15) with (2.10) it is obvious that this quasi-likelihood function is the sum of the univariate GARCH log-likelihood functions. Therefore, we can use the algorithm in section 2.2.2 to estimate the parameters $\phi = (\omega_1, \delta_1, \gamma_1, ..., \omega_n, \delta_n, \gamma_n)$

for each univariate GARCH process. Since the variance h_{it} for asset $i = 1,...n$ is estimated for $t \in [1,T]$, then also the element of the D_t matrix under the same time period is estimated.

3.2.2 step two

In the second step the correctly specified log-likelihood function is used to estimate $\varphi = (\alpha, \beta)$ given the estimated parameters $\hat{\phi} = (\hat{\omega}_1, \hat{\delta}_1, \hat{\gamma}_1, ..., \hat{\omega}_n, \hat{\delta}_n, \hat{\gamma}_n)$ from step one.

$$\ell_2\left(\varphi \middle| \hat{\phi}, r_t\right) = -\frac{1}{2}\sum_{t=1}^{T}\left(n\log(2\pi) + 2\log\left(|D_t|\right) + \log\left(|R_t|\right) + \varepsilon_t' R_t^{-1}\varepsilon_t\right) \tag{3.16}$$

In view of the fact that the two first terms in the log-likelihood are constants, the two last terms including R_t is of interest to maximize [13]

$$\ell_2 \propto \log\left(|R_t|\right) + \varepsilon_t' R_t^{-1}\varepsilon_t \tag{3.17}$$

The standardized disturbances are calculated according to equation (3.5) and $\overline{Q}$ is estimated as $\widehat{\overline{Q}} = \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t\varepsilon_t'$. [9]

Even in this case, variance targeting is used in the dynamic structure and therefore $\hat{Q}_0 = \varepsilon_0\varepsilon_0'$ and since the conditional correlation matrix also is the covariance matrix of the standardized residuals $\hat{R}_0 = \varepsilon_0\varepsilon_0'$. [9]

3.2.3 A third step?

The estimated parameters from step 2 are consistent, but not efficient. If a third step in which the Newton Rahpson method maximizes the log-likelihood function (3.14) is established, then the parameters are asymptotically efficient. In this step the Newton Rahpson is iterated once with the starting values $\hat{\varphi} = (\hat{\alpha}, \hat{\beta})$ [13]

4 Forecasting the DCC-model

When forecasting the covariance matrix of the DCC-model, the forecast of the diagonal matrix of time varying standard variation from univariate GARCH –processes D_t and the forecast of the conditional correlation matrix of the standardized disturbances R_t can be calculated separately. [27] The correlation coefficient is not itself forecast, but it is the ratio of the forecast of the covariance to the square root of the product of the forecasts of the variances. Thus unbiased forecasts can't easily be computed. The following forecasting method gives the least biased forecast according to Engle and Sheppard (2001).

4.1Forecasting GARCH(1,1)

Assume that the volatility for time *t* is estimated then what is the forecast for $t+k$? It follows directly from equation (2.5) that when $k=1$ the volatility is

$$h_{t+1} = \omega + \delta\eta_t^2 + \gamma h_t \tag{4.1}$$

Accordingly, the GARCH model itself generates volatility forecast for the very next point in time, which implies $E\left[h_{t+1}|\psi_t\right] = h_{t+1}$ for all *t*. To obtain the forecast $E\left[h_{t+k}|\psi_t\right]$ for $t+k$ when $k>1$ successive forward substitution at time $t+2$, $t+3$, $t+4$ and so on is used. Before presenting the formula for $t+k$, let's study the forecast of h_{t+2} and h_{t+3} given the information set at time *t*.

First, assume that k=2, then the forecast of h_{t+2} given the variance at time *t* is

$$E\left[h_{t+2}|\psi_t\right] = \omega + \gamma E\left[\eta_{t+1}^2|\psi_t\right] + \delta h_{t+1} \tag{4.2}$$

Since h_t and z_t is assumed to be independent it follows that

$$E\lfloor\eta_{t+1}^2|\psi_t\rfloor = E\lfloor h_{t+1}|\psi_t\rfloor E\lfloor z_{t+1}^2|\psi_t\rfloor = h_{t+1} \tag{4.3}$$

then

$$E\lfloor h_{t+2}|\psi_t\rfloor = \omega + (\delta+\gamma)h_{t+1} \tag{4.4}$$

Now assume that the k=3 instead then

$$E\lfloor h_{t+3}|\psi_t\rfloor = \omega + \gamma E\lfloor\eta_{t+2}^2|\psi_t\rfloor + \delta E\lfloor h_{t+2}|\psi_t\rfloor \tag{4.5}$$

With the same procedure as in equation (4.3) $E\lfloor\eta_{t+2}^2|\psi_t\rfloor = E\lfloor h_{t+2}|\psi_t\rfloor$ and therefore

$$\begin{aligned} E\lfloor h_{t+3}|\psi_t\rfloor &= \omega + (\delta+\lambda)(\omega\ \bar{h} + (\delta+\gamma)h_{t+1}) = \\ &= \omega(1+(\delta+\lambda)) + (\delta+\gamma)^2 h_{t+1} \end{aligned} \tag{4.6}$$

In formula (4.6) the weighted unconditional variance ω times $\delta+\gamma$ is added and also the $(\delta+\gamma)h_{t+1}$ is multiplied by $\delta+\gamma$ compared to (4.4) Continuing with the same approach as above for the next $k-3$ periods in time the following formula will be achieved, which is the forecast of h_{t+k} given information at time *t*. [8]

$$E\lfloor h_{t+k}|\psi_t\rfloor = h_{t+k|t} = \sum_{i=0}^{k-2}\omega(\delta+\gamma)^i + (\delta+\gamma)^{k-1}h_{t+1} \tag{4.7}$$

When $\omega = (1-\delta-\gamma)\bar{h}$ then (4.7) can be rewritten as following [1]

$$h_{t+k|t} = \bar{h} + (\delta+\gamma)^{k-1}(h_{t+1} - \bar{h}) \tag{4.8}$$

Consequently, the forecast of the GARCH*(1,1)* consists of two elements, the unconditional variance and the weighted variance of deviation between h_{t+1} and the unconditional variance . It is obvious, since $\delta+\gamma<1$, that the influence from the variance at time $t+1$ decreases when the forecast is further away from $t+1$. In the long run, when $k \to \infty$, the forecast will approach the unconditional variance $\overline{h}$

$$\lim_{k\to\infty} \overline{h} + (\gamma + \delta)^{k-1}(h_{t+1} - \overline{h}) = \overline{h} \tag{4.9}$$

How fast it will decay towards the unconditional variance depends on $\delta + \gamma$. The closer $\delta + \gamma$ is to one the greater the persistence, which implies that the variance will decay slowly towards the long run variance.

