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Taurai Chimanga

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Matematisk statistik
Matematiska institutionen
Stockholms universitet
106 91 Stockholm

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Abstract

The globilisation phenomena is causing an increasing interaction between different markets and sectors. This has led to the evolution of derivative instruments from "single asset" instruments to complex derivatives that have underlying assets from different markets, sectors and sub-sectors. These are the so-called hybrid products that have multi-assets as underlying instruments. This article focuses on interest rate hybrid products. In this article an analysis of the application of stochastic interest rate models and stochastic volatility models in pricing and hedging interest rate hybrid products will be explored.

*Postal address: Mathematical Statistics, Stockholm University, SE-106 91, Sweden. E-mail:t_chimanga@yahoo.com . Supervisor: Thomas Höglund.

“ *There is only one good, knowledge, and one evil, ignorance.*”
Socrates

I dedicate this thesis to my grandmother. Thank you for being my pillar of strength.

Contents

1 Introduction

Globilisation has created an increasing interaction between different markets, sectors and sub-sectors. It is typical that a single investor might have simultaneous open position in different markets or asset classes. This has prompted the financial engineering of complex financial products called hybrid products. A hybrid product is a financial instrument whose payout is linked to underlyings belonging to different, but usually correlated, markets. The focus of this paper is on interest rate hybrid products. In this article an analysis of the application of stochastic interest rate models and stochastic volatility models in pricing and hedging interest rate hybrid products will be explored.

In this paper we keep in mind that a complicated model is harder to implement in practice. We will thus analyse the impact of using stochastic interest rates and stochastic volatility on an interest rate hybrid product. These models will be dealt with in a manner to keep the problem tractable. Stochastic interest rates will be introduced first and thereafter stochastic volatility will be included. We will thus compare how the models perform based on how well they hedge the hybrid.

The rest of this paper will be arranged as follows. Section 2 will give a brief introduction of the interest rate products. Section 3 will look at the impact of stochastic interest rates in pricing hybrid products. The specific hybrid product to be analysed in this article will be introduced and other classes that can be combined with interest rates in creating hybrid products will be discussed. Section 4 will look at the inclusion of stochastic volatility models in the hybrid setting. Section 5 compares how the models perform based on how well they hedge the hybrid and will give concluding remarks.

2 A Primer on Interest Rate Products

In pricing derivatives, modelling is usually done under a risk neutral measure or a martingale measure $\mathbb{Q}$. Under $\mathbb{Q}$, the standard numeraire is the money account. The dynamics of the money account are governed by the evolution of the interest rate. Thus in valuing any contingent claim, interest rates play a vital role. We take for instance the price of a call option on a stock:

$$Price_{call}(t) = e^{-r(T-t)} E^{\mathbb{Q}}[(S_T - K)^+ | \mathcal{F}_t] \tag{1}$$

where r represents the interest rate.

If the derivative has the interest rate as the underlying eg. options on bonds, swaptions and captions, the modelling of the interest rate becomes increasingly important. As interest rate derivative prices are sensitive to the pricing of interest rate dependant assets, it would thus not make much sense to use a model to price the derivatives which hardly prices the underlying assets accurately. The simplest interest rate product is a zero coupon bond which pays its full face value at maturity T. The price of a zero coupon bond at time t, $P(t,T)$, is given by

$$P(t,T) = e^{-R(t,T)(T-t)} \tag{2}$$

where $R(t,T)$ is the continuously compounded spot rate.

2.1 Term Structure of Interest Rates

We try to model an arbitrage-free family of zero coupon bonds. We assume that under the objective probability measure $\mathbb{P}$, the short rate process follows the SDE

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)d\widetilde{W} \tag{3}$$

We assume the existence of an arbitrage free market and a market for T-bonds for every choice of T. Furthermore, we assume that the price of a T-bond has the form

$$P(t,T) = F(t, r_t, T) \tag{4}$$

where F is a smooth function of three variables with simple boundary condition

$$F(T, r, T) = 1 \quad \forall \quad r \tag{5}$$

In an arbitrage free bond market, F must satisfy the term structure equation:

$$F_t + (\mu - \sigma\lambda)F_r + \frac{1}{2}\sigma^2 F_{rr} - rF = 0 \tag{6}$$

$$F(T, r, T) = 1. \tag{7}$$

λ is exogenous and represents the market price of risk whereas F_r denotes the partial derivative of F with respect to variable r. The Feynman-Kač representation of F from (6) and (7) implies that the T-bond prices are given by

$$F(t, r, T) = E^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,\mathrm{d}s}\right] \tag{8}$$

where $\mathbb{Q}$ denotes that the expectation is taken under the martingale measure with the short rate following the SDE

$$dr_s = (\mu - \lambda\sigma)ds + \sigma dW \tag{9}$$

As there are many interest rate products, they are combined to form the yield curve usually expressed in terms of zero coupon bond prices. Structured interest products are usually replicated with simpler instruments. If the combination of the simpler instruments mimics the payoff of the structured product then under standard arbitrage arguments, the price of the structured product must be equal to the value of the combination of the simpler instruments. Other complex structures can not be replicated with simpler instruments thus numerical procedures are used for their valuations.

We will look at an example of an interest rate product called a *cap*. A *cap* is a portfolio of call options used to protect the holder from a rise in the interest rate. Each of the individual options constituting a cap is known as a caplet. At the exercise dates, if the reference rate rises above the strike price, the holder receives the difference between the strike price and the reference rate on the succesive coupon date.

As a cap is a portfolio of caplets, its value is equal to the value of the caplets. If the i^{th} caplet runs from T_{i-1} to T_i, exercise decision is made on date T_{i-1} and the payment is received on date T_i. Assuming that the reference rate is the LIBOR, K represents the strike price and δ_i represents the day count fraction of the i^{th} period. The value of the i^{th} caplet as seen on its exercise date is

$$c_i(T_{i-1}) = P(T_{i-1}, T_i)\delta_i\big(LIBOR_i - K\big)^+ \tag{10}$$

which is equivalent to a European call option on the LIBOR struck at K.

2.2 Combining Asset Classes

Interest rate hybrid products have claims which are contingent upon movements in the interest rate and other asset classes. Although interest rate hybrids can be constructed with more than two asset classes, we restrict our analysis to only two asset classes. The hybrid we will consider will thus depend on interest rates and another asset class from either equity, inflation, foreign currency exchange or credit.

In this article we will look at a particular hybrid product which has a coupon payment similar to that of a caplet. We look at the hybrid best-of products, which at time T_i pays coupons of the form

$$\mathbf{max}\{i_{rate}, a \cdot (V_{T_i}/V_{T_{i-1}} - 1)\} \tag{11}$$

where a represents the participation rate, i_{rate} represents the interest rate for the coupon period eg. 3 month LIBOR, determined at time T_{i-1} and V_t represents the price of another asset class other than interest rates at time t. We are interested in analysing the properties of this hybrid product under different assumptions. We assume that the hybrid will pay coupons quarterly ie $\delta_i = 0.25$. We will use the equity class for V, 100% participation rate and the 3 month LIBOR rate for i_{rate} for the rest of this article. As the interest component is known at T_{i-1} we can simplify the coupon payment at T_i as

$$\mathbf{max}\Big\{\delta_i LIBOR_i, S_{T_i}/S_{T_{i-1}} - 1\Big\} \tag{12}$$

3 Stochastic Interest Rates

3.1 Deterministic vs Stochastic Rates of Interest

Modeling interest rates as closely as possible to reality is important especially in the pricing of long-dated derivatives. For short-dated derivatives, a deterministic interest rate model can be applied. We will look at a figure showing the evolution of the 3 month Libor rate in US dollars for the period *Sept 2004 - Jan2011.*

