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Patrik Andersson*

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Abstract

We propose a model describing an economy where companies may default due to contagion. By using standard approximation results for stochastic process we are able to describe the features of the model. It turns out that the model reproduces the oscillations in the default rates that has been observed empirically. That is, we have an intrinsic oscillation in the economic system without applying any external macroeconomic force. These oscillations can be understood as cleansing of the unhealthy companies during a recession and the recession ending when sufficiently many of the unhealthy companies have left the economy. This is important both from a risk management perspective as well as from a policy perspective since it shows that contagious defaults may help to explain the oscillations of business cycles. We also investigate the first-passage times of the default process, using this as a proxy for the time to a recession.

KEY WORDS: Risk management; Credit risk; Contagion; Business cycle; Markov process; Ornstein-Uhlenbeck.

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1 Introduction

Banks and other financial institutions are facing many different types of risks in their everyday business. In particular, for banks with significant lending activities it is crucial to properly measure and manage *credit risk*, i.e. the risk that the bank experiences losses from an unexpectedly large number of its counterparties failing to fulfill their payment obligations.

Mathematical models for credit risk have been extensively studied in the literature, see e.g. [3] and the references therein. In the standard credit risk models, it is assumed that the economy under consideration is *static*, whereas in reality new companies are continuously entering the system, while other companies leave the system (through mergers, acquisitions, or defaults). In such a dynamic environment, not only the number of companies but also their credit quality may change considerably as time passes. In particular, during an upswing of the economy, start-up companies may find it easy to receive loans due to a more liberal granting process, while under adverse economic circumstances funding may be harder to raise.

Recently there has been research where the dependencies between companies have been investigated. In works like [4], [6], [7], [8] it is assumed that the probability of default for a particular company depends on both the current state of the economy and on the state of companies in its neighborhood. This is often referred to as default contagion.

The purpose of this work is to propose a modelling framework for credit defaults in a dynamically changing economy. In our model, companies enter the system at a given rate and is assigned a rating. Companies may then exit the system through default, adversely affecting the other companies, or through mergers and acquisitions, not affecting other companies adversely. The proposed framework is inspired by the work in modelling contagious endemic diseases, see e.g. [1], where a homogeneously mixing population is often assumed.

Since empirical observations show that there is a cyclicality in the default rates the credit risk literature often introduces default probabilities affected by *external* factors, either given in terms of macroeconomic variables or defined implicitly using latent variables. As it turns out, realistic choices of parameters in our model produce long-term behavior similar to the cycles in default rates observed empirically, without introducing any external macroeconomic force. This is also important since empirical research show co-cyclicality between GDP and defaults, see e.g. [10], and the model therefore shows that strong connections between companies may help to explain the oscillations of the business cycles.

2 Modelling framework and general results

2.1 Description of the model

We will design a dynamic system describing the evolution of an economy where companies may default. Time is continuous and measured in years. Today is $t = 0$. Let $X = \{X(t); t \geq 0\}$ be a continuous-time Markov process with state space $\mathbb{Z}_+^3$, where $\mathbb{Z}_+$ denotes the set of nonnegative integers. The companies in the economy can be in one of three states. They are either healthy, stressed or defaulted. The Markov process $X(t) = \big(H(t), S(t), D(t)\big)$ represents the number of companies in each state.

We define transition rates by the following device:

$$\mathbf{P}\left(X(t + \Delta t) = \xi + \ell \,|\, X(t) = \xi\right) = \Delta t N q_\ell(N^{-1}\xi) + o(\Delta t),$$

where N is a large parameter roughly corresponding to the number of companies in the economy and $\xi = (h, s, d)$. The possible transitions are classified as follows:

1. Inflow into the system: $q_{(0,1,0)}(\xi) = \theta$. New companies arrive to the economy in the stressed state.

2. Outflow of healthy companies from the system: $q_{(-1,0,0)}(\xi) = h\beta$. Healthy companies may leave the economy through mergers and acquisitions.

