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Bootstrapping for claims reserve uncertainty in general insurance

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Bootstrapping for claims reserve uncertainty in general insurance

Susanna Björkwall*

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Abstract

When England & Verrall (1999) and England (2002) introduced bootstrapping in claims reserving it soon became a popular method in practise as well as in the literature. However, even though bootstrapping has been hailed as a flexible tool to find the precision of complex reserve estimators, much focus so far has been on developing resampling schemes for, in particular, the chain-ladder method.
In this thesis we first develop the chain-ladder bootstrap to obtain a procedure that works for other development factor methods as well. This bootstrap procedure is then extended to be applicable for the separation method.

Keywords

Bootstrap; Chain-ladder; Development factor method; Development triangle; Inflation; Separation method; Stochastic claims reserving.

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"*...friends who suggested names more colorful than Bootstrap, including Swiss Army Knife, Meat Axe, Swan-Dive, Jack-Rabbit, and my personal favorite, the Shotgun, which, to paraphrase Tukey, "can blow the head off any problem if the statistician can stand the resulting mess"*."

Bradley Efron, 1979.
Bootstrap Methods: Another Look at the Jackknife.
The Annals of Statistics, vol. 7.

Acknowledgements

When I was eight or nine years old I made a friend play insurance company with me. I was the actuary and she had to be the CEO. As an actuary I was doing my home work in math in one room and as the CEO she had to sit at a desk in another room reading and considering a file containing her parents bills. I guess that I do not even have to mention that she hated that game. I, on the other hand, really enjoyed it and I was very disappointed that I only could play it once.

It is quite strange - or perhaps it can be considered as totally expected - that I many years later actually ended up as an actuary and, moreover, that I now have written two scientific papers with actuarial applications. However, I am for all time grateful to those who gave me this opportunity. Therefore I would like to thank my supervisors Ola Hössjer and Esbjörn Ohlsson, who have made my wish come true by teaching me how to make scientific research of practical issues relating to my job. I am so glad that I finally found someone to share my interest with and I have had so much fun!

I would also like to thank everyone else who has either helped me with practical things, supported me to reach my goal or showed interest in my research, since this has motivated me to work even harder.

List of papers

This thesis consists of two papers

I Björkwall, S., Hössjer, O. & Ohlsson, E. (2008): Non-parametric and parametric bootstrap techniques for arbitrary age-to-age development factor methods in stochastic claims reserving. Mathematical Statistics, Stockholm University, Research Report 2008:2. To appear in Scandinavian Actuarial Journal.

II Björkwall, S., Hössjer, O. & Ohlsson, E. (2009): Bootstrapping the separation method in claims reserving. Mathematical Statistics, Stockholm University, Research Report 2009:2. Submitted.

S. Björkwall has contributed with the simulations, the analysis of the results and most of the writing. The methodology was developed jointly.

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1 Introduction

One item appearing on the liability side of the non-life insurance company's balance sheet is the provision for outstanding claims – henceforth the claims reserve. The insurance company has put aside this amount for the future compensation of policy holders which is expected on the business written to date. It is indeed important that the claims reserve is carefully calculated; if it is underestimated the insurance company will not be able to fulfill its undertakings and if it is overestimated the insurance company unnecessarily holds the excess capital instead of using it for other purposes, e.g. for investments with higher risk and, hence, potentially higher return. Moreover, since the claims reserve usually constitutes a large share of the firm's total holdings even small miscalculations can imply considerable amounts of money.

On the basis of historical data the actuary can obtain estimates – or rather predictions – of the expected outstanding claims and the claims reserve. However, due e.g. to poor data quality, or sometimes even lack of data, unexpectedly large claim payments, changes in inflation regime or in the discount rate and even legal and political factors, the uncertainty of the *actuary's best estimate* can be quite high. Obviously, there is a risk that the claims reserve will not suffice to pay all claims in the end or, in the one year perspective, that we get a negative run-off result in the income statement the next accounting year. In order to monitor and manage this risk it is important that the actuary's best estimate is complemented by some measure of variability which can be followed up by the insurance company.

The literature provides an abundance of methods for the actuary to choose amongst for reserving purposes, see e.g. the Claims Reserving Manual by the Faculty and Institute of Actuaries (1997). The reserving methods used in practice are frequently deterministic. For instance, the claims reserve is often obtained according to case estimation of individual claims by claims handlers. A popular statistical method is *the chain-ladder method*, see

Taylor (2000), which originally was deterministic. Many ad hoc adjustments are applied as well, e.g. the projection of payments into the future can sometimes be done by extrapolating by eye. Hence, there is a long tradition of actuaries calculating reserve estimates without explicit reference to a stochastic model.

However, stochastic models are needed in order to assess the variability of the claims reserve. The standard statistical approach would be to first specify a model, then find an estimate of the outstanding claims under that model, e.g. by maximum likelihood, and finally the model could be used to find the precision of the estimate. As a compromise between this approach and the actuary's way of working without reference to a model the object of the research area called *stochastic claims reserving* has mostly been to first construct a model and a method that produces the actuary's best estimate and then use this model in order to assess the uncertainty of the estimate. In particular the object of several papers has been to find a model under which the best estimate is the one given by the chain-ladder method, see e.g. Verrall (2000), Mack & Venter (2000) and Verrall & England (2000).

Once the model has been chosen the variability of the claims reserve can be obtained either analytically or by simulation. For instance, the mean squared error of prediction for the chain-ladder method was first calculated analytically by Mack (1993). The reserve estimators are often complex functions of the observations and, hence, it might be difficult to derive analytical expressions. Therefore bootstrapping became a popular method when it was introduced for the chain-ladder by England & Verrall (1999) and England (2002). However, since the existing bootstrap techniques adopt the statistical assumptions in the literature, they have been constructed to give a measure of the precision of the actuary's best estimate *post festum*, i.e. without the possibility of changing the estimate itself

The purpose of Paper I is to develop a bootstrap technique which can be used in order to assess the variability of other *development factor methods* than the chain-ladder. This bootstrap technique is then extended in Paper II to be applicable for *the separation method*, see Taylor (1977).

2 Claims reserving

2.1 Data

Large insurance companies often have quite extensive data bases with historical information on incurred claims. Such information can include the numbers of claims reported and settled, the origin year of the events, the paid amounts, the year of the payments and case estimates. The actuary can regularly analyze the data in order to predict the outstanding claims and, hence, the claims reserve.

The analysis is typically done in the following way. To begin with, the actuary separates the data into risk homogenous groups such as lines of business, e.g. Motor, Property and Liability. A finer segmentation can be applied if the groups or the subgroups contain a sufficient number of observations. The actuary might also choose to divide some group according to the severity of the claims. The large claims can then be reserved according to case estimates while the subgroup consisting of smaller, but frequently occurring, claims can be reserved by some statistical method.

When the risk classification is established the actuary usually aggregates the data within the groups into development triangles. We now consider such an incremental triangle of paid claims $\{C_{ij}; i,j \in \nabla\}$, where the business has been observed during t years, i.e. $\nabla = \{i = 0, \dots, t; j = 0, \dots, t-i\}$. The suffixes i and j of the paid claims refer to the origin year and the payment year, respectively, see Table 2.1. In addition, the suffix $k = i + j$ is used for the calendar years, i.e. the diagonals of ∇.

If we assume that the claims are settled within the t observed years the actuary's goal is to predict the sum of the delayed claim amounts in the lower, unobserved future triangle $\{C_{ij}; i,j \in \Delta\}$, where $\Delta = \{i = 1, \dots, t; j = t - i + 1, \dots, t\}$, see Table 2.2. We write

	Development year					
Accident year	0	1	2	$\cdots$	$t-1$	t
0	C_{00}	C_{01}	C_{02}	$\cdots$	$C_{0,t-1}$	$C_{0,t}$
1	C_{10}	C_{11}	C_{12}	$\cdots$	$C_{1,t-1}$	
2	C_{20}	C_{21}	C_{22}	$\cdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$t-1$	$C_{t-1,0}$	$C_{t-1,1}$				
t	$C_{t,0}$					

Table 2.1: *The triangle ∇ of observed incremental payments.*

$R = \sum_{\Delta} C_{ij}$ for this sum, which is the outstanding claims for which the insurance company must hold a reserve.

	Development year					
Accident year	0	1	2	$\cdots$	$t-1$	t
0						
1						$C_{1,t}$
2					$C_{2,t-1}$	$C_{2,t}$
$\vdots$					$\vdots$	$\vdots$
$t-1$			$C_{t-1,2}$	$\cdots$	$C_{t-1,t-1}$	$C_{t-1,t}$
t		$C_{t,1}$	$C_{t,2}$	$\cdots$	$C_{t,t-1}$	$C_{t,t}$

Table 2.2: *The triangle Δ of unobserved future claim costs.*

Moreover, we assume that the actuary can sum up a triangle of incremental observations of the numbers of claims $\{N_{ij}; i, j \in \nabla\}$ corresponding to the same portfolio as in Table 2.1, i.e. the observations in Table 2.3. The ultimate number of claims relating to the period of origin year i is then

$$N_i = \sum_{j \in \nabla_i} N_{ij} + \sum_{j \in \Delta_i} N_{ij}\,, \tag{2.1}$$

where ∇_i and Δ_i denotes the rows corresponding to origin year i in the upper triangle ∇ and the lower triangle Δ, respectively.

When the paid amounts are presented as in Table 2.1 the payment patterns for the origin

	Development year					
Accident year	0	1	2	$\cdots$	$t-1$	t
0	N_{00}	N_{01}	N_{02}	$\cdots$	$N_{0,t-1}$	$N_{0,t}$
1	N_{10}	N_{11}	N_{12}	$\cdots$	$N_{1,t-1}$	
2	N_{20}	N_{21}	N_{22}	$\cdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$t-1$	$N_{t-1,0}$	$N_{t-1,1}$				
t	$N_{t,0}$					

Table 2.3: *The triangle ∇ of observed incremental numbers of reported claims.*

years emerge along the rows, while the columns provide the pattern for the accident years. Moreover, the diagonals show calendar year effects. Hence, regularities as well as irregularities become apparent to the actuary. For instance, occurrence of growth or run-off of the business, claims inflation or rare large claims can usually be detected in the development triangle and the actuary can then decide how to deal with these issues. If the business is growing or if it is in run-off the actuary can disregard the earliest origin years which have another payment pattern than the later ones. In case of inflation the payments can be adjusted to current value by some relevant index or a reserving method which models the inflation can be chosen. Claims originating from large events and catastrophes can be excluded from the triangle and treated separately.

Note that if observations are missing for some years the data in Table 2.1 will have another shape. Henceforth we assume that the data has the shape of a complete triangle. However, despite a complete triangle the information can still be inadequate if the business has not been observed during a sufficient time period. This is usually a problem for long-tailed lines of business, such as Motor TPL, where it can take several decades to settle the claims. We then have no origin year with finalized claims in Table 2.1. When needed, the model can be extended so that the unknown claims extend beyond t in a tail of length u, i.e. over the development years $t, t+1, \ldots, t+u$.

It is worth bearing in mind that sometimes the data quality may be increased and the reserving process may be refined, but only at a cost. In practise the amount of time and the cost of improving the processes have to be related to the benefits, but even if faster and cheaper approximations are chosen it is still important that the actuary is aware of e.g. imperfections in the data and how they affect the results.

2.2 The chain-ladder and other age-to-age development factor methods

The chain-ladder method is probably the most popular reserving technique in practise. According to Taylor (2000) its lineage can be traced to the mid-60's and the name should refer to the chaining of a sequence of age-to-age development factors into a ladder of factors by which one can climb from the observations to date to the predicted ultimate claim cost. The chain-ladder was originally deterministic, but in order to assess the variability of the estimate it has developed into a stochastic method. Taylor (2000) presents different derivations of the chain-ladder procedure; one of them is deterministic while another one is stochastic and based on the assumption that the incremental observations are Poisson distributed. Verrall (2000) provides several models which under maximum likelihood estimation reproduce the chain-ladder estimate.

The chain-ladder method operates on cumulative observations

$$A_{ij} = \sum_{\ell=0}^{j} X_{i\ell} \tag{2.2}$$

rather than incremental observations X_{ij}, where X_{ij} can be e.g. paid claims C_{ij} or the numbers of claims N_{ij}. Let $\nu_{ij} = E(A_{ij})$ and $\xi_{ij} = E(X_{ij})$. Development factors g_j are then estimated for $j = 0, 1, \ldots, t-1$ by

$$\hat{g}_j = \frac{\sum_{i=0}^{t-j-1} A_{i,j+1}}{\sum_{i=0}^{t-j-1} A_{ij}} \tag{2.3}$$

yielding the projections

$$\hat{\nu}_{ij} = A_{i,t-i}\, \hat{g}_{t-i}\, \hat{g}_{t-i+1} \cdots \hat{g}_{j-1} \tag{2.4}$$

and

$$\hat{\xi}_{i,j} = \hat{\nu}_{i,j} - \hat{\nu}_{i,j-1} \tag{2.5}$$

for Δ.

The actuary might want to make some ad hoc adjustments of the chain-ladder method in order to deal with the trends and occurrences of the influences discussed in Section 2.1. The reserving method is then usually referred to as an age-to-age development factor method and since it will be unique for the particular data set under analysis it is impossible to describe it in general terms. However, Paper I provides the following example of a procedure that might fit our scheme when a development triangle of paid claims is available.

We denote the cumulative claims by $D_{ij} = \sum_{\ell=1}^{j} C_{i\ell}$ and let $\mu_{ij} = E(D_{ij})$.

1. The chain-ladder method is used to produce development factors $\hat{f}_j$ that are estimates of $f_j = \mu_{i,j+1}/\mu_{ij}$, perhaps after excluding the oldest observations and/or sole outliers in ∇.

2. For $3 < j < t$, say, the $\hat{f}_j$'s are smoothed by some method, say exponential smoothing, i.e. they are replaced by estimates obtained from a linear regression of $\log(\hat{f}_j - 1)$ on j. By extrapolation in the linear regression, this also yields $\hat{f}_j$ for the tail $j = t, t+1 \ldots, t+u$. The original $\hat{f}_j$'s are kept for $j \leq 3$ and the smoothed ones used for all $j > 3$.

3. Now estimates $\hat{\mu}_{ij}$ for Δ are computed as in the standard chain-ladder method.

4. Estimates of $\hat{\mu}_{ij}$ for ∇ are obtained by the process of backwards recursion described in England & Verrall (1999).

5. Finally, the obtained claim values may be discounted by some interest rate curve, or inflated by assumed claims inflation. The latter of course requires that the observations where recalculated to fixed prices in the first place.

2.3 The separation method

In the Encyclopedia of Actuarial Science (2004) one can read that the separation method was developed by Taylor (1977) while he was employed at the Department of Trade, the supervisory authority in the UK. During the the mid-70's the inflation was high and unstable and the Department of Trade had been experimenting with the inflation-adjusted version of the chain-ladder, see e.g. Taylor (2000). However, the specification of the future inflation caused problems, since it was extremely controversial for a supervisory tool. As an attempt to forecast the inflation mechanically Taylor (1977) constructed the separation method on the basis of a technique introduced in the reinsurance context by Verbeek (1972).

Paper II provides a description of the separation method at a bit more detailed level than the one given in Taylor (1977). The original assumption underlying the method is

$$E\left(\frac{C_{ij}}{N_i}\right) = r_j\,\lambda_k\,, \tag{2.6}$$

where r_j is a parameter relating to the payment pattern for the development years, while λ_k is considered as an index that relates to the calendar year k during which the claims are paid. In this way the separation method separates the claim delay distribution from influences affecting the calendar years, e.g. claims inflation. Furthermore, it is assumed that the claims are fully paid by year t and we then have the constraint

$$\sum_{j=0}^{t} r_j = 1. \tag{2.7}$$

If N_i is estimated separately, e.g. by the chain-ladder if a triangle of claim counts is provided, it can be treated as known. Consequently, estimates $\hat{r}_j$ and $\hat{\lambda}_k$ can be obtained using the observed values

$$s_{ij} = \frac{C_{ij}}{\hat{N}_i}\,, \tag{2.8}$$

and the method of moments equations

$$s_{k0} + s_{k-1,1} + \ldots + s_{0k} = (\hat{r}_0 + \ldots + \hat{r}_k)\,\hat{\lambda}_k\,, \qquad k = 0, \ldots, t \tag{2.9}$$

for the diagonals of ∇ and

$$s_{0j} + s_{1j} + \ldots + s_{t-j,j} = (\hat{\lambda}_j + \ldots + \hat{\lambda}_t)\, \hat{r}_j\,, \qquad j = 0, \ldots, t \tag{2.10}$$

for the columns of ∇.

Taylor (1977) shows that the equations (2.9) - (2.10) have a unique solution under (2.7) which can be obtained recursively. This yields

$$\hat{\lambda}_k = \frac{\sum_{i=0}^{k} s_{i,k-i}}{1 - \sum_{j=k+1}^{t} \hat{r}_j}\,, \qquad k = 0, \ldots, t \tag{2.11}$$

and

$$\hat{r}_j = \frac{\sum_{i=0}^{t-j} s_{ij}}{\sum_{k=j}^{t} \hat{\lambda}_k}\,, \qquad j = 0, \ldots, t\,, \tag{2.12}$$

where $\sum_{j=k+1}^{t} \hat{r}_j$ is interpreted as zero when $k = t$.

Estimates $\hat{m}_{ij}$ of the expectations $m_{ij} = E(C_{ij})$ for cells in ∇ are now given by

$$\hat{m}_{ij} = \hat{N}_i\, \hat{r}_j\, \hat{\lambda}_k\,, \tag{2.13}$$

but in order to obtain the estimates of Δ it remains to predict λ_k for $t + 1 \leq k \leq 2\,t$ e.g. by extrapolation.

3 Bootstrapping for claims reserve uncertainty

3.1 Bootstrap techniques for the chain-ladder in the literature

When England & Verrall (1999) and England (2002) introduced bootstrapping in claims reserving it soon became a popular method in practise as well as in the literature. However, even though bootstrapping has been hailed as a flexible tool to find the precision of the complex reserve estimators it has developed to be the opposite in the literature. Instead of finding general techniques where the actuary can change and adjust the reserving method, the object of the research area has been to find techniques for, in particular, the chain-ladder. In practise this could be quite frustrating since the actuary then has to measure the uncertainty of her estimate by a bootstrap procedure fitted for chain-ladder even though she actually has used some other reserving method to calculate the claims reserve.

The bootstrap procedures in the literature are based on the resampling of residuals, see e.g. England & Verrall (1999), England (2002) and Pinheiro *et al.* (2003). In order to define the residuals some model assumption has to be adopted for the observations. The common choice is to use a generalized linear model (GLM) with an over-dispersed Poisson distribution (ODP) and a logarithmic link function for the incremental observations ∇C in Table 2.1, i.e.

$$\begin{aligned} &E(C_{ij}) = m_{ij} \quad \text{and} \quad \mathrm{Var}(C_{ij}) = \phi m_{ij} \\ &log(m_{ij}) = \eta_{ij} \\ &\eta_{ij} = c + \alpha_i + \beta_j, \quad \alpha_1 = \beta_1 = 0\,. \end{aligned} \tag{3.1}$$

The reason of the frequent use of this particular GLM is that Renshaw & Verrall (1998) have shown that it produces the same expected claims by maximum likelihood estimation of the parameters in the GLM as the chain-ladder method, provided that the column sums of the triangle are positive. Thus, the expectations of the claims can be obtained either by maximum likelihood estimation or by the chain-ladder, while the variances, which are needed for the residuals, are given by the assumption of the GLM. However, if the bootstrap procedure is constructed according to this particular model it only holds for the chain-ladder and, hence, the reserving algorithm cannot be changed.

In contrast to England & Verrall (1999) and England (2002), Pinheiro *et al.* (2003) adopts the model in (3.1) together with the plug-in-principle, see Efron & Tibshirani (1993), and, hence, the calculation of the estimators in the *real world* is repeated on the pseudo-data in the *bootstrap world*. This opens up for extended bootstrap procedures applicable to other reserving algorithms than the chain-ladder and therefore we focus on Pinheiro's method.

