English version

Baltic Way 2003 mathematical team contest

Riga, November 2, 2003

Working time: 4.5 hours.
Queries on the problem paper can be asked during the first 30 minutes.

1. Let $\mathbb{Q}_+$ be the set of positive rational numbers.
   Find all functions $f : \mathbb{Q}_+ \to \mathbb{Q}_+$ which for all $x \in \mathbb{Q}_+$ fulfill
   $(1) : f(\frac{1}{x}) = f(x)$
   $(2) : (1 + \frac{1}{x})f(x) = f(x+1)$

2. Prove that any real solution of
$$x^3 + px + q = 0$$
   satisfies the inequality $4qx \leq p^2$.

3. Let x, y and z be positive real numbers such that $xyz = 1$. Prove that
$$(1+x)(1+y)(1+z) \geq 2\left(1 + \sqrt[3]{\frac{y}{x}} + \sqrt[3]{\frac{z}{y}} + \sqrt[3]{\frac{x}{z}}\right).$$

4. Let a, b, c be positive real numbers. Prove that
$$\frac{2a}{a^2+bc} + \frac{2b}{b^2+ca} + \frac{2c}{c^2+ab} \leq \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}.$$

5. A sequence (a_n) is defined as follows: $a_1 = \sqrt{2}$, $a_2 = 2$, and $a_{n+1} = a_n a_{n-1}^2$ for $n \geq 2$. Prove that for every $n \geq 1$ we have
$$(1+a_1)(1+a_2)\dots(1+a_n) < (2+\sqrt{2})a_1 a_2 \dots a_n.$$

6. Let $n \geq 2$ and $d \geq 1$ be integers with $d \mid n$, and let $x_1, x_2, \dots, x_n$ be real numbers such that $x_1 + x_2 + \dots + x_n = 0$. Prove that there are at least $\binom{n-1}{d-1}$ choices of d indices $1 \leq i_1 < i_2 < \dots < i_d \leq n$ such that $x_{i_1} + x_{i_2} + \dots + x_{i_d} \geq 0$.

7. Let X be a subset of $\{1, 2, 3, \dots, 10000\}$ with the following property: if a, $b \in X$, $a \neq b$, then $a \cdot b \notin X$. What is the maximal number of elements in X?

8. There are 2003 pieces of candy on a table. Two players alternately make moves. A move consists of eating one candy or half of the candies on the table (the "lesser half" if there is an odd number of candies); at least one candy must be eaten at each move. The loser is the one who eats the last candy. Which player – the first or the second – has a winning strategy?

9. It is known that n is a positive integer, $n \leq 144$. Ten questions of type "Is n smaller than a?" are allowed. Answers are given with a delay: an answer to the i-th question is given only after the $(i+1)$-st question is asked, $i = 1, 2, \dots, 9$. The answer to the 10th question is given immediately after it is asked. Find a strategy for identifying n.

10. A *lattice point* in the plane is a point whose coordinates are both integral. The *centroid* of four points (x_i, y_i), $i = 1, 2, 3, 4$, is the point $(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4})$. Let n be the largest natural number with the following property: there are n distinct lattice points in the plane such that the centroid of any four of them is not a lattice point. Prove that $n = 12$.

11. Is it possible to select 1000 points in a plane so that at least 6000 distances between two of them are equal?

12. Let $ABCD$ be a square. Let M be an inner point on side BC and N be an inner point on side CD with $\angle MAN = 45^\circ$. Prove that the circumcenter of AMN lies on AC.

13. Let $ABCD$ be a rectangle and $BC = 2 \cdot AB$. Let E be the midpoint of BC and P an arbitrary inner point of AD. Let F and G be the feet of perpendiculars drawn correspondingly from A to BP and from D to CP. Prove that the points E, F, P, G are concyclic.

14. Let ABC be an arbitrary triangle and AMB, BNC, CKA regular triangles outward of ABC. Through the midpoint of MN a perpendicular to AC is constructed; similarly through midpoints of NK resp. KM perpendiculars to AB resp. BC are constructed. Prove that these 3 perpendiculars intersect at the same point.

15. Let P be the intersection point of the diagonals AC and BD in a cyclic quadrilateral. A circle through P touches the side CD in the midpoint M of this side and intersects the segments BD and AC in the points Q and R respectively. Let S be a point on the segment BD such that $BS = DQ$. The parallel to AB through S intersects AC at T. Prove that $AT = RC$.

16. Find all pairs of positive integers (a, b) such that $a - b$ is a prime and ab is a perfect square.

17. All the positive divisors of a positive integer n are stored into an array in increasing order. Mary has to write a program which decides for an arbitrarily chosen divisor $d > 1$ whether it is a prime. Let n have k divisors not greater than d. Mary claims that it suffices to check divisibility of d by the first $\lceil k/2 \rceil$ divisors of n: if a divisor of d greater than 1 is found among them, then d is composite, otherwise d is prime. Is Mary right?

18. Every integer is colored with exactly one of the colors BLUE, GREEN, RED, YELLOW. Can this be done in such a way that if a, b, c, d are not all 0 and have the same color, then $3a - 2b \neq 2c - 3d$?

19. Let a and b be positive integers. Prove that if $a^3 + b^3$ is the square of an integer, then $a + b$ is not a product of two different prime numbers.

20. Let n be a positive integer such that the sum of all positive divisors of n (except n) plus the number of these divisors is equal to n. Prove that $n = 2m^2$ for some integer m.