3. Migration from healthy to stressed: $q_{(-1,1,0)}(\xi) = h\alpha_h$. Healthy companies may become stressed.

4. Migration from stressed to healthy: $q_{(1,-1,0)}(\xi) = s\alpha_s$. Stressed companies may become healthy.

5. Default mechanism: $q_{(0,-1,1)}(\xi) = sd\lambda + s\alpha_d$. Stressed companies may default either through contagion or spontaneously.

6. Removal of defaults: $q_{(0,0,-1)}(\xi) = d\gamma$. After some time the defaulted companies cease to affect the economy.

All parameters above are defined to be strictly positive real numbers. The initial state is an arbitrary, deterministic or stochastic, vector in $\mathbb{Z}_+^3$. We may summarize the above in the transition chart shown in Figure 1.

The three possible states of a company in the economy can be viewed as a simplified version of the rating classes used by credit rating agencies. Of course the model is easily expanded to an arbitrary number of rating classes but for our purposes three states are sufficient. The difference between our model and models without contagion is the $\frac{\lambda}{N}sd$ term.

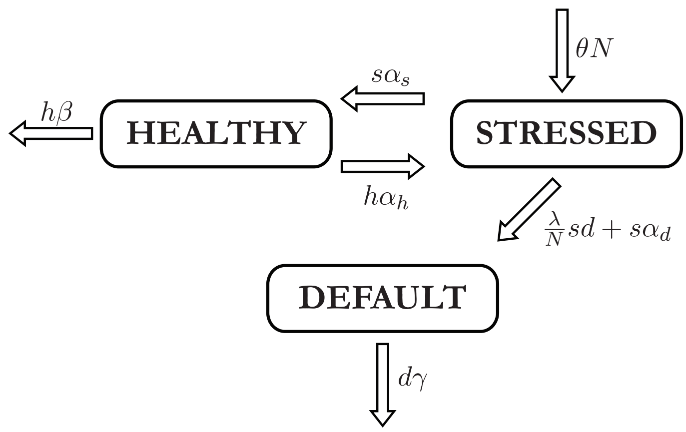


Figure 1: Transition chart.

We may interpret this as if every defaulted company causes a stressed company to default independently with probability $\frac{\lambda}{\lambda+\gamma N}$. An assumption is therefore that the economy is homogeneously mixing i.e. every company is equally likely to have connections to every other company and thereby affect it negatively in case of default. This is of course a great simplification as geographic or industrial aspects are not taken into account. However, this approach has been successfully used in for example the modelling of contagious diseases amongst humans so there is reason to believe that at least the qualitative behavior is preserved also in this context.

2.2 Law of large numbers

Now, let X^N denote a sequence of jump Markov processes as above, indexed by the parameter N. We may now show that the normalized sequence $x^N(t) = X^N(t)/N$ converges weakly in $\mathcal{D}[0,\infty)$, the space of càdlàg functions on $[0,\infty)$ equipped with the Skorohod topology, to a deterministic motion $x = \{x(t); t \geq 0\}$.

Towards this end let us define the drift vector

$$F(\xi) = \sum_{\ell} \ell q_{\ell}(\xi) = (-h\beta + s\alpha_s - h\alpha_h, \theta - s\alpha_s - sd\lambda - s\alpha_d + h\alpha_h, sd\lambda + s\alpha_d - d\gamma)\,. \quad (1)$$

If x satisfies the equation $\dot{x} = F(x)$ we have the following theorem by Kurtz [11].

Theorem 1 *Let q_ℓ be such that for every compact $K \subset \mathbb{R}^3$*

$$\sum_{\ell} |l| \sup_{\xi \in K} q_l(\xi) < \infty,$$

and there is $M_K > 0$ such that

$$|F(\xi) - F(\eta)| \leq M_K |\xi - \eta|, \;\; \xi, \eta \in K.$$

Then we have that

$$\lim_{N\to\infty} \sup_{s\le t} |x^N(s) - x(s)| = 0 \text{ a.s. for all } t > 0.$$

The applicability of this theorem in our case follows since for every ℓ we have that $q_\ell(\xi)$ is continuous which implies that $\sup_{\xi\in K} q_\ell(\xi) < \infty$, since K is compact, which gives us the first condition. The second condition is satisfied since $F(\xi)$ is continuously differentiable which implies that F is locally Lipschitz. But a locally Lipschitz function restricted to a compact set is Lipschitz which implies the second condition.

