Variants of categories with families and their formalization (work in progress)

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Overview of this talk

- (I) General goal and some obstacles
- (II) Categories with families some variants
- (III) Formalizing models of type theory in type theory

I. General goal and some obstacles

A longstanding goal: A constructive model theory of depedent type theory, more specifically Martin-Löf style type theories (MLTT).

- usable model-theoretic methods for obtaining effective results about theories and their relations, or to interpret richer type theories constructively.
- implement syntax and models in type theory, to be able to formally check correctness, facilitate reasoning and extract algorithms.

I. General goal and some obstacles

A longstanding goal: A constructive model theory of depedent type theory, more specifically Martin-Löf style type theories (MLTT).

- usable model-theoretic methods for obtaining effective results about theories and their relations, or to interpret richer type theories constructively.
- implement syntax and models in type theory, to be able to formally check correctness, facilitate reasoning and extract algorithms.

Obstacles:

- usual constructivity or predicativity issues.
- ▶ MLTT is a priori not as flexible as set theory as a meta-theory, and introduces coherence problems in the sense of category theory (Dybjer 1995). (What can MLTT + Univalence Axiom do to ameliorate this?)

Two kinds of models

Kind (α) : Interpretations of A=MLTT+Ax in B=MLTT*: one judgements in A are translated to possible several judgements, possibly of different kinds, in B, preferable so that definitional equalities in A are translated to definitional equalities in B.

Kind (β) : The model of A is a structure in B. In general MLTT* has to be a stronger theory than MLTT to succeed (by Gödel's theorem).

II. Categories with families

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- However, it is of interest to do the modeling in type theory itself, e.g. for the purpose of formal verification, and for foundational reasons. (Dybjer, *Internal Type Theory*, 1995 and onwards).

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$$A\{f\}$$
 for $Ty(f)(A)$.

(c) For each $A \in \mathrm{Ty}(\Gamma)$, an object $\Gamma.A$ in $\mathcal C$ and a morphism

$$p(A) = p_{\Gamma}(A) : \Gamma.A \longrightarrow \Gamma \text{ in } C.$$

This tells that each context can be extended by a type in the context, and that there is a projection from the extended context to the original one.

- (d) For each $A \in \mathrm{Ty}(\Gamma)$, there is a set $\mathrm{Tm}(\Gamma,A)$ thought of as the terms of type A. For $f:\Delta \longrightarrow \Gamma$ there is a function $\mathrm{Tm}(f):\mathrm{Tm}(\Gamma,A)\to\mathrm{Tm}(\Delta,A\{f\})$, where we write $a\{f\}$ for $\mathrm{Tm}(f)(a)$. It should satisfy the following
 - ▶ $a\{1_{\Gamma}\}=a \text{ for } a\in \mathrm{Tm}(\Gamma,A)$
 - ▶ $a\{f \circ g\} = a\{f\}\{g\} \text{ for } a \in \text{Tm}(\Gamma, A)$

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- (e) For each $A \in \mathrm{Ty}(\Delta)$ there is an element $v_A = v_{\Gamma,A} \in \mathrm{Tm}(\Delta.A, A\{p(A)\}).$
- (f) For any morphism $f: \Gamma \longrightarrow \Delta$ and $a \in Tm(\Gamma, A\{f\})$, there is

$$\langle f, a \rangle_A : \Gamma \longrightarrow \Delta.A.$$

This construction should satisfy

- ▶ $p(A) \circ \langle f, a \rangle_A = f$,
- $v_A\{\langle f,a\rangle_A\}=a,$
- $\langle p(A) \circ h, v_A\{h\} \rangle_A = h \text{ for any } h : \Gamma \longrightarrow \Delta.A.$

and moreover for any $g: \Theta \longrightarrow \Gamma$,

$$\langle f, a \rangle_A \circ g = \langle f \circ g, a\{g\} \rangle_A \tag{1}$$

Cwfs with named variables (projections)

Let V be an infinite discrete set. The set V is considered as the stock of variables. Let V^d denote the set of finite sequences

$$\mathbf{x} = x_1, x_2, \dots, x_n = ((\cdots(((), x_1), x_2), \cdots), x_n)$$

from V, where $x_i \neq x_i$ whenever $i \neq j$.

Definition A category with families and named variables (cwfn) consists of the following data

- (a1) [as before] A category C with a terminal object \top .
- (a2) A function $n: \mathrm{Ob}(\mathcal{C}) \longrightarrow V^d$ assigning each context a list of distinct variables (intended to be the names of projections). It is required that $n(\top) = ()$, the empty list. Define the fresh variables for Γ :

$$Fr(\Gamma) = \{x \in V : x \text{ is not in } n(\Gamma)\}.$$

- (b) [as before] Ty is a functor $C^{op} \longrightarrow Set$.
- (c) For each $A \in \mathrm{Ty}(\Gamma)$, and each $x \in \mathrm{Fr}(\Gamma)$, an object $\mathrm{Ext}(\Gamma,x,A)$ in \mathcal{C} , briefly written $\Gamma.(x:A)$ and a morphism $\mathrm{p}(x:A) = \mathrm{p}_{\Gamma}(x:A) : \Gamma.(x:A) \longrightarrow \Gamma$. It is required that

$$n(\operatorname{Ext}(\Gamma, x, A)) = (n(\Gamma), x).$$

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- (d) [as before] For each $A \in \mathrm{Ty}(\Gamma)$, there is a set $\mathrm{Tm}(\Gamma,A)$. It should be such that for $f:\Delta \longrightarrow \Gamma$ there is a function $\mathrm{Tm}(f):\mathrm{Tm}(\Gamma,A)\to\mathrm{Tm}(\Delta,A\{f\})$, where we write $a\{f\}$ for $\mathrm{Tm}(f)(a)$. It should satisfy the following
 - ▶ $a\{1_{\Gamma}\}=a \text{ for } a\in \mathrm{Tm}(\Gamma,A)$
 - ▶ $a\{f \circ g\} = a\{f\}\{g\} \text{ for } a \in \text{Tm}(\Gamma, A)$