3.2 Paper I: Non-parametric and parametric bootstrap techniques for arbitrary age-to-age development factor methods in stochastic claims reserving

The purpose of this paper is to find a reasonable model that fits the data instead of using a model which happens to reproduce a particular estimate for the bootstrap procedure. We therefore consider the log-additive assumption in (3.1) as unnecessary strong, but besides of that we continue to follow England & Verrall (1999), England (2002) and Pinheiro *et al.* (2003) assuming independent claims C_{ij} and a variance function in terms of the means, i.e.

$$E(C_{ij}) = m_{ij} \quad \text{and} \quad \mathrm{Var}(C_{ij}) = \phi\, m_{ij}^{p} \tag{3.2}$$

for some $p > 0$. We let the actuary's age-to-age development factor method implicitly specify the structure of all m_{ij} and produce estimates of $\hat{m}_{ij}$. Then, if the non-parametric bootstrap approach of Pinheiro *et al.* (2003) is used, it only remains to specify the variance function. We suggest that p is estimated for the particular data set under analysis and we provide a simple and straightforward way of doing it. Furthermore, since the standardized prediction errors in Pinheiro *et al.* (2003) sometimes are undefined in the bootstrap world we also investigate a bootstrap procedure which is based on the unstandardized prediction errors.

As a complement to the non-parametric predictive bootstrap we define a parametric version of Pinheiro's approach that requires more distributional assumptions. Hence, instead of resampling the residuals we sample pseudo-observations from a full distribution $F = F(m_{ij}, \phi\, m_{ij}^{p})$ consistently with (3.2).

3.3 Paper II: Bootstrapping the separation method in claims reserving

In this paper we adopt the parametric predictive bootstrap procedure in Paper I and extend it in order to handle ∇N as well as ∇C for the separation method. To this end, we introduce a parametric framework for the separation method where claim counts are Poisson distributed and claim amounts are gamma distributed *conditionally* on the ultimate claim counts. This enables joint resampling of claim counts and claim amounts.

Hence, we let $n_{ij} = E(N_{ij})$ and assume

$$N_{ij} \in Po(n_{ij}) \tag{3.3}$$

and

$$C_{ij}|N_i \in \Gamma\left(\frac{N_i}{\phi}, r_j\,\lambda_k\,\phi\right)\,. \tag{3.4}$$

We then have a model for the claim amounts where

$$E(C_{ij}|N_i) = N_i\,r_j\,\lambda_k\,, \tag{3.5}$$

which is consistent with the separation method assumption (2.6) when N_i is estimated separately. Moreover, we have

$$Var(C_{ij}|N_i) = \phi\,N_i\,(r_j\,\lambda_k)^2\,. \tag{3.6}$$

According to the parametric predictive bootstrap procedure in Paper I and the plug-in-principle we then let the separation method produce estimates of r_j and λ_k in the bootstrap world as in the real world.

The separation model is based on the assumption that N_i is considered as known at the moment when the reserving is being done, but in (3.3) N_i is a random variable. In order to get a view of how much uncertainty N_i contributes to the predictive distribution of the claims reserve we also consider the special case when N_i is treated as deterministic in (3.4), i.e. $\hat{N}_i \equiv N_i$.

4 Reserve risk in a business model

So far the insurance business as well as the authorities' supervision have been based on a general conservativeness regarding the liabilities to the policy holders. There are laws that dictate how much capital the firms must hold and how it may be invested, see e.g. Försäkringsrörelselagen by Sveriges Riksdag (1982) for the regulations applied in Sweden today. However, the current regulations rather consider the volume than the risk of the business in the calculation of the required amount of capital.

In order to capture the individual characteristics of the firms the regulations are being modernized within EU. According to the Solvency II Draft Framework Directive by EU Commission (2007), the required capital will instead be calculated by quantifying the risks of the firm under market-like assumptions. The authorities will provide a standard formula which consider the major risks that an insurance company is exposed to, but own internal models will also be allowed. For instance, the firms will have to quantify premium and reserve risk, catastrophe risk, market risks such as e.g. equity risk, interest rate risk and currency risk, counterparty default risk and operational risk. For Solvency II purposes the internal models will have to be stochastic, a one-year time perspective should be adopted and the risks should be measured according to a 99.5% confidence level. Furthermore, the purpose of an internal model is not only to be a supervisory tool - it has to be used in the business as well in order to show its trustworthiness. Potential areas of use could be e.g. business planning, investment strategies, reinsurance purchase and pricing.

The analysis of the business by such an internal simulation model is often referred to as Dynamic Financial Analysis (DFA) in general insurance. Kaufmann *et al.* (2001) gives an introduction to DFA and also provides an example of a basic model.

Thus, regarding the reserve risk for Solvency II purposes we have to model the amount of capital that the insurance company must hold in order to be able to handle a negative run-off result the next accounting year with 99.5% probability. The one year run-off result is defined as the difference between the opening reserve at the beginning of the year and the sum of payments during the year and the closing reserve of the same portfolio at the end of the year. Thus, if we at the end of year t want to make predictions of the run-off result at the end of the unobserved year $t+1$, and if we do not add neither a new accident year nor a new development year, we have to find the predictive distribution of

$$\hat{R}^t - (\sum_{i=2}^{t} C_{i,t+2-i} + \hat{R}^{t+1}) \,, \tag{4.1}$$

where $\hat{R}^t$ and $\hat{R}^{t+1}$ are the estimated reserves at the end of year t and $t+1$ respectively.

Paper I briefly discusses how the predictive distribution of the one year reserve risk can be obtained by bootstrapping, while Ohlsson & Lauzeningks (2008) provides more details for the one year reserve risk as well as the one year premium risk.

5 Discussion

5.1 Conclusions

In Paper I the parametric bootstrap procedure is numerically compared to Pinheiro's non-parametric procedure for the chain-ladder. The study shows that the two approaches give almost the same results. Moreover, in Paper II the parametric bootstrap procedure for the separation method is numerically compared to a parametric procedure for the chain-ladder for different assumptions of the future inflation rate. The study shows that the result is more affected by the assumption of the future claims inflation rate than the choice between the chain-ladder and the separation method.

The numerical analysis has revealed that the variability of the estimation error, when chain-ladder as well as the separation method is used, is much larger than the variability of the process error. Furthermore, the unstandardized bootstrap results in lower percentiles than the standardized one, seemingly due to the fact that the standardization makes the distribution more symmetric than the unstandardized case, where the predictive distribution is skewed to the left.

5.2 Future research

Several interesting topics for future research have been discovered during the development of the bootstrap procedures described in Paper I - II. For instance, estimation of the dispersion parameter p for the variance function and the feasible use of non-integer values should be analyzed further, the modeling of the future inflation rate of the separation method could be refined in order to improve the bootstrap procedure, the relative size of the estimation and process errors is indeed an interesting topic to explore and the double bootstrap, which is an improved version of the standardized bootstrap, should be investigated numerically. Furthermore, the bootstrap procedure could be extended for models which explicitly take into consideration the reporting year as well as the payment year of the claims, see e.g. Jessen *et al.* (2007). In the future it is also important to provide a guideline of how the actuary should choose between the standardized and the unstandardized bootstrap procedure.

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Paper I

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Non-parametric and parametric bootstrap techniques for arbitrary age-to-age development factor methods in stochastic claims reserving

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Abstract

In the literature, one of the the main objects of stochastic claims reserving is to find models underlying the chain-ladder method in order to analyze the variability of the outstanding claims, either analytically or by bootstrapping. In bootstrapping these models are used to find a full predictive distribution of the claims reserve, even though there is a long tradition of actuaries calculating the reserve estimate according to more complex algorithms than the chain-ladder, without explicit reference to an underlying model. In this paper we investigate existing bootstrap techniques and suggest two alternative bootstrap procedures, one non-parametric and one parametric, by which the predictive distribution of the claims reserve can be found for any age-to-age development factor method, using some rather mild model assumptions. For illustration, the procedures are applied to four different development triangles.

Keywords

Bootstrap, Chain-ladder, Development factor method, Development triangle, Dynamic financial analysis, Stochastic claims reserving.

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1 Introduction

The provision for outstanding claims – henceforth the claims reserve – is a major contributor to the total risk of an insurance company, especially for long-tailed lines of business. In order to estimate the risk that the provisions will not suffice to pay all claims in the end, the actuary's best estimate of the outstanding claims needs to be complemented by its predictive distribution; this is the ultimo perspective. For solvency control and risk management with Dynamic Financial Analysis we are also interested in a shorter period, say the one year risk. The reserving risk is then the risk of a negative run-off result, due to unexpectedly large claims payments, changes in inflation regime or in the discount rate in the simulated forecast year.

A well-known method for calculating the uncertainty of the claims reserve, obtained by chain-ladder, in meeting ultimate claims, or at least its mean squared error of prediction, is the one introduced by Mack (1993) and recently treated by Buchwalder *et al.* (2006) and Mack *et al.* (2006). Another popular method is bootstrapping, as introduced in this context by England & Verrall (1999) and England (2002). The latter method gives a full predictive distribution without further assumptions and can easily be used also for the purpose of finding the risk in the run-off result. Therefore, we focus on bootstrap methods here.

A standard statistical approach to claims reserving would be to first specify a model, then find an estimate of outstanding claims under the model, e.g. by maximum likelihood. Finally, the model could be used to find the precision of the estimate, possibly by bootstrapping if an analytic solution was untractable.

In practice, there is a long tradition of actuaries calculating reserve estimates without explicit reference to a model. The object of the research area called stochastic claims reserving, has mostly been to find a model and a method of giving a measure of the precision of the actuary's best estimate *post festum*, i.e. without the possibility of changing the estimate itself.

In particular the object of several papers on stochastic claims reserving has been to find a model under which the best estimate is the one given by the chain-ladder method; indeed, there has been a discussion of which method is underlying the chain-ladder, see in particular Verrall (2000), Mack & Venter (2000) and Verrall & England (2000). So even though the actuary did not use a model to pick her best estimate, these articles try to find a model that would make her work consistent with the standard approach of statistics: to specify the model before finding the estimate. In Verrall (2000) several underlying models, which produce the same reserve estimates as the chain-ladder method, are suggested, and it is also remarked on the importance of careful examination of the assumptions of the model and how the chosen model effects the outstanding claims.

In this paper we question the need to bootstrap an underlying model with claim distributions fully specified, which happens to reproduce the actuary's best estimate. Instead, we develop a bootstrap methodology for the data with as few model assumptions as possible, applicable to any age-to-age development factor method. We assume that the bootstrap procedure only depends on the mean and variance of the claims and that the chosen reserving algorithm implicitly specifies the mean structure and therefore the only additional assumption concerns the variance function. Furthermore, we discuss the non-parametric vs the parametric bootstrap and standardized vs unstandardized prediction errors. Finally, the suggested bootstrap procedures are applied to development triangles of different types.

Section 2 contains the definitions and gives an example of an age-to-age development factor method, that might be used in practise. In Section 3 the non-parametric bootstrap procedure of Pinheiro *et al.* (2003) is discussed and an alternative parametric procedure is suggested, as well as bootstrap procedures, which can be used to find the predictive distribution of any age-to-age development factor method. The double bootstrap is discussed, some details of the implementation of the bootstrap procedures are commented and finally the run-off result is defined and a sketch of a method of obtaining it's predictive distribution is provided. In Section 4 the bootstrap procedures are compared on four different development triangles.

2 A basic model

We consider data in the form of a triangle of n incremental observations $\{C_{ij}; i,j \in \nabla\}$, where ∇ denotes the upper, observational triangle $\nabla = \{i = 1, \ldots, t; j = 1, \ldots, t-i+1\}$ and C_{ij} is e.g. paid claims, the number of claims, claims incurred or some other quantity of interest of origin year i in development year j, see Table 2.1. For the time being we discuss paid claims. The actuary's goal is then to predict the sum of the delayed claim amounts in the lower, unobserved future triangle $\{C_{ij}; i,j \in \Delta\}$, where $\Delta = \{i = 2, \ldots, t; j = t-i+2, \ldots, t\}$, see Table 2.2. We write $R = \sum_{\Delta} C_{ij}$ for this sum, which is the outstanding claims for which the insurance company must hold a reserve.

	Development year					
Accident year	1	2	3	$\cdots$	$t-1$	t
1	C_{11}	C_{12}	C_{13}	$\cdots$	$C_{1,t-1}$	$C_{1,t}$
2	C_{21}	C_{22}	C_{23}	$\cdots$	$C_{2,t-1}$	
3	C_{31}	C_{32}	C_{33}	$\cdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$t-1$	$C_{t-1,1}$	$C_{t-1,2}$				
t	$C_{t,1}$					

Table 2.1: *The triangle ∇ of observed incremental payments.*

	Development year					
Accident year	1	2	3	$\cdots$	$t-1$	t
1						
2						$C_{2,t}$
3					$C_{3,t-1}$	$C_{3,t}$
$\vdots$					$\vdots$	$\vdots$
$t-1$			$C_{t-1,3}$	$\cdots$	$C_{t-1,t-1}$	$C_{t-1,t}$
t		$C_{t,2}$	$C_{t,3}$	$\cdots$	$C_{t,t-1}$	$C_{t,t}$

Table 2.2: *The triangle Δ of unobserved future claim costs.*

Above we have implicitly made the common assumption that claims are settled within the t observed years. In long-tailed business such as Motor TPL we often have no origin year with finalized claims; when needed, we extend the model so that the unknown claims extend beyond t in a tail of length u, i.e. over the development years $t, t+1, \ldots, t+u$, see Table 2.3. For simplicity, we use the notation Δ for the set of unobserved claims in this case, too.

In practice, the actuary has used some method to calculate an estimate of the outstanding claims R; in statistical terminology this is rather a *prediction* of R. We assume that the method gives estimates $\hat{m}_{ij}$ of the cell expectations $m_{ij} = E(C_{ij})$ for all claims in both ∇ and Δ and that these estimates are functions of our observations $\nabla C \doteq \{C_{ij}; i, j \in \nabla\}$ only. (We will use the notation ∇x to denote the ∇ collection of any variable x, and similar for Δx.) The estimate of outstanding claims is then $\hat{R} = \sum_\Delta \hat{m}_{ij}$. This is the case for age-to-age development factor methods. Note in particular that we do not assume that the reserving method is based on an explicit statistical model, in our experience this is seldom the case in practice.

Some reserving methods operate on cumulative claims $D_{ij} = \sum_{\ell=1}^{j} C_{i\ell}$ rather than incremental claims C_{ij}. Let $\mu_{ij} = E(D_{ij})$. Here is an example of an age-to-age development factor method that fits our scheme:

1. The chain-ladder method, see Taylor (2000), is used to produce development factors $\hat{f}_j$ that are estimates of $f_j = \mu_{i,j+1}/\mu_{ij}$, perhaps after excluding the oldest observations and/or sole outliers in ∇.

2. For $3 < j < t$, say, the $\hat{f}_j$'s are smoothed by some method, say exponential smoothing,

	Development year							
Accident year	1	2	3	$\cdots$	t	$t+1$	$\cdots$	$t+u$
1						$C_{1,t+1}$	$\cdots$	$C_{1,t+u}$
2					$C_{2,t}$	$C_{2,t+1}$	$\cdots$	$C_{2,t+u}$
3					$C_{3,t}$	$C_{3,t+1}$	$\cdots$	$C_{3,t+u}$
$\vdots$				$\vdots$	$\vdots$	$\vdots$		$\vdots$
$t-1$			$C_{t-1,3}$	$\cdots$	$C_{t-1,t}$	$C_{t-1,t+1}$	$\cdots$	$C_{t-1,t+u}$
t		$C_{t,2}$	$C_{t,3}$	$\cdots$	$C_{t,t}$	$C_{t,t+1}$	$\cdots$	$C_{t,t+u}$

Table 2.3: *The long tail case, with the triangle Δ of unobserved future claim costs extended with a rectangle beyond t.*

i.e. they are replaced by estimates obtained from a linear regression of $\log(\hat{f}_j - 1)$ on j. By extrapolation in the linear regression, this also yields $\hat{f}_j$ for the tail $j = t, t+1 \ldots, t+u$. The original $\hat{f}_j$'s are kept for $j \leq 3$ and the smoothed ones used for all $j > 3$.

3. Now estimates $\hat{\mu}_{ij}$ for Δ are computed as in the standard chain-ladder method.

4. Estimates of $\hat{\mu}_{ij}$ for ∇ are obtained by the process of backwards recursion described in England & Verrall (1999).

5. Finally, the obtained claim values may be discounted by some interest rate curve, or inflated by assumed claims inflation. The latter of course requires that the observations where recalculated to fixed prices in the first place.

We now have an estimator $\hat{R} = h(\nabla C)$ for some possibly quite complex function h, that might be specified only by an algorithm as in the example. Our primary object is to find the bootstrap estimate of the predictive distribution of $\hat{R}$.

3 Bootstrap methods

The basic idea of bootstrapping is to work with the *Bootstrap world* in order to make inference on the *Real world*, see Efron & Tibshirani (1993). This is done by investigating the result of B simulations in the bootstrap world and assuming that the conclusions from these are approximately valid in the real world; this is the so-called plug-in-principle, Efron & Tibshirani (1993). With the outstanding claims in consideration this means that a relation between the true outstanding claims R and its estimator $\hat{R}$ in the real world can be substituted in the bootstrap world by their bootstrap counterparts. This makes it possible to approximate the variance of the prediction error $R - \hat{R}$ as well as the predictive distribution of R.

Pinheiro *et al.* (2003) use the plug-in-principle to obtain the predictive distribution of R by a non-parametric bootstrap technique consistent with the statistical assumptions underlying the chain-ladder method in the literature. Our purpose is to modify it to a non-parametric bootstrap procedure which works for any age-to-age development factor method used in practise, e.g. the one described in the previous section. We also suggest a completely parametric approach consistent with, and as a complement to, the non-parametric procedure.

3.1 Bootstrapping data with a generalized linear model using standardized prediction errors

Some assumptions about the model structure of ∇C have to be imposed in order to bootstrap the data. In the literature a common choice is to use a generalized linear model, in particular an over-dispersed Poisson distribution with a logarithmic link function. A consequence of this underlying model is that the expected claims obtained by maximum likelihood estimation of the parameters in the generalized linear model equal the ones obtained by the chain-ladder method, if the column sums of the triangle are positive, see Renshaw & Verrall (1998). Thus, the expectations of the claims can be obtained either by maximum likelihood estimation or by the chain-ladder, while the variances, which are needed for the residuals, are given by the assumption of the generalized linear model. The bootstrap methods described by England & Verrall (1999), England (2002) and Pinheiro *et al.* (2003) are all based on generalized linear models.

The method discussed in Pinheiro *et al.* (2003) assumes the following log additive structure of the $n = t\,(t+1)/2$ incremental observations in ∇C

$$\begin{aligned} &E(C_{ij}) = m_{ij} \quad \text{and} \quad \mathrm{Var}(C_{ij}) = \phi m_{ij}^p \\ &log(m_{ij}) = \eta_{ij} \\ &\eta_{ij} = c + \alpha_i + \beta_j, \quad \alpha_1 = \beta_1 = 0 \end{aligned} \tag{3.1}$$

The fitted values $\nabla\hat{m}$ and the forecasts $\Delta\hat{m}$ are calculated by maximum quasi likelihood estimation of the $q = 2t-1$ model parameters c, α_i and β_j, e.g. under the assumption of an over-dispersed Poisson distribution, i.e. $p = 1$, or a gamma distribution, i.e. $p = 2$. Estimators of the outstanding claims are then obtained by summing per accident year $\hat{R}_i = \sum_{j\in\Delta_i}\hat{m}_{ij}$, where Δ_i denotes the row corresponding to accident year i in $\Delta\hat{m}$. The estimator of the grand total is $\hat{R} = \sum_{\Delta} \hat{m}_{ij}$.

The residuals are needed for the resampling process and the common choice is to use the Pearson residuals

$$r_{ij}^{P} = \frac{C_{ij} - \hat{m}_{ij}}{\sqrt{\hat{m}_{ij}^{p}}}\,, \tag{3.2}$$

which should have approximately zero mean and constant variance. Pinheiro *et al.* (2003), as well as England & Verrall (1999) and England (2002), work under the assumption that the residuals are independent and identically distributed, an assumption that can be questioned, see e.g. Larsen (2007) and Appendix 1. Nevertheless, we shall adhere to this assumption.

There are two ways of adjusting the Pearson residuals. England & Verrall (1999) and England (2002) use a global adjusting factor

$$r_{ij}^{PA} = \sqrt{\frac{n}{n-q}}\, r_{ij}^{P}, \tag{3.3}$$

whereas Pinheiro *et al.* (2003) argue that the hat matrix standardized Pearson residuals are a better choice. They are given by

$$r_{ij}^{PA} = \frac{r_{ij}^{P}}{\sqrt{1-h_{ij}}}, \tag{3.4}$$

where the h_{ij}:s are the diagonal elements of the n x n hat matrix H, which for generalized linear models is given by

$$H = X(X^{T}WX)^{-1}X^{T}W, \tag{3.5}$$

where X is an n x q design matrix and the generic elements $W_{ij,ij}$ of the n x n diagonal matrix W are

$$W_{ij,ij} = (V(m_{ij})(\frac{\partial\eta_{ij}}{\partial m_{ij}})^{2})^{-1} \tag{3.6}$$

and V is the variance function.