One question to regard is how the past observations in the GARCH*(1,1)* model will influence the forecast . To include the past observation from *J* earlier periods in time equation (2.6) is inserted in formula (4.7) then

$$h_{t+k|t} = \sum_{i=0}^{k-2}(1-\delta-\gamma)\overline{h}(\delta+\gamma)^i + (\delta+\gamma)^{k-1}\left(\omega\sum_{j=0}^{J-1}\gamma^j + \delta\sum_{i=1}^{J}\gamma^{i-1}\eta_{t+1-i}^2 + \gamma^J h_{t+1-J}\right) \tag{4.10}$$

For instant, assume that a single squared return shock η_{t-j}^2 occurs, what impact will that have on the forecast *n* days apart? In other words, how long is the memory of the shock? The answer is given by the ratio of the derivate of (4.9) with respect to η_{t-j}^2 at time *t*+*k*+*n* and *t*+*k* [8]

$$\frac{\partial h_{t+m+n|t} \big/ \partial \eta_{t-j}^2}{\partial h_{t+m|t} \big/ \partial \eta_{t-j}^2} = (\delta + \gamma)^n \tag{4.11}$$

where

$$\partial h_{t+k|t} \big/ \partial \eta_{t-j}^2 = \delta(\delta + \gamma)^{k-1}\gamma^j \tag{4.12}$$

The memory will decline with the exponential rate $\delta + \gamma$. Compared with empirical studies the GARCH*(1,1)* model has been criticized to have too short memory, especially with high frequency data. [7]

4.2 Forecasting the correlation matrix

Recall from section 3 that the structure of the conditional correlation matrix is the following non-linear GARCH like process.

$$Q_t = (1-\alpha-\beta)\overline{Q} + \alpha\varepsilon_{t-1}\varepsilon'_{t-1} + \beta Q_{t-1}$$

Under the assumption that $\overline{R} \approx \overline{Q}$ and $E[R_{t+i}|\psi_t] \approx E[Q_{t+i}|\psi_t]$ for $i = 1,...,k$ a similar approach as in the GARCH*(1,1)* case can be used to derive the formula for $E[R_{t+k}|\psi_t]$, since it is also known that $E[\varepsilon_{t+k-1}\varepsilon'_{t+k-1}|\psi_t] = E[R_{t+k-1}|\psi_t]$. Therefore the formula to forecast the model k step ahead is

$$E_t[R_{t+k}] = \sum_{i=0}^{k-2}(1-\alpha-\beta)\overline{R}(\alpha+\beta)^i + (\alpha+\beta)^{k-1}R_{t+1} \tag{4.13}$$

Even the forecast of the conditional correlation matrix will in the long run converge to the unconditional correlation matrix of the standardized residuals. Another evident feature of the forecast formula is that the influence from R_{t+1} will decay with ratio $(\alpha+\beta)$ for each future step ahead. [13]

5 Evaluate the forecast

Traditionally in the financial industry the covariance matrix has been indirectly evaluated by putting the out-of-sample forecast in the global minimum-variance portfolio (GMVP). The solution to the following minimization problem gives the portfolio weights w_{it} at time t for each asset $i = 1,...,n$

$$\min_{w_t} w'_t H_t w_t$$
$$\text{subject to} \sum_{i=1}^{n} w_{it} = 1 \tag{5.1}$$

Given the weights the variance σ_t of the portfolio at time *t* can be computed

$$\sigma_t^2 = w_t H_t w_t \tag{5.2}$$

The GMVP possesses a unique property, namely that the correct covariance leads to improved performance. Consequently, the portfolio with the most correct covariance will have the least variance σ_t^2 at time *t* and therefore the competing covariance forecasts can be compared by the GMVP. [22]

6 The DCC model in practice

To study how the DCC model works in practice data between 1990-01-05 and 2007-12-28 from sixteen different indices is used. Among them there are two credit bond indices OMRX Mortage index and Global Broad Non-Sovereign, six JP Morgan Broad Government Bond indices from different regions, seven stock indices from MSCI also from different regions and finally an index for real estates. All of these indices are hedged into Swedish crowns. One remark about the Global Broad Non Sovereign index that is merged with US MG Master between 1990-01-05 and 1996-12-31 to increase the time period. When data were missing the return from the day before was assumed to be the real value.

If daily returns are selected the information in the data may be misleading since the return significantly changes during a day and are asynchronous in different time zones. To prevent this, data is selected with four different frequencies, weekly as well as every second, third and fourth week. The time period 1990 to 2007 is divided into three subsections 1990 to 2005, 1991 to 2006 and finally 1992 to 2007.

6.1 The return series and normal distribution

Recall from section 2 and 3 that the return was assumed to be normally distributed with zero mean and variance h_t and H_t respectively. The question arises whether this assumption is durable for the return series. If the returns are normally distributed the histograms in figure 1 will be shaped as bells. From the histograms in figure 1, the returns do not seem to fit perfectly in the framework of the normal distribution. Both the JP Morgan Broad Government index over Japan and the MSCI stock index for Asia seem to exhibit extreme values.

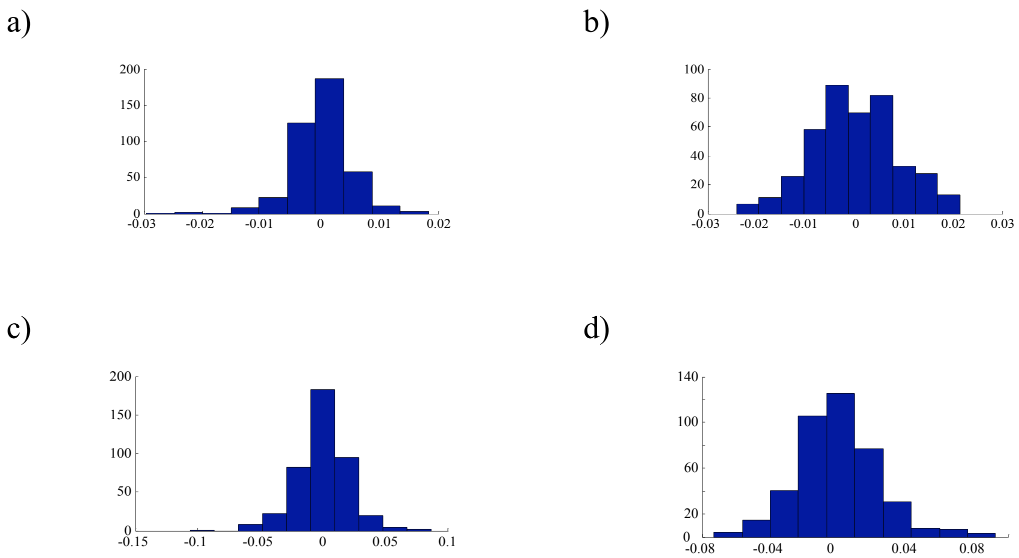

Fig.1. a) the return of the JP Morgan Broad Government index over Japan. b) the return of the JP Morgan Broad Government index over the world. c) the return of MSCI – stock index for Asia. d) the return of the Real Estate index. All of the indices above are from 1991-2006 with returns from every second week.

6.2 The parameters of the univariate GARCH(1,1)

Former studies of the univariate GARCH*(1,1)* models' parameters have shown that the approximate size of the parameters are $\delta = 0.05$ and $\gamma = 0.85$ which implies that $\omega = 0.1\hat{\bar{h}}$. In the end of this section table 1 to 4 present the values of the estimated parameters of the indices for each data frequency respectively.

In table 1, the estimates seem to follow the standard above. This is not the case for all parameters in table 2, where the frequency of the data has decreased from every week in table1 to every second week in table 2. The parameters of the Japanese stock index separate from the standard values during the period 1991-2006. Recall equation (2.6) and (2.7), which implies that if $\delta = 0$ the estimated conditional variance is constant over time. In figure 2 this behavior is shown. The conditional variance of the Japanese stock index has a slightly downward sloping trend. During the same time period the conditional variance of the United Kingdoms stock index is time varying. These estimates follow the standard size.