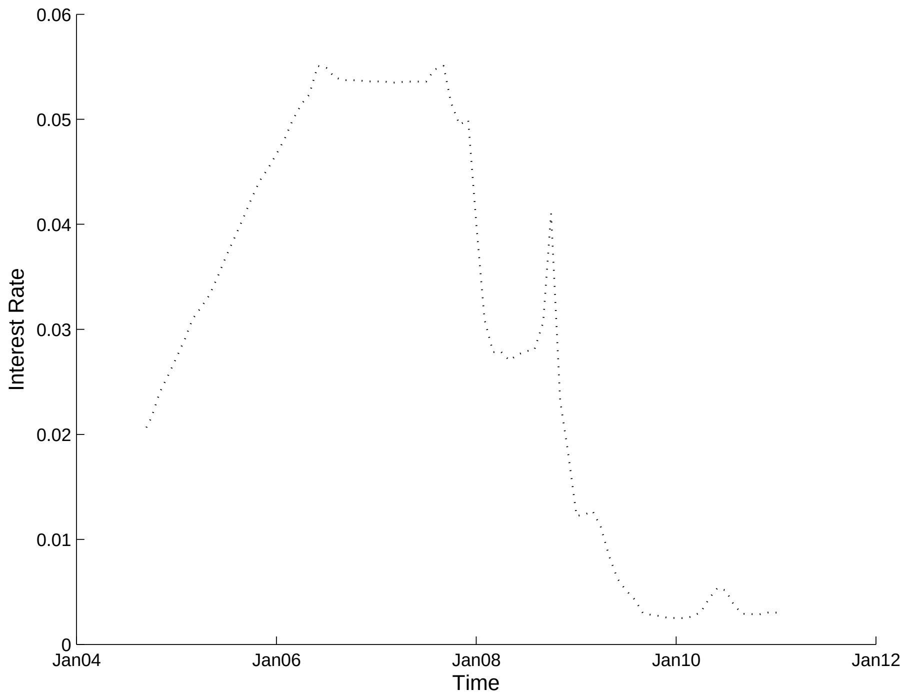

Figure 1: Evolution of the 3 Month Libor Rate in US Dollars

The stochastic nature of interest rates is apparent from Figure 1. The mean reverting characteristic is not clear from the plot because of the unstable period between *2007* and *2009* when the global economy experienced a recession. A recession is fortuitous and presents a higher level of volatility than usual in the market.

Our model in this section has the stock following a geometric brownian motion and the short rate modelled by the Hull-White model. The dynamics for the stock and the short-rate are:

$$dS_t = \mu_t S_t dt + \sigma_t^s S_t dW_t^s \tag{13}$$

$$dr_t = (\theta_t - \kappa_t r_t)dt + \sigma_t^r dW_t^r \tag{14}$$

where $\langle dW_t^r, dW_t^s \rangle = \rho dt$

3.2 Affine Term Structure

According to [4], if the term structure $\{p(t,T); 0 \leq t \leq T, T > 0\}$ has the form

$$p(t,T) = V(t, r_t, T) \tag{15}$$

where V has the form

$$V(t, r_t, T) = e^{A(t,T) - B(t,T)r_t} \tag{16}$$

and where A and B are deterministic functions, then the model is said to possess the affine term structure. We consider the Hull-White with constant volatility parameters, $\kappa_t = \kappa$ and $\sigma_t^r = \sigma^r$. According to [4], if the drift and volatility parameters for the short rate are time independent, a necessary condition for the existence of an affine term structure is that the drift and the volatility are affine in r. This implies that the Hull-White model with constant volatility parameters has an affine term structure with bond prices given by

$$p(t,T) = e^{A(t,T) - B(t,T)r_t}; \tag{17}$$

where

$$B(t,T) = \frac{1}{\kappa}\left\{1 - e^{-\kappa(T-t)}\right\} \tag{18}$$

$$A(t,T) = \int_t^T \left\{\frac{1}{2}\sigma^r B^2(t,T) - \theta_s B(s,T)\right\} \tag{19}$$

The yield curve is inverted by choosing θ such that the model matches initial bond prices. Choosing θ is equivalent to specifying a martingale measure as we have different martingale measures for different choices of the market price of risk, λ. The theoretical bond prices using the martingale measure $\mathbb{Q}$ are given by

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)} exp\left\{B(t,T)f^*(0,t) - \frac{\sigma_r^2}{4\kappa}B^2(t,T)(1 - e^{-2\kappa t}) - B(t,T)r_t\right\} \tag{20}$$

where variables with a superscript * are observed from the market.

3.3 Pricing a Cap

We use the affine term structure to price a cap. The value of a cap is equal to the value of the caplets. The i^{th} LIBOR is given by

$$L_i = \frac{1}{\delta_i}\left(\frac{1}{P(T_{i-1}, T_i)} - 1\right) \tag{21}$$

The value of the i^{th} caplet as seen on its exercise date is therefore:

$$c_i(T_{i-1}) = P(T_{i-1}, T_i)\delta_i(L_i - K)^+ \tag{22}$$

$$c_i(T_{i-1}) = (1 - P(T_{i-1}, P(T_i))(1 + K\delta_i))^+ \tag{23}$$

and using (20)

$$\Rightarrow c_i(T_{i-1}) = \Bigg(1 - (1 + K\delta_i)\frac{p^*(0, T_i)}{p^*(0, T_{i-1})}exp\big\{B(T_{i-1}, T_i)f^*(0, T_{i-1}) - \frac{\sigma_r^2}{4\kappa}B^2(T_{i-1}, T_i)(1 - e^{-2\kappa T_{i-1}}) - B(T_{i-1}, T_i)r_{T_{i-1}}\big\}\Bigg)^+ \tag{24}$$

where δ_i is the day count fraction corresponding to the i^{th} LIBOR period.

3.4 Calibration

Calibration is the process of determining the parameters that are used in the term structure model. In the Hull-White model, the parameters to be determined are κ and σ_t^r. The procedure is to choose the parameters such that the implementation of the term structure model replicates, as much as possible, liquid interest rate dependant instruments like floors, caps and swaptions. Usually the prices or volatilities of the options that are used to hedge the option in question are used for the calibration.

3.5 Analysing the Rate-Stock Correlation

Figure 2 does not show any relationship between the monthly 3M LIBOR rate and the monthly stock return between *Sept 2004 - Jan 2011*. However, low interest rates (close to zero) on the graph are consistent with the recovery of the global economy from the recession. We test the correlation between

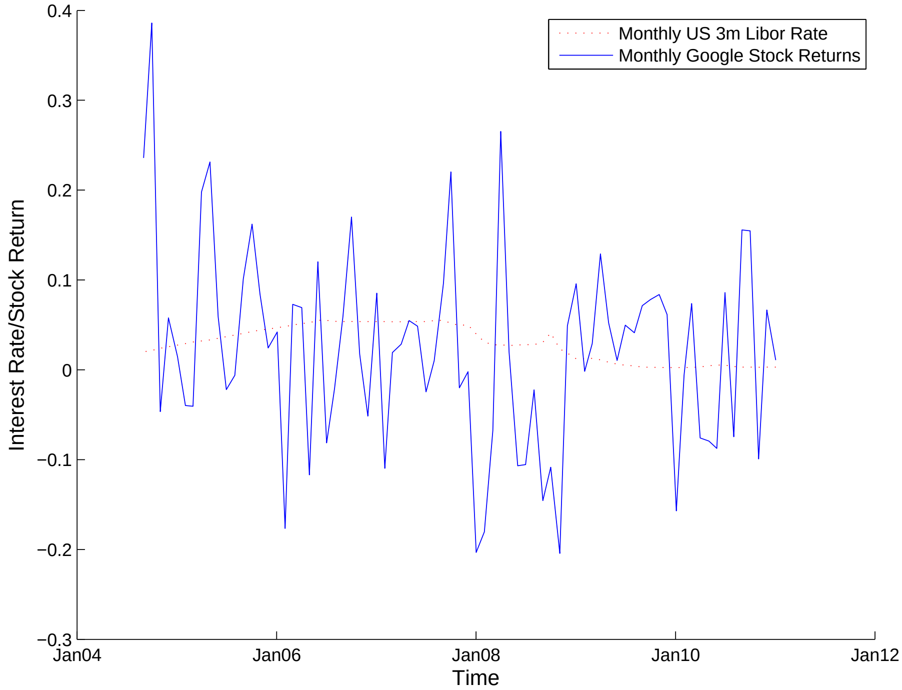

Figure 2: Plot showing 3M Libor Rate and Google Stock Returns

the interest rate and the stock return. The p-values for testing the null hypothesis that there is no correlation between the stock return and the 3M LIBOR rate against the alternative that there is a non-zero correlation are shown below.

method	correlation coefficient	p value
Pearson	-0.0274	0.8130
Spearman	-0.0295	0.7992
Kendall	-0.0195	0.8054

The p-values are too large, they are $\gg 0.05$ and thus for all the methods, we fail to reject the null hypothesis.

