CZF is not cowellpowered

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A set B is a *quotient* of a set A if and only if there is a surjective function $A \longrightarrow B$. In intuitionistic set theory IZF, the class of all quotients of a given set can represented by a set in the following sense:

> (CW) For every set A, there is a set S such that B is a quotient of A if, and only if, there is $b \in S$ and a bijection $\psi : b \longrightarrow B$.

Indeed we can take S to be the set

$$\{(A/\sim) : \ (\sim) \in \mathcal{P}(A \times A) \text{ is an equivalence relation on } A\}$$

which exists by the Power Set axiom of IZF. Here $(A/\sim) = \{[a]_\sim : a \in A\}$ where $[a]_\sim = \{b \in A : a \sim b\}$.

In constructive set theory CZF there is in general no such set. In fact we have:

Theorem 0.1 *In CZF, the axiom CW implies Power Set.*

Proof. Assume CW. It is enough to show that the power class $\mathcal{P}(\{0\})$ is a set, since by exponentiation $\mathcal{P}(\{0\})^X$ exists, and it is in bijection with $\mathcal{P}(X)$.

Let $2 = \{0, 1\}$ be the standard two element set. For every subset $p \subseteq \{0\}$ we define an equivalence relation $\sim_p$ on 2 by

$$x \sim_p y \iff x = y \text{ or } 0 \in p.$$

Form the quotient set $T(p) = 2/_{\sim_p}$. Note that

$$0 \in p \iff x = y, \text{ for all } x, y \in T(p). \tag{1}$$

Define a class function F by

$$F = \{(b, p) : p \subseteq \{0\} \text{ and } (0 \in p \Leftrightarrow (\forall u, v \in b) u = v)\}$$

It is functional since if $(b, p), (b, p') \in F$: if $x \in p$, then $x = 0$, so $0 \in p$ and $(\forall u, v \in b)u = v$. But then also $0 \in p'$ by definition of F. Hence $x \in p'$. Thus $p \subseteq p'$. Similarly $p' \subseteq p$, and hence $p = p'$.

Suppose that S is the set of representatives provided by CW for $A = 2$. By replacement $F[S]$ is a set. Clearly $F[S] \subseteq \mathcal{P}(\{0\})$. Assume now $p \in \mathcal{P}(\{0\})$. By CW there is a $b \in S$ and bijection $\psi : b \longrightarrow T(p)$. Then by (1) and the bijection we have

$$\begin{aligned} 0 \in p \quad &\Leftrightarrow \quad (\forall x, y \in T(p))x = y \\ &\Leftrightarrow \quad (\forall u, v \in b)u = v. \end{aligned}$$

Hence (b, p) in F, and $p \in F[S]$. Thus $F[S] = \mathcal{P}(\{0\})$, and we conclude that $\mathcal{P}(\{0\})$ is a set. □

The proof shows that not even the quotients of the finite set $\{0, 1\}$ can be represented by a set in CZF.