(e) For each $A \in \mathrm{Ty}(\Delta)$, and each $x \in \mathrm{Fr}(\Delta)$, there is an element

$$\mathbf{v}_{(x:A)} = \mathbf{v}_{\Delta,(x:A)} \in \mathrm{Tm}(\Delta.(x:A), A\{\mathbf{p}(x:A)\}).$$

(e) For each $A \in \mathrm{Ty}(\Delta)$, and each $x \in \mathrm{Fr}(\Delta)$, there is an element

$$\mathrm{v}_{(x:A)}=\mathrm{v}_{\Delta,(x:A)}\in\mathrm{Tm}(\Delta.(x:A),A\{\mathrm{p}(x:A)\}).$$

(f) For any morphism $f: \Gamma \longrightarrow \Delta$, $A \in \mathrm{Ty}(\Delta)$, and each $x \in \mathrm{Fr}(\Delta)$, and $a \in \mathrm{Tm}(\Gamma, A\{f\})$, there is

$$\langle f, a \rangle_{(x:A)} : \Gamma \longrightarrow \Delta.(x:A).$$

We use the suggestive notation $\langle f, x := a \rangle_A$ for this morphism. It should satisfy

- $p(x:A) \circ \langle f, x := a \rangle_A = f,$
- $v_{(x;A)}\{\langle f,x:=a\rangle_A\}=a,$

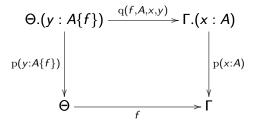
and moreover for any $g:\Theta\longrightarrow \Gamma$,

$$\langle f, x := a \rangle_A \circ g = \langle f \circ g, x := a \{g\} \rangle_A$$
 (2)

We define for $f: \Theta \longrightarrow \Gamma$, $A \in \mathrm{Ty}(\Gamma)$, $y \in \mathrm{Fr}(\Theta)$ and $x \in \mathrm{Fr}(\Gamma)$, a morphism $q(f, A, x, y) : \Theta.(y : A\{f\}) \longrightarrow \Gamma.(x : A)$ by

$$\mathrm{q}(f,A,x,y) =_{\mathrm{def}} \langle f \circ \mathrm{p}(y : A\{f\}), x := \mathrm{v}_{(y : A\{f\})} \rangle_{A}.$$

The following is a pullback square, satisfying well-known functoriality conditions in the parameter f:



Π -type construction in a cwfn

A cwfn supports Π -types if for $A \in \mathrm{Ty}(\Gamma)$, $x \in \mathrm{Fr}(\Gamma)$, and $B \in \mathrm{Ty}(\Gamma.(x:A))$ there is a type $\Pi_{\Gamma}(x,A,B) \in \mathrm{Ty}(\Gamma)$, and moreover for every $b \in \mathrm{Tm}(\Gamma.(x:A),B)$ there is an element $\lambda_{\Gamma,A,B}(x,b) \in \mathrm{Tm}(\Gamma,\Pi_{\Gamma}(x,A,B))$, and furthermore for any $c \in \mathrm{Tm}(\Gamma,\Pi_{\Gamma}(x,A,B))$ and any $a \in \mathrm{Tm}(\Gamma,A)$ there is an element $\mathrm{App}_{\Gamma,x,A,B}(c,a) \in \mathrm{Tm}(\Gamma,B\{\langle 1_{\Gamma},x:=a\rangle_A\})$, such that the following equations hold for any $f:\Theta \longrightarrow \Gamma$:

$$(\beta\text{-conv}) \operatorname{App}_{\Gamma,A,B}(\lambda_{\Gamma,A,B}(x,b),a) = b\{\langle 1_{\Gamma}, x := a \rangle_A\},\$$

- (Π-subst) $\Pi_{\Gamma}(x, A, B)\{f\} = \Pi_{\Theta}(y, A\{f\}, B\{q(f, A, x, y)\})$, provided $y \in Fr(\Theta)$
- (λ -subst) $\lambda_{\Gamma,A,B}(x,b)\{f\} = \lambda_{\Theta,A\{f\},B\{q(f,A,x,y)\}}(y,b\{q(f,A,x,y)\}),$ provided $y \in Fr(\Theta)$
- app-subst) $\operatorname{App}_{\Gamma,x,A,B}(c,a)\{f\} = \operatorname{App}_{\Theta,y,A\{f\},B\{q(f,A,x,y)\}}(c\{f\},a\{f\}),$ provided $y \in \operatorname{Fr}(\Theta)$.