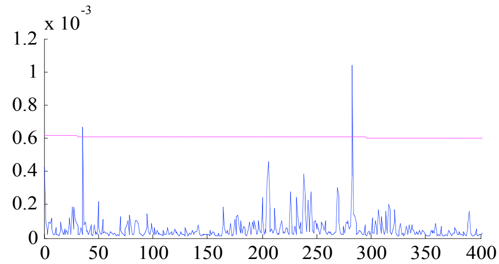

Fig.2 The conditional variance of the Japanese stock index (magenta) has a slightly downward sloping trend. The conditional variance of United Kingdoms stock index (blue) that varies all the time.

This result does not seem to be correct. In figure 3, both the return series for the United Kingdom and Japan stock index are plotted. Over the whole, it is obvious that both the United Kingdoms' and Japan's returns series vary with the same regularity. Therefore the result that the conditional variance for Japan should be constant is peculiar, when that of the UK varies over the time.

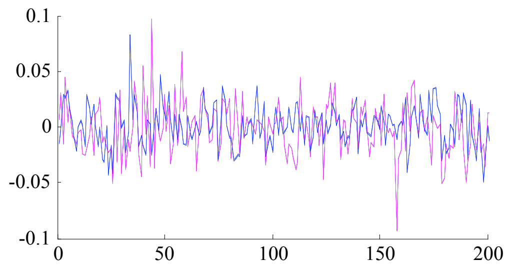

Fig.3 The return of the Japanese stock index (magenta) and of United Kingdoms stock index (blue) that varies all the time.

The same thing happens with the Japanese stock index for the period 1992-2007 in table 3.

Even when the frequency of the data is four weeks some of the estimated parameters once again have uncommon values. Surprisingly, it is not only the parameter values during the period 1990 to 2005 and 1992 to 2007 for Japan that differ but also the parameter values of the USA bond indices for all three periods and the values of UK bond index during the period of 1991 to 2006.

In figure 4 the US and European government bond indices variance are plotted. In real life the variance of these two assets are similar, which is not the case here. When $\gamma = 0$, as for the US government bond index, the conditional variance on a certain day only depends ω and δ times the squared return from the previous period.

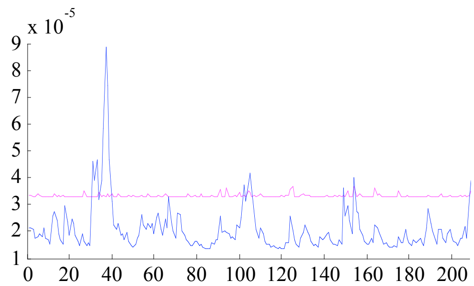

Fig.4 The conditional variance of the US government bond index (magenta) and the conditional variance of Europe government bond index (blue).

A reasonable explanation why some of the assets' estimated parameters deviate from the standard values when the frequency decreases is that the GARCH*(1,1)* model is not suitable.

Every week data			Period 1990-2005	Period 1991-2006	Period 1992-2007
Credit bonds	***Global Broad Non-sovereign***	ω	$1.943 \cdot 10^{-6}$	$1.261 \cdot 10^{-6}$	$1.589 \cdot 10^{-6}$
		δ	0.0804	0.0792	0.0812
		γ	0.8844	0.8881	0.8774
	OMRX Mortg BI	ω	$2.808 \cdot 10^{-7}$	$2.563 \cdot 10^{-7}$	$2 \cdot 10^{-7}$
		δ	0.1139	0.1193	0.1080
		γ	0.8736	0.8869	0.8817
JP MORGAN Broad Government Bond Index	***World***	ω	$9.821 \cdot 10^{-7}$	$9.490 \cdot 10^{-7}$	$1.242 \cdot 10^{-6}$
		δ	0.0424	0.0434	0.0494
		γ	0.9456	0.9452	0.9355
	USA	ω	$1.033 \cdot 10^{-6}$	$9.109 \cdot 10^{-7}$	$8.833 \cdot 10^{-7}$
		δ	0.0411	0.0451	0.0432
		γ	0.9338	0.9324	0.9355
	Europe	ω	$2.097 \cdot 10^{-6}$	$1.969 \cdot 10^{-6}$	$2.088 \cdot 10^{-6}$
		δ	0.0782	0.0742	0.0643
		γ	0.8303	0.8371	0.8426
	United Kingdom	ω	$7.482 \cdot 10^{-6}$	$8.348 \cdot 10^{-6}$	$8.779 \cdot 10^{-6}$
		δ	0.1210	0.1293	0.1179
		γ	0.7593	0.7250	0.7313
	Japan	ω	$4.182 \cdot 10^{-7}$	$4.094 \cdot 10^{-7}$	$4.196 \cdot 10^{-7}$
		δ	0.1605	0.1604	0.1763
		γ	0.8357	0.8345	0.8187
	Sweden	ω	$1.342 \cdot 10^{-6}$	$1.355 \cdot 10^{-6}$	$9.610 \cdot 10^{-7}$
		δ	0.0988	0.1328	0.1164
		γ	0.8711	0.8377	0.8624
MSCI-Stock Index	***North America***	ω	$1.898 \cdot 10^{-6}$	$1.943 \cdot 10^{-6}$	$2.457 \cdot 10^{-6}$
		δ	0.0566	0.0573	0.0585
		γ	0.9392	0.9381	0.9373
	Europe (United Kingdom excluded)	ω	$1.765 \cdot 10^{-5}$	$1.584 \cdot 10^{-5}$	$1.673 \cdot 10^{-5}$
		δ	0.1729	0.1588	0.1444
		γ	0.7982	0.8134	0.8267
	Sweden	ω	$4.178 \cdot 10^{-5}$	$3.974 \cdot 10^{-5}$	$3.911 \cdot 10^{-5}$
		δ	0.1492	0.8130	0.1471
		γ	0.8175	0.1536	0.8205
	United Kingdom	ω	$7.245 \cdot 10^{-6}$	$8.883 \cdot 10^{-6}$	$9.931 \cdot 10^{-6}$
		δ	0.0799	0.0877	0.0826
		γ	0.9053	0.8929	0.8950
	Japan	ω	$9.270 \cdot 10^{-5}$	$5.369 \cdot 10^{-5}$	$8.59 \cdot 10^{-5}$
		δ	0.1096	0.0697	0.0841
		γ	0.7640	0.8506	0.7893
	Asia	ω	$9.434 \cdot 10^{-6}$	$6.486 \cdot 10^{-6}$	$8.153 \cdot 10^{-6}$
		δ	0.1114	0.0888	0.0906
		γ	0.8724	0.8983	0.8935
	Emerging Markets	ω	$1.551 \cdot 10^{-5}$	$2.298 \cdot 10^{-5}$	$2.867 \cdot 10^{-5}$
		δ	0.0988	0.1254	0.1129
		γ	0.8833	0.8455	0.8571
	Real Estate	ω	$4.001 \cdot 10^{-5}$	$4.352 \cdot 10^{-5}$	$2.839 \cdot 10^{-5}$
		δ	0.1012	0.1129	0.0903
		γ	0.8303	0.8123	0.8658

Table 1:The estimates of the univariate GARCH(1,1)-model for each asset based on weekly data.