This choice of residual correction is in accordance with Davison & Hinkley (1997). The result of the comparison in Pinheiro *et al.* (2003) does not indicate a big difference to the correction in (3.3).

Note that the residuals are also used to produce the Pearson estimate of the unknown ϕ,

$$\hat{\phi} = \frac{1}{n-q} \sum_{\nabla} (r_{ij}^{P})^2 = \frac{1}{n} \sum_{\nabla} (r_{ij}^{PA})^2 \,, \tag{3.7}$$

where the last equality is exact when (3.3) is used and an approximation for (3.4).

The next step is to get B new triangles of residuals ∇r^* by drawing samples with replacement from the collection of residuals in (3.3) or (3.4). This procedure means sampling from the empirical distribution function of the approximately independent and identically distributed residuals r.

Then B pseudo-triangles ∇C^* are generated by computing

$$C_{ij}^{*} = \hat{m}_{ij} + r_{ij}^{*}\sqrt{\hat{m}_{ij}^{p}} \quad \text{for} \quad i,j \in \nabla \tag{3.8}$$

and for these B pseudo-triangles the future values $\Delta\hat{m}^*$ are forecasted by the same method as above, i.e. by estimating the parameters of the generalized linear model. Estimators for the outstanding claims in the bootstrap world are then derived by $\hat{R}_i^* = \sum_{j\in\Delta_i} \hat{m}_{ij}^*$ and $\hat{R}^* = \sum_{\Delta} \hat{m}_{ij}^*$.

In order to get the random outcome of the true outstanding claims in the bootstrap world, i.e. $R_i^{**} = \sum_{j\in\Delta_i} C_{ij}^{**}$ and $R^{**} = \sum_{\Delta} C_{ij}^{**}$, the resampling is done once more from the empirical distribution function of the residuals to get B triangles of Δr^{**} and then solving

$$C_{ij}^{**} = \hat{m}_{ij} + r_{ij}^{**}\sqrt{\hat{m}_{ij}^{p}} \quad \text{for} \quad i,j \in \Delta \tag{3.9}$$

to get ΔC^{**}.

The final step is to calculate the B prediction errors and in Pinheiro *et al.* (2003) this is done by the following equations

$$\mathrm{pe}_i^{**} = \frac{R_i^{**} - \hat{R}_i^{*}}{\sqrt{\widehat{Var}(R_i^{**})}} \qquad \text{and} \qquad \mathrm{pe}^{**} = \frac{R^{**} - \hat{R}^{*}}{\sqrt{\widehat{Var}(R^{**})}} \,. \tag{3.10}$$

The predictive distributions of the outstanding claims R_i and R are then obtained by plotting

$$\tilde{R}_i^{**} = \hat{R}_i + \mathrm{pe}_i^{**}\sqrt{\widehat{Var}(R_i)} \qquad \text{and} \qquad \tilde{R}^{**} = \hat{R} + \mathrm{pe}^{**}\sqrt{\widehat{Var}(R)} \tag{3.11}$$

for each B.

We tacitly assume that the mean and variance of all bootstrapped quantities are conditional on the observed data ∇C. For instance, the variance of the bootstrapped outstanding claims are

$$Var(R_i^{**}) = \hat{\phi} \sum_{j\in\Delta_i} \hat{m}_{ij}^p \qquad \text{and} \qquad Var(R^{**}) = \hat{\phi} \sum_{\Delta} \hat{m}_{ij}^p\,, \tag{3.12}$$

since the variance of the bootstrapped residuals conditional on ∇C is $\hat{\phi}$ according to (3.3), (3.4) and (3.7). Since Pinheiro *et al.* (2003), as well as England (2002), consider ϕ as constant for the data, the estimates of (3.12) appearing in (3.10) are

$$\widehat{Var}(R_i^{**}) = \hat{\phi} \sum_{j\in\Delta_i} \hat{m}_{ij}^{*p} \qquad \text{and} \qquad \widehat{Var}(R^{**}) = \hat{\phi} \sum_{\Delta} \hat{m}_{ij}^{*p} \tag{3.13}$$

and hence computable from the bootstrap world data ∇C^*. Nevertheless, ϕ is unknown and therefore

$$\widehat{Var}(R_i^{**}) = \hat{\phi}^* \sum_{j\in\Delta_i} \hat{m}_{ij}^{*p} \qquad \text{and} \qquad \widehat{Var}(R^{**}) = \hat{\phi}^* \sum_{\Delta} \hat{m}_{ij}^{*p} \tag{3.14}$$

should rather be used, see Davison & Hinkley (1997). This is in analogy with the estimated variances of the true claims reserves

$$\widehat{Var}(R_i) = \hat{\phi} \sum_{j\in\Delta_i} \hat{m}_{ij}^p \qquad \text{and} \qquad \widehat{Var}(R) = \hat{\phi} \sum_{\Delta} \hat{m}_{ij}^p, \tag{3.15}$$

which are computable from the real data ∇C, as opposed to $Var(R_i)$ and $Var(R)$.

As a complement to the non-parametric procedure described above we suggest a parametric approach. In addition to the assumptions in (3.1) we assume a full distribution F, parametrised by the mean and variance, so that we may write $F = F(m_{ij}, \phi m_{ij}^p)$. Instead of resampling the residuals, we draw C_{ij}^* from $F(\hat{m}_{ij}, \hat{\phi}\hat{m}_{ij}^p)$ for all $i, j \in \nabla$ and thereby we directly get the pseudo-triangles ∇C^*. The bootstrap estimates $\hat{R}_i^* = \sum_{j\in\Delta_i} \hat{m}_{ij}^*$ and $\hat{R}^* = \sum_\Delta \hat{m}_{ij}^*$ are then calculated for each simulation by estimating the parameters of the generalized linear model. In order to get $R_i^{**} = \sum_{j\in\Delta_i} C_{ij}^{**}$ and $R^{**} = \sum_\Delta C_{ij}^{**}$ we sample once again from $F(\hat{m}_{ij}, \hat{\phi}\hat{m}_{ij}^p)$ to get C_{ij}^{**} for all $i, j \in \Delta$. Finally, the B observations of (3.10) and (3.13) are inserted into (3.11) to yield the sought predictive distribution.

These methods of bootstrapping for claims reserve uncertainty are described in Figure 1 and are referred to as the non-parametric and the parametric standardized predictive bootstrap.

England & Verrall (1999) and England (2002) use other bootstrap approaches, which are described in Appendix 2. In England (2002) the bootstrap counterparts of the outstanding claims in the real world are obtained by another simulation conditional on the one in Substage 2.1 in Figure 1. In this way the *process error* $R - E(R)$ is bootstrapped differently from Substage 2.2, while Substage 2.1 bootstraps the *estimation error* $\hat{R} - E(R)$. Thus, B triangles $\Delta\hat{m}^{\dagger}$ are obtained by sampling a random observation $\hat{m}_{ij}^{\dagger}$ from a distribution with mean $\hat{m}_{ij}^{*}$ and variance $\phi\,\hat{m}_{ij}^{*}$ for all $i, j \in \Delta$. The predictive distribution of the outstanding claims R in real world is then obtained by plotting the B values of $\tilde{R}^{\dagger} = \sum_{\Delta} m_{ij}^{\dagger}$. England (2002) suggests using e.g. an over-dispersed Poisson distribution, a negative binomial or a Gamma distribution as the process distribution.

England & Verrall (2006) comment on the approach of including the process error by sampling from a separate distribution, by noting that the non-parametric standardized predictive bootstrap in Pinheiro *et al.* (2003) cannot give larger extremes of the process error than the most extreme residuals observed. Nevertheless, we see no reason to assume separate distributions for the process error and the estimation error. Either we believe in the chosen distribution on the whole and use a parametric predictive bootstrap or we do not and continue to use a non-parametric predictive bootstrap.

3.2 The double bootstrap

It would be preferable to use

$$\text{pe}^{**} = \frac{R^{**} - \hat{R}^{*}}{\sqrt{\widehat{Var}(R^{**} - \hat{R}^{*})}} \tag{3.16}$$

and

$$\tilde{R}^{**} = \hat{R} + \text{pe}^{**}\sqrt{\widehat{Var}(R - \hat{R})} \tag{3.17}$$

instead of (3.10) and (3.11), in particular if the estimation error is much larger than the process error. Although this is more complicated it can be achieved by means of a double bootstrap.

Stage 1 - Real world

· Estimate the parameters in the generalized linear model in equation (3.1).
· Forecast the future expected values $\Delta\hat{m}$ and calculate the fitted values $\nabla\hat{m}$.
· Calculate the residuals for the non-parametric bootstrap and $\hat{\phi}$ for the parametric bootstrap.
· Calculate the outstanding claims $\hat{R}_i = \sum_{j\in\Delta_i} \hat{m}_{ij}$ and $\hat{R} = \sum_{\Delta} \hat{m}_{ij}$.

Stage 2 - Bootstrap world

Substage 2.1 - The estimated outstanding claims

· Resample residuals (3.3) or (3.4) with replacement and insert into (3.8) together with $\nabla\hat{m}$ or sample from $F(\hat{m}_{ij}, \hat{\phi}\hat{m}_{ij}^p)$ for $i,j \in \nabla$ to obtain the pseudo-reality in ∇C^*.
· Estimate the parameters in the generalized linear model with the pseudo-triangles.
· Forecast the future expected values $\Delta\hat{m}^*$.
· Calculate the estimated outstanding claims $\hat{R}_i^* = \sum_{j\in\Delta_i} \hat{m}_{ij}^*$ and $\hat{R}^* = \sum_{\Delta} \hat{m}_{ij}^*$.

Substage 2.2 - The true outstanding claims

· Resample residuals (3.3) or (3.4) with replacement and insert into (3.9) together with $\Delta\hat{m}$ or sample from $F(\hat{m}_{ij}, \hat{\phi}\hat{m}_{ij}^p)$ for $i,j \in \Delta$ to obtain the pseudo-reality in ΔC^{**}.
· Calculate the true outstanding claims $R_i^{**} = \sum_{j\in\Delta_i} C_{ij}^{**}$ and $R^{**} = \sum_{\Delta} C_{ij}^{**}$.
· Store the prediction errors pe_i^{**} and pe^{**} in (3.10).
· Return to the beginning of the bootstrap loop and repeat it B times.

Stage 3 - Analysis of the simulations

· Obtain the predictive distribution of R_i and R, the true outstanding claims in the real world, by plotting the B values in (3.11).

Figure 1: *The procedure of the non-parametric and the parametric standardized predictive bootstrap.*

However, the computational complexity of this approach is quite prohibitive because of the nested bootstrap loop and therefore the double bootstrap is not included in our numerical study.

For each of the B bootstrap replicates, we generate $\tilde{B}$ double bootstrap claims reserves R^d and estimated claims reserves $\hat{R}^d$ in analogy with R^{**} and $\hat{R}^*$ in Section 3.1, the difference being that we use ∇C^* as our data rather than ∇C. Then

$$\widehat{Var}(R - \hat{R}) = Var(R^{**} - \hat{R}^* | \nabla C) \tag{3.18}$$

and

$$\widehat{Var}(R^{**} - \hat{R}^*) = Var(R^d - \hat{R}^d | \nabla C^*), \tag{3.19}$$

where the last variance is approximated by the sample variance of all $\tilde{B}$ double bootstrap replicates.

An alternative to (3.18) and (3.19) is to use the variance of the process and the estimation errors in (5.2) in Appendix 2, i.e.

$$\widehat{Var}(R - \hat{R}) = \widehat{Var}(R) + \widehat{Var}(\hat{R}) \tag{3.20}$$

and

$$\widehat{Var}(R^{**} - \hat{R}^*) = \widehat{Var}(R^{**}) + \widehat{Var}(\hat{R}^*), \tag{3.21}$$

where the process errors are estimated by

$$\widehat{Var}(R) = \hat{\phi} \sum_{\Delta} \hat{m}_{ij}^p \tag{3.22}$$

and

$$\widehat{Var}(R^{**}) = \hat{\phi}^* \sum_{\Delta} \hat{m}_{ij}^{*p}. \tag{3.23}$$

The estimation errors are approximated by the sample variance of the corresponding bootstrap replicates

$$\widehat{Var}(\hat{R}) = Var(\hat{R}^*) \tag{3.24}$$

and

$$\widehat{Var}(\hat{R}^*) = Var(\hat{R}^d). \tag{3.25}$$

3.3 Bootstrapping data with a simple underlying model and a reserving algorithm using unstandardized prediction errors

For the purpose of obtaining the predictive distribution of the claims reserve by bootstrapping, the assumption of a generalized linear model in (3.1) is unnecessarily strong. In practise the actuary seldom assumes any model for ∇C and ΔC, but only uses a reserving algorithm in order to estimate $\nabla\hat{m}$ and $\Delta\hat{m}$. Thus, when using the plug-in-principle we just need to make an assumption of the model that generates ∇C^* and ΔC^{**} from the data ∇C, while the reserving algorithm can be used in bootstrap world too in order to estimate $\Delta\hat{m}^*$.

We follow England & Verrall (1999), England (2002) and Pinheiro (2003) and assume independent claims C_{ij} and a variance function in terms of the means, i.e.

$$E(C_{ij}) = m_{ij} \quad \text{and} \quad \mathrm{Var}(C_{ij}) = \phi\, m_{ij}^p \tag{3.26}$$

for some $p > 0$. Thus the mean and variance of C_{ij} are still related as in (3.1), but m_{ij} need no longer satisfy the log-additive conditions in (3.1). Instead the chosen reserving algorithm implicitly specifies the structure of all m_{ij} and produces estimates of $\hat{m}_{ij}$. The bootstrap procedures are then performed as in Section 3.1 with the exception that the residuals (3.3) are used rather than (3.4). The interpretation of n and q as the number of observations and model parameters is still the same. Using the pure chain-ladder method together with the backwards recursive operation described in England & Verrall (1999) implies that $q = 2t - 1$, as for the generalized linear model in (3.1), since this procedure demands the estimation of $t-1$ development factors as well as the t starting values of the backwards recursive operation. Adding exponential smoothing of the development factors, like in the example in Section 2, can indeed complicate the determination of the number of model parameters but the correction factor in (3.3) can be considered as an approximation, although the number of parameters q typically depends on the amount of smoothing.

Standardized prediction errors may still be used, since (3.10) - (3.15) continue to hold. Indeed, it is well known that for many bootstrap procedures, resampling of standardized quantities often increases accuracy compared to using unstandardized quantities, see e.g. Hall (1995).

Nevertheless, the unstandardized prediction errors

$$\mathrm{pe}_i^{**} = R_i^{**} - \hat{R}_i^{*} \qquad \text{and} \qquad \mathrm{pe}^{**} = R^{**} - \hat{R}^{*} \tag{3.27}$$

are useful, in particular for the purpose of studying the estimation and the process errors, but also since they are always defined. On the contrary, the denominators of (3.10) may sometimes be non-positive, yielding undefined or imaginary standardized prediction errors, see Section 3.5. The predictive distributions of the outstanding claims R_i and R are then obtained by plotting

$$\tilde{R}_i^{**} = \hat{R}_i + \mathrm{pe}_i^{**} \qquad \text{and} \qquad \tilde{R}^{**} = \hat{R} + \mathrm{pe}^{**} \tag{3.28}$$

for each B. These prediction errors are used in Li (2006).

The alternative bootstrap procedures discussed above are described in detail in Figure 2 and are referred to as the non-parametric and the parametric unstandardized predictive bootstrap.

3.4 Estimation of p

In the literature the most frequent choice of dispersion parameter is $p = 1$ in order to reproduce the chain-ladder estimates under the assumption of a generalized linear model, but as indicated in the method example in Section 2, a pure chain-ladder is seldomly used in practise. Thus, another approach would be to choose the p that best fits the data.

A straightforward way of obtaining a suitable value of p is to use the unstandardized residuals

$$r_{ij} = \sqrt{\frac{n}{n-q}}\,(C_{ij} - \hat{m}_{ij}) \ . \tag{3.29}$$

The following relation then holds approximatively

$$E(r_{ij}^2) \approx Var(C_{ij}) = \phi\, m_{ij}^p \tag{3.30}$$

and minimizing the function

$$f(p,\phi) = \sum_{i,j} w_{ij}\,(r_{ij}^2 - \phi\,\hat{m}_{ij}^p)^2, \tag{3.31}$$

Stage 1 - Actuary's best estimate in the real world

· Choose an appropriate reserving method e.g. the one described in Section 2.
· Forecast the future expected values $\Delta\hat{m}$ and calculate the fitted values $\nabla\hat{m}$.
· Calculate the residuals for the non-parametric bootstrap and $\hat{\phi}$ for the parametric bootstrap.
· Calculate the outstanding claims $\hat{R}_i = \sum_{j\in\Delta_i}\hat{m}_{ij}$ and $\hat{R} = \sum_{\Delta}\hat{m}_{ij}$.

Stage 2 - Bootstrap world

Substage 2.1 - The estimated outstanding claims

· Resample residuals (3.3) with replacement and insert into (3.8) together with $\nabla\hat{m}$ or sample from $F(\hat{m}_{ij}, \hat{\phi}\hat{m}_{ij}^p)$ for $i, j \in \nabla$ to obtain the pseudo-reality in ∇C^*.
· Apply the chosen reserving method to ∇C^* to obtain the forecasts $\Delta\hat{m}^*$.
· Calculate the estimated outstanding claims $\hat{R}_i^* = \sum_{j\in\Delta_i}\hat{m}_{ij}^*$ and $\hat{R}^* = \sum_{\Delta}\hat{m}_{ij}^*$.

Substage 2.2 - The true outstanding claims

· Resample residuals (3.3) with replacement and insert into (3.9) together with $\Delta\hat{m}$ or sample from $F(\hat{m}_{ij}, \hat{\phi}\hat{m}_{ij}^p)$ for $i, j \in \Delta$ to obtain the pseudo-reality in ΔC^{**}.
· Calculate the true outstanding claims $R_i^{**} = \sum_{j\in\Delta_i} C_{ij}^{**}$ and $R^{**} = \sum_{\Delta} C_{ij}^{**}$.
· Store the prediction errors pe_i^{**} and pe^{**} in (3.27).
· Return to the beginning of the bootstrap loop and repeat it B times.

Stage 3 - Analysis of the simulations

· Obtain the predictive distribution of R_i and R, the true outstanding claims in the real world, by plotting the B values in (3.28).

Figure 2: *The procedure of the non-parametric and the parametric unstandardized bootstrap.*

where w_{ij} is a weight for observation C_{ij}, with respect to p and ϕ yields an estimator for p. Once a reasonable value of p is chosen and the residuals for the resampling process are defined, ϕ is estimated by (3.7). The simplest choice is to use uniform wights $w_{ij} \equiv 1$ in (3.31). Another possibility is inverse variance weighting, $w_{ij} = \widehat{Var}(r_{ij}^2)^{-1}$. In order to specify these weights, further model assumptions would be needed though.

3.5 Implementation details

There are some major problems with the process of resampling the residuals for the non-parametric bootstrap procedures. Firstly, the bootstrap world is hardly a good approximation of the real world if the claims triangle is small. Furthermore, the basic assumption of identically distributed residuals is certainly violated for $p = 1$, i.e. for an over-dispersed Poisson distribution, see Appendix 1. Depending on the chosen reserving method and the value of p, the standardized residuals in (3.2) sometimes imply a limitation of the set of triangles that can be analyzed, since the residual will be undefined or imaginary whenever a fitted value in $\nabla\hat{m}$ is non-positive. Finally, using the residuals to solve equation (3.8) sometimes results in undesirable negative increments in the pseudo-triangles.

Thus, if the claims triangle ∇C is small, a parametric bootstrap procedure seems preferable. On the other hand, if we know nothing about F and have a large triangle, a non-parametric bootstrap procedure would be our first choice. Note, however, that a parametric bootstrap procedure does not solve the problem with undefined residuals since they are needed in order to estimate ϕ as well. Furthermore, a parametric bootstrap procedure should be used if negative increments in the pseudo-triangles are unacceptable and a gamma distribution should particularly be used if it is undesirable that the increments only take on the values zero and multiples of ϕ, which is the case for the over-dispersed Poisson distribution.