3.6 Pricing the Hybrid

3.6.1 Analytical Solution

Our hybrid has call options embedded in it and thus we will price it as a portfolio of forward starting call options. The i^{th} coupon payment made at time T_i, with exercise decision made at T_{i-1} valued at time t_0 is:

$$\Pi_{t_0} = E^{\mathbb{Q}}\left[e^{-\int_{t_0}^{T_i} r_s ds}\mathbf{max}\left\{\delta_i L_i, \frac{S_{T_i}}{S_{T_{i-1}}} - 1\right\}\middle|\mathcal{F}_{t_0}\right] \tag{25}$$

$$\Pi_{t_0} = E^{\mathbb{Q}}\left[e^{-\int_{t_0}^{T_{i-1}} r_s ds}E^{\mathbb{Q}}\left[e^{-\int_{T_{i-1}}^{T_i} r_s ds}\mathbf{max}\left\{\delta_i L_i, \frac{S_{T_i}}{S_{T_{i-1}}} - 1\right\}\middle|\mathcal{F}_{T_{i-1}}\right]\middle|\mathcal{F}_{t_0}\right] \tag{26}$$

We first deal with the inner expectation which using (21) simplifies to

$$E^{\mathbb{Q}}\left[e^{-\int_{T_{i-1}}^{T_i} r_s ds}\left(\delta_i L_i + \mathbf{max}\{0, \frac{S_{T_i}}{S_{T_{i-1}}} - \frac{1}{P(T_{i-1},T_i)}\}\right)\middle|\mathcal{F}_{T_{i-1}}\right] \tag{27}$$

using that $S_{T_i} = S_{T_{i-1}}e^{\int_{T_{i-1}}^{T_i} r_s ds - \frac{1}{2}\sigma^2(T_i - T_{i-1}) + \sigma(W_{T_i} - W_{T_{i-1}})}$ (27) becomes

$$E^{\mathbb{Q}}\left[P(T_{i-1},T_i)\mathbf{max}\left\{0, e^{\int_{T_{i-1}}^{T_i} r_s ds - \frac{1}{2}\sigma^2(T_i - T_{i-1}) + \sigma(W_{T_i} - W_{T_{i-1}})} - \frac{1}{P(T_{i-1},T_i)}\right\}\middle|\mathcal{F}_{T_{i-1}}\right]$$

$$+ \quad 1 - P(T_{i-1},T_i) \tag{28}$$

$$= E^{\mathbb{Q}}\left[\mathbf{max}\left\{0, e^{-\frac{1}{2}\sigma^2(T_i - T_{i-1}) + \sigma(W_{T_i} - W_{T_{i-1}})} - 1\right\}\middle|\mathcal{F}_{T_{i-1}}\right] + 1 - P(T_{i-1},T_i) \tag{29}$$

$$= Call(S = 1, K = 1, \sigma, r = 0, \tau = T_i - T_{i-1}) + 1 - P(T_{i-1},T_i) \tag{30}$$

$Call(S = 1, K = 1, \sigma, r = 0, \tau = T_i - T_{i-1})$ is a call option valued in a world with zero interest rate. The volatility of the underlying is the unknown input and thus it will determine the price of the option. The call option is struck at the money thus using the Black Scholes formula we get the value of this option as:

$$Call(S = 1, K = 1, \sigma, r = 0, \tau = T_i - T_{i-1}) = N(d_+) - N(d_-) \tag{31}$$

where:
N($\cdot$) is the cumulative standard normal distribution function;

$d_+ = \big(log(S/K) + 0.5\sigma^2\tau\big)/(\sigma\sqrt{\tau})$
$d_- = d_+ - \sigma\sqrt{\tau}$

Inserting the inner expectation back to (26) yields:

$$\Pi_{t_0} = E^{\mathbb{Q}}\left[e^{-\int_{t_0}^{T_{i-1}} r_s ds}\left\{N(d_+) - N(d_-) + 1 - P(T_{i-1}, T_i)\right\}\middle|\mathcal{F}_{t_0}\right]$$
$$\Pi_{t_0} = P(t_0, T_{i-1})\left\{N(d_+) - N(d_-) + 1\right\} - P(t_0, T_i) \tag{32}$$

We notice that $P(t_0, T_{i-1})$ and $P(t_0, T_i)$ are observed from the market and thus the pricing of the hybrid is invariant under stochastic interest rates. The volatility of the underlying will thus determine the price of the hybrid.

3.7 Hedging

In this section, we let $N(d_+) - N(d_-) + 1 = c$. The interest rate is the only source of risk and thus to make our portfolio delta neutral, we have to hedge against interest rate movements. We use a T^* bond to hedge the interest rate risk where $T^* > T_i$. We thus seek to determine how many T^* bonds we require to hedge the interest rate delta. Let x be the number of T^* bonds required.

$$\frac{\partial}{\partial r}\left\{cP(t_0, T_{i-1}) - P(t_0, T_i)\right\} = \frac{\partial}{\partial r}\left\{xP(t_0, T^*)\right\} \tag{33}$$

We know that $P(t, T) = exp(-r(T - t))$ thus we get that

$$x = \frac{\partial}{\partial r}\left\{cP(t_0, T_{i-1}) - P(t_0, T_i)\right\}\Big/\frac{\partial}{\partial r}\left\{P(t_0, T^*)\right\} \tag{34}$$
$$= -\left\{\Big((T_i - t_0)P(t_0, T_i) - c(T_{i-1} - t_0)P(t_0, T_{i-1})\right\}\Big/\left\{(T^* - t_0)P(t_0, T^*)\right\} \tag{35}$$

4 Stochastic Volatility

In the Black-Scholes model, risk is quantified by a constant volatility parameter. Real market data for options suggests that volatility is not constant but dependant on the strike price. The volatility that is calculated from actual option prices is called the implied volatility. When the implied volatility is plotted against the strike price, a volatility smile results. In European option pricing, the volatility smile phenomena can be explained assuming that the volatility of the underlying follows a stochastic process such as that detailed in Heston(1993)[2]. In a stochastic volatility model, the volatility changes randomly, following the dynamics of a stochastic differential equation or some discrete random process. We will thus add stochastic volatility to our framework, assuming that the asset class other than that of the interest rate has volatility which follows a stochastic process. In our case, the other asset class is the equity class.

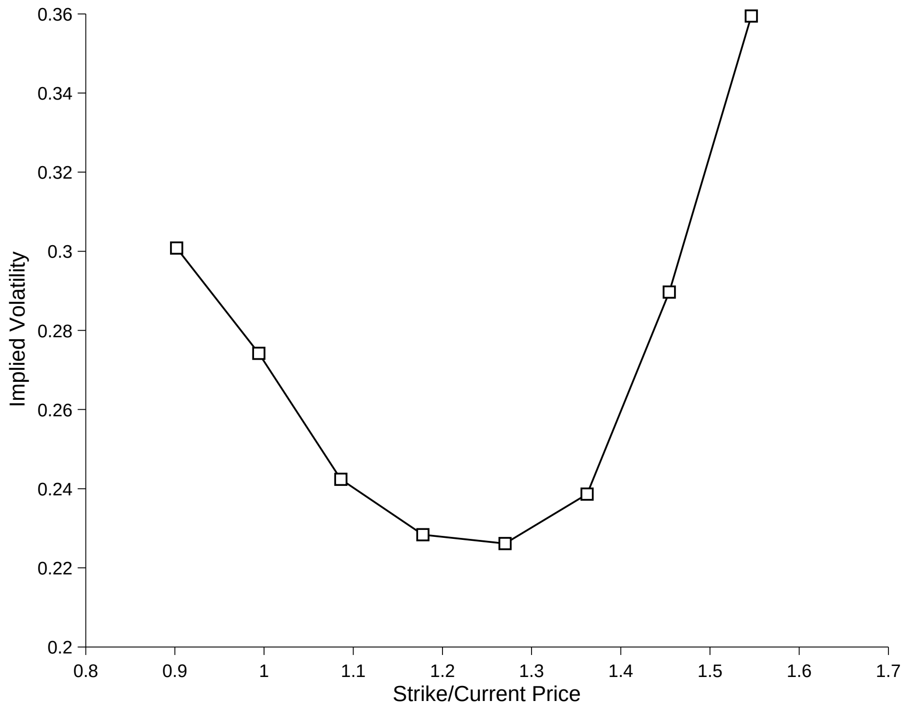

Figure 3: Volatility Smile as Observed from Google Call Options

In this section, we follow the Shöbel Zhu Hull White (SZHW) model. We present a slight change of notation to the dynamics of the stock and interest rate processes and assume that the volatility process follows an Ornstein-Uhlenbeck process. The dynamics for the stock process, volatility process and interest rate process are as follows:

$$dS_t = \mu_t S_t dt + v_t S_t dW_t^s \tag{36}$$

$$dv_t = \kappa[\omega - v_t]dt + \sigma_v dW_t^v \tag{37}$$

$$dr_t = (\zeta - \eta_t r_t)dt + \sigma_r dW_t^r \tag{38}$$

$$\begin{aligned} \langle dW_t^v, dW_t^s \rangle &= \rho_{sv} dt \\ \langle dW_t^r, dW_t^s \rangle &= \rho_{rs} dt \\ \langle dW_t^v, dW_t^r \rangle &= \rho_{rv} dt \end{aligned} \tag{39}$$

4.1 Change of Numeraire

We change the numeraire to a T-bond and thus change our measure from $\mathbb{Q}$ to a T-forward measure, $\mathbb{Q}^T$. By changing the numeraire, we hope to lose one variable and be left with two variables to deal with. We introduce the forward price

$$F_t = \frac{S_t}{P(t,T)} \tag{40}$$

Recalling the Hull-White affine term structure framework given in (20), the dynamics for the discount process under $\mathbb{Q}$ are given by

$$dP = r_t P dt - \sigma_r B(t,T) P dW_t^r \tag{41}$$

Applying Itô's lemma to (40) yields

$$dF = (\sigma_r^2 B_r^2(t,T) + \rho_{rs} v_t \sigma_r B(t,T)) F dt + v_t F dW_t^s + \sigma_r B(t,T) F dW_t^r \tag{42}$$

F_t is a martingale under $\mathbb{Q}^T$ and thus we have the following transformations from the $\mathbb{Q}$ measure to the $\mathbb{Q}^T$ measure:

$$\begin{aligned} dW_t^r &\mapsto dW_r^T(t) - \sigma_r B(t,T)dt \\ dW_t^s &\mapsto dW_s^T(t) - \rho_{rs}\sigma_r B(t,T)dt \\ dW_t^v &\mapsto dW_v^T(t) - \rho_{rv}\sigma_r B(t,T)dt \end{aligned}$$

Thus under $\mathbb{Q}^T$, v_t and F_t can be written as

$$dv(t) = \kappa[\omega - \frac{\rho_{rv}\sigma_r\sigma_v B(t,T)}{\kappa} - v_t]dt + \sigma_v dW_v^T(t) \tag{43}$$

$$dF(t) = v_t F dW_s^T(t) + \sigma_r B(t,T) F dW_r^T(t) \tag{44}$$

We can simplify (44) by using a log transformation and switching from $dW_r^T(t)$ and $dW_s^T(t)$ to $dW_F^T(t)$. We let $y(t) = log(F(t))$ and use Itô's lemma to get:

$$dv(t) = \kappa[\theta - v_t]dt + \sigma_v dW_v^T(t) \tag{45}$$

$$dy(t) = -\frac{1}{2}\varphi_F^2(t)dt + \varphi_F(t)dW_F^T(t) \tag{46}$$

with

$$\varphi_F^2(t) = v^2 + 2\rho_{rs}v_t\sigma_r B(t,T) + \sigma_r^2 B^2(t,T)$$

$$\theta = \omega - \frac{\rho_{rv}\sigma_r\sigma_v B(t,T)}{\kappa} \tag{47}$$

4.2 Pricing

According to the Meta Theorem in [4], a market is incomplete if the number of random sources in the model is greater than the number of traded assets. This implies that the model with stochastic volatility presents an incomplete market as there are at least two driving Weiner processes and only one traded asset. We now seek for a characteristic function for the forward log-asset price. We apply the Feynman-Kač theorem which transforms the problem into solving a PDE.

According to the the Feynman-Kač theorem, the characteristic function given by

$$f(t,y,v) = E^{\mathbb{Q}^T}\left[\exp(iuy(T))|\mathcal{F}_t\right] \tag{48}$$

is the solution to the PDE

$$\begin{aligned} 0 &= f_t - \frac{1}{2}\varphi_F^2(t)f_y + \kappa(\theta - v)f_v + \frac{1}{2}\varphi_F^2(t)f_{yy} \\ &\quad + (v\sigma_v\rho_s v + \rho_{rv}\sigma_v\sigma_r B(t,T))f_{yv} + \frac{1}{2}\sigma_v^2 f_{vv} \end{aligned} \tag{49}$$

$$f(T,y,v) = \exp(iuy(T)) \tag{50}$$

The solution to this problem is presented in [15]. We present the solution here and for proof, the reader is refered to the [15].

The characteristic function of a T-forward log-asset price of the SZHW model is given by the following closed form solution:

$$f(t,y,v) = \exp\left[A(u,t,T)+B(u,t,T)y(t)+C(u,t,T)v(t)+\frac{1}{2}D(u,t,T)v^2(t)\right], \tag{51}$$

where:

$$\begin{aligned}A(u,t,T) =& -\frac{1}{2}u(i+u)V(t,T)\\ &+\int_t^T\left[\kappa\omega+\rho_{rv}(iu-1)\sigma_v\sigma_r B_r(s,T)C(s)+\frac{1}{2}\sigma_v^2(C^2(s)+D(s))\right]ds\end{aligned} \tag{52}$$

$$B(u,t,T) = iu, \tag{53}$$

$$C(u,t,T) = -u(i+u)\frac{\left((\gamma_3-\gamma_4 e^{-2\gamma(T-t)})-(\gamma_5 e^{-a(T-t)}-\gamma_6 e^{-(2\gamma+a)(T-t)})-\gamma_7 e^{-\gamma(T-t)}\right)}{\gamma_1+\gamma_2 e^{-2\gamma(T-t)}}, \tag{54}$$

$$D(u,t,T) = -u(i+u)\frac{1-e^{-2\gamma(T-t)}}{\gamma_1+\gamma_2 e^{-2\gamma(T-t)}}, \tag{55}$$

with:

$$\begin{aligned}
\gamma &= \sqrt{(\kappa-\rho_{sv}\sigma_v iu)^2+\sigma_v^2 u(i+u)}, & \gamma_1 &= \gamma+(\kappa-\rho_{sv}\sigma_v iu), \\
\gamma_2 &= \gamma-(\kappa-\rho_{sv}\sigma_v iu), & \gamma_3 &= \frac{\rho_{sr}\sigma_r\gamma_1+\kappa\eta\omega+\rho_{rv}\sigma_r\sigma_v(iu-1)}{\eta\gamma}, \\
\gamma_4 &= \frac{\rho_{sr}\sigma_r\gamma_2-\kappa\eta\omega-\rho_{rv}\sigma_r\sigma_v(iu-1)}{\eta\gamma}, & \gamma_5 &= \frac{\rho_{sr}\sigma_r\gamma_1+\rho_{rv}\sigma_r\sigma_v(iu-1)}{\eta(\gamma-\eta)}, \\
\gamma_6 &= \frac{\rho_{sr}\sigma_r\gamma_2-\rho_{rv}\sigma_r\sigma_v(iu-1)}{\eta(\gamma+\eta)}, & \gamma_7 &= (\gamma_3-\gamma_4)-(\gamma_5-\gamma_6)
\end{aligned} \tag{56}$$

and:

$$V(t,T) = \frac{\sigma_r^2}{\eta^2}\left((T-t)+\frac{2}{\eta}e^{-\eta(T-t)}-\frac{1}{2\eta}e^{-2\eta(T-t)}-\frac{3}{2\eta}\right) \tag{57}$$

The variance process, v_t^2, can be derived using Itô's formula as

$$dv_t^2 = 2\kappa[\frac{\sigma_v^2}{2\kappa}+\omega v_t - v_t^2]dt + 2\sigma_v v_t dW_t^v \tag{58}$$

which can be written as the familiar square root process [used by Cox, Ingersoll, and Ross(1985)]

$$dv_t^* = \kappa^*[\theta^* - v_t^*]dt + \sigma_v^* \sqrt{v_t^*} dW_t^v \tag{59}$$

with

$$\begin{aligned} v_t^2 &= v_t^*, & \kappa^* &= 2\kappa \\ \theta^* &= \frac{\sigma_v^2}{2\kappa} + \omega v_t, & \sigma_v^* &= 2\sigma_v \end{aligned} \tag{60}$$

where κ^* is called the "speed of mean reversion", $\sqrt{\theta^*}$ the "long vol", σ_v^* the "vol of vol" and the initial value v_0^* the "short vol".According to [12] the vol of vol and the correlation can be thought as the parameters responsible for the skew whereas the other parameters control the term structure of the model. We can see from (59) that the Heston model is as special case of our model.