Analyzing the syntax

With the notations

$$\Pi(A,(x)B) = \Pi(x,A,B)$$
$$\lambda((x)b) = \lambda(x,b)$$

we can rewrite the term formers as the rules

$$\frac{\Gamma: \mathrm{Ctx} \quad x: \mathrm{Fr}(\Gamma) \quad A: \mathrm{Ty}(\Gamma) \quad B: \mathrm{Ty}(\Gamma.(x:A))}{\Pi(A,(x)B): \mathrm{Ty}(\Gamma)}$$

 $\frac{\Gamma: \mathrm{Ctx}, x: \mathrm{Fr}(\Gamma) \quad A: \mathrm{Ty}(\Gamma) \quad B: \mathrm{Ty}(\Gamma.(x:A)) \quad b: \mathrm{Tm}(\Gamma.(x:A), A)}{\lambda((x)b): \mathrm{Tm}(\Gamma, \Pi(A, (x)B))}$

► Γ : Ctx, x : Fr(Γ), A : Ty(Γ), B : Ty(Γ .(x : A)), c : Tm(Γ , Π (A, (x)B)), a : Tm(Γ , A) / App(c, a) : Tm(Γ , B{(Γ , x := a)A})

Demonstration of resulting syntax

We drop several of the subscripts and annotations in these examples. We also write

$$\langle x_1 := a_1, x_2 := a_2, \dots, x_n := a_n \rangle$$

= $\langle \dots \langle \langle !, x_1 := a_1 \rangle, x_2 := a_2 \rangle, \dots, x_n := a_n \rangle.$

Note that $\langle \rangle = !$ and $\langle x_1 := a_1 \rangle = \langle !, x_1 := a_1 \rangle$. For a context $\Gamma = \top .x_1 : A_1.x_2 : A_2....x_n : A_n$ we note that its identity is

$$\mathbf{1}_{\Gamma} = \langle x_1 := \mathrm{v}_{x_1}, x_2 := \mathrm{v}_{x_2}, \ldots, x_n := \mathrm{v}_{x_n} \rangle.$$

Consider a simple example with a binary relation R between two types and a function f.

Let $A, B \in \mathrm{Ty}(\top)$. Then $\top.x : A \in \mathcal{C}$ and $p = ! : \top.x : A \longrightarrow \top$. Thus $B\{p\} \in \mathrm{Ty}(\top.x : A)$. Form $\top.x : A.y : B\{p\} :\in \mathcal{C}$. Let $R \in \mathrm{Ty}(\top.x : A.y : B\{p\})$ and $f \in \mathrm{Tm}(\top.x : A, B\{p\})$. Now $1_{\top.x : A} : \top.x : A \longrightarrow \top.x : A$ gives

$$\langle 1_{\top.x:A}, y := f \rangle : \top.x : A \longrightarrow \top.x : A, y : B\{p\}$$

and hence

$$R\{\langle 1_{\top.x:A}, y := f \rangle\} \in \mathrm{Ty}(\top.x : A).$$

Rewriting the identity and using $f = f\{1_{\top,x:A}\}$ we get

$$R\{\langle x := v_x, y := f\{\langle x := v_x \rangle\}\rangle\} \in Ty(\top .x : A).$$

Thus

$$\Pi(A,(x)R\{\langle x:=v_x,y:=f\{\langle x:=v_x\rangle\}\rangle\})\in\mathrm{Ty}(\top).$$

Assume furthermore that

$$g \in \operatorname{Tm}(\top .x : A, R\{\langle x := v_x, y := f\{\langle x := v_x \rangle\} \rangle\}).$$

Rewriting the identity this becomes

$$g\{\langle x:=\mathrm{v}_x\rangle\}\in\mathrm{Tm}\big(\top.x:A,R\{\langle x:=\mathrm{v}_x,y:=f\{\langle x:=\mathrm{v}_x\rangle\}\rangle\}\big).$$

Thus

$$\lambda((x)g\{\langle x:=v_x\rangle\})\in\Pi(A,(x)R\{\langle x:=v_x,y:=f\{\langle x:=v_x\rangle\}\rangle\})\in\mathrm{Ty}(\top X)$$

where $(\lambda x : A)b$ is short for $\lambda_{\Gamma,A,B}(x,b)$. Writing $E[a_1,\ldots,a_n]$ for $E\{\langle x_1 := a_1,\ldots,x_n := a_n \rangle\}$ and E type or term in the context $\top .x_1 : A_1.x_2 : A_2.\ldots.x_n : A_n$, we can express this as

$$\lambda((x)g[v_x]) \in \Pi(A,(x)R[v_x,f[v_x]]) \in \mathrm{Ty}(\top),$$

We have a twist map

$$\langle x := v_x, y := v_y\{p\} \rangle : \top . y : B.x : A\{p\} \longrightarrow \top . x : A.y : B\{p\}$$

Thus

$$R\{\langle x:=\mathrm{v}_x,y:=\mathrm{v}_y\{p\}\rangle\}\in\mathrm{Ty}(\top.y:B.x:A\{p\}).$$

With the abbreviation principle above

$$R[v_x, v_y\{p\}] \in Ty(\top . y : B.x : A\{p\}).$$

Furthermore

$$\Pi(A\{p\},(x)R[v_x,v_y\{p\}]) \in Ty(\top y:B). \tag{3}$$

This suggests that projections p should be forgotten in writing out the formulas in order to produce the standard variableful presentation. The last would then be

$$\Pi(A,(x)R[v_x,v_y]) \in \mathrm{Ty}(\top.y:B).$$

Martin-Löf's substitution calculus - polymorphic, free cwfs?

The first-order part of the Substitution Calculus (Martin-Löf 1992) has the judgment forms

 Γ : context $\Gamma = \Delta$: context $\gamma : \Delta \longrightarrow \Gamma$ $\gamma = \delta : \Delta \longrightarrow \Gamma$ $\Gamma \Longrightarrow A : \text{type}$ $\Gamma \Longrightarrow A = B : \text{type}$ $\Gamma \Longrightarrow a : A$ $\Gamma \Longrightarrow a = b : A$

One can also add (as in Tasistro 1993) a judgement for *context end* extension

 $\Gamma \preceq \Delta : \mathrm{context}$ to explicate this notion.