The choice of prediction errors causes another problem. The standardized ones in (3.10) are sensitive to pseudo-triangles where the row sums of the outstanding claims are non-positive. An ad hoc solution is simply to cut out these pseudo-triangles from the simulation process if they are rare, another solution is to use the unstandardized prediction errors in (3.27) instead. The unstandardized ones, on the other hand, result in a predictive distribution which is more skewed to the left than the distribution obtained by the standardized prediction errors, see Section 4 for more details.

Since England & Verrall (1999), England (2002) and Pinheiro *et al.* (2003) replace the maximum likelihood estimation of the parameters in (3.1) by chain-ladder when $p = 1$, the same method is adopted here for the standardized predictive distribution in Figure 1, even though the non-positive column sums of the pseudo-triangles make the estimates disagree.

In this paper, the estimated value of p in Section 3.4 is just considered as an indicator of whether $p = 1$ or $p = 2$ should be used in the non-parametric bootstrap and whether an over-dispersed Poisson distribution or a gamma distribution should be chosen in a parametric bootstrap. The distributions of the residuals corresponding to different choices of $p \in (1, 2)$ should indeed be investigated, but this is beyond the scope of this paper.

3.6 Dynamic Financial Analysis and the one year run-off result

See Kaufmann *et al.* (2001) for an introduction to Dynamic Financial Analysis. Here the movements of the claims reserve are of particular interest. The one year run-off result is the change in the reserve during the financial year and is defined as the difference between the opening reserve at the beginning of the year and the sum of payments during the year and the closing reserve of the same portfolio at the end of the year. Thus, if we at the end of year t want to make predictions of the run-off result at the end of the unobserved year $t+1$, and if we do not add neither a new accident year nor a new development year, we have to find the predictive distribution of

$$\hat{R}^t - \left(\sum_{i=2}^{t} C_{i,t+2-i} + \hat{R}^{t+1}\right), \tag{3.32}$$

where $\hat{R}^t$ and $\hat{R}^{t+1}$ are the estimated outstanding claims at the end of year t and $t+1$ respectively.

One method to obtain the predictive distribution of the one year run-off result is to condition on the claims triangle ∇C. $\hat{R}^t$ is then considered fixed, while the predictive distribution of the payments corresponding to the forecast year $t+1$ is obtained by B times simulating the new diagonal $\{(i,j);\, i+j=t+2\}$ by one of the bootstrap procedures discussed above. This is done by storing e.g. the unstandardized prediction errors $\mathrm{pe}_{ij}^{**} = C_{ij}^{**} - \hat{C}_{ij}^{*}$ of each increment in the new diagonal and then adding them to the corresponding estimated values $\hat{C}_{ij}$ in the real world to obtain $\tilde{C}_{ij}^{**} = \hat{C}_{ij} + \mathrm{pe}_{ij}^{**}$. In this way B pseudo-triangles, consisting of the fixed triangle ∇C known at the end of year t and a new simulated diagonal $\tilde{C}^{**}$, are generated and the outstanding claims are then recalculated by the same reserving method as before, in order to obtain B records of $\hat{R}^{t+1\,*} = \hat{R}^{t+1}(\nabla C \cup \tilde{C}^{**})$. Finally the B values of

$$\hat{R}^t - (\sum_{i=2}^{t} \tilde{C}_{i,t+2-i}^{**} + \hat{R}^{t+1\,*})|\nabla C\,, \tag{3.33}$$

are investigated in order to estimate the predictive distribution of the one year run-off result.

De Felice & Moriconi (2003) use a similar method in order to analyze $\hat{R}^{t+1}$, but in the simulation process the oldest accident year is removed, while a new accident year, corresponding to the year $t+1$, is added to the pseudo-triangle.

4 Numerical study

The purpose of the numerical study is to compare the non-parametric and the parametric bootstrap procedures under different choices of p, F and prediction errors. Since the actuary chooses an age-to-age development factor method that fits the particular development triangle under analysis, it is difficult to find one single algorithm that works for all situations. Therefore we only use the pure chain-ladder method in the comparisons, even though the bootstrap procedures allow the use of any age-to-age development factor method as well. From now on $B = 10\,000$ simulations are used for each prediction. The upper 95 percent limits are studied due to higher robustness than, e.g., the 99.5 percentile, which is perhaps the most frequent choice in practise. The coefficients of variation are also presented.

4.1 The triangle from Taylor & Ashe (1983)

4.1.1 Comparison with Pinheiro *et al.* (2003)

First, the well-known triangle from Taylor & Ashe (1983), called Data 1 in Table 4.1, is analyzed by the non-parametric standardized predictive bootstrap procedure, i.e. the bootstrap procedure described in Pinheiro *et al.* (2003). The estimated reserves and the upper 95 percent limits for $p = 1$ and $p = 2$ are presented in Table 4.2. The second accident year is left out from the tabulation of results when $p = 1$ since a negative increment in the northeast corner of a pseudo-triangle causes a situation with an imaginary prediction error for that year. The remaining accident years are not as sensitive to negative increments as this year.

The results of the standardized predictive bootstrap procedure are in accordance with Pinheiro *et al.* (2003). As we can see, for earlier accident years, the $p = 2$ percentiles are smaller than the $p = 1$ percentiles, whereas the opposite is true for later accident years. This is natural, since most of the future claims C_{ij} of later years have large m_{ij} and hence larger variance for $p = 2$ than for $p = 1$.

	1	2	3	4	5	6	7	8	9	10
1	357 848	766 940	610 542	482 940	527 326	574 398	146 342	139 950	227 229	67 948
2	352 118	884 021	933 894	1 183 289	445 745	320 996	527 804	266 172	425 046	
3	290 507	1 001 799	926 219	1 016 654	750 816	146 923	495 992	280 405		
4	310 608	1 108 250	776 189	1 562 400	272 482	352 053	206 286			
5	443 160	693 190	991 983	769 488	504 851	470 639				
6	396 132	937 085	847 498	805 037	705 960					
7	440 832	847 631	1 131 398	1 063 269						
8	359 480	1 061 648	1 443 370							
9	376 686	986 608								
10	344 014									

Table 4.1: *Data 1 from Taylor & Ashe (1983).*

4.1.2 The choice of $\hat{\phi}$ or $\hat{\phi}^*$

We continue to use the non-parametric standardized predictive bootstrap and Data 1, but we now replace (3.13) with (3.14) in Substage 2.2 in Figure 1. Thus, we do not consider ϕ as constant for the data and therefore we replace $\hat{\phi}$ by $\hat{\phi}^*$. The results are presented in Table 4.3. As we can see, the replacement hardly affects the results.

Note that since $p = 1$ occasionally yields $\hat{m}^*_{ij} < 0$ the corresponding Pearson residuals in the bootstrap world are imaginary while $\hat{\phi}^*$ is real. Since the assumption of an over-dispersed Poisson distribution for the parametric procedure occasionally yields $\hat{m}^*_{ij} = 0$, the corresponding Pearson residuals in the bootstrap world are undefined and as a result, $\hat{\phi}^*$ is undefined as well. Thus, in the sequel we use (3.13) in all simulations.

Year	Estimated reserve	95% $p=1$	Estimated reserve	95% $p=2$
2	94 634		93 316	222 789
3	469 511	903 221	446 504	799 700
4	709 638	1 187 641	611 145	992 585
5	984 889	1 527 903	992 023	1 497 633
6	1 419 459	2 076 496	1 453 085	2 170 480
7	2 177 641	3 034 860	2 186 161	3 284 490
8	3 920 301	5 277 768	3 665 066	5 692 764
9	4 278 972	6 139 286	4 122 398	6 975 123
10	4 625 811	9 760 307	4 516 073	9 286 282
Total	18 680 856	23 681 062	18 085 772	23 033 968

Table 4.2: *The estimated reserves and the 95 percentiles of the non-parametric standardized predictive bootstrap with (3.13) used in Substage 2.2 of Figure 1 for Data 1. Chain-ladder is used for $p = 1$ and maximum likelihood estimation for $p = 2$.*

Year	Estimated reserve	95% $p=1$	Estimated reserve	95% $p=2$
2	94 634		93 316	216 698
3	469 511	889 639	446 504	796 146
4	709 638	1 186 623	611 145	978 315
5	984 889	1 533 399	992 023	1 497 722
6	1 419 459	2 082 287	1 453 085	2 136 423
7	2 177 641	3 041 716	2 186 161	3 290 061
8	3 920 301	5 290 749	3 665 066	5 738 496
9	4 278 972	6 181 331	4 122 398	6 795 927
10	4 625 811	9 328 277	4 516 073	9 476 343
Total	18 680 856	23 603 123	18 085 772	23 042 954

Table 4.3: *The estimated reserves and the 95 percentiles of the non-parametric standardized predictive bootstrap when (3.13) is replaced by (3.14) in Substage* 2.2 *in Figure 1 for Data 1. Chain-ladder is used for $p = 1$ and maximum likelihood estimation for $p = 2$.*

4.1.3 Maximum likelihood estimation vs chain-ladder when p = 2

The next step is to replace the maximum likelihood estimator of the model parameters by the chain-ladder for the non-parametric standardized predictive bootstrap when $p = 2$. (We already use the chain-ladder when $p = 1$, cf. Section 3.5.) Consequently, the estimated reserves in Table 4.4 are the same as when $p = 1$ in Table 4.2 whereas the percentiles in Table 4.4 are consistently higher than in Table 4.2.

This is an example of bootstrapping under a model that does not produce the estimator actually employed, a model which might nevertheless be quite realistic for paid claims. We will use the chain-ladder in all remaining numerical studies, since it is popular and simple.

Year	Estimated reserve	95% $p = 2$
2	94 634	236 850
3	469 511	875 382
4	709 638	1 156 050
5	984 889	1 503 685
6	1 419 459	2 141 470
7	2 177 641	3 308 805
8	3 920 301	6 199 841
9	4 278 972	7 646 140
10	4 625 811	10 698 797
Total	18 680 856	23 991 584

Table 4.4: *The estimated reserve and the 95 percentiles of the non-parametric standardized predictive bootstrap with (3.13) used in Substage* 2.2 *in Figure 1 for Data 1. Chain-ladder is used for* $p = 2$.

4.1.4 Non-parametric bootstrap vs parametric bootstrap

For the purpose of comparing the non-parametric and the parametric bootstrap procedures we continue to use the standardized predictive bootstrap with chain-ladder for Data 1. See Table 4.5 for the upper 95 percent limits and Table 4.6 for the coefficients of variation, i.e. $\sqrt{Var(\tilde{R_i^{**}})}/\hat{R}_i$ and $\sqrt{Var(\tilde{R^{**}})}/\hat{R}$. (In the tables ODP denotes the over-dispersed Poisson distribution.)

The results of the parametric bootstrap coincide well with the results of the non-parametric bootstrap except for the last accident year. It is well-known that the chain-ladder estimate of the outstanding claims for the last accident year is extremely sensitive to outliers in the south corner of the upper triangle. If C^*_{t1} happens to be small in the pseudo-triangle then the corresponding reserve $\hat{R}^*_t$ will be small compared to R^{**}_t, which affects the prediction error in (3.10). The parametric bootstrap generates more stable C^*_{t1}:s than the non-parametric bootstrap, consequently there is a discrepancy in the results of the last accident year for the non-parametric and the parametric bootstrap procedures in Tables 4.5 - 4.6. The conclusion is that the parametric bootstrap may be preferable in some cases.

Year	Estimated Reserve	Non-parametric $p = 1$	Parametric ODP	Non-parametric $p = 2$	Parametric Gamma
2	94 634			236 850	220 643
3	469 511	903 221	920 956	875 382	866 833
4	709 638	1 187 641	1 215 254	1 156 050	1 162 942
5	984 889	1 527 903	1 537 266	1 503 685	1 516 868
6	1 419 459	2 076 496	2 096 805	2 141 470	2 150 441
7	2 177 641	3 034 860	3 057 599	3 308 805	3 309 838
8	3 920 301	5 277 768	5 308 472	6 199 841	6 192 286
9	4 278 972	6 139 286	6 192 655	7 646 140	7 272 012
10	4 625 811	9 760 307	9 163 520	10 698 797	9 222 470
Total	18 680 856	23 681 062	23 685 724	23 991 584	24 095 302

Table 4.5: *The estimated reserve and the 95 percentiles of the non-parametric and the parametric standardized predictive bootstrap with (3.13) used in Substage* 2.2 *in Figures 1- 2 for Data 1. Chain-ladder is used in both cases.*

4.1.5 Standardized prediction errors vs unstandardized prediction errors

From now on the unstandardized predictive bootstrap procedures are used in all tables; the results for Data 1 are presented in Tables 4.7 - 4.8. As we can see, the percentiles for the unstandardized predictive bootstrap in Table 4.7 are lower than for the standardized predictive bootstrap in Table 4.5, and the same goes for the coefficients of variation. Note that there is a large discrepancy in the coefficients of variation, in Table 4.8, for the two choices of distribution for year 2. The reason for the extreme values, when $p = 1$ or an over-dispersed Poisson distribution is assumed, is discussed in Section 4.3.

In Figures 3 (c) - (d) and 4 (c) - (d) the predictive distributions of the total claims reserve are plotted when assuming $p = 1$ for the non-parametric bootstrap procedures and an over-dispersed Poisson distribution for the parametric bootstrap procedures. The predictive distribution obtained by the unstandardized bootstrap in (c) is slightly skewed to the left compared to the one obtained by the standardized bootstrap in (d), which is almost symmetric. This follows since the process component (Figures 3 (a) and 4 (a)) has smaller variability than the estimation component (Figures 3 (b) and 4 (b)), and the latter is slightly skewed to the right. This skewness is to a large extent removed for the standardized prediction errors (3.10), because of the denominator, but not for the unstandardized prediction errors (3.27). Furthermore, from Figures 3 (a) and 4 (a), it does not seem to matter whether we use a non-parametric or parametric approach for the process error, even though England & Verrall (2006) argue that the former choice cannot give larger extremes than the most extreme residual observed. The same holds for $p = 2$ or a gamma distribution (results not shown here).

4.1.6 Estimation of p

Estimation of p by minimizing the (unweighted) sum in (3.31) yields $p = 0.7280$. Thus, $p = 1$ or an over-dispersed Poisson distribution seems to be more reasonable for this development triangle.

Year	Estimated Reserve	Non-parametric $p = 1$	Parametric ODP	Non-parametric $p = 2$	Parametric Gamma
2	94 634			76	62
3	469 511	50	50	43	42
4	709 638	37	38	32	32
5	984 889	31	31	27	28
6	1 419 459	27	27	26	26
7	2 177 641	23	23	27	26
8	3 920 301	20	20	30	29
9	4 278 972	24	25	38	35
10	4 625 811	53	50	64	48
Total	18 680 856	16	16	15	16

Table 4.6: *The estimated reserve and the coefficients of variation of the simulations (in %) of the non-parametric and the parametric standardized predictive bootstrap with (3.13) used in Substage* 2.2 *in Figures 1- 2 for Data 1. Chain-ladder is used in both cases.*

Year	Estimated Reserve	Non-parametric $p=1$	Parametric ODP	Non-parametric $p=2$	Parametric Gamma
2	94 634	274 891	252 438	168 132	167 585
3	469 511	823 274	814 256	750 175	754 646
4	709 638	1 148 468	1 125 650	1 055 135	1 064 059
5	984 889	1 486 951	1 475 088	1 414 799	1 403 919
6	1 419 459	2 040 277	2 019 093	1 995 397	1 982 611
7	2 177 641	2 983 269	2 979 860	3 043 356	3 049 215
8	3 920 301	5 201 768	5 171 112	5 579 973	5 564 848
9	4 278 972	5 916 186	5 910 048	6 363 139	6 257 000
10	4 625 811	7 755 623	7 517 443	7 387 885	7 088 050
Total	18 680 856	23 264 493	23 122 056	23 109 992	23 107 180

Table 4.7: *The estimated reserve and the 95 percentiles of the non-parametric and the parametric unstandardized predictive bootstrap when chain-ladder is used for Data 1.*

Year	Estimated Reserve	Non-parametric $p=1$	Parametric ODP	Non-parametric $p=2$	Parametric Gamma
2	94 634	121	118	52	50
3	469 511	47	46	39	38
4	709 638	38	37	31	31
5	984 889	31	31	28	27
6	1 419 459	27	26	26	26
7	2 177 641	23	23	26	26
8	3 920 301	21	21	28	27
9	4 278 972	25	25	32	32
10	4 625 811	46	44	40	38
Total	18 680 856	17	16	17	16

Table 4.8: *The estimated reserve and the coefficients of variation of the simulations (in %) of the non-parametric and the parametric unstandardized predictive bootstrap when chain-ladder is used for Data 1.*

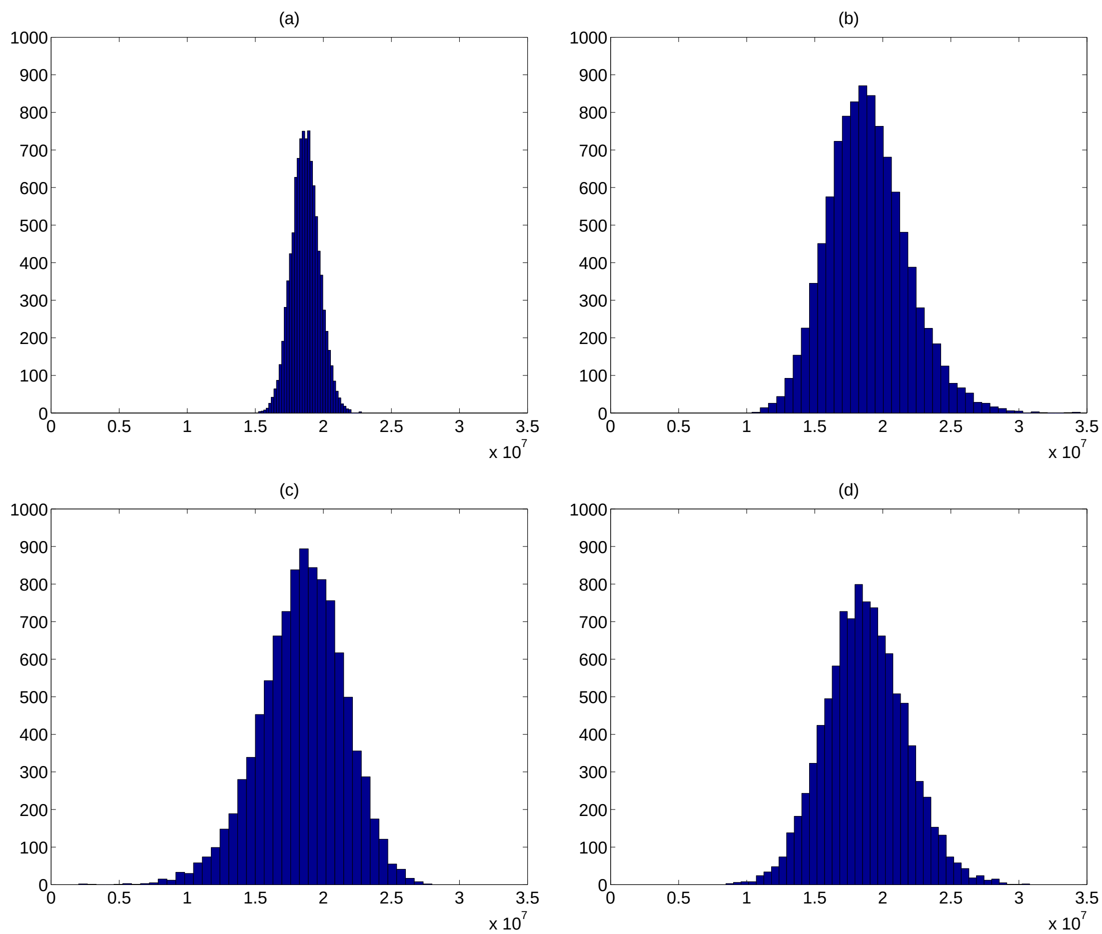

Figure 3: *Density charts of* R^{**} *(a),* $\hat{R}^{*}$ *(b) and* $\tilde{R}^{**}$ *for the unstandardized (c) and standardized (d) non-parametric predictive bootstrap procedures for Data 1 when* $p = 1$.