When pricing our hybrid, we have to price it as a forward starting option. We follow the method proposed by [8]. The value of the hybrid at time t_0 is given by:

$$\Pi_{t_0} = P(t,T_i)E^{Q^T}\left[\mathbf{max}\left\{\delta L_i, \frac{S_{T_i}}{S_{T_{i-1}}} - 1\right\}\middle|\mathcal{F}_{t_0}\right] \tag{61}$$

$$= P(t_0,T_{i-1})E^{Q^T}\left[P(T_{i-1},T_i)E^{Q^T}\left[\mathbf{max}\left\{\delta L_i, \frac{S_T}{S_{T_{i-1}}} - 1\right\}\middle|\mathcal{F}_{T_{i-1}}\right]\middle|\mathcal{F}_{t_0}\right] \tag{62}$$

$$= P(t_0,T_{i-1})E^{Q^T}\left[P(T_{i-1},T_i)E^{Q^T}\left[\delta L_i + \left\{\frac{S_{T_i}}{S_{T_{i-1}}} - \frac{1}{P(T_{i-1},T_i)}\right\}^+\middle|\mathcal{F}_{T_{i-1}}\right]\middle|\mathcal{F}_{t_0}\right] \tag{63}$$

$$= P(t_0,T_{i-1})E^{Q^T}\left[P(T_{i-1},T_i)\delta L_i + P(T_{i-1},T_i)E^{Q^T}\left\{\frac{S_{T_i}}{S_{T_{i-1}}} - \frac{1}{P(T_{i-1},T_i)}\right\}^+\middle|\mathcal{F}_{T_{i-1}}\right]\middle|\mathcal{F}_{t_0}\right] \tag{64}$$

$$= P(t_0,T_{i-1})E^{Q^T}\left[P(T_{i-1},T_i)\delta L_i\middle|\mathcal{F}_{t_0}\right] \tag{65}$$

$$+ P(t_0,T_i)E^{Q^T}\left[E^{Q^T}\left[\left\{\frac{S_{T_i}}{S_{T_{i-1}}} - \frac{1}{P(T_{i-1},T_i)}\right\}^+\middle|\mathcal{F}_{T_i}\right]\middle|\mathcal{F}_{t_0}\right] \tag{66}$$

We let the second part of the equation equal to Γ which is given by:

$$\Gamma_{t,T_{i-1},T_i} = P(t_0,T_i)E^{Q^T}\left[E^{Q^T}\left[\left\{\frac{S_{T_i}}{S_{T_{i-1}}} - \frac{1}{P(T_{i-1},T_i)}\right\}^+ \middle| \mathcal{F}_{T_{i-1}}\right] \middle| \mathcal{F}_{t_0}\right] \tag{67}$$

$$= P(t_0,T_i)E^{Q^T}\left[\left(\frac{S_{T_i}}{S_{T_{i-1}}} - \frac{1}{P(T_{i-1},T_i)}\right)^+ \middle| \mathcal{F}_{t_0}\right] \tag{68}$$

$$= P(t_0,T_i)E^{Q^T}\left[\frac{1}{P(T_{i-1},T_i)}\left(\frac{S_{T_i}}{S_{T_{i-1}}}P(T_{i-1},T_i) - 1\right)^+ \middle| \mathcal{F}_{t_0}\right] \tag{69}$$

$$= P(t_0,T_{i-1})E^{Q^T}\left[\frac{P(T_{i-1},T_i)}{P(T_{i-1},T_i)}\left(\frac{S_{T_i}}{S_{T_{i-1}}}P(T_{i-1},T_i) - 1\right)^+ \middle| \mathcal{F}_{t_0}\right] \tag{70}$$

$$= P(t_0,T_{i-1})E^{Q^T}\left[\left(\frac{S_{T_i}}{S_{T_{i-1}}}P(T_{i-1},T_i) - 1\right)^+ \middle| \mathcal{F}_{t_0}\right] \tag{71}$$

$$\tag{72}$$

We focus on the expectation as we recognise that it looks like a call option on the undelying $\frac{S_{T_i}}{S_{T_{i-1}}}P(T_{i-1},T_i)$ struck at 1. Our task is thus to price this call option and then we will come back to Γ.

In pricing the call option, we consider the function $z(T_{i-1},T_i)$ which is given by

$$z(T_{i-1},T_i) = log\left(\frac{S_{T_i}}{S_{T_{i-1}}}P(T_{i-1},T_i)\right) \tag{73}$$

We have already defined y as

$$y(T_{i-1}) = log(S_{T_{i-1}}) - log(P(T_{i-1},T_i)) \tag{74}$$

thus we can simplify $z(T_{i-1},T_i)$ to:

$$z(T_{i-1},T_i) = y(T_i) - y(T_{i-1}) \tag{75}$$

We thus need to find the forward characteristic function for $z(T_{i-1},T_i)$ which is given by:

$$\phi_{T_{i-1},T_i}(u) = E^{Q^T}\left[exp\left\{iu\big(y(T_i) - y(T_{i-1})\big)\right\} \middle| \mathcal{F}_t\right] \tag{76}$$

We know the T-forward characteristic function of log-asset price $y(T)$. We assume that $y(T)$ is a Markov chain and using the Markov chain property,

$y(T_{i-1})$ and $y(T_i)$ are independent given that $\exists\ t^*$ where $T_{i-1} < t^* < T_i$ s.t $y(t^*)$ exists. We assume that such a t^* exists. A characteristic function for the difference of two independent random variables x and y is given by:

$$\phi_{x-y}(u) = \phi_x(u)\phi_y(-u) \tag{77}$$

Thus the forward characteristic function for $z(T_{i-1}, T_i)$ is given by:

$$\begin{aligned}\phi_{T_{i-1},T_i}(u) &= E^{Q^T}\left[exp\left\{iuy(T_i)\right\}\middle|\mathcal{F}_t\right]E^{Q^T}\left[exp\left\{-iuy(T_{i-1})\right\}\middle|\mathcal{F}_t\right] &(78)\\ &= f(T_i, y, v, u)f(T_{i-1}, y, v, -u) &(79)\end{aligned}$$

Once we have the forward characteristic function, we use Fourier Fast Tranform(FFT) method proposed by [6]. We use a value of 1.25 for α for the modified call option given by:

$$c_T(k) = exp(\alpha k)P(t_0, T_i)E^{Q^T}\left[\left(e^{Z(T_{i-1},T_i)} - e^k\right)^+\right] \tag{80}$$

where
$k = log(K)$.

The transform of the call as given by [8] is:

$$\psi(t_0, T_{i-1}, T_i) = P(t_0, T)\frac{\phi_{T_{i-1},T_i}(u - (\alpha+1)i)}{(\alpha + iu)(\alpha + 1 + iu)} \tag{81}$$

We can thus calculate the price of the forward starting call using the inverse FFT. Let $C^{fwd}(t_0, T_{i-1}, T_i)$ denote the price of this forward starting option. Returning to Γ, we get that:

$$\Gamma_{t_0,T_{i-1},T_i} = P(t_0, T_{i-1})C^{fwd}(t_0, T_{i-1}, T_i) \tag{82}$$

In Section 3, we showed that

$$P(t, T_{i-1})E^{Q^T}\left[P(T_{i-1}, T_i)\delta L_i\middle|\mathcal{F}_t\right] = P(t_0, T_{i-1}) - P(t_0, T_i) \tag{83}$$

thus the price of the hybrid is given by:

$$\Pi_t = P(t_0, T_{i-1})\left\{1 + C^{fwd}(t_0, T_{i-1}, T_i)\right\} - P(t_0, T_i) \tag{84}$$

We note that the prices of the bonds $P(t_0, T_{i-1})$ and $P(t_0, T_i)$ are observed from the market.