To understand the implications of the above theorem we need to investigate the ODE $\dot{x} = F(x)$. Specifically we would like to find any stationary points, i.e. points $\hat{x} = (\hat{h}, \hat{s}, \hat{d})$ such that $F(\hat{x}) = 0$, since these determine the asymptotic behaviour of $x(t)$. Using (1) we get

$$\begin{aligned} -\hat{h}\beta + \hat{s}\alpha_s - \hat{h}\alpha_h &= 0, && (2)\\ \theta - \hat{s}\alpha_s - \hat{s}\hat{d}\lambda - \hat{s}\alpha_d + \hat{h}\alpha_h &= 0, && (3)\\ \hat{s}\hat{d}\lambda + \hat{s}\alpha_d - \hat{d}\gamma &= 0. && (4) \end{aligned}$$

It is easily shown that the above system has a unique strictly positive solution, given by

$$\begin{aligned} \hat{h} &= \frac{\alpha_h\alpha_d\gamma + \alpha_s\beta\gamma + \alpha_d\beta\gamma + \alpha_h\theta\lambda + \beta\theta\lambda - \sqrt{\Psi}}{2\beta(\alpha_h+\beta)\lambda}, && (5)\\ \hat{s} &= \frac{\alpha_h\alpha_d\gamma + \alpha_s\beta\gamma + \alpha_d\beta\gamma + \alpha_h\theta\lambda + \beta\theta\lambda - \sqrt{\Psi}}{2\alpha_s\beta\lambda}, && (6)\\ \hat{d} &= \frac{-\alpha_h\alpha_d\gamma - \alpha_s\beta\gamma - \alpha_d\beta\gamma + \alpha_h\theta\lambda + \beta\theta\lambda + \sqrt{\Psi}}{2(\alpha_h+\beta)\gamma\lambda}, && (7) \end{aligned}$$

where

$$\Psi = (-\alpha_h\alpha_d\gamma - \alpha_s\beta\gamma - \alpha_d\beta\gamma + \alpha_h\theta\lambda + \beta\theta\lambda)^2 + 4\alpha_d(\alpha_h+\beta)^2\gamma\theta\lambda.$$

The stability of the stationary point is determined by the Jacobian of F evaluated at the stationary point,

$$\partial\hat{F} = \begin{pmatrix} -\beta - \alpha_h & \alpha_s & 0 \\ \alpha_h & -\alpha_s - \hat{d}\lambda - \alpha_d & -\hat{s}\lambda \\ 0 & \hat{d}\lambda + \alpha_d & \hat{s}\lambda - \gamma \end{pmatrix}. \quad (8)$$

If the real-part of the eigenvalues are all negative then the matrix is stable, i.e. the stationary point is locally asymptotically stable. As it turns out, showing explicitly that the eigenvalues of $\partial\hat{F}$ are negative for all parameter values is difficult. Another method is verifying the so called Routh-Hurwitz conditions which in this case is also difficult. We therefore use a different equivalent method from [13].

Let $||\cdot||$ denote a vector norm on $\mathbb{R}^n$ and the matrix norm it induces. We may then define the Dahlquist-Lozinskiĭ measure[1] of a real $n \times n$ matrix, with respect to the norm $||\cdot||$, as

$$\mu(A) = \lim_{h \to 0^+} \frac{||I - hA|| - 1}{h},$$

I being the identity matrix. Denote the spectrum of A as $\sigma(A)$ and let

$$s(A) = \max \{\Re(\lambda) : \lambda \in \sigma(A)\}.$$

We may then use the fact that $s(A) \leq \mu(A)$ so that it is enough to show that $\mu(A) < 0$ for some Dahlquist-Lozinskiĭ measure to deduce stability for A. In general the Dahlquist-Lozinskiĭ measure of a matrix is hard to calculate but for certain choices of $||\cdot||$ there are explicit formulas. Below we will consider the Dahlquist-Lozinskiĭ measure with respect to the norm $||x||_\infty = \sup_i |x_i|$ for which we have the formula