Every second week data			**Period 1990-2005**	**Period 1991-2006**	**Period 1992-2007**
Credit bonds	***Global Broad Non-sovereign***	ω	$5.919\cdot10^{-6}$	$5.778\cdot10^{-6}$	$5.729\cdot10^{-6}$
		δ	0.0970	0.0871	0.0769
		γ	0.7455	0.7591	0.7713
	OMRX Mortg BI	ω	$5.699\cdot10^{-7}$	$4.179\cdot10^{-7}$	$3.739\cdot10^{-7}$
		δ	0.2458	0.2185	0.2220
		γ	0.7542	0.7815	0.7780
JP MORGAN Broad Government Bond Index	***World***	ω	$7.766\cdot10^{-7}$	$7.509\cdot10^{-7}$	$9.847\cdot10^{-7}$
		δ	0.0309	0.0286	0.0295
		γ	0.9600	0.9621	0.9577
	USA	ω	$6.801\cdot10^{-6}$	$3.981\cdot10^{-6}$	$3.042\cdot10^{-6}$
		δ	0.0690	0.0537	0.0500
		γ	0.7616	0.8462	0.8743
	Europe	ω	$3.153\cdot10^{-6}$	$2.849\cdot10^{-6}$	$2.916\cdot10^{-6}$
		δ	0.0920	0.0816	0.0764
		γ	0.7614	0.7825	0.7894
	United Kingdom	ω	$1.095\cdot10^{-5}$	$1.154\cdot10^{-5}$	$8.609\cdot10^{-6}$
		δ	0.0881	0.1092	0.0763
		γ	0.7351	0.6954	0.7766
	Japan	ω	$9.452\cdot10^{-7}$	$9.974\cdot10^{-7}$	$9.732\cdot10^{-7}$
		δ	0.2351	0.2413	0.2529
		γ	0.7577	0.7467	0.7361
	Sweden	ω	$2.991\cdot10^{-6}$	$2.319\cdot10^{-6}$	$1.904\cdot10^{-6}$
		δ	0.2265	0.2076	0.2006
		γ	0.7303	0.7564	0.7700
MSCI-Stock Index	***North America***	ω	$3.470\cdot10^{-6}$	$3.448\cdot10^{-6}$	$4.561\cdot10^{-6}$
		δ	0.0789	0.0794	0.0850
		γ	0.9136	0.9122	0.9079
	Europe *(United Kingdom excluded)*	ω	$4.078\cdot10^{-5}$	$4.243\cdot10^{-5}$	$4.402\cdot10^{-5}$
		δ	0.2757	0.2758	0.2433
		γ	0.6750	0.6696	0.6968
	Sweden	ω	$9.742\cdot10^{-5}$	$1.095\cdot10^{-4}$	$1.070\cdot10^{-4}$
		δ	0.2531	0.2574	0.2609
		γ	0.6970	0.6736	0.6753
	United Kingdom	ω	$1.056\cdot10^{-5}$	$1.279\cdot10^{-4}$	$1.295\cdot10^{-5}$
		δ	0.0919	0.0961	0.0854
		γ	0.8867	0.08764	0.8859
	Japan	ω	$8.473\cdot10^{-5}$	$2\cdot10^{-7}$	$4.906\cdot10^{-5}$
		δ	0.0279	0	0.0207
		γ	0.8421	0.9996	0.9013
	Asia	ω	$2.110\cdot10^{-5}$	$1.967\cdot10^{-5}$	$2.765\cdot10^{-5}$
		δ	0.1422	0.1365	0.1646
		γ	0.8144	0.8211	0.7843
	Emerging Markets	ω	$2.594\cdot10^{-5}$	$4.173\cdot10^{-5}$	$5.344\cdot10^{-5}$
		δ	0.0905	0.1399	0.1774
		γ	0.8719	0.8008	0.7577
	Real Estate	ω	$3.291\cdot10^{-5}$	$3.038\cdot10^{-5}$	$2.191\cdot10^{-5}$
		δ	0.0770	0.0749	0.0727
		γ	0.8681	0.8746	0.8950

Table 2:The estimates of the univariate GARCH(1,1)-model for each asset based on data from every second week.

Every third week data			**Period 1990-2005**	**Period 1991-2006**	**Period 1992-2007**
Credit bonds	***Global Broad Non-sovereign***	ω	$6.750 \cdot 10^{-6}$	$1.169 \cdot 10^{-6}$	$1.869 \cdot 10^{-6}$
		δ	0.1553	0.0431	0.0432
		γ	0.6822	0.9246	0.9056
	OMRX Mortg BI	ω	$1.265 \cdot 10^{-6}$	$2 \cdot 10^{-7}$	$4.072 \cdot 10^{-7}$
		δ	0.3350	0.0570	0.1441
		γ	0.6650	0.9236	0.8315
JP MORGAN Broad Government Bond Index	***World***	ω	$3.514 \cdot 10^{-6}$	$7.887 \cdot 10^{-7}$	$3.289 \cdot 10^{-6}$
		δ	0.0626	0.0349	0.0476
		γ	0.8938	0.9566	0.9129
	USA	ω	$8.645 \cdot 10^{-7}$	$2.234 \cdot 10^{-6}$	$1.653 \cdot 10^{-6}$
		δ	0.02583	0.0430	0.0469
		γ	0.9527	0.9053	0.9106
	Europe	ω	$6.496 \cdot 10^{-6}$	$1.205 \cdot 10^{-5}$	$3.772 \cdot 10^{-6}$
		δ	0.0558	0.0902	0.1985
		γ	0.6596	0.3822	0.6268
	United Kingdom	ω	$3.287 \cdot 10^{-5}$	$3.209 \cdot 10^{-5}$	$5.075 \cdot 10^{-6}$
		δ	0.0707	0.1574	0.0278
		γ	0.3223	0.4264	0.8561
	Japan	ω	$1.808 \cdot 10^{-6}$	$5.751 \cdot 10^{-7}$	$1.493 \cdot 10^{-6}$
		δ	0.2528	0.1721	0.1224
		γ	0.7235	0.8103	0.8067
	Sweden	ω	$1.703 \cdot 10^{-6}$	$2 \cdot 10^{-7}$	$5.940 \cdot 10^{-6}$
		δ	0.2088	0.0498	0.4097
		γ	0.7912	0.9438	0.4717
MSCI-Stock Index	***North America***	ω	$8.451 \cdot 10^{-6}$	$4.910 \cdot 10^{-6}$	$1.053 \cdot 10^{-4}$
		δ	0.1419	0.0989	0.4445
		γ	0.8418	0.8867	0.3904
	Europe *(United Kingdom excluded)*	ω	$6.449 \cdot 10^{-5}$	$1.316 \cdot 10^{-5}$	$1.198 \cdot 10^{-5}$
		δ	0.3981	0.0966	0.1152
		γ	0.5461	0.8214	0.8677
	Sweden	ω	$1.746 \cdot 10^{-4}$	$4.174 \cdot 10^{-5}$	$1.255 \cdot 10^{-5}$
		δ	0.2606	0.1361	0.3567
		γ	0.6036	0.8770	0.6080
	United Kingdom	ω	$2.176 \cdot 10^{-5}$	$2.493 \cdot 10^{-5}$	$1.185 \cdot 10^{-5}$
		δ	0.1560	0.0614	0.0613
		γ	0.8091	0.8694	0.9118
	Japan	ω	$4.282 \cdot 10^{-4}$	$1.426 \cdot 10^{-4}$	$7.308 \cdot 10^{-4}$
		δ	0.2297	0.0921	0
		γ	0.2247	0.6520	0.0251
	Asia	ω	$2.355 \cdot 10^{-5}$	$1.134 \cdot 10^{-5}$	$4.219 \cdot 10^{-5}$
		δ	0.0870	0.1059	0.0556
		γ	0.8666	0.8766	0.8614
	Emerging Markets	ω	$4.593 \cdot 10^{-5}$	$4.659 \cdot 10^{-5}$	$9.221 \cdot 10^{-5}$
		δ	0.0448	0.0891	0.1583
		γ	0.8869	0.8519	0.7118
	Real Estate	ω	$6.647 \cdot 10^{-5}$	$8.911 \cdot 10^{-5}$	$3.367 \cdot 10^{-5}$
		δ	0.1123	0.0744	0.0300
		γ	0.7566	0.7790	0.9093

Table 3: The estimates of the univariate GARCH(1,1)-model for each asset based on data from every third week.