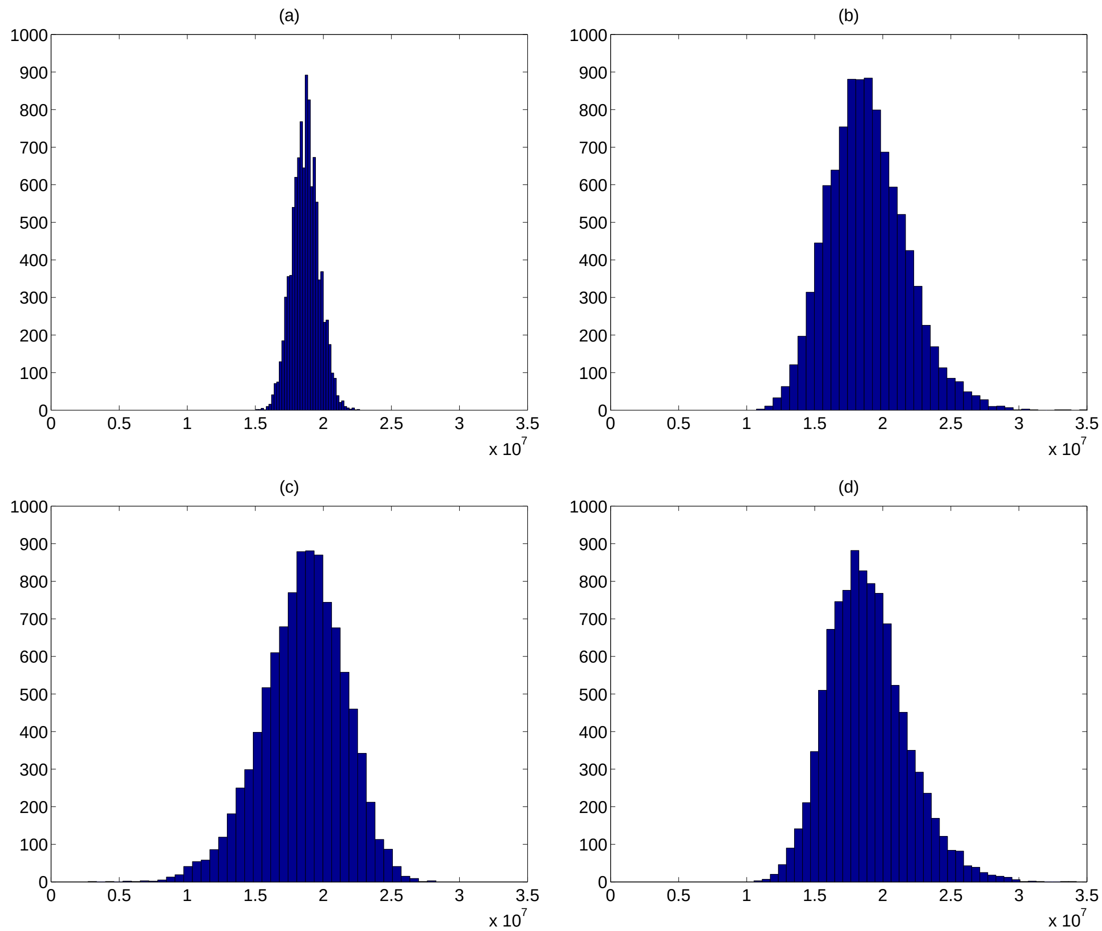

Figure 4: *Density charts of* R^{**} *(a),* $\hat{R}^{*}$ *(b) and* $\tilde{R}^{**}$ *for the unstandardized (c) and standardized (d) parametric predictive bootstrap procedures for Data 1 under the assumption of an over-dispersed Poisson distribution.*

4.2 A small triangle of claim counts

The non-parametric and the parametric unstandardized predictive bootstrap procedures are now compared on a triangle of claim counts appearing in Taylor (2000). Because of the shape of the data and in order to avoid non-positive column sums we use just the later part of the original triangle, see Table 4.9. This is reasonable since the claim counts from previous accident years are almost finalized.

	1	2	3	4	5	6	7
1989	589	210	29	17	12	4	9
1990	564	196	23	12	9	5	
1991	607	203	29	9	7		
1992	674	169	20	12			
1993	619	190	41				
1994	660	161					
1995	660						

Table 4.9: *Data 2 from Taylor (2000).*

Estimation of p yields $\hat{p} = 0.5596$, which indicates that $p = 1$ is a better choice than $p = 2$ for the non-parametric bootstrap and an over-dispersed Poisson distribution is preferable for the parametric bootstrap, as expected for claim counts. Nevertheless, the results for both choices are presented in Tables 4.10 - 4.11 and, as we can see, the results of the parametric bootstrap coincides well with the results of the non-parametric one.

The density charts of R^{**} and $\hat{R}^{*}$ are plotted in Figure 5. The variability of the estimation error is larger than the variability of the process error for Data 2 too, but the difference is not as extreme as for Data 1 in Figures 3 - 4.

4.3 A small triangle of paid claims from a short-tailed line of business

Table 4.12 shows a triangle of paid claims, provided by the Swedish insurance company *AFA Försäkring*, for the short-tailed line of business *Severance Grant*.

The results of the bootstrap procedures are presented in Tables 4.13 - 4.14. The percentiles for year 1996 are very different for the two choices of distribution. This is a consequence

Year	Estimated Reserve	Non-parametric $p=1$	Parametric ODP	Non-parametric $p=2$	Parametric Gamma
1990	8	19	18	14	14
1991	14	26	26	20	20
1992	24	40	39	34	34
1993	36	56	55	51	50
1994	65	90	89	91	90
1995	269	323	321	400	399
Total	417	500	496	555	554

Table 4.10: *The estimated reserve and the 95 percentiles of the non-parametric and the parametric unstandardized predictive bootstrap when chain-ladder is used for Data 2.*

Year	Estimated Reserve	Non-parametric $p=1$	Parametric ODP	Non-parametric $p=2$	Parametric Gamma
1990	8	74	71	43	42
1991	14	57	55	35	33
1992	24	40	39	29	28
1993	36	32	31	26	25
1994	65	23	22	25	25
1995	269	12	12	32	31
Total	417	12	12	22	21

Table 4.11: *The estimated reserve and the coefficients of variation of the simulations (in %) of the non-parametric and the parametric unstandardized predictive bootstrap when chain-ladder is used for Data 2.*

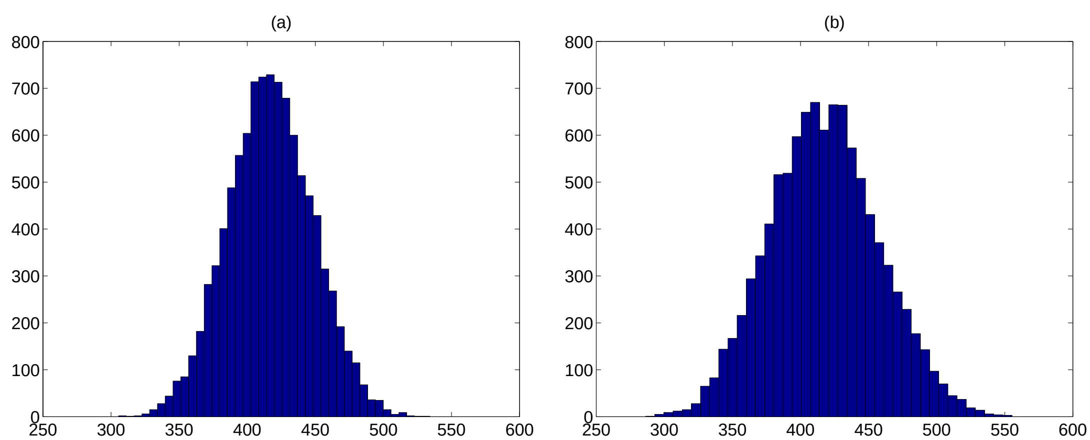

Figure 5: *Density charts of R^{**} (a) and $\hat{R}^{*}$ (b) for the unstandardized non-parametric predictive bootstrap procedures for Data 2 when $p = 1$.*

of occasional non-positive $\hat{m}^*_{ij}$ caused by the resampling process. Tables 4.15 - 4.16 show examples of pseudo-triangles when $p = 1$ for the non-parametric bootstrap procedure and an over-dispersed Poisson distribution is assumed for the parametric bootstrap procedure. By (3.27) and (3.28) these particular simulations yield $\tilde{R}^{**}_{1996} = 2\,614$ and $\tilde{R}^{**}_{1996} = 2\,876$, respectively, which is not reasonable. Thus, even though $\hat{p} = 1.1915$, a comparison of the results for $p = 1$ and $p = 2$ indicates that $p = 2$ might be a better choice for this triangle. Another alternative might be to use a truncated over-dispersed Poisson distribution to exclude zero values, but this is outside the scope of the present paper.

The density charts of R^{**} and $\hat{R}^*$ are plotted in Figure 6 and, as for previous data, the variability of the estimation error is larger than the variability of the process error.

4.4 A large triangle of paid claims from a long-tailed line of business

Finally the two bootstrap procedures are applied to a development triangle for Motor TPL, a typically long-tailed line of business, where there are still unreported claims. Due to an

	1	2	3	4	5	6	7
1995	48 545	56 786	32 659	12 973	4 005	1 696	490
1996	58 294	79 824	38 287	15 957	4 617	1 427	
1997	73 859	73 237	35 281	13 960	3 854		
1998	65 707	67 632	32 832	12 158			
1999	92 901	80 931	36 508				
2000	66 834	47 630					
2001	45 838						

Table 4.12: *Data 3 provided by the Swedish insurance company AFA Försäkring.*

Year	Estimated Reserve	Non-parametric $p = 1$	Parametric ODP	Non-parametric $p = 2$	Parametric Gamma
1996	621	2 369	2 124	873	862
1997	2 408	5 377	5 382	3 128	3 116
1998	6 317	10 763	10 823	8 027	7 960
1999	25 536	34 668	34 673	32 242	32 163
2000	46 196	59 249	58 820	58 910	58 395
2001	82 821	107 213	105 455	110 188	108 440
Total	163 898	195 586	195 097	195 876	193 573

Table 4.13: *The estimated reserve and the 95 percentiles of the non-parametric and the parametric unstandardized predictive bootstrap when chain-ladder is used for Data 3.*

Year	Estimated Reserve	Non-parametric $p=1$	Parametric ODP	Non-parametric $p=2$	Parametric Gamma
1996	621	173	169	26	25
1997	2 408	77	74	19	18
1998	6 317	44	42	17	16
1999	25 536	22	22	17	16
2000	46 196	17	17	17	17
2001	82 821	17	17	21	20
Total	163 898	12	12	12	12

Table 4.14: *The estimated reserve and the coefficients of variation of the simulations (in %) of the non-parametric and the parametric unstandardized predictive bootstrap when chain-ladder is used for Data 3.*

					757						-1 236
				2 007	-746					1 949	-1 367
			604	3 300	326				2 653	1 561	-1 095
		16 480	6 116	2 838	1 108			17 954	3 478	2 046	-1 435
	30 487	11 674	2 527	1 924	-948		26 073	11 364	2 201	1 295	-908
47 952	16 537	14 875	2 315	640	-368	40 247	18 232	7 947	1 539	905	-635

Table 4.15: *An example of pseudo-triangles when $p=1$; the left triangle is ΔC^{**} and the right triangle is $\Delta\hat{m}^{*}$.*

					2 255						0
				1 503	2 255					1 712	0
			3 758	3 006	0				4 919	1 716	0
		18 788	2 255	3 006	2 255			16 892	5 385	1 878	0
	30 061	12 024	3 006	752	752		28 929	12 041	3 838	1 339	0
42 085	27 055	9 770	2 255	2 255	752	55 292	28 841	12 004	3 827	1 335	0

Table 4.16: *An example of pseudo-triangles when an over-dispersed Poisson distribution is assumed; the left triangle is ΔC^{**} and the right triangle is $\Delta\hat{m}^{*}$.*

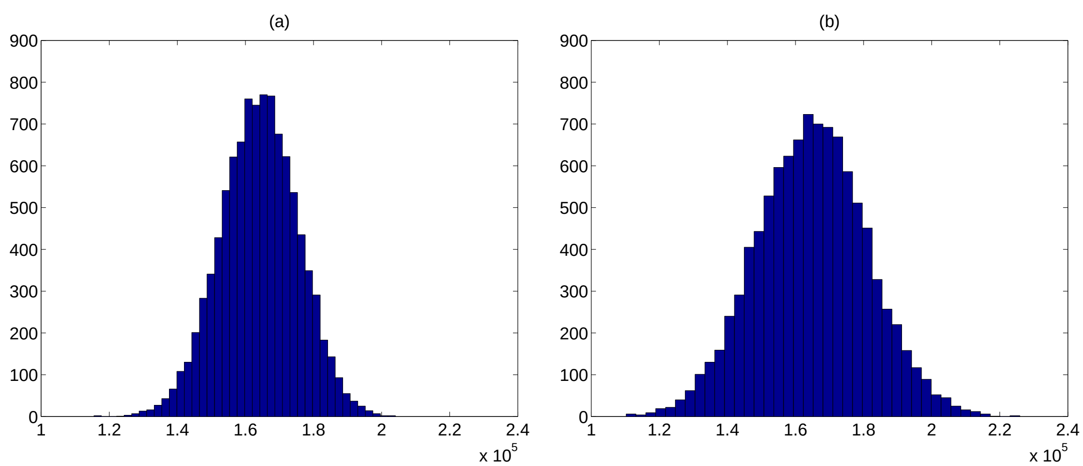

Figure 6: *Density charts of R^{**} (a) and $\hat{R}^{*}$ (b) for the unstandardized non-parametric predictive bootstrap procedures for Data 3 when $p=1$.*

outlier in the oldest accident year (1987) we exclude this year from the original triangle in Naziropoulou (2005), see Table 4.17 for Data 4.

Estimation of p yields $\hat{p} = 0.7773$ and the results of the bootstrap procedures are presented in Tables 4.18 - 4.19. The conclusions are the same as in the earlier examples. The density charts of R^{**} and $\hat{R}^{*}$ are plotted in Figure 7 and for Data 4 the variability of the estimation error is again larger than the variability of the process error.

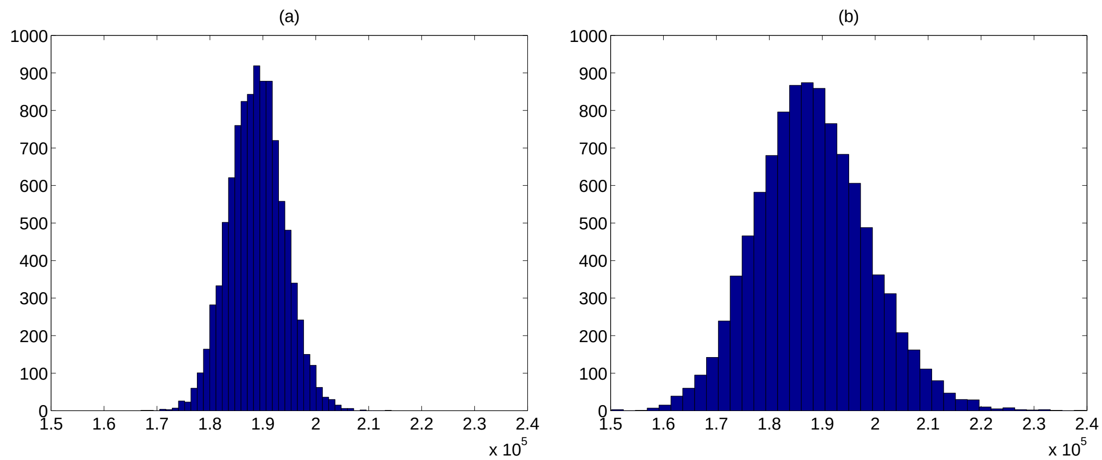

Figure 7: *Density charts of* R^{**} *(a) and* $\hat{R}^{*}$ *(b) for the unstandardized non-parametric predictive bootstrap procedures for Data 4 when* $p = 1$*.*

5 Conclusions

So far most papers concerning bootstrapping for claims reserve uncertainty focus on obtaining the predictive distribution for the chain-ladder method by assuming underlying models, which reproduce the chain-ladder estimates. However, the assumption of an underlying model is generally not made in practise for the purpose of estimating the claims reserve, since the actuary rather uses somewhat complex reserving algorithms, without reference to statistical models. In this paper we suggest using either a non-parametric or a parametric bootstrap methodology with as few model assumptions as possible in order to make the bootstrap pro-

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1988	13 286	9 064	2 260	1 271	1 295	1 007	1 484	150	1 110	598	780	1 262	1 470	350	881	496	170
1989	12 428	9 740	2 387	1 751	1 261	902	1 054	1 086	1 378	1 983	634	1 129	1 346	700	844	1 142	
1990	13 292	8 996	2 615	1 493	1 462	834	1 102	734	1 297	1 160	781	2 021	997	416	417		
1991	13 174	9 023	2 476	1 586	1 361	1 056	758	955	972	1 468	1 029	2 483	599	996			
1992	12 300	8 562	2 444	1 282	1 444	637	1 474	1 368	944	1 328	1 013	1 250	1 009				
1993	12 710	7 747	2 242	2 164	1 478	1 263	1 069	2 160	962	3 870	803	475					
1994	11 935	8 340	2 814	1 870	1 464	1 107	1 221	1 214	1 617	1 310	1 591						
1995	11 959	9 377	2 804	2 488	1 746	1 466	3 168	1 832	1 763	2 051							
1996	11 518	8 953	3 269	1 865	1 522	1 753	1 770	1 717	2 084								
1997	11 621	8 233	3 705	2 091	2 080	1 697	1 800	2 418									
1998	12 416	8 518	2 670	1 951	1 861	1 365	1 874										
1999	12 957	8 917	3 172	2 550	2 141	2 116											
2000	12 964	10 432	3 060	2 382	1 606												
2001	14 959	12 404	4 017	2 663													
2002	16 890	11 899	3 633														
2003	17 167	11 629															
2004	17 658																

Table 4.17: *Data 4 from Naziropoulou (2005).*

Year	Estimated Reserve	Non-parametric $p = 1$	Parametric ODP	Non-parametric $p = 2$	Parametric Gamma
1989	184	551	608	311	314
1990	1 000	1 785	1 810	1 528	1 538
1991	1 765	2 783	2 773	2 523	2 530
1992	2 250	3 401	3 386	3 084	3 080
1993	3 586	5 010	5 002	4 798	4 819
1994	4 947	6 611	6 563	6 576	6 580
1995	6 811	8 805	8 761	9 014	8 952
1996	8 245	10 607	10 523	10 902	10 886
1997	9 865	12 444	12 460	13 060	12 988
1998	10 797	13 455	13 493	14 131	14 245
1999	13 529	16 623	16 531	17 759	17 764
2000	14 933	18 240	18 179	19 716	19 661
2001	19 798	23 719	23 700	26 008	26 280
2002	22 920	27 141	27 176	30 525	30 771
2003	26 757	31 598	31 539	36 447	36 359
2004	40 854	48 032	48 283	61 070	60 315
Total	188 242	207 770	207 461	218 375	217 784

Table 4.18: *The estimated reserve and the 95 percentiles of the non-parametric and the parametric unstandardized predictive bootstrap when chain-ladder is used for Data 4.*

Year	Estimated Reserve	Non-parametric $p = 1$	Parametric ODP	Non-parametric $p = 2$	Parametric Gamma
1989	184	131	129	48	47
1990	1 000	49	49	35	34
1991	1 765	36	36	28	28
1992	2 250	31	31	24	24
1993	3 586	25	24	22	22
1994	4 947	21	21	22	21
1995	6 811	18	18	21	20
1996	8 245	16	16	20	20
1997	9 865	15	15	20	20
1998	10 797	14	15	20	20
1999	13 529	13	13	20	20
2000	14 933	13	13	21	20
2001	19 798	12	12	21	21
2002	22 920	11	11	22	22
2003	26 757	11	11	24	24
2004	40 854	11	11	34	33
Total	188 242	6	6	11	10

Table 4.19: *The estimated reserve and the coefficients of variation of the simulations (in %) of the non-parametric and the parametric unstandardized predictive bootstrap when chain-ladder is used for Data 4.*

cedures more consistent with the actuary's way of working. It is assumed that the bootstrap procedures only depend on the mean and variance of the claims, while the actuary's choice of reserving algorithm implicitly specifies the mean structure. Consequently, the suggested bootstrap procedures can be used to obtain the predictive distribution of any age-to-age development factor method. The non-parametric and the parametric bootstrap procedures are compared to techniques described in Pinheiro *et al.* (2003), as well as in England (2002), and finally they are applied to four development triangles of different types.

We have seen that the results of the parametric standardized predictive bootstrap are consistent with the results of its non-parametric counterpart in Pinheiro *et al.* (2003). Furthermore, the unstandardized predictive bootstrap procedures have revealed that the variability of the estimation error, when chain-ladder is used, is larger than the variability of the process error for all four investigated development triangles and for the two largest of them the difference is considerable. Finally, our simulation results are almost the same for the non-parametric and the parametric approach.