$$\mu(A) = \max_i \left\{ \sum_j |a_{ij}| \right\},$$

with $A = (a_{ij})$. For a given $n \times n$ invertible matrix Q we may construct a new norm by $||x||_Q = ||Qx||$ and therefore a new Dahlquist-Lozinskiĭ measure μ_Q which satisfies

$$\mu_Q(A) = \mu(QAQ^{-1}).$$

We will also need the spectral property of A's second additive compound matrix $A^{[2]}$ that if $\sigma(A) = \{\xi_i, i = 1, \ldots, n\}$ we have that $\sigma(A^{[2]}) = \{\xi_i + \xi_j : 1 \leq i < j \leq n\}$. These facts produce the following result, see [13].

Theorem 2 *Let A be an $n \times n$ matrix and assume that $(-1)^n \det(A) > 0$. Then A is stable if and only if $\mu(A^{[2]}) < 0$ for some Dahlquist-Lozinskiĭ measure μ.*

Straightforward calculations show that $\det(\partial \hat{F}) = -\sqrt{\Psi} < 0$.

For a 3×3 matrix the second additive compound matrix is,

$$A^{[2]} = \begin{pmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{pmatrix}.$$

Using (8) above this gives us

$$\partial \hat{F}^{[2]} = \begin{pmatrix} -\beta - \alpha_h - \alpha_s - \hat{d}\lambda - \alpha_d & -\hat{s}\lambda & 0 \\ \hat{d}\lambda + \alpha_d & -\beta - \alpha_h + \hat{s}\lambda - \gamma & \alpha_s \\ 0 & \alpha_h & \hat{s}\lambda - \gamma - \alpha_s - \hat{d}\lambda - \alpha_d \end{pmatrix}.$$

[1]Sometimes referred to as simply the Lozinskiĭ measure or as the logarithmic norm.

We let μ be the Dahlquist-Lozinskiĭ measure on 3×3 matrices with respect to the $||x||_\infty$ norm. Define the diagonal matrix Q by $Q = \operatorname{diag}(\hat{d}, -\hat{s}, \hat{h})$. We then have that

$$\begin{aligned}\mu_Q(\partial F^{[2]}) &= \mu(Q\partial F^{[2]}Q^{-1}) \\ &= \max\Bigg\{ -\beta - \alpha_h - \alpha_s - \alpha_d, \\ &\quad -\alpha_h - \frac{\hat{s}\alpha_s}{\hat{h}} - \frac{\hat{d}(\beta+\gamma) + \hat{s}\alpha_d}{\hat{d}}, -\frac{\hat{h}\alpha_h}{\hat{s}} - \gamma - \hat{d}\lambda - \alpha_s - \alpha_d + \hat{s}\lambda \Bigg\}.\end{aligned}$$

Now, from (4) we get that

$$-\frac{\hat{h}\alpha_h}{\hat{s}} - \gamma - \hat{d}\lambda - \alpha_s - \alpha_d + \hat{s}\lambda = -\frac{\hat{h}\alpha_h}{\hat{s}} - \hat{d}\lambda - \alpha_s - \alpha_d - \frac{\hat{s}\alpha_d}{\hat{d}} < 0,$$

so that $\mu_Q(\partial F^{[2]}) < 0$ and we may conclude that the stationary point $\hat{x} = (\hat{h}, \hat{s}, \hat{d})$ is indeed asymptotically stable. That is, if we let $x(0)$ be sufficiently close to $\hat{x}$ then $x(t)$ will remain close to $\hat{x}$ for all t and $x(t) \to \hat{x}$ as $t \to \infty$.