There are two rules for context formation

$$\frac{\Gamma : \text{context} \quad \Gamma \Longrightarrow A : \text{type}}{(\Gamma, x : A) : \text{context}}$$

The thinning rules ("context polymorphism") are:

$$\frac{\gamma : \Delta \longrightarrow \Gamma}{\gamma : \Theta \longrightarrow \Gamma} (\Delta \preceq \Theta) \qquad \frac{\gamma : \Delta \longrightarrow \Gamma}{\gamma : \Delta \longrightarrow \Theta} (\Theta \preceq \Gamma)$$

$$\frac{\gamma : \Delta \Longrightarrow A : \text{type}}{\gamma : \Theta \Longrightarrow A : \text{type}} (\Delta \preceq \Theta) \qquad \frac{\gamma : \Delta \Longrightarrow a : A}{\gamma : \Theta \Longrightarrow a : A} (\Delta \preceq \Theta)$$

The composition rules and associativity laws (omitted here). Unit laws:

$$\frac{\Gamma : \text{context}}{() : \Gamma \longrightarrow \Gamma}$$

$$\frac{\gamma : \Delta \longrightarrow \Gamma}{()\gamma = \gamma : \Delta \longrightarrow \Gamma}$$

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$$\frac{\Gamma \longrightarrow A : \text{type}}{\Gamma \longrightarrow A() = A : \text{type}}$$

$$\frac{\Gamma \longrightarrow a : A}{\Gamma \longrightarrow a() = a : A}$$

Updating laws and β , η , ξ -rules

$$\frac{\gamma : \Delta \longrightarrow \Gamma \quad \Delta \Longrightarrow a : A\gamma}{(\gamma, x = a) : \Delta \longrightarrow \Gamma, x : A}$$

$$\frac{\delta : \Theta \longrightarrow \Delta \quad \gamma : \Delta \longrightarrow \Gamma \quad \Delta \Longrightarrow a : A\gamma}{(\gamma, x = a)\delta = (\gamma\delta, x = a\delta) : \Theta \longrightarrow \Gamma, x : A}$$

$$\frac{\gamma : \Delta \longrightarrow \Gamma \quad \Delta \Longrightarrow a : A\gamma}{(\gamma, x = a) = \gamma : \Delta \longrightarrow \Gamma} \qquad \frac{\gamma : \Delta \longrightarrow \Gamma \quad \Delta \Longrightarrow a : A\gamma}{\Delta \Longrightarrow x(\gamma, x = a) = a : A\gamma}$$

$$\frac{\gamma : \Delta \longrightarrow ()}{\gamma = () : \Delta \longrightarrow ()} \qquad \frac{\gamma : \Delta \longrightarrow \Gamma, x : A}{\gamma = (\gamma, x = x\gamma) : \Delta \longrightarrow \Gamma, x : A}$$

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$$\frac{\gamma : \Delta \longrightarrow \Gamma, x : A \quad \delta : \Delta \longrightarrow \Gamma, x : A \quad \gamma = \delta : \Delta \longrightarrow \Gamma \quad \Delta \Longrightarrow x\gamma = x\delta}{\gamma = \delta : \Delta \longrightarrow \Gamma, x : A}$$

Variable law: derive $\Gamma \Longrightarrow x : A$ when x : A is declared in Γ .

Interpretation Substitution Calculus by a cwfn C?

- ▶ Contexts: $\Gamma \in Ob(C)$
- ▶ Context maps from Δ to Γ :

$$\Delta \succeq \Delta' {\stackrel{f}{\longrightarrow}} \, \Gamma' \succeq \Gamma$$

Types over Γ:

$$\Gamma \succeq \Gamma' \qquad A \in \mathrm{Ty}(\Gamma')$$

Elements of type A over Γ:

$$\Gamma \succeq \Gamma' \qquad \textit{A} \in \mathrm{Ty}(\Gamma') \qquad \textit{a} \in \mathrm{Tm}(\Gamma',\textit{A})$$

Interpreting equalities

$$(\Delta \succeq \Delta' \xrightarrow{f} \Gamma' \succeq \Gamma) \sim (\Delta \succeq \Delta'' \xrightarrow{g} \Gamma'' \succeq \Gamma) \text{ iff}$$

$$p_{\Gamma',\Gamma} \circ f \circ p_{\Delta,\Delta'} = p_{\Gamma'',\Gamma} \circ g \circ p_{\Delta,\Delta''}$$

$$(\Gamma \succeq \Gamma'; A \in \mathrm{Ty}(\Gamma')) \sim (\Gamma \succeq \Gamma''; B \in \mathrm{Ty}(\Gamma'')) \text{ iff}$$

$$A\{p_{\Gamma,\Gamma'}\} = B\{p_{\Gamma,\Gamma''}\}$$

$$(\Gamma \succeq \Gamma'; A \in \mathrm{Ty}(\Gamma'); a \in \mathrm{Tm}(\Gamma', A)) \sim$$

$$(\Gamma \succeq \Gamma''; B \in \mathrm{Ty}(\Gamma''); b \in \mathrm{Tm}(\Gamma'', B)) \text{ iff}$$

$$A\{\mathrm{p}_{\Gamma,\Gamma'}\}=B\{\mathrm{p}_{\Gamma,\Gamma''}\} \text{ and } a\{\mathrm{p}_{\Gamma,\Gamma'}\}=b\{\mathrm{p}_{\Gamma,\Gamma''}\}.$$

III. Formalizing models of type theory in type theory

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- An important example is Hofmann's model (Hofmann 1997), which is built from the syntax and judgements of intensional type theory, and is used to translate proofs in extensional type theory into proofs in the intensional theory.