Every fourth week data			**Period 1990-2005**	**Period 1991-2006**	**Period 1992-2007**
Credit bonds	***Global Broad Non-sovereign***	ω	$3.353 \cdot 10^{-6}$	$2.756 \cdot 10^{-6}$	$3.350 \cdot 10^{-6}$
		δ	0.1505	0.1508	0.1118
		γ	0.7422	0.7633	0.7818
	OMRX Mortg BI	ω	$1.774 \cdot 10^{-6}$	$4.865 \cdot 10^{-7}$	$5.962 \cdot 10^{-7}$
		δ	0.4800	0.2369	0.2926
		γ	0.5200	0.7631	0.7074
JP MORGAN Broad Government Bond Index	***World***	ω	$9.727 \cdot 10^{-5}$	$6.211 \cdot 10^{-6}$	$6.863 \cdot 10^{-6}$
		δ	0.0626	0.0394	0.0411
		γ	0.7885	0.8701	0.8581
	USA	ω	$3.197 \cdot 10^{-5}$	$3.184 \cdot 10^{-5}$	$3.273 \cdot 10^{-5}$
		δ	0.0247	0.0227	0.0145
		γ	$2.01 \cdot 10^{-12}$	0	0
	Europe	ω	$5.286 \cdot 10^{-6}$	$5.842 \cdot 10^{-6}$	$6.052 \cdot 10^{-6}$
		δ	0.1722	0.1761	0.1735
		γ	0.5504	0.5246	0.5351
	United Kingdom	ω	$1.233 \cdot 10^{-5}$	$3.740 \cdot 10^{-5}$	$1.327 \cdot 10^{-5}$
		δ	0.1118	0.2119	0.1179
		γ	0.6408	0	0.6136
	Japan	ω	$7.313 \cdot 10^{-6}$	$7.474 \cdot 10^{-6}$	$6.762 \cdot 10^{-6}$
		δ	0.3802	0.3977	0.4085
		γ	0.4197	0.3900	0.3878
	Sweden	ω	$1.313 \cdot 10^{-6}$	$7.868 \cdot 10^{-7}$	0.0101
		δ	0.1261	0.1050	0.1046
		γ	0.8487	0.8757	0.8680
MSCI-Stock Index	***North America***	ω	$7.334 \cdot 10^{-6}$	$7.955 \cdot 10^{-6}$	$1.401 \cdot 10^{-5}$
		δ	0.1136	0.1184	0.1290
		γ	0.8705	0.8650	0.8465
	Europe *(United Kingdom excluded)*	ω	$2.695 \cdot 10^{-5}$	$3.605 \cdot 10^{-5}$	$4.212 \cdot 10^{-5}$
		δ	0.2167	0.3445	0.2609
		γ	0.7650	0.6555	0.7080
	Sweden	ω	$4.486 \cdot 10^{-5}$	$3.063 \cdot 10^{-5}$	$4.351 \cdot 10^{-5}$
		δ	0.0782	0.0765	0.0921
		γ	0.8785	0.8903	0.8701
	United Kingdom	ω	$1.659 \cdot 10^{-5}$	$1.803 \cdot 10^{-5}$	$2.302 \cdot 10^{-5}$
		δ	0.0998	0.1025	0.1086
		γ	0.8624	0.8551	0.8443
	Japan	ω	$4.383 \cdot 10^{-4}$	$1.493 \cdot 10^{-4}$	$2.534 \cdot 10^{-4}$
		δ	0.1196	0.0752	0.1155
		γ	0.0856	0.6494	0.4469
	Asia	ω	$1.462 \cdot 10^{-5}$	$9.761 \cdot 10^{-6}$	$2.302 \cdot 10^{-5}$
		δ	0.1085	0.1227	0.1745
		γ	0.8543	0.8552	0.7872
	Emerging Markets	ω	$6.200 \cdot 10^{-5}$	$6.942 \cdot 10^{-5}$	$1.724 \cdot 10^{-4}$
		δ	0.1117	0.1015	0.3180
		γ	0.7831	0.7775	0.4416
	Real Estate	ω	$3.833 \cdot 10^{-5}$	$4.641 \cdot 10^{-5}$	$4.346 \cdot 10^{-5}$
		δ	0.0360	0.0364	0.0480
		γ	0.8982	0.8850	0.8855

Table 4:The estimates of the univariate GARCH(1,1)-model for each asset based on data from every fourth week.

6.3 The Parameters of the conditional correlation matrix

Up to now, the parameters of the univariate conditional variances have been discussed and it is time to move on to the parameters associated with the conditional correlation. From earlier studies these parameters have an approximate size of $\alpha = 0.01$ and $\beta = 0.97$.

Suppose that the conditional covariance of four different portfolios' are estimated, namely

- Portfolio 1 consisting of the two credit bond indices OMRX Mortage index and Global Broad Non-Sovereign,
- Portfolio 2 the six JP Morgan Broad Government Bond indices
- Portfolio 3 the seven stock indices from MSCI
- Portfolio 4 all of the sixteen indices

For each portfolio, table 5 to 8 in the end of this section present the estimated parameters of the conditional correlation. On the whole, the results of the estimated parameters agree with the approximate size above. However, there is never a rule without exceptions. The estimates in portfolio 1 *(see table 5)* differ during the period 1991-2006 for weekly data and 1990-2005 for data selected every third week. Also, in portfolio 3 the estimates sampled every fourth week deviate from the standard result during the period 1990-2005 and 1992-2007 respectively. *(see table7)*

The later estimated parameters imply that the conditional correlation matrix roughly equals the unconditional correlation. This result is not convincing since it is not realistic that the conditional correlation will be constant over time for a portfolio only consisting of stock indices. The reason is the poorly estimated parameters of the Japanese stock index in the univariate case. Excluding the Japanese stock index from portfolio 3, the estimates during 1992 to 2007 will be $\alpha = 0.0059$ and $\beta = 0.9705$ when data is selected every fourth week. During 1990 to 2005 the estimates without Japan will be $\alpha = 0.0070$ and $\beta = 0.9721$.

Portfolio 1	Period 1990-2005	Period 1991-2006	Period 1992-2007
Every week			
α	0.0540	0.1587	0.0594
β	0.9354	0.5669	0.9204
Every second week			
α	0.0309	0.0334	0.0323
β	0.9636	0.9565	0.9570
Every third week			
α	0.3064	0.0361	0.0631
β	0.2158	0. 9507	0.9097
Every fourth week			
α	0.0979	0.0685	0.0733
β	0.7811	0.8642	0.8315

Table 5: The parameters of the DCC-model for portfolio 1

Portfolio 2	Period 1990-2005	Period 1991-2006	Period 1992-2007
Every week			
α	0.0190	0.0190	0.0208
β	0.9800	0.9794	0.9767
Every second week			
α	0.0214	0.0213	0.0204
β	0.9765	0.9756	0.9765
Every third week			
α	0.0277	0.0255	0.0324
β	0.9620	0.9628	0.9544
Every fourth week			
α	0.0282	0.0225	0.0151
β	0.9553	0.9551	0.9393