2.3 Weak convergence

Now define the sequence of processes $\mathfrak{X}^N = \{\mathfrak{X}^N(t); t \geq 0\}$ by

$$\mathfrak{X}^N(t) = \sqrt{N}\left(x^N(t) - x(t)\right).$$

We start the process at time 0 and choose $x(0)$ sufficiently close to $\hat{x}$, in the above sense. Let x^N be such that $\lim_{N\to\infty} \sqrt{N}\left|x^N(0) - x(0)\right| = 0$ and let V be defined as

$$V(t) = \sum_\ell \ell W_\ell\left(\int_{t_0}^t q_\ell\left(x(s)\right) ds\right) + \int_{t_0}^t \partial F\left(x(s)\right) V(s) ds,$$

where the W_ℓ's are Wiener processes. Then we have a central limit theorem by Kurtz [11].
Theorem 3 *If $\sum_\ell |\ell|^2 \sup_\xi q_\ell(\xi) < \infty$, $\partial F(\xi)$ is a bounded, continuous function of ξ, then $\mathfrak{X}^N \Rightarrow V$.*

Because of the asymptotic stability of the stationary point $\hat{x}$ there is some compact set K such that the trajectory of $x(t)$ is contained in K. But the restriction of ∂F and q_ℓ to K is bounded, which is what we need for the theorem. We may write $V(t)$ as the solution of the linear SDE

$$dV = \partial F(x(t))V dt + G(x(t))^{\frac{1}{2}} dW,$$

where W is the 3-dimensional Wiener process and $G(\xi) = \sum_\ell \ell\ell^T q_\ell(\xi)$. As we let $t \to \infty$ we get that $x(t) \to \hat{x}$, $\partial F(x(t)) \to \partial F(\hat{x}) \equiv \partial\hat{F}$ and $G(x(t)) \to G(\hat{x}) \equiv \hat{G}$. Therefore

V approaches a stationary Ornstein-Uhlenbeck process. We will henceforth consider this process described by the SDE

$$dV = \partial\hat{F}V dt + \hat{G}^{\frac{1}{2}}dW, \tag{9}$$

where

$$\hat{G} = \sum_{\ell} \ell\ell^T q_\ell(\hat{x}) = \begin{pmatrix} \hat{h}(\beta+\alpha_h)+\hat{s}\alpha_s & -\hat{h}\alpha_h - \hat{s}\alpha_s & 0 \\ -\hat{h}\alpha_h - \hat{s}\alpha_s & \theta + \hat{h}\alpha_h + \hat{s}(\alpha_s+\alpha_d+\hat{d}\lambda) & -\hat{s}(\hat{d}\lambda+\alpha_d) \\ 0 & -\hat{s}(\hat{d}\lambda+\alpha_d) & \hat{s}(\hat{d}\lambda+\alpha_d)+\hat{d}\gamma \end{pmatrix}.$$

From the definition of $G(\xi)$ it easy to show that $z^T G z > 0 \; \forall \; z \in \mathbb{R}^3$, i.e. $G(\xi)$ is positive definite.

Introduce the matrix Σ by the equation

$$\partial\hat{F}\Sigma + \Sigma\partial\hat{F}^T = -\hat{G}. \tag{10}$$

In stability theory this is known as the Lyapunov equation and the previously established stability of $\partial\hat{F}$ and positive definiteness of $\hat{G}$ guarantees the existence of a unique positive-definite symmetric solution, see [12], and we may interpret Σ as the covariance matrix of the stationary distribution of V. It is hard to obtain an analytic solution but there are numerical algorithms available, see e.g. [2].

Having calculated Σ we get from theorem 5.6.7 in [9] that the covariance function of V is given by

$$\rho(t) = \Sigma e^{t\partial\hat{F}^T}. \tag{11}$$

3 Case study

3.1 Contagion

To exemplify the properties of the model we study a specific choice of parameters in depth. No effort is made to estimate the parameters but it is our view that the chosen parameters are reasonably realistic.