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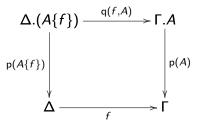
Categories with attributes is a variant of cwfs where the elements $\operatorname{Tm}(\Gamma,A)$ are defined as sections of projections $\operatorname{p}(A):\Gamma.A\longrightarrow \Gamma$. It seems slightly easier to formalize.

Definition 1. A category with attributes (cwa) consists of the data

- (a) A category C with a terminal object \top .
- (b) A functor $Ty : \mathcal{C}^{op} \longrightarrow Set$.
- (c) For each $A \in \mathrm{Ty}(\Gamma)$, an object $\Gamma.A$ in $\mathcal C$ and a morphism

$$p(A) = p_{\Gamma}(A) : \Gamma.A \longrightarrow \Gamma \text{ in } C.$$

(d) The final datum tells how substitutions interact with context extensions: For each $f: \Delta \longrightarrow \Gamma$ and $A \in \mathrm{Ty}(\Gamma)$, there is a morphism $\mathrm{q}(f,A) = \mathrm{q}_{\Gamma}(f,A) : \Delta.(A\{f\}) \longrightarrow \Gamma.A$ in $\mathcal C$ such that



is a pullback, and furthermore functoriality conditions

(d.1)
$$q(1_{\Gamma}, A) = 1_{\Gamma, A}$$

(d.2)
$$q(f \circ g, A) = q(f, A) \circ q(g, A\{f\})$$
 for $\Theta \xrightarrow{g} \Delta \xrightarrow{f} \Gamma$.

Example: The set-theoretic interpretation of the pullback diagram in (d) is then

$$\Sigma_{y \in \Delta}.A(f(y)) \xrightarrow{q} \Sigma_{x \in \Gamma}.A(x)$$

$$\downarrow^{p}$$

$$\Delta \xrightarrow{f} \Gamma$$

where
$$p(\langle u, v \rangle) = u$$
 and $q(\langle y, s \rangle) = \langle f(y), s \rangle$.

By assuming that the universe U is closed under further constructions one can verify axioms for type theoretic constructions like Π , W, I-types etc.

Indirect model in MLTT

In view of the fact that intensional Martin-Löf type theory interprets the universe V of CZF (Aczel 1978, 1986) we get an indirect interpretation of the extensional M-L type theory in the intensional one.

Aczel's interpretation has been formalized in various proof assistants: LEGO (N.P. Mendler, 1990), Agda 1 (M.Takeyama mid 1990s), Coq (P. and Wilander 2011). In the latter interpretation a full faithful functor

$$V \longrightarrow Setoids$$

is explicitly constructed.

However, we are interested in more direct interpretations.

Setoids

In type theories the notion of set is usually understood in the sense of Bishop as a type together with an equivalence relation, also called a setoid

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$$A = (|A|, =_A)$$

where |A| is a type and $=_A$ is an equivalence relation on |A|. An extensional function $f:A\longrightarrow B$ between setoids is a function $|A|\longrightarrow |B|$ which respects the equivalence relations, i.e.

$$(\forall x, y : |A|)[x =_A y \Longrightarrow f(x) =_B f(y)]$$

Two such functions f and g are extensionally equal $(f =_{\text{ext}} g)$ if $(\forall x : |A|)(f(x) =_B g(x))$.

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The properties of the category reflects the possibilities and limitations of constructions in the background theory.

The notion of e-category is a variant of the standard notion of category, but where no equality relation is required on objects.

An e-category $\mathcal C$ consists of a type $\operatorname{Ob} \mathcal C$ of objects, together with a setoid $\mathcal C(A,B)$ of morphisms for every pair of objects A and B. The composition is an extensional function

$$\circ: \mathcal{C}(B,C) \times \mathcal{C}(A,B) \longrightarrow \mathcal{C}(A,C)$$

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An e-functor is a functor where the object part is just a function between types. There is no equality of objects to respect.

E-cwas - cwas in type theory

Referring back to Definition 1, in order for the equations (d.1) and (d.2) to make sense, we need the *object equalities*

$$\Gamma.(A\{1_{\Gamma}\}) = \Gamma.A$$
 $p(A\{1_{\Gamma}\}) = p(A)$

and

$$\Theta.(A\{f\circ g\})=\Theta.(A\{f\}\{g\})$$

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This notion of cwa is not appropriate for categories $\mathcal C$ that lack object equality, like e-categories.

We modify the structure slightly to the setting of e-categories:

Definition 2. An e-category with attributes (ecwa) consists of the following data (a) - (d):

- (a) An e-category C with a terminal object \top .
- (b) An e-functor Ty : $C^{op} \longrightarrow Setoids$.
- (c) There is an e-functor $\Delta_{\Gamma} : \mathrm{Ty}(\Gamma)^{\#} \longrightarrow \mathcal{C}/\Gamma$. For $A \in \mathrm{Ob}\ \mathrm{Ty}(\Gamma)$, write $\Delta_{\Gamma}(A) = (p(A) : \Gamma.A \longrightarrow \Gamma)$.