Table 6: The parameters of the DCC-model for portfolio 2

Portfolio 3	Period 1990-2005	Period 1991-2006	Period 1992-2007
Every week			
α	0.0118	0.0083	0.0188
β	0.9811	0.9877	0.9860
Every second week			
α	0.0071	0.0047	0.0088
β	0.9808	0.9953	0.9834
Every third week			
α	0.0216	0.0214	0.0198
β	0.9066	0.8080	0.9720
Every fourth week			
α	0.0047	0.0058	0.0124
β	$2*10^{-7}$	0.9728	$2*10^{-7}$

Table 7: The parameters of the DCC-model for portfolio 3

Portfolio 4	Period 1990-2005	Period 1991-2006	Period 1992-2007
Every week			
α	0.0139	0.0070	0.0141
β	0.9813	0.9885	0.9808
Every second week			
α	0.0124	0.0045	0.0126
β	0.9695	0.9946	0.9695
Every third week			
α	0.0165	0.0149	0.0190
β	0.9191	0.9109	0.9491
Every fourth week			
α	0.0099	0.0115	0.0084
β	0.8107	0.8700	0.8470

Table 8: The parameters of the DCC-model for portfolio 4

6.4 The forecast of the DCC-model

Recall that from section 5 that the most correct covariance matrix will have the least variance σ_t^2 at time t and therefore the competing covariance forecasts can be compared by the GMVP. Figure 5 illustrates the variance of the GMVP for the covariance matrix forecast by the DCC model (magenta) and sample covariance matrix (blue) for portfolio 1.

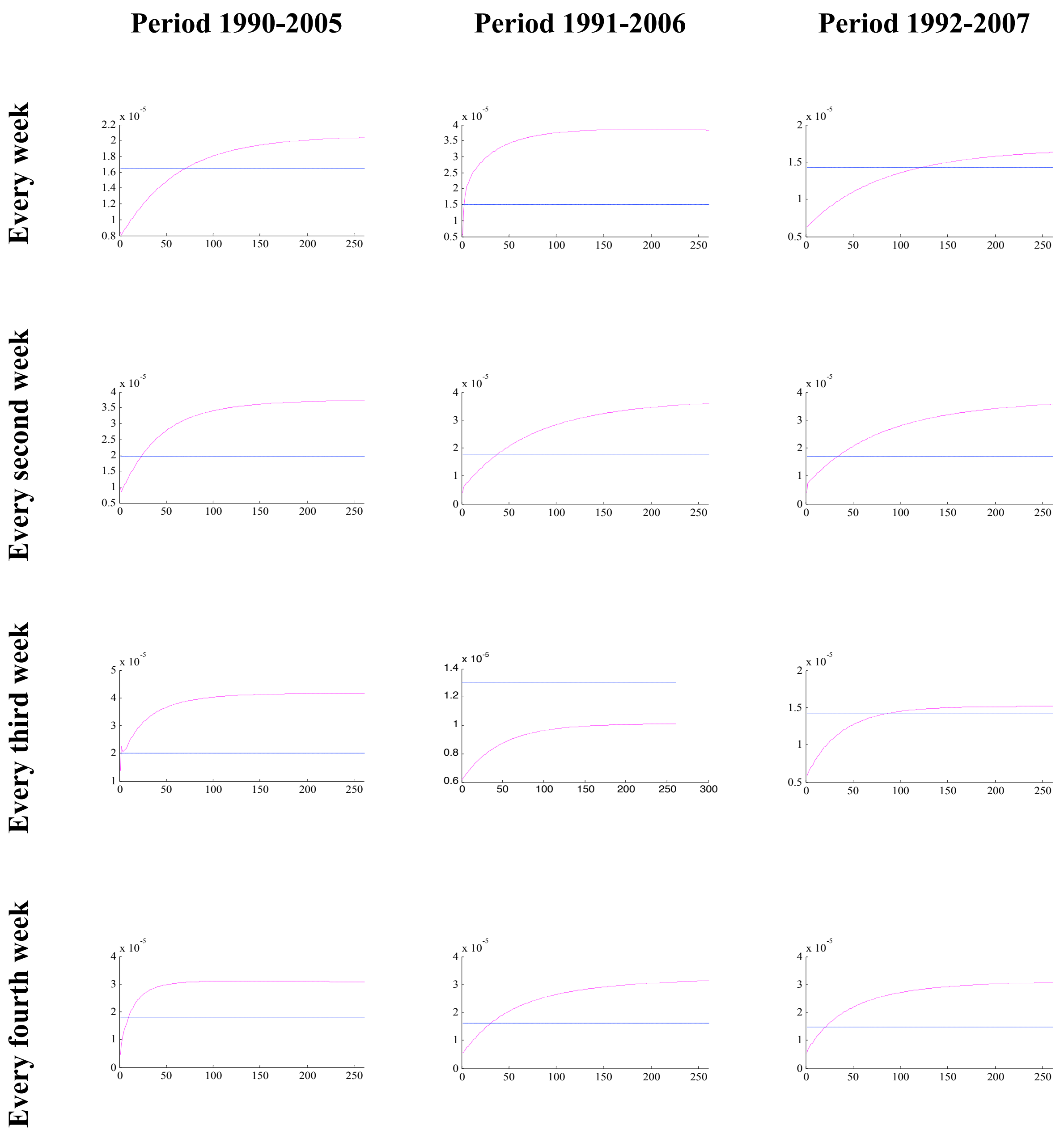

Fig 5 The variance of the GMVP based on the covariance matrix estimated by the DCC-multivariate GARCH model (magenta) and the sample variance (blue) in portfolio 1.

Clearly the DCC multivariate GARCH model forecasts the covariance matrix better than the sample covariance matrix in the short run, although in the long run the sample covariance matrix give the most correct forecast. Only the graph for period 1991 to 2006 with the frequency of every third week diverges from this result. For all time periods the variance of the GMVP for the DCC increases because the correctness of the forecast decline by time.

Another point of interest is that the predicted covariance matrix of the DCC model tends to be better than the covariance matrix based on historical data for a longer time when it is persistent. In other words, when $\alpha + \beta$ is close to one.

Generally speaking the assets in portfolio 2 leads to the same result as for portfolio 1. The result of the different frequencies of the data time periods for portfolio 2 is presented in figure 7. It is basically two graphs that disagree with the others, namely the one in period 1990 to 2005 based on data chosen at every third week and for period 1992 to 2007 for data selected after a period of every fourth week.

One possible explanation of this last mentioned graph is that the estimated parameters of the US government bond index differ. If forecasting the covariance matrix and excluding this bond the graph in fig. 6 will be completed.

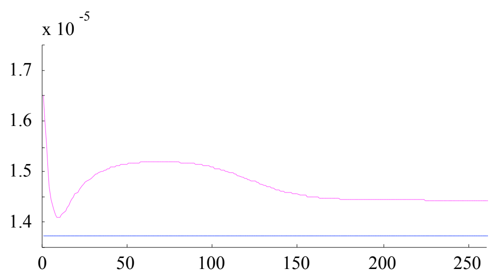

Fig 6 The variances of the GMVP for portfolio 2 excluding the US government bond index

The shape does not change but it is clearer that the covariance matrix based on historical data is better for all points in time.