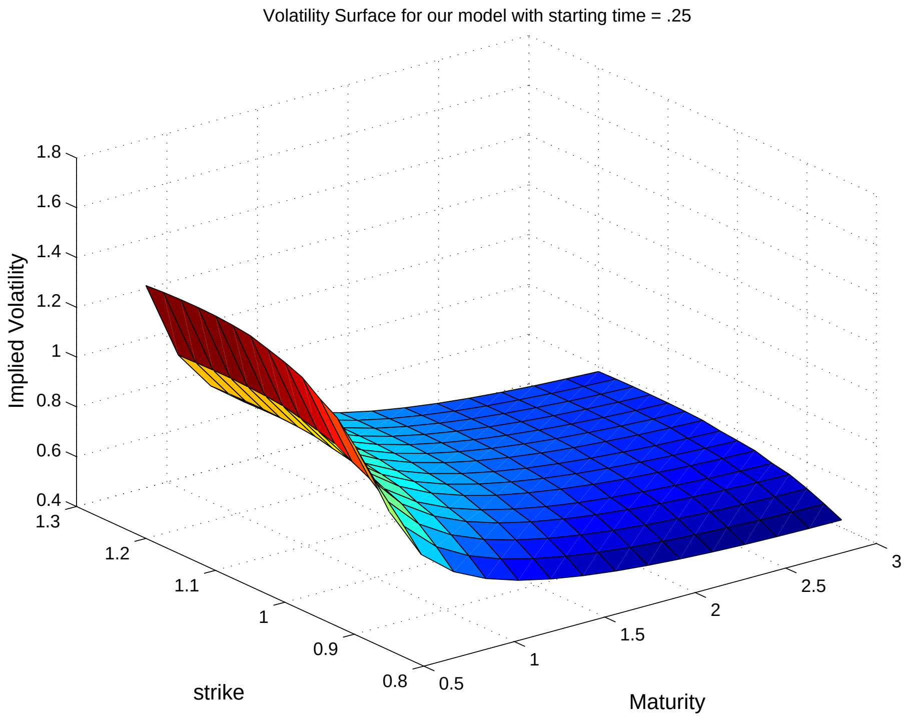

Figure 4: Volatility Surface for our Model with $T_{i-1} = .25yr, S_0 = 1, V_0 = .2, \kappa = 2, \eta = 2, \omega = .08, \sigma_r = .02, \rho_{sv} = .5, \rho_{sr} = .5, \rho_{rv} = .5, r_0 = .02$

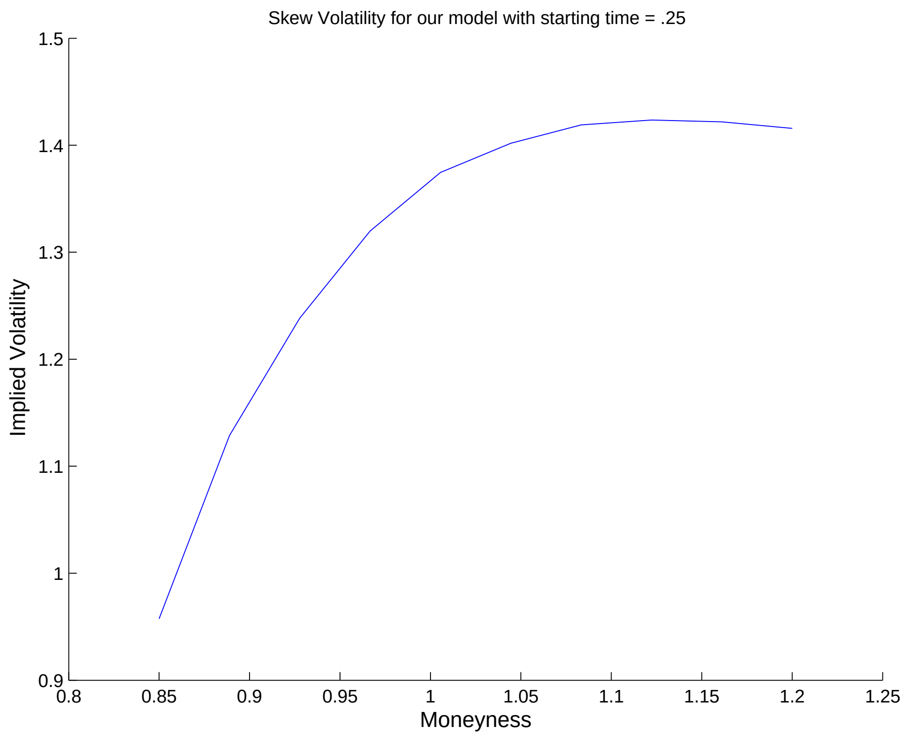

Figure 5: Skew Volatility for our Model with: $T_{i-1} = .25yr, S_0 = 1, V_0 = .2, \kappa = 2, \eta = 2, \omega = .08, \sigma_r = .02, \rho_{sv} = .5, \rho_{sr} = .5, \rho_{rv} = .5, r_0 = .02$

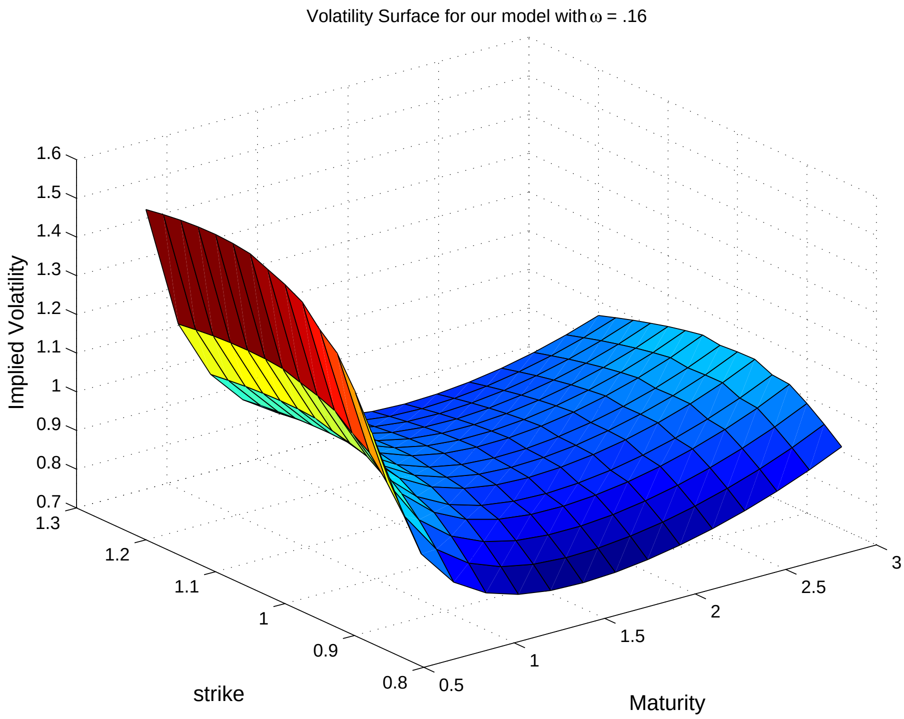

Figure 6: Volatility Surface for our Model with $T_{i-1} = .25yr, S_0 = 1, V_0 = .2, \kappa = 2, \eta = 2, \omega = .16, \sigma_r = .02, \rho_{sv} = .5, \rho_{sr} = .5, \rho_{rv} = .5, r_0 = .02$