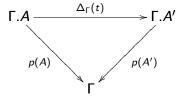
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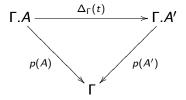
Here $S^{\#}$ denotes the discrete e-category induced by a setoid S. \mathcal{C}/Γ denotes the slice e-category of \mathcal{C} over Γ .

Thus for any proof object t of $A =_{\mathrm{Ty}(\Gamma)} A'$, $\Delta_{\Gamma}(t) : \Gamma.A \longrightarrow \Gamma.A'$ is an isomorphism such that



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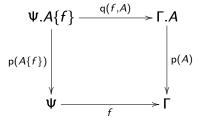
commutes. Moreover, $\Delta_{\Gamma}(t)$ is independent of t and

$$\Delta_{\Gamma}(t) = 1_{\Gamma.A} \qquad (t ext{ proof of } A =_{\mathrm{Ty}(\Gamma)} A)$$

$$\Delta_{\Gamma}(s \circ t) = \Delta_{\Gamma}(s) \circ \Delta_{\Gamma}(t)$$
 (t pf. of $A =_{\mathrm{Ty}(\Gamma)} A'$ and s pf. of $A' =_{\mathrm{Ty}(\Gamma)} A'$

(A particular feature of the slice of an e-category is that equalities of objects over the base turn into isomorphism.)

(d) For each $f: \Psi \longrightarrow \Gamma$ and $A \in \mathrm{Ty}(\Gamma)$, there is a morphism $q(f,A): \Psi.A\{f\} \longrightarrow \Gamma.A$ in $\mathcal C$ such that



is a pullback, and moreover these morphisms satisfy

- (d.1) $q(1_{\Gamma}, A) \circ \Delta_{\Gamma}(t) = 1_{\Gamma, A}$ where t is any pf. for $Ty(1_{\Gamma})(A) = A$.
- (d.2) $q(f \circ g, A) \circ \Delta_{\Theta}(t) = q(f, A) \circ q(g, A\{f\}) \text{ for } \Theta \xrightarrow{g} \Psi \xrightarrow{f} \Gamma$ and where t is any pf. for $A\{f \circ g\} =_{\mathrm{Ty}(\Theta)} A\{f\}\{g\}$.

Note the type correcting isomorphisms $\Delta(t)$.

Further in condition (d), note that if $f = f' : \Psi \longrightarrow \Gamma$, s is a proof of $A = A' \in \mathrm{Ty}(\Gamma)$ and t is a proof of $A\{f\} = A'\{f'\}$, then by the pullback properly,

$$\mathsf{q}(f,A)\circ\Delta_{\Psi}(t)=\Delta_{\Gamma}(s)\circ\mathsf{q}(f',A').$$

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Equality on finite contexts. Note that if $f: \Psi \longrightarrow \Gamma$ is an isomorphism, and $A \in \mathrm{Ty}(\Psi)$, $B \in \mathrm{Ty}(\Gamma)$ and t is proof of $A =_{\mathrm{Ty}(\Psi)} B\{f\}$, then get isomorphism:

$$q(f,B) \circ \Delta(t) : \Psi.A \longrightarrow \Gamma.B$$

Thus we can inductively define when two finite contexts are equal.

Interpretation

The definition of ecwas suggests introducing the following judgements about types

- ▶ $\Gamma \vdash A$ type meaning $A \in Ty(\Gamma)$
- ▶ $\Gamma \vdash A = A'$ meaning $A =_{\mathrm{Ty}(\Gamma)} A'$ where $A, A' \in \mathrm{Ty}(\Gamma)$.

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Define $\operatorname{Tm}(\Gamma, A)$, the elements of A in the context Γ , to be the setoid of sections of $p(A): \Gamma.A \longrightarrow \Gamma$.

Note that if r is a proof for $A =_{\mathrm{Ty}(\Gamma)} A'$, then $M \in \mathrm{Tm}(\Gamma, A)$ implies $\Delta_{\Gamma}(r) \circ M \in \mathrm{Tm}(\Gamma, A')$.

(Cf. canonical isomorphisms of Curien 1993.)

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- ▶ $\Gamma \vdash (a, A) : A'$ meaning $a \in \operatorname{Tm}(\Gamma, A)$ and $A =_{T(\Gamma)} A'$.
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 $\Gamma \vdash (a, A) : A''$ and $\Gamma \vdash (a, A') : A''$ and that there is a proof r of $A =_{\mathrm{Ty}(\Gamma)} A'$ such that $\Delta_{\Gamma}(r) \circ a =_{\mathrm{Tm}(\Gamma, A')} a'$.

The rules

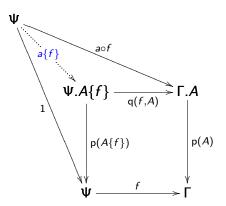
$$\frac{\Gamma \vdash s : A \qquad \Gamma \vdash A = B}{\Gamma \vdash s : B}$$

$$\frac{\Gamma \vdash s = t : A \qquad \Gamma \vdash A = B}{\Gamma \vdash s = t : B}$$

are immediately justified.

Substitution into terms

For $a \in \operatorname{Tm}(\Gamma, A)$ and $f : \Psi \longrightarrow \Gamma$, define $a\{f\} : \Psi \longrightarrow \Psi.A\{f\}$ as the unique morphism (see diagram below) with $\operatorname{p}(A\{f\})) \circ a\{f\} = 1_{\Psi}$ and $\operatorname{q}(f,A) \circ a\{f\} = a \circ f$. Thus $a\{f\} \in \operatorname{Tm}(\Psi, A\{f\})$.