Parameter	Value
N	5000
θ	0.1
β	0.01
α_h	0.03
α_s	0.03
λ	1.1
α_d	0.002
γ	2

The choice $N = 5000$ is arbitrary since it is only a matter of scale. Interpreting healthy and stressed as investment grade and speculative grade respectively we get from the transition matrices supplied by the rating agencies that 0.03 is a typical intensity for transitions between healthy and stressed, see e.g. [3]. Choosing $\gamma = 2$ represents a defaulted company affecting the economy for half a year on average. Since this is hard to observe in reality it is hard to motivate this on more than a qualitative level. It is however reasonable to assume that a defaulted company should affect the economy for more than a month but not much more than a year. The choice of θ can be seen as a choice of the scale of the system, appearing only as a multiplicative factor of N, and the value 0.1 is largely arbitrary. Having set β to 0.01 represents about 1% of the healthy companies disappearing from the economy through some other mechanism than default. Again this is hard to give any precise motivation other than that is seems to be a reasonable order of scale. Plugging these values into eqs. (5) to (7) and multiplying by N gives us

$$\begin{aligned}\hat{H} &= 6544.4,\\ \hat{S} &= 8725.8,\\ \hat{D} &= 217.28.\end{aligned}$$

Writing the transition intensity between the stressed and default state as $s\left(\frac{\lambda d}{N} + \alpha_d\right)$ we see that $\frac{\lambda \hat{d}}{N} + \alpha_d = 0.0498$ can be interpreted as the transition intensity between the stressed and default states per stressed company in equilibrium. This is fairly consistent with the 4% of investment grade companies defaulting annually reported in [5]. The SDE (9) now becomes

$$dV = \begin{pmatrix} -0.004 & 0.03 & 0 \\ 0.003 & -0.0798 & -1.9197 \\ 0 & 0.0498 & -0.0803 \end{pmatrix} V dt + \begin{pmatrix} 0.3019 & -0.1154 & -0.0159 \\ -0.1154 & 0.5057 & -0.0974 \\ -0.0159 & -0.0974 & 0.4051 \end{pmatrix} dW.$$

Solving eq. (10) in this case gives us

$$\Sigma = \begin{pmatrix} 2.1309 & 1.0960 & 0.2532 \\ 1.0960 & 21.4999 & -0.8041 \\ 0.2532 & -0.8041 & 0.5835 \end{pmatrix},$$

where we note in particular the negative correlation between the stressed and default states. We are now ready to calculate the correlation function by eq. (11). Plotting the autocorrelation for the three states gives us Figure 2. We see that the autocorrelation of the stressed and default states suggests oscillatory trajectories with a period of about 20 years. Using the number of defaults in an economy as an indicator of the business cycle this roughly agrees with the period of business cycles that has been observed historically.

To gain further understanding of the model we resort to simulation. We simulate the system with the above parameters using the gaussian approximation described above, starting the system at the equilibrium point. A sample trajectory is plotted in Figure 3, where the oscillatory behavior is evident.

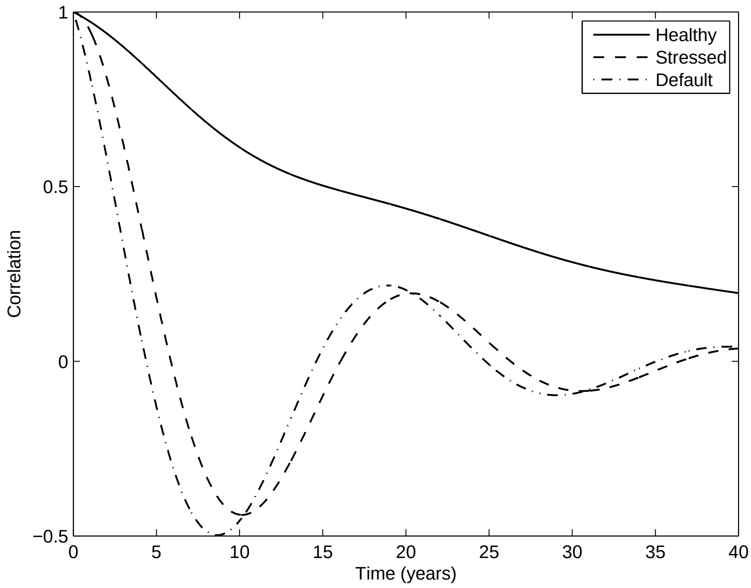


Figure 2: Autocorrelation function.