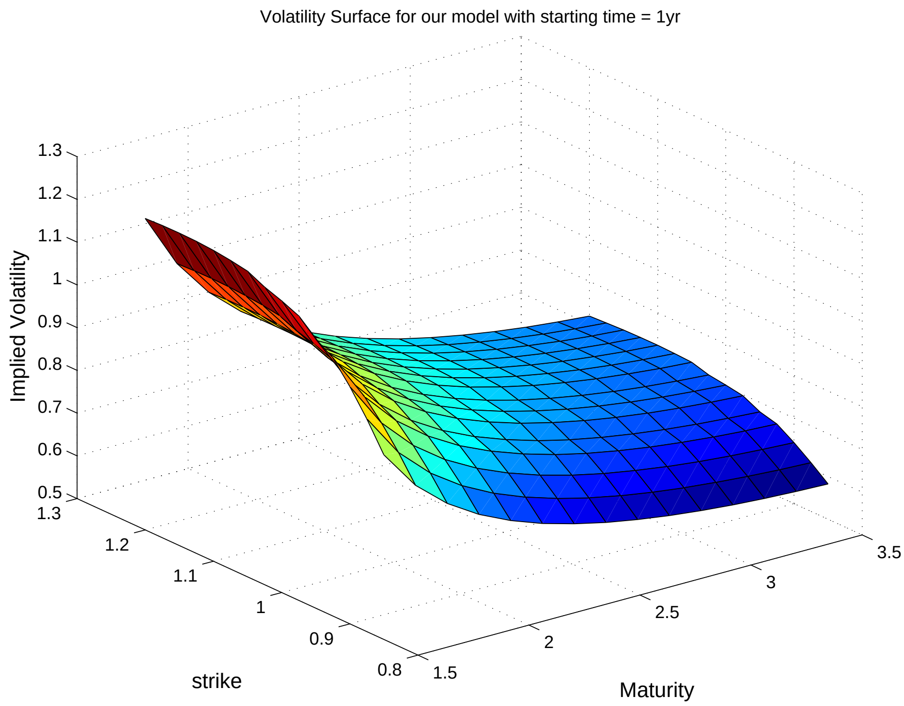

Figure 7: Volatility Surface for our Model with $T_{i-1} = 1yr, S_0 = 1, V_0 = .2, \kappa = 2, \eta = 2, \omega = .08, \sigma_r = .02, \rho_{sv} = .5, \rho_{sr} = .5, \rho_{rv} = .5, r_0 = .02$

4.3 Hedging

A model with stochastic volatility presents an incomplete market. In an incomplete market, a unique martingale measure does not exist and thus a derivative cannot be hedged perfectly by only the underlying asset and the money account. Hedging a derivative in an incomplete market model thus requires the addition of a benchmark derivative. We will call this benchmark derivative C. We create a risk neutral portfolio by:

1. Making the portfolio vega neutral by adding a position in C.
2. Making the portfolio rho neutral by adding a position in a bond.
3. Making the portfolio delta neutral by adding a stock position.

5 Hedging Accuracy Tests

In testing the performance of our models, we will evaluate how well the models hedges perform. We highlighted in the stochastic interest rates section that the volatility of the underlying is the only input into the model and thus determines the pricing of the hybrid. For the models to be comparable, we will use the implied volatility from the stochastic volatility and stochastic interest rate model as the input to get the stochastic interest rate price. In comparing the prices from the two different models, let $\Pi_{t0}(SISV)$ denote the price from the stochastic volatility and stochastic interest rate model and let $\Pi_{t0}(SI)$ denote the price from the stochastic interest rate model.

$$\Pi_t(SISV) = P(t_0, T_{i-1})\left\{1 + C^{fwd}(t_0, T_{i-1}, T_i)\right\} - P(t_0, T_i) \tag{85}$$

$$\Pi_{t_0}(SI) = P(t_0, T_{i-1})\left\{N(d_+) - N(d_-) + 1\right\} - P(t_0, T_i) \tag{86}$$

The two prices are similar and will be the same if and only if

$$N(d_+) - N(d_-) = C^{fwd}(t_0, T_{i-1}, T_i) \tag{87}$$

We compare how the hedges perform for a three month period where rebalancing is done weekly. We use $T_{i-1} = .25$ and $T_i = .5$. We first compare the two models seperately and then we compare the relative errors of the models. The data set used for the hedging tests is shown in the appendix.

In comparing the models, we get more information by comparing the standard errors of the error term. The summation of the squared relative errors give us the variance of the error term. Dividing the standard deviation of the errors by the square root of the number of data points used gives us the standard error. The table below shows the standard errors of the two models.

Model	Standard Error
SI	72.56%
SISV	47.74%

We note that the standard error is greater for the stochastic interest rate model. This implies that it is better to use the SISV holding all else constant.

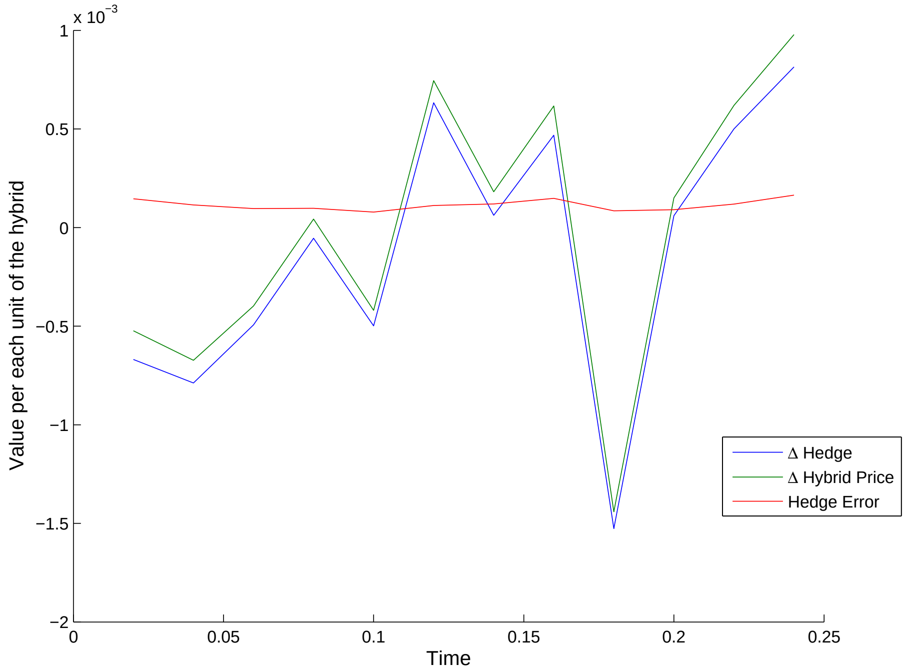

Figure 8: Stochastic interest rate model hedge error. The hybrid price is calculated as in Section 3.6 and the hedge as in Section 3.7

The drawbacks of using SISV is that vega is not easy to hedge. There is also more calibration required in SISV than in SI. The SI model presents a simple and straightforward way of getting an estimate of the hybrid's price. The skew volatility that we would have expected is that shown for the google share in Section 4 but our model has a different skew volatility which is like the inverted skew volatility that we would have expected.

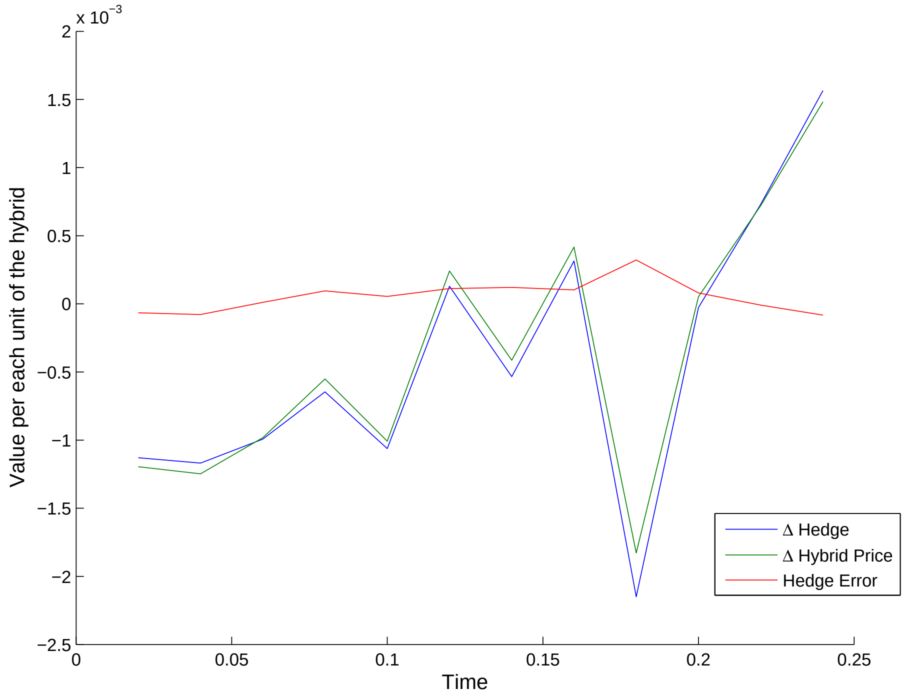

Figure 9: Stochastic interest rate and stochastic volatility model hedge error. The hybrid price is calculated as in Section 4.2 and the hedge as in Section 4.3