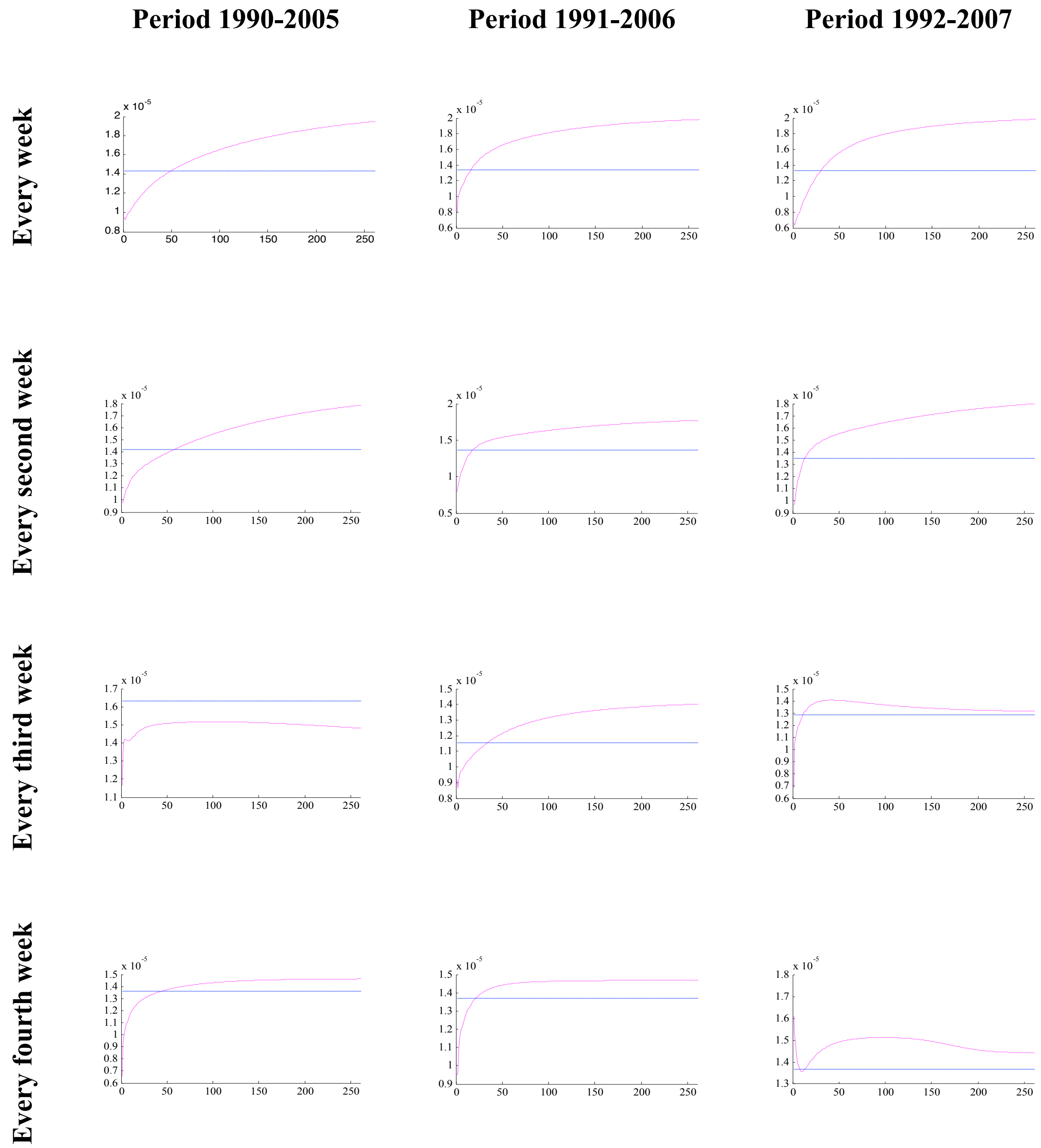

Fig 7 The variance of the GMVP based on the covariance matrix estimated by the DCC-multivariate GARCH model (magenta) and the sample variance (blue) in portfolio 2.

The next portfolio to examine is number three, only consisting of stock indices. This inspection gives no straight answer to the question; which model will give the most accurate forecast? A few graphs will support earlier results that the DCC-model forecast the covariance matrix better than the sample in the short run. The other proves the opposite. Although, it is important to look back at the estimates of the univariate GARCH*(1,1)*, which gives the impression that the Japanese stock index makes a lot of problems.

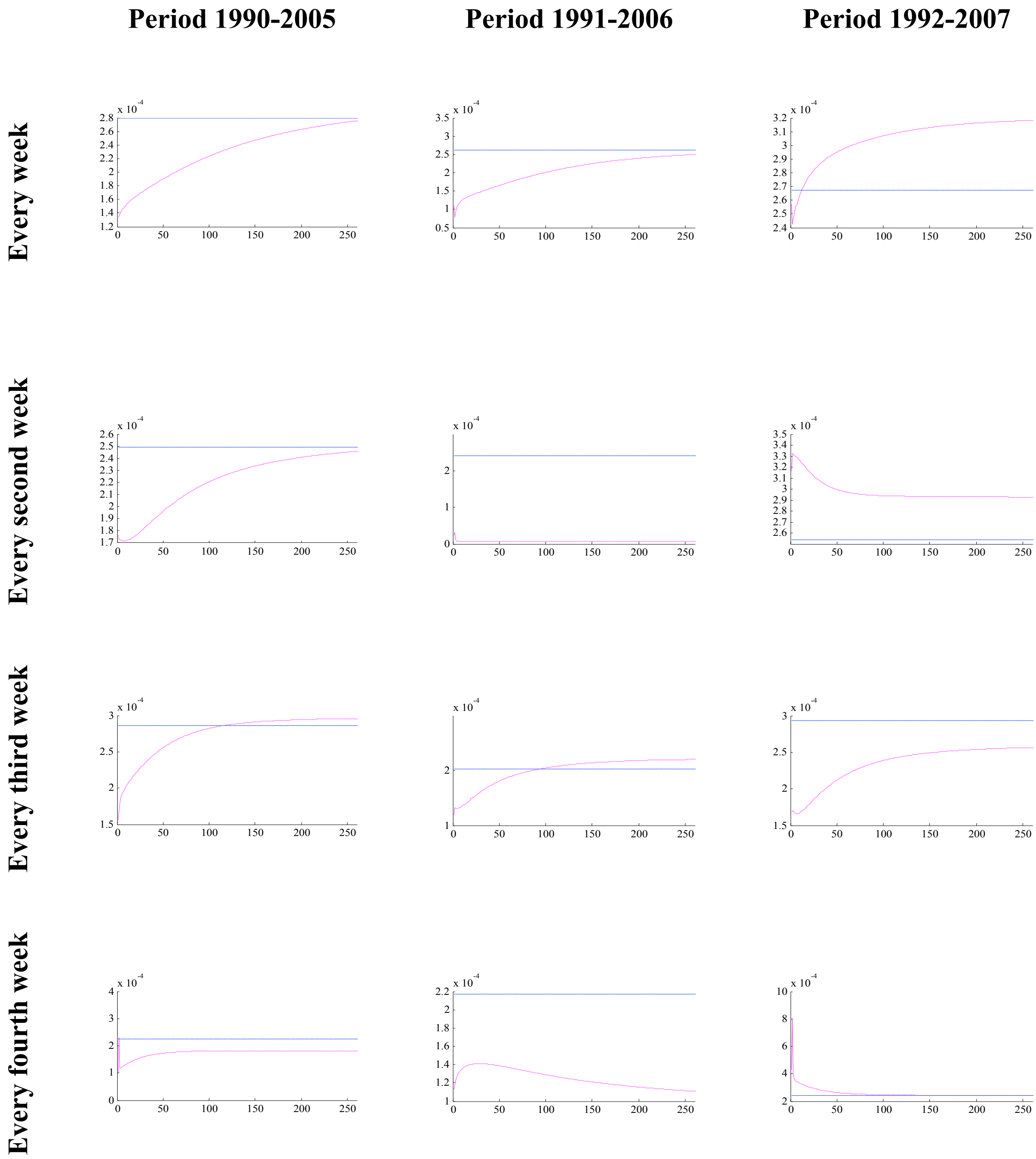

Fig 8 The variance of the GMVP based on the covariance matrix estimated by the DCC-multivariate GARCH model (magenta) and the sample variance (blue) in portfolio 3.