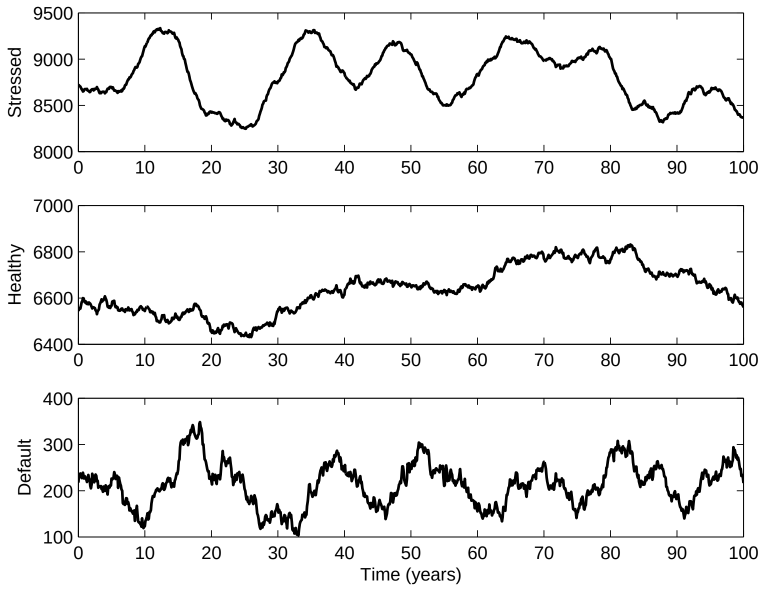


Figure 3: Sample trajectory from simulation, showing the oscillatory behavior of the stressed and default states.

We also investigate first-passage times of the default process. Again using the interpretation of the default process as an indicator of the business cycle we may view the first-passage time to a high level as representing the time to a recession. We start the simulation at the equilibrium point and plot the mean first-passage time as a function of the barrier height in Figure 4. For a 1D Ornstein-Uhlenbeck process we know from [14] that the distribution of the first-passage time approaches an exponential distribution as the barrier height grows. From [15] we also know that for a 1D Ornstein-Uhlenbeck process with infinitesimal drift $-\theta x$ and variance σ^2 the mean of the first-passage time to a barrier at height z is asymptotically $\sqrt{\frac{\pi\sigma^2}{z^2\theta^3}} \exp\left\{\frac{z^2\theta}{\sigma^2}\right\}$. The simulation seems to agree qualitatively with the one dimensional results, suggesting that the same asymptotic behavior can be used in our multivariate case. Also, the exponential distribution seems to fit well already at quite low barriers. It is important to note, though, that the approximating Ornstein-Uhlenbeck process is not adequate at very high levels. One would in this case need to invoke large deviation techniques in order to get more accurate estimates of the first-passage time.