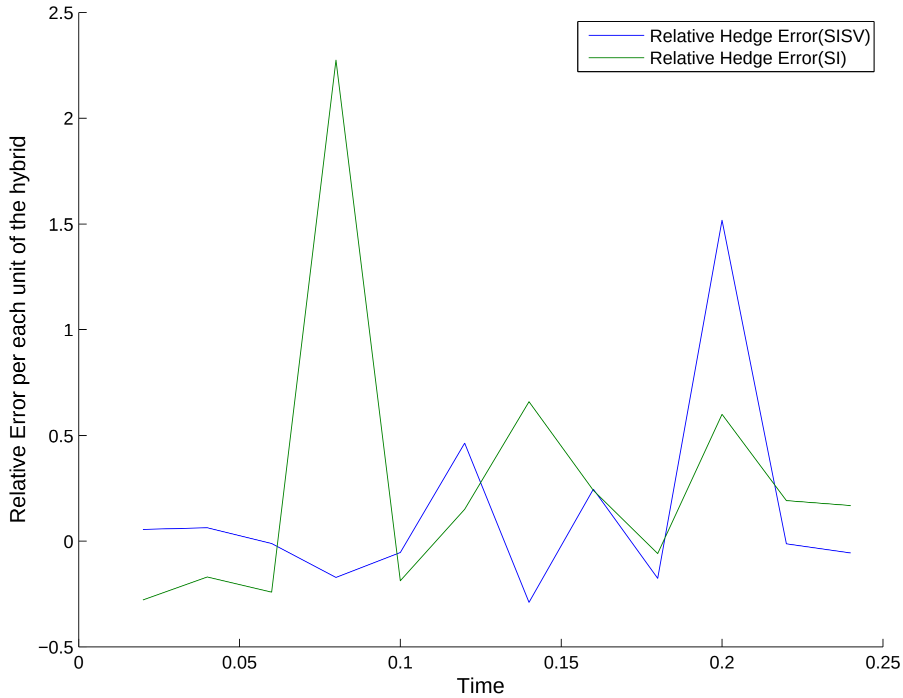

Figure 10: Relative hedge error for our models

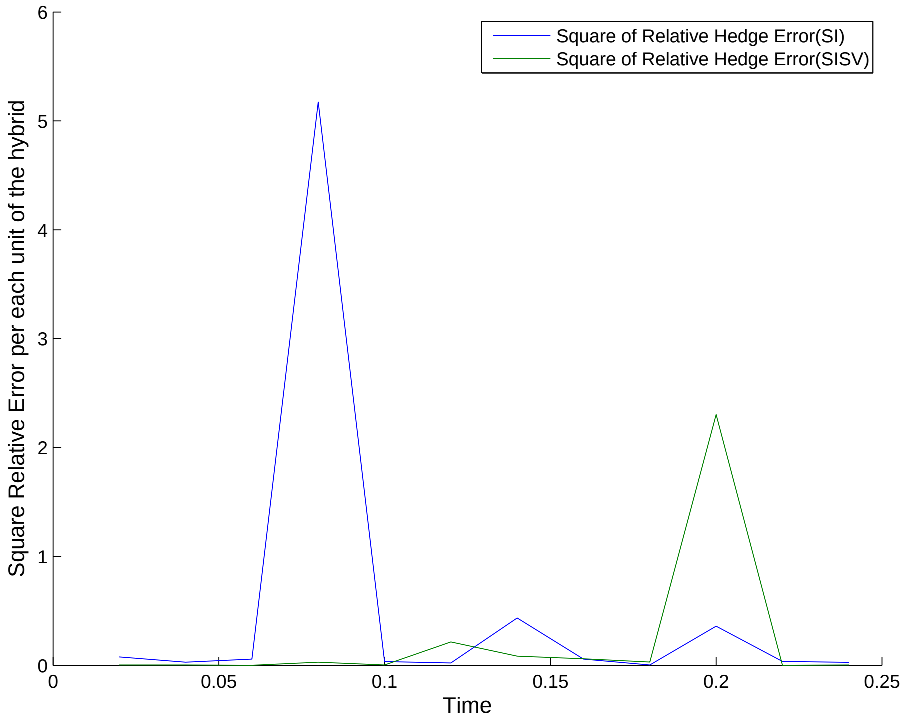

Figure 11: Square relative hedge error for our models

5.1 Conclusion

In this article we have discussed pricing methods for an interest rate hybrid product. We have priced the hybrid in two ways; using stochastic interest rates alone and also using stochastic interest rates and stochastic volatility. We have found that the pricing of the hybrid is invariant under stochastic interest rates. We have compared how the models used perform based on how well they hedge the hybrid. We have found that the stochastic interest rate model gives us a simple and straightforward solution but gives us a greater error than the model with stochastic interest rates and stochastic volatility. We have mentioned that although the stochastic interest rate and stochastic volatility model is attractive, it is harder to calibrate as well as hedge. Our analysis has a shortcoming in that we have priced a forward starting hybrid and have only looked at the time before the hybrid has started. Further research can thus be done to look at the pricing of the hybrid after it has started.

6 Data

Time	Current Stock Price	Current Instantaneous Volatility	Current Short Rate
0	1	0.2	0.02
0.02	1.115049126	0.200230098	0.016723563
0.04	1.06749221	0.2001447	0.012858947
0.06	0.599980722	0.19926816	0.010565438
0.08	0.492904166	0.198912533	0.010518325
0.1	0.506514443	0.198967457	0.008289288
0.12	0.59134628	0.199300691	0.011536484
0.14	0.877871633	0.200266364	0.01209704
0.16	0.886192738	0.200285347	0.014551937
0.18	0.923664935	0.200370036	0.008017769
0.2	0.990715299	0.200515488	0.008483938
0.22	1.224401187	0.20098845	0.010868651
0.24	0.667152074	0.200073718	0.014616501

References

[1] Kristina Andersson. Stochastic Volatility. 2003.

[2] Alexander Batchvarov. *Hybrid Products*. Risk Books, 2005.

[3] Eric Benhamou and Pierre Gauthier. Impact of Stochastic Interest Rates and Stochastic Volatility on Variable Annuities. 2009.

[4] Tomas Björk. *Arbitrage Theory in Continuous Time*. Oxford University Press, USA, 1999.

[5] Fischer Black and Myron Scholes. The Pricing of Options and Corporate Liabilities. *The Journal of Political Economy*, 81(3):637–654, 1973.

[6] Peter Carr and Dilip B. Madan. Option Valuation Using the Fast Fourier Transform. 1999.

[7] Anurag Gupta and Marti G. Subrahmanyam. Pricing and hedging interest rate options: Evidence from capfloor markets. *European of Banking and Finance*, 29:701–733, 2005.

[8] George Hong. Forward Smile and Derivative Pricing. 2004.

[9] Susanne Kruse. On the Pricing of Forward Starting Options under Stochastic Volatility. 2003.

[10] Nimalin Moodley. The Heston Model: A Practical Approach. 2005.

[11] Mikiyo Kii Niizeki. Option Pricing Models: Stochastic Interest Rates and Volatilities. 1999.

[12] Marcus Overhaus. *Equity Hybrid Derivatives*. Wiley, 2007.

[13] Rainer Schöbel and Jianwei Zhu. Stochastic Volatility With an OrnsteinUhlenbeck Process: Extension. *European Finace Review*, 3(1):23–46, 1999.

[14] Alexander van Haastrecht. Valuation of long-term hybrid equity-interest rate options. 2008.

[15] Alexander van Haastrecht, Roger Lord, Antoon Pelsser, and David Schrager. Pricing long-maturity equity and fx derivatives with stochastic interest rates and stochastic volatility. 2005.

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