Eliminating the Japanese stock index from portfolio 3, the following result in figure 10 for the period 1992 to 2007 with data from every fourth week is generated.

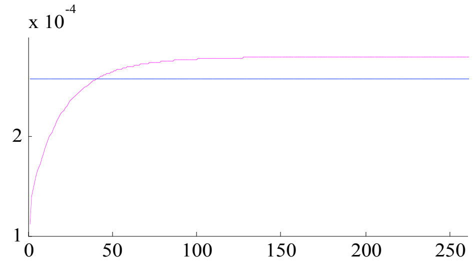

Fig. 9. The variance of the GMVP based on the covariance matrix estimated by the DCC-multivariate GARCH model (magenta) and the sample variance (blue). For the assets in portfolio 3 not including the Japanese stock index.

Without the Japanese stock index the result agrees with previous results of portfolio 1 and 2.

One important feature to observe is that earlier studies have shown that there is other univariate GARCH models such as GJR or EGARCH that will estimate and forecast the stock indices better than the GARCH*(1,1)*.

Finally, the last portfolio to study is the portfolio consisting of all sixteen assets. Also for this portfolio the result points out that the DCC-model give better forecasts in the short run while the sample covariance is better in the long run. Three periods and frequencies are an exception from this rule in figure 10.

Comparing the persistence in the DCC model with the graphs in figure 9 the same tendency as for portfolio 1 and 2 can be confirmed. That is to say when the model is more persistent then the DCC-model is a better model to forecast the covariance for a longer time period. However, in the long run the sample covariance wins the competition.

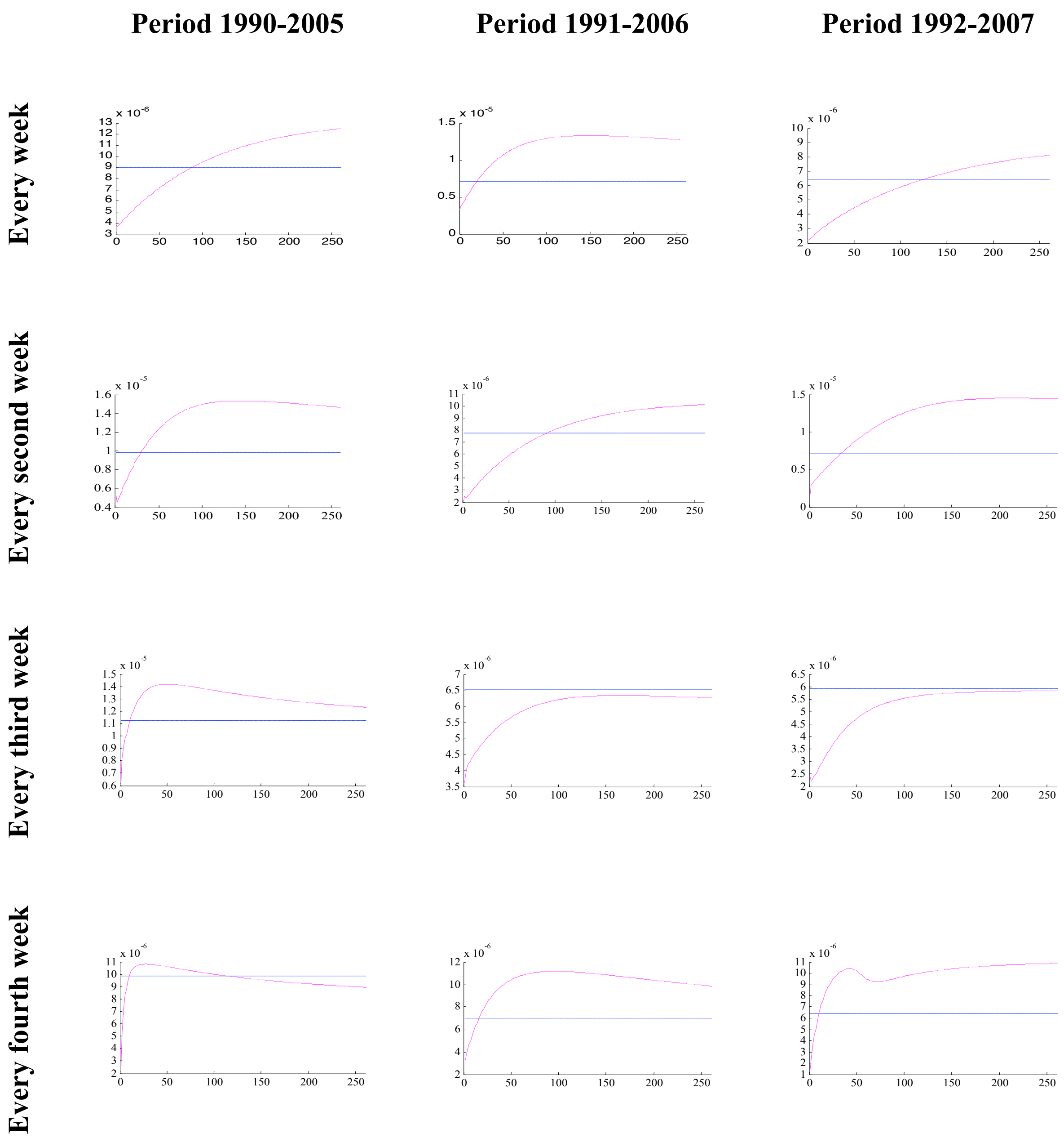

Fig 10. The variance of the GMVP based on the covariance matrix estimated by the DCC-multivariate GARCH model (magenta) and the sample variance (blue) in portfolio 4..

7 Conclusion

This thesis presented a strong tendency that the dynamic conditional correlation model forecasts covariance matrices better than the naive sample covariance in the short run.

In the long run the opposite relation tends to hold. There is no specific time horizon that is short or long, but the model seems to forecast better than the sample covariance for a longer time when the persistence is high.

If the GARCH*(1,1)* should be used to estimate and forecast the conditional variances for each index, then data with weekly or every second week frequency seem to be best. However, since the GARCH*(1,1)* model does not include the asymmetry in the returns it might be good to use other models such the EGARCH and GJR for more accurate estimations and forecasts of the conditional variance. The more correct specification of the univariate GARCH model the more correct estimation and forecast will be achieved from the DCC-model.

Another issue is the distribution of the returns. The assumption of the normally distributed returns makes the model easier to compute but might restrict the model too much. For instance, to capture the behavior of the returns a fat-tailed distribution to include the extreme values would be necessary. However, two questions need to be answered before introducing a new distribution; is this possible from a theoretical point of view? And also, will the opportunity cost be too high in practice when increasing the computational burden?

In the Black Litterman model, the covariance matrix in the benchmark portfolio is assumed to remain constant over a specific time horizon. This assumption is not met by the time varying GARCH models. There are two possible ways to overcome this problem. First, the benchmark portfolio can be rebalanced with the same frequency as in the data. Secondly, a more complicated approach is to use the same idea as for the integrated univariate GARCH model. In the end this will generate a constant covariance under a specific time horizon based on a mean value of the time varying covariance matrices from the DCC-model.

Finally, the DCC-model with suitably specified univariate GARCH-models is an appropriate model to use when forecasting the covariance matrix in the short run, since it is relatively easy to compute.

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