Stability under substitution

Thus if

$$\Gamma \vdash (a, A) : A'$$

and $f:\Psi\longrightarrow \Gamma$, we have $a\{f\}\in \mathrm{Tm}(\Psi,A\{f\})$ and $A\{f\}=_{\mathrm{Ty}(\Psi)}A'\{f\}$, so

$$\Psi \vdash (a\{f\}, A\{f\}) : A'\{f\}$$

Thus the rule

$$\frac{\Gamma \vdash s : A' \qquad f : \Psi \longrightarrow \Gamma}{\Psi \vdash s\{f\} : A'\{f\}}$$

is justified.

$$\Gamma \vdash (a, A) = (a', A') : A''$$

we have $\Delta_{\Gamma}(r) \circ a =_{\mathrm{Tm}(\Gamma, A')} a'$ for some proof r of $A =_{\mathcal{T}(\Gamma)} A'$.

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$$\Delta_{\Gamma}(r) \circ a \circ f = a' \circ f$$

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It follows by uniqueness that $\Delta_{\Psi}(s) \circ a\{f\} = a'\{f\}$, so indeed

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$$\Psi \vdash (a\{f\}, A\{f\}) = (a'\{f\}, A'\{f\}) : A''\{f\}.$$

This justifies also the rule:

$$\frac{\Gamma \vdash s = t : A'' \qquad f : \Psi \longrightarrow \Gamma}{\Psi \vdash s\{f\} = t\{f\} : A''\{f\}}$$

The notion of Π -types is similar for cwas as for cwfs, but note that elements are particular context maps. For ecwas we need to make appropriate changes.

An ecwa supports Π -types if for $A \in \mathrm{Ty}(\Gamma)$ and $B \in \mathrm{Ty}(\Gamma.A)$ there is a type $\Pi(A,B) \in \mathrm{Ty}(\Gamma)$, and moreover for every $b \in \mathrm{Tm}(\Gamma.A,B)$ there is an element $\lambda_{A,B}(b) \in \mathrm{Tm}(\Gamma,\Pi(A,B))$, and furthermore for any $M \in \mathrm{Tm}(\Gamma,\Pi(A,B))$ and any $a \in \mathrm{Tm}(\Gamma,A)$ there is an element $\mathrm{App}_{A,B}(c,a) \in \mathrm{Tm}(\Gamma,B\{a\})$, such that the following equations hold for any $f:\Psi \longrightarrow \Gamma$:

- $(\beta$ -red) App_{A,B} $(\lambda_{A,B}(b), a) =_{\operatorname{Tm}(\Gamma,B\{a\})} P\{a\}$, [as before]
- (Π -subst) $\Pi(A, B)\{f\} =_{\mathrm{Ty}(B)} \Pi(A\{f\}, B\{q(f, A)\})$, [as before]

This part of the definition has type adjustments:

 \triangleright (λ -subst)

$$\lambda_{A,B}(b)\{f\} =_{\operatorname{Tm}(B,\Pi(A,B)\{f\})} \Delta_B(t) \circ \lambda_{A\{f\},B\{q(f,A)\}}(b\{q(f,A)\}),$$

for any proof t of $\Pi(A\{f\},B\{q(f,A)\})=_{\mathrm{Ty}(B)}\Pi(A,B)\{f\}$,

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for any proof t of $\Pi(A\{f\}, B\{q(f,A)\}) =_{\mathrm{Ty}(B)} \Pi(A,B)\{f\}$,

(App-subst)

$$\mathsf{App}_{A,B}(c,a)\{f\} =_{\mathrm{Tm}(\cdots)} \Delta_{\Psi}(s) \circ \mathsf{App}_{A\{f\},B\{\mathsf{q}(f,A)\}}(\Delta_{\Psi}(t) \circ c\{f\},a\{f\})$$
 for any proof s of $B\{\mathsf{q}(f,A)\}\{a\{f\}\} =_{\mathrm{Ty}(\Psi)} B\{a\}\{f\}$ and

any proof t of $\Pi(A, B)\{f\} =_{\operatorname{Tv}(\Psi)} \Pi(A\{f\}, B\{q(f, A)\}).$

Furthermore there are the following extensionality conditions on Π , λ and App:

▶ (Π -cong) if s is a proof of $A =_{\mathrm{Ty}(\Gamma)} A'$ and for $B \in \mathrm{Ty}(\Gamma.A)$, $B' \in \mathrm{Ty}(\Gamma.A')$ with $B =_{\mathrm{Ty}(\Gamma.A)} B'\{\Delta(s)\}$, then

$$\Pi(A,B) =_{\mathrm{Ty}(\Gamma)} \Pi(A',B').$$

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• $(\lambda$ -cong) if $b =_{\mathrm{Tm}(\Gamma,A,B)} b'$, then

$$\lambda_{A,B}(b) =_{\mathrm{Tm}(\Gamma,\Pi(A,B))} \lambda_{A,B}(b')$$

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$$\lambda_{A,B}(b) =_{\mathrm{Tm}(\Gamma,\Pi(A,B))} \lambda_{A,B}(b')$$

▶ (App-cong) if $c =_{\operatorname{Tm}(\Gamma,\Pi(A,B))} c'$ and $a =_{\operatorname{Tm}(\Gamma,A)} a'$ then

$$\Delta_{\Gamma}(s) \circ \mathsf{App}_{A,B}(c,a) =_{\mathrm{Tm}(\Gamma,B\{a'\})} \mathsf{App}_{A,B}(c',a'),$$

where s is any proof of $B\{a\} =_{\mathrm{Tv}(\Gamma)} B\{a'\}$.