Since resampling of standardized quantities often increases accuracy compared to using unstandardized quantities, the standardized predictive bootstrap is in theory preferable to the unstandardized one. We have seen that the standardized case yields higher estimated risk, seemingly due to the fact that it makes the distribution more symmetric than the unstandardized case, where the predictive distribution is skewed to the left. A disadvantage of the standardized predictive bootstrap is that the denominators of (3.10) may sometimes be non-positive, yielding undefined or imaginary prediction errors. In principle, this could be corrected by the double bootstrap, which provides a better estimation of the variance since it includes the estimation error as well as the process error. Therefore, it would be interesting, in a future paper, to analyze the behaviour of the double bootstrap method both for simulated and real data sets.

Finally, a somewhat surprising result of the numerical studies is that the estimation error is consistently larger than the process error. This could be the case for further study.

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Appendix 1

The basic assumption of the resampling process of the non-parametric bootstrap is independent and identically distributed residuals. We will now motivate that the model in (3.1) gives approximately identically distributed residuals r_{ij} for the majority of residuals (3.2) or (3.3) in the upper triangle (not close to any of the corners) when $p = 2$ (gamma distribution), but not for $p = 1$ (over-dispersed Poisson distribution). By large triangles we mean that $t \to \infty$ and hence also $n \to \infty$. For each *fixed* ij, $\hat{m}_{ij}$ is a consistent estimate of m_{ij} as n grows, and $q/n \to 0$. Hence, for large n, the residuals can be written as

$$r_{ij} = \frac{C_{ij} - m_{ij}}{\sqrt{m_{ij}^p}}.$$

Since the moment generating function of a $\Gamma(\alpha, \beta)$ distribution is $M(t) = (1 - \beta\, t)^{-\alpha}$ and $p = 2$ is equivalent to $C_{ij} \in \Gamma(\frac{1}{\phi},\, \phi\, m_{ij})$, the residuals r_{ij} are identically distributed according to

$$M_{r_{ij}}(t) = e^{-t}\, M_{C_{ij}}(\frac{t}{m_{ij}}) = e^{-t}(1 - \phi\, t)^{-\frac{1}{\phi}}.$$

The moment generating function of a $Po(\lambda)$ distribution is $M(t) = e^{\lambda(e^t - 1)}$, but since $p = 1$ implies an over-dispersed Poisson distribution we need a help variable X_{ij} in order to find the distribution of the residuals. The underlying model is fulfilled if $C_{ij} = \phi\, X_{ij}$, $X_{ij} \in Po(\frac{m_{ij}}{\phi})$ and the residuals are distributed according to

$$M_{r_{ij}}(t) = e^{-t\sqrt{m_{ij}}}\, M_{C_{ij}}\,(\frac{t}{\sqrt{m_{ij}}}) = e^{-t\sqrt{m_{ij}}}\, M_{X_{ij}}\,(\frac{\phi\, t}{\sqrt{m_{ij}}}) = e^{-t\sqrt{m_{ij}}}\, e^{\frac{m_{ij}}{\phi}(e^{\frac{\phi\, t}{\sqrt{m_{ij}}}} - 1)}.$$

The distributions of the residuals r_{ij} depend on m_{ij} and consequently the residuals cannot be identically distributed.

Appendix 2

In order to find the variability of the claims reserve obtained by the chain-ladder method England & Verrall (1999) assume the model structure in (3.1) and on the basis of the standard error of prediction of a single future value C_{ij} in ΔC, i.e.

$$SEP(C_{ij}) = \sqrt{\widehat{E}(C_{ij} - \hat{C}_{ij})^2} \cong \sqrt{\widehat{Var}(C_{ij}) + \widehat{Var}(\hat{C}_{ij})}, \tag{5.1}$$

an expression for the standard error of prediction of the total claims reserve is derived as

$$\begin{aligned} SEP(R) &= \sqrt{\widehat{Var}(R - \hat{R})} \approx \sqrt{\widehat{Var}(R) + \widehat{Var}(\hat{R})} \qquad (5.2) \\ &= \sqrt{\widehat{Var}(\sum_{\Delta} C_{ij}) + \widehat{Var}(\sum_{\Delta} \hat{C}_{ij})} \\ &\approx \sqrt{\sum_{\Delta} \hat{\phi}\, \hat{m}_{ij}^{p} + \sum_{\Delta} \hat{m}_{ij}\, \widehat{Var}(\hat{\eta}_{ij}) + 2 \sum_{\Delta,\, i_1 j_1 \neq i_2 j_2} \hat{m}_{i_1 j_1}\, \hat{m}_{i_2 j_2}\, \widehat{Cov}(\hat{\eta}_{i_1 j_1}\, \hat{\eta}_{i_2 j_2})}\,, \end{aligned}$$

where $\hat{\eta}_{ij}$ is the estimate of η_{ij} appearing in (3.1). The first term provides for the variance of the process error and can easily be estimated analytically, while the two last terms, providing for the variance of the estimation error, can be obtained by bootstrapping. When $p = 1$, England & Verrall (1999) replace equation (5.2) by the bootstrap standard error of prediction

$$SEP_{bs}(R) = \sqrt{\hat{\phi}\, \hat{R} + (SE_{bs}(\hat{R}^*))^2}, \tag{5.3}$$

where $SE_{bs}(\hat{R}^*)$ is the standard error of the B simulated values of $\hat{R}^*$ obtained by the non-parametric standardized bootstrap procedure in Substage 2.1 in Figure 1. However, England & Verrall (1999) substitute the maximum likelihood estimates of the model parameters in Figure 1 by the chain-ladder method.

In order to obtain a complete predictive distribution England (2002) extended the method in England & Verrall (1999) by replacing the analytic calculation of the process error by another simulation conditional on the bootstrap simulation. The process error is included to the B triangles $\Delta\hat{m}^*$ by sampling a random observation from a process distribution with mean $\hat{m}_{ij}^*$ and variance $\phi\, \hat{m}_{ij}^*$ to obtain the future claims $\Delta m^\dagger$. The predictive distribution of the outstanding claims is then obtained by plotting the B values of $\tilde{R}^\dagger = \sum_{\Delta} m_{ij}^\dagger$ and finally the standard deviation of the simulations gives the standard error of prediction of the outstanding claims.

England (2002) presents no justification of this procedure, but sampling from over-dispersed Poisson distributions with mean $\hat{m}^*_{ij}$ and variance $\phi\,\hat{m}^*_{ij}$ will indeed provide us with a predictive distribution of R consistent with (5.3). Since

$$E(R^\dagger|\,\Delta\hat{m}^*) = \sum_{\Delta} E(m^\dagger_{ij}|\,\Delta\hat{m}^*) = \sum_{\Delta}\hat{m}^*_{ij} = \hat{R}^*$$

and

$$\mathrm{Var}(R^\dagger|\,\Delta\hat{m}^*) = \sum_{\Delta} Var(m^\dagger_{ij}|\,\Delta\hat{m}^*) = \sum_{\Delta}\hat{\phi}\,\hat{m}^*_{ij} = \hat{\phi}\,\hat{R}^*$$

the variance of the simulated predictive distribution is

$$\begin{aligned}
\mathrm{Var}(R^\dagger) &= E[\mathrm{Var}(R^\dagger|\,\Delta\hat{m}^*)] + \mathrm{Var}[E(R^\dagger|\,\Delta\hat{m}^*)] \\
&= E(\hat{\phi}\,\hat{R}^*) + \mathrm{Var}(\hat{R}^*) = \hat{\phi}\,E(\hat{R}^*) + \mathrm{Var}(\hat{R}^*) \approx \hat{\phi}\,\hat{R} + \mathrm{Var}(\hat{R}^*),
\end{aligned}$$

where, in the last step, we used $E(\hat{R}^*) \approx \hat{R}$ and (3.12).

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Bootstrapping the separation method in claims reserving

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Abstract

The separation method was introduced by Verbeek (1972) in order to forecast numbers of excess claims and it was developed further by Taylor (1977) to be applicable to the average claim cost. The separation method differs from the chain-ladder in that when the chain-ladder only assumes claim proportionality between the development years, the separation method also separates the claim delay distribution from influences affecting the calendar years, e.g. inflation. Since the inflation contributes to the uncertainty in the estimate of the claims reserve it is important to consider its impact in the context of risk management, too.
In this paper we present a method for assessing the prediction error distribution of the separation method. To this end we introduce a parametric framework within the separation model which enables joint resampling of claim counts and claim amounts. As a result, the variability of Taylor's predicted reserves can be assessed by extending the parametric bootstrap techniques of Björkwall *et al.* (2008). The performance of the bootstrapped separation method and chain-ladder is compared for a real data set.

Keywords

Bootstrap; Chain-ladder; Development triangle; Inflation; Separation method; Stochastic claims reserving.

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1 Introduction

One issue for the reserving actuary is how to deal with inflation, which contributes to the uncertainty in the estimate of the claims reserve. Even though some proper reserving techniques are suggested in the literature, little has been said about how to approach this issue when it comes to finding the variability of the actuary's best estimate either analytically or by bootstrapping.

Due to external forces the average cost per claim will change from one calendar year to another. Typically this *claims inflation* is specific to each line of business and depends on the economic inflation, which usually can be tied to some relevant index, as well as on factors like legislation and attitudes to policy holder compensation. The latter result in so called *superimposed claims inflation*.

The chain-ladder method makes implicit allowance for claims inflation since it projects the inflation present in the past data into the future, see e.g. Taylor (2000). Consequently, it only works properly when the inflation rate is constant. When the economic inflation rate is non-constant, the past paid losses can be converted to current value by some relevant index before they are projected into the future by the chain-ladder, but still there is no allowance for superimposed claims inflation.

Another approach of dealing with claims inflation is to incorporate it into the model underlying the reserving method. In this way the past inflation rate can be estimated and the future inflation rate can be predicted within the model. Verbeek (1972) introduced such a method in the reinsurance context and Taylor (1977) developed it further to be applicable to the average claim cost in a general context. The reserving technique is called *the separation method*. However, the separation method has, unlike its famous relative, remained quite anonymous in the literature on stochastic claims reserving. For instance, the mean squared error of prediction for the chain-ladder was analytically calculated by Mack (1993) and revisited by Buchwalder *et al.* (2006) and Mack *et al.* (2006) and a full predictive distribution was obtained for the chain-ladder by bootstrapping in England & Verrall (1999), England (2002) and Pinheiro *et al.* (2003). Recently the variability of other reserving methods has been investigated as well, e.g. the Bornhuetter-Ferguson method by analytical approximation in Mack (2008) and the Munich chain-ladder, see Quarg & Mack (2004), by bootstrapping of two correlated quantities in Liu & Verrall (2008).

The object of this paper is to analyze the variability of the separation method. Since bootstrapping easily gives a full predictive distribution and can also be used in risk management with Dynamic Financial Analysis (DFA) we develop a bootstrap procedure for the separation method. For this purpose we use an extended version of the parametric bootstrap technique described in Björkwall *et al.* (2008). To this end, we introduce a parametric framework within the separation model, in which claim counts are Poisson distributed and claim amounts are gamma distributed *conditionally* on the ultimate claim counts. This enables joint resampling of claim counts and claim amounts.

Section 2 contains the definitions and the theory behind the separation method. In Section 3 the suggested bootstrap methodology is discussed and it is studied numerically for the well-known data set from Taylor & Ashe (1983) in Section 4. Finally, Section 5 contains a discussion.

2 The separation method

Assume that we have a triangle of incremental observations of paid claims $\{C_{ij}; i, j \in \nabla\}$, where ∇ denotes the upper, observational triangle $\nabla = \{i = 0, \ldots, t; j = 0, \ldots, t-i\}$. The suffixes i and j refer to the origin year and the development year, respectively, see Table 2.1. In addition, the suffix $k = i + j$ is used for the calendar years, i.e. the diagonals of ∇. The purpose is to predict the sum of the delayed claim amounts in the lower, unobserved future triangle $\{C_{ij}; i, j \in \Delta\}$, where $\Delta = \{i = 1, \ldots, t; j = t - i + 1, \ldots, t\}$, see Table 2.2. We write $R = \sum_{\Delta} C_{ij}$ for this sum, which is the outstanding claims for which the insurance company must hold a reserve. Furthermore, assume that we have a triangle of the incremental observations of the numbers of claims $\{N_{ij}; i, j \in \nabla\}$ corresponding to the same portfolio as in Table 2.1, i.e. the observations in Table 2.3. The ultimate number of claims relating to period of origin year i is then

$$N_i = \sum_{j \in \nabla_i} N_{ij} + \sum_{j \in \Delta_i} N_{ij}\,, \tag{2.1}$$

where ∇_i and Δ_i denotes the rows corresponding to origin year i in the upper triangle ∇ and the lower triangle Δ, respectively.

The separation method is based on the assumption that N_i is considered as known. Since the number of claims is often finalized quite early even for long-tailed business, N_i may very well be estimated separately, e.g. by the chain-ladder if a triangle of claim counts

	Development year					
Accident year	0	1	2	$\cdots$	$t-1$	t
0	C_{00}	C_{01}	C_{02}	$\cdots$	$C_{0,t-1}$	$C_{0,t}$
1	C_{10}	C_{11}	C_{12}	$\cdots$	$C_{1,t-1}$	
2	C_{20}	C_{21}	C_{22}	$\cdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$t-1$	$C_{t-1,0}$	$C_{t-1,1}$				
t	$C_{t,0}$					

Table 2.1: *The triangle ∇ of observed incremental payments.*

	Development year					
Accident year	0	1	2	$\cdots$	$t-1$	t
0						
1						$C_{1,t}$
2					$C_{2,t-1}$	$C_{2,t}$
$\vdots$					$\vdots$	$\vdots$
$t-1$			$C_{t-1,2}$	$\cdots$	$C_{t-1,t-1}$	$C_{t-1,t}$
t		$C_{t,1}$	$C_{t,2}$	$\cdots$	$C_{t,t-1}$	$C_{t,t}$

Table 2.2: *The triangle Δ of unobserved future claim costs.*

is provided, and then be treated as known. Henceforth estimates $\hat{n}_{ij}$ of the expectations $n_{ij} = E(N_{ij})$ is obtained by the chain-ladder for all cells in both ∇ and Δ. The estimated ultimate number of claims relating to origin year i is then

$$\hat{N}_i = \sum_{j \in \nabla_i} N_{ij} + \sum_{j \in \Delta_i} \hat{n}_{ij} \,. \tag{2.2}$$

The chain-ladder method operates on cumulative claim counts

$$A_{ij} = \sum_{\ell=0}^{j} N_{i\ell} \tag{2.3}$$

rather than incremental claim counts N_{ij}. Let $\nu_{ij} = E(A_{ij})$. Development factors g_j are estimated for $j = 0, 1, \ldots, t-1$ by

$$\hat{g}_j = \frac{\sum_{i=0}^{t-j} A_{i,j+1}}{\sum_{i=0}^{t-j} A_{ij}} \tag{2.4}$$

yielding

$$\hat{\nu}_{ij} = A_{i,t-i}\, \hat{g}_{t-i}\, \hat{g}_{t-i+1} \ldots \hat{g}_{j-1} \tag{2.5}$$

and

$$\hat{n}_{i,j} = \hat{\nu}_{i,j} - \hat{\nu}_{i,j-1} \tag{2.6}$$

for Δ, while estimates of $\hat{\nu}_{ij}$ for ∇ are obtained by the process of backwards recursion described in England & Verrall (1999).

	Development year					
Accident year	0	1	2	$\cdots$	$t-1$	t
0	N_{00}	N_{01}	N_{02}	$\cdots$	$N_{0,t-1}$	$N_{0,t}$
1	N_{10}	N_{11}	N_{12}	$\cdots$	$N_{1,t-1}$	
2	N_{20}	N_{21}	N_{22}	$\cdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$t-1$	$N_{t-1,0}$	$N_{t-1,1}$				
t	$N_{t,0}$					

Table 2.3: *The triangle ∇ of observed incremental numbers of reported claims.*

While the chain-ladder only assumes claim proportionality between the development years, the separation method in Taylor (1977) separates the claim delay distribution from influences effecting the calendar years, e.g. inflation. In the separation model we first assume that the proportion of the average claim amount paid in development year j is constant over i; denote this proportion by r_j. If the claims are fully paid by year t we have the constraint

$$\sum_{j=0}^{t} r_j = 1. \tag{2.7}$$

We then make a further assumption that the claim amount is proportional to some index, say λ_k, that relates to the calendar year k during which the claims are paid. The expected claim cost for development year j and calendar year k is then proportional to $r_j \, \lambda_k$.

	Development year					
Accident year	0	1	2	$\cdots$	$t-1$	t
0	$r_0\,\lambda_0$	$r_1\,\lambda_1$	$r_2\,\lambda_2$	$\cdots$	$r_{t-1}\,\lambda_{t-1}$	$r_t\,\lambda_t$
1	$r_0\,\lambda_1$	$r_1\,\lambda_2$	$r_2\,\lambda_3$	$\cdots$	$r_{t-1}\,\lambda_t$	
2	$r_0\,\lambda_2$	$r_1\,\lambda_3$	$r_2\,\lambda_4$	$\cdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$t-1$	$r_0\,\lambda_{t-1}$	$r_1\,\lambda_t$				
t	$r_0\,\lambda_t$					

Table 2.4: *The triangle* ∇ *of expected paid claims.*

The separation model can be given the following formulation, which is at a bit more detailed level than the one given in Taylor (1977). Let C_{ijl} denote the amount paid during calendar year k for the l:th individual claim incurred in origin year i and assume that C_{ijl} are conditionally independent for all i, j and l given N_i. According to the discussion above we also assume that

$$E(C_{ijl}|N_i) = r_j \, \lambda_k. \tag{2.8}$$

Since the total amount paid during calendar year k for claims incurred in origin year i is

$$C_{ij} = \sum_{l=1}^{N_i} C_{ijl} \tag{2.9}$$

we obtain

$$E\left(\frac{C_{ij}}{N_i}\middle| N_i\right) = \frac{1}{N_i}\sum_{l=1}^{N_i} E\left(C_{ijl}|N_i\right) = \frac{1}{N_i}\sum_{l=1}^{N_i} r_j\,\lambda_k = r_j\,\lambda_k \tag{2.10}$$

for the conditional expectation of the average claim costs given the ultimate number of claims and this relation is the basic assumption of the separation method. The expectations in equation (2.10) now build up the triangle in Table 2.4.

If N_i is estimated separately by (2.2), it follows from (2.8) and (2.9) that

$$\begin{aligned} E\left(\frac{C_{ij}}{\hat{N}_i}\middle|\nabla N\right) &= \frac{E(E(C_{ij}|N_i,\nabla N)|\nabla N)}{\hat{N}_i} \\ &= r_j\,\lambda_k\,\frac{(\sum_{\nabla_i} N_{ij} + \sum_{\Delta_i} n_{ij})}{(\sum_{\nabla_i} N_{ij} + \sum_{\Delta_i} \hat{n}_{ij})} \\ &\approx r_j\,\lambda_k \end{aligned} \tag{2.11}$$

where in the last equality we used $\hat{n}_{ij} \approx n_{ij}$.

Estimates $\hat{r}_j$ and $\hat{\lambda}_k$ of the parameters in the triangle in Table 2.4 can now be obtained using the corresponding triangle ∇s of observed values

$$s_{ij} = \frac{C_{ij}}{\hat{N}_i}\,, \tag{2.12}$$

and the method of moments equations

$$s_{k0} + s_{k-1,1} + \ldots + s_{0k} = (\hat{r}_0 + \ldots + \hat{r}_k)\,\hat{\lambda}_k\,, \qquad k = 0,\ldots,t \tag{2.13}$$

for the diagonals of ∇ and

$$s_{0j} + s_{1j} + \ldots + s_{t-j,j} = (\hat{\lambda}_j + \ldots + \hat{\lambda}_t)\,\hat{r}_j\,, \qquad j = 0,\ldots,t \tag{2.14}$$

for the columns of ∇.

Taylor (1977) shows that the equations (2.13) - (2.14), with the side constraint (2.7), have a unique solution that can be obtained recursively, starting with $k = t$ in (2.13) to solve for $\hat{\lambda}_t$, then $j = t$ in (2.14) to solve for $\hat{r}_t$, $k = t-1$ in (2.13) to solve for $\hat{\lambda}_{t-1}$ and so on. This yields

$$\hat{\lambda}_k = \frac{\sum_{i=0}^{k} s_{i,k-i}}{1 - \sum_{j=k+1}^{t} \hat{r}_j}\,, \qquad k = 0,\ldots,t \tag{2.15}$$

$$\hat{r}_j = \frac{\sum_{i=0}^{t-j} s_{ij}}{\sum_{k=j}^{t} \hat{\lambda}_k}, \qquad j = 0, \ldots, t, \tag{2.16}$$

where $\sum_{j=k+1}^{t} \hat{r}_j$ is interpreted as zero when $k = t$.