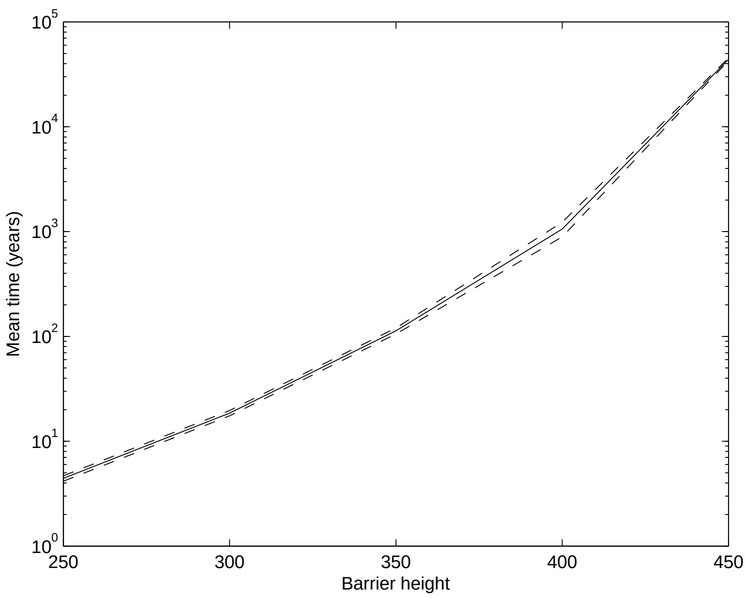


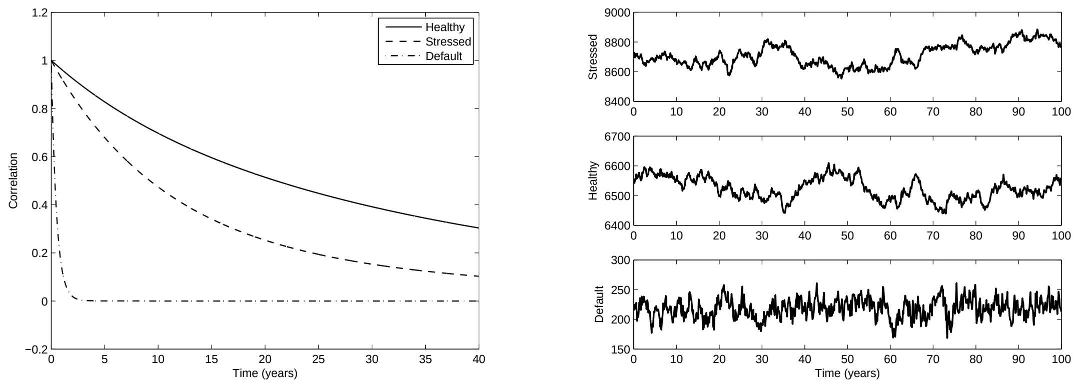
Figure 4: Mean first-passage time as a function of barrier height along with standard errors.

3.2 Without contagion

It is also illuminating to compare the model with contagion to a more standard model without contagion. Let us chose the same parameter values as above except for

Parameter	Value
λ	0
α_d	0.0498.

This produces the same equilibrium point and the same equilibrium transition intensities. However the behavior of the system is quite different. Again plotting the autocorrelation function and simulating a sample trajectory produces Figure 5. We may conclude from



(a) Autocorrelation function.

(b) Sample trajectory.

Figure 5: No contagion case

this that the no contagion case does not produce the oscillatory behavior obtained when contagion is present. This can also be shown to be a general result. Setting $\lambda = 0$ in (8) gives the following eigenvalues:

$$
\begin{aligned}
\zeta_1 &= -\gamma, \\
\zeta_{2,3} &= \frac{1}{2}\left(-\beta - \alpha_h - \alpha_s - \alpha_d \pm \sqrt{\left(-\beta - \alpha_h + \alpha_s + \alpha_d\right)^2 + 4\alpha_h\alpha_s}\right).
\end{aligned}
$$

Since the eigenvalues are real we get no oscillating trajectories.

4 Conclusions

The understanding of what drives the variation in the default intensities is important both from a risk management and a policy perspective. In this paper we have proposed a simple model of the default process inspired by research in the spread of disease among humans. Despite the simplicity of the model, by incorporating default contagion, we have been able to reproduce the cyclicality of default intensities empirically observed in the economy. This has been achieved without introducing any external macroeconomic force. These oscillations can be understood as cleansing of the unhealthy companies during a recession and the recession ending when sufficiently many of the unhealthy companies have left the economy. We have also shown that these oscillations is, in this model, a consequence of the contagion. We have made no attempt to estimate any of the parameters introduced in the model. Further research would be needed to do this in a consistent way. It is also possible to introduce additional states, corresponding to the rating classes used by the rating agencies.

This would however substantially complicate the analysis and is perhaps best suited for simulation studies. It would also be possible to relax the assumption of homogeneous mixing and instead introduce a network structure, taking into account geographic location and type of industry.

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