Bounded categories with attributes

Here all types are subobjects of a fixed setoid, the bounding setoid, thought of a setoid of raw types. This brings the possibility to eliminate the type correcting Δs in the formulation above.

A category (with equality) on objects can be formulated essentially algebraically in type theory with three setoids $\mathrm{Ob}(\mathcal{C})$, $\mathrm{Arr}(\mathcal{C})$ and $\mathrm{Cmp}(\mathcal{C})$ of object, arrows and composable arrows, respectively. There are extensional functions

$$1: Ob \longrightarrow Arr$$

$$dom, cod: Arr \longrightarrow Ob$$

$$cmp: Cmp \longrightarrow Arr \qquad fst, snd: Cmp \longrightarrow Arr$$
 satisfying familiar equations, with convention that

cod(fst(u)) = dom(snd(u)).

Let B be a setoid. The class of subobjects (or subsetoids) of B (denoted $\operatorname{Sub}(B)$) consists of all pairs $S=(\underline{S},m_S)$ where \underline{S} is a setoid, and $m_S:\underline{S}\longrightarrow S$ is an extensional injective function. An element $x\in B$ is a member of S if there is some $u\in\underline{S}$ with $m_S(u)=_B x$, and this case we write

$$x \in S$$
.

From this we get the usual derived relations \doteq and \subseteq between subobjects. A binary relation on B is a subobject R of $B \times B$. Composition $R \circ S$ of such binary relations is defined as expected. For subsetoids X, Y of B. We say that $R \in \operatorname{Sub}(B \times B)$ is relation from X to Y, if $(x, y) \in R$ implies $x \in X$ and $y \in Y$. The relation is total on X if $x \in X$ implies that $(x, y) \in R$ for some $y \in B$. A relation is functional if

$$(x,y) \in R, (x,z) \in R \Longrightarrow y =_B z.$$

A total functional relation from A to B is in a certain sense equivalent to a setoid extensional function $\underline{A} \longrightarrow \underline{B}$, but we shall not utilize this below.

Definition. A bounded category with attributes (bcwa) consists of the following data

- (a) A category C with a terminal object \top .
- (b1) A set B thought of a set of raw types.
- (b2) An extensional function $Ty : Ob(\mathcal{C}) \longrightarrow Sub(B)$.
- (b3) An extensional function $Ty : Arr(C) \longrightarrow Sub(B \times B)$ such that
 - ▶ for each $f \in Arr(C)$, Ty(f) is a total functional relation from Ty(cod(f)) to T(dom(f)),
 - for each $\Gamma \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{T}(1_{\Gamma})$ is the identity relation on $\mathrm{Ty}(\Gamma)$
 - ▶ for each $u \in \text{Cmp}(\mathcal{C})$,

$$\operatorname{Ty}(\operatorname{cmp}(u)) = \operatorname{Ty}(\operatorname{fst}(u)) \circ \operatorname{Ty}(\operatorname{snd}(u))$$

(c) For each $\Gamma \in \mathrm{Ob}(\mathcal{C})$, each $A \in B$, and each proof $q: (A \dot{\in} \mathrm{Ty}(\Gamma))$, an arrow $\mathsf{p}_{\Gamma}(A,q) \in \mathrm{Arr}(\mathcal{C})$ with $\mathrm{cod}(\mathsf{p}_{\Gamma}(A,q)) = \Gamma$ satisfying the extensionality condition

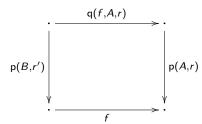
$$\Gamma = \Gamma', A = A' \Longrightarrow p_{\Gamma}(A, q) = p_{\Gamma'}(A', q').$$

Write $\operatorname{Ext}(\Gamma, A, q)$ or $\Gamma.A$ for $\operatorname{dom}(\mathsf{p}_{\Gamma}(A, q))$

(d) For each $f: \Delta \longrightarrow \Gamma$ and $A \in B$ with $r: (A \dot{\in} \operatorname{Ty}(\operatorname{dom}(f)))$, there is a $\operatorname{q}_{\Gamma}(f,A,r) \in \operatorname{Arr}(\mathcal{C})$ satisfying the extensionality condition

$$\Gamma = \Gamma', A = A', f = f' \Longrightarrow q_{\Gamma}(f, A, r) = q_{\Gamma'}(f', A', r').$$
and $q(f, A, r) = q_{\Gamma}(f, A, r) \in A_{\Gamma}(f)$ such that, for $P \in P$ a

and $q(f,A,r) = q_{\Gamma}(f,A,r) \in Arr(\mathcal{C})$ such that, for $B \in B$ and $r' : ((A,B) \dot{\in} Ty(f))$,



is well composed and is a pullback, and furthermore these functoriality conditions hold:

(d.1)
$$q(1_{\Gamma}, A, r) = 1_{\Gamma, A}$$

(d.2) $q(\operatorname{cmp}(u), A, r) = q(\operatorname{fst}(u), A, r') \circ q(\operatorname{snd}(u), B, r'')$ for $u \in \operatorname{Cmp}(\mathcal{C})$ and $((A, B) \in \mathcal{T}(\operatorname{fst}(u)))$

Conjecture: Every type generated ecwa can be turned in to a bcwa and this can be formalized in Coq.

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