Estimates $\hat{m}_{ij}$ of the expectations $m_{ij} = E(C_{ij})$ for cells in ∇ are now given by

$$\hat{m}_{ij} = \hat{N}_i \, \hat{r}_j \, \hat{\lambda}_k , \tag{2.17}$$

but in order to obtain the estimates of Δ it remains to predict λ_k for $t+1 \leq k \leq 2\,t$, which requires some inflation assumption.

If there is a trend in the inflation indexes $\hat{\lambda}_k$ for $k \leq t$ then smoothing and extrapolation could be used in order to forecast the future inflation. An alternative is to use an average of the past indexes. In any case, with an inflation assumption of, say, $K\%$, the forecasted λ_{k+1} can be obtained as $\hat{\lambda}_{k+1} = (1 + \frac{K}{100})\, \hat{\lambda}_k$ for $t \leq k \leq 2\,t - 1$. The cell expectations of ΔC_{ij} are then estimated by equation (2.17) and estimators of the outstanding claims are obtained by summing per accident year $\hat{R}_i = \sum_{j \in \Delta_i} \hat{m}_{ij}$. The estimator of the total reserve is $\hat{R} = \sum_{\Delta} \hat{m}_{ij}$.

The separation model described by Taylor (1977) is more general than the one discussed in this paper, since the original model do not presume that N_i is the number of claims; it could be some other exposure relating to origin year i as well. However, in this paper we stick to the number of claims.

3 A conditional parametric bootstrap approach

For the purpose of obtaining the predictive distribution of the claims reserve R estimated by the separation method the bootstrap technique described in Pinheiro *et al.* (2003) and, in particular, the parametric approach in Björkwall *et al.* (2008) is used. For the sampling process we model the paid claims C_{ij} conditionally on N_i in accordance with (2.11). We provide models for the assumption of stochastic N_i as well as for the case when N_i is considered as known. The former assumption demands that we develop the technique described in Björkwall *et al.* (2008) in order to handle ∇N as well as ∇C.

3.1 Stochastic Poisson distributed claim counts

Verbeek (1972) adopted a Poisson distribution for the claim counting variable, while the method described in Taylor (1977) is distribution-free. However, the assumption of independent and Poisson distributed claim counts

$$N_{ij} \in Po(n_{ij}) \tag{3.1}$$

yields a very reasonable model for the sampling process.

In addition we assume that the conditionally independent claims $C_{ijl}|N_i$ in (2.8) are gamma distributed. We use the notation

$$C_{ijl}|N_i \in \Gamma\left(\frac{1}{\phi}, r_j\,\lambda_k\,\phi\right)\,, \tag{3.2}$$

where $1/\phi$ is the so called index parameter and $r_j\,\lambda_k\,\phi$ is the scale, so that the expected value is $r_j\,\lambda_k$. Moreover, $\phi > 0$.

Recalling (2.9) and the independence of the C_{ijl} we find that

$$C_{ij}|N_i \in \Gamma\left(\frac{N_i}{\phi}, r_j\,\lambda_k\,\phi\right)\,, \tag{3.3}$$

which is consistent with (2.10) since

$$E(C_{ij}|N_i) = N_i\,r_j\,\lambda_k\,. \tag{3.4}$$

The variance of the amounts in (3.3) is

$$Var(C_{ij}|N_i) = \phi\,N_i\,(r_j\,\lambda_k)^2 = \phi\,\frac{E^2(C_{ij}|N_i)}{N_i}\,, \tag{3.5}$$

which corresponds to a weighted generalized linear model under the assumption of a logarithmic link function and a gamma distribution. We use a Pearson type estimate of ϕ, cf.

McCullagh & Nelder (1989),

$$\hat{\phi} = \frac{1}{|\nabla| - q} \sum_{\nabla} \hat{N}_i \, \frac{(C_{ij} - \hat{E}(C_{ij}|N_i))^2}{\hat{E}^2(C_{ij}|N_i)} = \frac{1}{|\nabla| - q} \sum_{\nabla} \hat{N}_i \, \frac{(C_{ij} - \hat{N}_i \, \hat{r}_j \, \hat{\lambda}_k)^2}{(\hat{N}_i \, \hat{r}_j \, \hat{\lambda}_k)^2}\,, \tag{3.6}$$

where $|\nabla| = (t+1)\,(t+2)/2$ is the number of observations in ∇C, the estimators $\hat{N}_i$, $\hat{\lambda}_j$ and $\hat{r}_j$ are obtained from (2.2), (2.15) and (2.16) and $q = 2\,t + 1$ is the number of parameters that have to be estimated by the separation method, i.e. r_j for $j = 0, 1, \ldots, t-1$ and λ_k for $k = 0, 1, \ldots, t$.

Notice that (3.3) could be interpreted as follows; given N_i claims we allocate claim amounts independently over the development years j according to the proportions $r_0, \ldots, r_t$ before the inflation is considered. According to (3.2) we not only allocate claim amounts but individual claims as well. This interpretation is consistent with the assumptions discussed in Section 2.

We adopt the bootstrap technique described in Pinheiro *et al.* (2003) and, in particular, the parametric approach in Björkwall *et al.* (2008). The relation between the true outstanding claims R and its estimator $\hat{R}$ in the real world is, by the plug-in-principle, substituted in the bootstrap world by their bootstrap counterparts. Hence, the process error is included in R^{**}, i.e. the true outstanding claims in the bootstrap world, while the estimation error is included in $\hat{R}^{*}$, i.e. the estimated outstanding claims in the bootstrap world. Henceforth we use the index $*$ for random variables or plug-in estimators in the bootstrap world which correspond to observations or estimators in the real world, while the index $**$ is used for random variables in the bootstrap world when the counterparts in the real world are unobserved.

The parametric bootstrap approach in Björkwall *et al.* (2008) can now be implemented for the separation method using (3.1) and (3.3) in the following way. We draw N^{*}_{ij} and N^{**}_{ij} from

$$N^{*}_{ij} \in Po(\hat{n}_{ij}) \quad \text{and} \quad N^{**}_{ij} \in Po(\hat{n}_{ij}) \tag{3.7}$$

B times for all $i, j \in \nabla$ and $i, j \in \Delta$, respectively. We thereby get the B pseudo-triangles

∇N^* and ΔN^{**}. The ultimate number of claims per origin year in the bootstrap world is then given by

$$N_i^{**} = \sum_{j \in \nabla_i} N_{ij}^* + \sum_{j \in \Delta_i} N_{ij}^{**} \tag{3.8}$$

according to (2.1).

Once N_i^{**} is calculated, C_{ij}^* is sampled B times from

$$C_{ij}^* | N_i^{**} \in \Gamma \left(\frac{N_i^{**}}{\hat{\phi}}, \hat{r}_j \, \hat{\lambda}_k \, \hat{\phi} \right) , \tag{3.9}$$

for all $i, j \in \nabla$ yielding the B pseudo-triangles ∇C^*. Here $\hat{\lambda}_k$ and $\hat{r}_j$ are obtained from (2.15) and (2.16).

The heuristic estimation process described in Section 2 is then repeated B times for each pair of pseudo-triangles. The claim counts are first forecasted by $\Delta \hat{n}^*$, obtained by the chain-ladder from ∇N^*, in order to estimate the ultimate number of claims per origin year

$$\hat{N}_i^* = \sum_{j \in \nabla_i} N_{ij}^* + \sum_{j \in \Delta_i} \hat{n}_{ij}^* \tag{3.10}$$

according to (2.2). The future payments are then forecasted by estimating $\Delta \hat{m}^*$ according to (2.12) - (2.17). Finally, estimators for the outstanding claims in the bootstrap world are obtained by $\hat{R}_i^* = \sum_{j \in \Delta_i} \hat{m}_{ij}^*$ and $\hat{R}^* = \sum_{\Delta} \hat{m}_{ij}^*$.

In order to generate a random outcome of the true outstanding claims in the bootstrap world, i.e. $R_i^{**} = \sum_{j \in \Delta_i} C_{ij}^{**}$ and $R^{**} = \sum_{\Delta} C_{ij}^{**}$, we sample once again from (3.9) for all $i, j \in \Delta$ to get B triangles ΔC^{**}.

The final step is to calculate the B prediction errors

$$\mathrm{pe}_i^{**} = \frac{R_i^{**} - \hat{R}_i^*}{\sqrt{\widehat{Var}(R_i^{**})}} \qquad \text{and} \qquad \mathrm{pe}^{**} = \frac{R^{**} - \hat{R}^*}{\sqrt{\widehat{Var}(R^{**})}} \, . \tag{3.11}$$

The predictive distributions of the outstanding claims R_i and R are then obtained by plotting

$$\tilde{R}_i^{**} = \hat{R}_i + \mathrm{pe}_i^{**} \sqrt{\widehat{Var}(R_i)} \qquad \text{and} \qquad \tilde{R}^{**} = \hat{R} + \mathrm{pe}^{**} \sqrt{\widehat{Var}(R)} \tag{3.12}$$

for each B.

The conditional independence of C_{ij} for all i and j given N_i (3.3) implies that

$$\begin{aligned} Var(R_i) &= E\left(Var\left(R_i|N_i\right)\right) + Var\left(E\left(R_i|N_i\right)\right) \\ &= E\left(\sum_{j\in\Delta_i} \phi N_i \left(r_j\lambda_k\right)^2\right) + Var\left(\sum_{j\in\Delta_i} N_i r_j \lambda_k\right) \\ &= \phi E\left(N_i\right) \sum_{j\in\Delta_i} \left(r_j\lambda_k\right)^2 + Var\left(N_i\right)\left(\sum_{j\in\Delta_i} r_j\lambda_k\right)^2 \\ &= \left(\phi \sum_{j\in\Delta_i} \left(r_j\lambda_k\right)^2 + \left(\sum_{j\in\Delta_i} r_j\lambda_k\right)^2\right)\left(\sum_{j\in\nabla_i\cup\Delta_i} n_{ij}\right) \end{aligned} \tag{3.13}$$

since

$$E(N_i) = Var(N_i) = \sum_{j\in\nabla_i\cup\Delta_i} n_{ij}\,. \tag{3.14}$$

By plugging in the estimates we find

$$\widehat{Var}\left(R_i\right) = \left(\hat{\phi} \sum_{j\in\Delta_i} \left(\hat{r}_j\hat{\lambda}_k\right)^2 + \left(\sum_{j\in\Delta_i} \hat{r}_j\hat{\lambda}_k\right)^2\right)\left(\sum_{j\in\nabla_i\cup\Delta_i} \hat{n}_{ij}\right) \tag{3.15}$$

and

$$\widehat{Var}\left(R\right) = \sum_i \left(\hat{\phi} \sum_{j\in\Delta_i} \left(\hat{r}_j\hat{\lambda}_k\right)^2 + \left(\sum_{j\in\Delta_i} \hat{r}_j\hat{\lambda}_k\right)^2\right)\left(\sum_{j\in\nabla_i\cup\Delta_i} \hat{n}_{ij}\right)\,. \tag{3.16}$$

Analogously, the variances appearing in (3.11) are

$$\widehat{Var}\left(R_i^{**}\right) = \left(\hat{\phi}^* \sum_{j\in\Delta_i} \left(\hat{r}_j^*\hat{\lambda}_k^*\right)^2 + \left(\sum_{j\in\Delta_i} \hat{r}_j^*\hat{\lambda}_k^*\right)^2\right)\left(\sum_{j\in\nabla_i\cup\Delta_i} \hat{n}_{ij}^*\right) \tag{3.17}$$

and

$$\widehat{Var}\left(R^{**}\right) = \sum_i \left(\hat{\phi}^* \sum_{j\in\Delta_i} \left(\hat{r}_j^*\hat{\lambda}_k^*\right)^2 + \left(\sum_{j\in\Delta_i} \hat{r}_j^*\hat{\lambda}_k^*\right)^2\right)\left(\sum_{j\in\nabla_i\cup\Delta_i} \hat{n}_{ij}^*\right) \tag{3.18}$$

where

$$\hat{\phi}^* = \frac{1}{|\nabla| - q} \sum_{\nabla} \hat{N}_i^* \frac{(C_{ij}^* - \hat{N}_i^*\,\hat{r}_j^*\,\hat{\lambda}_k^*)^2}{(\hat{N}_i^*\,\hat{r}_j^*\,\hat{\lambda}_k^*)^2}\,. \tag{3.19}$$

in accordance with (3.6).

It is remarked in Björkwall *et al.* (2008) that for many bootstrap procedures, resampling of standardized quantities often increases accuracy compared to using unstandardized quantities. Nevertheless, the unstandardized prediction errors

$$\mathrm{pe}_i^{**} = R_i^{**} - \hat{R}_i^{*} \qquad \text{and} \qquad \mathrm{pe}^{**} = R^{**} - \hat{R}^{*} \tag{3.20}$$

are useful, e.g. for the purpose of studying the estimation and the process errors, but also since they are always defined.

The predictive distributions of the outstanding claims R_i and R are then obtained by plotting the B quantities

$$\tilde{R}_i^{**} = \hat{R}_i + \mathrm{pe}_i^{**} \qquad \text{and} \qquad \tilde{R}^{**} = \hat{R} + \mathrm{pe}^{**}\,. \tag{3.21}$$

The parametric predictive bootstrap procedure is described in Figure 1 and according to Björkwall *et al.* (2008) we will refer to it as standardized or unstandardized depending on which prediction errors that are used.

3.2 Known claim counts

In Section 2 it was remarked that the separation model is based on the assumption that N_i is considered as known at the moment when the reserving is being done. This can often be a reasonable assumption since the numbers of claims are usually finalized quite early even for long-tailed business. In Section 3.1 N_i was a random variable; in order to get a view of how much uncertainty N_i contributes to the predictive distribution of the claims reserve we now consider the special case when N_i is treated as deterministic, in contrast to (3.1). Consequently, $\hat{N}_i \equiv N_i$ in all equations above.

Assumption (3.3) can still be used and $\hat{\phi}$ is estimated as in (3.6), but the sampling process changes. We do not have to generate pseudo-triangles of claim counts in the bootstrap

Stage 1 - Real world

Substage 1.1 - The triangle of claim counts ∇N

· Forecast the future expected values $\Delta\hat{n}$ and calculate the fitted values $\nabla\hat{n}$ by chain-ladder.
· Calculate the estimated ultimate claim count per origin year $\hat{N}_i$.

Substage 1.2 - The triangle of paid claims ∇C

· Use $\hat{N}_i$ from Substage 1.1 for the purpose of forecasting the future expected values $\Delta\hat{m}$ and calculating the fitted values $\nabla\hat{m}$ by the separation method.
· Calculate $\hat{\phi}$ for the sampling process.
· Calculate the outstanding claims $\hat{R}_i = \sum_{j\in\Delta_i}\hat{m}_{ij}$ and $\hat{R} = \sum_{\Delta}\hat{m}_{ij}$.

Stage 2 - Bootstrap world

Substage 2.1 - The estimated outstanding claims

Substage 2.1.1 - The pseudo-triangle of claim counts ∇N^*

· Sample from (3.7) for $i, j \in \nabla$ to obtain the pseudo-reality in ∇N^*.
· Forecast the future expected values $\Delta\hat{n}^*$ by chain-ladder.
· Calculate the estimated ultimate claim count per origin year $\hat{N}_i^*$.

Substage 2.1.2 - The pseudo-triangle of paid claims ∇C^*

· Sample from (3.7) for $i, j \in \Delta$ to obtain the pseudo-reality in ΔN^{**}.
· Calculate the ultimate claim count per origin year N_i^{**} using ∇N^* from Substage 2.1.1 and ΔN^{**}.
· Sample from (3.9) for $i, j \in \nabla$ to obtain the pseudo-reality in ∇C^* conditionally on ΔN_i^{**}.
· Use $\hat{N}_i^*$ from Substage 2.1.1. for the purpose of forecasting the future expected values $\Delta\hat{m}^*$ by the separation method.
· Calculate the estimated outstanding claims $\hat{R}_i^* = \sum_{j\in\Delta_i}\hat{m}_{ij}^*$ and $\hat{R}^* = \sum_{\Delta}\hat{m}_{ij}^*$.

Substage 2.2 - The true outstanding claims

· Sample from (3.9) for $i, j \in \Delta$ to obtain the pseudo-reality in ΔC^{**} conditionally on ΔN_i^{**}.
· Calculate the true outstanding claims $R_i^{**} = \sum_{j\in\Delta_i} C_{ij}^{**}$ and $R^{**} = \sum_{\Delta} C_{ij}^{**}$.
· Store either the standardized prediction errors in (3.11) or the unstandardized ones in (3.20).
· Return to the beginning of the bootstrap loop in Stage 2 and repeat B times.

Stage 3 - Analysis of the simulations

· Obtain the predictive distribution of R_i and R, the true outstanding claims in the real world, by plotting the B values in either (3.12) or (3.21).

Figure 1: *The procedure of the parametric predictive bootstrap for the separation method.*

world, i.e. ∇N^* and ΔN^{**}, since N_i is considered as known. Thus, we just draw C^*_{ij} from

$$C^*_{ij} \in \Gamma(\frac{N_i}{\hat{\phi}}, \hat{r}_j\, \hat{\lambda}_k\, \hat{\phi}) \tag{3.22}$$

B times for all $i, j \in \nabla$ yielding ∇C^*. The estimation process of the separation method is then repeated for each ∇C^* using N_i as the exposure in the bootstrap world as well. Finally, we sample once again B times from (3.22) for all $i, j \in \Delta$ to get ΔC^{**}.

The prediction errors and the predictive distributions are as earlier obtained by (3.11) and (3.12), respectively, but since $Var(N_i) = 0$, we obtain the estimators

$$\widehat{Var}(R_i) = \hat{\phi} N_i \sum_{\Delta_i} (\hat{r}_j \hat{\lambda}_k)^2 \tag{3.23}$$

and

$$\widehat{Var}(R) = \sum_i \hat{\phi} N_i \sum_{\Delta_i} (\hat{r}_j \hat{\lambda}_k)^2 \tag{3.24}$$

instead of (3.15) and (3.16).

Analogously, the estimators appearing in (3.11) are

$$\widehat{Var}(R^{**}_i) = \hat{\phi}^* N_i \sum_{\Delta_i} (\hat{r}^*_j \hat{\lambda}^*_k)^2 \tag{3.25}$$

and

$$\widehat{Var}(R^{**}) = \sum_i \hat{\phi}^* N_i \sum_{\Delta_i} (\hat{r}^*_j \hat{\lambda}^*_k)^2\,, \tag{3.26}$$

where $\hat{\phi}^*$ is estimated by (3.19).

The unstandardized prediction errors in (3.20) can of course be used as well. The predictive distributions are then obtained by (3.21).

This simplified approach is summarized in Figure 2.

4 Numerical study

The purpose of the numerical study is to illustrate the parametric bootstrap procedure for the separation method and to compare it to the approach for the chain-ladder described in

Stage 1 - Real world

Substage 1.1 - The triangle of claim counts ∇N

· Forecast the future expected values $\Delta \hat{n}$ by chain-ladder.
· Calculate the estimated ultimate claim count per origin year $\hat{N}_i$.

Substage 1.2 - The triangle of paid claims ∇C

· Use $\hat{N}_i$ from Substage 1.1 for the purpose of forecasting the future expected values $\Delta \hat{m}$ and calculating the fitted values $\nabla \hat{m}$ by the separation method.
· Calculate $\hat{\phi}$ for the sampling process.
· Calculate the outstanding claims $\hat{R}_i = \sum_{j \in \Delta_i} \hat{m}_{ij}$ and $\hat{R} = \sum_{\Delta} \hat{m}_{ij}$.

Stage 2 - Bootstrap world

Substage 2.1 - The estimated outstanding claims

· Sample from (3.22) for $i, j \in \nabla$ to obtain the pseudo-reality in ∇C^*.
· Use $\hat{N}_i$ for the purpose of forecasting the future expected values $\Delta \hat{m}^*$ by the separation method.
· Calculate the estimated outstanding claims $\hat{R}_i^* = \sum_{j \in \Delta_i} \hat{m}_{ij}^*$ and $\hat{R}^* = \sum_{\Delta} \hat{m}_{ij}^*$.

Substage 2.2 - The true outstanding claims

· Sample from (3.22) for $i, j \in \Delta$ to obtain the pseudo-reality in ΔC^{**}.
· Calculate the true outstanding claims $R_i^{**} = \sum_{j \in \Delta_i} C_{ij}^{**}$ and $R^{**} = \sum_{\Delta} C_{ij}^{**}$.
· Store either the standardized prediction errors in (3.11) or the unstandardized ones in (3.20).
· Return to the beginning of the bootstrap loop in Stage 2 and repeat B times.

Stage 3 - Analysis of the simulations

· Obtain the predictive distribution of R_i and R, the true outstanding claims in the real world, by plotting the B values in either (3.12) or (3.21).

Figure 2: *The procedure of the simplified parametric predictive bootstrap for the separation method.*

Björkwall *et al.* (2008). From now on $B = 10\,000$ simulations are used for each prediction. The upper 95 percent limits are studied and the coefficients of variation, i.e. $\sqrt{Var(\tilde{R}_i^{**})}/\hat{R}_i$ and $\sqrt{Var(\tilde{R}^{**})}/\hat{R}$, are presented as well.

We use the well-known data from Taylor & Ashe (1983), who also provide observations of number of claims. The triangles of paid claims ∇C and claim counts ∇N are presented in Table 4.1 and Table 4.2, respectively.

4.1 The estimate of the claims reserve and the payment pattern

The assumption of the future inflation rate has great impact on the claims reserve estimated by the separation method. The future inflation rate can of course be modeled by more refined approaches, but this is beyond the scope of this paper and we just consider a constant or the mean rate observed so far. In Table 4.3 the estimators are shown under three different assumptions. The inflation rate $11,01\%$ corresponds to the mean inflation rate observed so far, while 5% and 15% are chosen just for comparison. The estimated claims reserves obtained by the chain ladder are presented as well.

Table 4.4 shows the expected cumulative payment proportions

$$\hat{c}_j = \frac{\sum_{l=0}^{j}\sum_{i=0}^{t}\hat{m}_{il}}{\sum_{l=0}^{t}\sum_{i=0}^{t}\hat{m}_{il}} . \tag{4.1}$$

Obviously, a higher future inflation rate tends to delay the payments.

	0	1	2	3	4	5	6	7	8	9
0	357 848	766 940	610 542	482 940	527 326	574 398	146 342	139 950	227 229	67 948
1	352 118	884 021	933 894	1 183 289	445 745	320 996	527 804	266 172	425 046	
2	290 507	1 001 799	926 219	1 016 654	750 816	146 923	495 992	280 405		
3	310 608	1 108 250	776 189	1 562 400	272 482	352 053	206 286			
4	443 160	693 190	991 983	769 488	504 851	470 639				
5	396 132	937 085	847 498	805 037	705 960					
6	440 832	847 631	1 131 398	1 063 269						
7	359 480	1 061 648	1 443 370							
8	376 686	986 608								
9	344 014									

Table 4.1: *Observations of paid claims ∇C from Taylor & Ashe (1983).*

	0	1	2	3	4	5	6	7	8	9
0	40	124	157	93	141	22	14	10	3	2
1	37	186	130	239	61	26	23	6	6	
2	35	158	243	153	48	26	14	5		
3	41	155	218	100	67	17	6			
4	30	187	166	120	55	13				
5	33	121	204	87	37					
6	32	115	146	103						
7	43	111	83							
8	17	92								
9	22									

Table 4.2: *Observations of claim counts ∇N from Taylor & Ashe (1983).*

4.2 Predictive bootstrap results for the chain-ladder

In order to compare the separation method to the chain-ladder we summarize the results of the parametric predictive bootstrap procedures described in Björkwall *et al.* (2008), where data is bootstrapped according to the plug-in-principle under the assumption of a gamma distribution; see the reference for details. Tables 4.5 - 4.6 show the results for the standardized as well as the unstandardized approach.

Year i	Future inflation rate 5.00%	Future inflation rate 11.01%	Future inflation rate 15.00%	Chain-ladder
1	84 339	89 163	92 371	94 634
2	473 893	506 151	527 909	469 511
3	720 846	794 132	845 099	709 638
4	1 144 208	1 288 308	1 391 323	984 889
5	1 497 489	1 722 883	1 888 356	1 419 459
6	2 095 131	2 448 039	2 713 372	2 177 641
7	2 793 640	3 269 931	3 634 088	3 920 301
8	3 636 785	4 314 184	4 841 171	4 278 972
9	4 990 729	6 043 441	6 879 216	4 625 811
Total	17 437 060	20 476 232	22 812 905	18 680 856

Table 4.3: *The estimated claims reserves under the chain-ladder, compared to the separation method with different inflation assumptions. The mean inflation rate observed so far is 11, 01%.*

Development year j	Future inflation rate 5.00%	Future inflation rate 11.01%	Future inflation rate 15.00%	Chain-ladder
0	7.1	6.7	6.4	6.9
1	25.2	23.9	23.0	24.2
2	44.5	42.5	41.0	42.2
3	63.3	60.9	59.1	61.5
4	73.7	71.3	69.5	72.2
5	81.2	79.0	77.4	79.7
6	87.7	86.0	84.7	86.6
7	92.3	91.0	90.1	91.3
8	98.6	98.3	98.1	98.3
9	100.0	100.0	100.0	100.0

Table 4.4: *The expected cumulative payment proportion (in %) under the chain-ladder, compared to the separation method with different inflation assumptions. The mean inflation rate observed so far is* $11,01\%$.

Year i	Standardized Gamma	Unstandardized Gamma
1	219 178	168 756
2	861 781	756 634
3	1 169 041	1 062 783
4	1 519 540	1 409 034
5	2 127 947	1 975 222
6	3 358 037	3 038 732
7	6 253 164	5 562 133
8	7 386 412	6 284 020
9	9 247 043	7 148 120
Total	23 991 467	23 123 593

Table 4.5: *The 95 percentiles of the parametric predictive bootstrap procedures described in Björkwall* et al. *(2008) for the chain-ladder. We work under the assumption of a gamma distribution and the procedure is either standardized or unstandardized.*

4.3 The standardized predictive bootstrap for the separation method

The results for the procedure described in Section 3.1, when the standardized prediction errors are used, are presented in Table 4.7 for the three different assumptions of the future inflation rate. Two of these are mean inflation rates observed so far, either treated as a constant (11.01%) or as stochastic in the bootstrap world. According to the plug-in-principle the inflation rate should be treated as stochastic, i.e. recomputed from $\{\hat{\lambda}_k^*\}$ for each resample, but the former alternative is shown as well for comparison. Table 4.8 contains the coefficients of variation. Tables 4.7 - 4.8 also include the results obtained by the chain-ladder for comparison.

As we can see the results are strongly affected by the inflation assumption and the coefficients of variation are naturally higher when the mean inflation is treated as stochastic, in particular for the grand total. As expected the coefficients of variation of the latest origin year are lower for the separation method than for the chain-ladder, since the extreme sensitivity to outliers for the chain-ladder in the south corner of the upper triangle is removed for the separation method. Less expected is that the separation method has

Year i	Standardized Gamma	Unstandardized Gamma
1	65	50
2	41	38
3	32	31
4	28	27
5	26	25
6	27	25
7	29	27
8	35	32
9	47	38
Total	15	16

Table 4.6: *The coefficients of variation of the simulations (in %) of the parametric predictive bootstrap procedures described in Björkwall* et al. *(2008) for the chain-ladder. We work under the assumption of a gamma distribution and the procedure is either standardized or unstandardized.*

Year i	Inflation 5.00%	Inflation 11.01%	Inflation Mean	Inflation 15.00%	Chain Ladder Gamma
1	197 907	201 184	190 418	208 028	219 178
2	839 849	882 300	858 679	926 020	861 781
3	1 137 848	1 253 445	1 204 288	1 336 139	1 169 041
4	1 704 374	1 908 980	1 859 360	2 066 985	1 519 540
5	2 178 017	2 513 476	2 446 109	2 751 393	2 127 947
6	3 033 630	3 526 976	3 516 529	3 901 976	3 358 037
7	4 223 019	4 893 910	4 807 925	5 359 921	6 253 164
8	5 564 419	6 540 182	6 489 287	7 239 800	7 386 412
9	8 261 189	9 852 469	9 540 033	11 081 546	9 247 043
Total	23 412 570	27 442 696	27 659 095	30 692 578	23 991 467

Table 4.7: *The 95 percentiles of the standardized predictive bootstrap procedure under the chain-ladder, compared to the separation method with different inflation assumptions. Two of these are mean inflation rates observed so far, either treated as a constant (11.01 %) or as stochastic (Mean).*

Year i	Inflation 5.00%	Inflation 11.01%	Inflation Mean	Inflation 15.00%	Chain Ladder Gamma
1	63	61	57	60	65
2	38	38	37	38	41
3	30	29	30	30	32
4	26	25	27	25	28
5	24	24	27	24	26
6	24	23	28	23	27
7	26	26	31	25	29
8	28	27	32	26	35
9	33	32	37	31	47
Total	18	18	25	17	15

Table 4.8: *The coefficients of variation of the simulations (in %) of the standardized predictive bootstrap procedure under the chain-ladder, compared to the separation method with different inflation assumptions. Two of these are mean inflation rates observed so far, either treated as a constant (11.01 %) or as stochastic (Mean).*

lower coefficients of variation for years 1-3.

4.4 The unstandardized predictive bootstrap for the separation method

In order to study the estimation and the process error we also investigate the procedure described in Section 3.1 when the unstandardized prediction errors are used. The results are shown in Tables 4.9 - 4.10.

Year i	Inflation 5.00%	Inflation 11.01%	Inflation Mean	Inflation 15.00%	Chain Ladder Gamma
1	152 189	158 866	158 108	163 797	168 756
2	765 412	803 966	792 018	840 344	756 634
3	1 071 483	1 180 997	1 150 577	1 262 227	1 062 783
4	1 632 010	1 825 048	1 780 637	1 967 326	1 409 034
5	2 082 197	2 413 236	2 340 763	2 644 546	1 975 222
6	2 916 401	3 389 043	3 327 822	3 754 270	3 038 732
7	4 024 333	4 666 419	4 547 122	5 125 141	5 562 133
8	5 270 015	6 180 526	6 027 989	6 874 970	6 284 020
9	7 528 152	9 024 898	8 787 987	10 208 677	7 148 120
Total	22 281 683	26 091 962	26 145 893	29 117 165	23 123 593

Table 4.9: *The 95 percentiles of the unstandardized predictive bootstrap procedure under the chain-ladder, compared to the separation method with different inflation assumptions. Two of these are mean inflation rates observed so far, either treated as a constant (11.01 %) or as stochastic (Mean).*

As remarked in Björkwall *et al.* (2008) the percentiles of the unstandardized predictive bootstrap tend to be lower than for the standardized one. This was explained by the left skewness of the predictive distribution of the unstandardized bootstrap compared to the distribution obtained by the standardized bootstrap. According to Figure 3 this seems to hold for the separation method too. Figure 3 (c) - (d) show the predictive distributions of the total claims reserve under the assumption of a stochastic future inflation rate corresponding to the mean inflation rate observed so far. The predictive distribution obtained by the unstandardized bootstrap in (c) is skewed to the left compared to the one obtained by the standardized bootstrap in (d), which is slightly skewed to the right. This follows since the process component in Figure 3 (a) has smaller variability than the estimation component in Figure 3 (b), and the latter is skewed to the right. The left skewness is to a large extent removed for the standardized prediction errors (3.11), because of the denominator, but not for the unstandardized prediction errors (3.20).

Recomputing the future inflation rate from $\{\hat{\lambda}_k^*\}$ for each resample in the bootstrap world yields some rates which are unreasonably high. These rates affect the estimation component, which become more skewed to the right than for a constant future inflation rate. Consequently, the predictive distribution of the outstanding claims is more skewed to the left for the stochastic future inflation rate than for the constant. This explains why most of the percentiles in Tables 4.7 and 4.9 are lower for stochastic inflation.

4.5 Known claim counts

In Tables 4.11 - 4.12 we present the results of the simplified approach in Section 3.2 where we treat N_i as known. As expected the variability has decreased compared to the results in Tables 4.7 - 4.8, but the difference is notably small. This is consistent with the separation

Year i	Inflation 5.00%	Inflation 11.01%	Inflation Mean	Inflation 15.00%	Chain Ladder Gamma
1	49	48	50	48	50
2	36	36	38	35	38
3	29	29	33	29	31
4	25	25	32	25	27
5	24	24	34	24	25
6	24	23	36	23	25
7	26	26	39	25	27
8	27	26	43	26	32
9	33	32	52	31	38
Total	18	18	35	17	16

Table 4.10: *The coefficients of variation of the simulations (in %) of the unstandardized predictive bootstrap procedure under the chain-ladder, compared to the separation method with different inflation assumptions. Two of these are mean inflation rates observed so far, either treated as a constant (11.01 %) or as stochastic (Mean).*

method assumption that the numbers of claims usually are finalized early enough to be considered as known. This is interesting, since Table 4.2 reveals that the data here is actually an example when claim numbers are not finalized very fast. As expected, the difference is largest for the last origin year, i.e. where we predict the ultimate number of claims based on one single observation.

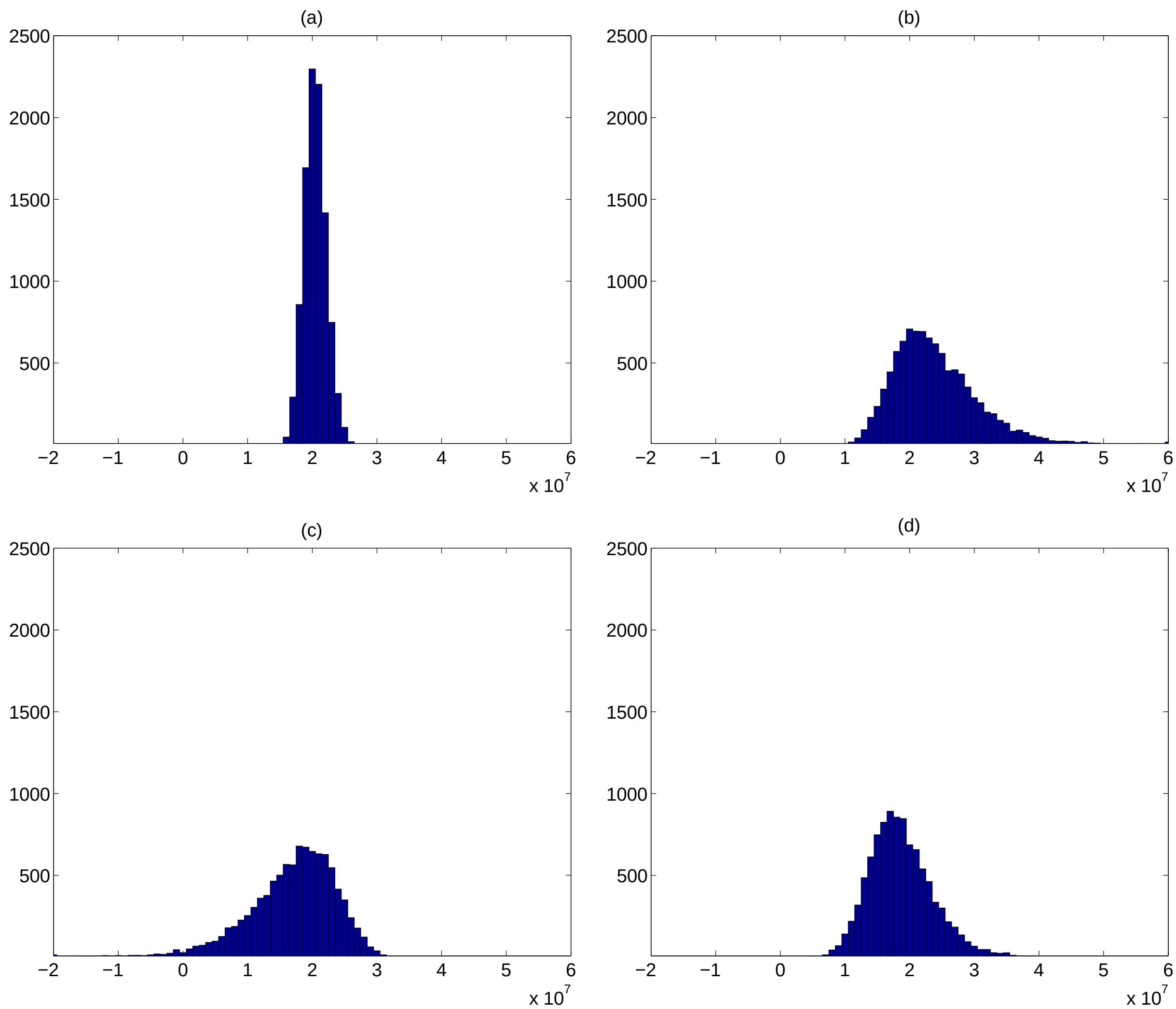

Figure 3: *Density charts of R^{**} (a), $\hat{R}^{*}$ (b) and $\tilde{R}^{**}$ for the unstandardized (c) and standardized (d) predictive bootstrap procedure under the assumption of a stochastic future inflation rate corresponding to the mean inflation rate observed so far.*

Year i	Inflation 5.00%	Inflation 11.01%	Inflation Mean	Inflation 15.00%	Chain Ladder Gamma
1	192 618	203 666	194 659	211 058	219 178
2	838 502	897 770	875 334	934 584	861 781
3	1 142 097	1 243 302	1 195 689	1 332 776	1 169 041
4	1 697 879	1 918 643	1 858 748	2 074 808	1 519 540
5	2 200 470	2 525 290	2 451 200	2 726 575	2 127 947
6	3 032 494	3 577 632	3 481 789	3 914 571	3 358 037
7	4 250 351	4 871 638	4 819 584	5 386 039	6 253 164
8	5 532 888	6 507 485	6 421 424	7 227 340	7 386 412
9	7 461 845	9 023 231	8 859 782	10 108 297	9 247 043
Total	23 398 840	27 417 470	27 783 761	30 304 007	23 991 467

Table 4.11: *The 95 percentiles of the simplified standardized predictive bootstrap procedure under the chain-ladder, compared to the separation method when N_i is considered as known. We work under different inflation assumptions. Two of these are mean inflation rates observed so far, either treated as a constant (11.01 %) or as stochastic (Mean).*

Year i	Inflation 5.00%	Inflation 11.01%	Inflation Mean	Inflation 15.00%	Chain Ladder Gamma
1	63	62	57	61	65
2	39	39	39	39	41
3	30	29	30	29	32
4	26	26	27	25	28
5	24	24	27	24	26
6	24	24	27	23	27
7	27	26	31	25	29
8	27	26	32	25	35
9	26	25	32	24	47
Total	17	17	24	17	15

Table 4.12: *The coefficients of variation of the simulations (in %) of the simplified standardized predictive bootstrap procedure under the chain-ladder, compared to the separation method when N_i is considered as known. We work under three different inflation assumptions. Two of these are mean inflation rates observed so far, either treated as a constant (11.01 %) or as stochastic (Mean).*

5 Conclusions

The separation method is a useful reserving technique for the purpose of modeling claims inflation, which contributes to the uncertainty of the claims reserve and therefore should be considered in risk management. This paper provides a parametric bootstrap procedure, which can be used to assess the uncertainty of the separation method. It is of course difficult to forecast the future inflation and in this paper simple assumptions have been used. We believe that the future inflation for real applications should be modeled by more refined approaches.

In one example we saw that whether we consider N_i as stochastic or known in the bootstrap procedure the results are still at the same level. Of course, the situation might be different in another example.

Furthermore, when we compare the percentiles obtained for the separation method with the ones for the chain-ladder in Tables 4.7 and 4.9 we can see that the result is more affected by the assumption of the future claims inflation rate than the choice between the chain ladder and the separation method. Since the separation method, under the assumption of a future inflation rate corresponding to the mean rate observed so far, indicates a higher risk than predicted by the chain-ladder the question of which method is preferable in a given situation immediately arises. Therefore, in a future paper, it would be interesting to compare the two methods in more situations than the one in Section 4 and in particular for long-tailed data.

The bootstrap approach for the separation method can also be used in a DFA context to simulate the reserve risk. However, as remarked by England & Verrall (2006), a DFA model usually includes an economic scenario generator (ESG), which simulates the future inflation, and it is important that the dependence between reserve risk and the inflation from the ESG is incorporated in the DFA model. Therefore, England & Verrall (2006) suggest that the data is adjusted to remove effects of the economic inflation before applying a reserving method, which use calendar year components to model superimposed claims inflation, is applied to forecast the future payments. Once the future payments has been simulated they are re-adjusted according to the inflation obtained from the ESG.

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